The Complex Fuzzy Measure

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Abstract In this paper, we define the concept of complex Fuzzy measure, which is different from the concept of complex Fuzzy measure in [2], and discuss its properties and theorems. On the basis of the concept of complex Fuzzy measurable function in [2], we study its convergence theorem. It builds the certain foundation for the research of complex Fuzzy integral.

Keywords Complex fuzzy measure · Complex fuzzy measurable function

1 Introduction

In 1990–1991, Buckley [1] proposed the concept of fuzzy complex numbers and fuzzy complex-valued function. In 1997, Qiu Jiqing [2–5] firstly proposed the concept of the complex fuzzy measure on the basis of classical measure theory method. Since 2000, According to this issue, Ma shengquan [6] has done some exploratory work, and made a series of achievement in this field. The theory of fuzzy complex valued measure is an important part of fuzzy complex analysis, which has a strong background of practical application [7]. For instance it can use in the fuzzy system identification, fuzzy control, multi-classifier system design and other fields. The development of theoretical research of fuzzy complex valued measure is slow,

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because it is much more complicated than Fuzzy real-valued measure. The complex Fuzzy measure which defined in this paper is different from that in paper [2], the concept of Fuzzy measure was redefined, which distinguished between the real and imaginary parts in order to facilitate research.

2 Complex Fuzzy Measure

$\hat{R}^+$ denote positive real set, $\hat{C}^+$ denote the set of complex number on $\hat{R}^+$ [8].

Definition 2.1 *Let X be a nonempty set, F be a $\sigma-$ algebra, comprising of the subset of X, the mapping μ:$F \to \hat{\mathrm{C}}^+$ is set function, satisfying:*

(1) $\mu(\emptyset) = 0$;
(2) (monotonicity) If $A, B \in F$ and $A \subseteq B$,then $Re(\mu(A)) \leq Re(\mu(B))$ and $Im(\mu(A)) \leq Im(\mu(B))$. Denote $\mu(A) \leq \mu(B)$
(3) if $A_n \in F(n = 1, 2, \cdots)$, $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

(4) if $A_n \in F(n = 1, 2, \ldots)$, $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, and $\exists n_0$ such that $Re(\mu(A_{n_0})) < \infty, Im(\mu(A_{n_0})) < \infty$ then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. Then μ is called as complex Fuzzy measure on F, (X,F,μ) is called as complex Fuzzy measure space.

Definition 2.2 *Fuzzy complex measure μ is said to be zero-additive, if for arbitrary $E, F \in F$, $\mu(F) = 0$ and $E \cap F = \varphi$, then $\mu(E \cup F) = \mu(E)$.*

Theorem 2.1 (X, F, μ) *is complex Fuzzy measure space, The following propositions are equivalence.*

(1) μ is zero-additive ;
(2) since $\mu(F) = 0$, then for arbitrary $E, F \in F$, such that $\mu(E \cup F) = \mu(E)$;
(3) since $\mu(F) = 0$, then for arbitrary $E, F \in F$, such that $\mu(E \backslash F) = \mu(E)$:

Proof. (1)⇒(2):
$E \cup F = E \cup (F\backslash E)$, since μ is nonnegative monotony, then for $\mu(F) = 0$, such that
$\mu(F\backslash E) \leq \mu(F) = 0$, therefore $\mu(F\backslash E) = 0$. Applying the zero-additive of μ, $\mu(F\backslash E) = 0$ and $E \cap (F\backslash E) = \varphi$, then

$$\mu(E \cup F) = \mu(E \cup (F\backslash E)) = \mu(E).$$

(2)⇒(3):

Due to $E = (E \backslash F) \cup (E \cap F)$, and $\mu(F) = 0$, then $\mu(E \cap F) = 0$.
We know $\mu(E) = \mu((E \backslash F) \cup (E \cap F)) = \mu(E \backslash F)$ from the proposition (2).(3)⇒(1):
Due to $E \cap F = \varphi$, then $E = (E \cup F) \backslash F$.
If $\mu(F) = 0$ and $E \cap F = \varphi$, we can know

$$\mu(E) = \mu((E \cup F) \backslash F) = \mu(E \cup F)$$

from the proposition (3).

