Flooding Edge Weighted Graphs

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Abstract. This paper characterizes floodings on edge weighted graphs. Of particular interest are the highest floodings of a graph below a ceiling function defined on the nodes. Two classes of algorithms for their construction are presented. The first are applied on the dendrogram representing the hierarchy associated to the edge weighted graph. The second consist in shortest distance algorithms on the graph itself.

1 Introduction

Edge weighted graphs are ubiquitous in the field of classification and image processing. A hierarchy is easily derived from an edge weighted graph: cutting all edges with a weight above some threshold produces a number of connected subgraphs, representing each one scale of a taxonomy. For higher thresholds less edges are cut, resulting in larger subgraphs, obtained by the union of smaller ones. If the nodes represent the different tiles of a partition, and the edge weights represent a dissimilarity between adjacent tiles, the hierarchy is a series of nested partitions which are coarser and coarser, each tile at a given level, being obtained by the union of tiles at lower levels. In the region adjacency graph for instance, the nodes represent the catchment basins of a topographic surface, edges link neighboring basins and their weights represents the altitudes of the pass points between neighboring basins. If the topographic surface is flooded, the flood passes from basin to basin through these pass points. The progression of the flood is thus the same on the topographic surface or on the RAG. The present paper defines floodings on arbitrary edge weighted graphs. Criteria are given characterizing physically valid floodings. We then study the extension of a lake containing a given node when its flooding level increases. The highest flooding below a ceiling function defined on the nodes is unique. It has a great interest in image segmentation and filtering. Various algorithms are proposed for its construction.

2 The Laws of Hydrostatics and Floodings

2.1 Criteria Characterizing a Flooding

Consider a non oriented node and edge weighted graph $G = [E, N]$, E representing the edges and N the nodes.

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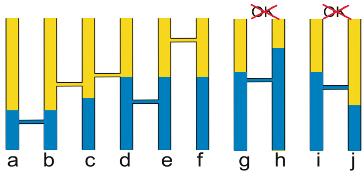

Fig. 1. Tank and pipe network:
- a and b form a regional minimum with $\tau_a = \tau_b = \lambda$; $e_{ab} \leq \lambda$; $e_{bc} > \lambda$
- b and c have unequal levels but are separated by a higher pipe.
- d and e form a full lake, reaching the level of its lowest exhaust pipe e_{cd}
- e and f have the same level ; however they do not form a lake, as they are linked by a pipe which is higher.
The distribution in the last four tanks is not compatible with the laws of hydrostatics.

In order to give a physical interpretation to our graph, we consider the nodes as vertical tanks of infinite height and depth. The weight τ_i represents the level of water in the tank i, equal to $-\infty$ if no water is present. Two neighboring tanks i and j are linked by a pipe at an altitude e_{ij} equal to the weight of the edge. We call such an edge weighted graph a tank network. Edge weights e and flooding levels τ take their values in $[-\infty, +\infty]$. We suppose that the laws of hydrostatics apply to our network of tanks and pipes:

* if the level τ_i in the tank i is higher than the pipe e_{ij}, then $\tau_i = \tau_j$.
* the level τ_i in the tank i cannot be higher than the level τ_j, unless $e_{ij} \geq \tau_i$.

In fact, this second condition implies the first one. We adopt it as a criterion defining valid floodings on a tank network.

Definition 1. *The distribution* τ *of water in the tanks of the graph* $G = [E, N]$ *is a flooding of this graph, i.e. is a stable distribution of fluid if it verifies the criterion { for any couple of neighboring nodes* (p, q) : $(\tau_p > \tau_q \Rightarrow e_{pq} \geq \tau_p)$ *(criterion* 1) *}*

Figure 1 presents a number of configurations compatible with the laws of hydrostatics and others which are not.

