Statistical Stability in Chaotic Dynamics

J.F. Alves and M. Soufi

Abstract We present some results on the existence and continuous variation of physical measures for families of chaotic dynamical systems. Quadratic maps and Lorenz flows will be considered in more detail. A brief idea on the proof of a recent theorem in Alves and Soufi (Nonlinearity 25:3527–3552, 2012) on the statistical stability of Lorenz flows will be given.

1 Introduction

The theory of Dynamical Systems started in the work of Poincaré on the three-body problem studies processes which evolve in time. The description of these processes may be given by flows (continuous time) or iterations of maps (discrete time). The main goals of this theory are: to describe the typical behavior of orbits as time goes to infinity; and to understand how this behavior changes under perturbations of the system and to which extent it is stable. In this work we are mostly concerned with the stability of dynamical systems.

Ergodic Theory deals with measure preserving processes in a measure space. In particular, it tries to describe the average time spent by typical orbits in different regions of the phase space. According to Birkoff's Ergodic Theorem, these times are well defined for almost all points, with respect to any invariant probability measure. However, the notion of typical orbit is usually meant in the sense of volume (Lebesgue measure), which is not always an invariant measure.

It is a fundamental open problem to understand under which conditions the behavior of typical (positive Lebesgue measure) orbits is well defined from the

J.F. Alves (✉) · M. Soufi
Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
e-mail: falves@fc.up.pt; msoufin@gmail.com

S. Ibáñez et al. (eds.), *Progress and Challenges in Dynamical Systems*, Springer Proceedings in Mathematics & Statistics 54,
DOI 10.1007/978-3-642-38830-9_2,

statistical point of view. In chaotic dynamical systems this can be precisely formulated by means of *Sinai-Ruelle-Bowen (SRB) measures*, which were introduced by Sinai for Anosov diffeomorphisms [23] and later extended by Ruelle and Bowen for Axiom A diffeomorphisms [21] and flows [9]. In trying to capture the persistence of the statistical properties of a dynamical system, Alves and Viana [4] proposed the notion of *statistical stability*, which expresses the continuous variation of SRB measures as a function of the dynamical system and is the problem that we address in this work.

2 Physical Measures

Let M be a compact connected Riemannian manifold, $f : M \to M$ a (possibly piecewise) C^k map, for some $k > 1$. We shall denote by λ the Lebesgue measure on the Borel sets of M.

The *basin* of an f-invariant probability measure μ on the Borel sets of M is the set of points $x \in M$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \quad \text{for any continuous } \varphi : M \to \mathbb{R}.$$

A probability measure μ is called a *physical measure* if its basin has positive Lebesgue measure.

Example 1. If $p \in M$ has an attracting periodic orbit of period $k \geq 1$, then the average of Dirac measures on the orbit of p,

$$\mu = \frac{1}{k}\left(\delta_p + \cdots + \delta_{f^{k-1}(p)}\right),$$

is a physical measure.

Example 2. It follows from Birkhoff's Ergodic Theorem that any ergodic invariant probability measure which is absolutely continuous with respect to Lebesgue measure is a physical measure. We shall refer to this special case of physical measures as *Sinai-Ruelle-Bowen (SRB) measures*.

A probability measure μ which is invariant under a flow $(X^t)_t$ is called a *physical measure* if the basin of μ, i.e. the set of points x such that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(X^t(x)) \, dt = \int \varphi \, d\mu, \quad \text{for any continuous } \varphi : M \to \mathbb{R},$$

has positive Lebesgue measure.

We shall refer to *statistical stability* as the continuous variation of the physical measures in the weak* topology as a function of the dynamical system. *Strong statistical stability* means the continuous variation of the densities (if they exist) of the physical measures in the L^1-norm as a function of the dynamical system.

3 Quadratic Maps

In this section we consider some results on the statistical stability of the family of quadratic maps, $f_a : [-1, 1] \to [-1, 1]$, with $a \in [0, 2]$, given for $x \in [-1, 1]$ by

$$f_a(x) = 1 - ax^2.$$

The dynamics of the maps in this family has been exhaustively studied by many authors in the last decades, serving as a model for many important results in the field.

3.1 Physical Measures

The first result on the existence of SRB measures is for parameters for which the critical point is nonrecurrent. Recall that a point is *nonrecurrent* if that point is not accumulated by its orbit.

Theorem 1 (Misiurewicz [19]). *If the orbit of* 0 *is nonrecurrent, then* f_a *has an SRB measure.*

The set $\mathcal{M}$ of *Misiurewicz parameters*, i.e. the set of parameters $a \in [0, 2]$ for which the orbit of 0 under f_a is nonrecurrent, has the cardinality of the continuum but zero Lebesgue measure. As the orbit of 0 is nonrecurrent for f_2, then $2 \in \mathcal{M}$ and f_2 has an SRB measure (f_2 is actually conjugated to the tent map by a diffeomorphism of $(-1,1)$).

The next result shows that SRB measures appear more generally from a measure theoretical point of view.

