Flexible and Simple Methods of Calculations on Fuzzy Numbers with the Ordered Fuzzy Numbers Model

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Abstract. The publication shows the way of implementing arithmetic operations on fuzzy numbers based on Ordered Fuzzy Numbers calculation model [12], [13], [14]. This model allows to perform calculations on fuzzy numbers in a way that the outcomes meet the same criteria as the outcomes of calculations on real numbers. In this text, to the four basic operations with Ordered Fuzzy Numbers, a logarithm and exponentiation was added. Several examples of the calculations are included, the results of which are obvious and typical of real numbers but not achievable with the use of conventional computational methods for fuzzy numbers. From these examples one can see that the use of Ordered Fuzzy Numbers allows to obtain outcomes for real numbers in spite of using the fuzzy values.

Keywords: fuzzy number, Ordered Fuzzy Numbers, arithmetic calculations on fuzzy numbers, logarithm of fuzzy numbers, exponent of fuzzy numbers.

1 Introduction

The abilities to analyze and process information is an important factor in the development in every field. In many real problems, however, we find cases where the data that we are able to obtain are imprecise or uncertain. People deal with such data by linguistic description such as heavy weight, cold, little water, far away, etc. The need for using more formal methods appears when one needs to save such data in a digital way in order to automate further processing.

More appropriate tool in such situations is the theory of fuzzy sets [1], which allows to describe imprecise information mathematically. Furthermore, in the case of modeling imprecise quantitative data such as: about 4, more or less 2, etc. fuzzy numbers are used.

There are many mathematical models for analyzing the data. A large number of numerical methods supporting processing the information were defined. Unfortunately, the existing computational model of fuzzy numbers makes it difficult to apply all the tools in the processing of imprecise data. This is due to the fact

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that in most applications numerical operations on fuzzy values, are based on the so-called Zadehs extension principle [2]. It introduces a formal apparatus for transferring operations (also arithmetic) from ordinary sets to fuzzy sets. Unfortunately, the use of the extension principle involves inconvenient consequences, particularly in the case of repeating series of actions. The main problem here is the expansion of the fuzzy number support with the subsequent actions. Regardless of whether we add or subtract numbers, they becoming "fuzzier". Consequently, after a few calculations we can receive impractical outcomes with a very wide support. These operations properties are also related to other negative consequences, such as the difference $A - A$ does not allow to obtain the neutral element of addition, which is the number zero. Another important drawback is the lack of a simple method for solving the elementary equation of $A + X = B$ (where A and B are non-singleton fuzzy numbers). These negative calculations characteristics impede the application of even basic calculations in the numerical analysis of the imprecise data, not to mention the more complex ones.

Remark 1. Crisp values can be represented as fuzzy numbers with the use of the so-called singletons. This publication focuses, however, on the general characteristic of fuzzy values, so any further discussion in the context of fuzzy numbers will apply to situations in which we are dealing with the general form of a fuzzy number, not its particular case - a singleton.

One of the commonly accepted models of fuzzy numbers is the proposal introduced by Dubois and Prade [3]. It is based on the fact of considering the membership function of fuzzy numbers as a pair of two shape functions describing the left and right fuzzy spread. This model is called (L, R) fuzzy numbers. (L, R) numbers gained great popularity due to good interpretation and relatively easy implementation of basic operations such as adding. Calculations are also based on the Zadehs extension principle, which as mentioned before is connected with several significant adverse consequences.

To improve the computational properties of fuzzy numbers several additional solutions were introduced. They are usually connected with defining additional operations or constraints ([6][7][11]).

An alternative solution is to use Ordered Fuzzy Numbers mathematical model [12] [13]. This publication focuses precisely on this model, and in particular on methods of carrying out calculations. A better understanding of the OFN model requires a new approach to modeling fuzzy values, which can cause some difficulties. However, an important benefit is the ability to perform all arithmetical calculations with the same relations between the results, as in the case of operations on real numbers.

2 Ordered Fuzzy Numbers (OFN)

In the series of papers [12], [13], [14],[16],[18],[19], [21], [22] were introduced and developed main concepts of the idea of Ordered Fuzzy Numbers. Following these papers fuzzy number will be identified with the pair of functions defined on the interval $[0, 1]$.

