Characterizations of α-Quasi Uniformity and Theory of α-P.Q. Metrics

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Abstract In Wu Fuzzy Systems and Mathematics 3:94–99, 2012, the author introduced concepts of α-remote neighborhood mapping and α-quasi uniform, and obtained many good results in α-quasi uniform spaces. This chapter will further investigate properties of α-remote neighborhood mapping, and give some characterizations of α-quasi uniforms. Based on this, this chapter also introduces concept of α-P.Q. metric, and establishes the relations between α-quasi uniforms and α-P.Q. metrics.

Keywords α-Quasi uniform · α-Homeomorphism · α-P.Q. metric · α-Remote neighborhood mapping

1 Introduction

Theory of quasi-uniformity in completely distributive lattices was firstly introduced by Erceg [1] and Hutton [2]. Then it was developed into various forms and was extended into different topological spaces [3–9]. In [10], the author introduced the concept of α-quasi uniform in α-layer order-preserving operator spaces, and revealed the relations between α-layer topological spaces and α-quasi uniform spaces. In this chapter, firstly, we further study properties of α-remote neighborhood mappings.

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Then we discuss some characterizations of α-quasi uniformities. Secondly, we introduce the concept of α-P.Q. metrics, and establish the relations between α-quasi uniforms and α-P.Q. metrics.

2 Preliminaries

In this chapter, X, Y will always denote nonempty crisp sets, A mapping $A : X \to L$ is called an L-fuzzy set. L^X is the set of all L-fuzzy sets on X. An element $e \in L$ is called an irreducible element in L, if $p \vee q = e$ implies $p = e$ or $q = e$, where $p, q \in L$. The set of all nonzero irreducible elements in L will be denoted by $M(L)$. If $x \in X, \alpha \in M(L)$, then x_α is called a molecule in L^X. The set of all molecules in L^X is denoted by $M^*(L^X)$. If $A \in L^X, \alpha \in M(L)$, take $A_{[\alpha]} = \{x \in X \mid A(x) \geq \alpha\}$ [3] and $A^\alpha = \vee\{x_\alpha \mid x_\alpha \not\leq A\}$ [11]. It is easy to check $(A_{[\alpha]})' = A^\alpha_{[\alpha]}$.

Let (L^X, δ) be an L-fuzzy topological space, $\alpha \in M(L)$. $\forall A \in L^X$, $D_\alpha(A) = \wedge\{G \in \delta' \mid G_{[\alpha]} \supset A_{[\alpha]}\}$. Then the operator D_α is a α-closure operator of some co-topology on L^X, denoted by $D_\alpha(\delta)$. We called α-layer topology. The pair $(L^X, D_\alpha(\delta))$ is called α-layer co-topological space [11]. An α-layer topological space$(L^X, D_\alpha(\delta))$ is called an α-C_{II} space, if there is a countable base $\mathscr{B}_\alpha$ of $D_\alpha(\delta)$.

A mapping $F_\alpha : L^X \to L^Y$ is called an α-mapping, if $F_\alpha(A)_{[\alpha]} = F_\alpha(B)_{[\alpha]}$ whenever $A_{[\alpha]} = B_{[\alpha]}$, and $F_\alpha(A) = 0_X$ whenever $A_{[\alpha]} = \emptyset$. The mapping $F_\alpha^{-1} : L^Y \to L^X$ is called the reverse mapping of F_α, if for each $B \in L^Y$, $F_\alpha^{-1}(B) = \vee\{A \in L^X \mid F_\alpha(A)_{[\alpha]} \subset B_{[\alpha]}\}$. Clearly, F_α^{-1} is also an α-mapping.

An α-mapping $F_\alpha : L^X \to L^Y$ is called an α-order-preserving homomorphism (briefly α-*oph*), iff both F_α and F_α^{-1} are α-union preserving mappings.

An α-mapping $F_\alpha : L^X \to L^Y$ is called an α-Symmetric mapping, if for every $A, B \in L^X$, we have

$$\exists C_{[\alpha]} \not\subset A^\alpha_{[\alpha]}, B_{[\alpha]} \not\subset F_\alpha(C)_{[\alpha]} \Leftrightarrow \exists D_{[\alpha]} \not\subset B^\alpha_{[\alpha]}, A_{[\alpha]} \not\subset F_\alpha(D)_{[\alpha]}.$$

An α-mapping $f_\alpha : L^X \to L^X$ is called an α-remote neighborhood mapping, if for each $A \in L^X$ with $A_{[\alpha]} \neq \emptyset$, we have $A_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}$. The set of all α-remote neighborhood mappings is denoted by $\mathscr{F}_\alpha(L^X)$, (briefly by $\mathscr{F}_\alpha$).

For $f_\alpha, g_\alpha \in \mathscr{F}_\alpha$, let's define

(1) $f_\alpha \leq g_\alpha \Leftrightarrow \forall A \in L^X, f_\alpha(A)_{[\alpha]} \subset g_\alpha(A)_{[\alpha]}$.
(2) $(f_\alpha \vee g_\alpha)(A) = f_\alpha(A) \vee g_\alpha(A)$.
(3) $(f_\alpha \odot g_\alpha)(A) = \wedge\{f_\alpha(B) \mid \exists B \in L^X, B_{[\alpha]} \not\subset g_\alpha(A)_{[\alpha]}\}$.

