

On the Complexity of Approximate Sum of Sorted List^{*}

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Abstract. We consider the complexity for computing the approximate sum $a_1 + a_2 + \dots + a_n$ of a sorted list of numbers $a_1 \leq a_2 \leq \dots \leq a_n$. We show an algorithm that computes an $(1 + \epsilon)$ -approximation for the sum of a sorted list of nonnegative numbers in an $O(\frac{1}{\epsilon} \min(\log n, \log(\frac{x_{max}}{x_{min}})) \cdot (\log \frac{1}{\epsilon} + \log \log n))$ time, where x_{max} and x_{min} are the largest and the least positive elements of the input list, respectively. We prove a lower bound $\Omega(\min(\log n, \log(\frac{x_{max}}{x_{min}})))$ time for every $O(1)$ -approximation algorithm for the sum of a sorted list of nonnegative elements. We also show that there is no sublinear time approximation algorithm for the sum of a sorted list that contains at least one negative number.

1 Introduction

Computing the sum of a list of numbers is a classical problem that is often found inside the high school textbooks. There is a famous story about Karl Friedrich Gauss who computed $1 + 2 + \dots + 100$ via rearranging these terms into $(1 + 100) + (2 + 99) + \dots + (50 + 51) = 50 \times 101$, when he was seven years old, attending elementary school. Such a method is considered an efficient algorithm for computing a class of lists of increasing numbers. Computing the sum of a list of elements has many applications, and is ubiquitous in software design. In the classical mathematics, many functions can be approximated by the sum of simple functions via Taylor expansion. This kind of approximation theories is in the core area of mathematical analysis. In this article we consider if there is an efficient way to compute the sum of a general list of nonnegative numbers with nondecreasing order.

Let ϵ be a real number at least 0. Real number s is an $(1 + \epsilon)$ -approximation for the sum problem a_1, a_2, \dots, a_n if $\frac{\sum_{i=1}^n a_i}{1+\epsilon} \leq s \leq (1+\epsilon) \sum_{i=1}^n a_i$. Approximate sum problem was studied in the randomized computation model. Every $O(1)$ -approximation algorithm with uniform random sampling requires $\Omega(n)$ time in the worst case if the list of numbers in $[0, 1]$ is not sorted. Using $O(\frac{1}{\epsilon^2} \log \frac{1}{\delta})$ random samples, one can compute the $(1 + \epsilon)$ -approximation for the mean, or decide

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if it is at most δ for a list numbers in $[0, 1]$ [9]. Canetti, Even, and Goldreich [3] showed that the sample size is tight. Motwani, Panigrahy, and Xu [14] showed an $O(\sqrt{n})$ time approximation scheme for computing the sum of n nonnegative elements. There is a long history of research for the accuracy of summation of floating point numbers (for examples, see [10, 2, 1, 4–6, 8, 11–13, 15, 16]). The efforts were mainly spent on finding algorithms with small rounding errors.

We investigate the complexity for computing the approximate sum of a sorted list. When we have a large number of data items and need to compute the sum, an efficient approximation algorithm becomes important. Har-Peled developed an coresets approach for a more general problem. The method used in his paper implies an $O(\frac{\log n}{\epsilon})$ time approximation algorithm for the approximate sum of sorted nonnegative numbers [7]. The coresets is a subset of numbers selected from a sorted input list, and their positions only depends on the size n of the list, and independent of the numbers. The coresets of a list of n sorted nonnegative numbers has a size $\Omega(\log n)$. This requires the algorithm time to be also $\Omega(\log n)$ under all cases.

We show an algorithm that gives an $(1 + \epsilon)$ -approximation for the sum of a list of sorted nonnegative elements in $O(\frac{1}{\epsilon} \min(\log n, \log(\frac{x_{max}}{x_{min}})) \cdot (\log \frac{1}{\epsilon} + \log \log n))$ time, where x_{max} and x_{min} are the largest and the least positive elements of the input list, respectively. This algorithm has an incomparable complexity with Har-Peled’s algorithm. Our algorithm is of sub-logarithm complexity when $\frac{x_{max}}{x_{min}} \leq n^{\frac{1}{(\log \log n)^{1+a}}}$ for any fixed $a > 0$. The algorithm is based on a different method, which is a quadratic region search algorithm, from the coresets construction used in [7].

