Homoclinic Spirals: Theory and Numerics

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Abstract In this paper we examine spiral structures in bi-parametric diagrams of dissipative systems with strange attractors. First, we show that the organizing center for spiral structures in a model with the Shilnikov saddle-focus is related to the change of the structure of the attractor transitioning between the spiral and screw-like types located at the turning point of a homoclinic bifurcation curve. Then, a new computational technique based on the symbolic description utilizing kneading invariants is proposed for explorations of parametric chaos in Lorenz like attractors. The technique allows for uncovering the stunning complexity and universality of the patterns discovered in the bi-parametric scans of the given models and detects their organizing centers – codimension-two T-points and separating saddles.

1 Introduction

Several analytic and experimental studies, including modeling simulations, have focused on the identification of key signatures to serve as structural invariants. Invariants would allow dynamically similar nonlinear systems with chaotic dynamics from diverse origins to be united into a single class. Among these key structures are various homoclinic and heteroclinic bifurcations of low codimensions that are the heart of the understanding of complex behaviors because of their roles as organizing centers of dynamics in parameterized dynamical systems.

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One computationally justified approach for studying complex dynamics capitalizes on the sensitivity of deterministic chaos. Sensitivity of chaotic trajectories can be quantified in terms of the divergence rate evaluated through the largest Lyapunov characteristic exponent. In several low-order dissipative systems, like the Rössler model, the computational technique based on the largest Lyapunov characteristic exponent reveals that they possess common, easily recognizable patterns involving spiral structures in bi-parametric planes [1,2]. Such patterns have turned out to be ubiquitously in various discrete and continuous-time systems [3], and they are easily located, as spiral patterns have regular and chaotic spiral "arms" in the systems with the Shilnikov saddle-focus [4–6].

Application of the Lyapunov exponents technique fails, in general, to reveal fine structures embedded in the bi-parametric scans of Lorenz-like systems. This implies that the instability of the Lorenz attractors does not vary noticeably as control parameters of the system are varied. This holds true when one attempts to find the presence of characteristic spiral structures that are known to theoretically exist in Lorenz-like systems [7], identified using accurate bifurcation continuation approaches [8, 9]. Such spirals in a bi-parametric parameter plane of a Lorenz-like system are organized around the T[erminal]-points; corresponding to codimension-two, closed heteroclinic connections involving two saddle-foci and a saddle at the origin, see Fig. 6. Such T-points have been located in various models of diverse origins including electronic oscillators and nonlinear optics.

Despite the overwhelming number of studies reporting the occurrence of spiral structures, there is still little known about the fine construction details and underlying bifurcation scenarios for these patterns. In this paper we study the genesis of the spiral structures in several low order systems and reveal the generality of underlying global bifurcations. We will start with the Rössler model and demonstrate that such parametric patterns are the key feature of systems with homoclinic connections involving saddle-foci meeting a single Shilnikov condition [10, 11]. The occurrence of this bifurcation causing complex dynamics is common for a plethora of dissipative systems, describing (electro)chemical reactions [12], population dynamics [13], electronic circuits [3, 14]. The other group is made of models with the Lorenz attractor. Here we present a computational toolkit capitalizing on the symbolic representation for the dynamics of Lorenz-like systems that employ kneading invariants [15, 16].

2 Spiral Structures: Homoclinic Loop

One of the most paradigmatic examples of low-dimensional deterministic chaos is the canonical Rössler system [17]:

$$\dot{x} = -(y + z), \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - c),$$

