The Resource Lambda Calculus Is Short-Sighted in Its Relational Model

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Abstract. Relational semantics is one of the simplest and categorically most natural semantics of Linear Logic. The co-Kleisli category MRel associated with its multiset exponential comonad contains a fully abstract model of the untyped λ-calculus. That particular object of MRel is also a model of the resource λ-calculus, deriving from Ehrhard and Regnier's differential extension of Linear Logic and related to Boudol's λ-calculus with multiplicities. Bucciarelli et al. conjectured that model to be fully-abstract also for the resource λ-calculus. We give a counter-example to the conjecture. As a by-product we achieve a context lemma for the resource λ-calculus.

Keywords: Full abstraction, resource λ-calculus, linear logic, nondeterminism.

1 Introduction

Rel. The category Rel of set and relations is known to model Linear Logic, and its construction is canonical from categorical point of view. Indeed, Rel can be seen as the free infinite biproduct completion of the Boolean ring seen as a category with one object and two morphisms (true and false), the conjunction being the identity [13]. The exponential modality ! of linear logic is given by the finite multisets comonad that precisely is the free commutative comonad in Rel [13]. Moreover, despite the biproduct, proofs are morally preserved, *i.e.* the interpretation of cut free proofs is injective up to isomorphism[1] [7].

This multiset comonoid $!A$ of a set A is the set of finite multisets of elements in A. Intuitively a finite multiset in $a \in !A$ is a resource that behaves as $\parr_{\alpha\in a}\alpha$, *i.e.* like a resource that must be used by a program exactly once per element in a (with multiplicities). This behavior enabling an interesting resource management, it was natural to develop a syntactical counterpart.

Resource λ-Calculus. A restricted version was previously introduced by Boudol in 1993 [1]. Boudol's resource λ-calculus extends the call-by-value λ-calculus with a special resource sensitive application (able to manage finite resources) that involves multisets of affine arguments each one used at most once. Independently from our considerations on Rel, this was seen as a natural

[1] Up to technical details, but the unrestricted injectivity is strongly conjectured.

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way to export resource sensitiveness into the functional setting. However, restricted by a fixed evaluation strategy, it was not fully explored. Later on, Ehrhard and Regnier, working on the implementation of behaviors discovered in Rel, came to a similar calculus, the differential λ-calculus [11], which enjoys many syntactical and semantical properties (confluence, Taylor expansion). In Ehrhard and Regnier's differential λ-calculus the resource-sensitiveness is obtained by adding to the λ-calculus a derivative operation $\frac{\partial M}{\partial x}(N)$ (will be implemented in our notations as the term $M\langle N/x\rangle$, see section 2). This operator syntactically corresponds to a substitution of exactly one occurrence of x by N in M (introducing non determinism on the choice of the substituted occurrence); confluence of the calculus is recovered, then, by performing all the possible choices at once. This linear substitution takes place when β-reducing specific applications where an argument is marked as linear, in order to be used exactly once. We will adopt the syntax of [16] that re-implements improvements from differential λ-calculus into Boudol's calculus, and we will call it resource λ-calculus or $\partial\lambda$-calculus.

MRel. For Rel as for most categorical models of Linear Logic, the interpretation of the exponential modality induces a comonad from which we can construct the Kleisli category that contains a model of the λ-calculus. In the case of Rel, this category, MRel, corresponds to the category whose objects are sets and whose morphisms from A to B are the relations from $\mathcal{N}\langle A\rangle$ (the set of finite multisets over A) to B. It is then a model of both λ and $\partial\lambda$-calculi. This construction being very natural, the reflexive objects of MRel are the most-studied models of the $\partial\lambda$-calculus.

MRel and $\partial\lambda$-Calculus. The depth of the connection between the reflexive objects of MRel and the $\partial\lambda$-calculus is precisely the purpose of our work. More precisely, we investigate the question of the full abstraction of $\mathcal{M}_\infty$, a reflexive object for the $\partial\lambda$-calculus [5]. We also endowed $\partial\lambda$-calculus with a particular choice of reduction that is the may-outer-reduction; this is not the only choice, but this corresponds to the intuition that conducts from Rel to Ehrhard-Regnier's differential calculus. Until now we knew that $\mathcal{M}_\infty$ was adequate for the $\partial\lambda$-calculus [3], *i.e.* that two terms carrying the same interpretations in $\mathcal{M}_\infty$ behave the same way in all contexts. But we did not know anything about the converse, the completeness, and thus about the full abstraction.

Full Abstraction. The full abstraction of $\mathcal{M}_\infty$ has been thoroughly studied. For lack of direct results, the full abstraction has been proved for restrictions and extensions of the $\partial\lambda$-calculus: for the untyped λ-calculus (which is the deterministic and linear-free fragment of the $\partial\lambda$-calculus); for the orthogonal bang-free restriction where the application only accepts bags of linear arguments; and for the extension with tests of [3], an extension with must non-determinism and with operators inspired by 0-ary *par* and *tensor product* that could be added freely in DiLL-proof nets (DiLL for Differential Linear Logic).

These studies were encouraging since they systematically showed MRel to be fully abstract for these calculi ([14] for untyped λ-calculus , [4] for the bang-free restriction and [3] for resource λ-calculus with tests). Therefore Bucciarelli *et.al.* [3] conjectured a full abstraction for the $\partial\lambda$-calculus.

The Counter-Example. The purpose of this article is to set out a highly unexpected counter-example to this conjecture. We will see how an untyped fixpoint and a may non-deterministic sum can combine to produce a term $\boldsymbol{A}$ (Equation 7) behaving like an infinite sum $\Sigma_{i\geq 1}\boldsymbol{B}_i$ where every $\boldsymbol{B}_i$ begins with $(i{+}1)$ λ-abstractions, put its $(i{+}1)^{th}$ argument in head position but otherwise behave as the identity in applicative contexts with exactly i arguments; that how $\boldsymbol{A}$ can be thought to have an arbitrary number of λ-abstractions. Such a term can thus look for an argument further than the length of any bounded applicative context. There lies the immediate interest of achieving a context lemma (which have not been done for this calculus, yet) in order to prove that the observational equivalence is so short-sighted. This will refute the inequational full abstraction since the relational semantics can sublimate this short-sightedness. More concretely we will see that $\boldsymbol{A}$ is observationally above the identity but not denotationally. It is not difficult, then, to refute the equational full abstraction.

We proceed in this order. Section 2 present the $\partial\lambda$-calculus and its properties. Section 3 describes MRel and its reflexive object $\mathcal{M}_\infty$, and see how it is related to $\partial\lambda$-calculus. Section 4 gives our results with the context lemma followed by the counter-example (Theorem 8). We will also discuss the generality of this counter-example in the conclusion and explain how it is representative of an unhealthy interaction between untyped fixpoints and *may-non-determinism* that can be reproduced in other calculi like the *may-non-deterministic* extension of λ-calculus.

Notation: We denote $\mathcal{N}\langle A\rangle$ for the set of finite multisetes of elements in the set A.