The following equivalences yield other useful criteria for recognizing flood distributions on tank networks:

$(\tau_p > \tau_q \Rightarrow e_{pq} \geq \tau_p) \Leftrightarrow (\text{not } (\tau_p > \tau_q) \text{ or } e_{pq} \geq \tau_p) \Leftrightarrow$
$(\tau_p \leq \tau_q \text{ or } \tau_p \leq e_{pq}) \Leftrightarrow (\tau_p \leq \tau_q \vee e_{pq}) \qquad (\textit{criterion } 2)$

2.2 The Algebra of Floodings

Lemma 1. *If* τ *and* ν *are two floodings of a tank network* G*, then* $\tau \vee \nu$ *and* $\tau \wedge \nu$ *also are floodings of* G*.*

2.3 Creation of Lakes

We first define the ultrametric flooding distance $\mathrm{ud}(p, q)$ between two nodes p and q on an edge weighted graph as the lowest value λ such that there exists a connected path between p and q with no edge higher than λ. The highest edge along the path has a weight λ. For a node p the closed ball of centre p and radius ρ is defined by $\overline{\text{Ball}}(p, \rho) = \{q \in N \mid \text{ud}(p, q) \leq \rho\}$. Such balls have strange properties:

- two closed balls with the same radius are either disjoint or identical.
- each element of a closed ball is a centre of this ball.
- the radius of a ball is equal to its diameter, that is the longest distance between two nodes in the ball.

Open balls $\text{Ball}(p, \rho) = \{q \in N \mid \text{ud}(p, q) < \rho\}$ have similar properties.

The following lemma presents the basic mechanism generating lakes.

Lemma 2. *If (p, q) are neighboring nodes of the flooded graph G, linked by an edge with weight $e_{pq} < \tau_p$, then $\tau_p = \tau_q$.*

Proof. Indeed the criterion $(\tau_p > \tau_q \Rightarrow e_{pq} \geq \tau_p)$ is equivalent with $(e_{pq} < \tau_p \Rightarrow \tau_p \leq \tau_q)$. Hence if $e_{pq} < \tau_p$, we have $\tau_p \leq \tau_q$; so we also have $e_{pq} < \tau_q$ implying $\tau_q \leq \tau_p$; finally $\tau_p = \tau_q$.

Consider now a node p with a flood level λ. In the open ball $\text{Ball}(p, \lambda)$ all neighboring nodes (s, t) are connected by an edge $e_{st} < \lambda$, hence $\tau_s = \tau_t = \lambda$ and the whole ball X is a lake with the same altitude λ as p.

Lemma 3. *If an open ball* $\text{Ball}(p, \rho)$ *has one node with a floding level $\lambda > \rho$, then its flooding level is uniform and equal to λ.*

By definition of an open ball, all edges in the cocycle of X have weights $\geq \lambda$. the smallest of them has a weight $\mu > \lambda$ or $\mu = \lambda$. Consider both cases separately.

Creation of a Lake Zone: $\boldsymbol{\mu = \lambda}$. There exists an edge with weight λ in the cocycle of X, linking a node s of X with a node t outside X. The node t does not belong to the open ball $\text{Ball}(p, \lambda)$ but to the closed ball $Y = \overline{\text{Ball}}(p, \lambda) \supset \text{Ball}(p, \lambda)$. What is the level of the flooding within Y ? Each node s of Y is linked with p by a path whose edges are lower or equal than λ. The criterion 2 characterizing floodings may then be applied to all pairs (u, v) of edges along this path: $\tau_v \leq \tau_u \vee e_{uv}$. If $\tau_u, e_{uv} \leq \lambda$, then also $\tau_v \leq \lambda$. This proves:

Lemma 4. *If there is at least one node with a weight $\leq \lambda$ in a closed ball* $\overline{\text{Ball}}(p, \lambda)$ *of level λ, all other nodes in this ball have a flooding level $\leq \lambda$.*

The diameter of Y is λ. Such a closed ball is called lake zone.

Creation of a Regional Minimum Lake: $\boldsymbol{\mu > \lambda}$. If the smallest edge of the cocycle is higher than λ, so are all edges of the cocycle. Hence X forms a lake with a uniform flooding level λ. As it is not possible to quit X without crossing an edge with a weight λ, we define it as a **regional minimum lake**.