An non-empty subfamily $\mathscr{D}_\alpha \subset \mathscr{F}_\alpha$ is called an α-quasi-uniform, if $\mathscr{D}_\alpha$ satisfies:
(α-U1) $\forall f_\alpha \in \mathscr{D}_\alpha, g_\alpha \in \mathscr{F}_\alpha$ with $f_\alpha \leq g_\alpha$, then $g_\alpha \in \mathscr{D}_\alpha$.
(α-U2) $\forall f_\alpha, g_\alpha \in \mathscr{D}_\alpha$ implies $f_\alpha \vee g_\alpha \in \mathscr{D}_\alpha$.
(α-U3) $\forall f_\alpha \in \mathscr{D}_\alpha$, then $\exists g_\alpha \in \mathscr{D}_\alpha$, such that $g_\alpha \odot g_\alpha \geq f_\alpha$.
$(L^X, \mathscr{D}_\alpha)$ is called an α-quasi-uniform space. A subset $\mathscr{B}_\alpha \subset \mathscr{D}_\alpha$ is called a base of $\mathscr{D}_\alpha$, if $\forall f_\alpha \in \mathscr{D}_\alpha$, there is $g_\alpha \in \mathscr{B}_\alpha$, such that $f_\alpha \leq g_\alpha$. A subset $\mathscr{A}_\alpha \subset \mathscr{D}_\alpha$ is

called a subbase of $\mathscr{D}_\alpha$, if all of finite unions of the elements in $\mathscr{A}_\alpha$ consist a base of $\mathscr{D}_\alpha$. An α-quasi-uniform $\mathscr{D}_\alpha$ is called an α-uniform, if $\mathscr{D}_\alpha$ possesses a base whose elements are α-symmetric. Usually, we call this base α-symmetric base.

In [10], the author discussed the relation between an α-quasi uniform space and an α-layer co-topological space as following:

Let $\mathscr{D}_\alpha$ be an α-quasi uniform on L^X. $\forall A \in L^X$, Let's take $c_\alpha(A) = \vee\{B \in L^X \mid \forall f_\alpha \in \mathscr{D}_\alpha, A_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]}\}$. Then c_α is an α-closure operator of some L-fuzzy co-topology, which is denoted by $\eta_\alpha(\mathscr{D}_\alpha)$. Each α-layer topological space $(L^X, D_\alpha(\delta))$ can be α-quasi uniformitale, i.e., there is an α-quasi uniform $\mathscr{D}_\alpha$, such that $D_\alpha(\delta) = \eta_\alpha(\mathscr{D}_\alpha)$.

Other definitions and notes not mentioned here can be seen in [12].

3 Properties of α-Remote Neighborhood Mappings

Theorem 3.1 *Let $f_\alpha, g_\alpha \in \mathscr{F}_\alpha$. Then*

(1) $f_\alpha \vee g_\alpha \in \mathscr{F}_\alpha$, $f_\alpha \odot g_\alpha \in \mathscr{F}_\alpha$.
(2) $f_\alpha \odot g_\alpha \le f_\alpha$, $f_\alpha \odot g_\alpha \le g_\alpha$.
(3) $(f_\alpha \odot g_\alpha) \vee h_\alpha = (f_\alpha \vee h_\alpha) \odot (g_\alpha \vee h_\alpha)$,

$(f_\alpha \vee g_\alpha) \odot h_\alpha = (f_\alpha \odot h_\alpha) \vee (g_\alpha \odot h_\alpha)$.

Theorem 3.2 *Let $f_\alpha \in \mathscr{F}_\alpha$. If for each $A \in L^X$,*

$$f_\alpha^{\nabla}(A) = \wedge\{f_\alpha(B) \mid A_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset B^\alpha_{[\alpha]}} f_\alpha(G)^\alpha_{[\alpha]}\},$$

and

$$f_\alpha^{\diamond}(A) = \wedge\{f_\alpha(B) \mid \bigcup_{G_{[\alpha]} \not\subset B^\alpha_{[\alpha]}} f_\alpha(G)^\alpha_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}\}.$$

Then

(1) $f_\alpha^{\nabla}, f_\alpha^{\diamond} \in \mathscr{F}_\alpha$.
(2) $f_\alpha^{\diamond} \le f_\alpha^{\nabla} \le f_\alpha$.
(3) $f_\alpha^{\diamond}(A) = \wedge\{f_\alpha^{\nabla}(B) \mid B_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}\} = (f_\alpha^{\nabla} \odot f_\alpha)(A)$.
(4) f_α^{∇} *is α-Symmetric.*

Proof (1) By $A_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset A^\alpha_{[\alpha]}} f_\alpha(G)^\alpha_{[\alpha]}$, we have $A_{[\alpha]} \not\subset f_\alpha^{\nabla}(A)_{[\alpha]}$. Thus $f_\alpha^{\nabla} \in \mathscr{F}_\alpha$. Again by $A_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}$, we get $\bigcup_{G_{[\alpha]} \not\subset A^\alpha_{[\alpha]}} f_\alpha(G)^\alpha_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}$. Therefore, $f_\alpha^{\diamond} \in \mathscr{F}_\alpha$.

(2) The proof is obvious.

