# Independent Domination: Reductions from Circular- and Triad-Convex Bipartite Graphs to Convex Bipartite Graphs*


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**Abstract.** An *independent dominating set* in a graph is a subset of vertices, such that no edge has both ends in the subset, and each vertex either itself is in the subset or has a neighbor in the subset. In a *convex bipartite* (*circular convex bipartite*, *triad convex bipartite*, respectively) graph, there is a linear ordering (a circular ordering, a triad, respectively) defined on one class of vertices, such that for every vertex in the other class, the neighborhood of this vertex is an interval (a circular arc, a subtree, respectively), where a *triad* is three paths with a common end. The problem of finding a minimum independent dominating set, called *independent domination*, is known $\mathcal{NP}$-complete for bipartite graphs and tractable for convex bipartite graphs. In this paper, we make polynomial time reductions for independent domination from triad- and circular-convex bipartite graphs to convex bipartite graphs.



**Keywords:** Independent domination, circular convex bipartite graph, triad convex bipartite graph, polynomial time reduction.


## 1 Introduction

An *independent dominating set* in a graph is a subset of vertices, such that the subset is an independent set, and every vertex in the graph either itself is in the subset or has a neighbor in the subset. The problem of finding a minimum independent dominating set, called *independent domination*, is $\mathcal{NP}$-complete for *chordal bipartite* graphs, but polynomial time solvable for *convex bipartite* graphs [3]. In a *convex bipartite* graph [7,3,2], there is a linear ordering defined on one class of vertices, such that for every vertex in another class, the neighborhood of this vertex is an interval. In a *chordal bipartite* graph [6], every cycle of length at least *six* has a chord, where a *chord* of a cycle on a graph is an edge between two vertices of the cycle but the edge itself is not a part of the cycle.


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Beside convex bipartite graphs and chordal bipartite graphs, there are other interesting bipartite graph classes, such as *circular convex bipartite* graphs [11] and *triad convex bipartite* [10,9] graphs, etc, see Figure 1.

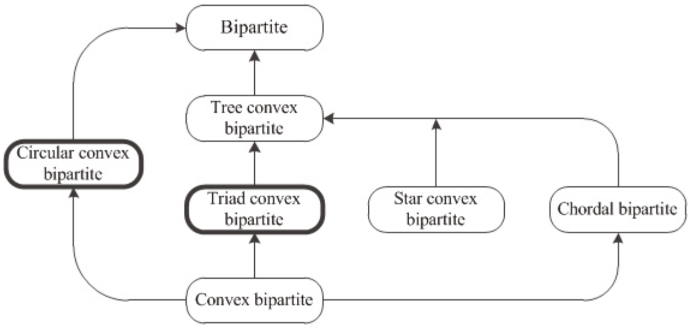


**Fig. 1.** Various bipartite graph classes and their inclusions

In a *circular convex bipartite* graph [11], there is a circular ordering defined on one class of vertices, such that for every vertex in another class, the neighborhood of this vertex is a circular arc. Circular convex bipartite graphs are natural models for scheduling problems. For example, the available working hours of a worker is usually a consecutive period of hours. A group of workers and their available hours can be modeled by a circular convex bipartite graph [11]. For a long time, complexity results for circular convex bipartite graphs are scarce. Maximum matching and Hamiltonian cycle and path are known linear time solvable for circular-convex bipartite graphs [11]. The complexity of independent domination for circular-convex bipartite graphs is unknown before. In this paper, we show that independent domination is *polynomial time* solvable for *circular convex bipartite* graphs.

In a *tree convex bipartite* graph [8,9], there is a tree defined on one class of vertices, such that for every vertex in another class, the neighborhood of this vertex is a subtree. When the tree is a star (a triad, respectively), the graph is called *star convex bipartite* [8,9] (*triad convex bipartite* [10,9], respectively), where a *triad* is three pathes with a common end. It is known that independent domination is $\mathcal{NP}$-complete for star convex bipartite graphs, but tractable for triad convex bipartite graphs in [14]. In this paper, we simplify the tractability proof in [14].