Definition 1. *The Ordered Fuzzy Number (OFN in short) A is an ordered pair of two continuous functions*

$$A = (f_A, g_A) \tag{1}$$

called the UP-part and the DOWN-part, respectively, both defined on the closed interval $[0, 1]$ *with values in* $\boldsymbol{R}$.

If the both functions f and g are monotonic (Fig.1 a)), they are also invertible and possess the corresponding inverse functions defined on a real axis with the values in $[0, 1]$. Now, if these two opposite functions are not connected, we linking them with constant function (with the value 1). In such way we receive an object which directly represents the classical fuzzy number. For the finalization of transformation, we need to mark an order of f and g with an arrow on the graph (see Fig.1 b)). Notice that pairs (f, g) and (g, f) are the two different Ordered Fuzzy Numbers, unless $f = g$. They differ by their orientation.

The interpretations for this orientation and its relations with the real world problems are explained in the [21] and [22].

It is worth to point out that a class of Ordered Fuzzy Numbers (OFNs) represents the whole class of convex fuzzy numbers ([4],[8],[9],[10],[17], with continuous membership functions.

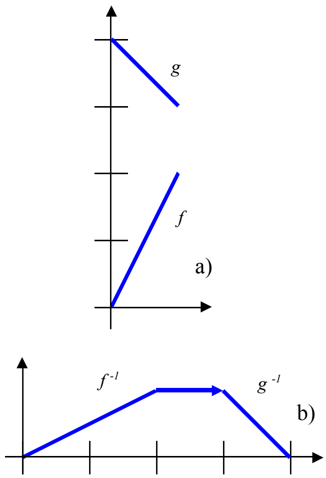

Fig. 1. a)Ordered Fuzzy Number from definition, b)Ordered Fuzzy Number as convex fuzzy number with an arrow

There are publications about OFNs, where propositions of the new methods for the fuzzy systems can be found. The papers [18],[19],[20] contains examples of the new inference methods based on the OFNs. The works [15],[23] are about defuzzyfication methods.

3 Arithmetic Operations

Operations on Ordered Fuzzy Numbers we define as operations with the UP and DOWN parts as follows:

Definition 2. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called Ordered Fuzzy Numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula*

$$f_C(y) = f_A(y) \star f_B(y) \qquad \wedge \qquad g_C(y) = g_A(y) \star g_B(y) \tag{2}$$

where "$\star$" replaces operations "$+$", "$-$", "$\cdot$", and "$/$". Moreover A/B s determined only if the OFN B does not contain zero. The $y \in [0, 1]$ is the domain of functions f and g.

It is also worth noting that the subtraction is equal to the addition of the opposite number, where the opposite number is obtained by multiplying the given value by the -1 (real number - singleton). By using the above-mentioned method in calculation of $A - A$ we obtain exact zero (crisp number).

Remark 2. Determining whether a given OFN contains (or not) r value (real number) is a mental shortcut. To be more precise, this phrase should be understood as a situation in which we consider whether any of the functions (UP or DOWN part) forming OFN have value r for any argument.

3.1 More Operations

Going further in the direction of arithmetical operations, we can offer definition of exponentiation and counting logarithms in the similar way as in basic four operations.

Definition 3. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are OFNs. The result of exponentation A raised to the power of B written A^B is defined by formula*

$$f_C = {f_A}^{f_B} \qquad \wedge \qquad g_C = {g_A}^{g_B}. \tag{3}$$

The logarithm of a number is the exponent by which another fixed value, the base, has to be raised to produce that number.

Definition 4. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are OFNs. The logarithm of a number A with respect to base B written log_B^A is defined by formula*

$$f_C = log_{f_B}(f_A) \qquad \wedge \qquad g_C = log_{g_B}(g_A). \tag{4}$$

Of course, as in the case of real numbers as with the OFNs, appropriate restrictions should be applied. During exponentiation when the exponent is not an integer, the main limitation is the exclusion as a base these OFNs that contain negative values. With logarithms, in turn, the limitations are as follows:

- OFN which is a base and an exponent can contain only non-negative values;
- in addition, the base of the logarithm cannot include number 1.

