Parameterized Approximability of Maximizing the Spread of Influence in Networks

Cristina Bazgan[1,3], Morgan Chopin[1], André Nichterlein[2], and Florian Sikora[1]

[1] PSL, Université Paris-Dauphine, LAMSADE UMR CNRS 7243, France
{bazgan,chopin,florian.sikora}@lamsade.dauphine.fr
[2] Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany
andre.nichterlein@tu-berlin.de
[3] Institut Universitaire de France

Abstract. In this paper, we consider the problem of maximizing the spread of influence through a social network. Here, we are given a graph $G = (V, E)$, a positive integer k and a threshold value $\mathrm{thr}(v)$ attached to each vertex $v \in V$. The objective is then to find a subset of k vertices to "activate" such that the number of activated vertices at the end of a propagation process is maximum. A vertex v gets activated if at least $\mathrm{thr}(v)$ of its neighbors are. We show that this problem is strongly inapproximable in fpt-time with respect to (w.r.t.) parameter k even for very restrictive thresholds. For unanimity thresholds, we prove that the problem is inapproximable in polynomial time and the decision version is W[1]-hard w.r.t. parameter k. On the positive side, it becomes $r(n)$-approximable in fpt-time w.r.t. parameter k for any strictly increasing function r. Moreover, we give an fpt-time algorithm to solve the decision version for bounded degree graphs.

1 Introduction

Optimization problems that involve a diffusion process in a graph are well studied [16,12,7,1,11,6,2,17]. Such problems share the common property that, according to a specified *propagation rule*, a chosen subset of vertices activates all or a fixed fraction of the vertices, where initially all but the chosen vertices are inactive. Such optimization problems model the spread of influence or information in social networks via word-of-mouth recommendations, of diseases in populations, or of faults in distributed computing [16,12,11]. One representative problem that appears in this context is the *influence maximization* problem introduced by Kempe *et al.* [12]. Given a directed graph, the task is to choose a vertex subset of size at most a fixed number such that the number of activated vertices at the end of the propagation process is maximized. The authors show that the problem is polynomial-time $(\frac{e}{e-1} + \varepsilon)$-approximable for any $\varepsilon > 0$ under some stochastic propagation models, but NP-hard to approximate within a ratio of $n^{1-\varepsilon}$ for any $\varepsilon > 0$ for general propagation rules.

In this paper, we use the following deterministic propagation model. We are given an undirected graph, a threshold value $\mathrm{thr}(v)$ associated to each vertex

v, and the following propagation rule: a vertex becomes active if at least $\mathrm{thr}(v)$ many neighbors of v are active. The propagation process proceeds in several rounds and stops when no further vertex becomes active. Given this model, finding and activating a minimum-size vertex subset such that all or a fixed fraction of the vertices become active is known as the *minimum target set selection* (MinTSS) problem introduced by Chen [7]. It has been shown NP-hard even for bipartite graphs of bounded degree when all thresholds are at most two [7]. Moreover, the problem was surprisingly shown to be hard to approximate within a ratio $O(2^{\log^{1-\varepsilon} n})$ for any $\varepsilon > 0$, even for constant degree graphs with thresholds at most two and for general graphs when the threshold of each vertex is half its degree (called *majority* thresholds) [7]. If the threshold of each vertex equals its degree (*unanimity* thresholds), then the problem is polynomial-time equivalent to the vertex cover problem [7] and, thus, admits a 2-approximation and is hard to approximate with a ratio better than 1.36 [9]. Concerning the parameterized complexity, the problem is shown to be W[2]-hard with respect to (w.r.t.) the solution size, even on bipartite graphs of diameter four with majority thresholds or thresholds at most two [14]. Furthermore, it is W[1]-hard w.r.t. each of the parameters "treewidth", "cluster vertex deletion number", and "pathwidth" [2,8]. On the positive side, the problem becomes fixed-parameter tractable w.r.t. each of the single parameters "vertex cover number", "feedback edge set size", and "bandwidth" [14,8]. If the input graph is complete, or has a bounded treewidth and bounded thresholds then the problem is polynomial-time solvable [14,2].

Here, we study the complementary problem of MinTSS, called *maximum k-influence* (Max$k$Inf) where the task is to maximize the number of activated vertices instead of minimizing the target set size. Since both optimization problems have the same decision version, the parameterized as well as NP-hardness results directly transfer from MinTSS to Max$k$Inf. We show that also Max$k$Inf is hard to approximate and, confronted with the computational hardness, we study the parameterized approximability of Max$k$Inf.

Our Results. Concerning the approximability of the problem, there are two possibilities of measuring the value of a solution: counting the vertices activated by the propagation process including or excluding the initially chosen vertices (denoted by MAX CLOSED k-INFLUENCE and MAX OPEN k-INFLUENCE, respectively). Observe that whether or not counting the chosen vertices might change the approximation factor. In this paper, we consider both cases and our approximability results are summarized in Table 1.