Our main contributions are making two explicit reductions for independent domination from circular- and triad-convex bipartite graphs respectively to convex bipartite graphs, instead of running modified algorithms such as in [14]. In fact, the second reduction can be viewed as a detailed proof for the correctness of the algorithm in [14], easier to understand with a better modularity. Moreover, our reductions are Cook reductions (i.e. polynomial time Turing reductions) [5], which call the known polynomial time algorithms of independent domination for convex bipartite graphs [3] many times, and also work for *weighted* circular- and triad-convex bipartite graphs, though the original algorithm in [3] only works for unweighted bipartite graphs. Before our works, only Karp reduction (i.e. polynomial many-one reduction) [5] from circular convex bipartite graphs to circular-arc graphs is used [11]. Thus, our methods may be of use to show more problems tractable for circular- and triad-convex bipartite graphs.

This paper is structured as follows. After introducing necessary definitions and notations mainly from graph theory (Section 2), polynomial time reductions for independent domination from circular-convex bipartite graphs (Section 3) and triad-convex bipartite graphs (Section 4) to convex bipartite graphs are shown respectively. Concluding remarks are at the last section (Section 5).

## 2 Preliminaries

A *graph* $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$. Each edge $e$ in $E$ is *incident* to two vertices, called its ends, and these two ends are called *adjacent* to each other. For each vertex $v$, its *neighborhood* $N(v) = \{u | v \text{ is adjacent to } u\}$, its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. For a subset $V'$ of vertices, $N(V') = \bigcup_{v \in V'} N(v)$. A *path* in a graph is a sequence of different vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, such that each two consecutive vertices are adjacent to each other. A *cycle* is a path where $v_{i_1}$ and $v_{i_k}$ are also adjacent to each other. A graph is *connected* if every two vertices are connected by a path. A *tree* is a connected graph without any cycle. For a subset $V'$ of vertices, the *induced subgraph* $G[V'] = (V', E')$, where $V' \subseteq V$ and $E' = \{e \in E | e \text{ has both ends in } V'\}$. An *independent set* is a subset of vertices whose induced subgraph has no edge.

In a *weighted* graph $G = (V, E, w)$, there is a function $w$ defined on $V$, such that each vertex $v$ has a weight $w(v)$. The weight of a vertex subset $V'$ is $w(V') = \sum_{v \in V'} w(v)$. When $w(v) = 1$ for all vertices $v$, the graph is called *unweighted.* In a *finite* graph, both $V$ and $E$ are finite sets. A *simple* graph has no loop and no parallel edges, where a loop has the same one vertex as its ends, and two parallel edges are incident to the same two ends. In a *bipartite* graph, denoted by $G = (A, B, E)$, the vertex set $V$ is divided into two classes $A$ and $B$, such that each edge is incident to a vertex in $A$ and a vertex in $B$ respectively. In this paper, we only consider finite simple bipartite graphs.

The *cardinality* of a set $X$, i.e. the number of elements in $X$, is denoted by $|X|$. The *difference* of two sets $X$ and $Y$ is denoted by $X \backslash Y = \{x \mid x \in X \text{ and } x \notin Y\}$. The *empty set* is denoted by $\emptyset$. An arbitrary ordering on a set is denoted by $\prec$.

**Definition 1.** *(*Independent Dominating Set*) In a graph $G = (V, E)$, an independent dominating set $D$ is a subset of $V$, such that $D$ is an independent set, and for each vertex $v$ in $V$, either $v \in D$ or $N(v) \cap D \neq \emptyset$.*

**Definition 2.** *(*Triad*) A $G = (V, E)$ is called a* triad*, if the vertex set $V$ can be partitioned into four parts, $V_1, V_2, V_3, \{v_0\}$, such that for $i = 1, 2, 3$, $V_i \cup \{v_0\}$ induces a path. The vertex $v_0$ is called* center*.*

**Definition 3.** *(*Circular Convex Bipartite Graphs *[11]) A bipartite graph $G = (A, B, E)$ is called* circular convex bipartite*, if there is a circular ordering $\prec$*

*defined on* $A = \{a_1, \ldots, a_n\}$, $a_1 \prec a_2 \prec \cdots \prec a_n \prec a_1$, *such that for each vertex* $b$ *in* $B$, *its neighborhood* $N(b)$ *is a circular arc under this circular ordering, that is, there are two (possibly equal) vertices* $a_i$ *and* $a_j$, *where* $1 \leq i \leq j \leq n$, *such that* $N(b) = \{a_i, a_{i+1}, \ldots, a_j\}$ *or* $N(b) = \{a_j, a_{j+1}, \ldots, a_n, a_1, \ldots, a_i\}$.