We also prove a lower bound $\Omega(\min(\log n, \log(\frac{x_{max}}{x_{min}})))$ for this problem. We first derive an $O(\log \log n)$ time approximation algorithm that finds an approximate region of the list for holding the items of size at least a threshold b . Our approximate sum algorithm is derived with it as a submodule. We also show an $\Omega(\log \log n)$ lower bound for approximate region algorithms for the sum of a sorted list with only nonnegative elements.

In Section 2, we present an algorithm that computes $(1 + \epsilon)$ -approximation for the sum of a sorted list of nonnegative numbers. In Section 3, we present lower bounds related to the sum of sorted list. In Section ??, we show the experimental results for the implementation of our algorithm in Section 2.

2 Algorithm for Approximate Sum of Sorted List

In this section, we show a deterministic algorithm for the sorted elements. We first show an approximation to find an approximate region of a sorted list with elements of size at least threshold b .

A crucial part of our approximate algorithm for the sum of sorted list is to find an approximate region with elements of size at least a threshold b . We develop a method that is much faster than binary search and it takes $O(\log \frac{1}{\delta} + \log \log n)$ time to find the approximate region. We first apply the square function to expand the region and use the square root function to narrow down to a region that only

has $(1 + \delta)$ factor difference with the exact region. The parameter δ determines the accuracy of approximation.

Definition 1. For $i \leq j$, let $|[i, j]|$ be the number of integers in the interval $[i, j]$.

If both i and j are integers with $i \leq j$, we have $|[i, j]| = j - i + 1$.

Definition 2. A list X of n numbers is represented by an array $X[1, n]$, which has n numbers $X[1], X[2], \dots, X[n]$. For integers $i \leq j$, let $X[i, j]$ be the sublist that contains elements $X[i], X[i + 1], \dots, X[j]$. For an interval $R = [i, j]$, denote $X[R]$ to be $X[i, j]$.

Definition 3. For a sorted list $X[1, n]$ with nonnegative elements by nondecreasing order and a threshold b , the b -region is an interval $[n', n]$ such that $X[n', n]$ are the numbers at least b in $X[1, n]$. An $(1 + \delta)$ -approximation for the b -region is a region $R = [s, n]$ such that at least $\frac{|R|}{1+\delta}$ numbers in $X[s, n]$ are at least b , and $[s, n]$ contains all every position j with $X[j] \geq b$.

2.1 Approximate Region

The approximation algorithm for finding an approximate b -region to contain the elements at least a threshold b has two loops. The first loop searches the region by increasing the parameter m via the square function. When the region is larger than the exact region, the second loop is entered. It converges to the approximate region with a factor that goes down by a square root each cycle. Using the combination of the square and square root functions makes our algorithm much faster than the binary search.

In order to simplify the description of the algorithm `Approximate-Region(.)`, we assume $X[i] = -\infty$ for every $i \leq 0$. It can save the space for the boundary checking when accessing the list X . The description of the algorithm is mainly based on the consideration for its proof of correctness. For a real number a , denote $\lfloor a \rfloor$ to be the largest integer at most a , and $\lceil a \rceil$ to be the least integer at least a .

Algorithm `Approximate-Region`(X, b, δ, n)

Input: $X[1, n]$ is a sorted list of n numbers by nondecreasing order; n is the size of $X[1, n]$; b is a threshold in $(0, +\infty)$; and δ is a parameter in $(0, +\infty)$.

