Disjoint Small Cycles in Graphs and Bipartite Graphs*

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Abstract. Let k be a positive integer and let G be a graph of order $n \geq 3k+1$, X be a set of any k distinct vertices of G. It is proved that if $d(x) + d(y) \geq n + 2k - 2$ for any pair of nonadjacent vertices $x, y \in V(G)$, then G contains k disjoint cycles $T_1, \cdots, T_k$ such that each cycle contains exactly one vertex in X, and $|T_i| = 3$ for each $1 \leq i \leq k$ or $|T_k| = 4$ and the rest are all triangles. We also obtained two results about disjoint 6-cycles in a bipartite graph.

Keywords: Degree condition, Vertex-disjoint, Triangles.

1 Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges and we use Bondy and Murty [1] for terminology and notation not defined here. Let $G = (V, E)$ be a graph, the order of G is $|G| = |V|$ and its size is $E(G) = |E|$. A set of subgraphs is said to be vertex-disjoint or independent if no two of them have any common vertex in G, and we use disjoint or independent to stand for vertex-disjoint throughout this paper. Let G_1 and G_2 be two subgraphs of G or a subsets of $V(G)$. If G_1 and G_2 have no any common vertex in G, we define $E(G_1, G_2)$ to be the set of edges of G between G_1 and G_2, and let $E(G_1, G_2) = |E(G_1, G_2)|$. Let H be a subgraph of G and $u \in V(G)$ a vertex of G, $N(u, H)$ is the set of neighbors of u contained in H. We let $d(u, H) = |N(u, H)|$. Clearly, $d(u, G)$ is the degree of u in G, we often write $d(x)$ to replace $d(x, G)$. The minimum degree of G will be denoted by $\delta(G)$. If there is no fear of confusion, we often identify a subgraph H of G with its vertex set $V(H)$, for a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. If H is a subgraph in G, we define $d_H(U) = \sum_{x \in U} d_H(x)$. Let C and P be a cycle and a path, respectively, we use $l(C)$ and $l(P)$ to denote the length of C and P, respectively. That is, $l(C) = |C|$ and $l(P) = |P| - 1$. A Hamiltonian cycle of G is a cycle which contains all vertices of G, and a Hamiltonian path of G is a path of G which contains every vertex in G. Let $v_1, \ldots, v_k$ be k distinct vertices in G, and let $C_1, \ldots, C_k$ be k disjoint cycles passing through $v_1, \ldots v_k$, respectively, in G. Then we say that G has k disjoint cycles $C_1, \ldots, C_k$ with respect to $\{v_1, \ldots v_k\}$. A cycle of

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length 3 is called a triangle and a cycle of length 4 is called a quadrilateral. For a cycle C with $l(C) = k$, we call that C be a k-cycle. Let v be a vertex and H is a subgraph of G, we say H is a v-subgraph if $v \in V(H)$. In particular, a v-cycle or a v-path is a cycle or path passing through v, respectively. For a graph G, we define

$$\sigma_2(G) = \min\{d(x) + d(y) \,|xy \notin E(G)\}$$

When G is a complete graph, we define $\sigma_2(G) = \infty$.In 1963, Corrádi and Hajnal [2] proved Erdös's conjecture in the early 1960s which concerns independent cycles in a graph. They proved that if G is a graph of order $n \geq 3k$ with $\delta(G) \geq 2k$, then G contains k disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k disjoint triangles. In the same year, Dirac [3] obtained the following result.

Theorem 1. *(Dirac [3]) Let k, n be two positive integers and let G be a graph of order $n \geq 3k$. If $\delta(G) \geq (n + k)/2$, then G contains k independent triangles.*

Many people have studied the problems of cyclability, which concerns that for a given subset S of vertices, whether there exists a cycle or several independent cycles that covering S.Which motivated us to be interested in the following problem, for any k independent vertices $v_1, \ldots, v_k$, what ensures that there exist k disjoint triangles $C_1, \ldots, C_k$ with respect to $\{v_1, \ldots, v_k\}$, such that each C_i $(i \in \{1, 2, \ldots, k\})$ contains exactly one vertex of v_i $(i \in \{1, 2, \ldots, k\})$. For the disjoint triangles covering, Li et al. [6] have obtained the following result.