**Definition 4.** *(*Triad Convex Bipartite Graphs *[8,9]) A bipartite graph* $G = (A, B, E)$ *is called* triad convex bipartite, *if these is a triad* $T = (A, F)$ *defined on* $A$, *such that for each vertex* $b$ *in* $B$, *its neighborhood* $N(b)$ *is a subtree of* $T$.

*Remark 1.* The adjacent matrices of circular convex (convex, respectively) bipartite graphs have the so-called *circular* (*consecutive*, respectively) *ones property*, which are recognizable in *linear* time [4]. Tree convex bipartite graphs are also recognizable in *linear* time [1]. The associated circular orderings (trees, respectively) are all constructible in *linear* time, thus can safely be assumed as part of the inputs. Chordal bipartite graphs are recognizable in *square* time.

We refer to [5] for the notions of *polynomial time*, *reductions*, and $\mathcal{NP}$*-completeness*.

## 3 Reduction from Circular-Convex Bipartite Graphs

In this section, we show that independent domination is polynomial time solvable for circular-convex bipartite graphs, by a polynomial time reduction for this problem from circular-convex bipartite graphs to convex bipartite graphs.

**Theorem 1.** *For circular convex bipartite graphs* $G = (A, B, E)$ *with a circular ordering on* $A$, *independent domination is* $O\left(|A|(|A| + |B|)^3\right)$ *time solvable.*

*Proof.* Without loss of generality, we assume that $G$ contains *no* isolated vertex, since isolated vertices are trivially in every independent dominating set.

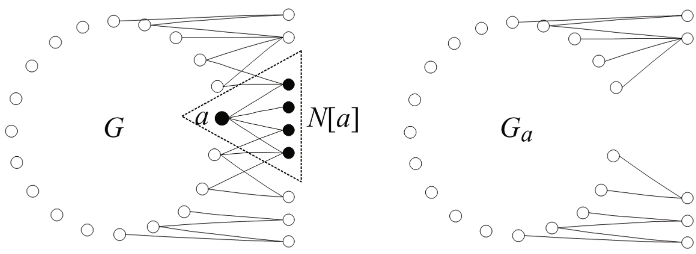


**Fig. 2.** Removing vertices in $N[a]$ from graph $G$ results in graph $G_a$

First, for each vertex $a$ in $A$, we define a graph $G_a$ as follows, see Figure 2.

$$G_a = (A_a, B_a, E_a), \text{ where } A_a = A \setminus \{a\}, B_a = B \setminus N(a), \text{ and}$$
$$E_a = \{e \in E \mid e \text{ is } \textit{not} \text{ incident to any vertex in } N[a]\}.$$

**Lemma 1.** *For each $a \in A$, $G_a$ is convex bipartite.*

*Proof.* We prove by definition of convex bipartite graphs. After removing $\{a\} \cup N(a)$ and the incident edges from $G$, no vertex in $B_a = B \setminus N(a)$ is adjacent to vertex $a$. Since $G$ is circular convex bipartite, for each vertex in $B_a$, its neighborhood is a circular arc contained in $A_a = A \setminus \{a\}$. Thus, we can restrict the circular ordering on $A$ to a linear ordering on $A_a$, such that for each vertex in $B_a$, its neighborhood is an interval under this linear ordering. □

**Lemma 2.** *For each $a \in A$, if $D$ is an independent dominating set of $G$ containing $a$, then $D \setminus \{a\}$ is an independent dominating set of $G_a$.*

*Proof.* We prove by definition of independent dominating sets. Since $a \in D$, $N(a) \cap D = \emptyset$. For each vertex $a' \in A_a$, either $a' \in D$ or $N(a') \cap D \neq \emptyset$. Since $a \notin N[a']$, either $a' \in D \setminus \{a\}$ or $N(a') \cap (D \setminus \{a\}) \neq \emptyset$. For each vertex $b' \in B_a$, either $b' \in D$ or $N(b') \cap D \neq \emptyset$. Since $a \notin N[b']$, either $b' \in D \setminus \{a\}$ or $N(b') \cap (D \setminus \{a\}) \neq \emptyset$. □

**Lemma 3.** *For each $a \in A$, if $D'$ is an independent dominating set of $G_a$, then $D' \cup \{a\}$ is an independent dominating set of $G$.*

*Proof.* We prove again by definition. Since $G_a$ is resulted by removing $N[a]$ from $G$, $a$ is not adjacent to any vertex in $G'$, $D' \cup \{a\}$ is an independent set. Since $D'$ is an independent dominating set of $G_a$ and each vertex in $N(a)$ is adjacent to $a$, $D' \cup \{a\}$ is an independent dominating set of $G$. □

Next, we define a set $\mathcal{S}$ as follows.

$$\mathcal{S} = \{B\} \cup \{D_a \cup \{a\} \mid a \in A \text{ and } D_a \text{ is a minimum independent dominating set in } G_a\}.$$

*Remark 2.* For each $a$, $G_a$ is unique, but for each $G_a$, $D_a$ may not be unique. For our purpose, however, for each $a$, we only need one such $D_a$ in $\mathcal{S}$, see proof of Lemma 5 below.

