Efficient Cryptosystems from 2^k-th Power Residue Symbols⋆

Marc Joye and Benoît Libert

Technicolor 975 avenue des Champs Blancs, 35576 Cesson-Sévigné Cedex, France
{marc.joye,benoit.libert}@technicolor.com

Abstract. Goldwasser and Micali (1984) highlighted the importance of randomizing the plaintext for public-key encryption and introduced the notion of semantic security. They also realized a cryptosystem meeting this security notion under the standard complexity assumption of deciding quadratic residuosity modulo a composite number. The Goldwasser-Micali cryptosystem is simple and elegant but is quite wasteful in bandwidth when encrypting large messages. A number of works followed to address this issue and proposed various modifications.

This paper revisits the original Goldwasser-Micali cryptosystem using 2^k-th power residue symbols. The so-obtained cryptosystems appear as a very natural generalization for $k \geq 2$ (the case $k = 1$ corresponds exactly to the Goldwasser-Micali cryptosystem). Advantageously, they are efficient in both bandwidth and speed; in particular, they allow for fast decryption. Further, the cryptosystems described in this paper inherit the useful features of the original cryptosystem (like its homomorphic property) and are shown to be secure under a similar complexity assumption. As a prominent application, this paper describes the most efficient lossy trapdoor function based on quadratic residuosity.

Keywords: Public-key encryption, quadratic residuosity, Goldwasser-Micali cryptosystem, homomorphic encryption, standard model.

1 Introduction

Encryption is arguably one of the most fundamental cryptographic primitives. Although it seems an easy task to identify properties that a good encryption scheme must fulfill, it turns out that rigorously defining the right security notion is not trivial at all. Security is context sensitive. Merely requiring that the plaintext cannot be recovered from the ciphertext is not enough in most applications. One may require that the knowledge of some *a priori* information on the plaintext does not help the adversary to obtain any new information, that is, beyond what can be obtained from the *a priori* information. This intuition is formally captured by the notion of *semantic security*, introduced by Goldwasser and Micali in their seminal paper [20]. They also introduced the equivalent notion of *indistinguishability of encryptions*, which is usually easier to work with.

⋆ Full version available at http://eprint.iacr.org/

Given the encryption of any two equal-length (distinct) plaintexts, an adversary should not be able to distinguish the corresponding ciphertexts.

Clearly, the latter notion is only achievable by probabilistic public-key encryption schemes. One such cryptosystem was also presented in [20]. It achieves ciphertext indistinguishability under the *Quadratic Residuosity* (QR) assumption. Informally, this assumption says that it is infeasible to distinguish squares from non-squares in $\mathbb{J}_N$ (*i.e.*, the set of elements in $\mathbb{Z}_N^*$ whose Jacobi symbol is 1) where $N = pq$ is an RSA-type modulus of unknown factorization.

The Goldwasser-Micali cryptosystem is simple and elegant. The public key comprises an RSA modulus $N = pq$ and a non-square $y \in \mathbb{J}_N$ while the private key is the secret factor p. The encryption of a bit $m \in \{0,1\}$ is given by $c = y^m x^2 \bmod N$ for a random $x \in \mathbb{Z}_N^*$. The message m is recovered using p, by checking whether c is a square: $m = 0$ if so, and $m = 1$ otherwise —observe that a non-square $y \in \mathbb{J}_N$ is also a non-square modulo p. The encryption of a string $m = (m_{k-1}, \ldots, m_0)_2$, with $m_i \in \{0,1\}$, proceeds by forming the ciphertexts $c_i = y^{m_i} x^2 \bmod N$, for $0 \le i \le k - 1$. The scheme is computationally efficient but somewhat wasteful in bandwidth as $k \cdot \log_2 N$ bits are needed to encrypt a k-bit message. Several proposals were made to address this issue.

A first attempt is due to Blum and Goldwasser [8]. They achieve a better ciphertext expansion: the ciphertext has the same length as the plaintext plus an integer of the size of modulus. The scheme is proved semantically secure assuming the unpredictability of the output of the Blum-Blum-Shub's pseudorandom generator [6,7] which resides on the factorisation hardness assumption. Details about this scheme can be found in [21].

Another direction, put forward by Benaloh and Fischer [12,5], is to use a k-bit prime r such that $r \mid p - 1$, $r^2 \nmid p - 1$ and $r \nmid q - 1$. The scheme also requires $y \in \mathbb{Z}_N^*$ such that $y^{\phi(N)/r} \not\equiv 1 \pmod{N}$, where $\phi(N) = (p-1)(q-1)$ denotes Euler's totient function. A k-bit message m (with $m < r$) is encrypted as $c = y^m x^r \bmod N$, where $x \in_R \mathbb{Z}_N^*$. It is recovered by searching over the entire message space, $[0, r) \subseteq \{0,1\}^k$, for the element m satisfying $(y^{\phi(N)/r})^m \equiv c^{\phi(N)/r} \pmod{N}$. The scheme is shown to be secure under the *prime-residuosity assumption* (which generalizes the quadratic residuosity assumption). With the Benaloh-Fischer cryptosystem, the ciphertext corresponding to a k-bit message is short but the decryption process is now demanding. In practice, the scheme is therefore limited to small values of k, say $k < 40$.

The Benaloh-Fischer cryptosystem was subsequently extended by Naccache and Stern [39]. They observe that the decryption can be sped up by rather considering a product of small (odd) primes $R = \prod_i r_i$ such that $r_i \mid \phi(N)$ but $r_i^2 \nmid \phi(N)$ for each prime r_i. Given a ciphertext, the plaintext m is reconstructed from $m_i := m \bmod r_i$ through Chinese remaindering. The advantage is that each m_i is searched in the subspace $[0, r_i)$ instead of the entire message space. A variant of this technique was used by Groth [22].

Other generalizations and extensions of the Goldwasser-Micali cryptosystem but without formal security analysis can be found in [53,32,44]. More recently, Monnerat and Vaudenay developed applications using the more general theory

of characters [38,37], specifically with characters of order ≤ 4. Related cryptosystems are described in [49,48]. Yet another, different approach was proposed by Okamoto and Uchiyama [42], who suggested to use moduli of the form $N = p^2q$. This allows encrypting messages of size up to $\log_2 p$ bits. This was later extended by Paillier [43] to the setting $N = p^2q^2$. In 2005, Boneh, Goh and Nissim [10] showed an additively homomorphic system also supporting one multiplication.

A useful application of additive homomorphic encryption schemes resides in the construction of *lossy trapdoor functions* (or LTDFs in short). These functions, as introduced by Peikert and Waters [45], are function families wherein injective functions are computationally indistinguishable from *lossy* functions, which lose many bits of information about their input. LTDFs have proved to be very powerful and versatile in the cryptographer's toolbox. They notably imply chosen-ciphertext-secure public-key encryption [45], deterministic encryption [2,9] as well as cryptosystems that retain some security in the absence of reliable randomness [3] or in the presence of selective-opening adversaries [4].