2 Syntax

2.1 $\partial\lambda$-Calculus

In this section we give some background on the $\partial\lambda$-calculus, a lambda calculus with resources. The grammar of its syntax is the following:

$$\begin{array}{lllllll}
\text{(terms)} & \Lambda: & L,M,N ::= & x & \mid \ \lambda x.M & \mid \ M\ P & \\
\text{(bags)} & \Lambda^b: & P,Q ::= & 1 & \mid \ [M] & \mid \ [M^!] & \mid \ P\cdot Q \\
\text{(sums)} & \mathbb{\Lambda},\mathbb{\Lambda}^b: & & \mathbb{L},\mathbb{M}\in\mathcal{N}\langle\Lambda\rangle & & \mathbb{P},\mathbb{Q}\in\mathcal{N}\langle\Lambda^b\rangle &
\end{array}$$

Fig. 1. Grammar of the $\partial\lambda$-calculus

The $\partial\lambda$-calculus extends the standard λ-calculus in two directions. First, it is a non deterministic λ-calculus. The argument of an application is a superposition of inputs, called *bag of resources* and denoted by a multiset in multiplicative notation (namely $P{\cdot}Q$ is the disjoint union of P and Q). Symmetrically, the result of a reduction step is a superposition of outputs denoted by a multiset in additive notation (namely $\mathbb{L}+\mathbb{M}$ is the disjoint union of $\mathbb{L}$ and $\mathbb{M}$). We also have empty multisets, expressing an absence of available inputs (denoted by 1) or of results (denoted by 0).

Second, the $\partial\lambda$-calculus distinguishes between *linear* and *reusable* resources. The formers will never suffer any duplication or erasing regardless the reduction strategy. A reusable resource will be denoted by a *banged* term $M^!$ in a bag, e.g. $[N^!, L, L]$ is a bag of two linear occurrences of the resource L and a reusable occurrence of the resource N. We use the notation $N^{(!)}$ whenever we do not set out whether M occurs linearly or not in a bag. A bag with no *banged resources* will be said *linear* and one with only *banged resources* will be said *exponential*.

Finally, keeping all possible results of a reduction step (with multiplicities) into a finite multiset $\Sigma_i M_i$ of outcomes allows to have a confluent rewriting system in such a non-deterministic setting [16].

Small Latin letters $x, y, z, ...$ will range over an infinite set of λ-calculus variables. Capital Latin letters L, M, N (resp. P, Q, R) are meta-variables for terms (resp bags). Initial capital Latin letters E, F will denote indifferently terms and bags and will be called *expressions*. Finally, the meta-variables $\mathbb{L}, \mathbb{M}, \mathbb{N}$ (resp $\mathbb{P}, \mathbb{Q}, \mathbb{R}$) vary over sums (*i.e.* multisets in additive notation) of terms (resp. bags). Bags and sums are multisets, so we are assuming associativity and commutativity of the disjoint union and neutrality of the empty multiset.

Notice that the sum operator is always at the top level of the syntax trees. This is a design choice taken from [16] allowing for a lighter syntax. However, it is sometimes convenient to write sums inside an expression as a short notation for the expression obtained by distributing the sums following the conventions:

$$\begin{array}{ll} \lambda x.(\Sigma_i M_i) := \Sigma_i(\lambda x.M_i) & (\Sigma_i M_i)\ (\Sigma_j P_j) := \Sigma_{i,j}(M_i\ P_j) \\ [(\Sigma_i M_i)^!]\cdot P\ :=\ [M_1^!, \dots, M_n^!]\cdot P & [\Sigma_i M_i]\cdot P\ :=\ \Sigma_i [M_i]\cdot P \end{array}$$

Notice that every construct is (multi)-linear but the bang $(\cdot)^!$, where we apply the linear logic equivalence $[(M+N)^!] = [M^!]\cdot[N^!]$ which is reminiscent of the standard exponential rule $e^{a+b} = e^a \cdot e^b$. Notice moreover that the 0-ary version of those rules also hold.

Since we have two kinds of resources, we need two different substitutions: the usual one, denoted $\{.\}$, and the linear one, denoted $\langle . \rangle$. Supposing that $x \neq y$, and $x \neq z$, and $z \notin \mathrm{FV}(N)$ (FV denoting free variables):

$$\begin{array}{c} x\langle N/x\rangle := N \qquad y\langle N/x\rangle := 0 \qquad (\lambda z.M)\langle N/x\rangle := \lambda z.(M\langle N/x\rangle) \\ (M\ P)\langle N/x\rangle := (M\langle N/x\rangle\ P) + (M\ P\langle N/x\rangle) \\ [M^!]\langle N/x\rangle := [M\langle N/x\rangle, M^!] \qquad [M]\langle N/x\rangle := [M\langle N/x\rangle] \\ (P\cdot Q)\langle N/x\rangle := (P\langle N/x\rangle)\cdot Q + P\cdot(Q\langle N/x\rangle) \qquad 1\langle N/x\rangle := 0 \end{array}$$

Notice that in the above definition we are heavily using the natural convention of the distributing sums. For example, the bag $P = [x^!, y]\langle N/x\rangle$ reduces $P = [x\langle N/x\rangle, x^!, y] + [x^!, y\langle N/x\rangle] = [N, x^!, y] + [x^!, 0] = [N, x^!, y] + 0 = [N, x^!, y]$.

Substitutions enjoy the following commutation properties:

Lemma 1 ([16]). *For an expression E and terms M, N, if $x \notin \mathrm{FV}(N)$ and if $y \notin \mathrm{FV}(M)$ (potentially $x=y$) then:*

$$E\langle M/x\rangle\langle N/y\rangle = E\langle N/y\rangle\langle M/x\rangle \qquad E\{(M+x)/x\}\langle N/y\rangle = E\langle N/y\rangle\{(M+x)/x\}$$
$$E\{(M+x)/x\}\{(N+y)/y\} = E\{(N+y)/y\}\{(M+x)/x\}\}$$

Hence the notion of substitution of variables by bags, denoted $\langle\langle s\rangle\rangle$ (where s is a list of substitutions P/x), may be defined as follows (if $x \notin \mathrm{FV}(N) \cup \mathrm{FV}(P)$):

$$M\langle\langle 1/x\rangle\rangle := M\{0/x\} \qquad M\langle\langle [N^!]\cdot P/x\rangle\rangle := M\{(x+N)/x\}\langle\langle P/x\rangle\rangle$$
$$M\langle\langle [N]\cdot P/x\rangle\rangle := M\langle N/x\rangle\langle\langle P/x\rangle\rangle \qquad M\langle\langle s_1; s_2\rangle\rangle := M\langle\langle s_1\rangle\rangle\langle\langle s_2\rangle\rangle$$

2.2 Beta and *Outer Reduction*

Reduction is defined essentially as the contextual closure of the β-rule.

$$\frac{}{(\lambda x.M)\ P \to M\langle\langle P/x\rangle\rangle}\,\beta \qquad \frac{M \to \mathbb{M}}{M\ P \to \mathbb{M}\ P}\,\text{left} \qquad \frac{M \to \mathbb{M}}{\lambda x.M \to \lambda x.\mathbb{M}}\,\text{abs}$$
$$\frac{N \to \mathbb{N}}{M\ [N]\cdot P \to M\ [\mathbb{N}]\cdot P}\,\text{lin} \qquad \frac{N \to \mathbb{N}}{M\ [N^!]\cdot P \to M\ [\mathbb{N}^!]\cdot P}\,!$$
$$\frac{M \to \mathbb{M}' \qquad \mathbb{N} \to \mathbb{N}'}{M+\mathbb{N} \to \mathbb{M}'+\mathbb{N}'}\,s_1 \qquad \frac{M \to \mathbb{M}'}{M+\mathbb{N} \to \mathbb{M}'+\mathbb{N}}\,s_2$$

Fig. 2. Reduction rules

Rules s_1 and s_2 allow to reduce one or more terms of a sum in a single step (this is used in Theorem 1).