**Lemma 4.** *$\mathcal{S}$ contains a minimum independent dominating set of $G$.*

*Proof.* Let $D$ be a minimum independent dominating set of $G$. We consider the following two cases.

**Case 1**: $D \cap A = \emptyset$.

Since $D$ is an independent dominating set, for each vertex $b$ in $B$, either $b \in D$ or $N(b) \cap D \neq \emptyset$. Since $D \cap A = \emptyset$, $G$ is bipartite and $N(b) \subseteq A$, we have $N(b) \cap D = \emptyset$ and thus $b \in D$. So in Case 1 we have $D = B$ and thus $D \in \mathcal{S}$.

**Case 2**: $D \cap A \neq \emptyset$.

Assume that $a \in D \cap A$. For any minimum independent dominating set $D_a$ of $G_a$, by Lemma 2, $|D_a| \leq |D| - 1$, and by Lemma 3, $|D| \leq |D_a| + 1$, thus $|D| = |D_a| + 1 = |D_a \cup \{a\}|$. By Lemma 3 and the minimality of $D$ in $G$, $D_a \cup \{a\}$ is a *minimum* independent dominating set of $G$. □

**Lemma 5.** *$\mathcal{S}$ is computable in $O\left(|A|(|A|+|B|)^3\right)$ time.*

*Proof.* By Lemmm 1, for each $a \in A$, $G_a$ is convex bipartite, thus we can compute a minimum independent dominating set $D_a$ of $G_a$ by the known $O\left((|A|+|B|)^3\right)$ time algorithm in [3]. As remarked in Remark 2, for each $a$, we only need one such $D_a$ in $\mathcal{S}$. Thus, by an *enumeration* of all $|A|$ vertices $a$ in $A$, we can compute $\mathcal{S}$ in $O\left(|A|(|A|+|B|)^3\right)$ time. □

Finally, by Lemmas 4 and 5, we can find a minimum independent dominating set of $G$ in $O\left(|A|(|A|+|B|)^3\right)$ time.

This finishes the proof of Theorem 1. □

*Remark 3.* The above reduction also works for *weighted* independent domination. The only change is in replacing $|D| = |D_a| + 1$ by $w(D) = w(D_a) + w(a)$ in proof of Lemma 4. However, the known polynomial time algorithm in [3] only works for *unweighted* independent domination.

## 4 Reduction from Triad-Convex Bipartite Graphs

In this section, we show that independent domination is polynomial time solvable for triad-convex bipartite graphs, by a polynomial time reduction for this problem from triad-convex bipartite graphs to convex bipartite graphs. Due to space limitation, we omit some details in this section.

**Theorem 2.** *For triad convex bipartite graphs $G = (A, B, E)$ with a triad $T$ defined on $A$, independent domination is $O\left(|A|^3(|A|+|B|)^3\right)$ time solvable.*

*Proof.* Without loss of generality, we assume that $G$ contains *no* isolated vertex, since isolated vertices are trivially in every independent dominating set.

We assume that $A$ is divided into four parts, $A_1, A_2, A_3, \{a_0\}$, such that for $i = 1, 2, 3$, $A_i \cup \{a_0\}$ induces a path of $T$. To be specific, we assume that $A_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}\}$, where $\sum_{i=1}^{3} n_i = |A| - 1$ and $a_0 a_{i,1} a_{i,2} \cdots a_{i,n_i}$ are three paths of $T$ with a common end $a_0$.