Our Contributions

New Homomorphic Cryptosystem. We suggest an improvement of the original Goldwasser-Micali cryptosystem. It can be seen as a follow-up of the earlier works due to Benaloh and Fischer [12] and Naccache and Stern [39]. Before discussing it, we quote from [39]:

> *"Although the question of devising new public-key cryptosystems appears much more difficult [. . .] we feel that research in this direction is still in order: simple yet efficient constructions may have been overlooked."*

It is striking that the generalized cryptosystem in this paper was not already proposed because, as will become apparent (cf. Section 3), it turns out to be a very natural generalization. Our approach consists in considering n^{th}-power residues modulo N with $n = 2^k$ (the Goldwasser-Micali system corresponds to the case $k = 1$). This presents certain advantages. First, the resulting cryptosystem is bandwidth-efficient. Only $\log_2 N$ bits are needed for encrypting a k-bit message in typical applications (e.g., using the KEM/DEM paradigm). Second, the decryption process is very fast, even faster than in the Naccache-Stern cryptosystem. Searches are no longer needed (not even in smaller subspaces) in the decryption algorithm as plaintext messages can be recovered bit by bit. Third, the underlying complexity assumption is similar. The proposed cryptosystem is shown to be secure under the quadratic residuosity assumption for RSA moduli $N = pq$ such that $p, q \equiv 1 \pmod{2^k}$.

We also note that, similarly to the Goldwasser-Micali cryptosystem, our generalized cryptosystem enjoys an additive property known as *homomorphic encryption*. If c_1 and c_2 denote two ciphertexts corresponding to k-bit plaintexts m_1 and m_2, respectively, then $c_1 \cdot c_2 \pmod{N}$ is an encryption of the message $m_1 + m_2 \pmod{2^k}$. This reveals useful in several applications like voting schemes. An interesting extension would be to thresholdize it as was done in [29].

As another useful property, the new scheme also inherits the selective opening security[1] [16,4] of the Goldwasser-Micali system (in the sense of a simulation-based definition given in [4]). We actually prove its semantic security by showing that its public key is indistinguishable from a so-called *lossy* key for which encryptions reveal nothing about the encrypted message.

We thus believe our system to provide an interesting competitor to Paillier's cryptosystem for certain applications. As a salient example, we show that it provides a dramatically improved lossy trapdoor function based on a quadratic residuosity assumption.

NEW EFFICIENT LOSSY TRAPDOOR FUNCTIONS. The initial LTDF realizations [45] were based on the Decision Diffie-Hellman and Learning-with-Error [47] assumptions. More efficient examples based on the Composite Residuosity assumption were given in [9,17,18] while Kiltz *et al.* [30] showed that the RSA permutation provides a lossy function. Under the quadratic residuosity assumption, three distinct constructions were put forth in [23,17,18,51]. Those of Freeman *et al.* [17,18] and of Wee [51] must be used in combination with the results of Mol and Yilek [36] as they only lose single bits of information about the input. Hemenway and Ostrovsky [23] suggested a more efficient realization, of which Wee's framework [51] is a generalization. While their QR-based LTDF has found applications in the design of deterministic encryption schemes [11], it is conceptually very similar to the Peikert-Waters matrix-based schemes and suffers from similarly large outputs and descriptions.

We show that our variant of the Goldwasser-Micali cryptosystem drastically improves the efficiency of the Hemenway-Ostrovsky LTDF. Specifically, it reduces the length of the output (resp. the description of the function) by a factor of $O(\kappa)$ (resp. $O(\kappa^2)$), where κ is the security parameter. By appropriately selecting the parameters, we obtain evaluation keys and outputs consisting of a constant number of $\mathbb{Z}_N^*$ elements (and thus $O(\kappa)$ bits, instead of $O(\kappa^2)$ or $O(\kappa^3)$ as in the previous constructions). We thus obtain a QR-based LTDF, whose efficiency is competitive with Paillier-based realizations [9,17,18]. These improvements carry over to the deterministic encryption setting, when the Hemenway-Ostrovsky LTDF is used as a building block of the Brakerski-Segev system [11].

2 Background

We review some useful background and fix the notation. In particular, we define the *n*-th power residue symbol. We refer the reader to [25,50,52] for further details on (quadratic) residuosity. More information about encryption schemes can be found in textbooks in cryptography; e.g. [21,28].

---

[1] This notion refers to an attack scenario where the adversary is given t encryptions of possibly correlated messages, opens $t/2$ out of these (and thereby obtains the messages *and* encryption coins) before attempting to harm the security of remaining ciphertexts.

2.1 n^{th}-Power Residues

Let $N \in \mathbb{N}$. For each integer $n \geq 2$, we define $(\mathbb{Z}_N^*)^n = \{x^n \mid x \in \mathbb{Z}_N^*\}$ the set of nth-*power residues modulo N*. If the relation $a = x^n$ has no solution in $\mathbb{Z}_N^*$ then a is called a nth-*power non-residue modulo N*. Suppose that p is an odd prime. For any integer a with $\gcd(a, p) = 1$, it is easily verified that a is a nth-power residue modulo p if and only if

$$a^{\frac{p-1}{\gcd(n, p-1)}} \equiv 1 \pmod{p} \ .$$

When $n = 2$ (and so $\gcd(n, p-1) = 2$), this is known as Euler's criterion. It allows one to distinguish quadratic residues from quadratic non-residues. This defines the *Legendre symbol*. There are several ways to generalize the Legendre symbol (see [33]). In this paper, we consider the n-th power residue symbol for a divisor n of $(p-1)$, as presented in [52, Definition 1.6.21].

Definition 1. *Let p be an odd prime and let $n \geq 2$ such that $n \mid p - 1$. Then the symbol*

$$\left(\frac{a}{p}\right)_n = a^{\frac{p-1}{n}} \text{ mods } p$$

is called the n-th power residue symbol modulo p, where $a^{\frac{p-1}{n}} \text{ mods } p$ represents the absolute smallest residue of $a^{\frac{p-1}{n}}$ modulo p (namely, the complete set of absolute smallest residues are: $-(p-1)/2, \ldots, -1, 0, 1, \ldots, (p-1)/2$).

2.2   Quadratic Residuosity

Let $N = pq$ be the product of two (odd) primes p and q. For an integer a co-prime to N, the *Jacobi symbol* is the product of the corresponding Legendre symbols, namely $\left(\frac{a}{N}\right) = \left(\frac{a}{p}\right)\left(\frac{a}{q}\right)$. This gives rise to the multiplicative group $\mathbb{J}_N$ of integers whose Jacobi symbol is 1, $\mathbb{J}_N = \{a \in \mathbb{Z}_N^* \mid \left(\frac{a}{N}\right)_2 = 1\}$. A relevant subset of $\mathbb{J}_N$ is the set of quadratic residues modulo N, $\mathbb{QR}_N = \{a \in \mathbb{Z}_N^* \mid \left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)_2 = 1\}$.