(3) For each $B \in L^X$,

$$\bigcup_{G_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]} \Leftrightarrow \exists D_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset A^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]}, D_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]}$$
$$\Leftrightarrow \exists D_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]}, D_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]}.$$

So

$$f^{\diamond}_\alpha(A) = \wedge\{f^{\nabla}_\alpha(D) \mid D_{[\alpha]} \not\subset f_\alpha(D)_{[\alpha]}\} = (f^{\nabla}_\alpha \odot f_\alpha)(A).$$

(4) $\forall D, E \in L^X$. If there is $A_{[\alpha]} \not\subset D^{\alpha}_{[\alpha]}$, such that

$$E_{[\alpha]} \not\subset f^{\nabla}_\alpha(A)_{[\alpha]} = \cap\{f_\alpha(B)_{[\alpha]} \mid A_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]}\}.$$

Then there is $B \in L^X$, satisfying $E_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]}$ and $A_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset (B^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]}$. Clearly, $\bigcup_{G_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]} \not\subset D^{\alpha}_{[\alpha]}$, i.e., $D_{[\alpha]} \not\subset \bigcap_{G_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}} f_\alpha(G)_{[\alpha]}$. Thus, there is $C_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}$, such that $D_{[\alpha]} \not\subset f_\alpha(C)_{[\alpha]}$. Conclusively, we have

$$D_{[\alpha]} \subset f_\alpha(B)^{\alpha}_{[\alpha]} \subset \bigcup_{G_{[\alpha]} \not\subset B^{\alpha}_{[\alpha]}} f_\alpha(G)^{\alpha}_{[\alpha]}.$$

and

$$D_{[\alpha]} \not\subset \cap\{f_\alpha(C)_{[\alpha]} \mid D_{[\alpha]} \subset \bigcup_{H_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(H)^{\alpha}_{[\alpha]}\} = f^{\nabla}_\alpha(D)_{[\alpha]}.$$

Therefore, f^{∇}_α is α-Symmetric.

Theorem 3.3 *Let f_α be an α-Symmetric remote neighborhood mapping, then*

(1) $C_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]} \Rightarrow A_{[\alpha]} \subset \bigcup_{D_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)^{\alpha}_{[\alpha]}$.

(2) $A_{[\alpha]} \subset \bigcup_{D_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)^{\alpha}_{[\alpha]} \Rightarrow C_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}$.

Proof (1) Since f_α is an α-Symmetric mapping. $\forall D_{[\alpha]} \not\subset A^{\alpha}_{[\alpha]}$, there is $B_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}$, satisfying $D_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]}$. Therefore,

$$A^{\alpha}_{[\alpha]} \supset \bigcap_{B_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(B)_{[\alpha]}, \quad i.e., A_{[\alpha]} \subset \bigcup_{D_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)^{\alpha}_{[\alpha]}.$$

(2) By $A_{[\alpha]} \subset \bigcup_{D_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)^{\alpha}_{[\alpha]}$, we have $A^{\alpha}_{[\alpha]} \supset \bigcap_{D_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)_{[\alpha]}$. So $\forall x \notin A^{\alpha}_{[\alpha]}$, there is $D_{[\alpha]} \not\subset C^{\alpha}_{[\alpha]}$, such that $x \notin f_\alpha(D)_{[\alpha]}$. Since f_α is α-Symmetric.

There is $B^x_{[\alpha]} \not\subset x^\alpha_{[\alpha]}$, such that $C_{[\alpha]} \not\subset f_\alpha(B^x)_{[\alpha]}$. Take $E = \vee\{B^x \mid x \notin A^\alpha_{[\alpha]}\}$, then $x \not\subset E^\alpha_{[\alpha]}$. This implies $E^\alpha_{[\alpha]} \subset A^\alpha_{[\alpha]}$, i.e., $A_{[\alpha]} \subset E_{[\alpha]}$. Furthermore, we can conclude $C_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}$. Otherwise, if $C_{[\alpha]} \subset f_\alpha(A)_{[\alpha]} \subset f_\alpha(E)_{[\alpha]}$, then it contradicts with the statement: for each $B^x \leq E$, $C_{[\alpha]} \not\subset f_\alpha(B^x)_{[\alpha]}$.

Theorem 3.4 *Let $f_\alpha \in \mathscr{F}_\alpha$. Then*

$$(f_\alpha^\nabla)_\alpha^\nabla \leq f_\alpha^\nabla \odot f_\alpha^\nabla \leq f_\alpha^\diamond \leq f_\alpha \odot f_\alpha.$$

Proof $\forall A \in L^X$,

$$\begin{aligned}(f_\alpha^\nabla)_\alpha^\nabla(A) &= \wedge\{f_\alpha^\nabla(B) \mid A_{[\alpha]} \subset \bigcup_{G_{[\alpha]}\not\subset B^\alpha_{[\alpha]}} f_\alpha^\nabla(G)^\alpha_{[\alpha]}\}\\ &\leq \wedge\{f_\alpha^\nabla(C) \mid C_{[\alpha]} \not\subset f_\alpha^\nabla(A)_{[\alpha]}\}\\ &= (f_\alpha^\nabla \odot f_\alpha^\nabla)(A).\end{aligned}$$

By Theorem 2 (2), $f_\alpha^\nabla \leq f_\alpha$, we have

$$\begin{aligned}(f_\alpha^\nabla \odot f_\alpha^\nabla)(A) &= \wedge\{f_\alpha^\nabla(C) \mid C_{[\alpha]} \not\subset f_\alpha^\nabla(A)_{[\alpha]}\}\\ &\leq \wedge\{f_\alpha^\nabla(C) \mid C_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}\}\\ &= f_\alpha^\diamond(A).\end{aligned}$$

Therefore

$$f_\alpha^\diamond(A) = \wedge\{f_\alpha^\nabla(C) \mid C_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}\} \leq \wedge\{f_\alpha(C) \mid C_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}\} = (f_\alpha \odot f_\alpha)(A).$$

Theorem 3.5 *Let $f_\alpha \in \mathscr{F}_\alpha$. Then*

(1) $f_\alpha^\nabla \leq f_\alpha \odot f_\alpha$.
(2) $f_\alpha \odot f_\alpha \odot f_\alpha = f_\alpha^\diamond$.