Theorem 2. *(H.Li [6]) Let k, n be two positive integers and let G be a graph of order $n \geq 3k$, X a set of any k distinct vertices of G. If the minimum degree $\delta(G) \geq (n + 2k)/2$, then G contains k disjoint triangles such that each triangle contains exactly one vertex of X.*

In this paper, we obtain the following result.

Theorem 3. *Let k be a positive integer and let G be a graph of order $n \geq 3k + 1$, X a set of any k distinct vertices of G. If $\sigma_2(G) \geq n + 2k - 2$, then G contains k disjoint cycles $T_1, \ldots T_k$ such that each cycle contains exactly one vertex in X, and $|T_i| = 3$ for each $1 \leq i \leq k$ or $|T_k| = 4$ and the rest are all triangles.*

Remark 1. The condition $n \geq 3k + 1$ in Theorem 1.3 is necessary since there exists a graph G with $|V(G)| \geq 4k$ and $\sigma_2(G) = |V(G)| + k - 1$ such that G does not even contain k vertex disjoint triangles (see [5]).

The remainder of this paper is organized as follows. In Section 2, we list several lemmas which will be used to prove Theorem 1.3 in Section 3. In Section 4, we obtain two results about disjoint 6-cycles in a bipartite graph and conclude this paper in Section 5 by proposing two related problems.

2 Lemmas

Lemma 1. *[4] Let* $P = u_1u_2 \ldots u_s\,(s \geq 2)$ *be a path in* G*,* $u \in V(G) - V(P)$*, when* $uu_1 \notin E(G)$*, if* $d(u, P) + d(u_s, P) \geq s$*, then* G *has a path* P' *with vertex set* $V(P') = V(P) \cup \{u\}$ *whose end vertices are* u *and* u_1*. When* $uu_1 \in E(G)$*, if* $d(u, P) + d(u_s, P) \geq s+1$*, then* G *has a path* P' *with vertex set* $V(P') = V(P) \cup \{u\}$ *whose end vertices are* u *and* u_1*.*

Lemma 2. *[4] Let* $P = u_1u_2 \ldots u_s$ *be a path with* $s \geq 3$ *in* G*. If* $d(u_s, P) + d(u_1, P) \geq s$*, then* G *has a cycle* C *with* $V(C) = V(P)$*.*

3 Proof of Theorem 1.3

Proof. Suppose that G does not contain k disjoint cycles $T_1, \ldots, T_k$ such that each cycle contains exactly one vertex in X and $|T_k| = 4$ and the rest are triangles. We prove that G contains k disjoint triangles $T_1, \ldots T_k$ such that each cycle contains exactly one vertex in X. Suppose this is false, let G be an edge-maximal counterexample. Since a complete graph of order $n \geq 3k + 1$ contains k disjoint triangles such that each triangle contains exactly one vertex of X, thus, G is not a complete graph. Let u and v be nonadjacent vertices of G and define $G' = G + uv$, the graph obtained from G by adding the edge uv. Then G' is not a counterexample by the maximality of G, that is, for any $X = \{v_1, \ldots, v_k\} \subseteq V(G)$, G' contains k disjoint triangles $T_1, \ldots T_k$ with respect to $\{v_1, \ldots, v_k\}$.

Claim. $k \geq 2$

Proof. Otherwise, suppose $k = 1$. By the classical result of Ore, G contains a Hamiltonian cycle $C = y_1y_2 \cdots y_ny_1$. We may assume that $v_1 = y_1$, otherwise, we can relabel the index of C.

We consider the path $P = y_1y_2y_3y_4$. Then $y_1y_3 \notin E(G)$, $y_2y_4 \notin E(G)$, $N(y_1, C - V(P)) \cap N(y_3, C - V(P)) = \phi$ and $N(y_2, C - V(P)) \cap N(y_4, C - V(P)) = \phi$. Then it follows that $2n \leq \Sigma_{x \in V(P)} d(x, G) \leq 6 + 2(n - 4) = 2n - 2$, a contradiction.