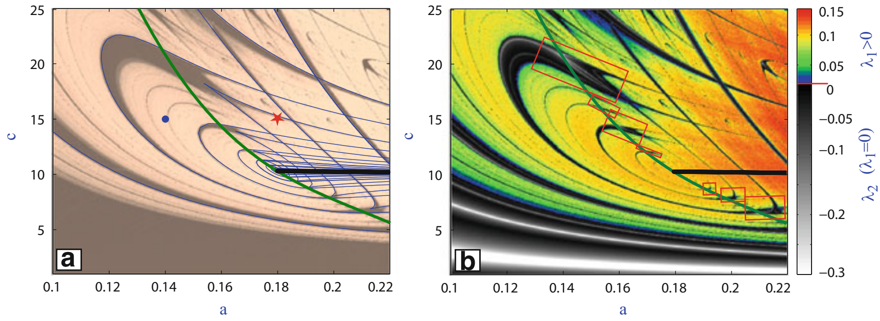

Fig. 1 Spirals and "shrimps" in the 1,000 × 1,000 grid biparametric bifurcation diagrams for the Rössler model. The F-point of the hub is located at $(a, c) = (0.1798, 10.3084)$. The color bars for the Lyapunov exponent range identify the regions of chaotic and regular dynamics. *Left* monochrome panels are superimposed with bifurcation curves: *thin blue* for saddle-nodes, and *thick black* for homoclinic bifurcations of saddle-foci. The *medium-thick green* boundary determines a change in the topological structure of chaotic attractors from spiral (at •) to screw-shaped (at ⋆)

with two bifurcation parameters a and c (we fix $b = 0.2$). For $c^2 > 4ab$ the model has two equilibrium states, $P_{1,2}(ap_\pm, -p_\pm, p_\pm)$, where $p_\pm = (c \pm \sqrt{c^2 - 4ab})/2a$. This classical model exhibits the spiral and screw chaotic attractors after a period doubling cascade followed by the Shilnikov bifurcations of the saddle-focus P_2.

Bi-parametric screening of the Rössler model unveils a stunning universality of the periodicity hubs in the bifurcation diagrams shown in Fig. 1 [5]. The diagram is built on a dense grid of 1,000 × 1,000 points in the $(a\ c)$-parameter plane. Solutions of the model were integrated using the high precision ODE solver TIDES [18]. The color is related with the Lyapunov exponents, where dark and light colors discriminate between the regions of regular and chaotic dynamics corresponding to a zero and positive maximal Lyapunov exponent λ_1, respectively. The figure reveals the characteristic spiral patterns due to variations of the Lyapunov exponents.

The chaotic-regular regions spiral around a F[ocal] point [3] located at $(a, c) = (0.1798, 10.3084)$. This F-point terminates the bifurcation curve (black) corresponding to the formation of a homoclinic loop of the saddle-focus, P_2, in the phase space of the Rössler model [2]. Another curve (medium green) passes through the F-point: crossing it rightward the chaotic attractor in the phase space of the model changes the topological structure from spiral to screw-shaped.

The dark bifurcation curve in Fig. 1 corresponds to a formation of the homoclinic orbits to the saddle-focus, P_2, of topological type (1,2), i.e. with 1D stable and 2D unstable manifolds, in the Rössler model. Depending on the magnitudes of the characteristic exponents of the saddle-focus, the homoclinic bifurcation can give rise to the onset of either rich complex or trivial dynamics in the system [10, 11, 19]. The cases under considerations meet the Shilnikov conditions and hence the existence of a single homoclinic orbit implies chaotic dynamics in the models within the parameter range in the presented diagrams. We remark that the homoclinic