Proof (1) By Theorem 2, we have

$$\begin{aligned}f_\alpha^\nabla(A) &= \wedge\{f_\alpha(B) \mid A_{[\alpha]} \subset \bigcup_{G_{[\alpha]}\not\subset B^\alpha_{[\alpha]}} f_\alpha(G)^\alpha_{[\alpha]}\}\\ &\leq \wedge\{f_\alpha(B) \mid A_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}\}\\ &= f_\alpha \odot f_\alpha(A).\end{aligned}$$

(2) By Theorem 3 (2), we have

$$\bigcup_{D_{[\alpha]}\not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)^{\alpha}_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]} \Rightarrow \exists B_{[\alpha]} \subset \bigcup_{D_{[\alpha]}\not\subset C^{\alpha}_{[\alpha]}} f_\alpha(D)^{\alpha}_{[\alpha]}, B_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}$$
$$\Rightarrow \exists B_{[\alpha]} \not\subset f_\alpha(A)_{[\alpha]}, C_{[\alpha]} \not\subset f_\alpha(B)_{[\alpha]}$$
$$\Rightarrow C_{[\alpha]} \not\subset f_\alpha \odot f_\alpha(A)_{[\alpha]}.$$

This shows $f_\alpha \odot f_\alpha \odot f_\alpha \leq f_\alpha^{\diamond}$. On the other hand, by Theorem 2 (2) and (1) above, it is easy to find $f_\alpha^{\diamond} \leq f_\alpha^{\nabla} \odot f_\alpha \leq f_\alpha \odot f_\alpha \odot f_\alpha$. Therefore, (2) holds.

4 Characterizations of α-Quasi Uniformities

Theorem 4.1 *An non-empty subfamily $\mathscr{D}_\alpha \subset \mathscr{F}_\alpha$ is an α-uniform, iff $\mathscr{D}_\alpha$ satisfies:*

(1) $f_\alpha \in \mathscr{D}_\alpha, g_\alpha \in \mathscr{F}_\alpha$ with $f_\alpha \leq g_\alpha$, then $g_\alpha \in \mathscr{D}_\alpha$.
(2) $f_\alpha, g_\alpha \in \mathscr{D}_\alpha$ implies $f_\alpha \vee g_\alpha \in \mathscr{D}_\alpha$.
(3) $f_\alpha \in \mathscr{D}_\alpha$, then $\exists g_\alpha \in \mathscr{D}_\alpha$, such that $g_\alpha^{\diamond} \geq f_\alpha$.

Proof Necessity. Since $\mathscr{D}_\alpha$ is an α-uniform, (1) and (2) hold. Furthermore, if $\mathscr{B}_\alpha \subset \mathscr{D}_\alpha$ is an α-symmetric base. So for every $f_\alpha \in \mathscr{D}_\alpha$, there is $g_\alpha \in \mathscr{B}_\alpha$, such that $g_\alpha \odot g_\alpha \odot g_\alpha \odot g_\alpha \geq f_\alpha$. By Theorem 5 (2), $g_\alpha^{\diamond} = g_\alpha \odot g_\alpha \odot g_\alpha \geq g_\alpha \odot g_\alpha \odot g_\alpha \odot g_\alpha \geq f_\alpha$.

Sufficiency. If $\mathscr{B}_\alpha = \{g_\alpha^{\nabla} \mid g_\alpha \in \mathscr{D}_\alpha\}$. For every $f_\alpha \in \mathscr{D}_\alpha$, there is $g_\alpha \in \mathscr{D}_\alpha$, such that $g_\alpha^{\diamond} \geq f_\alpha$. By Theorem 4, $g_\alpha \odot g_\alpha \geq g_\alpha^{\diamond} \geq f_\alpha$. Then $\mathscr{D}_\alpha$ is an α-quasi-uniform. By Theorem 2 (2)and (4), we know $g_\alpha^{\nabla} \geq g_\alpha^{\diamond} \geq f_\alpha$. This shows $\mathscr{B}_\alpha$ is an α-symmetric base of $\mathscr{D}_\alpha$. Therefore $\mathscr{D}_\alpha$ is an α-uniform.

Theorem 4.2 *An non-empty subfamily $\mathscr{D}_\alpha \subset \mathscr{F}_\alpha$ is an α-uniform, iff $\mathscr{D}_\alpha$ satisfies:*

(1) $f_\alpha \in \mathscr{D}_\alpha, g_\alpha \in \mathscr{F}_\alpha$ with $f_\alpha \leq g_\alpha$, then $g_\alpha \in \mathscr{D}_\alpha$.
(2) $f_\alpha, g_\alpha \in \mathscr{D}_\alpha$ implies $f_\alpha \vee g_\alpha \in \mathscr{D}_\alpha$.
(3) $f_\alpha \in \mathscr{D}_\alpha$, then $\exists g_\alpha \in \mathscr{D}_\alpha$, such that $g_\alpha^{\nabla} \geq f_\alpha$.

Proof Necessity. Since $\mathscr{D}_\alpha$ is an α-uniform, (1) and (2) hold. By Theorem 6, For every $f_\alpha \in \mathscr{D}_\alpha$, there is $g_\alpha \in \mathscr{D}_\alpha$, such that $g_\alpha^{\diamond} \geq f_\alpha$. So according to Theorem 2 (2), $g_\alpha^{\nabla} \geq g_\alpha^{\diamond} \geq f_\alpha$. Thus (3) holds.