While MinTSS is both constant-approximable in polynomial time and fixed-parameter tractable for the unanimity case, this does not hold anymore for our problem. Indeed, we prove that, in this case, MAX CLOSED k-INFLUENCE (resp. MAX OPEN k-INFLUENCE) is strongly inapproximable in polynomial-time and the decision version, denoted by (k, ℓ)-INFLUENCE, is W[1]-hard w.r.t. the combined parameter (k, ℓ). However, we show that MAX CLOSED k-INFLUENCE (resp. MAX OPEN k-INFLUENCE) becomes approximable if we are allowed to use fpt-time and (k, ℓ)-INFLUENCE gets fixed-parameter tractable w.r.t combined parameter (k, Δ), where Δ is the maximum degree of the input graph.

Table 1. Table of the approximation results for MAX OPEN k-INFLUENCE and MAX CLOSED k-INFLUENCE

Thresholds	Bounds	MAX OPEN k-INFLUENCE		MAX CLOSED k-INFLUENCE	
		poly-time	fpt-time	poly-time	fpt-time
General	Upper	n	n	n	n
	Lower	$n^{1-\varepsilon},\forall\varepsilon>0$	$n^{1-\varepsilon},\forall\varepsilon>0$	$n^{1-\varepsilon},\forall\varepsilon>0$	$n^{1-\varepsilon},\forall\varepsilon>0$
Constant	Upper	n	n	n	n
	Lower	$n^{\frac{1}{2}-\varepsilon},\forall\varepsilon>0$	$n^{\frac{1}{2}-\varepsilon},\forall\varepsilon>0$	$n^{\frac{1}{2}-\varepsilon},\forall\varepsilon>0$	$n^{\frac{1}{2}-\varepsilon},\forall\varepsilon>0$ [Th. 2]
Majority	Upper	n	n	n	n
	Lower	$n^{1-\varepsilon},\forall\varepsilon>0$	$n^{1-\varepsilon},\forall\varepsilon>0$	$n^{1-\varepsilon},\forall\varepsilon>0$	$n^{1-\varepsilon},\forall\varepsilon>0$[Th. 1]
Unanimity	Upper	2^k [Th. 5]	$r(n),\forall r$ [Th. 2]	2^k	$r(n),\forall r$
	Lower	$n^{1-\varepsilon},\forall\varepsilon>0$ [Th. 4]	?	$1+\varepsilon$ [Th. 7]	?

Our paper is organized as follows. In Section 2, after introducing some preliminaries, we establish some basic lemmas. In Section 3 we study MAX OPEN k-INFLUENCE and MAX CLOSED k-INFLUENCE with majority thresholds and thresholds at most two. In Section 4 we study the case of unanimity thresholds in general graphs and in bounded degree graphs. Conclusions are provided in Section 5. Due to space limitation, some proofs are deferred to a full version.

2   Preliminaries and Basic Observations

In this section, we provide basic backgrounds and notation used throughout this paper, give the statements of the studied problems, and establish some lemmas.

Graph Terminology. Let $G = (V, E)$ be an *undirected graph*. For a subset $S \subseteq V$, $G[S]$ is the subgraph induced by S. The *open neighborhood* of a vertex $v \in V$, denoted by $N(v)$, is the set of all neighbors of v. The *closed neighborhood* of a vertex v, denoted $N[v]$, is the set $N(v) \cup \{v\}$. Furthermore, for a vertex set $V' \subset V$ we set $N(V') = \bigcup_{v \in V'} N(v)$ and $N[V'] = \bigcup_{v \in V'} N[v]$. The set $N^k[v]$, called the k-neighborhood of v, denotes the set of vertices which are at distance at most k from v (thus $N^1[v] = N[v]$). The *degree* of a vertex v is denoted by $\deg_G(v)$ and the *maximum degree* of the graph G is denoted by Δ_G. We skip the subscript if G is clear from the context. Two vertices are *twins* if they have the same neighborhood. They are called *true twins* if they are moreover neighbors, *false twins* otherwise.

Cardinality Constrained Problem. The problems studied in this paper are cardinality constrained. We use the notations and definitions from Cai [4]. A cardinality constrained optimization problem is a quadruple $A = (\mathcal{B}, \Phi, k, obj)$, where $\mathcal{B}$ is a finite set called solution base, $\Phi : 2^{\mathcal{B}} \to \{0, 1, 2, \ldots\} \cup \{-\infty, +\infty\}$ an objective function, k a non-negative integer and $obj \in \{min, max\}$. The goal is then to find a solution $S \subseteq \mathcal{B}$ of cardinality k so as to maximize (or minimize) the objective value $\Phi(S)$. If S is not a feasible solution we set $\Phi(S) = -\infty$ if $obj = max$ and $\Phi(S) = +\infty$ otherwise.