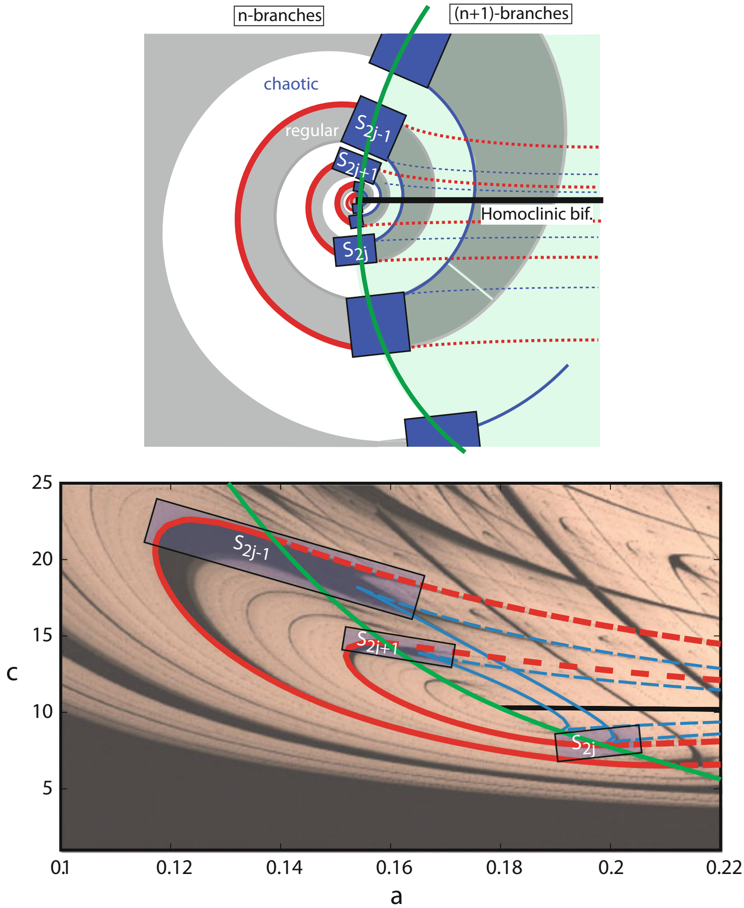

Fig. 2 Outline of the spiral structures: (*Top*) phenomenological sketch of the spiral hub formed by the “shrimps.” (*Bottom*) Magnification of the bifurcation portrait of the spiral hub, overlaid with principal folded (*thick red*) and cusp-shaped (*thin blue*) bifurcation curves setting the boundaries for largest “shrimps” in the Rössler model

bifurcation curve in the diagram (Fig. 1) has actually two branches, although very close each other. This curve has a U-shape with the turn at the F-point.

Figure 2 outlines the structure of the bifurcation unfolding around the spiral hub [2, 5]. The picture depicts a number of the identified folded and cusp-shaped saddle-node bifurcation curves of periodic orbits, toward the spiral hub in the (a, c)-parameter plane for the Rössler model. Note that none of these curves is actually a spiral – the overall spiral structure must be supported by homoclinic bifurcations. At this B-point, the saddle with real characteristic exponents becomes a saddle-focus for smaller values of the parameter a.

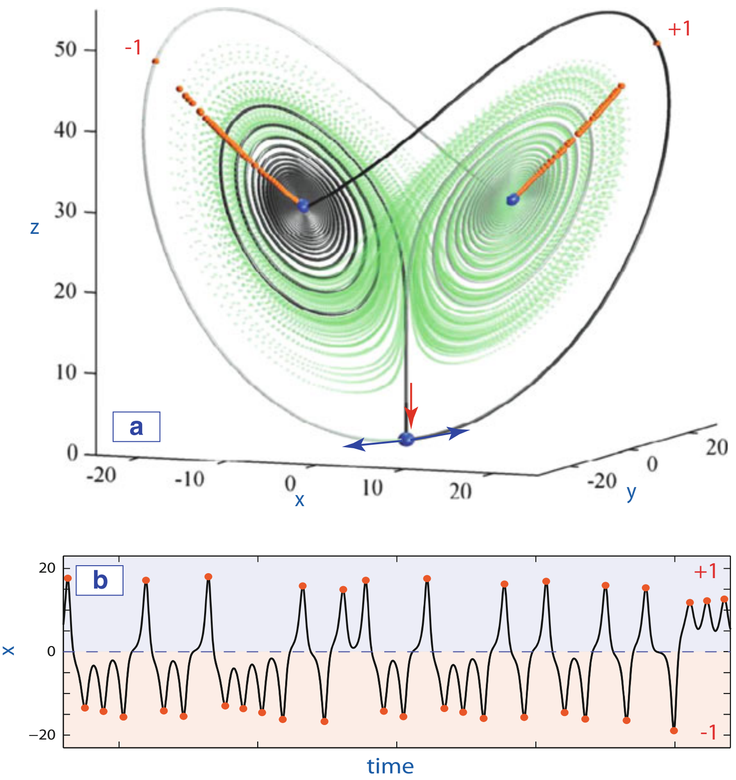

Fig. 3 (**a**) Heteroclinic connection (in *dark color*) between the saddle at the origin and two saddle-foci (*blue spheres*) being overlaid with the strange attractor (*green light color*) on the background at the primary T-point ($r = 30.38$, $\sigma = 10.2$) in the Lorenz model. *Orange spheres* on the butterfly wings indicating the turning points around the *right* and *left* saddle-foci define the kneading sequence entries, $\{\pm 1\}$, respectively. (**b**) A typical time evolution of either symmetric coordinate of the *right* separatrix of the saddle