Theorem 2 (Jakobson [14]). *There is a positive Lebesgue measure set* $\mathcal{J}$ *of parameters* $a \in [0, 2]$ *for which* f_a *has an SRB measure.*

In particular, for a in the set of Jakobson parameters, f_a has no attracting periodic orbit.

Theorem 3 (Lyubich [17]). *For Lebesgue almost every* $a \in [0, 2]$ *the map* f_a *has either a periodic attracting orbit or an SRB measure.*

Let $\mathcal{P}$ be the set of parameters in $[0, 2]$ which have a physical measure. By Lyubich Theorem, the set $\mathcal{P}$ has full Lebesgue measure in $[0, 2]$.

3.2 Statistical Instability

Now we address ourselves to the statistical stability on the set of parameters whose dynamics has a physical measure. The first result shows that we do not have statistical stability inside that family.

Theorem 4 (Hofbauer-Keller [13]). *The map $\mathscr{P} \ni a \to \mu_a$ is not (weak*) continuous at $a = 2$.*

As mentioned before, f_2 has an SRB measure. On the other hand, Hofbauer and Keller showed that $a = 2$ is accumulated by a sequence of parameters a_n for which f_{a_n} has a physical measure which is a Dirac measure supported on a repelling fixed point.

The next result shows that statistical instability is not a so uncommon phenomenum.

Theorem 5 (Thunberg [24]).

1. *$\mathscr{P} \ni a \mapsto \mu_a$ is not (weak*) continuous at any Misiurewicz parameter;*
2. *$\mathscr{P} \ni a \mapsto \mu_a$ is not (weak*) continuous at any full Lebesgue measure subset of [0,2].*

3.3 Statistical Stability

Here we see some results showing that when restricted to some special sets of parameters we can have statistical stability, even in a strong sense. The first result is for Misiurewicz parameters.

Theorem 6 (Rychlik-Sorets [22]). *The map $\mathscr{M} \ni a \to \mu_a$ is (strongly) continuous.*

The next result shows that we can have a similar result in a wider context.

Theorem 7 (Tsujii [25]). *There is $\mathscr{T} \subset [0,2]$ with positive Lebesgue measure and positive density at 2 such that $\mathscr{T} \ni a \to \mu_a$ is (weak*) continuous.*

For the proof of the next result, Benedicks and Carleson developed a strategy that enabled them to extend some results on the chaotic behavior of many parameters (positive Lebesgue measure) in the quadratic family to the family of two-dimensional Hènon diffeomorphisms; see [6].

Theorem 8 (Benedicks-Carleson [6]). *There is $\mathscr{BC} \subset [0,2]$ with positive Lebesgue measure and positive density at 2 and constants $c, \alpha > 0$ such that for each $a \in \mathscr{BC}$*

1. $|(f_a^n)'(f_a(0))| \geq e^{cn}, \quad \forall n \in \mathbb{N};$
2. $|f_a^n(0) - 0| \geq e^{-\alpha n}, \quad \forall n \in \mathbb{N}.$

The proof on the existence of SRB measures for these parameters is stated in the next result.

Theorem 9 (Benedicks-Young [7]). *For each $a \in \mathscr{BC}$, the map f_a has a unique SRB measure.*

However, we do not have statistical stability for the parameters $\mathscr{BC}$ when we consider the full set of parameters $\mathscr{P}$ which have a physical measure.

Theorem 10 (Thunberg [24]). *The map $\mathscr{P} \ni a \to \mu_a$ is not (weak*) continuous at any $a \in \mathscr{BC}$.*

On the other hand, the next shows that restricting ourselves to the $\mathscr{BC}$ parameters we do have statistical stability.

Theorem 11 (Freitas [11]). *The map $\mathscr{BC} \ni a \to \mu_a$ is (strongly) continuous.*

In order to prove this result, Freitas showed that, for each $a \in \mathscr{BC}$, the expansion and slow recurrence conditions stated by Benedicks and Carleson Theorem for the critical orbit hold for Lebesgue almost every point in the phase space:

1. f_a is *nonuniformly expanding*:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(f_a'(f_a^i(x))) > c, \quad \text{Lebesgue a.e. } x$$

2. f_a has *slow recurrence to the critical set*: for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log d_\delta(f_a^i(x), 0) > -\epsilon, \quad \text{Lebesgue a.e. } x$$

where d_δ is the δ-truncated distance, defined as

$$d_\delta(x, y) = \begin{cases} |x-y| & \text{if} \quad |x-y| \le \delta, \\ 1 & \text{if} \quad |x-y| > \delta. \end{cases}$$

This allows us to introduce the *expansion time*

$$E_a(x) = \min\left\{ N \ge 1 : \frac{1}{n} \sum_{i=0}^{n-1} \log f_a'(f_a^i(x)) > c, \forall n \ge N \right\},$$

the *recurrence time*

$$R_a(x) = \min\left\{ N \ge 1 : \frac{1}{n} \sum_{i=0}^{n-1} \log d_\delta(f_a^i(x), 0) > -\epsilon, \forall n \ge N \right\},$$

and the *tail set* (at time n)

$$\Gamma_a^n = \{x \in I : E_a(x) > n \quad \text{or} \quad R_a(x) > n\}.$$

Freitas [11] proved that there are $C, \gamma > 0$ such that $|\Gamma_a^n| \le Ce^{-\gamma n}$ for all $n \ge 1$ and $a \in \mathscr{BC}$, which together with the following result gives the conclusion of Theorem 11.