Sufficiency. If $\mathscr{B}_\alpha = \{g_\alpha^{\nabla} \mid g_\alpha \in \mathscr{D}_\alpha\}$. By (3), For every $f_\alpha \in \mathscr{D}_\alpha$, there is $g_\alpha \in \mathscr{D}_\alpha$, such that $g_\alpha^{\nabla} \geq f_\alpha$. Again, there is $h_\alpha \in \mathscr{D}_\alpha$, such that $h_\alpha^{\nabla} \geq g_\alpha$. By Theorem 4, we get

$$h_\alpha \odot h_\alpha \geq h_\alpha^{\nabla} \odot h_\alpha^{\nabla} \geq (h_\alpha^{\nabla})_\alpha^{\nabla} \geq g_\alpha^{\nabla} \geq f_\alpha.$$

So $\mathscr{B}_\alpha$ is an α-symmetric base of $\mathscr{D}_\alpha$. Therefore, $\mathscr{D}_\alpha$ is an α-uniform.

5 α-P.Q. Metric and its Properties

A binary mapping $d_\alpha : L^X \times L^X \to [0, +\infty)$ is called an α-mapping, if $\forall (A, B), (C, D) \in L^X \times L^X$ satisfying $A_{[\alpha]} = C_{[\alpha]}$ and $B_{[\alpha]} = D_{[\alpha]}$, then $d_\alpha(A, B) = d_\alpha(A, B)$.

Definition 5.1 An α-mapping $d_\alpha : L^X \times L^X \to [0, +\infty)$ is called an α-P.Q metric on L^X, if

(α-M1) $d_\alpha(A, A) = 0$.
(α-M2) $d_\alpha(A, C) \leq d_\alpha(A, B) + d_\alpha(B, C)$.
(α-M3) $d_\alpha(A, B) = \bigwedge_{C_{[\alpha]} \subset B_{[\alpha]}} d_\alpha(A, C)$.

Theorem 5.1 *Let d_α be an α-P.Q. metrics on L^X. $\forall r \in (0, +\infty)$, a mapping $P_r : L^X \to L^X$ is defined by $\forall A \in L^X$,*

$$P_\alpha^r(A) = \vee\{B \in L^X \mid d_\alpha(A, B) \geq r\}.$$

Then

(1) P_α^r is α-symmetric mapping.
(2) $\forall A, B \in L^X$, $B_{[\alpha]} \subset P_\alpha^r(A)_{[\alpha]} \Leftrightarrow d_\alpha(A, B) \geq r$.
(3) $\forall A \in L^X, r > 0$, $A_{[\alpha]} \not\subset P_\alpha^r(A)_{[\alpha]}$.
(4) $\forall r, s \in (0, \infty)$, $P_\alpha^r \odot P_\alpha^s \geq P_\alpha^{r+s}$.
(5) $\forall A \in L^X, r > 0$, $P_\alpha^r(A)_{[\alpha]} = \bigcap_{s<r} P_\alpha^s(A)_{[\alpha]}$.
(6) $\forall A \in L^X$, $\bigcap_{r>0} P_\alpha^s(A)_{[\alpha]} = \emptyset$.

Proof Since d_α is an α-mapping, (1), (5) and (6) are easy.

(2) Clearly, $d_\alpha(A, B) \geq r \Rightarrow B_{[\alpha]} \subset P_\alpha^r(A)_{[\alpha]}$. Conversely. $\forall x \in B_{[\alpha]}$, $\exists x \in D_x \in L^X$, such that $d_\alpha(A, D_x) \geq r$. So $d_\alpha(A, \{x_\alpha\}) \geq d_\alpha(A, D_x) \geq r$. Thereby $d_\alpha(A, B) = \bigwedge_{x \in B_{[\alpha]}} d_\alpha(A, \{x_\alpha\}) \geq r$.

(3) Suppose $A_{[\alpha]} \subset P_\alpha^r(A)_{[\alpha]}$. By (2), we get $d_\alpha(A, A) \geq r$. This is a contradiction with (α-M1).

(4) $\forall A, B \in L^X$, if $B_{[\alpha]} \not\subset P_\alpha^r \odot P_\alpha^s(A)_{[\alpha]}$, then there is $D \in L^X$, such that $D_{[\alpha]} \not\subset P_\alpha^s(A)_{[\alpha]}$ and $B_{[\alpha]} \not\subset P_\alpha^r(D)_{[\alpha]}$. By (2), we have $d_\alpha(A, D) < s$, $d_\alpha(D, B) < r$. Then by (α-M2), we gain $d_\alpha(A, B) < r + s$. Therefore $B_{[\alpha]} \not\subset P_\alpha^{r+s}(A)_{[\alpha]}$. This means $P_\alpha^r \odot P_\alpha^s \geq P_\alpha^{r+s}$.

Theorem 5.2 *If a family of α-mappings $\{P_\alpha^r \mid P_r : L^X \to L^X, r > 0\}$ satisfies the conditions (2)–(5) in Theorem 8. For each $A, B \in L^X$, let's denote*

$$d_\alpha(A, B) = \wedge\{r \mid B_{[\alpha]} \not\subset P_\alpha^r(A)_{[\alpha]}\}.$$

Then

(1) $d_\alpha(A, B) < r \Leftrightarrow B_{[\alpha]} \not\subset P_\alpha^r(A)_{[\alpha]}$.
(2) d_α *is* α*-P.Q. metric on* L^X.

Proof (1) By Theorem 8 (2),(5), we have

$$d_\alpha(A, B) < r \Leftrightarrow \exists s < r, B_{[\alpha]} \not\subset P_\alpha^s(A)_{[\alpha]} \Leftrightarrow B_{[\alpha]} \not\subset P_\alpha^r(A)_{[\alpha]}.$$

(2) $\forall A, B \in L^X, r, s > 0$, if $d_\alpha(A, B) > r + s$, then

$$B_{[\alpha]} \subset P_\alpha^{r+s}(A)_{[\alpha]} \subset (P_\alpha^r \odot P_\alpha^s(A))_{[\alpha]}.$$

Thus $\forall C \in L^X$, we have $C_{[\alpha]} \not\subset P_\alpha^s(A)_{[\alpha]}$ and $B_{[\alpha]} \not\subset P_\alpha^r(C)_{[\alpha]}$. this implies $d_\alpha(A, C) > s$, and $d_\alpha(A, B) > r$. Hence $d_\alpha(A, B) + d_\alpha(A, C) > r + s$. Consequently, we obtain $d_\alpha(A, B) + d_\alpha(A, C) \geq d_\alpha(A, B)$.