From a lake zone to the next As any two nodes of X are linked by a path of altitude $< \lambda, \operatorname{diam}(X) = \mu < \lambda$. In a closed ball, diameter and radius are equal, hence X is also a closed ball of radius μ, i.e. a lake zone. According to the preceding lemma, if only one node of X has a flooding level equal to μ, then all its other nodes have a flooding level $\leq \mu$. But as soon one of its nodes has a flooding level $\lambda > \mu$, then all nodes of X have the same flooding level λ. And X is a regional minimum lake as long as its flooding level is lower than the lowest edge ν in its cocycle. As the flooding level within X reaches ν, X remains an open ball $\operatorname{Ball}(p, \nu)$ with a uniform flooding level ν, included in a closed ball $\overline{\operatorname{Ball}}(p, \nu)$ with a flooding level lower or equal than ν outside X.

An Increasing Series of Lakes Containing a Node p. We define the operator $\varepsilon_{ne}p$ which computes the weight of lowest adjacent edge of p ; similarly, $\varepsilon_{Xe}Y$ is the operator which computes the weight of the lowest edge in the cocycle of Y. We now describe the extension of the successive lakes containing a given node p for increasing levels η of flooding.

- for $\eta < \varepsilon_{ne}p$, i.e. a flooding level below $\varepsilon_{ne}p$, the extension of the lake is $X_0 = \{p\}$ and is a regional minimum lake. Hence for $\eta < \varepsilon_{ne}p : X_0 = \{p\}$.

- for $\eta = \varepsilon_{ne}p$, the lake containing p is included in a lake zone $X_1 = B(p, \varepsilon_{ne}p)$. The flood level is equal to η on X_0 and $\leq \eta$ everywhere else on X_1. We have $\operatorname{diam}(X_1) = \varepsilon_{ne}p = \varepsilon_{Xe}X_0$.

- for $\operatorname{diam}(X_1) < \eta < \varepsilon_{Xe}X_1$, the lake is a regional minimum lake with the extension X_1.

- for $\eta = \varepsilon_{Xe}X_1$, the lake containing p is included in a lake zone $X_2 = B(p, \varepsilon_{Xe}X_1)$. The flood level is equal to η on X_1 and $\leq \eta$ everywhere else on X_2. We have $\operatorname{diam}(X_2) = \varepsilon_{Xe}X_1$

- ...

- for $\operatorname{diam}(X_n) < \eta < \varepsilon_{Xe}X_n$, the lake is a regional minimum lake with the extension X_n.

- for $\eta = \varepsilon_{Xe}X_n$, i.e. a flooding level equal to the lowest adjacent edge of X_n, the lake containing p is included in a lake zone $X_{n+1} = B(p, \varepsilon_{Xe}X_n)$. The flood level is equal to η on X_n and $\leq \eta$ everywhere else on X_{n+1}. We have $\operatorname{diam}(X_{n+1}) = \varepsilon_{Xe}X_n$.

- the alternating series of regional minima lakes and lake zones goes on until all nodes of N are flooded.

Dendrogram Structure of the Lake Zone. For $p \in Y$, we define the operator κ_p by $\kappa_p(Y) = B(p, \varepsilon_{Xe}Y)$ and its iteration: $\kappa_p^{(n)}(Y) = \kappa_p \kappa_p^{(n-1)}(Y)$ Starting with the set $X_0 = \{p\}$ we obtain a series of lake zones: $X_0 = \{p\}$, $X_1 = \kappa_p\{p\}$, ..., $X_n = \kappa_p(X_{n-1}) = \kappa_p^{(n)}\{p\}$.

Obviously, for each node $q \in \kappa_p^{(n)}\{p\}$, there exists a number m such that $\kappa_p^{(n)}\{p\} = \kappa_q^{(m)}\{q\}$.

The sets $\kappa_p^{(n)}\{p\}$ for all n and all nodes p form a hierarchy. Its sets may be organized as a dendrogram. The leaves of the dendrogram are the nodes of G.