Theorem 12 (Alves [1]). *Let $\mathscr{A}$ be a set of parameters for which there are $C > 0$ and $\gamma > 1$ with $|\Gamma_a^n| \le Cn^{-\gamma}$, for all $n \ge 1$ and all $a \in \mathscr{A}$. Then each f_a with $a \in \mathscr{A}$ has an SRB measure μ_a and $\mathscr{A} \ni a \longmapsto d\mu_a/d\lambda$ is continuous.*

4 Lorenz Flow

Lorenz [16] studied numerically the vector field X defined by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx - y - xz, \\ \dot{z} = xy - cz, \end{cases}$$

for the parameters $a = 10$, $b = 28$ and $c = 8/3$. The following properties are well known for this vector field:

1. X has a *singularity* at the origin with eigenvalues

$$0 < -\lambda_3 \approx 2.6 < \lambda_1 \approx 11.83 < -\lambda_2 \approx 22.83;$$

2. There is *trapping region* U such that $\Lambda = \cap_{t>0} X^t(U)$ is an attractor and the origin is the unique singularity contained in U;
3. The divergence of X is negative:

$$\text{div}X = \partial\dot{x}/dx + \partial\dot{y}/dy + \partial\dot{z}/dz = -(a + 1 + c) < 0,$$

Thus X contracts volume, and in particular Λ has zero volume.

In experimental computations, Lorenz realized that the flow has *sensitivity to the initial conditions*, i.e. even a small initial error lead to enormous differences in the outcome.

4.1 Geometric Model

In the 1970s, Guckenheimer and Williams [12] introduced a *geometric model* for the Lorenz flow. The vector field X is linear in a neighborhood of the singularity $(0, 0, 0)$ whose eigenvalues satisfy

$$0 < -\lambda_3 < \lambda_1 < -\lambda_2.$$

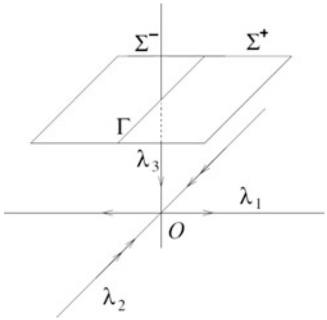

Fig. 1 The cross-section

X has a cross-section $\Sigma = \{z = 1, |x| \leq 1/2, |y| \leq 1/2\}$ intersecting the (two-dimensional) stable manifold of the singularity 0 along a curve Γ which divides Σ into two regions Σ^+ and Σ^- (Fig. 1).

An easy calculation shows that the time needed to go from $\Sigma^\pm$ to $S^\pm = \{x = \pm 1, |y| \leq 1, 0 \leq z \leq 1\}$ is given by

$$\tau_0(x, y, 1) = -\frac{1}{\lambda_1} \log |x|.$$

To complete the geometric model of Lorenz flow, it is assumed that the flow from $S^\pm$ reaches Σ in finite time T_0. Hence, the return time from Σ to itself is

$$\tau_0(x, y, 1) = -\frac{1}{\lambda_1} \log |x| + T_0.$$

Poincaré Return Map

The return map $P : \Sigma \setminus \Gamma \to \Sigma$ admits a *stable foliation* $\mathscr{F}$ on Σ with the following properties (Fig. 2):

- *Invariant:* the image under P of a stable leaf ξ (distinct from Γ) is contained in another stable leaf;
- *Contracting:* the diameter of $P^n(\xi)$ goes to zero when $n \to \infty$, uniformly over all leaves;
- *Quotient map:* P induces a map f on the quotient space $\Sigma/\mathscr{F} \sim [-1/2, 1/2] = I$.