In the following example and all along this article we denote:

$$\boldsymbol{\omega} := \lambda x.x[x^!] \qquad \boldsymbol{I} := \lambda x.x \qquad \boldsymbol{\Delta} := \lambda gu.u\ [(g\ [g^!]\ [u^!])^!] \qquad \boldsymbol{\Theta} := \boldsymbol{\Delta}[\boldsymbol{\Delta}^!]$$

Example 1.

$$\boldsymbol{I}\ [u^!, v^!] \to u+v \qquad (\lambda x.y\ [(x\ [y])^!])\ [u, v^!] \to y\ [u\ [y], (v\ [y])^!] \tag{1}$$
$$(\lambda x.x\ [x, x^!])\ [u, v^!] \to (u\ [v, v^!])+(v\ [u, v^!])+(v\ [v, u, v^!]) \tag{2}$$
$$u\ [\boldsymbol{I}\ 1] \to 0 \qquad u\ [(\boldsymbol{I}\ 1)^!] \to u\ 1 \tag{3}$$
$$\boldsymbol{\omega}\ [\boldsymbol{\omega}^!] \to \boldsymbol{\omega}\ [\boldsymbol{\omega}^!] \qquad \boldsymbol{\omega}\ [\boldsymbol{\omega}] \to 0 \tag{4}$$
$$\boldsymbol{\Theta}\ [v^!] \to^2 (v\ [(\boldsymbol{\Theta}\ [v^!])^!]) \tag{5}$$
$$\boldsymbol{\Theta}\ [u, v^!] \to^2 (u\ [(\boldsymbol{\Theta}\ [v^!])^!]) + (v\ [(\boldsymbol{\Theta}[u, v^!]), (\boldsymbol{\Theta}[v^!])^!]) \tag{6}$$

As customary, a notion of convergence will be used for relating the operational and denotational semantics of the $\partial\lambda$-calculus.

In this paper, we consider the *may-outer convergence* of [16]. The attribute *may* refers to an angelic notion of non-determinism, hence $M+N$ will converge whenever at least one of the two converges. Indeed, the demonic (must) convergence is also of great interest, however it is harder to deal with (see [17]), in fact the demonic non-determinism does not interact well with the Taylor expansion, which is a crucial tool in our analysis (Section 2.3). Moreover, the attribute *outer* refers to the fact that we reduce only redexes not under the scope of a bang. This turns out to be the analogous of the head-reduction in the λ-calculus.

Definition 1 (*onf* and *monf*). *A term is in* outer-normal form*,* onf *for short, iff it has no redexes but under a* !*, that is a term of the form:*

$$\lambda x_1,\dots,x_m.y\ [N_{1,1}^{(!)},\dots,N_{1,k_1}^{(!)}]\cdots[N_{n,1}^{(!)},\dots,N_{n,k_n}^{(!)}]$$

Where every $N_{i,j}^{(!)}$ *are either* banged *or in* outer-normal form*.*
A sum of terms is in may-outer-normal form*,* monf *for short, iff at least one of its addends is in* outer-normal form *(in particular* 0 *is not a* monf*).*

This notion generalizes the one of head-normal form of the untyped lambda calculus. Asking for linear terms of a bag to be in *monf* is a way of expressing that $x\ [\boldsymbol{\omega}\ [\boldsymbol{\omega}^!]]$ diverges while $x\ [(\boldsymbol{\omega}\ [\boldsymbol{\omega}^!])^!]$ is an *onf*. *Monf*s correspond to *may-solvability* [17] in the same way as head-normal-forms correspond to solvability in untyped λ-calculus. From previous examples only contracta of (3.1), (4.1) and (4.2) are not *monf*, and only (3.2)'s redex is.
The restricted reduction leading to the (principal) *monf* of a term is the following:

Definition 2. *The* outer reduction*, denoted* $\to_o$ *is defined by the rules of Figure 2 but the rule* !*, which is omitted. We denote by* $\to^*$ *and* $\to_o^*$ *the reflexive and transitive closures of* $\to$ *and* $\to_o$*, respectively.*

In the Example 1, all reductions but the (3.2) are *outer reductions*.

Lemma 2 ([16]). *If* $M \to^* \mathbb{M}$ *and* $\mathbb{M}$ *is in* monf*, then there exists a* monf $\mathbb{N}$ *such that* $M \to_o^* \mathbb{N} \to^* \mathbb{M}$*. Thus the convergence to a* monf *and the* outer convergence *to a* monf *coincide.*

We will write $M \Downarrow_n$ if there exists a *monf* $\mathbb{M}$ and an *outer reduction* sequence from M to $\mathbb{M}$ of length at most n. We will write $M \Downarrow$ if there exists n such that $M \Downarrow_n$ and say that M *outer converges*. Finally we will write $M \Uparrow$ for the *outer-divergence* on M.
The two rules s_1 and s_2 of Figure 2 allow the followings:

Theorem 1. *If* $M \to \mathbb{M}_1$ *and* $M \to \mathbb{M}_2$ *for* $\mathbb{M}_1, \mathbb{M}_2 \neq 0$*, then there is* $\mathbb{N}$ *such that* $\mathbb{M}_1 \to \mathbb{N}$ *and* $\mathbb{M}_2 \to \mathbb{N}$*.*

Corollary 1. *If* $M \Downarrow_{n+1}$ *and* $M \to_o \mathbb{N}$ *then exists* $N \in \mathbb{N}$ *such that* $N \Downarrow_n$*.*

This is due to the trivial divergence of the case $M = 0$. Notice moreover that M is not a sum.

2.3 Taylor Expansion

A natural restriction of the $\partial\lambda$-calculus is the fragment $\partial\lambda^\ell$ which is obtained by removing the bang construction $[M^!]$ in Figure 1. This restriction has a very limited computational power, for instance it enjoys the following theorem.

Theorem 2 ([Folklore]). *The reduction $\to$ in $\partial\lambda^\ell$ is strongly normalizing.*

Proof. We set an order $\sqsubseteq$ on the finite multisets of terms generated by $\mathbb{M} \sqsubseteq \mathbb{N}$ if $\mathbb{M} = \mathbb{M}'+\mathbb{L}$, $\mathbb{N} = \mathbb{N}'+\mathbb{L}$ and there exists $N \in \mathbb{N}'$ such that for all $M \in \mathbb{M}'$, the inequality $|M| \leq |N|$ (where $|M|$ is the structural size of M) holds. Then, $\to$ is strictly decreasing in this well founded order. □

The main interest of $\partial\lambda^\ell$ comes with the Taylor expansion. The Taylor expansion of a λ-term M has been developed in [11,12] and it recalls the usual decomposition of an analytic function:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(f)(0)x^n$$

In this paper, we are interested only in the support of the Taylor expansion of a $\partial\lambda$-term M defined in [11,12], *i.e.* in the set M^o of the $\partial\lambda^\ell$-terms appearing in the Taylor expansion of M with non-null coefficient. Such a set can be defined as follows.