Each node $\kappa_p^{(n)}\{p\}$ is linked by an edge with its unique immediate successor $\kappa_p^{(n+1)}\{p\}$ as illustrated in fig.2A.

3 Dominated Floodings

3.1 Lake Level and Lake Extension at a Node *p*

The preceding section has described how the lake containing a given node is extended as its flooding level increases. Many flooding distributions are physically possible. However there is only one if we consider the highest flooding below a ceiling function ω defined on each node. We consider all floodings of G whose flooding level is lower than the function ω on all its nodes. The supremum of all these floodings also is a valid flooding of G and is the highest flooding of G below ω. We define the ceiling function $\omega(X)$ as the smallest value taken by ω on a node of X.

What will be the level of the flooding and the extension of the lake containing a given node p ? As shown above, the possible lakes containing the node p form an increasing series of nested sets $\kappa^{(n)}\{p\}$, the smallest being $\{p\}$, the largest being the root $\kappa^{(m)}\{p\}$ of the dendrogram.

The operator $\omega(X)$ is decreasing and the operator $\operatorname{diam}(X)$ increasing with X. As the series $\kappa^{(n)}\{p\}$ is increasing with n, we get a series of decreasing values $\omega(\kappa^{(n)}\{p\})$ and a series of increasing values $\operatorname{diam}(\kappa^{(n)}\{p\})$:

- as the set $\{p\}$ has no inside edge, we have $\operatorname{diam}(\kappa^{(0)}\{p\}) = \operatorname{diam}\{p\} = -\infty$. Hence $\omega\{p\} > \operatorname{diam}\{p\} = -\infty$

- if at the root we still have $\omega(\kappa^{(m)}\{p\}) > \operatorname{diam}(\kappa^{(m)}\{p\})$, i.e. the ceiling of p is higher than the root of the dendrogram, then $\tau_p = \operatorname{diam}(\kappa^{(m)}\{p\})$, the lowest flooding value covering the whole domain $\kappa^{(m)}\{p\}$.

- if on the contrary $\omega(\kappa^{(m)}\{p\}) \leq \operatorname{diam}(\kappa^{(m)}\{p\})$, let $k \leq m$ be the smallest index for which $\omega(\kappa^{(k)}\{p\}) \leq \operatorname{diam}(\kappa^{(k)}\{p\})$ (rel. 1). Hence $\omega(\kappa^{(k-1)}\{p\}) > \operatorname{diam}(\kappa^{(k-1)}\{p\})$ (rel. 2), which implies that on $\kappa^{(k-1)}\{p\}$ the flooding level is uniform and higher than $\operatorname{diam}(\kappa^{(k-1)}\{p\})$. On the other hand Rel.1 implies that on $\kappa^{(k)}\{p\}$ the maximal flooding level is $\operatorname{diam}(\kappa^{(k)}\{p\})$. Two possibilities are compatible with both relations:

* if $\omega(\kappa^{(k-1)}\{p\}) \leq \operatorname{diam}(\kappa^{(k)}\{p\})$, then $\tau_p = \tau_{\kappa^{(k-1)}\{p\}} = \omega(\kappa^{(k-1)}\{p\})$

* if $\omega(\kappa^{(k-1)}\{p\}) > \operatorname{diam}(\kappa^{(k)}\{p\})$, then $\tau_p = \tau_{\kappa^{(k-1)}\{p\}} = \operatorname{diam}(\kappa^{(k)}\{p\})$.

3.2 Illustration

Determination of the Flooding Level at the Node *c*. The ceiling function ω is equal to ∞ on all nodes excepting the nodes $\omega(c) = 6$ and $\omega(h) = 1$. We represent inside a yellow dot the function ω on each node of the dendrogram in fig.2A.