The *Quadratic Residuosity* (QR) assumption says that, given a random element $a \in \mathbb{J}_N$, it is hard to decide whether $a \in \mathbb{QR}_N$ if the prime factors of N are unknown. To emphasize that this should hold for moduli $N = pq$ with $p, q \equiv 1 \pmod{2^k}$, we will refer to it as the k-QR *assumption*. Formally, we have:

Definition 2 (Quadratic Residuosity Assumption). *Let* RSAGen *be a probabilistic algorithm which, given a security parameter κ, outputs primes p, q such that $p \equiv q \equiv 1 \pmod{2^k}$, and their product $N = pq$. The* Quadratic Residuosity (QR) *assumption asserts that the function* $\mathbf{Adv}_{\mathcal{D}}^{\text{QR}}(1^\kappa)$, *defined as the distance*

$$\left| \Pr[\mathcal{D}(x, N) = 1 \mid x \xleftarrow{R} \mathbb{QR}_N] - \Pr[\mathcal{D}(x, N) = 1 \mid x \xleftarrow{R} \mathbb{J}_N \setminus \mathbb{QR}_N] \right|$$

is negligible for any probabilistic polynomial-time distinguisher $\mathcal{D}$; the probabilities are taken over the experiment of running $(N, p, q) \leftarrow$ RSAGen(1^κ) and choosing at random $x \in \mathbb{QR}_N$ and $x \in \mathbb{J}_N \setminus \mathbb{QR}_N$.

3   A New Public-Key Encryption Scheme

We generalize the Goldwasser-Micali cryptosystem so that it can efficiently support the encryption of larger messages while remaining additively homomorphic.

3.1   Description

The setting is basically the same as for the Goldwasser-Micali cryptosystem. The only additional requirement is that primes p and q are chosen congruent to 1 modulo 2^k where k denotes the bit-size of the messages being encrypted.

   In more detail, our encryption scheme is the tuple (KeyGen, Encrypt, Decrypt) defined as follows.

KeyGen(1^κ) Given a security parameter κ, KeyGen defines an integer $k \geq 1$, randomly generates primes $p, q \equiv 1 \pmod{2^k}$, and sets $N = pq$. It also picks $y \in \mathbb{J}_N \setminus \mathbb{QR}_N$. The public and private keys are $pk = \{N, y, k\}$ and $sk = \{p\}$.

Encrypt(pk, m) Let $\mathcal{M} = \{0,1\}^k$. To encrypt a message $m \in \mathcal{M}$ (seen as an integer in $\{0, \ldots, 2^k - 1\}$), Encrypt picks a random $x \in \mathbb{Z}_N^*$ and returns the ciphertext $c = y^m x^{2^k} \bmod N$.

Decrypt(sk, c) Given $c \in \mathbb{Z}_N^*$ and the private key $sk = \{p\}$, the algorithm first computes $z = \left(\frac{c}{p}\right)_{2^k}$ and then finds $m \in \{0, \ldots, 2^k - 1\}$ such that the relation

$$\left[\left(\frac{y}{p}\right)_{2^k}\right]^m = z \quad (\text{mods } p)$$

holds. An efficient method to recover message m in a bit-by-bit fashion is detailed in the next section (§ 3.2).

The correctness is easily verified by observing that $\alpha := \left(\frac{y}{p}\right)_{2^k}$ has order 2^k as an element in $\mathbb{Z}_p^*$. Indeed, letting $n = \text{ord}_p(\alpha)$ the order of α, we have $n \mid 2^k$ since, by definition, $\alpha \equiv y^{\frac{p-1}{2^k}} \pmod{p}$. But n cannot be equal to $2^{k'}$ for some $k' < k$ because $\alpha^{2^{k'}} \equiv 1 \pmod{p}$ would imply $y^{\frac{p-1}{2}} \equiv 1 \pmod{p}$, which contradicts the assumption that $y \in \mathbb{J}_N \setminus \mathbb{QR}_N \iff \left(\frac{y}{p}\right) = \left(\frac{y}{q}\right) = -1$. The decryption algorithm recovers the unique $m \in \{0, \ldots, 2^k - 1\}$ such that $\alpha^m \equiv z \pmod{p}$.

Remark 1. We notice that the case $k = 1$ corresponds to the Goldwasser-Micali cryptosystem. Indeed, the 2^k-th power residue symbol is then the classical Legendre symbol and the assumption $p, q \equiv 1 \pmod{2^k}$ is trivially verified.

3.2   Fast Decryption

At first glance, from the above description, it seems that the decryption process amounts to a search through the entire message space $\{0, 1\}^k$, similarly to some earlier cryptosystems. But we can do better. One of the main advantages of the proposed cryptosystem is that it provides an efficient way to recover the message.

Hence, it remains practical, even for large values of k. The decryption algorithm proceeds similarly to the Pohlig-Hellman algorithm [46] and is detailed below.

Algorithm 1. Decryption algorithm

Input: Ciphertext c, private key p (and public-key elements y and k)
Output: Plaintext $m = (m_{k-1}, \ldots, m_0)_2$

1: $m \leftarrow 0; B \leftarrow 1$
2: **for** $i = 1$ to k **do**
3:     $z \leftarrow \left(\frac{c}{p}\right)_{2^i}; t \leftarrow \left(\frac{y}{p}\right)_{2^i}^m \text{ mods } p$
4:     **if** $(t \neq z)$ **then** $m \leftarrow m + B$
5:     $B \leftarrow 2B$
6: **end for**
7: **return** m

The message $m \in \{0, 1\}^k$ is viewed as a k-bit integer given by its binary expansion $m = \sum_{i=0}^{k-1} m_i \, 2^i$, with $m_i \in \{0, 1\}$. Given $c = y^m x^{2^k} \bmod N$, we have

$$
\left(\frac{c}{p}\right)_{2^i} = \left(\frac{y^m x^{2^k}}{p}\right)_{2^i} = \left(\frac{y^{\sum_{j=0}^{i-1} m_j \, 2^j}}{p}\right)_{2^i} = \left(\frac{y}{p}\right)_{2^i}^{\sum_{j=0}^{i-1} m_j \, 2^j} \qquad (\text{mods } p)
$$

since $y^m x^{2^k} = y^{\sum_{j=0}^{i-1} m_j \, 2^j} \cdot \left(y^{\sum_{j=i}^{k-1} m_j \, 2^{j-i}} x^{2^{k-i}}\right)^{2^i}$, for $1 \leq i \leq k$. As a result, m can be recovered bit by bit using p, starting from the rightmost bit. The algorithm uses an accumulator B which contains the successive powers of 2.

3.3 Security Analysis

We prove that the scheme provides indistinguishable encryptions under the k-QR assumption. The case $k = 1$ corresponds to the Goldwasser-Micali cryptosystem and the standard Quadratic Residuosity assumption. So, we henceforth assume $k \geq 2$. In this case, since $p, q \equiv 1 \pmod{2^k}$, we know that $p, q \equiv 1 \pmod 4$ and $\left(\frac{-1}{p}\right) = \left(\frac{-1}{q}\right) = 1$. This implies that the square roots of an element in $\mathbb{QR}_N$ all have the same Jacobi symbol.