Assuming *the strong dissipative condition* at the singularity

$$-\frac{\lambda_2}{\lambda_1} > -\frac{\lambda_3}{\lambda_1} + 2,$$

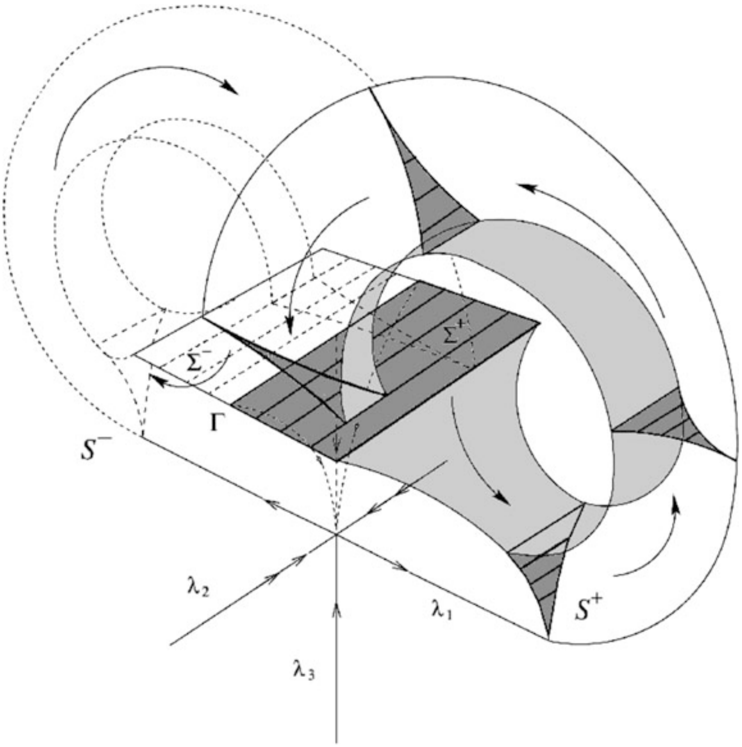

Fig. 2 The return map

the foliation $\mathscr{F}$ is C^2, and the one-dimensional quotient map f is C^2 away from the singularity. The map f has the following properties:

1. f is discontinuous at $x = 0$: $\lim_{x\to 0^+} f(x) = -1/2$ and $\lim_{x\to 0^-} f(x) = 1/2$;
2. f is differentiable on $I \setminus \{0\}$ and $f'(x) > \sqrt{2}$ for all $x \in I \setminus \{0\}$;
3. $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = +\infty$ (Fig. 3).

Attractor

The geometric model X admits a *strange attractor*, i.e. there is a compact set Λ such that:

1. Λ is invariant under the flow;
2. Λ contains a dense orbit;
3. Λ contains the singularity O;
4. There is an open neighborhood U of Λ such that $\Lambda = \cap_{t>0} X_t(U)$;
5. The flow has sensitive dependence on the initial conditions in U.

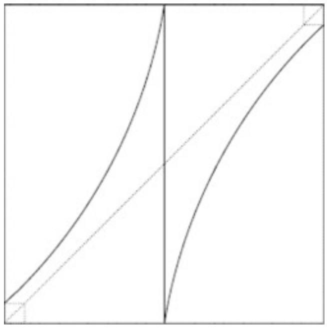

Fig. 3 The one-dimensional map

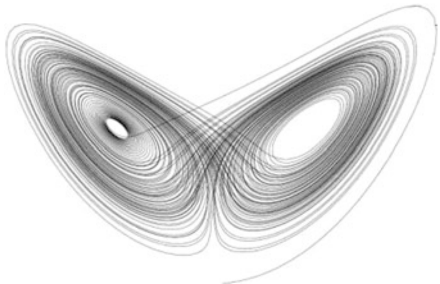

Fig. 4 Lorenz attractor

The next result shows that the classical values of Lorenz equations do indeed support a strange attractor (Fig. 4).

Theorem 13 (Tucker [26]). *For the classical parameter values, Lorenz equations support a robust strange attractor.*

Robustness

Observe that all the properties in the geometric model for the Lorenz flow remain valid under small perturbations. We consider a *family* $\mathscr{L}$ *of Lorenz flows* as a C^2 neighborhood of X such that for each $Y \in \mathscr{L}$:

1. The maximal forward invariant set Λ_Y inside U is an attractor containing a hyperbolic singularity;
2. Σ is a cross-section for the flow with a return time τ_Y and a Poincaré map P_Y;
3. P_Y admits a C^2 uniformly contracting invariant foliation $\mathscr{F}_Y$ on Σ with projection along the leaves of $\mathscr{F}_Y$ onto I given by a map π_Y;
4. The quotient map f_Y on I is a C^2 piecewise expanding with two branches; moreover, $f_Y' > \sqrt{2}$ except at the unique discontinuity point c_Y and $\lim_{x\to c_Y^{\pm}} f_Y'(x) = +\infty$;
5. There is some constant $C > 0$ such that for each $Y \in \mathscr{L}$

$$\tau_Y(x) \leq -C \log|\pi_Y(x) - c_Y|.$$

4.2 Physical Measures

It is well know that a one-dimensional map with the properties of f_Y above, for $Y \in \mathscr{L}$, has some SRB measure. Actually, we have the following folklore result; see e.g. [27].

Theorem 14. *For each $Y \in \mathscr{L}$, the map f_Y has a unique ergodic acip $\bar{\mu}_Y$ whose density (wrt Lebesgue measure) has bounded variation.*

The statistical stability for the maps f_Y with $Y \in \mathscr{L}$ was proved by Keller.

Theorem 15 (Keller [15]). *For each $Y \in \mathscr{L}$, the map f_Y is strongly statistically stable.*

It actually follows from the proof of this last result that the densities of the SRB measures are uniformly bounded: there is $C > 0$ such that for any $Y \in \mathscr{L}$

$$\left\| \frac{d\bar{\mu}_Y}{d\lambda} \right\|_\infty \le C.$$

Given $Y \in \mathscr{L}$, let $P_Y : \Sigma \setminus \Gamma_Y \to \Sigma$ be its Poincaré map and $\bar{\mu}_Y$ be the physical measure for the quotient map. Given a bounded function $\varphi : \Sigma \to \mathbb{R}$, define

$$\varphi_+(x) := \sup_{y \in \xi(x)} \varphi(y) \qquad \text{and} \qquad \varphi_-(x) := \inf_{y \in \xi(x)} \varphi(y),$$

where $\xi(x)$ is the leaf in the foliation $\mathscr{F}_Y$ which contains x. The proof of the next lemma follows standard arguments in hyperbolic dynamics; see e.g. [8].