Theorem 2.2 *Suppose* μ *is complex Fuzzy measure of zero-additive,* $A \in F$*, there is a descending sequence of* $\{B_n\} \subset F$*,* $(B_1 \supseteq B_2 \supseteq \cdots)$*, if* $\mu(B_n) \to 0$*, then*

(1) $\mu(A \backslash B_n) \to \mu(A)$*;*
(2) whereupon $Re(\mu(A)) < \infty$*, and* $Im(\mu(A)) < \infty$ *, and if exists* $Re(\mu(A \cup B_{n_0})) < \infty$ *and* $Im(\mu(A \cup B_{n_0})) < \infty$ *, therefore* $\mu(A \cup B_n) \to \mu(A)$*.*

Proof. (1) since $\{A \backslash B_n\}$ is ascending sequence and if μ is lower-continuous, we can know $\lim_{n\to\infty} \mu(A \backslash B_n) = \mu(\bigcup_{n=1}^{\infty} (A \backslash B_n)) = \mu(A \backslash \bigcap_{n=1}^{\infty} B_n)$.
Applying μ is upper-continuous, $\lim_{n\to\infty} \mu(B_n) = \mu(\bigcap_{n=1}^{\infty} B_n) = 0$, since μ is zero-additive, from the Theorem 1, then $\lim_{n\to\infty} \mu(A \backslash B_n) = \mu(A)$.
(2) $\{A \cup B_n\}$ is descending sequence, $Re(\mu(A)) < \infty$, and $Im(\mu(A)) < \infty$, and exist $Re(\mu(A \cup B_{n_0})) < \infty$ and $Im(\mu(A \cup B_{n_0})) < \infty$, due to μ is upper-continuous,

$$\lim_{n\to\infty} \mu(A \cup B_n) = \mu(\bigcap_{n=1}^{\infty} (A \cup B_n)) = \mu(A \cup (\bigcap_{n=1}^{\infty} B_n)).$$

Since μ is zero-additive, and $\mu(\bigcap_{n=1}^{\infty} B_n) = 0$, so we know

$$\lim_{n\to\infty} \mu(A \cup B_n) = \mu(A)$$

according to Theorem 1.

Definition 2.3 *Let (X,F) be measurable space, the mapping* $\mu{:}F \to \hat{\mathrm{C}}^{+}$ *is set function, Complex fuzzy measure* μ *is upper-self-continuous, if for arbitrary* $A, B_n \in F$*, and* $A \cap B_n = \Phi$*,* $\lim_{n\to\infty} B_n = 0$*, then*

$$\lim_{n\to\infty} \mu(A \backslash B_n) = \mu(A).$$

Complex fuzzy measure μ *is lower-self-continuous, if for arbitrary* $A, B_n \in F$ *and* $B_n \subseteq A$*,* $\lim_{n\to\infty} B_n = 0$*, then* $\lim_{n\to\infty} \mu(A \backslash B_n) = \mu(A)$*.*

Complex fuzzy measure μ is self-continuous, if and only if μ is not only upper-self-continuous but also lower-self-continuous.

Theorem 2.3 *Suppose μ is complex Fuzzy measure on (X,F), then*

(1) μ is upper-self-continuous if and only if $\lim_{n\to\infty} \mu(A\cup B_n) = \mu(A)$ if for arbitrary $A, B_n \in F$ and $\lim_{n\to\infty} B_n = 0$.
(2) μ is lower-self-continuous if and only if $\lim_{n\to\infty} \mu(A\backslash B_n) = \mu(A)$ if for arbitrary $A, B_n \in F$ and $\lim_{n\to\infty} B_n = 0$.