The k-QR assumption states that, without knowing the factorization of N, random elements of $\mathbb{QR}_N$ are computationally indistinguishable from random elements of $\mathbb{J}_N \setminus \mathbb{QR}_N$. Here, it will be convenient to consider a *gap* variant of the k-QR assumption. We chose the terminology "gap" (not to be confused with computational problems which have an easy decisional counterpart [41]) by analogy with certain lattice problems, where not every instance is a YES or NO instance since a gap exists between these.

Definition 3 (Gap 2^k-Residuosity Assumption). *Let $N = pq$ be the product of two large primes p and q with $p, q \equiv 1 \pmod{2^k}$. The* Gap 2^k-Residuosity *($\mathsf{Gap\text{-}2^k\text{-}Res}$) problem in $\mathbb{Z}_N^*$ is to distinguish the distribution of the following two sets given only $N = pq$:*

$$V_0 = \{x \in \mathbb{J}_N \setminus \mathbb{QR}_N\} \quad and \quad V_1 = \{y^{2^k} \bmod N \mid y \in \mathbb{Z}_N^*\} \ .$$

The Gap 2^k-Residuosity *assumption posits that the advantage $\mathbf{Adv}_{\mathcal{D}}^{\mathsf{Gap\text{-}2^k\text{-}Res}}(1^\kappa)$ of any PPT distinguisher $\mathcal{D}$, defined as the distance*

$$\left| \Pr[\mathcal{D}(x, k, N) = 1 \mid x \xleftarrow{R} V_0] \ - \Pr[\mathcal{D}(x, k, N) = 1 \mid x \xleftarrow{R} V_1] \right|$$

where probabilities are taken over all coin tosses, is negligible.

The latter assumption was independently considered by Abdalla, Ben Hamouda and Pointcheval [1] who used it to provide tighter security proofs for forward-secure signatures. Our result thus implies that their tighter reduction holds under the more standard k-QR assumption.

In the above definition, we explicitly give k to the distinguisher and remark that this information should be of little help considering that it can always be guessed with non-negligible probability. Also observe that from $p, q \equiv 1 \pmod{2^k}$, it follows that $2^k \mid N - 1$.

Theorem 1 (k-QR $\implies$ $\mathsf{Gap\text{-}2^k\text{-}Res}$). *The Quadratic Residuosity assumption implies the Gap 2^k-Residuosity assumption. More precisely, for any PPT distinguisher $\mathcal{B}_0$ against the former, there exists a QR distinguisher $\mathcal{B}_1$ with comparable running time and for which $\mathbf{Adv}_{\mathcal{B}_0}^{\mathsf{Gap\text{-}2^k\text{-}Res}}(1^\kappa) \leq 4 \cdot k \cdot \mathbf{Adv}_{\mathcal{B}_1}^{k\text{-}\mathsf{QR}}(1^\kappa).$*

Proof. The proof is given in the full version of the paper.   □

It is not hard to see that the semantic security of the scheme is equivalent to the $\mathsf{Gap\text{-}2^k\text{-}Res}$ assumption. We thus obtain the following theorem as a corollary.

Theorem 2. *The scheme is semantically secure under the k-QR assumption. More precisely, for any $\mathsf{IND\text{-}CPA}$ adversary $\mathcal{A}$, we have a k-QR distinguisher $\mathcal{B}$ such that $\mathbf{Adv}_{\mathcal{A}}^{\mathsf{ind\text{-}cpa}}(1^\kappa) \leq 4 \cdot k \cdot \mathbf{Adv}^{k\text{-}\mathsf{QR}}(\mathcal{B}).$*

Proof. The proof proceeds by simply changing the distribution of the public key. Under the $\mathsf{Gap\text{-}2^k\text{-}Res}$ assumption, instead of picking y uniformly in $\mathbb{J}_N \setminus \mathbb{QR}_N$, we can choose it in the subgroup of 2^k-th residue without the adversary noticing. However, in this case, the ciphertext carries no information about the message and the $\mathsf{IND\text{-}CPA}$ adversary has no advantage.   □

Interestingly, the proof of Theorem 2 implicitly shows that, like the original Goldwasser-Micali system, our scheme is a *lossy* encryption scheme [4] (*i.e.*, it admits an alternative distribution of public keys for which encryptions statistically hide the plaintext), which provides security guarantees against selective-opening attacks [16]. Moreover, for a lossy key (y, N), there exists an efficient algorithm that opens a given ciphertext c to any arbitrary plaintext m (by finding random coins that explain c as an encryption of m). It implies that our scheme satisfies the simulation-based definition [4] of selective-opening security.

4   Implementation and Performance

We detail here some implementation aspects. We explain how to select the parameters involved in the system set-up and key generation. Finally, we discuss the ciphertext expansion and give a comparison with previous schemes.

4.1   Parameter Selection

The key generation (cf. §3.1) requires two primes p and q such that $p, q \equiv 1$ (mod 2^k) and an element $y \in \mathbb{J}_N \setminus \mathbb{QR}_N$, where $N = pq$. The condition $y \in \mathbb{J}_N \setminus \mathbb{QR}_N$ is equivalent to $\left(\frac{y}{p}\right) = \left(\frac{y}{q}\right) = -1$. So, we need to generate an element $y \in \mathbb{Z}_N^*$ such that (i) $y \bmod p$ is primitive in $\mathbb{Z}_p^*$, and (ii) $y \bmod q$ is primitive in $\mathbb{Z}_q^*$. Finding a primitive element modulo a prime number p is not difficult when the factorization of $p - 1$ is known. Therefore, we suggest to select prime p as a *k-quasi-safe prime*, that is, $p = 2^k p' + 1$ for some prime p' (likewise for prime q, we take $q = 2^k q' + 1$ for some prime q'). An efficient algorithm for generating k-quasi-safe primes is discussed in [27, Section 4.2].

Consider now the primitive 2^k-th root of unity $\zeta_{2^k} = e^{2i\pi/2^k}$ with $i = \sqrt{-1}$. It generates a cyclic group of order 2^k under multiplication. In our case, the key observation is that, when p is 2^k-quasi-safe prime, if y is a square modulo p then $\zeta_{2^k} y$ is not. Indeed, we have

$$\left(\frac{\zeta_{2^k} y}{p}\right) = \left(\frac{\zeta_{2^k}}{p}\right)\left(\frac{y}{p}\right) \equiv \zeta_{2^k}^{\frac{p-1}{2}} \left(\frac{y}{p}\right) \equiv (e^{i\pi})^{p'} \left(\frac{y}{p}\right) = -\left(\frac{y}{p}\right) \pmod{p}$$

since p' is odd. This leads to the following algorithm.