Lemma 1. *Given any continuous function $\varphi : \Sigma \to \mathbb{R}$, both limits*

$$\lim_{n\to\infty} \int (\varphi \circ P_Y^n)_- d\bar{\mu}_Y \quad \textit{and} \quad \lim_{n\to\infty} \int (\varphi \circ P_Y^n)_+ d\bar{\mu}_Y$$

exist and they coincide.

Using Riesz-Markov Representation Theorem we easily deduce the next result.

Corollary 1. *There is a P_Y-invariant probability measure $\tilde{\mu}_Y$ on Σ such that*

$$\int \varphi \, d\tilde{\mu}_Y = \lim_{n\to\infty} \int (\varphi \circ P_Y^n)_- d\bar{\mu}_Y = \lim_{n\to\infty} \int (\varphi \circ P_Y^n)_+ d\bar{\mu}_Y,$$

for each continuous function $\varphi : \Sigma \to \mathbb{R}$.

One can check that $\tilde{\mu}_Y$ is a physical measure for the Poincaré map P_Y and using more or less standard arguments we build a measure for Y; see e.g. [5].

Theorem 16. *For $Y \in \mathscr{L}$, the physical measure μ of Y on U is given by*

$$\int \varphi \, d\mu = \frac{1}{\int \tau d\tilde{\mu}} \int \int_0^{\tau(x)} \varphi(X(x,t))dt \, d\tilde{\mu}(x)$$

for any continuous $\varphi : U \to \mathbb{R}$. The measure μ_Y is a unique physical measure supported on the attractor Λ_Y.

The next result gives the statistical stability of Lorenz flows.

Theorem 17 (Alves-Soufi [3]). *Each $Y \in \mathscr{L}$ is statistically stable.*

Our goal in the subsections below is to give a brief idea about the proof of this theorem.

Remark 1. As the physical measure is supported on attractor Λ_Y, which has zero Lebesgue measure, it makes no sense to talk about strong statistical stability. In the sections below we shall give an idea of the proof of this last result.

4.3 Statistical Stability

Let $X_n \in \mathscr{L}$ is a sequence converging to $X_0 \in \mathscr{L}$ in the C^2 topology. We shall simplify our notations and write $\tilde{\mu}_n = \tilde{\mu}_{X_n}$ and $\mu_n = \mu_{X_n}$ for all $n \geq 0$. We also let λ denote Lebesgue measure.

The next result gives the statistical stability of the Poincaré maps.

Proposition 1. *$\tilde{\mu}_n \longrightarrow \tilde{\mu}_0$ as $n \to \infty$ in the weak* topology.*

Proof. We need to show that for any continuous $\varphi : \Sigma \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int \varphi d\tilde{\mu}_n = \int \varphi d\tilde{\mu}_0.$$

By definition

$$\lim_{n\to\infty} \int \varphi d\tilde{\mu}_n = \lim_{n\to\infty} \lim_{m\to\infty} \int (\varphi \circ P_n^m)_- \, d\bar{\mu}_n.$$

We have

$$\left| \int (\varphi \circ P_n^m)_- d\bar{\mu}_n - \int (\varphi \circ P_0^m)_- d\bar{\mu}_0 \right| \leq$$
$$\left| \int (\varphi \circ P_n^m)_- d\bar{\mu}_n - \int (\varphi \circ P_0^m)_- d\bar{\mu}_n \right|$$
$$+ \left| \int (\varphi \circ P_0^m)_- d\bar{\mu}_n - \int (\varphi \circ P_0^m)_- d\bar{\mu}_0 \right|.$$

The second term tends to zero since it is less that $\|\varphi\|_\infty \|d\bar{\mu}_n/d\lambda - d\bar{\mu}_0/d\lambda\|_1$ and

$$d\bar{\mu}_n/d\lambda \xrightarrow{L^1} \bar{\mu}_0/d\lambda, \quad \text{as } n \to \infty.$$

We are left to prove that the first term converges to zero when $n \to \infty$.

$$\begin{aligned}
&\left|\int (\varphi \circ P_n^m)_- d\bar{\mu}_n - \int (\varphi \circ P_0^m)_- d\bar{\mu}_n\right| \\
&\quad = \left|\int (\varphi \circ P_n^m)_- \frac{d\bar{\mu}_n}{d\lambda} d\lambda - \int (\varphi \circ P_0^m)_- \frac{d\bar{\mu}_n}{d\lambda} d\lambda\right| \\
&\quad \le \int |(\varphi \circ P_n^m)_- - (\varphi \circ P_0^m)_-| \left|\frac{d\bar{\mu}_n}{d\lambda}\right| d\lambda \\
&\quad \le C \int |(\varphi \circ P_n^m)_- - (\varphi \circ P_0^m)_-| d\lambda
\end{aligned}$$