Let us compute the lake level and the extension of the node c. The smallest index for which $\omega(\kappa^{(k)}\{c\}) \leq \operatorname{diam}(\kappa^{(k)}\{c\})$, is $k = 3$, with $\kappa^{(3)}\{c\} = [b, c, d, e, f]$ having a diameter 7, whereas $\omega(\kappa^{(3)}\{c\}) = 6$. For $k = 2$, we get

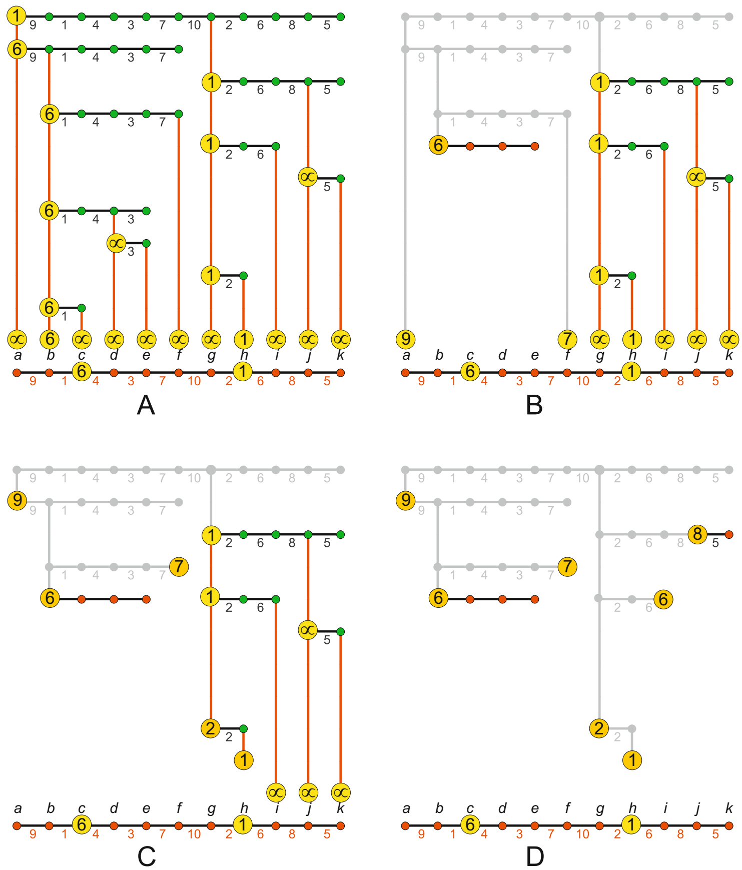

Fig. 2. A: Dendrogram asssociated to the edge weighted graph (red nodes linked by weighted edges). The yellow disks contain the ceiling level of each node of the dendrogram.
B: The lake containing the nodes c also contains the nodes (b, c, d, e) at a flooding level 6.
C: The ancestors of (b, c, d, e) are suppressed and its uncles becom the roots of sub-dendrograms which may be processed separately.
D: Final dendrogram with the flooding levels of the various nodes.

$\kappa^{(2)}\{c\} = [b, c, d, e]$ having a diameter 4, whereas $\omega(\kappa^{(2)}\{c\}) = 6$. According to the preceding analysis the flooding level of $\kappa^{(2)}\{c\} = [b, c, d, e]$ is $\tau_c = \tau_{\kappa^{(2)}\{c\}} = \omega(\kappa^{(2)}\{c\}) = 6$ (see fig.2B).

Pruning the Dendrogram. Fig.2 presents how the upstream of each flooded node is pruned. As the level of $\kappa^{(2)}\{c\}$ is known, the dendrogram may be pruned by discarding all ancestors of $\kappa^{(2)}\{c\}$. For $k > 2$, $\kappa^{(k)}\{c\}$ is an ancestor of c, the flooding level of all its immediate successors which are not ancestors of c, that is, brothers of $\kappa^{(k-1)}\{c\}$ is lower or equal than $\text{diam}(\kappa^{(k)}\{c\})$. The edge linking each brother Y of $\kappa^{(k-1)}\{c\}$ with its father $\kappa^{(k)}\{c\}$ is cut ; like that Y becomes the root of a sub-dendrogram ; as its flooding level is lower or equal than $\text{diam}(\kappa^{(k)}\{c\})$, one sets $\omega(Y) = \omega(Y) \wedge \text{diam}(\kappa^{(k)}\{c\})$. On the same time all ancestors of $\kappa^{(2)}\{c\}$ and the edges linking them are suppressed.