4 Calculations on OFNs

In this chapter a number of examples showing the arithmetic sequences of calculations will be presented. Examples focus on such transformations, which in cases of real numbers could be reduced and brought to one of the numbers that is a part of the transformation. However, because OFN was used in the calculations, partial results will be presented in order to make it easier to keep track of what is happening at each stage of the calculations.

4.1 Solving Equations

These examples show the solution of the equation $X = A + B$ where A and B are known values. Here ones attention should be drawn to two options:

a) when B has a grater support than A (Fig.2),
b) when A has a greater support than B (Fig.3).

It is inasmuch important that in the set of convex fuzzy numbers for option a) there is a solution, although it cannot be obtained by a simple arithmetic operation. However, for the option b) the solution does not exist, because there is no such fuzzy number, which could be added to the value A, to obtain the outcome of the number with "narrow" support. With the use of the OFN model both options are resolved in the same manner, by simple calculation of $X = B - A$, which is presented in the examples on Fig.2 and Fig.3.

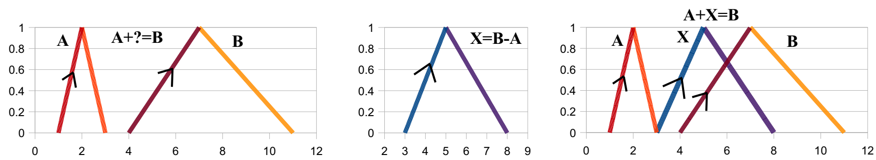

Fig. 2. Equation where B is wider than A

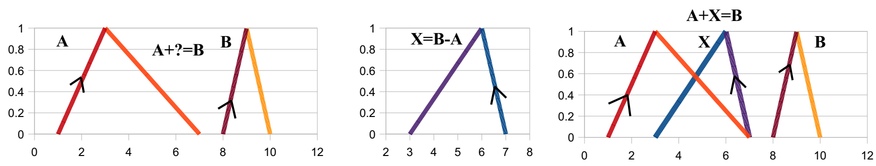

Fig. 3. Equation where A is wider than B

4.2 Distributivity

The following two examples should be considered together. The first one (Fig.4) shows as the result of calculation $A(B - C)$, and the next $AB - AC$ (Fig.5). With the use of OFN model the two results are the same, which corresponds to similar computation on real numbers. However, it would not provide us such results with the use of typical calculations with fuzzy numbers.

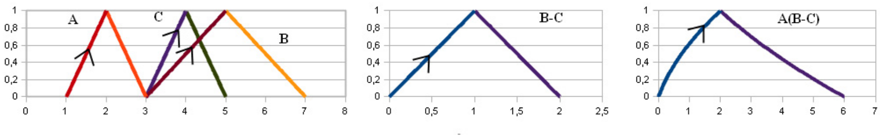

Fig. 4. Result of $A(B - C)$

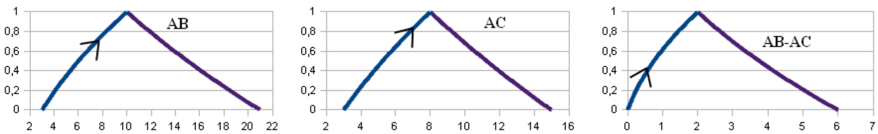

Fig. 5. Result of $AB - AC$ is the same like in Fig.4

4.3 Sum of Fractions

In this example (Fig.6), we see a situation where the sum of fractions $\frac{B}{C} + \frac{(C-B)}{C}$ comes down to the number 1 with the use of real numbers. The same thing also happens when in the same way we operate with OFNs.