Definition 3. *The Taylor expansion E^o of an expression E is a (possibly infinite) set of linear expressions defined by structural induction:*

$$(\lambda x.M)^o := \{\lambda x.M' | M' \in M^o\} \qquad (M\ P)^o := \{M'P' | M' \in M^o,\ P' \in P^o\}$$
$$[M]^o := \{[M'] | M' \in M^o\} \qquad (P{\cdot}Q)^o := \{P'{\cdot}Q' | P' \in P^o,\ Q' \in Q^o\}$$
$$[M^!]^o := \{[M_1, \dots, M_n] | n \geq 0,\ M_1, \dots, M_n \in M^o\} \quad 1^o := \{1\} \quad x^o := \{x\}$$

In the following we use set inclusion for comparing a finite multiset $\mathbb{N}$ with a set $\mathbb{M}^o$. This mean that the support (*i.e.* the set of element appearing in $\mathbb{N}$ with a nonzero multiplicity) of $\mathbb{N}$ is a subset of $\mathbb{M}^o$.

Lemma 3. *For any sum $\mathbb{M}$ and for any $\mathbb{N} \subseteq \mathbb{M}^o$, if $\mathbb{N}$ converges to a normal form $\mathbb{N}'$ then there exists $\mathbb{M}'$ such that $\mathbb{M} \to^* \mathbb{M}'$ and $\mathbb{N}' \subseteq \mathbb{M}'^o$.*

Proof. By induction on the length of the longest path of reduction for $\mathbb{N} \to^* \mathbb{N}'$ (indeed such a path exists by Theorem 2). The case $\mathbb{N} = \mathbb{N}'$ is trivial. We thus asume that $\mathbb{N} \to \mathbb{N}'' \to^* \mathbb{N}'$. Notice that $\mathbb{N} \to \mathbb{N}''$ does not use the rule s_1, otherwise there would be a longest reduction sequence from $\mathbb{N}$ to $\mathbb{N}'$ which contradicts the hypothesis that we have already choosen the largest path of reduction. This means that $\mathbb{N} \to \mathbb{N}''$ reduces a single redex. This redex being the image of a redex in $\mathbb{M}$ (not necessarily an outer redex), we can perform the corresponding reduction $\mathbb{M} \to \mathbb{M}''$. And by reducing all corresponding redexes on $\mathbb{N}''$ (the duplictions from the Taylor expension), we have $\mathbb{N}'' \to^* \mathbb{N}''' \subseteq \mathbb{M}''^o$. Then we conclude by induction hypothesis. □

3 Model

3.1 Categorical Construction of the Model

We recall the interpretation of the $\partial\lambda$-calculus into the reflexive object $\mathcal{M}_\infty$ of MRel used in [3]. MRel is the Cartesian closed category resulting from the co-Kleisli construction associated with the multiset exponential comonad of the category Rel of sets and relations, which is a well-known model of Linear Logic (and Differential Linear Logic). We refer to [9] for a detailed exposition, here we briefly present MRel and the object $\mathcal{M}_\infty$.

The objects of MRel are the sets. Its morphisms from A to B are the relations from the set of the finite multi-sets of A, namely $\mathcal{N}\langle A\rangle$, to the set B; *i.e.* $\mathbf{MRel}(A,B) := \mathcal{P}(\mathcal{N}\langle A\rangle \times B)$.

The composition of $g \in \mathbf{MRel}(B,C)$ and $f \in \mathbf{MRel}(A,B)$ is given by $f;g = \{(a,\gamma) \in \mathcal{N}\langle A\rangle \times C \mid \exists (a_1,\beta_1),...,(a_n,\beta_n) \in f,\ a = \Sigma_i a_i \text{ and } ([\beta_1,\dots,\beta_n],\gamma) \in g\}$

The identities are $\mathrm{id}_A := \{([\alpha],\alpha) | \alpha \in A\}$. Given a family $(A_i)_{i\in I}$, its Cartesian product is $\&_{i\in I} A_i := \{(i,\alpha) | i \in I, \alpha \in A_i\}$; with the projections $\pi_i := \{([(i,\alpha)],\alpha) | \alpha \in A_i\}$. The terminal object is the empty set. And the exponential object internalizing $\mathbf{MRel}(A,B)$ is $A \Rightarrow B := \mathcal{N}\langle A\rangle \times B$. Then the adjuction $\mathbf{MRel}(A \,\&\, B, C) \simeq \mathbf{MRel}(A, B \Rightarrow C)$ holds since $\mathcal{N}\langle \&_{i\leq n} A_i\rangle \simeq \prod_{i\leq n} \mathcal{N}\langle A_i\rangle$.

The reflexive object we choose is the simplest stratified object[2] of [14]. It can be recursively defined by (see [5]):

$$\mathcal{M}_0 := \emptyset \qquad \mathcal{M}_{n+1} := \mathcal{N}\langle \mathcal{M}_n\rangle^{(\omega)} \qquad \mathcal{M}_\infty := \bigcup_n \mathcal{M}_n$$

Where $\mathcal{N}\langle M\rangle^{(\omega)}$ is the list of almost everywhere empty multisets over M. Its elements can be generated by:

(elements)	$\mathcal{M}_\infty$:	$\alpha,\beta,\gamma ::=$	$* \mid a{::}\alpha$
(multisets)	$\mathcal{M}_\infty^b$:	$a,b,c ::=$	$[\alpha_1,\dots,\alpha_n]$

Where $*$, the unique element of $\mathcal{M}_1$, namely the infinite list of empty multisets, enjoys the equation:

$$* = []{::}*$$

The linear morphisms $\boldsymbol{app} \in \mathbf{MRel}(\mathcal{M}_\infty, \mathcal{M}_\infty \Rightarrow \mathcal{M}_\infty)$ and $\boldsymbol{abs} \in \mathbf{MRel}(\mathcal{M}_\infty \Rightarrow \mathcal{M}_\infty, \mathcal{M}_\infty)$ are defined by:

$$\boldsymbol{app} := \{([a{::}\alpha],(a,\alpha)) | (a,\alpha) \in \mathcal{M}_\infty\} \quad \boldsymbol{abs} := \{([(a,\alpha)], a{::}\alpha) | (a,\alpha) \in \mathcal{M}_\infty\}$$

One can easily check that $\boldsymbol{abs};\boldsymbol{app} = \mathrm{Id}_{\mathcal{M}_\infty \Rightarrow \mathcal{M}_\infty}$ (and even $\boldsymbol{app};\boldsymbol{abs} = \mathrm{Id}_{\mathcal{M}_\infty}$).

We could have interpreted the terms of the $\partial\lambda$-calculus by using the categorical structure of MRel. However, we prefer to give a description of such an

[2] Any other stratified object will also be subject to the counter-example since they share the crucial element $*$.

interpretation, using a non-idempotent intersection type system, following [10]. This type system has been introduced in [8].

The usual grammar of non-idempotent intersection types corresponds exactly to the grammar of $\mathcal{M}_\infty$. The cons operator (::) replaces the arrow and the multisets notation replaces the intersection notation. We will use the second one for uniformity consideration. The multisets of $\mathcal{M}_\infty^b$ will be denoted multiplicatively.