Algorithm 2. Generation of y

Input: Modulus $N = pq$ (with $p = 2^k p' + 1$ and $q = 2^k q' + 1$), primes p, q, p', q', and integer $k \geq 1$
Output: $y \in \mathbb{J}_N \setminus \mathbb{QR}_N$

1: Pick at random $y_p \in \mathbb{Z}_p^*$ and $y_q \in \mathbb{Z}_q^*$
2: **if** $\left(\frac{y_p}{p}\right) = 1$ **then** $y_p \leftarrow \zeta_{2^k} y_p \bmod p$
3: **if** $\left(\frac{y_q}{q}\right) = 1$ **then** $y_q \leftarrow \zeta_{2^k} y_q \bmod q$
4: Set $y \leftarrow y_p + p\big(p^{-1}(y_q - y_p) \bmod q\big)$
5: **return** y

The primes p and q are chosen so that $p, q \equiv 1$ (mod 2^k). Sharing common factors for $(p - 1)$ and $(q - 1)$ was used already in several other systems; see e.g. [19,34]. Letting r denote a common factor of $(p-1)$ and $(q-1)$, a baby-step giant-step approach developed by McKee and Pinch [35] can factor RSA modulus $N = pq$ in essentially $O(N^{1/4}/r)$ operations. In our case, we have $r = 2^k$. For security it is therefore necessary that $\frac{1}{4} \log_2 N - k > \kappa$, or equivalently,

$$k < \tfrac{1}{4} \log_2 N - \kappa$$

where κ is the security parameter.

A powerful LLL-based technique due to Coppersmith [13,14] also bounds the size of k to at most $\frac{1}{2}\min(\log_2 p, \log_2 q)$ bits as, otherwise, the factors of N would be revealed. Going beyond polynomial-time attacks, one should add an extra security margin to take into account exhaustive searches [40]. RSA moduli being balanced (i.e., $\frac{1}{2}\min(\log_2 p, \log_2 q) = \frac{1}{4}\log_2 N$), we so end up with the same upper bound as for the McKee-Pinch's approach: $k < \frac{1}{4}\log_2 N - \kappa$.

In practice, this restriction on k is not a limitation because, as described in the next section, long messages can be encrypted using the KEM/DEM paradigm. For example, a specific parameter choice is $k = 128$ and $\log_2 N = 2048$.

4.2   Ciphertext Expansion

Hybrid encryption allows designing efficient asymmetric schemes, as suggested by Shoup in the ISO 18033-2 standard for public-key encryption [26]. An asymmetric cryptosystem is used to encrypt a secret key that is then used to encrypt the actual message. This is the so-called *KEM/DEM paradigm*.

The next table compares the ciphertext expansion in the encryption of k-bit messages for different generalized Goldwasser-Micali cryptosystems. Only cryptosystems with a formal security analysis are considered. Further, the value of k is assumed to be relatively small (e.g., 128 or 256) as the "message" being encrypted is typically a symmetric key (for example a 128- or 256-bit AES key) in a KEM/DEM construction.

Table 1. Ciphertext expansion in a typical encryption

Encryption scheme	Assumption	Ciphertext size
Goldwasser-Micali [20]	Quadratic Residuosity (QR)	$k \cdot \log_2 N$
Benaloh-Fisher [12]	Prime residuosity (PR)	$\left\lceil \frac{k}{\log_2 r} \right\rceil \cdot \log_2 N$
Naccache-Stern [39]	Prime residuosity (PR)	$\log_2 N$
Okamoto-Uchiyama [42]	p-subgroup	$\log_2 N$
Paillier [43]	N-th residuosity	$2\log_2 N$
This paper	Quadratic residuosity (k-QR)	$\log_2 N$

It appears that the Goldwasser-Micali cryptosystem has the higher ciphertext expansion but its semantic security relies on the standard quadratic residuosity assumption. The ciphertext expansion of Benaloh-Fischer cryptosystem is similar to that of Naccache-Stern cryptosystem for *small* messages; i.e., when $k \leq \log_2 r$. For larger messages, the Naccache-Stern cryptosystem should be preferred. It also offers the further advantage of providing a faster decryption procedure. The same is true for the Okamoto-Uchiyama cryptosystem. The Paillier cryptosystem produces twice larger ciphertexts.

The encryption scheme proposed in this paper has the same ciphertext expansion as in the Naccache-Stern cryptosystem. Moreover, its decryption algorithm is fast (it is even faster than in the Naccache-Stern cryptosystem), requires less memory, and the security relies on a quadratic residuosity assumption.

5   More Efficient Lossy Trapdoor Functions from the k-Quadratic Residuosity Assumption

In this section, we show that our homomorphic cryptosystem allows constructing a lossy trapdoor function based on the k-QR assumption with much shorter outputs and keys than in previous QR-based examples.

In comparison with the function of Hemenway and Ostrovsky [23], it compresses function values by a factor of k when we work with a modulus $N = pq$ such that $p \equiv q \equiv 1 \pmod{2^k}$. Moreover, the size of the evaluation key is decreased by a factor of $O(k^2)$ while increasing the lossiness by $2k$ more bits. Finally, our inversion trapdoor has constant size, whereas [23] uses a trapdoor of size $O(n)$ to recover n-bit inputs. Our function also compares favorably with the QR-based function of Freeman *et al.* [17,18], which only loses a single bit.

In fact, by appropriately tuning our construction, we obtain the first QR-based lossy trapdoor function with short outputs and keys that loses many input bits. Among known lossy trapdoor functions based on traditional number-theoretic assumptions [45,9,17,18,30,23,36], this appears as a rare efficiency tradeoff. To the best of our knowledge, it has only been achieved under the Composite Residuosity assumption [9,17,18] so far.

Interestingly, our LTDF provides similar efficiency improvements to the QR-based deterministic encryption scheme of Brakerski and Segev [11], which also builds on the Hemenway-Ostrovsky LTDF. Note that the scheme of [11] is important in the deterministic encryption literature since it is one of the only known schemes providing security in the auxiliary input setting in the standard model.

5.1   Description and Security Analysis

We start by recalling the following definition.

Definition 4 ([45]). *Let $\kappa \in \mathbb{N}$ be a security parameter and $n : \mathbb{N} \to \mathbb{N}$, $\ell : \mathbb{N} \to \mathbb{R}$ be non-negative functions of κ. A collection of (n, ℓ)-lossy trapdoor functions (LTDF) is a tuple of efficient algorithms (InjGen, LossyGen, Eval, Invert) with the following specifications.*

- Sampling an injective function: *Given a security parameter κ, the randomized algorithm* InjGen(1^κ) *outputs the index ek of an injective function of the family and an inversion trapdoor t.*
- Sampling a lossy function: *Given a security parameter κ, the probabilistic algorithm* LossyGen(1^κ) *outputs the index ek of a lossy function.*
- Evaluation: *Given the index of a function ek —produced by either* InjGen *or* LossyGen— *and an input $x \in \{0, 1\}^n$, the evaluation algorithm* Eval *outputs $F_{ek}(x)$ such that:*
  - *If ek is an output of* InjGen, *then $F_{ek}(\cdot)$ is an injective function.*
  - *If ek was produced by* LossyGen, *then $F_{ek}(\cdot)$ has image size $2^{n-\ell}$. In this case, the value $n - \ell$ is called* residual leakage.