By the choice of G, there exists $v \in \{v_1, v_2 \ldots, v_k\}$ such that G contains $k-1$ triangles $T_1, \ldots T_{k-1}$ with respect to $\{v_1, v_2 \ldots, v_k\} - \{v\}$, $v \notin V\left(\bigcup_{i=1}^{k-1} T_i\right)$. Subject to this, we choose $v \in \{v_1, v_2 \ldots, v_k\}$ and $k - 1$ triangles $T_1, \ldots, T_{k-1}$ with respect to $\{v_1, v_2 \ldots, v_k\} - \{v\}$ such that

$$\textit{The length of the longest v-path in } G - V\left(\bigcup_{i=1}^{k-1} T_i\right). \tag{1}$$

Let $P = u_1 \ldots u_s$ be a longest v-path in $G - V\left(\bigcup_{i=1}^{k-1} T_i\right)$. Subject to (1), we choose $v \in \{v_1, v_2 \ldots, v_k\}$, $k - 1$ vertex disjoint triangles $T_1, \ldots, T_{k-1}$ with respect to $\{v_1, v_2 \ldots, v_k\} - \{v\}$ and P such that

$$\lambda(v, P). \tag{2}$$

Without loss of generality, suppose that $v = v_k$ and $v_i \in V(T_i)$ for each $i \in \{1, 2, \ldots, k-1\}$. Let $H = \bigcup_{i=1}^{k-1} T_i$, $D = G - H$ and $|D| = d$. Clearly, $d \geq 4$ as $n \geq 3k+1$. Furthermore, by the choice of G, $s \geq 3$.

Claim. P is a Hamiltonian path of D.

Proof. Suppose $s \leq d$. We choose an arbitrary vertex $x_0 \in D - V(P)$. Clearly, $x_0 u_1 \notin E(G)$ and $x_0 u_s \notin E(G)$. Note $s \geq 3$, by (1) and Lemma 2.1, $d(x_0, P) + d(u_1, P) \leq s - 1$. Since $d(x_0, D - V(P)) \leq d - s - 1$ and $d(u_1, D - V(P)) = 0$, it follows that $d(x_0, D) + d(u_1, D) \leq d - 2$. By the assumption on the degree condition of G, we have

$$d(x_0, H) + d(u_1, H) \geq (n + 2k - 2) - (d - 2) = 5(k-1) + 2$$

This implies that there exists $T_i \in H$, say T_1, such that $d(x_0, T_1) + d(u_1, T_1) = 6$. Let $T_1 = v_1 w_1 w_2 v_1$. If we replace T_1 with $x_0 w_2 v_1 x_0$, we obtain a path $P' = w_1 u_1 \ldots u_s$ with $|P'| = |P| + 1$, contradicting (1).

Claim. If $\lambda(v_k, P) = 0$ or 1, then D is Hamiltonian.

Proof. By Claim 3.2, D contains a Hamiltonian path $P = u_1 \ldots u_d$ passing through v_k. If $u_1 u_d \in E(G)$, then we have nothing to prove. So, $u_1 u_d \notin E(G)$. By symmetry, if $\lambda(v_k, P) = 0$, we may assume that $v_k = u_1$. If $\lambda(v_k, P) = 1$, we assume that $v_k = u_2$.

If there exists $T_i \in H_T$ such that $d(u_1, T_i) + d(u_d, T_i) = 6$, then there exists $w \in V(T_i)$ with $u_1 w \in E(G)$ such that $T_i - w + u_d$ contains a triangle $T_i{}'$ passing through v_i. If we replace T_i with $T_i{}'$, we see that D contains a v_k-path $P' = P - u_d + w$. However, $\lambda(v_k, P') = \lambda(v_k, P) + 1$, contradicting (2) while (1) still maintains. Hence,$d(u_1, T_i) + d(u_d, T_i) \leq 5$ for each $T_i \in H$ and so $d(u_1, H) + d(u_d, H) \leq 5(k-1)$. It follows that

$$d(u_1, D) + d(u_d, D) \geq n + 2k - 2 - 5(k-1) = d.$$

By Lemma 2.2, D contains a Hamiltonian cycle. This proves the claim.

Case 1. $d = 4$.

By Claim 3.3, D contains a Hamiltonian cycle C. Then G contains $k-1$ disjoint triangles $T_1, T_2, \ldots, T_{k-1}$ and a quadrilateral C with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction.

Case 2. $d \geq 5$.