Theorem 5.3 *An* α*-mapping* $d_\alpha : L^X \times L^X \to [0, +\infty)$ *satisfies* $(\alpha - M1)$,$(\alpha - M2)$ *and* $(\alpha - M3^*)$*, then for each* $C_{[\alpha]} \subset B_{[\alpha]}$, $d_\alpha(C, B) = 0$.
$(\alpha - M3^*) \forall A \in L^X, r > 0, A_{[\alpha]} \not\subset P_\alpha^r(A)_{[\alpha]}$.

Proof By (α-M2), for each $B, C \in L^X$, satisfying $C_{[\alpha]} \subset B_{[\alpha]}$, we have $d_\alpha(A, B) < d_\alpha(A, C) + d_\alpha(C, B)$. Here we can conclude $d_\alpha(C, B) = 0$. Otherwise, if $d_\alpha(C, B) = s > 0$, then $B_{[\alpha]} \subset P_\alpha^s(C)_{[\alpha]}$. it contracts with $C_{[\alpha]} \not\subset P_\alpha^s(C)_{[\alpha]}$ according to (α-M3*). Therefore $d_\alpha(C, B) = 0$.

Theorem 5.4 *An* α*-mapping* $d_\alpha : L^X \times L^X \to [0, +\infty)$ *is an* α*-P.Q metric on* L^X*, iff* d_α *satisfies (*α*-M1),(*α*-M2) and (*α*-M3*).*

Proof We only need to prove (α-M3)$\Leftrightarrow$(α-M3*). By Theorem 8 (2), it is easy to check (α-M3)$\Rightarrow$(α-M3*). If the converse result is not true, then there are $r, s > 0$, such that

$$d_\alpha(A, B) < s < r \leq \bigcap_{C_{[\alpha]} \subset B_{[\alpha]}} d_\alpha(A, C).$$

so for each $C_{[\alpha]} \subset B_{[\alpha]}$,

$$r \leq d_\alpha(A, C) \leq d_\alpha(A, B) + d_\alpha(B, C) < s + d_\alpha(B, C).$$

Thus, $0 < r - s < d_\alpha(B, C)$, which implies $C_{[\alpha]} \subset P_\alpha^{r-s}(B)_{[\alpha]}$. As a result

$$B_{[\alpha]} = (\vee\{C \mid C_{[\alpha]} \subset B_{[\alpha]}\})_{[\alpha]} \subset P_\alpha^{r-s}(B)_{[\alpha]}.$$

However, it contradicts with (α-M3*). Therefore (α-M3) holds.

Theorem 5.5 d_α *is* α*-P.Q. metric on* L^X*. Then* $\{P_\alpha^r \mid r > 0\}$ *satisfying (3)–(5) in Theorem 8 is an* α*-base of some* α*-uniform, which is called the* α*-uniform induced by* d_α.

Given an α-quasi-uniform $\mathscr{D}_\alpha$, if we say $\mathscr{D}_\alpha$ is metricable, we mean there is an α-P.Q. metric d_α, such that $\mathscr{D}_\alpha$ is induced by d_α.

Theorem 5.6 *An α-quasi-uniform spaces $(L^X, \mathscr{D}_\alpha)$ is α-P.Q. metricable iff it has a countable α-base.*

Proof By Theorem 12, the necessity is obvious. Let's prove the sufficiency.

Let $\mathscr{D}_\alpha$ be an α-uniformity on L^X, which has a countable α-base $\mathscr{B}_\alpha = \{P_\alpha^n \mid n \in N\}$. Let's take $g_\alpha^1 = P_\alpha^1$, then there is $g_\alpha^2 \in \mathscr{B}_\alpha$, such that $g_\alpha^2 \odot g_\alpha^2 \odot g_\alpha^2 \geq g_\alpha^1 \vee P_\alpha^2$. In addition, there is $g_\alpha^3 \in \mathscr{B}_\alpha$, such that $g_\alpha^3 \odot g_\alpha^3 \odot g_\alpha^3 \geq g_\alpha^2 \vee P_\alpha^3$. The process can be repeated again and again, then $\{g_\alpha^n \in n \in N\}$ is also an α-base. Obviously, $g_\alpha^{n+1} \odot g_\alpha^{n+1} \odot g_\alpha^{n+1} \geq g_\alpha^n$. Let's take $\varphi_\alpha : L^X \to L^X$, defined by: $\forall A \in L^X$,

$$\varphi_\alpha^r(A) = \begin{cases} g_\alpha^n(A), & \frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}, \\ 0_X, & r > 1. \end{cases}$$

Clearly, $\forall r > 0$, $A_{[\alpha]} \not\subset \varphi_\alpha^r(A)_{[\alpha]}$. And $\forall \frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$, we have

$$\varphi_\alpha^r \odot \varphi_\alpha^r \odot \varphi_\alpha^r = g_\alpha^n \odot g_\alpha^n \odot g_\alpha^n \geq g_\alpha^{n-1} = \varphi_\alpha^{2r}.$$

Let's define

$$f_\alpha^r(A) = \wedge \left\{ (\varphi_\alpha^{r_1} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}(A)) \mid \sum_{i=1}^{k} r_i = r \right\}.$$

Then $\mathscr{B}_\alpha = \{f_\alpha^r \mid r > 0\} \subset \mathscr{D}_\alpha$.