The result of the pruning is illustrated by fig.2B,C. The set $\kappa^{(2)}\{c\} = [b, c, d, e]$ got its flooding level 6 and its upstream is pruned:

- $\kappa^{(3)}\{c\} = [b, c, d, e, f]$ is suppressed and the node $\{f\}$ becomes the root of a sub-dendrogram, with a ceiling value $\omega(\{f\}) = \omega(\{f\}) \wedge \text{diam}(\kappa^{(3)}\{c\}) = 7$. As the sub-dendrogram is reduced to a node, its ceiling value is its flooding value, 7.

- $\kappa^{(4)}\{c\} = [a, b, c, d, e, f]$ is suppressed and the node $\{a\}$ becomes the root of a sub-dendrogram, with a ceiling value $\omega(\{a\}) = \omega(\{a\}) \wedge \text{diam}(\kappa^{(4)}\{c\}) = 9$. As the sub-dendrogram is reduced to a node, its ceiling value is its flooding value, 9.

- $\kappa^{(5)}\{c\}$, the root, is suppressed and the node $[g, h, i, j, k]$ becomes the root of a sub-dendrogram, with a ceiling value $\omega([g, h, i, j, k]) = \omega([g, h, i, j, k]) \wedge \text{diam}(\kappa^{(5)}\{c\}) = 1$.

In summary, as soon a node Y of the dendrogram gets its flooding level, the dendrogram may be pruned, suppressing all ancestors of Y, transforming each uncle Z of Y into the root of a sub-dendrogram, with a ceiling value $\omega(Z) = \omega(Z) \wedge \text{diam}\, \kappa(Z)$.

The final result is obtained by processing each sub-dendrogram separately and is illustrated in fig.2D.

An Algorithm Based on Edge Contractions. The following algorithm constructs the dendrogram and computes the flooding levels by iteratively contracting the edges of each lake. We define $\Lambda(p)$ as a collection of nodes with the same flooding level τ_p as the node p. The result of the algorithm is a list Λ of records of the type $(\Lambda(p), \tau_p)$. The algorithms proceeds by processing the edges still present in the graph in the order of increasing altitudes. Initially $\Lambda = \varnothing$ and for $p \in N : \Lambda(p) = [p]$ and $p =$ "*unflooded*":

As long there are edges to process, let (p, q) be the lowest edge to process:
if $p =$ "*isflooded*" and $q =$ "*isflooded*" take the next edge, else
$X = \overline{\text{Ball}}(p, e_{pq})$; $X_1 = \{x \in X \mid x = \text{"}isflooded\text{"}\}$; $X_2 = \{x \in X \mid \omega(x) \leq e_{pq}$ and $x = \text{"}unflooded\text{"}\}$
If $X_1 \cup X_2 = \varnothing : \Lambda(p) = \bigcup_{x \in X} \Lambda(p)$; $\omega(p) = \omega(X)$; contract (o, q) on p
else for each $x \in X/X_1$:

$\tau_x = \omega(x) \wedge e_{pq}$
$\Lambda = \text{append}[\Lambda, (\Lambda(x), \tau_x)]$
contract X on p
$p =$ "*isflooded*"

Illustration

- $e_{bc} = 1; X_1 \cup X_2 = \varnothing; \Lambda(c) = [b, c]\,; \omega(c) = 6$; contract (b, c) on c
- $e_{gh} = 2; X_2 = [h]\,; \tau_h = 1; \tau_g = 2$; $\Lambda = \text{append}[\Lambda, (h, 1), (g, 2)]$; contract (g, h) on h; $h =$ "*isflooded*"
- $e_{de} = 3; X_1 \cup X_2 = \varnothing; \Lambda(e) = [d, e]\,; \omega(e) = \infty$; contract (d, e) on e