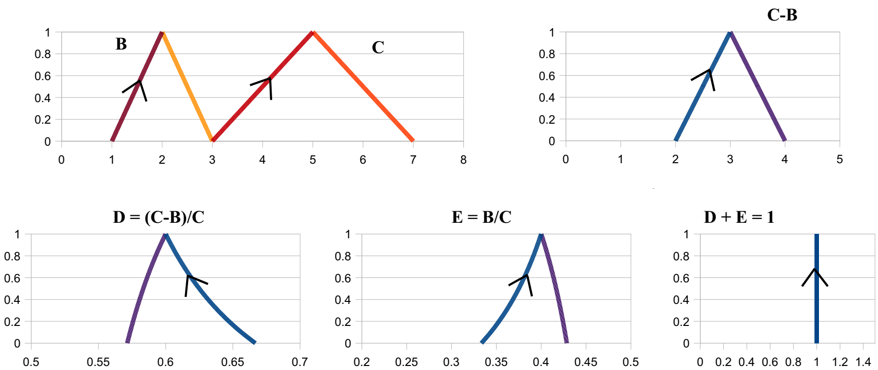

Fig. 6. Example when sum of the OFNs fractions is equal to crisp 1

4.4 Multiplication of Fractions

The next example (Fig.7) shows the calculations $\frac{3A}{2} \cdot \frac{2B}{3A}$ for a situation, in which the actual outcome with real numbers can be achieved by reducing repeated numerators and denominators without any calculations. After investigating the specific actions we can see that by using OFNs we could also simplify the calculations.

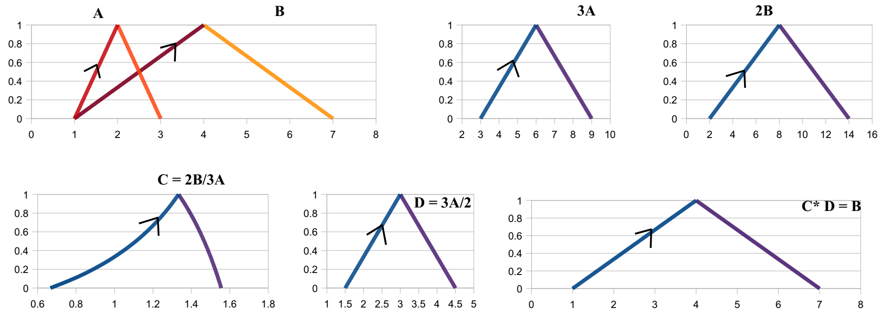

Fig. 7. Result of $\frac{3A}{2} \cdot \frac{2B}{3A}$

4.5 Exponentiation

This example shows (Fig.8) the calculations of exponential function A^B where the base and the exponent is a fuzzy number.

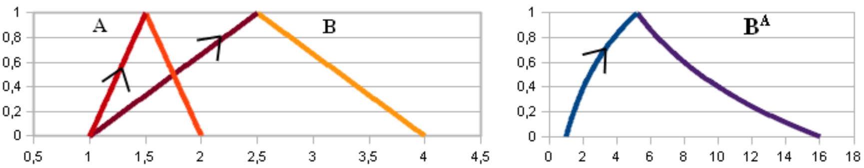

Fig. 8. OFN to the power of OFN

4.6 Exponentiation and Multiplying

The following two examples can be considered together. The first example shows the calculation, in which for the real numbers as a result we get a base of operation raising number to the power $A^{1/4} \cdot A^{3/4}$. As anyone can see on the Fig.9 the same is obtained when A is OFN.

Next example refers to the previous one but here are OFNs in the place of real numbers. Between exponents, there is the following relationship: the first exponent is $\frac{B}{C}$, and the other is $\frac{(C-B)}{C}$. In one of the previous examples (see Fig.6) it has already been shown that their sum is 1 (crisp). Additionally we

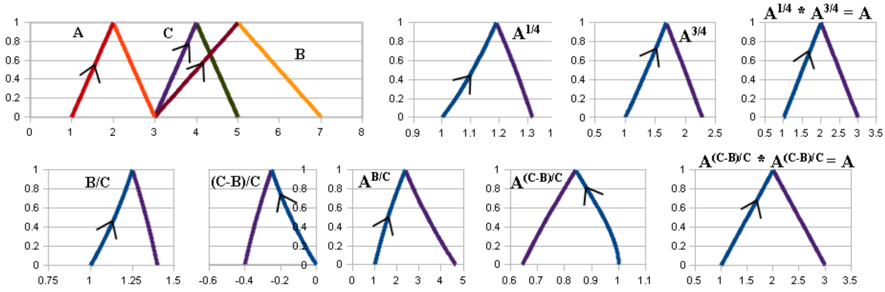

Fig. 9. Multiplying exponents which sum is equal 1

expect that, an every number raised to the power 1 gives in a result the base of exponentiation. We see (Fig.9) that is true also, with multiplication of the OFNs, where we have the same base and the exponents sum to unity, the outcome is the base, as with real numbers.