- Inversion: *For any pair (ek, t) produced by InjGen and any input $x \in \{0,1\}^n$, the inversion algorithm Invert returns $F_{ek}^{-1}(t, F_{ek}(x)) = x$.*
- Security: *The two ensembles $\{ek \mid (ek, t) \leftarrow \mathsf{InjGen}(1^\kappa)\}_{\kappa \in \mathbb{N}}$ and $\{ek \mid ek \leftarrow \mathsf{LossyGen}(1^\kappa)\}_{\kappa \in \mathbb{N}}$ are computationally indistinguishable.*

Our construction goes as follows.

SAMPLING AN INJECTIVE FUNCTION. Given a security parameter κ, let $\ell_N(\kappa)$ and $k(\kappa)$ be security parameters determined by κ. Let also $n(\kappa)$ be the desired input length. Algorithm InjGen defines $m = n/k$ (we assume that k divides n for simplicity) and conducts the following steps.

1. Generate a modulus $N = pq > 2^{\ell_N}$ such that $p = 2^k p' + 1$ and $q = 2^k q' + 1$ for primes p, q and odd co-prime integers p', q'. Choose $y \xleftarrow{R} \mathbb{J}_N \setminus \mathbb{QR}_N$.
2. For each $i \in \{1, \ldots, m\}$, pick h_i in the subgroup of order $p'q'$, by setting $h_i = g_i^{2^k} \bmod N$ for a randomly chosen $g_i \xleftarrow{R} \mathbb{Z}_N^*$.
3. Choose $r_1, \ldots, r_m \xleftarrow{R} \mathbb{Z}_{p'q'}$ and compute a matrix $Z = \big(Z_{i,j}\big)_{i,j \in \{1,\ldots,m\}}$ given by

$$Z = \begin{pmatrix} y^{z_{1,1}} \cdot h_1^{r_1} \bmod N \ \ldots\ldots\ y^{z_{1,m}} \cdot h_m^{r_1} \bmod N \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ y^{z_{m,1}} \cdot h_1^{r_m} \bmod N \ldots\ldots\ y^{z_{m,m}} \cdot h_m^{r_m} \bmod N \end{pmatrix},$$

where $(z_{i,j})_{i,j \in \{1,\ldots,m\}}$ denotes the identity matrix.

The evaluation key is $ek := \big(N, (Z_{i,j})_{i,j \in \{1,\ldots,m\}}\big)$ and the trapdoor is $t := p$.

SAMPLING A LOSSY FUNCTION. The process followed by LossyGen is identical to the above one but the matrix $(z_{i,j})_{i,j \in \{1,\ldots,m\}}$ is replaced by the all-zeroes $m \times m$ matrix.

EVALUATION. Given $ek = \big(N, (Z_{i,j})_{i,j \in \{1,\ldots,m\}}\big)$, algorithm Eval parses the input $x \in \{0,1\}^n$ as a vector of k-bit blocks $\tilde{x} = (x_1, \ldots, x_m)$, with $x_i \in \mathbb{Z}_{2^k}$ for each i. Then, it computes and returns $\tilde{y} = (y_1, \ldots, y_m)$, with $y_j \in \mathbb{Z}_N^*$, where

$$\tilde{y} = \Big(\prod_{i=1}^m Z_{i,1}^{x_i} \bmod N, \ldots, \prod_{i=1}^m Z_{i,m}^{x_i} \bmod N\Big)$$
$$= \Big(y^{\sum_{i=1}^m z_{i,1} x_i} \cdot h_1^{\sum_{i=1}^m r_i x_i} \bmod N, \ldots, y^{\sum_{i=1}^m z_{i,m} x_i} \cdot h_m^{\sum_{i=1}^m r_i x_i} \bmod N\Big).$$

INVERSION. Given $t = p$ and $\tilde{y} = (y_1, \ldots, y_m) \in \mathbb{Z}_N^m$, Invert applies the decryption algorithm of §3.2 to each y_j, for $j = 1$ to m. Observe that when $(z_{ij})_{i,j \in \{1,\ldots,m\}}$ is the identity matrix, $\left(\frac{y_j}{p}\right)_{2^k} \equiv \left[\left(\frac{y}{p}\right)_{2^k}\right]^{x_j} \pmod{p}$. From the resulting vector of plaintexts $\tilde{x} = (x_1, \ldots, x_m) \in \mathbb{Z}_{2^k}^m$, it recovers the input $x \in \{0,1\}^n$.

The Hemenway-Ostrovsky construction of [23] is slightly different in that, as in the DDH-based construction of Peikert and Waters [45], the evaluation key

includes a vector of the form $G = (g^{r_1}, \ldots, g^{r_m})^T$, where $g \in \mathbb{QR}_N$, and the trapdoor is $t = (\log_g(h_1), \ldots, \log_g(h_m))$. In their scheme, the evaluation algorithm additionally computes $\prod_{i=1}^{m} (g^{r_i})^{x_i}$ while the inversion algorithm does not use the factorization of N but rather performs a coordinate-wise ElGamal decryption. Here, explicitly using the factorization of N in the inversion algorithm makes it possible to process k-bit blocks at once.

Theorem 3. *The above construction is a $(n(\kappa), n(\kappa) - \log_2(p'q'))$-LTDF if the k-QR assumption holds.*

Proof. The proof is given in the full version of the paper. $\square$

It is worth noting that, with $N = pq$ such that $p \equiv q \equiv 1 \pmod{2^k}$, a side effect of working in the subgroup of odd order is an improved lossiness. Indeed, we lose $n - \log_2(p'q')$ bits in comparison with $n - \log_2 \phi(N)$ in [23].

Using the techniques of Peikert and Waters [45], it is easy to construct an equally efficient all-but-one trapdoor function providing the same amount of lossiness under the QR assumption. A difference is that, in order to enable inversion, the resulting all-but-one function handles $k/2$ bits (instead of k) in each chunk. The details are given in the full version of the paper.

More importantly, the dimension m of the matrix and the output vector can be reduced to a fairly small constant, as illustrated below.

5.2   Efficiency

Here, we consider chosen-ciphertext security as the targeted application.