A *typing context* is a finite partial function from variables into multisets in $\mathcal{M}_\infty^b$, we denote $(x_i : a_i)_{i\in I}$ the context associating x_i to a_i for $i \in I$. We have two kinds of typing judgments, depending whether we type terms or bags: the former are typed by elements in $\mathcal{M}_\infty$ and the latter by multisets in $\mathcal{M}_\infty^b$.

$$\frac{\Gamma \vdash M : \alpha}{x:1, \Gamma \vdash M : \alpha} \qquad \frac{\Gamma \vdash P : a}{x:1, \Gamma \vdash P : a} \qquad \frac{}{x:[\alpha] \vdash x : \alpha} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash M + \mathbb{M} : \alpha}$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a{::}\alpha} \qquad \frac{(x_i : a_i)_{i\in I} \vdash M : b{::}\alpha \qquad (x_i : a'_i)_{i\in I} \vdash P : b}{(x_i : a_i{\cdot}a'_i)_{i\in I} \vdash M\ P : \alpha}$$

$$\frac{}{\vdash 1 : 1} \qquad \frac{(x_i : a_i)_{i\in I} \vdash P : b \qquad (x_i : a'_i)_{i\in I} \vdash Q : c}{(x_i : a_i{\cdot}a'_i)_{i\in I} \vdash P{\cdot}Q : b{\cdot}c}$$

$$\frac{(x_i : a_i)_{i\in I} \vdash L : \beta}{(x_i : a_i)_{i\in I} \vdash [L] : [\beta]} \qquad \frac{(x_i : a_i^j)_{i\in I} \vdash L : \beta_j \qquad \text{for } j \leq m}{(x_i : \Pi_{j\leq m} a_i^j)_{i\in I} \vdash [L^!] : [\beta_1, \ldots, \beta_m]}$$

The usual presentation of the interpretation can be recovered with:

$$[\![\mathbb{M}]\!]^{x_1,\ldots,x_n} := \{((a_1,\ldots,a_n),\beta) | (x_i : a_i)_i \vdash \mathbb{M} : \beta\} \quad \in \mathbf{MRel}(\&_{i=1}^{n} \mathcal{M}_\infty, \mathcal{M}_\infty)$$

$$[\![\mathbb{P}]\!]^{x_1,\ldots,x_n} := \{((a_1,\ldots,a_n),b) | (x_i : a_i)_i \vdash \mathbb{P} : b\} \quad \in \mathbf{MRel}(\&_{i=1}^{n} \mathcal{M}_\infty, \mathcal{M}_\infty^b)$$

Theorem 3. *If* $\mathbb{M} \to \mathbb{N}$ *then* $[\![\mathbb{M}]\!]^{x_1,\ldots,x_n} = [\![\mathbb{N}]\!]^{x_1,\ldots,x_n}$.

An important characteristic of this model that seems to make it particularly suitable for our original purpose is that it models the Taylor expansion:

Theorem 4 ([15]). *For any term* M, $[\![M]\!]^{\bar{x}} = \bigcup_{N\in M^\circ} [\![N]\!]^{\bar{x}}$.

3.2 Observational Order and Adequacy

A first important result relating syntax and semantic is the sensibility theorem, a corollary of [3], but here reproved focussing on the role of the Taylor expansion.

Theorem 5. $\mathcal{M}_\infty$ *is sensible for* may-outer-convergence *of the* $\partial\lambda$*-calculus,* i.e.

$$\forall M, \quad M \Downarrow \ \Leftrightarrow \ [\![M]\!] \neq \emptyset.$$

Proof. The left-to-right side is trivial since any *monf* has a non-empty interpretation.

Conversely, assume $(\bar{a},\alpha) \in [\![M]\!]^{\bar{x}}$, by Theorem 4 there exists $N \in M^o$ such that $(\bar{a},\alpha) \in [\![N]\!]^{\bar{x}}$. Any single term of $\partial\lambda^\ell$-calculus converges either to 0 or to a normal form $N_0+\mathbb{N}$ (by Theorem 2). Since $[\![0]\!] = \emptyset$, N converges into a normal form. By applying Lemma 3, we thus have $M \to^* M_0+\mathbb{M}$ with $N_0 \in M_0^o$. Since the Taylor expansion conserves every redexes, M_0 is *outer-normal* and M is *may-outer converging.* □

Corollary 2. *A term may-outer converges iff one of the elements of its Taylor expansion may-outer converges:* $M\Downarrow \Leftrightarrow \exists N{\in}M^o\!,N\Downarrow$. *Equivalently, a term* may-outer diverges *iff any element of its Taylor expansion reduces to 0.*

Proof. For any closed term M, using Theorems 4 and 5:
$M\Downarrow \quad \Leftrightarrow_{\text{th}5} \quad [\![M]\!]{\neq}\emptyset \quad \Leftrightarrow_{\text{th}4} \quad \exists N{\in}M^o\!, [\![N]\!]{\neq}\emptyset \quad \Leftrightarrow_{\text{th}5} \quad \exists N{\in}M^o\!, N\Downarrow.$ □

In the following we use contexts, *i.e.* terms with holes that will be filled by terms. Contexts can be described by the grammar:

(contexts)	$\Lambda(\!\lvert.\rvert\!)$:	$C(\!\lvert.\rvert\!) ::=$	$(\!\lvert.\rvert\!) \mid M \mid \lambda x.C(\!\lvert.\rvert\!) \mid C(\!\lvert.\rvert\!)\ P(\!\lvert.\rvert\!)$
(bag-contexts)	$\Lambda^b(\!\lvert.\rvert\!)$:	$P(\!\lvert.\rvert\!) ::=$	$[C_1(\!\lvert.\rvert\!)^{(!)},\ldots,C_n(\!\lvert.\rvert\!)^{(!)}]$

We define the notions of observational preorder and equivalence using as basic observation the *may-outer-convergence* of terms. This is not the only possibility (*must* or *inner* declensions); we discuss this issue in the conclusion.

Definition 4. *We say that a term M is observationally below another term N (denoted $M \leq_o N$), if for all contexts $C(\!\lvert.\rvert\!)$:*

$$C(\!\lvert M\rvert\!)\Downarrow \;\Rightarrow\; C(\!\lvert N\rvert\!)\Downarrow$$

They are observationally equivalent (denoted $M \equiv_o N$) if $M \leq_o N$ and $N \leq_o M$.

Using sensibility we thus assert our adequation.

Theorem 6. $\mathcal{M}_\infty$ *is inequationally adequate for $\partial\lambda$-calculus,*

$$\forall M,N, \quad [\![M]\!] \subseteq [\![N]\!] \quad\Rightarrow\quad M \leq_o N.$$

Proof. Assume that $[\![M]\!] \subseteq [\![N]\!]$ and $C(\!\lvert M\rvert\!)\Downarrow$. Then since $[\![.]\!]$ is defined by structural induction we have $[\![C(\!\lvert N\rvert\!)]\!] \supseteq [\![C(\!\lvert M\rvert\!)]\!] \neq \emptyset$ and $C(\!\lvert N\rvert\!)\Downarrow$. □

4 Failure of the Full Abstraction

The main result of this paper is the refutation of the full abstraction conjecture:

Conjecture 1 ([3]). $\mathcal{M}_\infty$ is fully abstract for $\partial\lambda$-calculus. *i.e.* the denotational and the observational equivalences are identical:

$$\forall M,N, \quad [\![M]\!] = [\![N]\!] \quad\Leftrightarrow\quad M \equiv_o N$$

Its refutation (Theorem 8) proceeds as follows. First, we define a term $\boldsymbol{A}$ (Equation 7) and we prove that $\boldsymbol{I} \leq_o \boldsymbol{A}$ (Lemma 7, which uses a context lemma: Theorem 7), but $[\![\boldsymbol{I}]\!] \not\subseteq [\![\boldsymbol{A}]\!]$ (Lemma 9). This results in the refutation of the stronger conjecture:

Conjecture 2 ([3]). $\mathcal{M}_\infty$ is inequationally fully abstract for $\partial\lambda$-calculus. *i.e.* the denotational and the observational preorders are identical:

$$\forall M, N, \quad [\![M]\!] \subseteq [\![N]\!] \quad \Leftrightarrow \quad M \leq_o N$$

Only then will we consider the term $\boldsymbol{A'} := \boldsymbol{I}\ [\boldsymbol{A}^!, \boldsymbol{I}^!]$ and prove that $\boldsymbol{A'}$ and $\boldsymbol{A}$ yield a counter-example to Conjecture 1 (Theorem 8).

