Homogeneity at Infinity of Stationary Solutions of Multivariate Affine Stochastic Recursions

Yves Guivarc'h and Émile Le Page

Abstract We consider a d-dimensional affine stochastic recursion of general type corresponding to the relation

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \quad X_0 = x. \tag{S}$$

Under natural conditions, this recursion has a unique stationary solution R, which is unbounded. If $d > 2$, we sketch a proof of the fact that R belongs to the domain of attraction of a stable law which depends essentially of the linear part of the recursion. The proof is based on renewal theorems for products of random matrices, radial Fourier analysis in the vector space $\mathbb{R}^d$, and spectral gap properties for convolution operators on the corresponding projective space. We state the corresponding simpler result for $d = 1$.

1 Notations and Main Result

Let $V = \mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the scalar product $< x, y >= \sum_{i=1}^{d} x_i y_i$ and the corresponding norm $|x| = (\sum_{i=1}^{d} |x_i|^2)^{1/2}$. We denote by $G = GL(V)$ (resp. H =Aff(V)) the linear (resp. affine) group of V and we fix a probability measure μ (resp. λ) on G (resp. H) such that μ is the projection of λ. We consider the affine stochastic recursion (S) on V defined by

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \quad X_0 = x, \tag{S}$$

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where (A_n, B_n) are i.i.d. random variables with law λ, hence A_n (resp. B_n) are i.i.d. random matrices (resp. vectors). We assume that (S) has a stationary solution R which satisfies in distribution

$$R = AR^1 + B$$

where R^1 has the same law as R and is independent of (A, B). We are interested in the "asymptotic shape" of the law ρ of R. Our focus will be on the case $d > 1$. For $d = 1$, corresponding results are described in Sect. 4.

We denote by $\eta * \theta$ the convolution of a probability measure η on H with a positive Radon measure θ on V i.e. $\eta * \theta = \int \delta_{hx}\, d\eta(h)\, d\theta(x)$. Also η^n denotes the nth convolution iterate of η. With these notations, the law ρ_n of X_n is given by $\rho_n = \lambda^n * \delta_x$, and a λ-stationary (probability) measure ρ satisfies $\lambda * \rho = \rho$.

We denote by Ω the product space $H^{\otimes\mathbb{N}}$, by $\mathbb{P}$ the product measure $\lambda^{\otimes\mathbb{N}}$ on Ω, and by $\mathbb{E}$ the corresponding expectation operator. Provided that

$$\mathbb{E}(|\log |A||) + \mathbb{E}(|\log |B||) < \infty,$$

it is well known (see [12] for example) that a λ-stationary measure ρ exists and is unique if the top Lyapunov exponent

$$L_\mu = \lim_{n\to\infty} \frac{1}{n}\, \mathbb{E}(\log |A_n \dots A_1|)$$

is negative. For informations on products of random matrices we refer to [2,5,9].

Since the properties of μ play a dominant role for the "shape" of ρ, we give now a few corresponding notations. Let S (resp. T) be the closed subsemigroup of G (resp. H) generated by the support suppμ (resp. suppλ) of μ (resp. λ) and write

$$S_n = A_n \dots A_1, \quad k(s) = \lim_{n\to\infty} (\mathbb{E}(|S_n|^s))^{1/n} \ (s \geq 0).$$

Then $\log k(s)$ is a convex function on $I_\mu = \{s \geq 0; k(s) < \infty\}$ and we write $s_\infty = \sup\{s \geq 0; k(s) < \infty\}$.

We denote by $\mathbb{S}^{d-1}$ (resp. $\mathbb{P}^{d-1}$) the unit sphere (resp. projective space) of V and observe that in polar coordinates:

$$V\backslash\{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_+^*.$$

If $\dot{V}$ denotes the factor space of $V\backslash\{0\}$ by the group $\{\pm Id\}$, we have also

$$\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*.$$

For $x \in V\backslash\{0\},\ g \in G$, we write $\tilde{x}$ (resp. $\bar{x}$) for the projection of $x \in V$ on $\mathbb{S}^{d-1}$ (resp. $\mathbb{P}^{d-1}$), $g \cdot \tilde{x}$ (resp. $g \cdot \bar{x}$) for the projection of gx on $\mathbb{S}^{d-1}$ (resp. $\mathbb{P}^{d-1}$).

For some moment conditions on μ, the quantity $\gamma(g) = \sup(|g|, |g^{-1}|)$ will be used. The dual map g^* of $g \in GL(V)$ is defined by $< g^*x, y >=< x, gy >$ and the push-forward of μ by $g \to g^*$ will be denoted μ^*.

From now on, we assume $d > 1$. An element $g \in G$ is said to be *proximal* if g has a unique simple dominant eigenvalue $\lambda_g \in \mathbb{R}$ with $|\lambda_g| = \lim_{n\to\infty} |g^n|^{1/n}$. In this case we have the decomposition $V = \mathbb{R}w_g \oplus V_g^<$ where w_g is a dominant eigenvector and $V_g^<$ a g-invariant hyperplane. We say that a semigroup of G satisfies *condition* $i - p$ if this semigroup contains a proximal element and does not leave any finite union of subspaces invariant.

One can observe that, if $d > 1$, the set of probability measures μ on G such that S satisfies condition $i - p$ is open and dense in the weak topology. Also, condition $i - p$ is satisfied for S if and only if it is satisfied for the closed subgroup $Zc(S)$, the Zariski closure of S, which is a Lie group with a finite number of components. Thus condition $i - p$ is in particular satisfied, if $Zc(S) = G$.

It is known that, if the probability measure μ satisfies $\mathbb{E}(|\log |A||) < \infty$ and suppμ generates a closed semigroup S satisfying condition $i - p$, then the top Lyapunov exponent of μ is simple (see [2]). In this case $\log k(s)$ is strictly convex and analytic on $[0, s_\infty[$ (see [9]). Also the set S^{prox} of proximal elements in S is open and the set of corresponding positive dominant eigenvalues generates a dense subgroup of $\mathbb{R}_+^*$. Furthermore, the action of μ on $\mathbb{P}^{d-1}$ has a unique μ-stationary measure ν and suppν is the unique S-minimal subset of $\mathbb{P}^{d-1}$; the set $\Lambda(S) = \text{supp}\nu$ is the closure of $\{\bar{w}_g; g \in S^{\text{prox}}\}$ and has positive Hausdorff dimension.