Parameterized Complexity. A parameterized problem (I, k) is said *fixed-parameter tractable* (or in the class FPT) w.r.t. parameter k if it can be solved in $f(k) \cdot |I|^c$ time, where f is any computable function and c is a constant (one can see [10,15]). The parameterized complexity hierarchy is composed of the classes FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \cdots \subseteq$ W[P]. A W[1]-hard problem is not fixed-parameter tractable (unless FPT $=$ W[1]) and one can prove W[1]-hardness by means of a *parameterized reduction* from a W[1]-hard problem. This is a mapping of an instance (I, k) of a problem A_1 in $g(k) \cdot |I|^{O(1)}$ time (for any computable g) into an instance (I', k') for A_2 such that $(I, k) \in A_1 \Leftrightarrow (I', k') \in A_2$ and $k' \leq h(k)$ for some h.

Approximation. Given an optimization problem Q and an instance I of this problem, we denote by $|I|$ the size of I, by $opt_Q(I)$ the optimum value of I and by $val(I, S)$ the value of a feasible solution S of I. The *performance ratio* of S (or *approximation factor*) is $r(I, S) = \max \left\{ \frac{val(I,S)}{opt_Q(I)}, \frac{opt_Q(I)}{val(I,S)} \right\}$. The *error* of S, $\varepsilon(I, S)$, is defined by $\varepsilon(I, S) = r(I, S) - 1$. For a function f (resp. a constant $c > 1$), an algorithm is a $f(n)$-approximation (resp. a c-approximation) if for any instance I of Q it returns a solution S such that $r(I, S) \leq f(n)$ (resp. $r(I, S) \leq c$). An optimization problem is polynomial-time *constant approximable* (resp. has a *polynomial-time approximation scheme*) if, for some constant $c > 1$ (resp. every constant $\varepsilon > 0$), there exists a polynomial-time c-approximation (resp. $(1 + \varepsilon)$-approximation) for it. An optimization problem is $f(n)$-*approximable in fpt-time w.r.t. parameter k* if there exists an $f(n)$-approximation running in time $g(k) \cdot |I|^c$, where k is a positive integer depending on I, g is any computable function and c is a constant [13]. For a cardinality constrained problem a possible choice for the parameter is the cardinality of the solutions.

Problems definition. Let $G = (V, E)$ be an undirected graph and a threshold function thr : $V \to \mathbb{N}$. In this paper, we consider majority thresholds *i.e.* thr$(v) = \lceil \frac{\deg(v)}{2} \rceil$ for each $v \in V$, unanimity thresholds *i.e.* thr$(v) = \deg(v)$ for each $v \in V$, and constant thresholds *i.e.* thr$(v) \leq c$ for each $v \in V$ and some constant $c > 1$. Initially, all vertices are not activate and we select a subset $S \subseteq V$ of k vertices. The propagation unfolds in discrete steps. At time step 0, only the vertices in S are activated. At time step $t + 1$, a vertex v is activated if and only if the number of its activated neighbors at time t is at least thr(v). We apply the rule iteratively until no more activations are possible. Given that S is the set of initially activated vertices, *closed activated vertices*, denoted by $\sigma[S]$ is the set of all activated vertices at the end of the propagation process and *closed activated vertices*, denoted by $\sigma(S)$, is the set $\sigma[S] \cup S$. The optimization problems we consider are then defined as follows.

MAX OPEN k-INFLUENCE
Input: A graph $G = (V, E)$, a threshold function thr : $V \to \mathbb{N}$, and an integer k.
Output: A subset $S \subseteq V$, $|S| \leq k$ such that $|\sigma(S)|$ is maximum.

Similarly, the MAX CLOSED k-INFLUENCE problem asks for a set S such that $|\sigma[S]|$ is maximum. The corresponding decision version (k, ℓ)-INFLUENCE is also studied. Notice that, in this case, considering either the open or closed activated vertices is equivalent.

(k, ℓ)-INFLUENCE

Input: A graph $G = (V, E)$, a threshold function thr $: V \to \mathbb{N}$, and two integers k and ℓ.
Output: Is there a subset $S \subseteq V$, $|S| \leq k$ such that $|\sigma(S)| \geq \ell$?

Basic results. In the following, we state and prove some lemmas that will be used later in the paper.