1. if $(X[n] < b)$, return \emptyset ;
2. if $(X[n - 1] < b)$, return $[n, n]$;
3. if $(X[1] \geq b)$, return $[1, n]$;
4. let $m := 2$;
5. while $(X[n - m^2 + 1] \geq b)$ {
6. let $m := m^2$;
7. };
8. let $i := 1$;
9. let $m_1 := m$;

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10.   let  $r_1 := m$ ;
11.   while  $(m_i \geq 1 + \delta)$  {
12.       let  $m_{i+1} := \sqrt{m_i}$ ;
13.       if  $(X[n - \lfloor m_{i+1}r_i \rfloor + 1] \geq b)$ , then let  $r_{i+1} := m_{i+1}r_i$ ;
14.       else  $r_{i+1} := r_i$ ;
15.       let  $i := i + 1$ ;
16.   };
17.   return  $[n - \lfloor m_i r_i \rfloor + 1, n]$ ;

```

End of Algorithm

Lemma 1. *Let δ be a parameter in $(0, 1)$. Given an element b , and a list A of sorted n elements, Algorithm Approximate-Region(.) finds an $(1 + \delta)$ -approximate b -region in $O((\log \frac{1}{\delta}) + (\log \log n))$ time.*

Proof. After the first phase (lines 1 to 7) of the algorithm, we obtain number m such that

$$X[n - m + 1] \geq b, \quad \text{and} \tag{1}$$

$$X[n - m^2 + 1] < b. \tag{2}$$

As we already assume $X[i] = -\infty$ for every $i \leq 0$, there is no boundary problem for assessing the input list. The variable m is an integer in the first phase. Thus, the boundary point for the region with numbers at least the threshold b is in $[n - m^2 + 1, n - m + 1]$. The variable m can be expressed as 2^{2^k} for some integer $k \geq 0$ after executing k cycles in the first phase. Thus, the first phase takes $O(\log \log n)$ time because m is increased to m^2 at each cycle of the first while loop, and $2^{2^k} \geq n$ for $k \geq \log \log n$.

In the second phase (lines 8 to 17) of the algorithm, we can prove that $X[n - \lfloor r_i \rfloor + 1] \geq b$ and $X[n - \lfloor m_i r_i \rfloor + 1] < b$ at the end of every cycle (right after executing the statement at line 15) of the second loop (lines 11 to 16). Thus, the boundary point for the region with elements at the threshold b is in $[n - \lfloor m_i r_i \rfloor + 1, n - \lfloor r_i \rfloor + 1]$. The variable m_i is not an integer after $m_i < 2$ in the algorithm. It can be verified via a simple induction. It is true before entering the second loop (lines 11 to 16) by inequalities (1) and (2). Assume that at the end of cycle i ,

$$X[n - \lfloor r_i \rfloor + 1] \geq b; \quad \text{and} \tag{3}$$

$$X[n - \lfloor m_i r_i \rfloor + 1] < b. \tag{4}$$

Let us consider cycle $i + 1$ at the second loop. Let $m_{i+1} = \sqrt{m_i}$.

- Case 1: $X[n - \lfloor m_{i+1}r_i \rfloor + 1] \geq b$. Let $r_{i+1} = m_{i+1}r_i$ according to line 13 in the algorithm. Then $X[n - \lfloor r_{i+1} \rfloor + 1] = X[n - \lfloor m_{i+1}r_i \rfloor + 1] \geq b$. By inequality (4) in the hypothesis, $X[n - \lfloor m_{i+1}r_{i+1} \rfloor + 1] = X[n - \lfloor \sqrt{m_i}\sqrt{m_i}r_i \rfloor + 1] = X[n - \lfloor m_i r_i \rfloor + 1] < b$.

2. Case 2: $X[n - \lfloor m_{i+1}r_i \rfloor + 1] < b$. Let $r_{i+1} = r_i$ according to line 14 the algorithm. We have $X[n - \lfloor r_{i+1} \rfloor + 1] = X[n - \lfloor r_i \rfloor + 1] \geq b$ by inequality (3) in the hypothesis. By inequality (4) in the hypothesis, $X[n - \lfloor m_{i+1}r_{i+1} \rfloor + 1] = X[n - \lfloor m_{i+1}r_i \rfloor + 1] < b$ by the condition of this case.