By Claims 3.2 and 3.3, for each v_k-path P' of length 4 in D, we may assume that $\lambda(v_k, P') = 2$. Let $P' = y_1 y_2 y_3 y_4 y_5$ be an arbitrary v_k-path of length 4, then $v_k = y_3$. Since D does not contain a triangle passing through $y_3 = v_k$, hence, $y_1 y_3 \notin E(G)$ and $y_2 y_4 \notin E(G)$. Let $P'' = P' - y_5$. Since D contains no quadrilateral passing through $v_k = y_3$, then $N(y_1, D - V(P'')) \cap N(y_3, D - V(P'')) = \phi$ and

$N(y_2, D - V(P'')) \cap N(y_4, D - V(P'')) = \phi$. So, $\sum_{x \in V(P'')} d(x, D) \leq 2(d-4) + 6 = 2d - 2$. This gives that

$$\sum_{x \in V(P'')} d(x, H) \geq 2(n + 2k - 2) - (2d - 2) = 10(k-1) + 2$$

This implies that there exists $T_i \in H$, say T_1, such that $E(P'', T_1) \geq 11$. That is, there is at most one edge absent between P'' and T_1. Let $T_1 = v_1 w_1 w_2 v_1$.

We claim that $d(y_3, T_1) = 2$. Otherwise, say $d(y_3, T_1) = 3$. If $G[\{y_1, y_2, v_1\}]$ contains a triangle, denoted by $T_1{}'$, then G contains k disjoint triangles $T_1{}', T_2, \ldots T_{k-1}$, $y_3 w_1 w_2 y_3$ with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction. Hence, $E(v_1, y_1 y_2) \leq 1$ and $E(P'', T_1) \leq 11$, which yields to $d(y_4, T_1) = 3$ and $y_2 w_2 \in E(G)$. Consequently, G contains k disjoint triangles $y_4 w_1 v_1 y_4, T_2, \ldots, T_{k-1}, y_2 y_3 w_2 y_2$ with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction.

Since $d(y_3, T_1) = 2$, without loss of generality, say $y_3 w_2 \in E(G)$. Furthermore, we have $d(y_i, T_1) = 3$ for each $i \in \{1, 2, 4\}$. Then G contains k disjoint triangles $y_1 y_2 v_1 y_1, T_2, \ldots T_{k-1}$ and $y_3 y_4 w_2 y_3$ with respect to $\{v_1, v_2, \ldots, v_k\}$, a contradiction. This completes the proof of Case 2 and the proof of Theorem 3.

4 Bipartite Graph

In this section, we consider the disjoint 6-cycles in a bipartite graph. We list several useful lemmas.

Lemma 3. *Let C be a 6-cycle of G. Let $x \in V_1$ and $y \in V_2$ be two distinct vertices not on C. If $d(x, C) + d(y, C) \geq 5$, then there exists $z \in V(C)$ such that $C - z + x$ is a 6-cycle and $yz \in E(G)$.*

Proof. Without loss of generality, let $C = x_1 x_2 \ldots x_6 x_1$ with $x_1 \in V_1$. If $d(x, C) = 3$, since $d(y, C) \geq 2$, take any neighbor of $N(y, C)$ as z, the lemma is obvious. Hence, we may assume that $d(x, C) = 2$ and $d(y, C) = 3$. If $N(x, C) = \{x_2, x_4\}$, then $C - x_3 + x$ is a 6-cycle with $yx_3 \in E(G)$, we are done. By symmetry, we have $N(x, C) = \{x_2, x_6\}$. Then $C - x_1 + x$ is a 6-cycle with $yx_1 \in E(G)$. This proves the lemma.

Lemma 4. *[7] Let C be a 6-cycle, P_1, P_2 and P_3 be three paths in G with $l(P_1) = l(P_2) = l(P_3) = 1$. Suppose that C, P_1, P_2 and P_3 are disjoint and $E(C, P_1 \cup P_2 \cup P_3) \geq 13$, then $G[V(C \cup P_1 \cup P_2 \cup P_3)]$ contains a 6-cycle C' and a path P of order 6 such that C' and P are disjoint.*

Lemma 5. *[7] Let P_1 and P_2 be two disjoint paths in G with $l(P_1) = l(P_2) = 5$. If $E(P_1, P_2) \geq 7$, then $G[V(P_1 \cup P_2)]$ contains a 6-cycle.*