The contraction factor on the stable leaves of the cross-section Σ can be taken the same for all vector fields in $\mathscr{L}$. So, the last expression can be made uniformly small. □

Proposition 2. *$\mu_n \longrightarrow \mu_0$ as $n \to \infty$ in the weak* topology.*

To prove this result we need to show that for any uniformly continuous and bounded function $\varphi : U \to \mathbb{R}$,

$$\left|\int \varphi \, d\mu_n - \int \varphi \, d\mu_0\right| \tag{1}$$

can be made small when we take large n. Observe that (1) is bounded by the sum of the terms

$$\left|\frac{1}{\int \tau_n d\tilde{\mu}_n} - \frac{1}{\int \tau_0 d\tilde{\mu}_0}\right| \int \int_0^{\tau_0(x)} |\varphi(X_0(x,t))| dt \, d\tilde{\mu}_0(x), \tag{*}$$

and

$$\frac{1}{\int \tau_n d\tilde{\mu}_n} \left|\int \int_0^{\tau_n(x)} \varphi(X_n(x,t)) dt \, d\tilde{\mu}_n - \int \int_0^{\tau_0(x)} \varphi(X_0(x,t)) dt \, d\tilde{\mu}_0\right| \tag{**}$$

Lemma 2. $\lim_{n\to\infty} \int \tau_n \, d\tilde{\mu}_n = \int \tau_0 \, d\tilde{\mu}_0.$

Proof. Let A be a small rectangle containing the singular curve of τ_0 and τ_n. Observe that

$$\left|\int \tau_n \, d\tilde{\mu}_n - \int \tau_0 \, d\tilde{\mu}_0\right| = \left|\int_A \tau_n \, d\tilde{\mu}_n - \int_A \tau_0 \, d\tilde{\mu}_0\right| + \left|\int_{A^c} \tau_n \, d\tilde{\mu}_n - \int_{A^c} \tau_0 \, d\tilde{\mu}_0\right|.$$

Since τ_0 and τ_n are both integrable, then first term can be made small by taking A small enough. The second term is bounded by

$$\left|\int_{A^c} \tau_n \, d\tilde{\mu}_n - \int_{A^c} \tau_0 \, d\tilde{\mu}_n\right| + \left|\int_{A^c} \tau_0 \, d\tilde{\mu}_n - \int_{A^c} \tau_0 \, d\tilde{\mu}_0\right|.$$

In A^c we have bounded return time and so $|\tau_n - \tau_0|$ can be made arbitrarily small by taking n large enough. The second term converges to zero as n goes to ∞, because $\tilde{\mu}_n \xrightarrow{\text{w}^*} \tilde{\mu}_0$. □

The previous lemma gives the convergence of (*) to zero as $n \to \infty$:

$$\left|\frac{1}{\int \tau_n \, d\tilde{\mu}_n} - \frac{1}{\int \tau_0 \, d\tilde{\mu}_0}\right| \int\int_0^{\tau_0(\xi)} |\varphi(X_0(x,t))| dt \, d\tilde{\mu}_0(x)$$
$$\leq \left|\frac{1}{\int \tau_n \, d\tilde{\mu}_n} - \frac{1}{\int \tau_0 \, d\tilde{\mu}_0}\right| \|\varphi\|_\infty \int \tau_0 \, d\tilde{\mu}_0.$$

For (**) we use the next result.

Lemma 3. $\lim_{n\to+\infty} \int\int_0^{\tau_n(x)} \varphi(X_n(x,t)) dt \, d\tilde{\mu}_n(x) = \int\int_0^{\tau_0(x)} \varphi(X_0(x,t)) dt \, d\tilde{\mu}_0(x).$

Proof. Again let A be a small rectangle containing the singular curve of τ_0 and τ_n. The difference

$$\left|\int\int_0^{\tau_n(x)} \varphi(X_n(x,t)) dt \, d\tilde{\mu}_n(x) - \int\int_0^{\tau_0(x)} \varphi(X_0(x,t)) dt \, d\tilde{\mu}_0(x)\right|$$

is bounded by the sum of

$$\left|\int_A \int_0^{\tau_n(x)} \varphi(X_n(x,t)) dt \, d\tilde{\mu}_n(x) - \int_A \int_0^{\tau_0(x)} \varphi(X_0(x,t)) dt \, d\tilde{\mu}_0(x)\right|, \quad (2)$$

and

$$\left|\int_{A^c} \int_0^{\tau_n(x)} \varphi(X_n(x,t)) dt \, d\tilde{\mu}_n(x) - \int_{A^c} \int_0^{\tau_0(x)} \varphi(X_0(x,t)) dt \, d\tilde{\mu}_0(x)\right|. \quad (3)$$

Since $\int_0^{\tau_n(x)} \varphi(X_n(x,t))dt$ and $\int_0^{\tau_0(x)} \varphi(X_0(x,t))dt$ are respectively $\tilde{\mu}_n$ and $\tilde{\mu}_0$ integrable, (2) can be made arbitrarily small by taking A small enough. Observe that (3) is bounded by the sum of

$$\int_{A^c} \left| \int_0^{\tau_n(x)} \varphi(X_n(x,t))dt - \int_0^{\tau_0(x)} \varphi(X_0(x,t))dt \right| d\tilde{\mu}_n(x),$$

and

$$\left| \int_{A^c} \int_0^{\tau_0(x)} \varphi(X_0(x,t))dt\, d\tilde{\mu}_n(x) - \int_{A^c} \int_0^{\tau_0(x)} \varphi(X_0(x,t))dt\, d\tilde{\mu}(x) \right|.$$

As the return times are bounded on A^c, by the continuous variation of trajectories in finite periods of time, we can make the integrand in the first term small for large n. The second term converges to zero as n goes to ∞, because $\tilde{\mu}_n \xrightarrow{w^*} \tilde{\mu}_0$.