Therefore, $X[n - \lfloor r_{i+1} \rfloor + 1] \geq b$ and $X[n - \lfloor m_{i+1}r_{i+1} \rfloor + 1] < b$ at the end of cycle $i + 1$ of the second while loop.

Every number in $X[n - r_i + 1, n]$, which has r_i entries, is at least b , and $X[n - m_i r_i + 1, n]$ has $m_i r_i$ entries and $m_i \leq 1 + \delta$ at the end of the algorithm. Thus, the interval $[n - m_i r_i + 1, n]$ returned by the algorithm is an $(1 + \delta)$ -approximation for the b -region.

It takes $O(\log \log n)$ steps for converting m to be at most 2, and additional $\log \frac{1}{\delta}$ steps to make m to be at most $1 + \delta$. When $m_i < 1 + \delta$, we stop the loop, and output an $(1 + \delta)$ -approximation. This step takes at most $O(\log \frac{1}{\delta} + \log \log n)$ time since m_i is assigned to $\sqrt{m_i}$ at each cycle of the second loop. This proves Lemma 1.

After the first loop of the algorithm Approximate-Region(.), the number m is always of the format 2^{2^k} for some integer k . In the second loop of the algorithm Approximate-Region(.), the number m is always of the format 2^{2^k} when m is at least 2. Computing its square root is to convert 2^{2^k} to $2^{2^{k-1}}$, where k is an integer. Since $(1 + \frac{1}{2^i}) \cdot (1 + \frac{1}{2^i}) > (1 + \frac{1}{2^{i-1}})$, we have that $(1 + \frac{1}{2^i})$ is larger than the square root of $(1 + \frac{1}{2^{i-1}})$. We may let variable m_i go down by following the sequence $\{(1 + \frac{1}{2^i})\}_{i=1}^\infty$ after $m_i \leq 2$. In other words, let $g(\cdot)$ be an approximate square root function such that $g(1 + \frac{1}{2^i}) = 1 + \frac{1}{2^{i+1}}$ for computing the square root after $m \leq 2$ in the algorithm. It has the property $g(m) \cdot g(m) \geq m$. The assignment $m_{i+1} = \sqrt{m_i}$ can be replaced by $m_{i+1} = g(m_i)$ in the algorithm. It can simplify the algorithm by removing the computation of square root while the computational complexity is of the same order.

2.2 Approximate Sum

We present an algorithm to compute the approximate sum of a list of sorted nonnegative elements. It calls the module for the approximate region, which is described in Section 2.1.

The algorithm for the approximate sum of a sorted list X of n nonnegative numbers generates a series of disjoint intervals $R_1 = [r_1, r'_1], \dots, R_t = [r_t, r'_t]$, and a series of thresholds b_1, \dots, b_t such that each R_i is an $(1 + \delta)$ -approximate b_i -region in $X[1, r'_i]$, $r'_1 = n$, $r'_{i+1} = r_i - 1$, and $b_{i+1} \leq \frac{b_i}{1 + \delta}$, where $\delta = \frac{3\epsilon}{4}$ and $1 + \epsilon$ is the accuracy for approximation. The sum of numbers in $X[R_i]$ is approximated by $|R_i|b_i$. As the list $b_1 > b_2 > \dots > b_t$ decreases exponentially, we can show that $t = O(\frac{1}{\epsilon} \log n)$. The approximate sum for the input list is $\sum_{i=1}^t |R_i|b_i$. We give a formal description of the algorithm and its proof below.

Algorithm Approximate-Sum(X, ϵ, n)

Input: $X[1, n]$ is a sorted list of nonnegative numbers (by nondecreasing order) and n is the size of $X[1, n]$, and ϵ is a parameter in $(0, 1)$ for the accuracy of approximation.

1. if $(X(n) = 0)$, return 0;
2. let $\delta := \frac{3\epsilon}{4}$;