Under condition $i - p$ and for the action of S on $\mathbb{S}^{d-1}$, there are two cases I, II, say, regarding the existence of a convex S-invariant cone in V. In case I (non-existence), the inverse image $\tilde{\Lambda}(S)$ of $\Lambda(S)$ in $\mathbb{S}^{d-1}$ is the unique S-minimal invariant set in $\mathbb{S}^{d-1}$. In case II (existence), $\tilde{\Lambda}(S)$ splits into two symmetric S-minimal subsets $\tilde{\Lambda}_+(S)$ and $\tilde{\Lambda}_-(S)$.

Returning to the affine situation, we need to consider the compactification $V \cup \mathbb{S}_\infty^{d-1}$ of V by the sphere at infinity $\mathbb{S}_\infty^{d-1}$ and we identify $\tilde{\Lambda}(S)$ (resp. $\tilde{\Lambda}_+(S), \tilde{\Lambda}_-(S)$) with the corresponding subset $\tilde{\Lambda}^\infty(S)$ (resp. $\tilde{\Lambda}_+^\infty(S), \tilde{\Lambda}_-^\infty(S)$) of $\mathbb{S}_\infty^{d-1}$. We observe that $\mathbb{S}_\infty^{d-1}$ is H-invariant and the corresponding H-action reduces to the G-action.

If $h = (g, b)$ is such that $|g| < 1$, then h has a fixed point $x \in V$, and this point is attractive, i.e. for any $y \in V$, $\lim_{n\to\infty} h^n y = x$. The set $\Delta_a(T)$ of such attractive fixed points of elements $h \in T$ plays an important role in the description of supp ρ, for ρ λ-stationary.

On the other hand, if for some $s > 0$ we have $k(s) > 1$ and condition $i - p$ is satisfied, then one can show the existence of $g \in S$ with $\lim_{n\to\infty} |g^n|^{1/n} > 1$. This implies that supp$\rho$ is unbounded, if suppλ has no fixed point in V.

We have the following basic (see [10], Proposition 5.1)

Proposition 1. *Assume* $\mathbb{E}(\log \gamma(A)) + \log |B|) < \infty$ *and*

$$L_\mu = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log |S_n|) < 0.$$

Then $R_n = \sum_1^n A_1 \dots A_{k-1} B_k$ *converges* $\mathbb{P}$*-a.e. to*

$$R = \sum_1^\infty A_1 \dots A_{k-1} B_k,$$

and for any $x \in V$, X_n *converges in law to* R. *If* $\beta \in I_\mu$ *satisfies* $k(\beta) < 1$, $\mathbb{E}(|B|^\beta) < \infty$, *then* $\mathbb{E}(|R|^\beta) < \infty$.

The law ρ *of* R *is the unique* λ*-stationary measure on* V. *The closure* $\overline{\Delta_a(T)} = \Lambda_a(T)$ *in* V *is the unique* T*-minimal subset of* V *and* $\Lambda_a(T) = \operatorname{supp}\rho$. *If the semigroup* S *satisfies condition* $i-p$ *and* $\operatorname{supp}\lambda$ *has no fixed point in* V, *then* $\rho(W) = 0$ *for any affine subspace* W. *Furthermore, if* T *contains an element* (g,b) *with* $\lim_{n\to\infty} |g^n|^{1/n} > 1$, *then* $\Lambda_a(T)$ *is unbounded.*

The first part of the proposition is well known (see for example [12]).

For $s \geq 0$, we denote by l^s (resp. h^s) the s-homogeneous measure (resp. function) on $\mathbb{R}_+^*$ given by $l^s(dt) = t^{-(s+1)}dt$, $l^0 = l$ (resp. $h^s(t) = t^s$). We observe that the cone of Radon measures on $\dot{V}$ which are of the form $\eta \otimes l^s$ with η a positive measure on $\mathbb{P}^{d-1}$ is G-invariant. Also $g(\eta \otimes l^s) = (\rho_s(g)\eta) \otimes l^s$ with

$$\rho_s(g)\eta = \int |gx|^s \delta_{g\cdot x} d\eta(x).$$

One can show that if the subsemigroup S associated to μ satisfies condition $i-p$ and $s \in I_\mu$, there exists a unique probability measure ν^s on $\mathbb{P}^{d-1}$ such that equation $\mu * (\nu^s \otimes l^s) = k(s)\nu^s \otimes l^s$ is satisfied. Furthermore ν^s gives mass zero to any projective hyperplane and $\operatorname{supp}\nu^s = \Lambda(S)$.

We denote by $\tilde{\nu}^s$ the unique symmetric positive measure on $\mathbb{S}^{d-1}$ with projection ν^s on $\mathbb{P}^{d-1}$ and (in case II) by $\tilde{\nu}_+^s$, $\tilde{\nu}_-^s$ its normalized restrictions to $\tilde{\Lambda}_+(S)$, $\tilde{\Lambda}_-(S)$ hence $\tilde{\nu}^s = \frac{1}{2}(\tilde{\nu}_+^s + \tilde{\nu}_-^s)$. Then we have

$$\mu * (\tilde{\nu}^s \otimes l^s) = k(s)\tilde{\nu}^s \otimes l^s$$

and

$$\mu * (\tilde{\nu}_+^s \otimes l^s) = k(s)\tilde{\nu}_+^s \otimes l^s, \quad \mu * (\tilde{\nu}_- \otimes l^s) = k(s)\tilde{\nu}_-^s \otimes l^s.$$

If there exists $\alpha \in I_\mu$ such that $k(\alpha) = 1$, the measures $\tilde{\nu}^\alpha \otimes l^\alpha$, $\tilde{\nu}_+^\alpha \otimes l^\alpha$, $\tilde{\nu}_-^\alpha \otimes l^\alpha$ enter in an essential way in the description of the "shape" of ρ. We need first to discuss the action of S on $\mathbb{S}_\infty^{d-1}$, if supp ρ is unbounded. In this case $\overline{\operatorname{supp}\rho} \cap \mathbb{S}_\infty^{d-1}$ is a non trivial closed S-invariant set, hence three cases can occur, in view of the above discussion of minimality.

CASE I: S has no proper convex invariant cone and $\overline{\Lambda_a(T)} \supset \tilde{\Lambda}^\infty(S)$.

CASE II': S has a proper convex invariant cone and $\overline{\Lambda_a(T)} \supset \tilde{\Lambda}^\infty(S)$.