Proof. The necessity is easily proved from the definition.2.3

(1) Sufficiency: Let $E_n = B_n\backslash A$, we know $\mu(E_n) \le \mu(B_n)$, therefore $E_n \cap A = \varphi$, and $\lim_{n\to\infty} \mu(E_n) = 0$, then $\lim_{n\to\infty} \mu(A \cup E_n) = \mu(A)$, so μ is upper-self-continuous.
(2) Sufficiency: Let $E_n = B_n \cap A$, we know $\mu(E_n) \le \mu(B_n)$, therefore $E_n \subseteq A$, and $\lim_{n\to\infty} \mu(E_n) = 0$, then $\lim_{n\to\infty} \mu(A\backslash E_n) = \mu(A)$, so μ is lower-self-continuous

Definition 2.4 *Let (X, F) be measurable space, the mapping μ:F $\to$ $\hat{C}^+$ is set function,*

(1) Suppose for arbitrary $\varepsilon_i > 0$, exists $\delta_i = \delta(\varepsilon_i) > 0 (i = 1, 2)$, where $\varepsilon = \varepsilon_1 + i\varepsilon_2, \delta = \delta_1 + i\delta_2$, Complex fuzzy measure μ is uniform-upper-self-continuous, if for arbitrary $A, B \in F$ and $\mu(B) \le \delta$, then $\mu(A \cup B) \le \mu(A) + \varepsilon$.
(2) Suppose for arbitrary $\varepsilon_i > 0$, exists $\delta_i = \delta(\varepsilon_i) > 0 (i = 1, 2)$, where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\delta = \delta_1 + i\delta_2$, Complex fuzzy measure μ is uniform-lower-self- continuous, if for arbitrary $A, B \in F$ and $\mu(B) \le \delta$, then $\mu(A) - \varepsilon \le \mu(A\backslash B)$.
(3) Complex fuzzy measure μ is uniform-self-continuous, if and only if μ is not only uniform-upper-self-continuous but also uniform-lower-self-continuous.

Theorem 2.4 *Suppose set function μ is uniform-upper-self-continuous (uniform-lower-self-continuous), then μ is upper-self-continuous (lower-self- continuous).*

Proof. It is obvious.

Theorem 2.5 *Suppose μ is complex Fuzzy measure on (X,F), The following propositions are equivalence.*

(1) μ is uniform-self-continuous;
(2) μ is uniform-upper-self-continuous;
(3) μ is uniform-lower-self-continuous.

Proof. (1) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (3): Since μ is uniform-upper-self-continuous, so if for arbitrary $\varepsilon_i > 0, \exists\delta_i = \delta(\varepsilon_i) > 0(i = 1, 2)$, where $\varepsilon = \varepsilon_1 + i\varepsilon_2, \delta = \delta_1 + i\delta_2$, and for arbitrary $A', B' \in$ F, $\mu(B') \le \delta$,then $\mu(A') - \varepsilon \le \mu(A' \cup B') \le \mu(A') + \varepsilon$. if for arbitrary $A, B \in$ F, $\mu(B) \le \delta$, Let$A' = A\backslash B$, $B' = A \cap B$, $\mu(B') \le \mu(B) \le \delta$,

if $\mu(A\backslash B)-\varepsilon \le \mu(A'\cup B') \le \mu(A\backslash B)+\varepsilon$, then $\mu(A)-\varepsilon \le \mu(A\backslash B) \le \mu(A)+\varepsilon$. It means μ is uniform-lower-self-continuous.
(3) $\Rightarrow$ (1): Since μ is uniform-lower-self-continuous, if for arbitrary $\varepsilon_i > 0, \exists\delta_i = \delta(\varepsilon_i) > 0 (i = 1, 2)$, where $\varepsilon = \varepsilon_1 + i\varepsilon_2, \delta = \delta_1 + i\delta_2$, and for arbitrary $A', B' \in$ F, $\mu(B') \le \delta$, then $\mu(A') - \varepsilon \le \mu(A'\backslash B') \le \mu(A') + \varepsilon$. if for arbitrary
$A, B \in$ F, $\mu(B) \le \delta$, Let$A' = A \cup B, B' = A \cap B, \mu(B') \le \mu(B) \le \delta$, therefore

$$\mu(A'\backslash B\prime) \ge \mu(A') - \varepsilon \ge \mu(A) - \varepsilon.$$