3 Homoclinic Spirals: Kneading Invariants for Lorenz Like Systems

Chaos can be quantified by several means. One customary way is through the evaluation of topological entropy. The greater the value of topological entropy, the more developed and unpredictable the chaotic dynamics become. Another practical approach for measuring chaos in simulations capitalizes on evaluations of the largest (positive) Lyapunov exponent of a long yet finite-time transient on the chaotic attractor.

A trademark of any Lorenz-like system is the strange attractor of the iconic butterfly shape, such as shown in Fig. 3. The “wings” of the butterfly are marked with two symmetric “eyes” containing equilibrium states, stable or not, isolated from the trajectories of the Lorenz attractor. This attractor is structurally unstable [20, 21] as it bifurcates constantly as the parameters are varied. The primary cause of structural and dynamic instability of chaos in the Lorenz equations and similar models is the singularity at the origin – a saddle with two one-dimensional outgoing separatrices. Both separatrices densely fill the two spatially symmetric wings of the Lorenz attractor in the phase space. The Lorenz attractor undergoes a homoclinic bifurcation when the separatrices of the saddle change the alternating pattern of switching between the butterfly wings centered around the saddle-foci. At such a change, the separatrices comes back to the saddle thereby causing a homoclinic explosions in phase space [22, 23]. Other important points, that act as organizing centers, are the codimension-two T-points and separating saddles, but these points cannot be detected by using Lyapunov exponents as we can see in Fig. 4, where on the top we show the maximum Lyapunov exponent that cannot reveal the hidden structures inside the chaotic region. Among these structures is a T-point, that form a "kind" of "diamonds-mine" (middle pic) formed by homoclinic spirals as it can be shown by detailed bifurcation analysis (bottom pic). Therefore, we focus on presenting a new computational tool that locates automatically "all" the T-points.

The time progression of the "right" (or symmetrical "left") separatrix of the origin can be described geometrically and categorized in terms of the number of alternations around the nonzero equilibrium states in the phase space of the Lorenz-like system (Fig. 3). Alternatively, the description can be reduced to the time-evolution of a coordinate of the separatrix, as shown in panel b of Fig. 3. The sign-alternation of the x-coordinate suggests the introduction of a $\{\pm 1\}$-based alphabet for the symbolic description of the separatrix. Namely, whenever the right separatrix turns around O_1 or O_2, we record $+1$ or -1, respectively. For example, the time series shown in panel b generates the following kneading sequence starting with $\{+1, -1, -1, -1, +1, -1, -1, +1, -1, \ldots\}$.

We introduce and demonstrate a new computational toolkit for the analysis of chaos in the Lorenz-like models. The toolkit is inspired by the idea of kneading invariants introduced in [15]. A kneading invariant is a quantity that is intended to uniquely describe the complex dynamics of the system that admit a symbolic description using two symbols, here $+1$ and -1.

The kneading invariant for either separatrix of the saddle equilibrium state of the Lorenz attractor can be defined in the form of a formal power series:

$$P(q) = \sum_{n=0}^{\infty} \kappa_n \, q^n. \tag{1}$$

Letting $q \in (0, 1)$ guarantees the series is convergent. The smallest zero, q^*, if any, of the graph of (1) in the interval $q \in (0, 1)$ yields the topological entropy, $h(T) = \ln(1/q^*)$.