Lemma 1. *Let r be any computable function. If* MAX OPEN k-INFLUENCE *is $r(n)$-approximable then* MAX CLOSED k-INFLUENCE *is also $r(n)$-approximable where n is the input size.*

Proof. Let A be an $r(n)$-approximation algorithm for MAX OPEN k-INFLUENCE. Let I be an instance of MAX CLOSED k-INFLUENCE and $opt(I)$ its optimum value. When we apply A on I it returns a solution S such that $|\sigma(S)| \geq \frac{opt(I)-k}{r(n)}$ and then $|\sigma[S]| = k + |\sigma(S)| \geq \frac{opt(I)}{r(n)}$. $\qquad\square$

Lemma 2. *Let I be an instance of a cardinality constrained optimization problem $A = (\mathcal{B}, \Phi, k, obj)$. If A is $r_1(k)$-approximable in fpt-time w.r.t. parameter k for some strictly increasing function r_1 then it is also $r_2(|\mathcal{B}|)$-approximable in fpt-time w.r.t. parameter k for any strictly increasing function r_2.*

Proof. Let r_1^{-1} and r_2^{-1} be the inverse functions of r_1 and r_2, respectively. We distinguish the following two cases.

Case 1: $k \leq r_1^{-1}(r_2(|\mathcal{B}|))$. In this case, we apply the $r_1(k)$-approximation algorithm and directly get the $r_2(|\mathcal{B}|)$-approximation in time $f(k) \cdot |\mathcal{B}|^{O(1)}$ for some computable function f.

Case 2: $k > r_1^{-1}(r_2(|\mathcal{B}|))$. We then have $|\mathcal{B}| < r_2^{-1}(r_1(k))$. In this case, we solve the problem exactly by brute-force. If $obj = \max$ (resp. $obj = \min$) then try all possible subset $S \subseteq \mathcal{B}$ of size k and take the one that maximizes (resp. minimizes) the objective value $\Phi(S)$. The running time is then $O(|\mathcal{B}|^k) = O(r_2^{-1}(r_1(k))^k)$.

The overall running time is $O(\max\{r_2^{-1}(r_1(k))^k, f(k) \cdot |\mathcal{B}|^{O(1)}\})$, that is, fpt-time. $\qquad\square$

It is worth pointing out that a problem which is proven inapproximable in fpt-time obviously implies that it is not approximable in polynomial time with the same ratio. Therefore, fpt-time inapproximability can be considered as a "stronger" result than polynomial-time inapproximability.

3 Parameterized Inapproximability

In this section, we consider the parameterized approximability of both Max Closed k-Influence and Max Open k-Influence. We show that these problems are W[2]-hard to approximate within $n^{1-\varepsilon}$ and $n^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon \in (0,1)$ for majority thresholds and thresholds at most two, respectively. To do so, we use the following construction from Dominating Set as the starting point. The Dominating Set problem asks, given an undirected graph $G = (V, E)$ and an integer k, whether there is a vertex subset $S \subseteq V$, $|S| \le k$, such that $N[S] = V$.

Basic Reduction. Given an instance $(G = (V, E), k)$ of Dominating Set we construct a bipartite graph $G' = (V', E')$ as follows. For each vertex $v \in V$ we add two vertices v^t and v^b (t and b respectively standing for *top* and *bottom*) to V'. For each edge $\{u, v\} \in E$ add the edge $\{v^t, u^b\}$. Finally, set $\mathrm{thr}(v^t) = \deg_{G'}(v^t)$ and $\mathrm{thr}(v^b) = 1$ for every top vertex v^t and every bottom vertex v^b, respectively. Clearly, the construction can be computed in polynomial time and, furthermore, it has the following property.

Lemma 3. *Let $G' = (V', E')$ be the graph obtained from a graph G using the above construction. Then G admits a dominating set of size k if and only if G' admits a subset $S' \subseteq V'$ of size k such that $\sigma[S'] = V'$.*

Inapproximability Results. We are now ready to prove the main results of this section.

Theorem 1. *For any $\varepsilon \in (0,1)$, Max Closed k-Influence and Max Open k-Influence with majority thresholds cannot be approximated within $n^{1-\varepsilon}$ in fpt-time w.r.t. parameter k even on bipartite graphs, unless FPT = W[2].*

Proof. By Lemma 1, it suffices to show the result for Max Closed k-Influence. We construct a polynomial-time reduction from Dominating Set to Max Closed $(k+1)$-Influence with majority. In this reduction, we will make use of the ℓ-*edge* gadget, for some integer ℓ. An ℓ-edge between two vertices u and v consists of ℓ vertices of threshold one adjacent to both u and v.