Again let $A'' = (A \cup B)\backslash(A \cap B), B'' = B\backslash A$, then $A''\backslash B'' = A\backslash B$, and $\mu(B'') \le \mu(B) \le \delta$, so

$$\mu(A'') - \varepsilon \le \mu(A''\backslash B'') = \mu(A\backslash B) \Rightarrow \mu(A\backslash B) + \varepsilon \le \mu(A) + \varepsilon.$$

Again let $A = E, B = F\backslash E$, then $(A \cup B)\backslash(A \cap B) = E \cup F, \mu(B) \le \mu(F) \le \delta$, so $\mu(E)-\varepsilon \le \mu(E\cup F) \le \mu(E)+\varepsilon$. It means μ is uniform-upper-self-continuous. So μ is uniform-self-continuous.

Definition 2.5 *Let (X,F) be measurable space, the mapping μ:F $\to \hat{\text{C}}^+$ is set function, If for arbitrary $\{B_n\} \subseteq A, B_1 \supseteq B_2 \supseteq \cdots$, if $\exists n_0, \forall n > n_0, Re(\mu(B_n)) < \infty$, $Im(\mu(B_n)) < \infty$ and $\bigcap_{n=1}^{\infty} B_n = \varphi$, there must be $\lim_{n\to\infty} \mu(B_n) = 0$, so μ is called zero-upper-continuous.*

Theorem 2.6 *μ is nonnegative monotonic ascending set function, and zero-upper-continuous, Then if μ is upper-self- continuous, then μ is upper- continuous; if μ is limit and lower-self- continuous, then μ is lower- continuous.*

Proof. (1) If $\{A_n\} \subseteq$ F, $A_1 \supseteq A_2 \supseteq \cdots$, exist $Re(\mu(B_n)) < \infty, Im(\mu(B_n)) < \infty$, let
$A = \bigcap_{n=1}^{\infty} A_n, B_n = A_n\backslash A \quad (n = 1, 2, \cdots)$,then $B_1 \supseteq B_2 \supseteq \cdots$, and $\bigcap_{n=1}^{\infty} B_n = \varphi$.if for
arbitrary

$$n > n_0, Re(\mu(B_n)) \le Re(\mu(B_{n_0})) \le Re(\mu(A_{n_0})) < \infty,$$

$\text{Im}(\mu(B_n)) \le \text{Im}(\mu(B_{n_0})) \le \text{Im}(\mu(A_{n_0})) < \infty$. Since μ is zero-upper-continuous, we know $\lim_{n\to\infty} \mu(B_n) = 0$. Due to $A_n = A \cup B_n, A \cap B_n = \varphi$, and μ is upper-self-continuous, if $\mu(A_n) = \mu(A \cup B_n)$, then $\mu(A) = \mu(\bigcap_{n=1}^{\infty} A_n)$.So μ is upper-continuous.
(2)The proof is similar to (1) above.

3 Complex Fuzzy Measurable Function

R denote real set, C denote the set of complex number on R.

Definition 3.1 *[2] Suppose* (X,F,μ) *is complex Fuzzy measure space, the mapping* $\tilde{f} : X \to C$ *is called complex Fuzzy measurable function, if for arbitrary* $a+bi \in C$, *then* $\{x \in X \,\big|\, Re[\tilde{f}(x)] \geq a, Im[\tilde{f}(x)] \geq b\} \in F$.

Definition 3.2 *Suppose* (X,F,μ) *is complex Fuzzy measure space,* $\tilde{f}_n(n = 1, 2, \cdots)$, $\tilde{f}$ *is complex fuzzy measurable function, for arbitrary* $A \in F$,