Lemma 6. *[7] Let C be a 6-cycle, P_1 and P_2 be two paths in G with $l(P_1) = l(P_2) = 5$. Suppose that C, P_1 and P_2 are disjoint and $E(C, P_1 \cup P_2) \geq 25$, then $G[V(C \cup P_1 \cup P_2)]$ contains two disjoint 6-cycles.*

Theorem 4. *Let k be a positive integer and $G = (V_1, V_2; E)$ a bipartite graph with $|V_1| = |V_2| = 3k$. Suppose $\delta(G) \geq 2k$, then G contains $k-1$ disjoint 6-cycles and a path of order 6 such that all of them are disjoint.*

Proof. Suppose on the contrary, G does not contain k disjoint 6-cycles and a path of order 6. Let G be an edge-maximal counterexample. Since a complete bipartite graph with $|V_1| = |V_2| = 3k$ contains k disjoint 6-cycles and a path of order 6 such that all of them are disjoint. Thus, G is not a complete bipartite graph. Take any nonadjacent pair $u \in V_1$ and $v \in V_2$, $G+xy$ contains k disjoint 6-cycles and a path of order 6 such that all of them are disjoint. This implies G contains $k-2$ disjoint 6-cycle $Q_1, Q_2, \ldots, Q_{k-1}$ and a subgraph D. We divide the proof into two cases:

Case 1. D contains a 6-cycle, denoted by Q_k.

Let $H = \cup_{i=1}^{k} Q_i$ and $D = G - V(H)$. We may assume that D contains at least one edge. Otherwise, take any pair of $u \in V_1 \cap D$ and $v \in V_2 \cap D$. Then $uv \notin E(G)$ and $d(u, D) + d(v, D) = 0$. Consequently, $d(u, H) + d(v, H) \geq 4(k-1) + 4$, which implies that there exists $Q_i \in H$ such that $d(u, Q_i) + d(v, Q_i) \geq 5$. By Lemma 3, $G[V(Q_i) \cup \{u, v\}]$ contains a 6-cycle $Q_i{}'$ and edge e such that $Q_i{}'$ and e are disjoint. Replace Q_i with $Q_i{}'$, we see D contains an edge.

Let uv be an edge in D with $u \in V_1 \cap D$. An argument analogous to the process of above can lead to D contains three disjoint edges. Denoted by uv, xy and ab with $\{u, x, a\} \subseteq V_1$. Now we will prove that D contains a path of order 6. Otherwise, $E(D) \leq 5$ and so we have

$$\sum_{x \in V(D)} d(x, H) \geq 12k - 10 = 12(k-1) + 2$$

This implies that there exists $Q_i \in H$ such that $\sum_{x \in V(D)} d(x, Q_i) \geq 13$. By Lemma 4, $G[V(Q_i \cup D)]$ contains a 6-cycle $Q_i{}'$ and a path P of order 6 such that C' and P are disjoint. Replace Q_i with $Q_i{}'$, we see that D contains a path of order 6, a contradiction.

Case 2. D contains two disjoint paths of order 6, denoted by P_1 and P_2.

In this case, G contains $k-2$ disjoint 6-cycles $Q_1, Q_{2,\ldots} Q_{k-2}$. Let $H' = \cup_{i=1}^{k-2} Q_i$. By Case 1, we may assume that $G[V(P_1 \cup P_2)]$ contains no 6-cycle. This leads to $E(P_1) \leq 7$ and $E(P_2) \leq 7$ and $E(P_1, P_2) \leq 6$ by Lemma 5. Consequently, we obtain

$$\sum_{x \in V(P_1 \cup P_2)} d(x, H) \geq 24k - 40 = 24(k-2) + 8$$

This implies that there exists $Q_i \in H'$ such that $\sum_{x \in V(P_1 \cup P_2)} d(x, Q_i) \geq 25$. By Lemma 6, $G[V(Q_i \cup P_1 \cup P_2)]$ contains two disjoint 6-cycles. By Case 1, we obtain a contradiction. This completes the proof of Theorem 4.