3. let $r'_1 := n$;
4. let $s := 0$;
5. let $i := 1$;
6. let $b_1 := \frac{X[n]}{1+\delta}$;
7. while $(b_i \geq \frac{\delta X[n]}{3n})$ {
8. let $R_i := \text{Approximate-Region}(X, b_i, \delta, r'_i)$;
9. let $r'_{i+1} := r_i - 1$ for $R_i = [r_i, r'_i]$;
10. let $b_{i+1} := \frac{X[r'_{i+1}]}{1+\delta}$;
11. let $s_i := \lfloor [r_i, r'_i] \rfloor \cdot b_i$;
12. let $s := s + s_i$;
13. let $i := i + 1$;
14. };
15. return s ;

End of Algorithm

Theorem 1. *Let ϵ be a positive parameter. Then there is an $O(\frac{1}{\epsilon} \min(\log n, \log(\frac{x_{max}}{x_{min}}))) \cdot (\log \frac{1}{\epsilon} + \log \log n)$ time algorithm to compute $(1 + \epsilon)$ -approximation for the sum of sorted list of nonnegative numbers, where x_{max} and x_{min} are the largest and the least positive elements of the input list, respectively.*

Proof. Assume that there are t cycles executed in the while loop of the algorithm Approximate-Sum(.). Let regions R_1, R_2, \dots, R_t be generated. In the first cycle of the loop, the algorithm finds a region $R_1 = [r_1, n]$ of the elements of size at least $\frac{X[n]}{1+\delta}$. In the second cycle of the loop, the algorithm finds region $R_2 = [r_2, r_1 - 1]$ for the elements of size at least $\frac{X[r_1-1]}{1+\delta}$. In the i -th cycle of the loop, it finds a region $R_i = [r_i, r_{i-1} - 1]$ of elements of size at least $\frac{X[r_{i-1}-1]}{1+\delta}$. By the algorithm, we have

$$j \in R_1 \cup R_2 \cup \dots \cup R_t \text{ for every } j \text{ with } X[j] \geq \frac{\delta X[n]}{3n}. \tag{5}$$

Since each R_i is an $(1 + \delta)$ -approximation of $\frac{X[r_{i-1}-1]}{1+\delta}$ -region in $X[1, r_{i-1} - 1]$, $X[R_i]$ contains at least $\frac{|R_i|}{1+\delta}$ entries of size at least $\frac{X[r_{i-1}-1]}{1+\delta}$ in $X[1, r_{i-1} - 1]$, R_i also contains every entry of size at least $\frac{X[r_{i-1}-1]}{1+\delta}$ in $X[1, r_{i-1} - 1]$. Thus,

$$\frac{s_i}{1 + \delta} = \frac{|R_i|}{1 + \delta} \cdot \frac{X[r_{i-1} - 1]}{1 + \delta} \leq \sum_{j \in R_i} X[j] \leq |R_i| X[r_{i-1} - 1] = (1 + \delta) s_i.$$

Thus,

$$\frac{s_i}{1 + \delta} \leq \sum_{j \in R_i} X[j] \leq (1 + \delta)s_i.$$

We have

$$\frac{1}{1 + \delta} \sum_{j \in R_i} X[j] \leq s_i \leq (1 + \delta) \sum_{j \in R_i} X[j]. \tag{6}$$

Thus, s_i is an $(1 + \delta)$ -approximation for $\sum_{j \in R_i} X[j]$. We also have $\sum_{X[i] < \frac{\delta X[n]}{3n}} X[i] < \frac{\delta X[n]}{3}$ since $X[1, n]$ has only n numbers in total. Therefore, we have the following inequalities:

$$\sum_{X[i] \geq \frac{\delta X[n]}{3n}} X[i] = \sum_{i=1}^n X[i] - \sum_{X[i] < \frac{\delta X[n]}{3n}} X[i] \tag{7}$$

$$\geq \sum_{i=1}^n X[i] - \frac{\delta}{3} \sum_{i=1}^n X[i] \tag{8}$$

$$= (1 - \frac{\delta}{3}) \sum_{i=1}^n X[i]. \tag{9}$$

We have the inequalities:

$$s = \sum_{i=1}^t s_i \tag{10}$$

$$\geq \frac{1}{1 + \delta} \sum_{X[i] \geq \frac{\delta X[n]}{3n}} X[i] \quad (\text{by inequality (6)}) \tag{11}$$

$$\geq \frac{(1 - \frac{\delta}{3})}{1 + \delta} \sum_{i=1}^n X[i] \quad (\text{by inequality (9)}) \tag{12}$$

$$= \frac{1}{\frac{1+\delta}{1-\frac{\delta}{3}}} \sum_{i=1}^n X[i] \tag{13}$$

$$= \frac{1}{1 + \frac{4\delta}{1-\frac{\delta}{3}}} \sum_{i=1}^n X[i] \tag{14}$$