4.7 Logarithm

Here we have two examples also. First example (Fig.10) shows calculations on OFNs with the use of logarithmic function. In the next example we see (Fig.11) a series of transformations that for real numbers should generate an outcome which equals number A. The same thing is obtained when we use the OFN model.

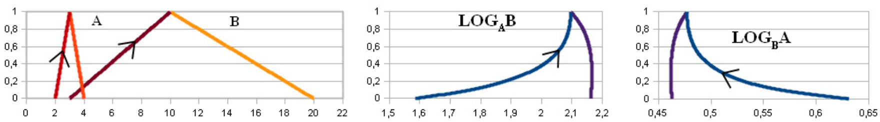

Fig. 10. Example of logarithm

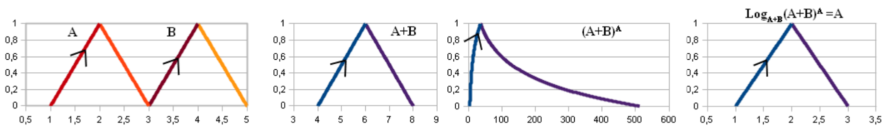

Fig. 11. Properties of logarithm preserved with OFNs

4.8 Comments for Examples

It is worth noting that the presentation of these few examples does not intend to prove mathematical relationships between calculations on OFN model and calculations on real numbers. The purpose of these examples is to show OFNs computational mechanisms and the fact that the outcomes are also fuzzy numbers, which can be interpreted as imprecise data.

However, when it comes to the accordance of the operations in OFN model with the ones on real numbers, it is a consequence of the definition of mathematical operations. Such results can be achieved because when introducing a new model, operations on fuzzy numbers were moved directly to operations on real numbers. After a closer investigation of the definitions (Def.2 and Def.3) it can be noted that the operations on the parts of OFN are executed through operations on functions representing these parts. The operations on the functions are, in turn, operations on their values. Thus, if the space of function value is the space of real numbers (as with all OFNs), then, in fact, operations on functions are carried out through operations on those numbers.

4.9 Improper OFNs

It is worth noting that the objects as shown in the figure (Fig.12) are also consistent with the definition of OFN. As one can see, their shape can not be defined as a function. They are called improper OFNs. In case when there is a need to read the membership values in the form of classical fuzzy number, one can use the definition of membership function for OFNs (see [22]), which indicates a clear solution for such structures. Here it should be also noted that despite the unusual (as for fuzzy numbers) shape, such OFNs still contain important information needed for the calculations. Moreover, according to the interpretation proposed in [22], this information may have a broader meaning depending on the context of carried out operations.

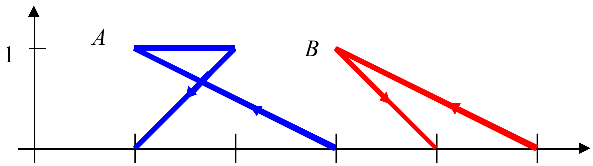

Fig. 12. Examples of results

5 Summary

OFNs constitute an important step in the development of calculation apparatus for uncertain or imprecise data. The new model is a tool allowing to transfer calculation characteristics from real numbers to fuzzy numbers, without defining separate follow-up actions, which are not in real numbers set, and avoiding the continuous expansion of the support with subsequent arithmetic operations.

Presented examples demonstrate the easiness and flexibility of calculations that can be applied to imprecise data processing. Although we operate on objects representing fuzzy numbers, we obtain the same relationships between the calculations and the outcomes of operations as with real numbers.

Such calculation properties allow taking the next step in processing imprecise values. With the use of OFNs we can transfer known mathematical models created for describing the world with the help of precise figures under the circumstances when we have data available only in the form of fuzzy numbers. Calculations of such relationships, apart from introducing a new model do not require further actions to improve the outcomes or to define new actions.

To sum up, it can be stated that by using the OFN model in terms of calculations we bring together possibilities of processing precise and imprecise data.

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