Given an instance $I = (G = (V, E), k)$ of Dominating Set with $n = |V|$, $m = |E|$, we define an instance I' of Max Closed $(k+1)$-Influence. We start with the *basic reduction* and modify G' and the function thr as follows. Replace every edge $\{v^t, v^b\}$ by an $(k+2)$-edge between v^t and v^b. Moreover, for a given constant $\beta = \frac{8-5\varepsilon}{\varepsilon}$, let $L = \lceil n^\beta \rceil$ and we add nL more vertices $x_1^1, \ldots, x_n^1, \ldots, x_1^L, \ldots, x_n^L$. For $i = 1, \ldots, n$, vertex x_i^1 is adjacent to all the bottom vertices. Moreover, for any $j = 2, \ldots, L$, each x_i^j is adjacent to x_k^{j-1}, for any $i, k \in \{1, \ldots, n\}$. We also add a vertex w and an $n + (k+2)(\deg_G(v) - 1)$-edge between w and v^b, for any bottom vertex v^b. For $i = 1, \ldots, n$, vertex x_i^1 is adjacent to w. For $i = 1, \ldots, n$ add n pending-vertices (*i.e.* degree one vertices) adjacent to x_i^L. For any vertex v^t add $(\deg_G(v) + 1)(k+2)$ pending-vertices adjacent to v^t. Add also
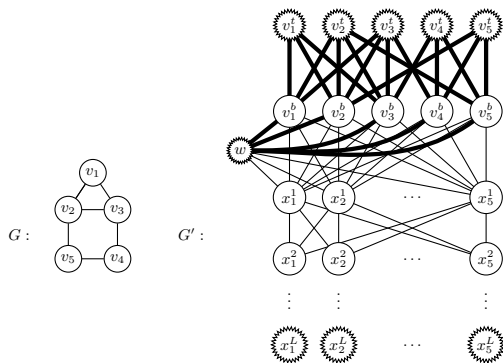
Fig. 1. The graph G' obtained after carrying out the modifications of Theorem 1. A thick edge represents an ℓ-edge for some $\ell > 0$. A "star" vertex v represents a vertex adjacent to $\frac{\deg_{G'}(v)}{2}$ pending-vertices.

$n + n^2 + (k+2)(2m-n)$ pending-vertices adjacent to w. All vertices of the graph G' have the majority thresholds (see also Figure 1).

We claim that if I is a *yes*-instance then $opt(I') \geq nL \geq n^{\beta+1}$; otherwise $opt(I') < n^4$. Let $n' = |V'|$, notice that we have $n' \leq n^4 + nL$.

Suppose that there exists a dominating set $S \subseteq V$ in G of size at most k. Consider the solution S' for I' containing the corresponding top vertices and vertex w. After the first round, all vertices belonging to the edge gadgets which top vertex is in S' are activated. Since S is a dominating set in G, after the second round, all the bottom vertices are activated. Indeed $\deg_{G'}(v^b) = 2(n + (k+2)\deg_G(v))$ and after the first round v^b has at least $k+2$ neighbors activated belonging to an $(k+2)$-edge between v^b and some $u^t \in V$ and $n+(k+2)(\deg_G(v)-1)$ neighbors activated belonging to an $n + (k+2)(\deg_G(v) - 1)$-edge between v^b and w. Thus, every vertex x_i^1 gets active after the third round, and generally after the jth round, $j = 4, \ldots, L+2$ the vertices x_i^{j-2} are activated, and at the $(L+3)$th round all pending-vertices adjacent to x_i^L are activated. Therefore, the size of an optimal solution is at least $nL \geq n^{\beta+1}$.

Suppose that there is no dominating set in G of size k. Without loss of generality, we may assume that no pending-vertices are in a solution of I' since they all have threshold one. If w does not take part of a solution in I', then no vertex x_i^1 could be activated and in this case $opt(I')$ is less than $n' - nL \leq n^4$. Consider now the solutions of I' of size $k + 1$ that contain w. Observe that if a top-vertex v^t gets active through bottom-vertices then v^t can not activate any other bottom-vertices. Indeed, as a contradiction, suppose that v^t is adjacent to a non-activated bottom-vertex. It follows that v^t could not have been activated because of its threshold and that no pending-vertices are part of the solution, a contradiction. Notice also that it is not possible to activate a bottom vertex by selecting some x_i^1 vertices since of their threshold. Moreover, since there is no dominating set of size k, any subset of k top vertices cannot activate all bottom vertices, therefore no vertex x_i^k, $i = 1, \ldots, n, k = 1, \ldots, L$ can be activated.

Hence, less than $n' - nL$ vertices can be activated in G' and the size of an optimal solution is at most n^4.

Assume now that there is an fpt-time $n^{1-\varepsilon}$-approximation algorithm A for MAX CLOSED $(k+1)$-INFLUENCE with majority threshold. Thus, if I is a *yes*-instance, the algorithm gives a solution of value $A(I') \geq \frac{n^{\beta+1}}{(n')^{1-\varepsilon}} > \frac{n^{\beta+1}}{n^{(1-\varepsilon)(\beta+5)}} = n^4$ since $n' \leq n^4 + nL < n^5 L$. If I is a *no*-instance, the solution value is $A(I') < n^4$. Hence, the approximation algorithm A can distinguish in fpt-time between *yes*-instances and *no*-instances for DOMINATING SET implying that $\mathsf{FPT} = \mathsf{W}[2]$ since this last problem is $\mathsf{W}[2]$-hard [10].   □

Theorem 2. *For any $\varepsilon \in (0, \frac{1}{2})$, MAX CLOSED k-INFLUENCE and MAX OPEN k-INFLUENCE with thresholds at most two cannot be approximated within $n^{\frac{1}{2}-\varepsilon}$ in fpt-time w.r.t. parameter k even on bipartite graphs, unless $\mathsf{FPT} = \mathsf{W}[2]$.*

Using Lemma 2, Theorem 1, and Theorem 2 we can deduce the following corollary.