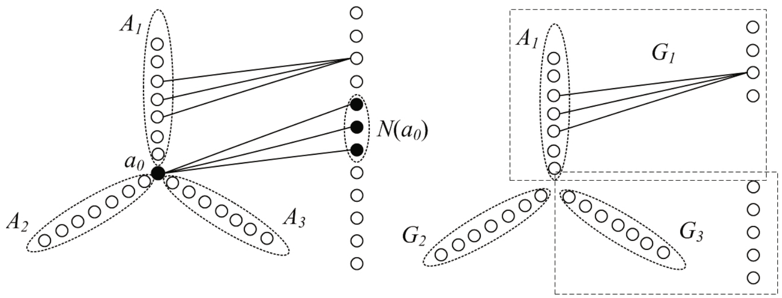


**Fig. 3.** Removing vertices in $N[a_0]$ from graph $G$ results in three graphs $G_1$,$G_2$,$G_3$

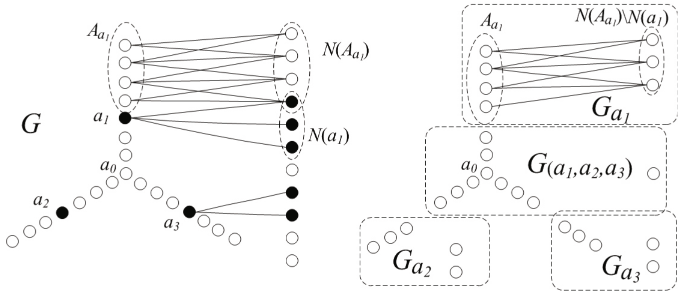


**Fig. 4.** Removing vertices in $\bigcup_{i=1}^{3}(\{a_i\} \cup N(a_i))$ from graph $G$ results in four graphs $G_{a_1}$,$G_{a_2}$,$G_{a_3}$ and $G_{(a_1,a_2,a_3)}$

First, we define three graphs $G_1, G_2, G_3$ as follows, see Figure 3.

$$\begin{aligned} G_i &= (A_i, N(A_i) \setminus N(a_0), E_i), \text{ for } i = 1,2,3, \text{ where} \\ E_i &= \{e \in E \mid e \text{ is incident to a vertex in } A_i \\ &\qquad \text{but not incident to a vertex in } N(a_0)\}, \end{aligned}$$

For $i = 1,2,3$, for each vertex $a_i = a_{i,j_i}$ in $A_i$, we define four graphs $G_{a_1}$, $G_{a_2}$, $G_{a_3}$, $G_{(a_1,a_2,a_3)}$ as follows, see Figure 4.

$$\begin{aligned} G_{a_i} &= (A_{a_i}, B_{a_i}, E_{a_i}), \text{ where} \\ A_{a_i} &= \{a_{i,j_i+1}, \ldots, a_{i,n_i}\}, B_{a_i} = N(A_{a_i}) \setminus N(a_i), \\ E_{a_i} &= \{e \in E \mid e \text{ is incident to a vertex in } B_{a_i}\}, \text{ and} \end{aligned}$$

$$\begin{aligned} G_{(a_1,a_2,a_3)} &= (A_{(a_1,a_2,a_3)}, B_{(a_1,a_2,a_3)}, E_{(a_1,a_2,a_3)}), \text{ where} \\ A_{(a_1,a_2,a_3)} &= A \setminus \bigcup_{i=1}^{3}(A_{a_i} \cup \{a_i\}), B_{(a_1,a_2,a_3)} = B \setminus \bigcup_{i=1}^{3}(B_{a_i} \cup \{N(a_i)\}), \text{ and} \\ E_{(a_1,a_2,a_3)} &= \{e \in E \mid e \text{ is incident to a vertex in } B_{(a_1,a_2,a_3)}\}. \end{aligned}$$

We define graphs $G_{(a_1,a_2,*)}$ as follows, $G_{(*,a_2,a_3)}, G_{(a_1,*,a_3)}$ are similar.

$$\begin{aligned} G_{(a_1,a_2,*)} &= (A_{(a_1,a_2,*)}, B_{(a_1,a_2,*)}, E_{(a_1,a_2,*)}), \text{ where} \\ A_{(a_1,a_2,*)} &= A \setminus \bigcup_{i\in\{1,2\}}(A_{a_i} \cup \{a_i\}), B_{(a_1,a_2,*)} = B \setminus \left(N(a_i) \bigcup_{i\in\{1,2\}} B_{a_i}\right), \text{ and} \\ E_{(a_1,a_2,*)} &= \{e \in E \mid e \text{ is incident to a vertex in } A_{(a_1,a_2,*)} \\ &\qquad \text{and a vertex in } B_{(a_1,a_2,*)}\}. \end{aligned}$$

We define graphs $G_{(a_1,*,*)}$ as follows, $G_{(*,a_2,*)}, G_{(*,*,a_3)}$ are similar.