(1) $\{\tilde{f}_n\}$ *almost everywhere converge to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{a.e.}{\to} \tilde{f}$, *if there exists* $B \in F$, *such that* $\mu(B) = 0$, *then* $\{\tilde{f}_n\}$ *with pointwise convergence to* $\tilde{f}$ *on* $A\backslash B$.
(2) $\{\tilde{f}_n\}$ *pseudo-almost everywhere converge to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{p.a.e.}{\to} \tilde{f}$, *if there exists* $B \in F$, *such that* $\mu(A\backslash B) = \mu(A)$, *then* $\{\tilde{f}_n\}$ *with pointwise convergence to* $\tilde{f}$ *on* $A\backslash B$.
(3) $\{\tilde{f}_n\}$ *almost everywhere uniformly converge to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{a.e.u.}{\to} \tilde{f}$, *if there exists* $B \in F$, *such that* $\mu(B) = 0$, *then* $\{\tilde{f}_n\}$ *with pointwise uniform convergence to* $\tilde{f}$ *on* $A\backslash B$.
(4) $\{\tilde{f}_n\}$ *pseudo-almost everywhere uniformly converge to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{p.a.e.u.}{\to} \tilde{f}$, *if there exists* $B \in F$, *such that* $\mu(A\backslash B) = \mu(A)$, *then* $\{\tilde{f}_n\}$ *with pointwise uniform convergence to* $\tilde{f}$ *on* $A\backslash B$.
(5) $\{\tilde{f}_n\}$ *almost uniformly converge to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{a.u.}{\to} \tilde{f}$, *if there exists set sequence*$\{E_k\}$ *on F, such that* $\tilde{\mu}(E_k) \to 0$ *,and for arbitrary k, then* $\{\tilde{f}_n\}$ *with pointwise uniform convergence to* $\tilde{f}$ *on* $A\backslash E_k$.
(6) $\{\tilde{f}_n\}$ *pseudo-almost uniformly converge to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{p.a.u.}{\to} \tilde{f}$, *if there exists set sequence* $\{E_k\}$ *on F, such that* $\mu(A\backslash E_k) \to \mu(A)$, *and for arbitrary k, then* $\{\tilde{f}_n\}$ *with pointwise uniform convergence to* $\tilde{f}$ *on* $A\backslash E_k$.
(7) $\{\tilde{f}_n\}$ *converge in complex fuzzy measure* μ *to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{u.}{\to} \tilde{f}$, *if for arbitrary* $\varepsilon = \varepsilon_1 + i\varepsilon_2, \varepsilon_1, \varepsilon_2 > 0$, *such that*

$$\lim_{n\to\infty} \mu(\{x \,\Big|\, Re\left|\tilde{f}_n - \tilde{f}\right| \geq \varepsilon_1, Im\left|\tilde{f}_n - \tilde{f}\right| \geq \varepsilon_2\} \cap A) = 0.$$

(8) $\{\tilde{f}_n\}$ *converge in pseudo complex fuzzy measure* $\tilde{\mu}$ *to* $\tilde{f}$ *on A, denote* $\tilde{f}_n \overset{p.u.}{\to} \tilde{f}$, *if for arbitrary* $\varepsilon > 0$, *such that*

$$\lim_{n\to\infty} \mu(\{x \,\Big|\, Re\left|\tilde{f}_n - \tilde{f}\right| < \varepsilon_1, Im\left|\tilde{f}_n - \tilde{f}\right| < \varepsilon_2\} \cap A) = \mu(A).$$

Theorem 3.1 *Suppose* (X,F,μ) *is complex Fuzzy measure space,* $\tilde{f}_n(n = 1, 2, \cdots)$, $\tilde{f}$ *is complex fuzzy measurable function, for arbitrary* $A \in F$,

(1) $\tilde{f}_n \overset{a.e.}{\to} \tilde{f}$ *if and only if* $\{\tilde{f}_n\}$ *converge to* $\tilde{f}$ *on* $A\backslash E_k$, *if there exists set sequence* $\{E_k\}$ *on F, such that* $\mu(E_k) \to 0$, *and for arbitrary k.*

(2) $\tilde{f}_n \overset{p.a.e.}{\to} \tilde{f}$ *if and only if* $\{\tilde{f}_n\}$ *converge to* $\tilde{f}$ *on* $A \backslash E_k$, *if there exists set sequence* $\{E_k\}$ *on F, such that* $\mu(A \backslash E_k) \to \mu(A)$, *and for arbitrary k.*

(3) $\tilde{f}_n \overset{a.e.u.}{\to} \tilde{f}$ *if and only if* $\left|Re(\tilde{f} - \tilde{f}_k)\right| < \varepsilon_1$ *and* $\left|Im(\tilde{f} - \tilde{f}_k)\right| < \varepsilon_2$, *if there exists set sequence* $\{E_k\}$ *on F, such that* $\mu(E_k) \to 0$, *and for arbitrary* $\varepsilon_i > 0, (i = 1, 2)$, *where* $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\exists n_0, \forall n > n_0, \forall k, \forall x \in A \backslash E_k$.