CASE II": S has a proper convex invariant cone and $\overline{\Lambda_a(T)}$ contains only one of the sets $\tilde{\Lambda}_+^\infty(S)$, $\tilde{\Lambda}_-^\infty(S)$, say $\tilde{\Lambda}_+^\infty(S)$ hence $\tilde{\Lambda}_-^\infty(S) \cap \overline{\Lambda_a(T)} = \emptyset$.

The push-forward of a measure η on V by the dilation $x \to tx$ $(t > 0)$ will be denoted $t.\eta$. For $d > 1$, our main result in [10] is the following

Theorem 1. *With the above notations, assume that S satisfies condition $i - p$, that T has no fixed point in V, that $L_\mu < 0$, and that there exists $\alpha > 0$ with $k(\alpha) = 1$ and $\mathbb{E}(|A|^\alpha \gamma^\delta(A)) < \infty$, $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$ for some $\delta > 0$. Then* supp ρ *is unbounded and we have the following vague convergence on $V \backslash \{0\}$:*

$$\lim_{t \to 0+} t^{-\alpha}(t.\rho) = C(\sigma^\alpha \otimes l^\alpha) = \Lambda,$$

where $C > 0$, σ^α is a probability on $\mathbb{S}^{d-1}$ and the Radon measure Λ satisfies $\mu * \Lambda = \Lambda$. Moreover,

$$\sigma^\alpha = \begin{cases} \tilde{\nu}^\alpha & \textit{in Case I}, \\ C_+\tilde{\nu}_+^\alpha + C_-\tilde{\nu}_-^\alpha \textit{ for some } C_+, C_- > 0 & \textit{in Case II'}, \\ \tilde{\nu}_+^\alpha & \textit{in Case II''}. \end{cases}$$

The measures $\tilde{\nu}^\alpha \otimes l^\alpha$ (case I), $\tilde{\nu}_+^\alpha \otimes l^\alpha$ and $\tilde{\nu}_-^\alpha \otimes l^\alpha$ (cases II', II") are minimal μ-harmonic measures.

The above convergence is valid on any Borel function f with $\sigma^\alpha \otimes l^\alpha$-negligible set of discontinuities such that $|w|^{-\alpha}|\log|w||^{1+\varepsilon}|f(w)|$ is bounded for some $\epsilon > 0$, hence

$$\lim_{t \to 0+} t^{-\alpha}\mathbb{E}(f(tR)) = \Lambda(f).$$

The theorem shows that ρ belongs to the domain of attraction of a stable law, a fact conjectured by F. Spitzer. It plays a basic role in the study of slow diffusion for random walk in a random medium on $\mathbb{Z}$ (see [7]), and also in extreme value theory for GARCH processes. The proof of the theorem shows that the above convergence is valid on the sets $H_w^+ = \{x \in V; < x, w > 1\}$ for $w \in V \backslash \{0\}$ under the weaker hypothesis $\mathbb{E}(|A|^\alpha \log \gamma(A) + |B|^{\alpha+\delta}) < \infty$. Then, using [1], it follows that the theorem is valid if $\alpha \notin \mathbb{N}$. Actually, [1] implies also the validity of the theorem under the above weaker hypothesis, in the following situations:

CASE I and $\alpha \notin 2\mathbb{N}$,
CASE II" and $\alpha > 0$,
CASE II' and $C_+ = C_-$, $\alpha \notin 2\mathbb{N}$.

As observed in [14], the condition $C_+ = C_-$ occurs if ρ is symmetric, in particular if the law of B is symmetric (for example if B is Gaussian). In the context of extreme value theory the convergence stated in the theorem says that

ρ has "multivariate regular variation". This property is basic for the development of the theory for "ARCH processes" (see [14]).

The proof given in [10] (Theorem 6) is long. For a short survey of earlier work, see [8]. Here we will give a sketch of a few main points of the proof.

2 Some Tools for the Proof of the Theorem

2.1 *The Renewal Theorem for Products of Random Matrices ($d > 1$)*

We use the notations already introduced above: μ is a probability measure on $G = GL(V)$, S the closed subsemigroup of G generated by $\operatorname{supp}\mu$, L_μ the top Lyapunov exponent of μ, ν the μ-stationary measure on $\mathbb{P}^{d-1}$ etc. Under condition $i-p$, the following is the d-dimensional analog of the classical renewal theorem (see [4]) and follows from the general renewal theorem of Kesten [13] for Markov random walks on $\mathbb{R}$.

Theorem 2. *Assume that the semigroup S associated with μ satisfies condition $i-p$, that $\log\gamma(g)$ is μ-integrable, and that $L_\mu = \lim_{n\to\infty}\frac{1}{n}\int\log|g|d\mu^n(g) > 0$. Then, for any $w \in V$, $\sum_0^\infty \mu^k * \delta_w$ is a Radon measure on $\dot{V}$ and we have*

$$\lim_{w\to 0}\sum_0^\infty \mu^k * \delta_w = \frac{1}{L_\mu}\nu^0 \otimes l.$$

in the sense of vague convergence. This convergence is also valid on any bounded continuous function f on $\dot{V}$ with $\sum_{-\infty}^{\infty}\sup\{|f(w)|; 2^l \le |w| \le 2^{l+1}\} < \infty$.

As proved in [10], if S satisfies $i-p$, $s \in I_\mu$ and $\int |g|^s \log\gamma(g)d\mu(g) < \infty$, then the top Lyapunov exponent $L_\mu(s) = \lim_{n\to\infty}\frac{1}{n}\int|g|^s\log|g|d\mu^n(g)$ exists, is simple and satisfies $L_\mu(s) = \frac{k'(s_-)}{k(s)} < \infty$. Also there exists a unique positive function e^s on $\mathbb{P}^{d-1}$ such that $\nu^s(e^s) = 1$ and

$$\mu * \delta_w(e^s \otimes h^s) = k(s)(e^s \otimes h^s)(w).$$

Then, using [13] again, we have the following result which includes information on the fluctuations of $S_n w$:

Theorem 3. *Assume that $L_\mu < 0$, $\alpha \in I_\mu$ exists with $\alpha > 0$, $k(\alpha) = 1$, $\int|g|^\alpha\log\gamma(g)d\mu(g) < \infty$, and S satisfies condition $i-p$. Then we have the following vague convergence on $\dot{V}$, for any $w \in \dot{V}$*

$$\lim_{t \to 0+} t^{-\alpha} \sum_{0}^{\infty} \mu^k * \delta_{tw} = \frac{(e^\alpha \otimes h^\alpha)(w)}{L_\mu(\alpha)} \nu^\alpha \otimes l^\alpha.$$

This convergence is actually valid on any continuous function f on $\dot{V}$ such that $f_\alpha(w) = |w|^{-\alpha} f(w)$ is bounded and $\sum_{-\infty}^{\infty} \sup\{f_\alpha(w); 2^l \le |w| \le 2^{l+1}\} < \infty$.