Corollary 1. *For any strictly increasing function r, MAX CLOSED k-INFLUENCE and MAX OPEN k-INFLUENCE with thresholds at most two or majority thresholds cannot be approximated within $r(k)$ in fpt-time w.r.t. parameter k unless $\mathsf{FPT} = \mathsf{W}[2]$.*

4   Unanimity Thresholds

For the unanimity thresholds case, we will give some results on general graphs before focusing on bounded degree graphs and regular graphs.

4.1   General Graphs

In this section, we first show that, in the unanimity case, (k, ℓ)-INFLUENCE is $\mathsf{W}[1]$-hard w.r.t. parameter $k + \ell$ and MAX OPEN k-INFLUENCE is not approximable within $n^{1-\varepsilon}$ for any $\varepsilon \in (0, 1)$ in polynomial time, unless $\mathsf{NP} = \mathsf{ZPP}$. However, if we are allowed to use fpt-time then MAX OPEN k-INFLUENCE with unanimity is $r(n)$-approximable in fpt-time w.r.t. parameter k for any strictly increasing function r.

Theorem 3. *(k, ℓ)-INFLUENCE with unanimity thresholds is $\mathsf{W}[1]$-hard w.r.t. the combined parameter (k, ℓ) even for bipartite graphs.*

Theorem 4. *For any $\varepsilon \in (0, 1)$, MAX OPEN k-INFLUENCE with unanimity thresholds cannot be approximated within $n^{1-\varepsilon}$ in polynomial time, unless $\mathsf{NP} = \mathsf{ZPP}$.*

Theorem 5. *MAX OPEN k-INFLUENCE and MAX CLOSED k-INFLUENCE with unanimity thresholds are 2^k-approximable in polynomial time.*

Using Lemma 2 and Theorem 5 we directly get the following.

Corollary 2. *For any strictly increasing function r, MAX OPEN k-INFLUENCE and MAX CLOSED k-INFLUENCE with unanimity thresholds are $r(n)$-approximable in fpt-time w.r.t. parameter k.*

For example, MAX OPEN k-INFLUENCE is $\log(n)$-approximable in time $O^*(2^{k2^k})$.

Finding dense subgraphs. In the following we show that MAX OPEN k-INFLUENCE with unanimity thresholds is at least as difficult to approximate as the DENSEST k-SUBGRAPH problem, that consists of finding in a graph a subset of vertices of cardinality k that induces a maximum number of edges. In particular, any positive approximation result for MAX OPEN k-INFLUENCE with unanimity would directly transfers to DENSEST k-SUBGRAPH.

Theorem 6. *For any strictly increasing function r, if MAX OPEN k-INFLUENCE with unanimity thresholds is $r(n)$-approximable in fpt-time w.r.t. parameter k then DENSEST k-SUBGRAPH is $r(n)$-approximable in fpt-time w.r.t. parameter k.*

Using Theorem 6 and Corollary 2, we have the following corollary, independently established in [3].

Corollary 3. *For any strictly increasing function r, DENSEST k-SUBGRAPH is $r(n)$-approximable in fpt-time w.r.t. parameter k.*

4.2   Bounded Degree Graphs and Regular Graphs

We show in the following that MAX OPEN k-INFLUENCE and thus MAX CLOSED k-INFLUENCE are constant approximable in polynomial time on bounded degree graphs with unanimity thresholds. Moreover, MAX CLOSED k-INFLUENCE and then MAX OPEN k-INFLUENCE have no polynomial-time approximation scheme even on 3-regular graphs if $P \neq NP$. Moreover, we show that (k, ℓ)-INFLUENCE is in FPT w.r.t. parameter k.

Lemma 4. MAX OPEN k-INFLUENCE *and* MAX CLOSED k-INFLUENCE *with unanimity thresholds on bounded degree graphs are constant approximable in polynomial time.*

Theorem 7. MAX OPEN k-INFLUENCE *and* MAX CLOSED k-INFLUENCE *with unanimity thresholds have no polynomial-time approximation scheme even on 3-regular graphs for $k = \theta(n)$, unless $P = NP$.*

In Theorem 3 we showed that (k, ℓ)-INFLUENCE with unanimity thresholds is W[1]-hard w.r.t. parameters k and ℓ. In the following we give several fixed-parameter tractability results for (k, ℓ)-INFLUENCE w.r.t. parameter k on regular graphs and bounded degree graphs with unanimity thresholds. First we show that using results of Cai *et al.* [5] we can obtain fixed-parameter tractable algorithms. Then we establish an explicit and more efficient combinatorial algorithm. Using [5] we can show:

Theorem 8. (k, ℓ)-INFLUENCE *with unanimity thresholds can be solved in* $2^{O(k\Delta^3)}n^2 \log n$ *time where* Δ *denotes the maximum degree and in* $2^{O(k^2 \log k)}n \log n$ *time for regular graphs.*

While the previous results use general frameworks to solve the problem, we now give a direct combinatorial algorithm for (k, ℓ)-INFLUENCE with unanimity thresholds on bounded degree graphs. For this algorithm we need the following definition and lemma.