The kneading sequence $\{\kappa_n\}$ composed of only $+1$s corresponds to the "right" separatrix of the saddle converging to an ω-limit set with $x(t) > 0$, such as a stable focus or stable periodic orbit. The corresponding kneading invariant is maximized

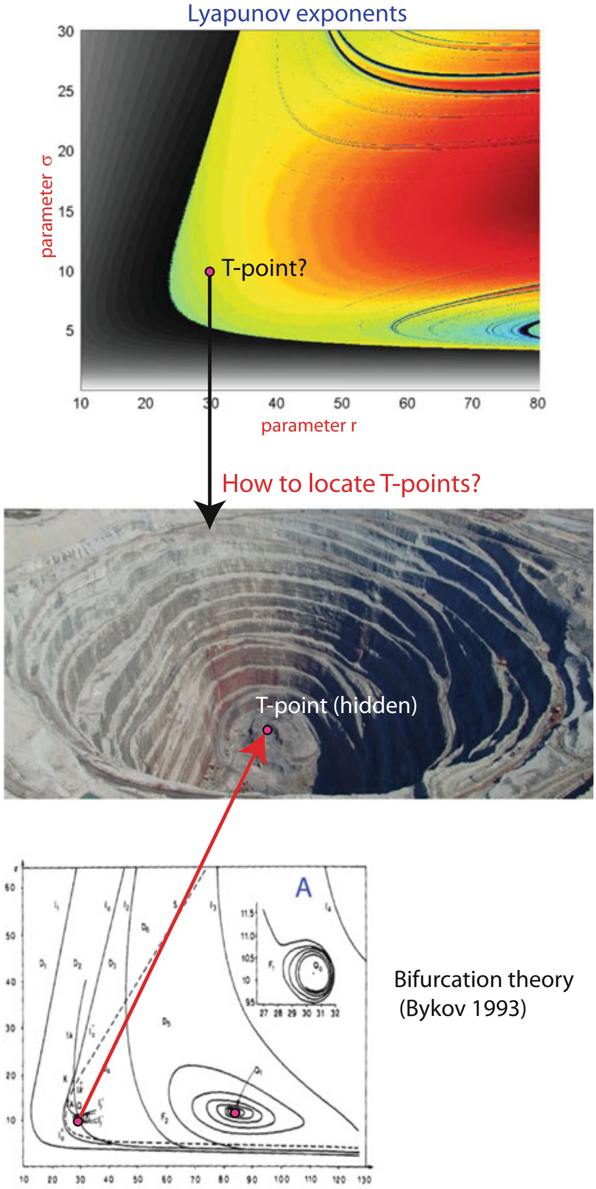

Fig. 4 (*Top*) Finite-time largest-Lyapunov exponent, $L_{\max}$, scan of the Lorenz equation showing no sign of spiral structures in the (r, σ)-parameter plane. The *dark region* corresponds to trivial attractors, where $L_{\max} \leq 0$, while the *red color* indicates $L_{\max} > 0$ in chaotic regions. The *red dot* points out the position of the primary T-point. (*Bottom*) Original bifurcation diagram of the Lorenz equation depicting the two detected T-points and primary homoclinic bifurcation curves from [7]

at $\{P_{\max}(q)\} = 1/(1-q)$. When the right separatrix converges to an attractor with $x(t) < 0$, then the kneading invariant is given by $\{P_{\min}(q)\} = 1 - q/(1-q)$ because the first entry $+1$ in the kneading sequence is followed by infinite -1s. Thus, $[\{P_{\min}(q)\}, \{P_{\max}(q)\}]$ yield the range of the kneading invariant values; for instance $[\{P_{\min}(1/2)\} = 0, \{P_{\max}(1/2)\} = 2]$.

In computational studies of the models below, we will consider a partial kneading power series truncated to the first 20 entries: $P_{20}(q) = \sum_{n=0}^{20} \kappa_n\, q^n$. The choice of the number of entries is not motivated by numerical precision, but by simplicity, as well as by resolution of the bitmap mappings for the bi-parametric scans of the models. One has also to determine the proper value of q: setting it too small makes the convergence fast so that the tail of the series has a little significance and hence does not differentiate the fine dynamics of the Lorenz equation for longer kneading sequences.