By combining the lossy and all-but-one trapdoor function, a CCA-secure encryption scheme can be obtained using the construction of [45]. We argue that $m = O(1)$ suffices for this purpose. Recall that the scheme of [45] combines a pairwise independent hash function $H : \{0, 1\}^n \to \{0, 1\}^\tau$, an (n, ℓ)-lossy function and an (n, ℓ')-all-but-one function such that $\ell + \ell' \geq n + \nu$ and $\tau \geq \nu - 2 \log_2(1/\varepsilon)$, for some $\nu \in \omega(\log n)$ and where ε is the statistical distance in the modified Leftover Hash Lemma used in [15]. If we choose $\varepsilon \approx 2^{-\kappa}$ and $\tau = k$ in order to encrypt k-bit messages, we can set $\nu = k + 2\kappa$. Setting $\ell = \ell' = n - \log_2(p'q')$, the constraint $\ell + \ell' \geq n + \nu$ translates into $n - 2\log_2(p'q') \geq \nu$. If we set $k = \frac{1}{4} \log_2 N - \kappa$, we have $\log_2(p'q') = \log_2 \phi(N) - 2k \approx 4(k + \kappa) - 2k = 2k + 4\kappa$, which yields $n \geq 3k + 6\kappa$. If $k > \kappa$, it is sufficient to set $n \geq 9k$. If we take into account the fact that our all-but-one function processes blocks of $k/2$ bits, we find that $m = 2n/k = 18$ suffices here.

As it turns out, when the Peikert-Waters construction [45, §4.3] of CCA-secure encryption is instantiated with our lossy and all-but-one trapdoor functions, it only requires a constant number of exponentiations while retaining constant-size public keys and ciphertexts.

With the exception of [24] (which relies on a weaker assumption), to the best of our knowledge, it yields the only known CCA-secure QR-based cryptosystem combining the aforementioned efficiency properties. Up to now, the most efficient chosen-ciphertext-secure cryptosystem strictly based on the QR assumption was the one of Kiltz *et al.*[31], where $O(\kappa)$ exponentiations are needed to

encrypt and the public key contains $O(\kappa)$ group elements. On the other hand, our construction requires more specific moduli than [31].

6   Conclusion

This paper introduced a new generalization of the Goldwasser-Micali cryptosystem. The so-obtained cryptosystems are shown to be secure under the quadratic residuosity assumption. Further, they enjoy a number of useful features including fast decryption, optimal ciphertext expansion, and homomorphic property. We believe that our proposal is the most natural yet efficient generalization of the Goldwasser-Micali cryptosystem. It keeps the nice attributes and properties of the original scheme while improving the overall performance.

When applied to the Peikert-Waters framework for building lossy trapdoor functions, it yields a practical construction based on quadratic residuosity, with companion deterministic encryption scheme and CCA-secure cryptosystem.

References

1. Abdalla, M., Ben Hamouda, F., Pointcheval, D.: Tighter reductions for forward-secure signature schemes. In: Kurosawa, K., Hanaoka, G. (eds.) PKC 2013. LNCS, vol. 7778, pp. 292–311. Springer, Heidelberg (2013)
2. Bellare, M., Boldyreva, A., O'Neill, A.: Deterministic and efficiently searchable encryption. In: Menezes, A. (ed.) CRYPTO 2007. LNCS, vol. 4622, pp. 535–552. Springer, Heidelberg (2007)
3. Bellare, M., Brakerski, Z., Naor, M., Ristenpart, T., Segev, G., Shacham, H., Yilek, S.: Hedged public-key encryption: How to protect against bad randomness. In: Matsui, M. (ed.) ASIACRYPT 2009. LNCS, vol. 5912, pp. 232–249. Springer, Heidelberg (2009)
4. Bellare, M., Hofheinz, D., Yilek, S.: Possibility and impossibility results for encryption and commitment secure under selective opening. In: Joux, A. (ed.) EUROCRYPT 2009. LNCS, vol. 5479, pp. 1–35. Springer, Heidelberg (2009)
5. Benaloh, J.D.C.: Verifiable Secret-Ballot Elections. PhD thesis, Yale University, New Haven, CT, USA (1987)
6. Blum, L., Blum, M., Shub, M.: Comparison of two pseudo-random number generators. In: Chaum, D., Rivest, R.L., Sherman, A.T. (eds.) Advances in Cryptology: Proceedings of CRYPTO 1982, pp. 61–78. Plenum Press (1983)
7. Blum, L., Blum, M., Shub, M.: A simple unpredictable pseudo-random number generator. SIAM J. Comput. 15(2), 363–383 (1986)
8. Blum, M., Goldwasser, S.: An efficient probabilistic public-key encryption scheme which hides all partial information. In: Blakely, G.R., Chaum, D. (eds.) CRYPTO 1984. LNCS, vol. 196, pp. 289–299. Springer, Heidelberg (1985)
9. Boldyreva, A., Fehr, S., O'Neill, A.: On notions of security for deterministic encryption, and efficient constructions without random oracles. In: Wagner, D. (ed.) CRYPTO 2008. LNCS, vol. 5157, pp. 335–359. Springer, Heidelberg (2008)

10. Boneh, D., Goh, E.-J., Nissim, K.: Evaluating 2-DNF formulas on ciphertexts. In: Kilian, J. (ed.) TCC 2005. LNCS, vol. 3378, pp. 325–341. Springer, Heidelberg (2005)
11. Brakerski, Z., Segev, G.: Better security for deterministic public-key encryption: The auxiliary-input setting. In: Rogaway, P. (ed.) CRYPTO 2011. LNCS, vol. 6841, pp. 543–560. Springer, Heidelberg (2011)
12. Cohen, J.D., Fischer, M.J.: A robust and verifiable cryptographically secure election scheme. In: 26th Annual Symposium on Foundations of Computer Science (FOCS 1985), pp. 372–382. IEEE Computer Society (1985)
13. Coppersmith, D.: Finding a small root of a bivariate integer equation; factoring with high bits known. In: Maurer, U.M. (ed.) EUROCRYPT 1996. LNCS, vol. 1070, pp. 178–189. Springer, Heidelberg (1996)
14. Coppersmith, D.: Small solutions to polynomial equations, and low exponent RSA vulnerabilities. J. Cryptology 10(4), 233–260 (1997)
15. Dodis, Y., Reyzin, L., Smith, A.: Fuzzy extractors: How to generate strong keys from biometrics and other noisy data. In: Cachin, C., Camenisch, J.L. (eds.) EUROCRYPT 2004. LNCS, vol. 3027, pp. 523–540. Springer, Heidelberg (2004)
16. Dwork, C., Naor, M., Reingold, O., Stockmeyer, L.: Magic functions. Journal of the ACM 50(6), 852–921 (2003)
17. Freeman, D.M., Goldreich, O., Kiltz, E., Rosen, A., Segev, G.: More constructions of lossy and correlation-secure trapdoor functions. In: Nguyen, P.Q., Pointcheval, D. (eds.) PKC 2010. LNCS, vol. 6056, pp. 279–295. Springer, Heidelberg (2010)
18. Freeman, D.M., Goldreich, O., Kiltz, E., Rosen, A., Segev, G.: More constructions of lossy and correlation-secure trapdoor functions. J. Cryptology (2012)
19. Girault, M.: An identity-based identification scheme based on discrete logarithms modulo a composite number. In: Damgård, I.B. (ed.) EUROCRYPT 1990. LNCS, vol. 473, pp. 481–486. Springer, Heidelberg (1991)
20. Goldwasser, S., Micali, S.: Probabilistic encryption. J. Comput. Syst. Sci. 28(2), 270–299 (1984)
21. Golreich, O.: Foundations of Cryptography, vol. II. Cambridge University Press (2004)
22. Groth, J.: Cryptography in subgroups of Z_n^*. In: Kilian, J. (ed.) TCC 2005. LNCS, vol. 3378, pp. 50–65. Springer, Heidelberg (2005)
23. Hemenway, B., Ostrovsky, R.: Lossy trapdoor functions from smooth homomorphic hash proof systems. In: Electronic Colloquium on Computational Complexity, ECCC (2009)
24. Hofheinz, D., Kiltz, E.: Practical chosen ciphertext secure encryption from factoring. In: Joux, A. (ed.) EUROCRYPT 2009. LNCS, vol. 5479, pp. 313–332. Springer, Heidelberg (2009)
25. Ireland, K., Rosen, M.: *A Classical Introduction to Modern Number Theory*, 2nd edn. Graduate Texts in Mathematics, vol. 84. Springer (1990)
26. ISO/IEC 18033-2. Information technology – Security techniques – Encryption algorithms – Part 2: Asymmetric ciphers. International Organization for Standardization (May 2006)
27. Joye, M., Paillier, P.: Fast generation of prime numbers on portable devices: An update. In: Goubin, L., Matsui, M. (eds.) CHES 2006. LNCS, vol. 4249, pp. 160–173. Springer, Heidelberg (2006)
28. Katz, J., Lindell, Y.: Introduction to Modern Cryptography. CRC Press (2007)
29. Katz, J., Yung, M.: Threshold cryptosystems based on factoring. In: Zheng, Y. (ed.) ASIACRYPT 2002. LNCS, vol. 2501, pp. 192–205. Springer, Heidelberg (2002)