$$\geq \frac{1}{1 + \frac{4\delta}{3}} \sum_{i=1}^n X[i] \tag{15}$$

$$= \frac{1}{1 + \epsilon} \sum_{i=1}^n X[i]. \tag{16}$$

As R_1, R_2, \dots are disjoint each other, we also have the following inequalities:

$$s = \sum_{i=1}^t s_i \tag{17}$$

$$\leq \sum_{i=1}^t (1 + \delta) \sum_{j \in R_i} X[j] \quad (\text{by inequality (6)}) \tag{18}$$

$$\leq (1 + \delta) \sum_{j=1}^n X[j] \tag{19}$$

$$\leq (1 + \epsilon) \sum_{j=1}^n X[j]. \tag{20}$$

Therefore, the output s returned by the algorithm is an $(1 + \epsilon)$ -approximation for the sum $\sum_{i=1}^n X[i]$. By Lemma 1, each cycle in the while loop of the algorithm takes $O((\log \frac{1}{\delta} + \log \log n))$ time for generating R_i . For the descending chain $r'_1 > r'_2 > \dots > r'_t$ with $X[r'_i] \leq \frac{X[r'_{i+1}]}{1 + \delta}$ and $b_i = X[r'_i] \geq \frac{\delta X[n]}{3n}$ for each i , we have that the number of cycles t is at most $O(\frac{1}{\delta} \log n)$. This is because $X[r'_t] \leq \frac{x_{max}}{(1 + \delta)^t} \leq \frac{\delta X[n]}{3n}$ for some $t = O(\frac{1}{\delta} \log n)$. Similarly, the number of cycles t is at most $O(\frac{1}{\delta} \log(\frac{x_{max}}{x_{min}}))$ because $X[r'_t] \leq \frac{x_{max}}{(1 + \delta)^t} \leq x_{min}$ for some $t = O(\frac{1}{\delta} \log(\frac{x_{max}}{x_{min}}))$.

Therefore, there are most $t = O(\frac{1}{\delta} \min(\log n, \log \frac{x_{max}}{x_{min}}))$ cycles in the while loop of the algorithm. Therefore, the total time is $O(\frac{1}{\delta} \min(\log n, \log(\frac{x_{max}}{x_{min}}))(\log \frac{1}{\delta} + \log \log n)) = O(\frac{1}{\epsilon} \min(\log n, \log(\frac{x_{max}}{x_{min}}))(\log \frac{1}{\epsilon} + \log \log n))$. This proves Theorem 1.

3 Lower Bounds

In this section, we show several lower bounds about approximation for the sum of sorted list. The $\Omega(\min(\log n, \log(\frac{x_{max}}{x_{min}})))$ lower bound is based on the general computation model for the sum problem. The lower bound $\Omega(\log \log n)$ for finding an approximate b -region shows that upper bound is optimal if using the method developed in Section 2. We also show that there is no sublinear time algorithm if the input list contains one negative element.

3.1 Lower Bound for Computing Approximate Sum

In this section, we show a lower bound for the general computation model, which almost matches the upper bound of our algorithm. This indicates the algorithm in Section 2 can be improved by at most $O(\log \log n)$ factor.

The lower bound is proved by a contradiction method. In the proof of the lower bound, two lists L_1 and L_2 are constructed. For an algorithm with $o(\log n)$ queries, the two lists will have the same answers to all queries. Thus, the approximation outputs for the two inputs L_1 and L_2 are the same. We let the gap of the sums from the two lists be large enough to make them impossible to share the same constant factor approximation.

Theorem 2. *For every positive constant $d > 1$, every d -approximation algorithm for the sum of a sorted list of nonnegative numbers needs at least*

$\Omega(\min(\log n, \log \frac{x_{max}}{x_{min}}))$ (adaptive) queries to the list, where γ is an arbitrary small constant in $(0, 1)$, where x_{max} and x_{min} are the largest and the least positive elements of the input list, respectively..