At the first stage of the routine, we perform a bi-parametric scan of the model within a specific range in the parameter plane. The resolution of scans is set by using mesh grids of [1,000 × 1,000] equally distanced points. Next by integrating the same separatrix of the saddle point we identify and record the sequences $\{\kappa_n\}_{20}$ for each point of the grid in the parameter plane. The mapping is then colorized in Matlab by using various built-in functions ranging between to $P_{20}^{\min}$ and $P_{20}^{\max}$, respectively. In the mapping, a particular color in the spectrum is associated with a persistent value of the kneading invariant on a level curve. Such level curves densely foliate the bi-parametric scans. Now, we repeat the study for the Lorenz model with the new technique in Fig. 5. Now we can observe that the T-points are revealed automatically.

Besides, we examine the kneading-based bi-parametric scanning of the Shimizu-Morioka model [8, 24]:

$$\dot{x} = y, \quad \dot{y} = x - \lambda y - xz, \quad \dot{z} = -\alpha z + x^2; \tag{2}$$

with α and λ being positive bifurcation parameters. The Z_2-symmetric model has three equilibrium states: a simple saddle, with one-dimensional separatrices, at the origin, and two symmetric stable-foci which can become saddle-foci through a supercritical Andronov-Hopf bifurcation.

This model was originally introduced to examine a pitch-fork bifurcation of the stable figure-8 periodic orbit that gives rise to multiple cascades of period doubling bifurcations in the Lorenz equation at large values of the Reynolds number. It was proved in [9] that the Eqs. (2) would be a universal normal form for several codimension-three bifurcations of equilibria and periodic orbits on Z_2-central manifolds. The model turned out to be very rich dynamically: it exhibits various interesting global bifurcations [25] including T-points for heteroclinic connections.

The structure of the bifurcation set of the Shimizu-Morioka is very complex. The detailed bifurcation diagram is shown in the top-left panel of Fig. 6. It reveals several T-points, and multiples curves corresponding to an Andronov-Hopf (AH), pitch-fork (PF), period doubling (PD) and homoclinic (H) bifurcations that shape the existence region of the Lorenz attractor in the model. The detailed description of the bifurcation structure of the Shimizu-Morioka model is out of scope of this paper. The reader can find a wealth of information on bifurcations of the Lorenz attractor in the original papers [9, 25].

The panels b and c of Fig. 6 is a de-facto proof of the new kneading invariant mapping technique. The panel represents the color bi-parametric scan of the dynamics of the Shimizu-Morioka model that is based on the evaluation of the first