30. Kiltz, E., O'Neill, A., Smith, A.: Instantiability of RSA-OAEP under chosen-plaintext attack. In: Rabin, T. (ed.) CRYPTO 2010. LNCS, vol. 6223, pp. 295–313. Springer, Heidelberg (2010)
31. Kiltz, E., Pietrzak, K., Stam, M., Yung, M.: A new randomness extraction paradigm for hybrid encryption. In: Joux, A. (ed.) EUROCRYPT 2009. LNCS, vol. 5479, pp. 590–609. Springer, Heidelberg (2009)
32. Kurosawa, K., Katayama, Y., Ogata, W., Tsujii, S.: General public key residue cryptosystems and mental poker protocols. In: Damgård, I.B. (ed.) EUROCRYPT 1990. LNCS, vol. 473, pp. 374–388. Springer, Heidelberg (1991)
33. Lemmermeyer, F.: Reciprocity Laws. Springer Monographs in Mathematics. Springer (2000)
34. Lim, C.H., Lee, P.J.: Security and performance of server-aided RSA computation protocols. In: Coppersmith, D. (ed.) CRYPTO 1995. LNCS, vol. 963, pp. 70–83. Springer, Heidelberg (1995)
35. McKee, J., Pinch, R.: Further attacks on server-aided RSA cryptosystems (1998) (unpublished manuscript)
36. Mol, P., Yilek, S.: Chosen-ciphertext security from slightly lossy trapdoor functions. In: Nguyen, P.Q., Pointcheval, D. (eds.) PKC 2010. LNCS, vol. 6056, pp. 296–311. Springer, Heidelberg (2010)
37. Monnerat, J., Vaudenay, S.: Generic homomorphic undeniable signatures. In: Lee, P.J. (ed.) ASIACRYPT 2004. LNCS, vol. 3329, pp. 354–371. Springer, Heidelberg (2004)
38. Monnerat, J., Vaudenay, S.: Undeniable signatures based on characters: How to sign with one bit. In: Bao, F., Deng, R., Zhou, J. (eds.) PKC 2004. LNCS, vol. 2947, pp. 69–85. Springer, Heidelberg (2004)
39. Naccache, D., Stern, J.: A new public key cryptosystem based on higher residues. In: ACM Conference on Computer and Communications Security (CCS 1998), pp. 59–66. ACM Press (1998)
40. Nguyen, P.Q.: Public-key cryptanalysis. In: Luengo, I. (ed.) Recent Trends in Cryptography, Contemporary Mathematics. AMS–RSME (2009)
41. Okamoto, T., Pointcheval, D.: The gap-problems: A new class of problems for the security of cryptographic schemes. In: Kim, K.-C. (ed.) PKC 2001. LNCS, vol. 1992, pp. 308–318. Springer, Heidelberg (2001)
42. Okamoto, T., Uchiyama, S.: A new public-key cryptosystem as secure as factoring. In: Nyberg, K. (ed.) EUROCRYPT 1998. LNCS, vol. 1403, pp. 308–318. Springer, Heidelberg (1998)
43. Paillier, P.: Public-key cryptosystems based on composite degree residuosity classes. In: Stern, J. (ed.) EUROCRYPT 1999. LNCS, vol. 1592, pp. 223–238. Springer, Heidelberg (1999)
44. Park, S.J., Lee, B.Y., Won, D.H.: A probabilistic encryption using very high residuosity and its applications. In: Global Telecommunications Conference (GLOBECOM 1995), pp. 1179–1182. IEEE Press (1995)
45. Peikert, C., Waters, B.: Lossy trapdoor functions and their applications. In: Dwork, C. (ed.) 40th Annual ACM Symposium on Theory of Computing (STOC 2008), pp. 187–196. ACM Press (2008)
46. Pohlig, S.H., Hellman, M.E.: An improved algorithm for computing logarithms over $GF(p)$ and its cryptographic significance. IEEE Tran. Inf. Theory 24(1), 106–110 (1978)
47. Regev, O.: On lattices, learning with errors, random linear codes, and cryptography. In: 37th Annual ACM Symposium on Theory of Computing (STOC 2005), pp. 84–93. ACM Press (2005)

48. Scheidler, R.: A public-key cryptosystem using purely cubic fields. J. Cryptology 11(2), 109–124 (1998)
49. Scheidler, R., Williams, H.C.: A public-key cryptosystem utilizing cyclotomic fields. Des. Codes Cryptography 6(2), 117–131 (1995)
50. Shoup, V.: A Computational Introduction to Number Theory and Algebra, 2nd edn. Cambridge University Press (2010)
51. Wee, H.: Dual projective hashing and its applications — lossy trapdoor functions and more. In: Pointcheval, D., Johansson, T. (eds.) EUROCRYPT 2012. LNCS, vol. 7237, pp. 246–262. Springer, Heidelberg (2012)
52. Yan, S.Y.: Number Theory for Computing, 2nd edn. Springer (2002)
53. Zheng, Y., Matsumoto, T., Imai, H.: Residuosity problem and its applications to cryptography. Trans. IEICE E-71(8), 759–767 (1988)