Finally, let's prove $\mathscr{B}_\alpha$ is an α-base of $\mathscr{D}_\alpha$ satisfying (3)-(6) in Theorem 8.

Step 1. For $f_\alpha \in \mathscr{D}$, there is $n \in N$, such that $g_\alpha^n \geq f_\alpha$. So if $r \in (\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, then $\varphi_\alpha^{2r} = g_\alpha^n$. Besides, it is easy to check, $\varphi_\alpha^{r_1} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k} \geq \varphi_\alpha^{2r}$ whenever $\sum_{i=1}^{k} r_i = r$. Thus $f_\alpha^r \geq \varphi_\alpha^{2r} = g_\alpha^n \geq f_\alpha$. This means $\mathscr{B}_\alpha$ be an α-base of $\mathscr{D}_\alpha$.

Step 2. Obviously, $f_\alpha \in \mathscr{B}$ satisfies (3) and (6) in Theorem 8. Furthermore, for $r, s > 0$, $A \in L^X$, if there is $B_{[\alpha]} \not\subset (f_\alpha^r \odot f_\alpha^s(A))_{[\alpha]}$. Then there is $C \in L^X$, such that $B_{[\alpha]} \not\subset f_\alpha^r(C)_{[\alpha]}$ and $C_{[\alpha]} \not\subset f_\alpha^s(A)_{[\alpha]}$. So there are $\sum_{i=1}^{k} r_i = r$ and $\sum_{i=1}^{m} s_i = s$, such that

$$B_{[\alpha]} \not\subset (\varphi_\alpha^{r_1} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}(C))_{[\alpha]}$$

and

$$C_{[\alpha]} \not\subset (\varphi_\alpha^{s_1} \odot \varphi_\alpha^{s_2} \odot \cdots \odot \varphi_\alpha^{s_m}(A))_{[\alpha]}.$$

Thus

$$B_{[\alpha]} \not\subset (\varphi_\alpha^{r_1} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}) \odot (\varphi_\alpha^{s_1} \odot \varphi_\alpha^{s_2} \odot \cdots \odot \varphi_\alpha^{s_m})(A)_{[\alpha]}.$$

As a result, $B_{[\alpha]} \not\subset f_\alpha^{r+s}(A)_{[\alpha]}$. Consequently, $f_\alpha^{r+s} \leq f_\alpha^r \odot f_\alpha^s$. This is the proof of (4) in Theorem 8.

Step 3. for $r > s > 0$, $A \in L^X$,

$$\begin{aligned} f_\alpha^r(A) &= \wedge \left\{ (\varphi_\alpha^{r_1} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}(A) \mid \sum_{i=1}^{k} r_i = r \right\} \\ &\leq \wedge \left\{ (\varphi_\alpha^{s_1} \odot \varphi_\alpha^{s_2} \odot \cdots \odot \varphi_\alpha^{s_m} \odot \varphi_\alpha^{r-s}(A)) \mid \sum_{i=1}^{m} s_i = m \right\} \\ &\leq \wedge \left\{ (\varphi_\alpha^{s_1} \odot \varphi_\alpha^{s_2} \odot \cdots \odot \varphi_\alpha^{s_m}(A)) \mid \sum_{i=1}^{m} s_i = m \right\} = f_\alpha^s(A). \end{aligned}$$

Hence $f_\alpha^r \leq \bigwedge_{s<r} f_\alpha^s$.

Conversely. Let's prove the reverse result.

If $B \in L^X$, $B_{[\alpha]} \not\subset f_\alpha^r(A)_{[\alpha]}$, then there is $\sum_{i=1}^{k} r_i = r$, such that $B_{[\alpha]} \not\subset \varphi_\alpha^{r_1} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}(A)_{[\alpha]}$. So there is $C \in L^X$, such that $C_{[\alpha]} \not\subset (\varphi_\alpha^{r_2} \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}(A)_{[\alpha]}$ and $B_{[\alpha]} \not\subset \varphi_\alpha^{r_1}(C)_{[\alpha]}$. By $\varphi_\alpha^{r_1} = \bigwedge_{t<r_1} \varphi_\alpha^t$, there is $t < r_1$, such that $B_{[\alpha]} \not\subset \varphi_\alpha^t(C)_{[\alpha]}$, i.e., $B_{[\alpha]} \not\subset \varphi_\alpha^t \odot \varphi_\alpha^{r_2} \odot \cdots \odot \varphi_\alpha^{r_k}(A)_{[\alpha]}$. Let's take $s = t + \sum_{i=2}^{k} r_i$, we have $s < r$ and $B_{[\alpha]} \not\subset f_\alpha^s(A)_{[\alpha]}$. Therefore $f_\alpha^r \geq \bigwedge_{s<r} f_\alpha^s$. Therefore (5) in Theorem 8 holds.

Theorem 5.7 *Each α-C_{II} α-layer topological space is P.Q.-metriclizable.*

Proof Let $(L^X, D_\alpha(\delta))$ be an α-C_{II} space, $\{P_n \mid n \in N\}$ be an α-base. $\forall n \in N$, $f_\alpha^{P_n} : L^X \to L^X$ is defined as: $\forall A \in L^X$,

$$f_\alpha^{P_n}(A) = \begin{cases} 0_X, & A_{[\alpha]} \subset P_{n[\alpha]}. \\ P_n, & A_{[\alpha]} \not\subset P_{n[\alpha]}. \end{cases}$$

Let's take $\mathscr{D}^* = \{f_\alpha^{P_n} \mid n \in N\}$ and

$$\mathscr{B}_\alpha = \{f_\alpha \mid \exists f_\alpha^{P_{n_i}} \in \mathscr{D}^*, i = 1, 2, \cdots, m, f_\alpha = \bigvee_{i=1}^{m} f_\alpha^{P_{n_i}}\}.$$

Then $\mathscr{B}_\alpha$ is an α-base of some uniform denoted by $\mathscr{D}_\alpha$ and clearly, $\eta_\alpha = \eta_\alpha(\mathscr{D}_\alpha)$. Furthermore, since $\mathscr{B}_\alpha$ is countable, we know $(L^X, D_\alpha(\delta))$ is P.Q.-metriclizable.