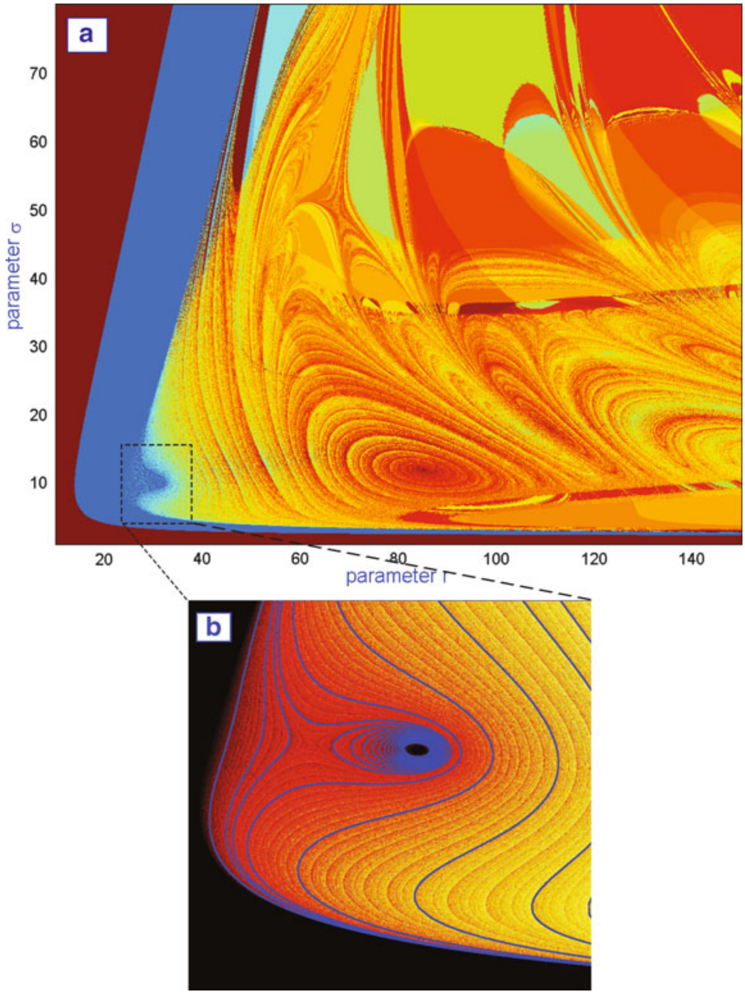

Fig. 5 (**a**) Kneading-based biparametric scan revealing multiple T-points and saddles that organize globally complex chaotic dynamics of the Lorenz equation in the (r, σ) parameter plane. *Solid-color regions* associated with constant values of the kneading invariant correspond to simple dynamics dominated by stable equilibria or stable periodic orbits. The border line between the *brown* and *blue region* corresponds to the bifurcation curve of the homoclinic butterfly. The border line between the *blue* and *yellow-reddish* region corresponds to the formation of the Lorenz attractor (below $\sigma \simeq 50$). (**b**) Zoom of the vicinity of the primary T-point at $(r = 30.4, \sigma = 10.2)$ to which a homoclinic bifurcation curve spirals onto (Data for the homoclinic curves (in *blue*) are courtesy of Yu. Kuznetsov)

20 kneadings of the separatrix of the saddle on the grid of $1{,}000 \times 1{,}000$ points in the (α, λ)-parameter region. Getting the mapping took a few hours on a high-end workstation without any parallelization efforts. The color scan reveals a plethora of primary, large, and small scale T-points as well as the saddles separating spiral structures.