In particular for some $A > 0$ and any $w \in V$

$$\lim_{t \to \infty} t^\alpha \, \mathbb{P}\{\sup_{n \ge 1} |S_n w| > t\} = A(e^\alpha \otimes h^\alpha)(w).$$

The last formula is the so-called Cramér estimate of ruin in collective risk systems if $d = 1$ [4].

For the convergence proof in Theorem 1, we will need an analogue of Theorem 3 with $\dot{V}$ replaced by $V \setminus \{0\}$. For $u \in \mathbb{S}^{d-1}$, the function $e^\alpha(u)$ can be lifted to $\mathbb{S}^{d-1}$ and we have

$$\int |gu|^\alpha \frac{e^\alpha(g.u)}{e^\alpha(u)} \, d\mu^n(g) = 1$$

for any $n \in \mathbb{N}$. Hence the family of probability measures $|gu|^\alpha \frac{e^\alpha(g.u)}{e^\alpha(u)} d\mu^{\otimes n}(g)$ with $g = g_1 \dots g_n$ defines a projective system on the spaces $G^{\otimes n}$ and one can consider the projective limit $\mathbb{Q}_u^\alpha$ on $G^{\otimes \mathbb{N}}$. Referring again to [13], we get the following

Theorem 4. *Assume μ and α are as in Theorem 3. Then, for any $u \in \mathbb{S}^{d-1}$, we have the vague convergence*

$$\lim_{t \to 0+} t^{-\alpha} \sum_{0}^{\infty} \mu^k * \delta_{tu} = \frac{1}{L_\mu(\alpha)} e^\alpha(u) \, \tilde{\nu}_u^\alpha \otimes l^\alpha,$$

where $\tilde{\nu}_u^\alpha$ is a probability measure on $\mathbb{S}^{d-1}$ and $\tilde{\nu}_u^\alpha \otimes l^\alpha$ is a μ-harmonic Radon measure on $V \setminus \{0\}$. The convergence is valid on any continuous function f such that $f_\alpha(w) = |w|^{-\alpha} f(w)$ is bounded and satisfies

$$\sum_{-\infty}^{\infty} \sup\{|f_\alpha(w)|; 2^l \le |w| \le 2^{l+1}\} < \infty.$$

There are two cases:

Case I: $\tilde{\nu}_u^\alpha = \tilde{\nu}$ has support $\tilde{\Lambda}(S)$.
Case II: $\tilde{\nu}_u^\alpha = p_+^\alpha(u)\tilde{\nu}_+^\alpha + p_-^\alpha(u)\tilde{\nu}_-^\alpha$, where $p_+^\alpha(u)$ (resp. $p_-^\alpha(u)$) is the entrance probability under $\mathbb{Q}_u^\alpha$ of $S_n \cdot u$ into the convex envelope of $\tilde{\Lambda}_+(S)$ (resp. $\tilde{\Lambda}_-(S)$).

These results improve earlier ones by Kesten [12] and Le Page [16].

2.2 A Spectral Gap Property for Convolution Operators ($d > 1$)

As above we consider the operator P on $\dot{V}$ defined by $Pf(w) = (\mu * \delta_w)(f)$ and its action on s-homogeneous functions. The Euclidean norm on V extends to a norm on the wedge product $\bigwedge^2 V$: For $x, y, x', y' \in V$, we put

$$< x \wedge y, x' \wedge y' > := \det \begin{pmatrix} < x, x' > & < x, y' > \\ < y, x' > & < y, y' > \end{pmatrix}.$$

This allows to consider the distance δ on $\mathbb{P}^{d-1}$ defined by $\delta(x, y) = |x \wedge y|$, where x, y correspond to unit vectors $\tilde{x}, \tilde{y}$ in $\mathbb{S}^{d-1}$. We will denote by $H_\varepsilon(\mathbb{P}^{d-1})$ the space of ε-Hölder functions on $\mathbb{P}^{d-1}$ with respect to the distance δ. We write

$$[\varphi]_\varepsilon = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\delta(x, y)^\varepsilon}, \quad |\varphi| = \sup_x |\varphi(x)|, \quad |\varphi|_\varepsilon = [\varphi]_\varepsilon + |\varphi|,$$

and we observe that $\varphi \to |\varphi|_\varepsilon$ defines a norm on $H_\varepsilon(\mathbb{P}^{d-1})$.

If $z \in \mathbb{C}$, $z = s + it$, and the z-homogeneous function f on $\dot{V}$ is of the form $f = \varphi \otimes h^z$, with $\varphi \in H_\varepsilon(\mathbb{P}^{d-1})$, the action of P on f defines an operator P^z on φ by

$$Pf = P^z\varphi \otimes h^z, \quad \text{i.e.} \quad P^z\varphi(x) = \int \varphi(g \cdot x)\, |gx|^z \, d\mu(g).$$

Then we have the following (see [10], Theorem A)

Theorem 5. *Let $d > 1$ and assume that the closed subsemigroup S generated by* $\operatorname{supp}\mu$ *satisfies condition $i-p$. For $s \in I_\mu$, assume $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$. Then, for any $\varepsilon > 0$ sufficiently small, the operator P^s on $H_\varepsilon(\mathbb{P}^{d-1})$ has a spectral gap, with dominant eigenvalue $k(s)$:*

$$P^s = k(s)(\nu^s \otimes e^s + U_s),$$

where $\nu^s \otimes e^s$ is the projection on $\mathbb{C}e^s$ defined by ν^s, e^s and U_s is an operator with spectral radius less than 1 which commutes with $\nu^s \otimes e^s$. Furthermore, if $\Im z = t \neq 0$, $z = s + it$, then the spectral radius of P^z is less than $k(s)$.