Theorem 5.8 *An α-layer co-topology $D_\alpha(\delta)$ on L^X can be α-P.Q. metriclizable iff there is a Sequence of α-remote neighborhood mappings $\{f_\alpha^n\}_{n\in N}$ satisfying*

(1) $\forall n \in N$, $f_\alpha^n \leq f_\alpha^{n+1} \odot f_\alpha^{n+1} \odot f_\alpha^{n+1}$,
(2) $\forall a \in M^*(L^X)$, $\{f_\alpha^n(a)\}_{n\in N}$ *is the α-remote neighborhood family of a.*

Proof Necessary. If $D_\alpha(\delta)$ can be α-P.Q. metriclizable, there is an α-P.Q. metric, say d_α. Let's take $f_\alpha^n = P_\alpha^{\frac{1}{3^n}}$. By Theorem 8, it is clear that (1) and (2) hold.

Sufficiency. If $\{f_\alpha^n\}_{n\in N}$ satisfies (1) and (2). Clearly, $\{f_\alpha^n\}_{n\in N}$ is countable. So by Theorem 12, it is an α-base of some α-quasi uniform $\mathscr{D}_\alpha$. Therefore $D_\alpha(\delta) = \eta_\alpha(\mathscr{D}_\alpha)$. This means $D_\alpha(\delta)$ is α-P.Q. metriclizable.

Theorem 5.9 *An α-layer co-topology $D_\alpha(\delta)$ on L^X can be α-P.Q. metriclizable iff there is a Sequence of α-symmetric remote neighborhood mappings $\{f_\alpha^n\}_{n\in N}$ satisfying*

(1) $\forall n \in N$, $f_\alpha^n \leq f_\alpha^{n+1} \odot f_\alpha^{n+1} \odot f_\alpha^{n+1}$,
(2) $\forall a \in M^*(L^X)$, $\{f_\alpha^n(a)\}_{n\in N}$ *is the α-remote neighborhood family of a.*

Theorem 5.10 *An α-layer co-topology $D_\alpha(\delta)$ on L^X can be α-P.Q. metriclizable iff there is a Sequence of α-symmetric remote neighborhood mappings $\{f_\alpha^n\}_{n\in N}$ satisfying*

(1) $\forall n \in N$, $f_\alpha^n \leq (f_\alpha^{n+1})^\diamond \leq f_\alpha^{n+1}$,
(2) $\forall a \in M^*(L^X)$, $\{f_\alpha^n(a)\}_{n\in N}$ *is the α-remote neighborhood family of a.*

Proof Necessity. It is similar to that of Theorem 14.

Sufficiency. If $\{f_\alpha^n\}_{n\in N}$ satisfies (1) and (2). By Theorem 2, $\forall n \in N$, $f_\alpha^n \leq (f_\alpha^{n+1})^\diamond \leq (f_\alpha^{n+1})^\nabla \leq f_\alpha^{n+1}$. By Theorem 4, $\forall n \in N$, $f_\alpha^n \leq (f_\alpha^{n+1})^\diamond \leq f_\alpha^{n+1} \odot f_\alpha^{n+1}$.

Therefore $\forall n \in N$,

$$f_\alpha^n \leq f_\alpha^{n+1} \odot f_\alpha^{n+1} \leq (f_\alpha^{n+2})^\nabla \odot (f_\alpha^{n+2})^\nabla \leq (f_\alpha^{n+2})^\nabla.$$

Again, by Theorem 4,

$$\begin{aligned} f_\alpha^n &\leq (f_\alpha^{n+2})^\nabla \odot (f_\alpha^{n+2})^\nabla \leq ((f_\alpha^{n+4})^\nabla)^\nabla \odot ((f_\alpha^{n+4})^\nabla)^\nabla \\ &\leq (f_\alpha^{n+4})^\nabla \odot (f_\alpha^{n+4})^\nabla \odot (f_\alpha^{n+4})^\nabla \odot (f_\alpha^{n+4})^\nabla \\ &\leq (f_\alpha^{n+4})^\nabla \odot (f_\alpha^{n+4})^\nabla \odot (f_\alpha^{n+4})^\nabla. \end{aligned}$$

$\forall n \in N$, $g_\alpha^n = (f_\alpha^{4n-3})^\nabla$. Then $\{g_\alpha^n\}_{n\in N}$ is a family of α-symmetric remote neighborhood mappings. It is clear that $\{g_\alpha^n(a)\}_{n\in N}$ is the α-remote neighborhood family of a.

Theorem 5.11 *An α-layer co-topology $D_\alpha(\delta)$ on L^X can be α-P.Q. metriclizable iff there is a Sequence of α-remote neighborhood mappings $\{f_\alpha^n\}_{n\in N}$ satisfying*

(1) $\forall n \in N$, $f_\alpha^n \leq (f_\alpha^{n+1})^\nabla \leq f_\alpha^{n+1}$,

(2) $\forall a \in M^*(L^X)$, $\{f_\alpha^n(a)\}_{n\in N}$ *is the* α*-remote neighborhood family of* a.

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