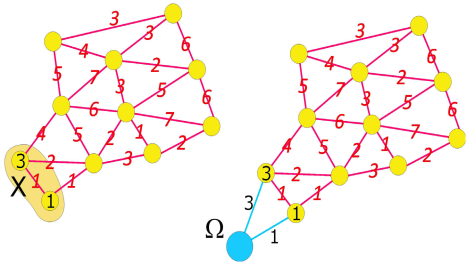

Fig. 3. Adding a dummy node linked to each node x in X by an edge weighted by the offset at x

- $e_{ce} = 4; X_1 \cup X_2 = \varnothing; \Lambda(e) = [b, c, d, e]\,; \omega(e) = 6$; contract (c, e) on e
- $e_{jk} = 5; X_1 \cup X_2 = \varnothing; \Lambda(e) = [j, k]\,; \omega(k) = \infty$; contract (j, k) on k
- $e_{hi} = 6; X_1 = [h]\,; \tau_i = 6$; $\Lambda = \text{append}[\Lambda, (i, 6)]$; contract (h, i) on i; $i =$ "$isflooded$"
- $e_{ef} = 7; X_1 = [e]\,; \tau_f = 7$; $\Lambda = \text{append}[\Lambda, (f, 7)]$; contract (e, f) on f; $f =$ "$isflooded$"
- $e_{ik} = 8; X_1 = [i]\,; \tau_k = 8$; $\Lambda = \text{append}[\Lambda, (k, 8)]$; contract (i, k) on k; $k =$ "$isflooded$"
- $e_{af} = 9; X_1 = [e]\,; \tau_a = 9$; $\Lambda = \text{append}[\Lambda, (a, 9)]$; contract (a, f) on f; $f =$ "$isflooded$"
- $e_{f,k} = 9; p =$ "$isflooded$" and $q =$ "$isflooded$", there is no further edge : end

4 Constrained Highest Floodings on Edge Weighted Graphs as Shortest Distances in an Augmented Graph

4.1 Highest Floodings and Shortest Distances

According to criterion 2, any flooding θ verifies the relation: $\theta_p \leq \theta_q \vee e_{pq}$, for each neighbor q of p. As this relation is to be true for all neighbors of p, we have $\theta_p \leq \bigwedge_{q \text{ neighbor of } p} (\theta_q \vee e_{pq})$ Simultaneously $\theta_p \leq h_p$. So $\theta_p \leq h_p \wedge \bigwedge_{q \text{ neighbor of } p} (\theta_q \vee e_{pq})$ and the highest of them, τ verifies
$\tau_p = h_p \wedge \bigwedge_{q \text{ neighbor of } p} (\tau_q \vee e_{pq})\,.$

If we add to the graph G_e a dummy node Ω with a weight $\tau_\Omega = 0$ linked by a dummy edge (Ω, p) with each node p and holding a weight equal to h_p, we get an augmented graph $\widehat{G_e}$. Relation (6) may be rewritten as $\tau_p = (\tau_\Omega \vee e_{\Omega p}) \wedge \bigwedge_{q \text{ neighbor of } p} (\tau_q \vee e_{qp})\,.$ This formula expresses that the shortest path for the ultrametric flooding distance between Ω and p is $e_{\Omega p} = h_p$ if the path is simply the edge (Ω, p) or is equal to $(\tau_s \vee e_{ps})$ if the path passes through the neighbor s of p, $(\tau_q \vee e_{qp})$ taking its smallest value for $q = s$.

Theorem 1. *The highest flooding of the graph G below a function h defined on the nodes is the shortest ultrametric flooding distance of each node to Ω .*

This theorem permits to use any shortest path algorithm for computing this highest flooding. The simplest recursively applies the relation (6) until stability is reached.