If $s = 0$, P^s reduces to convolution by μ on $\mathbb{P}^{d-1}$ and convergence to the unique μ-stationary measure $\nu^0 = \nu$ was a basic property studied in [5]. In this case the spectral gap property is a consequence of the simplicity of the top Lyapunov exponent of μ (see [2, 9]). The spectral gap properties of P^s are basic ingredients for the study of precise large deviations for the product of random matrices $S_n = A_n \dots A_1$ (see [16, 18]). Here the theorem will be used for the study of s-homogeneous P-eigenmeasures on $\dot{V}$ and $V\setminus\{0\}$. In the context of $V\setminus\{0\}$ we need to replace $\mathbb{P}^{d-1}$ by $\mathbb{S}^{d-1}$ and to use an analogous theorem (see [10]).

2.3 A Choquet-Deny Property for Markov Walk

Here (S, δ) is a compact metric space and P is a Markov kernel on $S\times\mathbb{R} = Y$ which commutes with the $\mathbb{R}$-translations and acts continuously on the space $C_b(S\times\mathbb{R})$ of continuous bounded functions on $S\times\mathbb{R}$. Such a set of datas will be called a Markov walk on $\mathbb{R}$. We define for $t \in \mathbb{R}$ the Fourier operator P^{it} on $C(S)$ by

$$P^{it}\varphi(x) = P(\varphi \otimes e_{it})(x, 0),$$

where e_{it} is the Fourier exponential on $\mathbb{R}$, $e_{it}(r) = e^{itr}$. For $t = 0$, $P^{it} = P^0$ is equal to $\bar{P}$, the factor operator on S defined by P. We assume that for $\varepsilon > 0$ P^{it} preserves the space of ε-Hölder functions $H_\varepsilon(S)$ on (S, δ) and is a bounded operator on $H_\varepsilon(S)$.

We denote $[\varphi]_\varepsilon = \sup_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{\delta(x,y)^\varepsilon}$, $|\varphi| = \sup_x |\varphi(x)|$ for $\phi \in C(S)$. Moreover, we assume that P^{it} and P satisfy the following condition D:

1. For any $t \in \mathbb{R}$, one can find $n_0 \in \mathbb{N}$, $\rho(t) \in [0, 1[$ and $C(t) \geq 0$ for which

$$[(P^{it})^{n_0}\varphi]_\varepsilon \leq \rho(t)[\varphi]_\varepsilon + C(t)|\varphi|.$$

2. For any $t \in \mathbb{R}$, the equation $P^{it}\varphi = e^{i\theta}\varphi$, $\varphi \in H_\varepsilon(S)$, $\varphi \neq 0$, has only the trivial solution $e^{i\theta} = 1$, $t = 0$, $\varphi =$ constant.
3. For some $\delta > 1$: $M_\delta = \sup_{x\in S} \int |a|^\delta P((x, 0), d(y, a)) < \infty$.

Conditions 1 and 2 above imply that $\bar{P}$ has a unique stationary measure π and the spectrum of $\bar{P}$ in $H_\varepsilon(S)$ is of the form $\{1\} \cup \Delta$, where Δ is a compact subset of the open unit disk (see [11]). They imply also that for any $t \neq 0$, the spectral radius of P^{it} is less than one.

If $Y = \dot{V}$, P is the convolution operator by μ on $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+$ $(d > 1)$, hence $S = \mathbb{P}^{d-1}$ and $\mathbb{R}^*_+ = \exp\mathbb{R}$. Theorem 5 implies that condition D is satisfied if $I_\mu \neq 0$ and condition $i - p$ is valid.

Furthermore, for $s \in I_\mu$ one can also consider the Markov operator Q_s on $\dot{V}$ defined by

$$Q_s f = \frac{1}{k(s)e^s \otimes h^s} P(f e^s \otimes h^s).$$

If for some $\delta > 0$, $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$, Theorem 5 implies that conditions D are also satisfied by Q_s.

We will say that a Radon measure θ on $Y = S \times \mathbb{R}$ is translation-bounded if for any compact $K \subset Y$ there exists $C(K) > 0$ such that $\theta(K + t) \leq C(K)$ for any $t \in \mathbb{R}$, where $K + t$ is the set obtained from K by translation with t. Then we have the following Choquet-Deny type property

Theorem 6. *With the above notations if the Markov operator P on $Y = S \times \mathbb{R}$ satisfies the condition D. Then any translation-bounded P-harmonic measure on Y is proportional to $\pi \otimes l$ with $l = dt$.*

This theorem can be used for $Y = \dot{V}$ and $P = Q_\alpha$ if $0 < \alpha < s_\infty$.

2.4 A Weak Renewal Theorem

As in the Sect. 2.3, we consider a Markov walk P on $\mathbb{R}$ with compact factor space S, a probability ν on S such that $\nu \otimes l$ is P-invariant. A path starting from S for this Markov chain will be denoted (X_n, V_n) with $X_n \in S,\ V_n \in \mathbb{R}$ and the canonical probability measure on the paths starting from $x \in S$ will be denoted by ${}^a\mathbb{P}_x$. We write also ${}^a\mathbb{P}_\nu = \int {}^a\mathbb{P}_x d\nu(x)$.

For a non negative Borel function on $S \times \mathbb{R}$, we write $U\psi = \sum_0^\infty P^k \psi$. We observe that if $(x,t) \in S \times \mathbb{R}$, $\psi = 1_K$, then $U\psi(x,t)$ is the expected number of visits to K starting from $(x,t) \in S \times \mathbb{R}$. In other words $U\psi(x,t) = \mathbb{E}_x\left(\sum_0^\infty \psi(X_k, t + V_k)\right)$. Then we have the following weak analogue of the renewal theorem.

Proposition 2. *Suppose that ψ is a bounded, non-negative and compactly supported Borel function on $S \times \mathbb{R}$. Further suppose that the potential $U\psi = \sum_0^\infty P^k \psi$ is locally bounded and that, for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} {}^a\mathbb{P}_\nu \left\{ \left| \frac{V_n}{n} - \gamma \right| > \varepsilon \right\} = 0 \quad \textit{with} \quad \gamma < 0$$

holds true. Then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t ds \int_s U\psi(x,s) d\nu(x) = \frac{1}{|\gamma|} \int \int_{-\infty}^{\infty} \psi(x,s) d\nu(x) ds.$$

If ψ is a non-negative Borel function on S such that $\lim_{t\to\infty} U\psi(x,t) = 0$ *ν-a.e., then $\psi = 0$ $\nu \otimes l$-a.e.*

3 Elements of Proof of Theorem 1

3.1 Convergence for Radon Transforms

For a finite measure η on V we write $\hat{\eta}(w) = \eta(H_w^+)$ where $u = tw,\ t > 0,\ u \in \mathbb{S}^{d-1},\ H_w^+ = \{x \in V; < x, w >> 1\}$. We observe that $\hat{\eta}$ can be considered as an integrated form of the Radon transform of η. Observe that $\widehat{\mu * \eta}(w) = (\mu^* * \delta_w)(\hat{\eta})$, hence convolution equations on $G \times V$ can be transformed to functional equations for Radon transforms.