$$G_{(a_1,*,*)} = (A_{(a_1,*,*)}, B_{(a_1,*,*)}, E_{(a_1,*,*)}), \text{ where}$$

$$A_{(a_1,*,*)} = A \setminus (A_{a_1} \cup \{a_1\}), B_{(a_1,*,*)} = B \setminus (N(a_1) \cap B_{a_1}), \text{ and}$$
$$E_{(a_1,*,*)} = \{e \in E \mid e \text{ is incident to a vertex in } A_{(a_1,*,*)} \text{ and a vertex in } B_{(a_1,*,*)}\}.$$

**Definition 5.** *For $i = 1, 2, 3$ and for each $a_i \in A_i$, a triple $(a_1, a_2, a_3)$ is called* good*, if $B_{(a_1,a_2,a_3)}$ is an independent dominating set of $G_{(a_1,a_2,a_3)}$.*

*Similarly for $(a_1, a_2, *)$, $(a_1, *, a_3)$,$(*, a_2, a_3)$,$(a_1, *, *)$,$(*, a_2, *)$,$(*, *, a_3)$.*

*Remark 4.* A triple $(a_1, a_2, a_3)$ is *good*, if and only if there is an independent dominating set $D$ of $G$, such that $D$ contains $\{a_1, a_2, a_3\}$ and $a_i$ is the first vertex of $D$ on the path $a_0 a_{i,1} \cdots a_{i,n_i}$ of the triad $T$ for $i = 1, 2, 3$. Similarly for $(a_1, a_2, *)$, $(a_1, *, a_3)$, $(*, a_2, a_3)$, $(a_1, *, *)$, $(*, a_2, *)$, $(*, *, a_3)$. A star $*$ on the $i$-th place of a triple means that no vertex in $\{a_0\} \cup A_i$ is in $D$ for $i = 1, 2, 3$.

**Lemma 6.** *For $i = 1, 2, 3$ and for each $a_i \in A_i$, $G_i$ and $G_{a_i}$ are convex bipartite.*

*Proof.* We prove by definition of convex bipartite graphs for $G_{a_i}$, the proof for $G_i$ is similar and thus omitted. After removing $N(a_i)$ and the incident edges from $G$, no vertex in $B_{a_i} = N(A_{a_i}) \setminus N(a_i)$ is adjacent to vertex $a_i$. Since $G$ is triad convex bipartite, for each vertex in $B_{a_i}$, its neighborhood is a path of $T$ on $A_{a_i} = \{a_{i,j_i+1}, \ldots, a_{i,j_n}\}$. Thus, we can define a linear ordering $\prec_i$ on $A_{a_i}$, $a_{i,j_i+1} \prec_i \cdots \prec_i a_{i,j_n}$, such that for each vertex in $B_{a_i}$, its neighborhood is an interval under this linear ordering $\prec_i$. □

**Lemma 7.** *For each triple $(a_1, a_2, a_3)$, if $D$ is an independent dominating set of $G$ containing $a_i$ for $i = 1, 2, 3$, then $D \cap (A_{a_i} \cup B_{a_i})$ is an independent dominating set of $G_{a_i}$ for $i = 1, 2, 3$. Similarly for $(a_1, a_2, *)$, $(a_1, *, a_3)$, $(*, a_2, a_3)$, $(a_1, *, *)$, $(*, a_2, *)$, $(*, *, a_3)$.*

*Proof.* We prove by definition of independent dominating sets. Since $a_i \in D$, $N(a_i) \cap D = \emptyset$. For each vertex $a' \in A_{a_i}$, either $a' \in D$ or $N(a') \cap D \neq \emptyset$. Since $a_i \notin \{a'\} \cup N(a')$, either $a' \in D \cap A_{a_i}$ or $N(a') \cap (D \cap B_{a_i}) \neq \emptyset$. For each vertex $b' \in B_{a_i}$, either $b' \in D$ or $N(b') \cap D \neq \emptyset$. Since $a_i \notin \{b'\} \cup N(b')$, either $b' \in D \cap B_{a_i}$ or $N(b') \cap (D \cap A_{a_i}) \neq \emptyset$. Similarly for $(a_1, a_2, *)$, $(a_1, *, a_3)$, $(*, a_2, a_3)$, $(a_1, *, *)$, $(*, a_2, *)$, $(*, *, a_3)$. □