(4) $\tilde{f}_n \overset{p.a.e.u.}{\to} \tilde{f}$ *if and only if* $\left|Re(\tilde{f} - \tilde{f}_k)\right| < \varepsilon_1$ *and* $\left|Im(\tilde{f} - \tilde{f}_k)\right| < \varepsilon_2$, *if there exists set sequence* $\{E_k\}$ *on F, such that* $\mu(A \backslash E_k) \to \mu(A)$, *and for arbitrary* $\varepsilon_i > 0, (i = 1, 2)$, *where* $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\exists n_0, \forall n > n_0, \forall k, \forall x \in A \backslash E_k$.

Proof. (1) If $\tilde{f}_n \overset{a.e.}{\to} \tilde{f}$, then exists $B \in$F, $\mu(B) = 0$, such that $\tilde{f}_n \to \tilde{f}$ on $A \backslash B$. Let $E_k = B (k = 1, 2, \cdots)$, we know $\mu(E_k) \to 0$, and $\{\tilde{f}_n\}$ converge to $\tilde{f}$ on $A \backslash E_k$, for arbitrary k.
Otherwise, if there exists $\{E_k\} \subseteq$F,$\mu(E_k) \to 0$, such that $\tilde{f}_n \to \tilde{f}$ on $A \backslash E_k$. Let $B_k = \bigcap_{i=1}^{k} E_i$, $B = \bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} E_k$, so $\mu(B_k) \leq \mu(E_k)$ and $B_1 \supseteq B_2 \supseteq \cdots$. Due to $\mu(E_k) \to 0$, there exists

$$\mathrm{Re}(\mu(B_{k_0})) \leq \mathrm{Re}(\mu(E_{k_0})) < \infty, \text{and } \mathrm{Im}(\mu(B_{k_0})) \leq \mathrm{Im}(\mu(E_{k_0})) < \infty.$$

Applying the upper-continuity of μ, we know that $\mu(B) = \lim_{k \to 0} \mu(B_k) = 0$. If for arbitrary $x \in A \backslash B = \bigcup_{k=1}^{\infty} (A \backslash E_k)$, $\exists k_0, x \in A \backslash E_{k_0}$, then $\tilde{f}_n \to \tilde{f}$, therefore $\{\tilde{f}_n\}$ converge to $\tilde{f}$ on $A \backslash B$, denote $\tilde{f}_n \overset{a.e.}{\to} \tilde{f}$.

The proof of (2),(3),(4) is similar to (1), we omit here.
Inference 3.1 If $\tilde{f}_n \overset{a.e.u.}{\to} \tilde{f}$, then $\tilde{f}_n \overset{a.u.}{\to} \tilde{f}$;If $\tilde{f}_n \overset{p.a.e.u.}{\to} \tilde{f}$, then $\tilde{f}_n \overset{p.a.u.}{\to} \tilde{f}$.

Theorem 3.2 *Suppose complex Fuzzy measure* μ *is lower-self- continuity, for* $A \in$*F,*
If $\tilde{f}_n \overset{a.e.}{\to} \tilde{f}$ *, then* $\tilde{f}_n \overset{p.a.e.}{\to} \tilde{f}$.
If $\tilde{f}_n \overset{a.e.u.}{\to} \tilde{f}$ *, then* $\tilde{f}_n \overset{p.a.e.u.}{\to} \tilde{f}$.
If $\tilde{f}_n \overset{a.u.}{\to} \tilde{f}$ *then* $\tilde{f}_n \overset{p.a.u.}{\to} \tilde{f}$.