□

This completes the proof of Proposition 2.

Remark 2. The previous lemmas are essentially consequence of the following two key ingredients:

1. $\tau_n(x,y,1) \sim -\log|x - c_n| \in L^1(\lambda)$, where the c_n is the discontinuity point of the quotient map f_{X_n}; and
2. There exists $C > 0$ such that $d\bar{\mu}_n/d\lambda \le C$ for all n.

This implies that

$$\int \tau_n d\bar{\mu}_n$$

uniformly bounded. By Hölder Inequality, the same arguments can be carried out under the assumption that $d\bar{\mu}_n/d\lambda$ is uniformly bounded in $L^p(\lambda)$, for some $p < \infty$, as long as we have $\tau_n \in L^q(\lambda)$ for all $q > 1$.

5 Contracting Lorenz Flow

Now we consider a geometric Lorenz vector field X_0 where replace the usual *expanding condition* $\lambda_3 + \lambda_1 > 0$ in the Lorenz vector field by the *contracting condition*

$$\lambda_3 + \lambda_1 < 0.$$

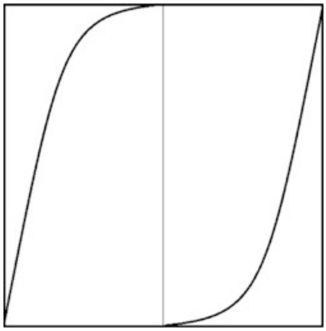

Fig. 5 The map f_0

Under these conditions there is still a trapping region U for X_0 on which

$$\Lambda = \cap_{t\geq 0} X_0^t(U)$$

is a singular-hyperbolic attractor (*Rovella attractor*).

Rovella attractor is not robust. However, Rovella proved in [20] that the chaotic attractor persists in a measure theoretical sense: there exists a one parameter family $(X_a)_{a\in[0,1]}$ of C^3 close vector fields to X_0 which have a transitive non-hyperbolic attractor.

5.1 Rovella Maps

As in the classical Lorenz flow, there is a Poincaré section for X_0 whose return map preserves a stable foliation. Under reasonable conditions, Rovella shows that the quotient map $f_0 : I \setminus \{0\} \to I$ satisfies (Fig. 5):

1. $\lim_{x\to 0^{\pm}} f_0(x) = \mp 1$;
2. ± 1 are pre-periodic and repelling;
3. f_0 is C^3 on $I \setminus \{0\}$ with negative Schwarzian derivative.

The next result shows that for many parameters in the family $(X_a)_a$ the corresponding one-dimensional quotient map behaves as the quadratic maps for the parameters in the set $\mathscr{BC}$.

Theorem 18 (Rovella [20]). *There is a set $\mathscr{R} \subseteq [0,1]$ with full density at 0 such that:*

1. For all $a \in \mathscr{R}$, f_a is C^3 on $I \setminus \{0\}$ and satisfies

$$K_2|x|^{s-1} \leq f_a'(x) \leq K_1|x|^{s-1};$$

2. *There exists* $c > 0$ *such that for all* $a \in \mathscr{R}$

$$(f_a^n)'(\pm 1) > e^{cn}, \quad \textit{for all } n \geq 0;$$

3. *There is* $\alpha > 0$ *such that for all* $a \in \mathscr{R}$

$$|f_a^{n-1}(\pm 1)| > e^{-\alpha n}, \quad \textit{for all } n \geq 1.$$

The existence of physical measures for the one-dimensional Rovella maps can also be proved in this case.

Theorem 19 (Metzger [18]). *For each* $a \in \mathscr{R}$, *the map* f_a *admits a unique SRB measure* μ_a.

We have statistical stability when restricted to set of Rovella parameters.

Theorem 20 (Alves-Soufi [2]). *Rovella maps are nonuniformly expanding with slow recurrence to the critical point. Moreover, there are constants* $C, \tau > 0$ *such that for all* $n \in \mathbb{N}$ *and* $a \in \mathscr{R}$,

$$\left|\Gamma_a^n\right| \leq C e^{-\tau n}.$$

The next corollary follows immediately from Theorems 12 and 20.