Theorem 5. *Let k be a positive integer and $G = (V_1, V_2; E)$ a bipartite graph with $|V_1| = |V_2| = 3k$. Suppose $\delta(G) \geq 2k$, then G contains k disjoint 6-cycles or $k-1$ disjoint 6-cycles and a quadrilateral such that all of them are disjoint.*

Proof. Suppose that G does not contain $k-1$ disjoint 6-cycles and a quadrilateral such that all of them are disjoint, we will show that G contains k disjoint 6-cycles. Suppose that this is not true. Let G be a edge-maximal counterexample. That is, for any pair of nonadjacent vertices $u \in V_1$ and $v \in V_2$, $G + uv$ contains k disjoint 6-cycles. This implies G contains $k-1$ disjoint 6-cycles $Q_1, \ldots Q_{k-1}$ and a path P of order 6. Denote $H = \cup_{i=1}^{k} Q_i$.

Since $G\left[V\left(P\right)\right]$ does not contain a quadrilateral or a 6-cycle, we see that $E\left(P\right) = 5$. Therefore, $\sum_{x \in V(P)} d\left(x, H\right) \geq 12k - 10 = 12\left(k-1\right) + 2$. This implies that there exists $Q_i \in H$ such that $\sum_{x \in V(P)} d\left(x, Q_i\right) \geq 13$. By our assumption, $G\left[V\left(Q_i \cup P\right)\right]$ contains two disjoint 6-cycles or two disjoint cycles with one quadrilateral and the other 6-cycle.

Without loss of generality, say $Q_i = a_1 a_2 \ldots a_6 a_1$ and $P = x_1 x_2 \ldots x_6 x_1$ with $\{a_1, x_1\} \subseteq V_1$. Set $d\left(a_1, P\right) = \max\left\{d\left(a_i, P\right) | i \in \{1,2,3,4,5,6\}\right\}$. Clearly, $d\left(a_1, P\right) \geq \left\lceil \frac{13}{6} \right\rceil = 3$, which means $N\left(x_1, P\right) = \{x_2, x_4, x_6\}$. Furthermore, we observe that $d\left(x_j, Q_i\right) \leq 1$ for each $j \in \{1,3,5\}$. Otherwise, without loss of generality, say $d\left(x_1, Q_i\right) \geq 2$. If $\{a_4, a_6\} \subseteq N\left(x_1, Q_i\right)$, then $G\left[V\left(Q_i \cup P\right)\right]$ contains a 6-cycle $P - x_1 + a_1$, which disjoint from a quadrilateral $x_1 a_4 a_5 a_6 x_1$, a contradiction. Hence, by symmetry, we may assume that $\{a_2, a_6\} \subseteq N\left(x_1, Q_i\right)$. Then $G\left[V\left(Q_i \cup P\right)\right]$ contains two disjoint 6-cycles $P - x_1 + a_1$ and $C - a_1 + x_1$, a contradiction. Consequently, it follows that $E\left(Q_i, P\right) \leq 3 + 3 + 6 = 12$, which contradicts the fact that $E\left(Q_i, P\right) \geq 13$, a final contradiction.

Remark 2. The degree condition in Theorem 4 is sharp in general. To see this, we construct bipartite graph G for positive integer as follows. Let $G_1 = (A, B; E_1)$ and $G_2 = (X, Y; E_2)$ be two independent complete bipartite graph with $|A| = |Y| = 2\left(k-1\right)$ and $|B| = |X| = k + 2$. Then G consists of G_1, G_1 and a set of $k+2$ independent edges between B and X, and finally, join every vertex in A to every vertex in Y. Clearly, $(A \cup X, B \cup Y)$ is a bipartition of G. It is easy to see that G does not contain $k-1$ disjoint 6-cycles and a path of order 6 such that all of them are disjoint. We see that the minimum degree of G is $2k-1$.

5 Conjectures

To conclude this paper, we propose the following two conjectures for readers to discuss.

Conjecture 1. Let $k \geq 2$, n be two positive integers and let G be a graph of order $n \geq 3k+1$, X a set of any k distinct vertices of G. If $\sigma_2\left(G\right) \geq n + 2k - 1$, G contains k disjoint triangles $T_1, \ldots T_k$ such that each triangle contains exactly one vertex in X.

Conjecture 2. Let k be a positive integer and $G = (V_1, V_2; E)$ a bipartite graph with $|V_1| = |V_2| = 3k$. Suppose $\delta\left(G\right) \geq 2k$, G contains k disjoint 6-cycles.

If Conjecture 2 is true, the degree condition is also sharp by Remark 2.

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