**Lemma 8.** *For each good triple $(a_1, a_2, a_3)$, if $D_{a_i}$ is an independent dominating set of $G_{a_i}$ for $i = 1, 2, 3$, then $B_{(a_1,a_2,a_3)} \cup \bigcup_{i=1}^{3} (D_{a_i} \cup \{a_i\})$ is an independent dominating set of $G$. Similarly for $(a_1, a_2, *)$, $(a_1, *, a_3)$, $(*, a_2, a_3)$, $(a_1, *, *)$, $(*, a_2, *)$, $(*, *, a_3)$.*

*Proof.* We prove by definition. Since for $i = 1, 2, 3$, $G_{a_i}$ is resulted by removing $\{a_i\} \cup N(a_i)$ from $G$, $a_i$ is not adjacent to any vertex in $G_{a_i}$ and $G_{(a_1,a_2,a_3)}$, $B_{(a_1,a_2,a_3)} \cup \bigcup_{i=1}^{3} (D_{a_i} \cup \{a_i\})$ is an independent set. Since the triple $(a_1, a_2, a_3)$ is good, $B_{a_1,a_2,a_3}$ is an independent dominating set of $G_{(a_1,a_2,a_3)}$. Since $D_{a_i}$ is an independent dominating set of $G_{a_i}$ and each vertex in $N(a_i)$ is adjacent to $a_i$, $B_{(a_1,a_2,a_3)} \cup \bigcup_{i=1}^{3} (D_{a_i} \cup \{a_i\})$ is an independent dominating set of $G$. Similarly for $(a_1, a_2, *)$, $(a_1, *, a_3)$, $(*, a_2, a_3)$, $(a_1, *, *)$, $(*, a_2, *)$, $(*, *, a_3)$. □

Next, we define a set $\mathcal{S} = \{B, \{a_0\} \cup D_1 \cup D_2 \cup D_3\} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ as follows.

$$\begin{aligned}\mathcal{S}_1 = \{&B_{(a_1,*,*)} \cup D_{a_1} \cup \{a_1\}, B_{(*,a_2,*)} \cup D_{a_2} \cup \{a_2\}, B_{(*,*,a_3)} \cup D_{a_3} \cup \{a_3\} \mid \\ &a_i \in A_i \text{ and } D_{a_i} \text{ is a minimum independent dominating set in } G_{a_i} \\ &\text{for } i = 1,2,3\},\\ \mathcal{S}_2 = \{&B_{(a_1,a_2,*)} \cup \bigcup_{i\in\{1,2\}} (D_{a_i} \cup \{a_i\}), B_{(*,a_2,a_3)} \cup \bigcup_{i\in\{2,3\}} (D_{a_i} \cup \{a_i\}),\\ &B_{(a_1,*,a_3)} \cup \bigcup_{i\in\{1,3\}} (D_{a_i} \cup \{a_i\}) \mid a_i \in A_i \text{ and } D_{a_i} \text{ is a minimum}\\ &\text{independent dominating set in } G_{a_i} \text{ for } i = 1,2,3\},\\ \mathcal{S}_3 = \{&B_{(a_1,a_2,a_3)} \cup \bigcup_{i\in\{1,2,3\}} (D_{a_i} \cup \{a_i\}) \mid (a_1,a_2,a_3) \text{ is good and } D_{a_i} \text{ is a}\\ &\text{minimum independent dominating set in } G_{a_i} \text{ for } i = 1,2,3\},\end{aligned}$$

where for $i = 1,2,3$, $D_i$ is a minimum dominating set of $G_i$ and $G_i$ is resulted by removing $N[a_0]$ from $G$.

*Remark 5.* For each triple $(a_1,a_2,a_3)$, $G_{a_i}$ is unique, but for each $G_{a_i}$, $D_{a_i}$ may not be unique. For our purpose, however, for each $(a_1,a_2,a_3)$, we only need one triple $(D_{a_1}, D_{a_2}, D_{a_3})$ in $\mathcal{S}$, see proof of Lemma 10 below. Similarly for $(a_1,a_2,*)$, $(a_1,*,a_3)$, $(*,a_2,a_3)$, $(a_1,*,*)$, $(*,a_2,*)$, $(*,*,a_3)$.

**Lemma 9.** *$\mathcal{S}$ contains a minimum independent dominating set of $G$.*

*Proof.* Let $D$ be a minimum independent dominating set of $G$. We consider the following *five* cases.