Corollary 2. *The map* $\mathscr{R} \ni a \longmapsto d\mu_a/d\lambda$ *is (strongly) continuous.*

5.2 A Final Remark

We obtain physical measures for the Rovella flows exactly as in the classical case. However, the statistical stability for the Rovella flows is still an open problem. We believe that at least when restricted to those flows corresponding to Rovella parameters we have statistical stability. Comparing to the classical case, the main difficulty lies in the fact that the density $d\mu_a/d\lambda$ is no longer (uniformly) bounded. In this direction, the following result has been obtained recently.

Theorem 21 (Cui-Ding [10]). *For* $a \in \mathscr{R}$, *the density of* μ_a *with respect to the Lebesgue measure belongs to some* $L^p(\lambda)$ *with* $p > 1$, *where* p *depends only on the (side) orders of the critical point.*

As observed in Remark 2, statistical stability for the contracting Lorenz flows in the Rovella parameters can be deduced, once we can show that $d\mu_a/d\lambda$ is uniformly bounded in $L^p(\lambda)$, for some $p > 1$. That is an interesting open problem.

Acknowledgements The authors were partially supported by Fundação Calouste Gulbenkian, by CMUP, by the European Regional Development Fund through the Programme COMPETE and by FCT under the projects PTDC/MAT/099493/2008 and PEst-C/MAT/UI0144/2011.

References

1. Alves, J.F.: Strong statistical stability of non-uniformly expanding maps. Nonlinearity **17**(4), 1193–1215 (2004)
2. Alves, J.F., Soufi, M.: Statistical stability and limit laws for Rovella maps. Nonlinearity **25**, 3527–3552 (2012)
3. Alves, J.F., Soufi, M.: Statistical stability of Lorenz attractors (2012, preprint)
4. Alves, J.F., Viana, M.: Statistical stability for robust classes of maps with non-uniform expansion. Ergod. Theory Dyn. Syst. **22**(1), 1–32 (2002)
5. Araujo, V., Pacifico, M.J., Pujals, E.R., Viana, M.: Singular-hyperbolic attractors are chaotic. Trans. Am. Math. Soc. **361**(5), 2431–2485 (2009)
6. Benedicks, M., Carleson, L.: The dynamics of the hénon map. Ann. Math. (2) **133**(1), 73–169 (1991)
7. Benedicks, M., Young, L.-S.: Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps. Ergod. Theory Dyn. Syst. **12**(1), 13–37 (1992)
8. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Mathematics, vol. 470. Springer, Berlin (1975)
9. Bowen, R., Ruelle, D.: The ergodic theory of Axiom A flows. Invent. Math. **29**(3), 181–202 (1975)
10. Cui, H., Ding, Y.: Invariant measures for interval maps with different one-side orders. Ergod. Theory Dyn. Syst. (to appear). doi:10.1017/etds.2013.62
11. Freitas, J.M.: Continuity of SRB measure and entropy for Benedicks-Carleson quadratic maps. Nonlinearity **18**(2), 831–854 (2005)
12. Guckenheimer, J., Williams, R.F.: Structural stability of Lorenz attractors. Inst. Hautes Études Sci. Publ. Math. **50**, 59–72 (1979)
13. Hofbauer, F., Keller, G.: Quadratic maps without asymptotic measure. Commun. Math. Phys. **127**(2), 319–337 (1990)
14. Jakobson, M.V.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. **81**(1), 39–88 (1981)
15. Keller, G.: Stochastic stability in some chaotic dynamical systems. Monatsh. Math. **94**(4), 313–333 (1982)
16. Lorenz, E.N.: Deterministic non-periodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
17. Lyubich, M.: Almost every real quadratic map is either regular or stochastic. Ann. Math. (2) **156**(1), 1–78 (2002)
18. Metzger, R.J.: Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows. Ann. Inst. Henri Poincaré Anal. Non Linéaire **17**(2), 247–276 (2000)
19. Misiurewicz, M.: Absolutely continuous measures for certain maps of an interval. Inst. Hautes Études Sci. Publ. Math. **53**, 17–51 (1981)
20. Rovella, A.: The dynamics of perturbations of the contracting Lorenz attractor. Bol. Soc. Brasil. Mat. (N.S.) **24**(2), 233–259 (1993)
21. Ruelle, D.: A measure associated with Axiom-A attractors. Am. J. Math. **98**(3), 619–654 (1976)
22. Rychlik, M., Sorets, E.: Regularity and other properties of absolutely continuous invariant measures for the quadratic family. Commun. Math. Phys. **150**(2), 217–236 (1992)
23. Sinai, J.G.: Gibbs measures in ergodic theory. Uspehi Mat. Nauk **27**(4), 21–64 (1972)
24. Thunberg, H.: Unfolding of chaotic unimodal maps and the parameter dependence of natural measures. Nonlinearity **14**(2), 323–337 (2001)

25. Tsujii, M.: On continuity of Bowen-Ruelle-Sinai measures in families of one-dimensional maps. Commun. Math. Phys. **177**(1), 1–11 (1996)
26. Tucker, W.: The Lorenz attractor exists. C. R. Acad. Sci. Paris Sér. I Math. **328**(12), 1197–1202 (1999)
27. Viana, M.: Stochastic dynamics of deterministic systems. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro (1997). 22o Colóquio Brasileiro de Matemática. (22th Brazilian Mathematics Colloquium)