We will not be able to apply directly the renewal Theorem 4 to the convolution equation $\lambda * \rho = \rho$ corresponding to $R = AR^1 + B$ but rather to functional equations for $\hat{\rho}$ and μ^*. We denote by ρ_1 the law of $R - B$ and we begin with the

Proposition 3. *With the hypothesis of Theorem 4, we denote by* ${}^*\tilde{\nu}_u^\alpha$ *the positive kernel on* $\mathbb{S}^{d-1}$ *given by Theorem 4 and associated with* μ^**. Then one has the equations on* $V \setminus \{0\}$

$$\rho = \sum_0^\infty \mu^k * (\rho - \rho_1), \qquad \hat{\rho}(w) = \sum_0^\infty ((\mu^*)^k * \delta_w)(\hat{\rho} - \hat{\rho}_1).$$

For $u \in \mathbb{S}^{d-1}$, *if* $\alpha \in]0, s_\infty[$, $k(\alpha) = 1$, *the function* $t \to t^{\alpha-1}(\hat{\rho} - \hat{\rho}_1)(u,t)$ *is Riemann-integrable on* $]0,\infty[$ *and one has, with* $r_\alpha(u) = \int_0^\infty t^{\alpha-1}(\hat{\rho} - \hat{\rho}_1)(u,t)dt$

$$\lim_{t\to\infty} t^\alpha \hat{\rho}(u,t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)}\, {}^*\tilde{\nu}_u^\alpha(r_\alpha) = C(\sigma^\alpha \otimes l^\alpha)(H_u^+),$$

where $C \geq 0$ *and the probability* σ^α *on* $\tilde{\Lambda}(S)$ *satisfies* $\mu * (\sigma^\alpha \otimes l^\alpha) = \sigma^\alpha \otimes l^\alpha$. *There exists* $b > 0$ *such that* $\mathbb{P}\{|R| > t\} \leq bt^{-\alpha}$. *Furthermore* $\operatorname{supp} \rho$ *is unbounded and: In case I:* $\sigma^\alpha = \tilde{\nu}^\alpha$, *in case II:* $C\sigma^\alpha = C_+\tilde{\nu}_+^\alpha + C_-\tilde{\nu}_-^\alpha$, $C_+, C_- \geq 0$.

Sketch of Proof

Since $|g^*| = |g|$, the function $k(s)$ is equal to the corresponding function for μ^*, condition $i - p$ is satisfied for μ^* and $L_{\mu^*}(\alpha) = L_\mu(\alpha)$. We observe that the stationarity equation $R - B = AR^1$ can be written in distribution as $\rho - \rho_1 = \rho - \mu * \rho$. Also $\rho(\{0\}) = 0$, hence we get

$$\rho = \sum_0^\infty \mu^k * (\rho - \rho_1), \qquad \hat{\rho}(w) = \sum_0^\infty ((\mu^*)^k * \delta_w)(\hat{\rho} - \hat{\rho}_1)$$

on $V \setminus \{0\}$.

In order to use Theorem 4, we need to regularize $\hat{\rho} - \hat{\rho}_1$ by multiplicative convolution on $\mathbb{R}_+^*$ with $1_{[0,1]}$, hence to consider

$$r^\alpha(u,t) = \frac{1}{t}\int_0^t x^{\alpha-1}(\hat{\rho} - \hat{\rho}_1)(u,x)dx.$$

Clearly $|r^\alpha(u,t)| \leq \alpha^{-1}t^{\alpha-1}$. By using the conditions $\mathbb{E}(|A|^{\alpha+\delta}) < \infty$ and $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$, one can show the existence of $\delta' > 0$, $c(\delta') > 0$ such that for $t \geq 1$,

$$|r^\alpha(u,t)| \leq c(\delta')t^{-\delta'}.$$

Then Theorem 4 can be applied to $f_\alpha(w) = r^\alpha(u,t)$, whence, by a Tauberian argument as in [6], we get the convergence of $t^\alpha \hat{\rho}(u,t)$ towards $\frac{1}{L_\mu(\alpha)} {}^*e^\alpha(u)^* \tilde{\nu}_u^\alpha(r_\alpha)$. From the existence of $\alpha \in]0, s_\infty[$ with $k(\alpha) = 1$, one can deduce the existence of $g \in S$ with $|g| > 1$, hence supp ρ is unbounded.

The above formulae and the description of ${}^*e^\alpha$, σ^α in terms of $\tilde{\nu}^\alpha, \tilde{\nu}_+^\alpha, \tilde{\nu}_-^\alpha$ give the harmonicity equation $\mu*(\sigma^\alpha \otimes l^\alpha) = \sigma^\alpha \otimes l^\alpha$. The boundedness of $t^\alpha P\{|R| > t\}$ follows from the convergence of $t^\alpha \hat{\rho}(u,t)$.

3.2 Homogeneity at Infinity of ρ

The boundedness of $t^\alpha P\{|R| > t\}$ stated in Proposition 3 implies that the family of Radon measures $\{t^{-\alpha}(t.\rho); t \in \mathbb{R}_+\}$ is relatively compact in the vague topology.

Proposition 4. *Given the situation of Theorem 1, assume that η is a vague limit of a sequence $t_n^{-\alpha}(t_n \cdot \rho)$ as $t_n \to \infty$. Then η is translation-bounded and satisfies $\mu * \eta = \eta$. If η and $\sigma \otimes l^\alpha$ satisfy*

$$\eta(H_u^+) = (\sigma \otimes l^\alpha)(H_u^+),$$

for any $u \in \mathbb{S}^{d-1}$ and some positive measure σ on $\mathbb{S}^{d-1}$, then $\eta = \sigma \otimes l^\alpha$.

This proposition is based on the moment conditions satisfied by R, A, B, and on Theorem 6. Using furthermore Propositions 4 and 3, we get the

Theorem 7. *With the hypothesis of Theorem 1, we have the following vague convergence*

$$\lim_{t \to 0+} t^{-\alpha}(t.\rho) = \Lambda = C(\sigma^\alpha \otimes l^\alpha),$$

where $C \geq 0$.