4.1 Context Lemma

Definition 5. Linear contexts *are contexts with exactly one hole and with this hole in linear position:*

(linear contexts) $\Lambda(\!|.|\!)_l :\ D(\!|.|\!) ::= \ (\!|.|\!) \ \mid\ \lambda x.D(\!|.|\!) \ \mid\ D(\!|.|\!)\ P \ \mid\ M\ [D(\!|.|\!)]{\cdot}P$

The applicative contexts *are particular linear contexts of the form* $K(\!|.|\!) = (\lambda x_1 \dots x_n.(\!|.|\!))\ P_1 \cdots\ P_k$

Lemma 4. *For any term M and any bags P, Q, there exists a decomposition $P = P^{l_1}{\cdot}P^{l_2}{\cdot}P^e$ such that $P^{l_1}{\cdot}P^{l_2}$ is linear, P^e exponential, and if the convergence $(M\ Q)\langle\!\langle P/x\rangle\!\rangle \Downarrow_n$ holds then $M\langle\!\langle P^{l_1}{\cdot}P^e/x\rangle\!\rangle\ Q\langle\!\langle P^{l_2}{\cdot}P^e/x\rangle\!\rangle \Downarrow_n$*

Proof. By definition of the may convergence, since $(M\ Q)\langle\!\langle P/x\rangle\!\rangle = \Sigma_{P=P^{l_1}.P^{l_2}.P^e} M\langle\!\langle P^{l_1}{\cdot}P^e/x\rangle\!\rangle\ Q\langle\!\langle P^{l_2}{\cdot}P^e/x\rangle\!\rangle$ □

Lemma 5 (Linear context lemma). *For any terms M and N, if there is a linear context $D(\!|.|\!)$ such that $D(\!|M|\!) \Downarrow$ and $D(\!|N|\!) \Uparrow$ then there is an applicative context that does the same.*

Proof. We will prove the following stronger property:
For every terms M, N, every bags $P_1, \dots, P_{p+q}$, and every variables $x_1, ..., x_p \notin \bigcup_{1\leq i\leq p+q} \mathrm{FV}(P_i)$, if $\langle\!\langle s\rangle\!\rangle := \langle\!\langle P_1/x_1; ...; P_p/x_p\rangle\!\rangle$ and if a linear context $D(\!|.|\!)$ is such that $(D(\!|M|\!)\langle\!\langle s\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Downarrow_n$ and $(D(\!|N|\!)\langle\!\langle s\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Uparrow$ then there exists an applicative context $K(\!|.|\!)$such that $K(\!|M|\!) \Downarrow$ and $K(\!|N|\!) \Uparrow$.
By cases, making induction on the lexicographically ordered pair $(n, D(\!|.|\!))$:

- If $D(\!|.|\!) = (\!|.|\!)$:
  $K(\!|.|\!) = (\lambda x_1, ..., x_p(\!|.|\!))\ P_1\ \cdots\ P_{p+q}$
- If $D(\!|.|\!) = \lambda z.D'(\!|.|\!)$:
  - If $q = 0$:
    The hypothesis gives $D'(\!|M|\!)\langle\!\langle s\rangle\!\rangle \Downarrow_n$ and $D'(\!|N|\!)\langle\!\langle s\rangle\!\rangle \Uparrow$, thus we can directly apply our induction hypothesis on $D'(\!|.|\!)$. That gives directly the required $K(\!|.|\!)$.

- Otherwise:
  By assuming that z does not appear in $P_{p+2}, ..., P_{p+q}$:
  The hypothesis and Corollary 1 apply to $D(\!|M|\!)\langle\!\langle s\rangle\!\rangle\ P_{p+1}\cdots P_{p+q}$ gives $(D'(\!|M|\!)\langle\!\langle P_{p+1}/z; s\rangle\!\rangle\ P_{p+2}\cdots P_{p+q}) \Downarrow_{n-1}$. Moreover $(D'(\!|N|\!)\langle\!\langle P_{p+1}/z; s\rangle\!\rangle\ P_2\cdots P_q) \Uparrow$.
  Then the induction hypothesis directly gives the required $K(\!|.|\!)$.

– If $D(\!|.|\!) = L\ [D'(\!|.|\!)]\cdot Q$:
  By assuming that $x_i \notin \mathrm{FV}(P_j)$ for $i \leq j \leq p$ and by Lemma 4, there exists, for all $i \leq p$, a decomposition $P_i = P_i^{l_1}\cdot P_i^{l_2}\cdot P_i^e$ such that if $\langle\!\langle s_j\rangle\!\rangle := \langle\!\langle P_1^{l_j}\cdot P_1^e/x_1; ...; P_p^{l_j}\cdot P_p^e/x_p\rangle\!\rangle$ (for all $j \in \{1,2\}$), there is $L' \in L\langle\!\langle s_1\rangle\!\rangle$ with $(L'\ ([D'(\!|M|\!)]\cdot Q)\langle\!\langle s_2\rangle\!\rangle\ P_1\cdots\ P_q) \Downarrow_n$ and $(L'\ ([D'(\!|N|\!)]\cdot Q)\langle\!\langle s_2\rangle\!\rangle\ P_1\cdots\ P_q) \Uparrow$.
  Then there are two cases. Either $L' \to_o \mathbb{L}$ and there is $L'' \in \mathbb{L}$ such that $((L''\ [D'(\!|M|\!)]\cdot Q)\langle\!\langle s_2\rangle\!\rangle\ P_1\cdots\ P_q) \Downarrow_{n-1}$ (using Corollary 1) that allow us to apply the induction hypothesis that result in the wanted $K(\!|.|\!)$. Or L' is in *outer-normal form*:
  - if $L' = \lambda z.L''$:
    Let $D''(\!|.|\!) = L''\langle\!\langle [D'(\!|.|\!)]\cdot Q/z\rangle\!\rangle$.
    We have $(D''(\!|M|\!)\langle\!\langle s_2\rangle\!\rangle\ P_1\cdots\ P_q) \Downarrow_{n-1}$ and $(D''(\!|N|\!)\langle\!\langle s_2\rangle\!\rangle\ P_1\cdots\ P_q) \Uparrow$.
    Then we can apply our induction hypothesis on $D'(\!|.|\!)$ that is still a linear context since $D'(\!|.|\!)$ was not under a "!". This gives directly the required applicative context.
  - if $L' = y\ Q_1\cdots Q_r$ with $y \neq x_i$ for all i:
    There exists, for all $i \leq p$, a multiset $P_i^{l_3} \subseteq P_i^{l_2}$ such that $D'(\!|M|\!)\langle\!\langle P_1^{l_3}\cdot P_1^e/x_1; ...; P_p^{l_3}\cdot P_p^e/x_p\rangle\!\rangle \Downarrow_n$ and $D'(\!|N|\!)\langle\!\langle P_1^{l_3}\cdot P_1^e/x_1; ...; P_p^{l_3}\cdot P_p^e/x_p\rangle\!\rangle \Uparrow$. Then we can apply the induction hypothesis on $D'(\!|.|\!)$ and obtain the wanted $K(\!|.|\!)$.