3.2 Lower Bound for Computing Approximate Region

We give an $\Omega(\log \log n)$ lower bound for the deterministic approximation scheme for a b -region in a sorted input list of nonnegative numbers. The method is that if there is an algorithm with $o(\log \log n)$ queries, two sorted lists L_1 and L_2 of 0, 1 numbers are constructed. They reply the same answer the each the query from the algorithm, but their sums have large difference. This lower bound shows that it is impossible to use the method of Section 2, which iteratively finds approximate regions via a top down approach, to get a better upper bound for the approximate sum problem.

Definition 4. For a sorted list $X[1, n]$ with 0, 1 numbers by nondecreasing order, an d -approximate 1-region is a region $R = [s, n]$, which contains the last position n of $X[1, n]$, such that at least $\frac{|R|}{d}$ numbers in $X[s, n]$ are 1, and $X[s, n]$ contains all the positions j with $X[j] = 1$, where $|R|$ is the number of integers i in R .

Theorem 3. For any parameter $d > 1$, every deterministic algorithm must make at least $\log \log n - \log \log(d + 1)$ adaptive queries to a sorted input list for the d -approximate 1-region problem.

Corollary 1. For any constant $\epsilon \in (0, 1)$, every deterministic $O(1)$ -approximation algorithm for 1-region problem must make at least $(1 - \epsilon) \log \log n$ adaptive queries.

3.3 Lower Bound for Sorted List with Negative Elements

We derive a theorem that shows there is not any factor approximation sublinear time algorithm for the sum of a list of elements that contains both positive and negative elements.

Theorem 4. Let ϵ be an arbitrary positive constant. There is no algorithm that makes at most $n - 1$ queries to give $(1 + \epsilon)$ -approximation for the sum of a list of n sorted elements that contains at least one negative element.

Proof. Consider a list of element $-m(m + 1), 2, \dots, 2m$. This list contains $n = m + 1$ elements. If there is an algorithm that gives $(1 + \epsilon)$ -approximation, then there is an element, say $2k$, that is not queried by the algorithm.

We construct another list that is identical to the last list except $2k$ being replaced by $2k + 1$.

The sum of the first list is zero, but the sum of the second list is 1. The algorithm gives the same result as the element $2k$ in the first list and the element $2k + 1$ in the second list are not queried (all the other queries are the of the same answers). This brings a contradiction.

Similarly, in the case that $-m(m + 1)$ is not queried, we can bring a contradiction after replacing it with $-m(m + 1) + 1$.

4 Conclusions and Open Problems

We studied the approximate sum in a sorted list with nonnegative elements. For a fixed ϵ , there is a $\log \log n$ factor gap between the upper bound of our algorithm, and our lower bound. An interesting problem of further research is to close this gap. Another interesting problem is the computational complexity of approximate sum in the randomized computational model, which is not discussed in this paper.

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