Initialisation: $\tau_p^{(0)} = h_p$

Repeat until $\tau_p^{(m)} = \tau_p^{(m-1)}$: $\tau_p^{(n)} = h_p \wedge \bigwedge_{q \text{ neighbor of } p} \left(\tau_q^{(n-1)} \vee e_{pq}\right)$

Stability is necessarily reached after a number n of iteration as the values of τ decrease and have a lower ceiling equal to 0. As $\tau_p^{(n)} \leq \tau_p^{(n-1)} \leq h_p$, we get an equivalent algorithm with the following sequence: $\tau_p^{(n)} = \tau_p^{(n-1)} \wedge \bigwedge_{q \text{ neighbor of } p} \left(\tau_q^{(n-1)} \vee e_{pq}\right)$.

4.2 The Moore Dijkstra Shortest Path Algorithm [5]

This famous greedy algorithm takes as many steps as there are nodes. At any step, S represents the subset of nodes for which the shortest path is known. For any neighboring node of S, the length of the shortest path for which all edges but the last belong to S constitutes an overestimation of this length. The node with the lowest guess is correctly estimated.

Initialization:

$S = \Omega$; $\overline{S} = N$; for each node p in N : $\tau_p = h_p$

Flooding:

While $\overline{S} \neq \varnothing$ repeat:

Select $j \in \overline{S}$ for which $\tau_j = \min_{i \in \overline{S}} [\tau_i]$

$\overline{S} = \overline{S} \backslash \{j\}$

For any neighbor i of j in $\overline{S}$ do $\tau_i = \min [\tau_i, \tau_j \vee e_{ji}]$

End While

Remark. The dummy node plays no role, nor the nodes with an infinite ceiling value. Without dummy node, the initialisation becomes $S = \varnothing$; for each node p in N verifying $h_p < \infty$, do $\tau_p = h_p$.

The nodes are processed in an increasing order of flooding. If we keep the edges linking each node with the node through which it has been flooded in the algorithm we get a tree. Along each edge of this tree, the level of the flood also is never decreasing.

5 Conclusion

We have given an axiomatic definition of floodings on edge weighted graph. The highest flooding under a ceiling function is a morphological opening of this ceiling function (increasing, anti-extensive and idempotent). The criteria characterizing this flooding permit to express it either as a shortest distance problem on an augmented graph for an ultrametric flooding distance or as a pruning of a dendrogram. The first expression permits to use the shortest path algorithm which is best adapted to each particular problem. The second permits to imagine extremely fast implementations as the dendrogram rapidly splits in sub-dendrograms which may be processed independently.

These results may be transposed on floodings for images [4]. An image f defined on a grid, may be considered as a node weighted graph G; the pixels becoming the nodes, their grey tones the node weights ; the edges connect neighboring pixels/nodes and are not weighted. It may be shown that any flooding τ of an image f is a flooding of the graph G, on which the edges get weights $e_{pq} = f_p \vee f_q$. This results permits to transpose on images all results established on tank networks.

References

1. Marcotegui, B., Beucher, S.: Fast implementation of waterfalls based on graphs. In: ISMM05: Mathematical Morphology and its Applications to Signal Processing, pp. 177–186 (2005)
2. Cousty, J., Bertrand, G., Najman, L., Couprie, M.: Watershed cuts: Minimum spanning forests and the drop of water 1principle. IEEE Transactions on Pattern Analysis and Machine Intelligence 31, 1362–1374 (2009)
3. Meyer, F.: Minimal spanning forests for morphological segmentation. In: ISMM94: Mathematical Morphology and its Applications to Signal Processing, pp. 77–84 (1994)
4. Meyer, F.: Flooding and segmentation. In: Goutsias, J., Vincent, L., Bloomberg, D.S. (eds.) Mathematical Morphology and its Applications to Image and Signal Processing. Computational Imaging and Vision, vol. 18, pp. 189–198 (2002)
5. Moore, E.F.: The shortest path through a maze. In: Proc. Int. Symposium on Theory of Switching, vol. 30, pp. 285–292 (1957)