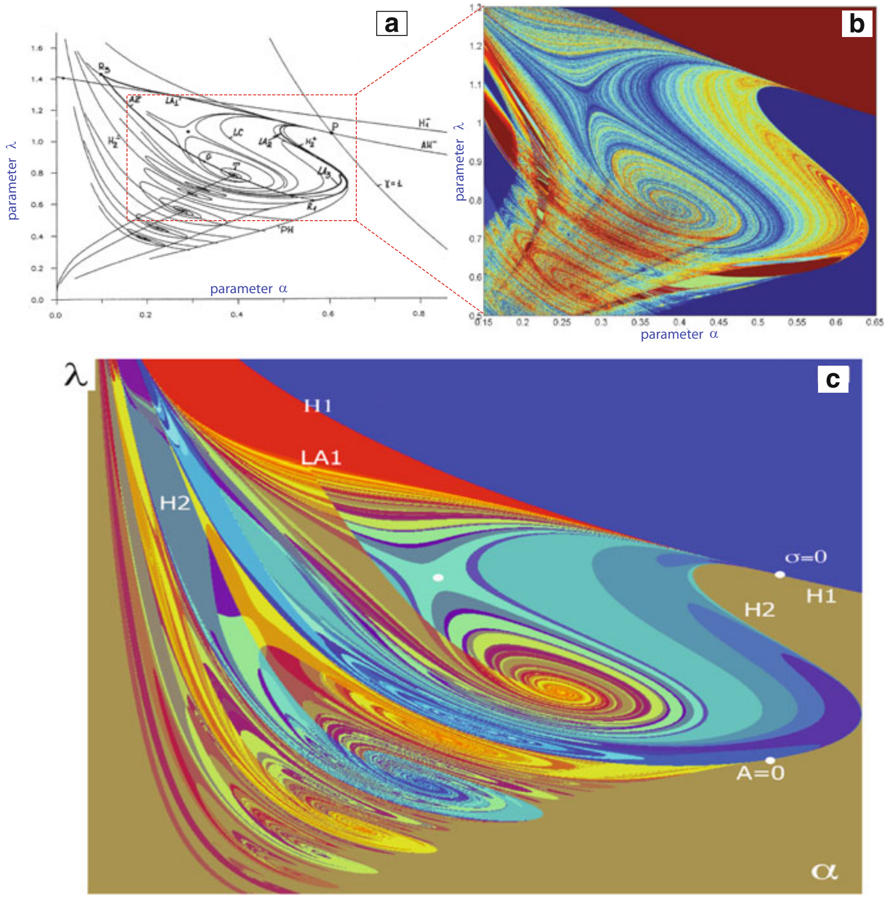

Fig. 6 (*Top-left*) Detailed (α, λ)-parameter plane of the Shimizu-Morioka model obtained by the parameter continuation method (Courtesy of [9]). Legend: AH stands for a supercritical Andronov-Hopf bifurcation, H_1 stands for the homoclinic butterfly made of two separatrix loops; the codimension-two points corresponding to the resonant saddle P on H_1 organizes the bifurcation unfolding of the model; cod-2 point $R_1 0$ stands for an orbit-flip bifurcation for the double-loop homoclinics on H_2. The *thick line* demarcates, with good precision, the existence region of the Lorenz attractor bounded by LA_1 and LA_2. (*Top-right* and *bottom*) Two kenading scans revealing multiple T-points and saddles that globally organize complex chaotic dynamics of the Shimizu-Morioka model. *Solid-color regions* associated with constant values of the kneading invariant correspond to simple dynamics dominated by stable equilibria (*brown*) or stable periodic orbits (*blue*). The border between the *brown* and *blue regions* corresponds to the bifurcation curve of the homoclinic butterfly

The solid-color zones in the mapping correspond to simple dynamics in the model. Such dynamics are due to either the separatrix converging to the stable equilibria or periodic orbits with the same kneading invariant (blue region), or to the symmetric and asymmetric stable figure-8 periodic orbits (brown region).

The borderlines between the simple and complex dynamics in the Shimizu-Morioka model are clearly demarcated. On the top is the curve, LA_1, (see the top-left panel of Fig. 6). The transition from the stable 8-shaped periodic orbits to the Lorenz attractor (through the boundary, LA_2) is similar though more complicated as it involves a pitch-fork bifurcation and bifurcations of double-pulsed homoclinics, see [9, 25] for details.

One can clearly see the evident resemblance between both diagrams found using the bifurcationaly exact numerical methods and by scanning the dynamics of the model using the proposed kneading invariant technique. The latter reveals a richer structure providing finer details. The structure can be enhanced further by examining longer tails of the kneading sequences. This allows for the detection of smaller-scale spiral structures within scrolls of the primary T-vortices, as predicted by the theory.

4 Conclusions

We have examined two formation mechanisms of spiral structures in biparametric mappings of systems with the Shilnikov saddle-focus and with the Lorenz attractor. The feature of the spiral hubs in the Rössler model is that the F[ocal]-point gives rise to the alternation of the topological structure of the chaotic attractor transitioning between the spiral and screw-like types, as well as terminates the primary homoclinic curves of the saddle-focus equilibrium state influencing the forward-time dynamics of the model. The findings let us hypothesize about universality of the structure of the spiral hubs in similar systems with chaotic attractors due to homoclinics of the Shilnikov saddle-focus.

We have demonstrated a new computational toolkit for thorough explorations of chaotic dynamics in models with the Lorenz attractor. The algorithmically simple yet powerful toolkit is based on the scanning technique that maps the dynamics of the system onto the bi-parametric plane. The core of the approach is the evaluation of the kneading invariants for regularly or chaotically varying alternating patterns of a single trajectory – the separatrix of the saddle singularity in the system. In the theory, the approach allows two systems with structurally unstable Lorenz attractors to be conjugated with a single number – the kneading invariant. The kneading scans unambiguously reveal the key features in Lorenz-like systems such as a plethora of underlying spiral structures around T-points, separating saddles in intrinsically fractal regions corresponding to complex chaotic dynamics. We point out that no other techniques, including approaches based on the Lyapunov exponents, can reveal the discovered parametric chaos with such stunning clarity and beauty.

The method should be beneficial for detailed studies of other systems admitting a reasonable symbolic description.

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