Proof. (1) If $\tilde{f}_n \overset{a.e.}{\to} \tilde{f}$ on A, then there exists $B \in$ F, $\mu(B) = 0$, such that $\{\tilde{f}_n\}$ converge to $\tilde{f}$ on $A \backslash B$, therefore $\tilde{f}_n \overset{p.a.e.}{\to} \tilde{f}$ on A.

The proof of (2),(3) is similar to (1), we omit here.

Theorem 3.3 *Suppose for arbitrary* $A \in$*F, and complex fuzzy measurable function* $\tilde{f}$ *and* $\tilde{f}_n (n = 1, 2, \cdots)$, *if* $\tilde{f}_n \overset{u.}{\to} \tilde{f}$, *then* $\tilde{f}_n \overset{p.u.}{\to} \tilde{f}$ *if and only if complex fuzzy measure* μ *is lower-self- continuity.*

Proof. (necessity) if $\tilde{f}_n \overset{u.}{\to} \tilde{f}$, then $\tilde{f}_n \overset{p.u.}{\to} \tilde{f}$, so $\lim_{n \to \infty} \mu(B_n) = 0$, for arbitrary $A \in$ F and $\{B_n\} \subseteq$ F
Let

$$\tilde{f}_n(x) = \begin{cases} 1 & x \in B_n \\ 0 & x \notin B_n \end{cases}$$

So

$$\lim_{n\to\infty} \mu(\{x \left|\left|\mathrm{Re}(\tilde{f}_n - 0)\right| \geq \varepsilon_1, \left|\mathrm{Im}(\tilde{f}_n - 0)\right| \geq \varepsilon_2\} \cap A) = \lim_{n\to\infty} \mu(B_n) = 0,\right.$$

where for arbitrary $\varepsilon_i > 0, (i = 1, 2), \varepsilon = \varepsilon_1 + i\varepsilon_2$. So on A, if $\tilde{f}_n \xrightarrow{u.} 0$, then $\tilde{f}_n \xrightarrow{p.u.} 0$.

Suppose for $\varepsilon_i < 1$, then $\mu(A \backslash B_n) = \mu(\{x \left|\left|\mathrm{Re}(\tilde{f}_n - 0)\right| < \varepsilon_1, \left|\mathrm{Im}(\tilde{f}_n - 0)\right| < \varepsilon_2\} \cap A) = \mu(A)\right.$,

So μ is lower-self- continuity.

(Sufficiency): If $\tilde{f}_n \xrightarrow{u.} \tilde{f}$ on A, then $\lim_{n\to\infty} \mu(\{x \left|\left|Re(\tilde{f}_n - 0)\right| \geq \varepsilon_1, \left|Im(\tilde{f}_n - 0)\right| \geq \varepsilon_2\} \cap A) = 0\right.$, where for arbitrary $\varepsilon_i > 0, (i = 1, 2), \varepsilon = \varepsilon_1 + i\varepsilon_2$. Let

$$B_n = \{x \left|\left|Re(\tilde{f}_n - \tilde{f})\right| \geq \varepsilon_1, \left|Im(\tilde{f}_n - \tilde{f})\right| \geq \varepsilon_2\} \cap A,\right.$$

then $\{B_n\} \subseteq$ A and $\lim_{n\to\infty} \mu(B_n) = 0$. Due to μ is lower-self- continuity, so

$$\mu(A \cap \{x \left|\left|Re(\tilde{f}_n - \tilde{f})\right| < \varepsilon_1, \left|Im(\tilde{f}_n - \tilde{f})\right| < \varepsilon_2\}) = \mu(A \backslash B_n) \to \mu(A).\right.$$

Therefore $\tilde{f}_n \xrightarrow{p.u.} \tilde{f}$ on A.

4 Conclusion

On the basis of the concept of complex Fuzzy measurable function in [2], we study its convergence theorem. It builds the certain foundation for the research of complex Fuzzy integral. Provide a strong guarantee for the complex fuzzy integral development, enrichment and development of complex fuzzy Discipline.

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