– If $D(\!|.|\!) = D'(\!|.|\!)\ Q$:
  By Lemma 4, there exists $P_i^{\ell_1}\cdot P_i^{\ell_1}\cdot P_i^e = P_i$ such that, if we denote $\langle\!\langle s_j\rangle\!\rangle := \langle\!\langle P_1^{\ell_i}\cdot P_1^e/x_1; ...; P_p^{\ell_i}\cdot P_p^e/x_p\rangle\!\rangle$:

$$(D'(\!|M|\!)\langle\!\langle s_1\rangle\!\rangle\ Q\langle\!\langle s_2\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Downarrow_n \qquad (D'(\!|N|\!)\langle\!\langle s_1\rangle\!\rangle\ Q\langle\!\langle s_2\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Uparrow$$

  The induction hypothesis on $D'(\!|.|\!)$ (with $Q\langle\!\langle s_2\rangle\!\rangle$ seen as one of the P_i's) results in the required applicative context. □

Theorem 7 (Context lemma). *For any terms M and N, if there is a context $C(\!|.|\!)$ such that $C(\!|M|\!) \Downarrow$ and $C(\!|N|\!) \Uparrow$ then there is an applicative context that does the same.*

Proof. Let $C(\!|.|\!)$ be such a context.
Let $\{x_1, ..., x_n\} = \mathrm{FV}(M) \cup \mathrm{FV}(N)$ be the free variables of M and N.
Let $L = \lambda u.C(\!|u\ [x_1^!]\ \cdots\ [x_n^!]|\!)$, $D(\!|.|\!) = \lambda x_1...x_n(\!|.|\!)$ and $C'(\!|.|\!) = L\ [D(\!|.|\!)^!]$.
Notice that $C'(\!|M|\!) \to^* C(\!|M|\!)$ and $C'(\!|N|\!) \to^* C(\!|N|\!)$. Hence, the hypothesis and Lemma 9 gives $C'(\!|M|\!) \Downarrow$ and $C'(\!|N|\!) \Uparrow$. Moreover, we have that $C'(\!|M|\!) = \bigcup_{n\geq 0}(L\ [D(\!|M|\!)^n)^o$; thus, by applying twice Corollary 2 we have an $n \in \mathcal{N}$ such

that $L\ [D(\!|M|\!)^n] \Downarrow$. Also, since $(L\ [D(\!|N|\!)^n])^o \subseteq C'(\!|N|\!)^o$, the same corollary and the hypothesis $C'(\!|N|\!) \Uparrow$ gives $L\ [D(\!|N|\!)^n] \Uparrow$
Since $L\ [D(\!|N|\!)^k, D(\!|M|\!)^{n-k}]$ converges for $k = 0$ and diverges for $k = n$ there exists $k_0 < n$ such that it converges for $k = k_0$ and diverges for $k = K_0+1$. Thus by applying Lemma 5 on the *linear context* $C''(\!|.|\!) = L\ [D(\!|N|\!)^{k_0}, D(\!|M|\!)^{n-k_0-1}, D(\!|.|\!)]$ we can conclude. □

4.2 Counter Example

We first exhibit a term $\boldsymbol{A}$ that is observationally above the identity $\boldsymbol{I}$, but whose interpretation will not contain $[*]{::}*$ in order to break Conjecture 2. We would like to have $\boldsymbol{A}$ somehow respecting:

$$\boldsymbol{A} \simeq \Sigma_{n\geq 1}\boldsymbol{B}_n \quad \text{with for } n \geq 1: \quad \boldsymbol{B}_n = \lambda v_1 \dots v_n w.w\ [\boldsymbol{I}\ [v_1^!]\ [v_2^!]\ \cdots\ [v_n^!]]$$

This term will converge on any applicative context that converges on the identity (take $\boldsymbol{B}_n$ with n greater than the number of applications), and thus is observationally above the identity. On the other side, its semantic will be independent to the semantics of the identity since none of the $[\![B_i]\!]$ contains $[*]{::}* \in [\![\boldsymbol{I}]\!]$.
Such an infinite sum $\Sigma_{n\geq 1}\boldsymbol{B}_n$ does not exists in our syntax so we have to represent it by using a fix point combinator and a bag of linear and non-linear resources. We define:

$$\boldsymbol{A} := \boldsymbol{\Theta}\ [\boldsymbol{G}, \boldsymbol{F}^!] \tag{7}$$

where $\boldsymbol{G}$ and $\boldsymbol{F}$ are defined by:

$$\boldsymbol{G} := \lambda uvw.w\ [\boldsymbol{I}\ [v^!]] \qquad\qquad \boldsymbol{F} := \lambda uv_1v_2.u\ [\boldsymbol{I}\ [v_1^!]\ [v_2^!]]$$

$\boldsymbol{A}$ seems quite complex, but, it can be seen as a non deterministic *while* that recursively apply $\boldsymbol{F}$ until it chooses (non-deterministically) to apply $\boldsymbol{G}$, giving one of the $\boldsymbol{B}_i$:

Lemma 6.

1. $\boldsymbol{G}[x^!] \to_o^* \boldsymbol{B}_1$
2. *For all* i, $\boldsymbol{F}\ [\boldsymbol{B}_i^!] \to_o^* \boldsymbol{B}_{i+1}$
3. $\boldsymbol{A} \equiv_\beta \boldsymbol{B}_1 + \boldsymbol{F}[\boldsymbol{A}]$

In particular, for every $i \geq 1$*, we have* $\boldsymbol{A} \equiv_\beta \boldsymbol{F}^i[\boldsymbol{A}] + \Sigma_{j=1}^{i-1}\boldsymbol{B}_j$*,*
where $\boldsymbol{F}^1[\boldsymbol{A}] := \boldsymbol{F}[\boldsymbol{A}]$ *and* $\boldsymbol{F}^{i+1} := \boldsymbol{F}^i[\boldsymbol{F}[\boldsymbol{A}]]$

Proof. Item 1 is trivial. Item 2 is just a one-step unfolding of $\boldsymbol{\Theta}$. Item 3 is obtained via the reduction $\boldsymbol{A} \to^* (\boldsymbol{G}[(\boldsymbol{\Theta}[\boldsymbol{F}^!])^!]) + (\boldsymbol{F}[\boldsymbol{A}, (\boldsymbol{\Theta}[\boldsymbol{F}^!])^!]) \equiv_\beta \boldsymbol{B}_1 + (\boldsymbol{F}[\boldsymbol{A}])$ the last step using the linearity of $\boldsymbol{F}$ on its first variable (thus in a context of the kind $[U, V^!]$ only U matters). □

Lemma 7. *For all contexts* $C(\!|.|\!)$ *of the* $\partial\lambda$*-calculus, if* $C(\!|\boldsymbol{I}|\!)$ *converges then* $C(\!|\boldsymbol{A}|\!)$ *converges,* i.e. $\boldsymbol{I} \leq_o \boldsymbol{A}$

Proof. Let $C(\!|.|\!)$ be a context that converges on $\boldsymbol{I}$.