The above convergence is also valid on any Borel function f such that the set of discontinuities of f is $(\sigma^\alpha \otimes l^\alpha)$-negligible and such that for some $\varepsilon > 0$, the function $|w|^{-\alpha}|\log|w||^{1+\varepsilon}|f(w)|$ is bounded.

3.3 Positivity of C_+, C_-

We need to consider processes (dual to X_n) and taking values in $(V\setminus\{0\}) \times \mathbb{R}$ or $\mathbb{S}^{d-1} \times \mathbb{R}$ and we write

$$S'_n = A^*_n \dots A^*_1.$$

Let M be a S^*-minimal subset of $\mathbb{S}^{d-1}$ i.e. $M = \tilde{\Lambda}(S^*)$ in case I and $M = \tilde{\Lambda}_+(S^*)$ (or $(\tilde{\Lambda}_-(S^*))$ in case II. We denote by $\Lambda^*_a(T)$ the set of $u \in \mathbb{S}^{d-1}$ such that the projection of ρ on the line $\mathbb{R}u(u \in \mathbb{S}^{d-1})$ is unbounded in direction u. The following is the essential step in the discussion of positivity.

Proposition 5. *With the hypothesis of Theorem 1, if $\Lambda^*_a(T) \supset M$, then for any $u \in M$*

$$C_M(u) = \lim_{t\to\infty} t^\alpha \mathbb{P}\{< R, u >> t\} > 0.$$

In order to explain the main points of the proof, we need to introduce some notations. We observe that R_n satisfies the recursion

$$< R_{n+1}, w >=< R_n, w > + < B_{n+1}, S'_n w >,$$

hence $(S'_n w, r+ < R_n, w >)$ is a Markov walk on $V\setminus\{0\} \times \mathbb{R}$ based on $\mathbb{S}^{d-1} \times \mathbb{R}$. If we write

$$t' = r^{-1},\ w = |w|u,\ p = r|w|^{-1}$$

with $u \in \mathbb{S}^{d-1}$ this Markov walk can be expressed on $(\mathbb{S}^{d-1} \times \mathbb{R}) \times \mathbb{R}^*$ as

$$u_{n+1} = g^*_{n+1}.u_n,\ p_{n+1} = \frac{p_n+ < b_{n+1}, u_n >}{|g^*_{n+1}u_n|},\ t'_{n+1} = t'_n(|g^*_{n+1}u_n|p_{n+1}p_n^{-1})^{-1}.$$

We denote by ${}^*\hat{P}$ the corresponding Markov kernel. Since $(S'_n w, r+ < R_n, w >)$ has equivariant projection $S'_n w$ on $V\setminus\{0\}$, we have ${}^*\hat{P}({}^*e^\alpha \otimes h^\alpha) ={}^* e^\alpha \otimes h^\alpha$, hence we can consider the new relativized kernel ${}^*\hat{P}_\alpha$ and the corresponding Markov walk (u_n, p_n, t'_n) over the chain $(u_n, p_n) \in X = M \times \mathbb{R}$.

We denote

$${}^*q^\alpha(u, g) = |g^*u|^\alpha \frac{{}^*e^\alpha(g^*.u)}{{}^*e^\alpha(u)}$$

and for $h = (g, b) \in H$,

$$h^u p = \frac{1}{|g^*u|}(p+ < b, u >);$$

then the Markov kernel ${}^*\hat{Q}^\alpha$ of the chain (u_n, p_n) is given by

$$ {}^*\hat{Q}^\alpha\varphi(u,p) = \int \varphi(g^*\cdot u, h^u p)^* q^\alpha(u,g) d\lambda(h). $$

We have $L_\mu(\alpha) > 0$ and M is minimal, hence it is easy to show that ${}^*\hat{Q}^\alpha$ has a unique stationary measure κ on X, and with respect to the Markov measure ${}^*\hat{\mathbb{Q}}_u^\alpha$ on $X \times \Omega$ we have $\mathbb{E}_u^\alpha(\log^+|p|) < \infty$ and $\limsup_{n\to\infty}|S_n' u||p_n| = \infty$. We observe that, since $L_\mu(\alpha) > 0$, the Markov walk (u_n, p_n, t_n') on $X \times \mathbb{R}^*$ has negative drift, in additive notation.

The condition $\Lambda_a^*(T) \supset M$ implies

$$ \kappa(M\times]0,\infty[) > 0, \ \limsup_{n\to\infty} |S_n' u| p_n = \infty, $$

for $p > 0$.

We now consider the following $\mathbb{N}\cup\{\infty\}$-valued stopping time τ on $X \times \Omega$ defined by

$$ \tau = Inf\{n > 1; p^{-1} < R_n, u >> 0\}, $$

and we observe that, by definition of p_n :

$$ \tau = Inf\{n > 1; p^{-1} p_n |S_n' u| > 1\}, $$

hence $p^{-1}p_\tau > 0$. Hence τ (resp. $p^{-1}p_\tau|S_\tau' u||$) can be interpreted as the first ladder epoch (resp. height) of the Markov walk $p^{-1}p_n|S_n' u|$ (see [4]).

Using Poincaré's recurrence theorem and $\lim_{n\to\infty}|S_n' u| = \infty$ ${}^*\hat{\mathbb{Q}}_u^\alpha$-a.e. we infer that $\tau < \infty$ ${}^*\hat{\mathbb{Q}}_\kappa^\alpha$-a.e.

Let ${}^*\hat{P}^\tau$, ${}^*\hat{Q}^\tau$ be the stopped kernels of ${}^*\hat{P}$, ${}^*\hat{Q}$, respectively, defined by τ and let ${}^*\hat{P}_\alpha^\tau$, ${}^*\hat{Q}^{\alpha,\tau}$ be the corresponding relativised Markovian kernels. Then we have the

Lemma 1. *With* $tw = u \in \mathbb{S}^{d-1}$, $t > 0$, *we write on* $X \times \mathbb{R}^*$

$$ \psi(w,p) = \mathbb{P}\{p^{-1} < R, u >> t\},\ \psi_\tau(v,p) = \mathbb{P}\{t < p^{-1} < R, u >< t + p^{-1} < R_\tau, u >\} $$

$$ \psi^\alpha = ({}^*e^\alpha \otimes h^\alpha)^{-1}\psi,\ \psi_\tau^\alpha = ({}^*e^\alpha \otimes h^\alpha)^{-1}\psi_\tau. $$

Then $\psi = \sum_0^\infty({}^*\hat{P}^\tau)^k\psi_\tau$, $\psi^\alpha = \sum_0^\infty({}^*\hat{P}_\alpha^\tau)^k\psi_\tau^\alpha$.