Definition 1. *Let* (α, β) *be a pair of positive integers,* $G = (V, E)$ *an undirected graph with unanimity thresholds, and* $v \in V$ *a vertex. We call* v *a realizing vertex for the pair* (α, β) *if there exists a vertex subset* $V' \subseteq N^{2\alpha-1}[v]$ *of size* $|V'| \leq \alpha$ *such that* $|\sigma(V')| \geq \beta$ *and* $\sigma[V']$ *is connected. Furthermore, we call* $\sigma[V']$ *a realization of the pair* (α, β).

We show first that in bounded degree graphs the problem of deciding whether a vertex is a realizing vertex for a pair of positive integers (α, β) is fixed-parameter tractable w.r.t. parameter α.

Lemma 5. *Checking whether a vertex* v *is a realizing vertex for a pair of positive integers* (α, β) *can be done in* $\Delta^{O(\alpha^2)}$ *time, where* Δ *is the maximum degree.*

Consider in the following the CONNECTED (k, ℓ)-INFLUENCE problem that is (k, ℓ)-INFLUENCE with the additional requirement that $G[\sigma[S]]$ has to be connected. Note that with Lemma 5 we can show that CONNECTED (k, ℓ)-INFLUENCE is fixed parameter tractable w.r.t. parameter k on bounded degree graphs. Indeed, observe that two vertices in $\sigma(S)$ cannot be adjacent since we consider unanimity thresholds. From this and the requirement that $G[\sigma[S]]$ is connected, it follows that $G[\sigma[S]]$ has a diameter of at most $2k$. Hence, the algorithm for CONNECTED (k, ℓ)-INFLUENCE checks for each vertex $v \in V$ whether v is a realizing vertex for the pair (k, ℓ). By Lemma 5 this gives an overall running time of $\Delta^{O(k^2)} \cdot n$.

We can extend the algorithm for the connected case to deal with the case where $G[\sigma[S]]$ is not connected. The general idea is as follows. For each connected component C_i of $G[\sigma[S]]$ the algorithm guesses the number of vertices in $S \cap C_i$ and in $\sigma(S) \cap C_i$. This gives an integer pair (k_i, ℓ_i) for each connected component in $G[\sigma[S]]$. Similar to the connected case, the algorithm will determine realizations for these pairs and the union of these realizations give S and $\sigma(S)$. Unlike the connected case, it is not enough to look for just one realization of a pair (k_i, ℓ_i) since the realizations of different pairs may be not disjoint and, thus, vertices may be counted twice as being activated. To avoid the double-counting we show that if there are "many" different realizations for a pair (k_i, ℓ_i), then there always exist a realization being disjoint to all realizations of the other pairs. Now consider only the integer pairs that do not have "many" different realizations. Since there are only "few" different realizations possible, the graph induced by all the vertices contained in all these realizations is "small". Thus, the algorithm can guess the realizations of the pairs having

Algorithm 1. The pseudocode of the algorithm solving the decision problem (k, ℓ)-INFLUENCE. The guessing part in the algorithm behind Lemma 5 is used in Line 7 as subroutine. The final check in Line 19 is done by brute force checking all possibilities.

```
 1: procedure SOLVEINFLUENCE(G, thr, k, ℓ)
 2:     Guess x ∈ {1, . . . , k}                    ▷ x: number of connected components of G[σ[S]]
 3:     Guess (k₁, ℓ₁), . . . , (kₓ, ℓₓ) such that ∑ˣᵢ₌₁ kᵢ = k and ∑ˣᵢ₌₁ ℓᵢ = ℓ
 4:     Initialize c₁ = c₂ = . . . = cₓ ← 0      ▷ one counter for each integer pair (kᵢ, ℓᵢ)
 5:     for each vertex v ∈ V do                       ▷ determine realizing vertices
 6:         for i ← 1 to x do
 7:             if v is a realizing vertex for the pair (kᵢ, ℓᵢ) then        ▷ see Lemma 5
 8:                 cᵢ ← cᵢ + 1
 9:                 T(v, i) = "yes"
10:             else
11:                 T(v, i) = "no"
12:     initialize X ← ∅                       ▷ X stores all pairs with "few" realizations
13:     for i ← 1 to x do
14:         if cᵢ ≤ 2 · x · Δ⁴ᵏ then
15:             X ← X ∪ {i}
16:     for each vertex v ∈ V do         ▷ remove vertices not realizing any pair in X
17:         if ∀i ∈ X : T(v, i) = "no" then
18:             delete v from G.
19:     if all pairs (kᵢ, ℓᵢ), i ∈ X, can be realized in the remaining graph then
20:         return 'YES'
21:     else
22:         return 'NO'
```

only "few" realizations and afterwards add greedily disjoint realizations of pairs having "many" realizations. See Algorithm 1 for the pseudocode.