With the context lemma (Theorem 7), and since neither $\boldsymbol{I}$ nor $\boldsymbol{A}$ has free variables, we can assume that $C(\!|.|\!) = (\!|.|\!)\ P_1\ \cdots\ P_k$ (where $P_1, ..., P_k$ are bags). Thus by Lemma 6, we have $\boldsymbol{A} \to^* \boldsymbol{C}_k + \boldsymbol{B}_k$ with $\boldsymbol{C}_k := \boldsymbol{F}^k[\boldsymbol{A}^!] + \Sigma_{j=1}^{k-1}\boldsymbol{B}_j$ and the following converges:

$$C(\!|\boldsymbol{A}|\!) \to^* C(\!|\boldsymbol{C}_k|\!) + \lambda w.w\ [\boldsymbol{I}\ P_1\ \cdots\ P_k] = C(\!|\boldsymbol{C}_k|\!) + \lambda w.w\ [C(\!|\boldsymbol{I}|\!)] \qquad \square$$

We will now compare $\boldsymbol{A}$ and $\boldsymbol{I}$ at the denotational level.

Lemma 8. *We have*

$$[\![\boldsymbol{A}]\!] = \bigcup_i [\![\boldsymbol{B}_i]\!]$$

Proof. $[\![\boldsymbol{A}]\!] \supseteq \bigcup_i [\![\boldsymbol{B}_i]\!]$ is a corollary of the Lemma 6 (the interpretation is stable by reduction), so we have to prove that $[\![\boldsymbol{A}]\!] \subseteq \bigcup_i [\![\boldsymbol{B}_i]\!]$:
Let $\alpha \in [\![\boldsymbol{A}]\!]$. By Theorem 4, there exists $M \in \boldsymbol{A}^o$ such that $\alpha \in [\![M]\!]$. By Theorem 2: $M \to^* \mathbb{N}$, with every element of $\mathbb{N}$ outer-normal. And trivially there is $N \in \mathbb{N}$ such that $\alpha \in [\![N]\!]$. By application of Lemma 3, there exists L such that $\boldsymbol{A} \to^* L + \mathbb{L}$ and $N \in L^o$ (thus $\alpha \in [\![L]\!]$). Since the Taylor expansion conserves all *outer-redexes*, necessary L is outer-normal. We conclude by Lemma 6 that one of the $\boldsymbol{B}_i$ is reducing to L. $\square$

Lemma 9. $[*]{::}* \notin [\![\boldsymbol{A}]\!]$, *while* $[*]{::}* \in [\![\boldsymbol{I}]\!]$

Proof. Because of Lemma 8, we just have to prove that $[*]{::}*$ is not in any $\boldsymbol{B}_i$, which is trivial since the elements of $[\![\boldsymbol{B}_i]\!]$ must be of the form $a_1{::}\cdots{::}a_i{::}[a_1{::}\cdots{::}a_i{::}\alpha]{::}\alpha$, for $i \geq 1$. $\square$

Hence, we have refuted the Conjecture 2 concerning the equality between the observational and denotational orders. We will now refute the Conjecture 1:

Theorem 8. $\mathcal{M}_\infty$ *is not fully abstract for the λ-calculus with resources.*
In particular $\boldsymbol{A'} := \boldsymbol{I}\ [\boldsymbol{A}^!, \boldsymbol{I}^!] \equiv_o \boldsymbol{A}$ *but* $[*]{::}* \in [\![\boldsymbol{A'}]\!]$ *and* $[*]{::}* \notin [\![\boldsymbol{A}]\!]$

Proof. Since $\boldsymbol{A'} \to \boldsymbol{A} + \boldsymbol{I}$, we have $\boldsymbol{A'} \geq_o \boldsymbol{A}$ and $\boldsymbol{A'} \leq_o \boldsymbol{A} + \boldsymbol{A} = \boldsymbol{A}$. But in the same time $[\![\boldsymbol{A'}]\!] = [\![\boldsymbol{A}]\!] \cup [\![I]\!] \ni [*]{::}*$ $\square$

5 Conclusion

Literature (on resource sensitive natural constructions from Linear Logic) are especially focussing on two objects, one in the semantical world, $\mathcal{M}_\infty$, and the other in the syntactical one, ∂λ-calculus. But they appeared not to respect full abstraction.

This unexpected result leads to questions on its generalization. For example, the idea can be applied to refute the full abstraction of $\mathcal{M}_\infty$ for the may-non-deterministic λ-calculus (an extension with a non deterministic operator endowed with a may-convergence operational semantic). Indeed, we can set $\boldsymbol{A}_0 = \lambda x.\boldsymbol{\Theta}\ (\lambda xy.x + \lambda xy.y)$ playing the role of $\boldsymbol{A}$. Such an $\boldsymbol{A}_0$ behaves as the infinite sum $\Sigma_{i=1}^{\infty} \lambda x_1...x_n y.y$, that is a top in its observational order but whose interpretation is not above the identity.

It can even be extended to other models since we can refute the full abstraction of Scott's $\mathcal{D}_\infty$ for the same may-non-deterministic λ-calculus (restriction of $\partial\lambda$-calculus to terms with only banged bags) or the may-must-non-deterministic λ-calculus (λ-calculus with both a may and a must non determinism), using A' in the same way. One can notice that the last case refutes a conjecture of [6].

More generally this counter-example describe the ill-behaved interaction between fixpoints and may-non-determinism that can tests any non-adequacy between the sights of the observation and of the model. We can thus conclude by giving the four key-points that leads to this kind of counter-examples:

– **short-sightedness of the contexts:** Calculi that offer control operators behaving as infinite applicative contexts like the resource λ-calculus with tests [3] are free of these considerations. This traduce the importance of the context lemma in our proof.
– **good sight of the model:** It is our better hope to find a fully abstract model for $\partial\lambda$-calculus but no known interesting algebraic models seems to break this property. Models tend indeed to approximate the condition "for any contexts of any size" into "for any infinite contexts".
– **Untyped fixpoints:** It is the first constructor that is necessary to construct a term that have a non bounded range. Thus, calculi with no fixpoints like the bang-free fragment of $\partial\lambda$-calculus will not suffer such troubles. But those calculi have limited expressive power.
– **may-non-determinism:** The second constructor, that is the most important part and the most interesting one since it can change our view of these calculi. To get rid of this problem without loosing the non determinism one can imagine a finer observation that discriminate the non idempotence of the sum, like the one provided by a probabilistic calculus.

Finally one may be disappointed by the "magic" resolution of Lemma 9. It was unclear, seeing $\boldsymbol{A}$, that this result would arise, and it needed quite a number of nontrivial lemmas. In this point lies a relation with tests mechanisms of [3], in this system $\tau((\!|.|\!)\ \bar{\tau}(\epsilon))$ *outer*-converges on $\boldsymbol{I}$ but not on $\boldsymbol{A}$, the calculus being inequationally fully abstract this gives Lemma 9 for free. That remark was the base of the previous (unpublished but cited) version of this article [2]. From our point of view the relation with tests is even deeper and essential. Indeed the counter-example was discovered naturally from a trial to prove full abstraction from reducing the one from the calculus with tests into the calculus without. This will be subject to an incoming paper.

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