The proof is analogous to the first part of Proposition 3, in order to get the Poisson equation $\psi_\tau = \psi -{}^* \hat{P}^\tau\psi$. Since $p^{-1}p_\tau > 0$, the operator ${}^*\hat{Q}^{\alpha,\tau}$ preserves $X_+ = M\times]0,\infty[$. If $\Lambda_a^*(T) \supset M$, then $\kappa(X_+) > 0$. Since $\mathbb{E}_\kappa^\alpha(\log^+|p|) < \infty$, one can show that the Markov kernel ${}^*\hat{Q}_x^{\alpha,\tau}$ has an ergodic stationary measure κ_+^τ which is absolutely continuous with respect to $1_{X_+}\kappa$. Also we have, using the interpretation of τ as a return time in the dynamical system associated with ${}^*\hat{\mathbb{Q}}_x^\alpha$ and the bilateral shift,

$$\mathbb{E}_0^\alpha(\tau) = \int \mathbb{E}_u^\alpha(\tau) d\kappa_+^\tau(u,p) < \infty,\ \gamma_\tau^\alpha = \mathbb{E}_{\kappa_+^\tau}^\alpha(\log(p^{-1}p_\tau|S'_\tau u|)) \in]0,\infty[$$

with $\gamma_\tau^\alpha = L_\mu(\alpha)\mathbb{E}_0^\alpha(\tau)$.

Now we can consider the Markov walk defined by ${}^*\hat{P}_\alpha^\tau$ on $X_+ \times \mathbb{R}_+^*$. In view of the above observations we can apply Proposition 2 to ${}^*\hat{P}_\alpha^\tau$ and $\kappa_+^\tau \otimes l$. We recall that, in additive notation, this Markov walk has negative drift $-\gamma_\tau^\alpha < 0$. If for some $u \in M$ we have $C_M(u) = 0$, then for $p > 0$ and $u = tw$ $(t > 0)$ $\lim_{t\to\infty} \psi^\alpha(w,p) = 0$.

Using Proposition 3 we get $\lim_{t\to\infty} \psi^\alpha(w,p) = 0$ for any $u = tw \in M$. In particular, this is valid κ_+^τ-a.e., hence Proposition 2 implies $\psi_\tau^\alpha = 0$ $\kappa_+^\tau \otimes l$-a.e., i.e.

$$\mathbb{P}\{t < p^{-1} < R, u >< t + p^{-1} < R_\tau, u >\} = 0.$$

Since $p^{-1} < R_\tau, u >> 0$, we get $p^{-1} < R, u > \leq 0$ $\kappa_+^\tau \otimes \mathbb{P}$-a.e., i.e. $< R, u > \leq 0$ $\mathbb{P}$-a.e. This contradicts $\Lambda_a^*(T) \supset M$. One can show that $\Lambda_a^*(T) = \mathbb{S}^{d-1}$ in cases I, II' and $\Lambda_a^*(T) \supset \tilde{\Lambda}_+(S^*)$ in case II'', hence $C_+ > 0$.

4 The One-Dimensional Case

If $d = 1$, the notations and definitions introduced in Sect. 1 make sense. Then $G = \mathbb{R}^*$ and $H = H_1$ is the affine group "$ax + b$" of the line. Condition $i - p$ is always satisfied for any probability μ on $\mathbb{R}^*$, and the analogue of Proposition 1 is valid verbatim. For the analogue of Theorem 1 one needs to consider the possibility that S resp. μ are arithmetic, i.e. S is contained in a subset of $\mathbb{R}^*$ of the form $\{\pm a^n\}$ for some $a > 0$. The function $k(s)$ has the explicit form

$$k(s) = \int |a|^s d\mu(a).$$

Also $L_\mu = \int \log|a| d\mu(a) = k'(0)$. Then, Theorem 1 has the following analogue, with weaker moment conditions.

Theorem 8. *Assume that the probability measure λ on H_1 and μ on $\mathbb{R}^*$ satisfy the following conditions*

(a) $\mathbb{E}(\log|A|) < 0,\ k(\alpha) = 1,$ *for some* $\alpha > 0$.
(b) S *is non arithmetic and* T *has no fixed point.*
(c) $\mathbb{E}(|B|^\alpha) < \infty$ *and* $\mathbb{E}|A|^\alpha|\log|A|| < \infty$.

Then one has the following convergences:

$$\lim_{t\to\infty} t^{\alpha}\mathbb{P}\{R > t\} = C_+$$

$$\lim_{t\to\infty} |t|^{\alpha}\mathbb{P}\{R < -t\} = C_-.$$

Either $\mathrm{supp}\rho = \mathbb{R}$ *and then* $C_+, C_- > 0$ *or* $\mathrm{supp}\rho$ *is a half-line* $[c, \infty[$ *(resp.* $]-\infty, c]$*) and then* $C_+ > 0,\ C_- = 0$ *(resp.* $C_- > 0, C_+ = 0$*).*

With respect to [6], the main new situation occurs for the discussion of positivity of C_+, if $A_n > 0$ and the r.v. B_n may have arbitrary sign. The proof [17] uses only the classical renewal theorem and a spectral gap property for the Markov chain p_n on $\mathbb{R}$. If $\mathrm{supp}\lambda$ does not preserve a half-line $]-\infty, c]$, one considers τ as the entrance time of p_n into $]0, \infty[$. The spectral gap property gives the finiteness of $\mathbb{E}^{\alpha}_{p}(\tau)$ for any $p \in \mathbb{R}$; using Wald's identity for the random walk $\log|S_n|$, one gets the finiteness and positivity of $\log|S_\tau|$ and then one concludes as for $d > 1$. Under stronger assumptions, the positivity of C_+ has been obtained also in the more general context of [3], using a complex analytic method for Mellin transform due to E. Landau, and familiar in analytic number theory. The positivity of $C_+ + C_-$ was obtained in [6], using P. Levy's symmetrisation method. For an analytic proof of these facts, using also Wiener-Ikehara theorem, see ([10], Appendix). In contrast to Theorem 1 and due to the Diophantine character of the hypothesis, the convergences stated in Theorem 8 are not robust under perturbation of λ in the weak topology. From that point of view, the respective roles of stable laws and of the Gaussian law are different for $d = 1$ and for $d > 1$.

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