**Case 1**: $D \cap A = \emptyset$. In this case we have $D = B$, which is in $\mathcal{S}$.

**Case 2**: $a_0 \in D$. In this case, similar to the reasoning process in Case 5 below, we have $|D| = |\{a_0\} \cup \bigcup_{i\in\{1,2,3\}} D_i|$, thus $\{a_0\} \cup \bigcup_{i\in\{1,2,3\}} D_i$ is a minimum independent dominating set of $G$, which is in $\mathcal{S}$.

**Case 3**: $D \cap A_i \neq \emptyset$ for some $i \in \{1,2,3\}$ but $D \cap (\{a_0\} \cup A_j) = \emptyset$ for $j \neq i$. In this case, similar to Case 5 below, a minimum dominating set of $G$ is in $\mathcal{S}_1$.

**Case 4**: $D \cap (\{a_0\} \cup A_i) = \emptyset$ for some $i \in \{1,2,3\}$ but $D \cap A_j \neq \emptyset$ for $j \neq i$. In this case, similar to Case 5 below, a minimum dominating set of $G$ is in $\mathcal{S}_2$.

**Case 5**: $D \cap A_i \neq \emptyset$ for $i = 1,2,3$. Assume that $a_i \in D \cap A_i$ for $i = 1,2,3$ and the triple $(a_1,a_2,a_3)$ is good. For any minimum independent dominating sets $D_{a_i}$ of $G_{a_i}$ for $i = 1,2,3$, by Lemma 7, $\sum_{i=1}^{3} |D_{a_i}| \le |D| - |B_{(a_1,a_2,a_3)}| - 3$, and by Lemma 8, $|D| \le \sum_{i=1}^{3} |D_{a_i}| + |B_{(a_1,a_2,a_3)}| + 3$, thus $|D| = \sum_{i=1}^{3} |D_{a_i}| + |B_{(a_1,a_2,a_3)}| + 3 = |B_{(a_1,a_2,a_3)} \cup \bigcup_{i=1}^{3}(D_{a_i} \cup \{a_i\})|$. By Lemma 8 and the minimality of $D$ in $G$, $B_{(a_1,a_2,a_3)} \cup \bigcup_{i\in\{1,2,3\}}(D_{a_i} \cup \{a_i\})$ is a *minimum* independent dominating set of $G$, which is in $\mathcal{S}_3$. □

**Lemma 10.** *$\mathcal{S}$ is computable in $O\left(|A|^3(|A|+|B|)^3\right)$ time.*

*Proof.* By Lemmm 6, for $i = 1,2,3$ and for each $a_i \in A_i$, $G_{a_i}$ is convex bipartite, thus we can compute a minimum independent dominating set $D_{a_i}$ of $G_{a_i}$ by the known $O\left((|A|+|B|)^3\right)$ time algorithm in [3]. As remarked in Remark 5, for each good triple $(a_1, a_2, a_3)$, we only need one such triple $(D_{a_1}, D_{a_2}, D_{a_3})$ in $\mathcal{S}$. Thus, by an *enumeration* of all $|A_1||A_2||A_3|$ triples $(a_1, a_2, a_3)$, we can compute $\mathcal{S}_3$ in $O\left(|A|^3(|A|+|B|)^3\right)$ time. Similarly for $\mathcal{S}_1$ and $\mathcal{S}_2$. □

Finally, by Lemmas 9 and 10, we can find a minimum independent dominating set of $G$ in $O\left(|A|^3(|A|+|B|)^3\right)$ time.

This finishes the proof of Theorem 2. □

*Remark 6.* The above reduction also works for *weighted* independent domination. The only changes are in replacing $|D| = \sum_{i=1}^{3} |D_{a_i}| + |B_{(a_1,a_2,a_3)}| + 3$ by $w(D) = \sum_{i=1}^{3}(w(D_{a_i}) + w(a_i)) + w(B_{(a_1,a_2,a_3)})$ and so on in proof of Lemma 9.

## 5 Concluding Remarks

We have shown that independent domination is polynomial time reducible from circular- and triad-convex bipartite graphs to convex bipartite graphs. As in [12], we make Cook reductions from circular convex bipartite graphs to convex bipartite graphs. Our methods may be of use to show more problems tractable for circular- and triad-convex bipartite graphs. It would be interesting to find real applications of these results.

Recently, maximum non-crossing matching for convex bipartite graphs is studied [2]. Whether the results in [2] carry over for circular- and triad-convex bipartite graphs is still unknown.

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