Theorem 9. *Algorithm 1 solves* (k, ℓ)-INFLUENCE *with unanimity thresholds in* $2^{O(k^2 \log(k\Delta))} \cdot n$ *time, where* Δ *is the maximum degree of the input graph.*

5 Conclusions

We established results concerning the parameterized complexity as well as the polynomial-time and fpt-time approximability of two problems modeling the spread of influence in social networks, namely MAX OPEN k-INFLUENCE and MAX CLOSED k-INFLUENCE.

In the case of unanimity thresholds, we show that MAX OPEN k-INFLUENCE is at least as hard to approximate as DENSEST k-SUBGRAPH, a well-studied problem. We established that DENSEST k-SUBGRAPH is $r(n)$-approximable for any strictly increasing function r in fpt-time w.r.t. parameter k. An interesting open question consists of determining whether MAX OPEN k-INFLUENCE is constant approximable in fpt-time. Such a positive result would improve the approximation in fpt-time for DENSEST k-SUBGRAPH. In the case of thresholds

bounded by two we excluded a polynomial time approximation scheme for MAX CLOSED k-INFLUENCE but we did not found any polynomial-time approximation algorithm. Hence, the question arises, whether this hardness result can be strengthened. Another interesting open question is to study the approximation of min target set selection problem in fpt-time.

References

1. Aazami, A., Stilp, K.: Approximation algorithms and hardness for domination with propagation. SIAM J. Discrete Math. 23(3), 1382–1399 (2009)
2. Ben-Zwi, O., Hermelin, D., Lokshtanov, D., Newman, I.: Treewidth governs the complexity of target set selection. Discrete Optim. 8(1), 87–96 (2011)
3. Bourgeois, N., Giannakos, A., Lucarelli, G., Milis, I., Paschos, V.T.: Exact and approximation algorithms for densest k-subgraph. In: Ghosh, S.K., Tokuyama, T. (eds.) WALCOM 2013. LNCS, vol. 7748, pp. 114–125. Springer, Heidelberg (2013)
4. Cai, L.: Parameterized complexity of cardinality constrained optimization problems. Comput. J. 51(1), 102–121 (2008)
5. Cai, L., Chan, S.M., Chan, S.O.: Random separation: A new method for solving fixed-cardinality optimization problems. In: Bodlaender, H.L., Langston, M.A. (eds.) IWPEC 2006. LNCS, vol. 4169, pp. 239–250. Springer, Heidelberg (2006)
6. Chang, C.-L., Lyuu, Y.-D.: Spreading messages. Theor. Comput. Sci. 410(27-29), 2714–2724 (2009)
7. Chen, N.: On the approximability of influence in social networks. SIAM J. Discrete Math. 23(3), 1400–1415 (2009)
8. Chopin, M., Nichterlein, A., Niedermeier, R., Weller, M.: Constant thresholds can make target set selection tractable. In: Even, G., Rawitz, D. (eds.) MedAlg 2012. LNCS, vol. 7659, pp. 120–133. Springer, Heidelberg (2012)
9. Dinur, I., Safra, S.: The importance of being biased. In: Proc. of STOC, pp. 33–42. ACM (2002)
10. Downey, R.G., Fellows, M.R.: Parameterized Complexity. Springer (1999)
11. Dreyer, P.A., Roberts, F.S.: Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. Discrete Appl. Math. 157(7), 1615–1627 (2009)
12. Kempe, D., Kleinberg, J., Tardos, É.: Maximizing the spread of influence through a social network. In: Proc. of KDD, pp. 137–146. ACM (2003)
13. Marx, D.: Parameterized complexity and approximation algorithms. Comput. J. 51(1), 60–78 (2008)
14. Nichterlein, A., Niedermeier, R., Uhlmann, J., Weller, M.: On tractable cases of target set selection. Soc. Network Anal. Mining (2012) (online available)
15. Niedermeier, R.: Invitation to Fixed-Parameter Algorithms. Oxford University Press (2006)
16. Peleg, D.: Local majorities, coalitions and monopolies in graphs: a review. Theor. Comput. Sci. 282, 231–257 (2002)
17. Reddy, T.V.T., Rangan, C.P.: Variants of spreading messages. J. Graph Algorithms Appl. 15(5), 683–699 (2011)