Adjunctions on the Lattice of Dendrograms

Fernand Meyer

CMM-Centre de Morphologie Mathématique,
Mathématiques et Systèmes, MINES ParisTech, France
fernand.meyer@mines-paristech.fr

Abstract. Dendrograms are used in hierarchical classification. They also are useful structures in image processing, for segmentation or filtering purposes. The structure of a hierarchy is univocally expressed by a ultrametric ecart. The hierarchies form a complete lattice on which two adjunctions will be defined.

1 Introduction

Hierarchies are the classical structures for representing a taxonomy. The most famous taxonomy is the Linnaean system. Each genus is the union of all species it contains, which in turn is the union of animals it contains.

As hierarchies are nested partitions of a domain, they are also encountered in image segmentation. Multiple segmentations of increasing coarseness are produced. Each level of the hierarchy contains a partition of the image and from level to level only fusions of regions take place [4].

Partitions are thus the simplest hierarchies, with only one level. The algebraic structure of partitions has been studies by Heijmans, Serra and Ronse [2], [11], [7]. Often one is not interested in partitioning the total domain of an image, but one wants to get the masks of some objects of interest. These masks are disjoint sets but do not partition the domain ; they constitute a partial partition as introduced by Ch. Ronse [5].

A series of nested partitions, where each coarser partition is obtained by merging regions of finer partitions, constitutes a hierarchy. We have a partial hierarchy or dendrogram, if the lowest level contains particular interest zones ; in higher levels, some preexisting regions become larger, eventually merge and others appear. Consider a topographic surface which is flooded such that all lakes have the same altitude. For each flooding level, the lakes form a partial partition. From one level to a higher level, the extension of a lake may grow, new lakes appear, existing lakes merge. The corresponding partial hierarchy is called min-tree [8] and is often used for image filtering.

The paper is organized as follows. A first part gives an axiomatic definition of dendrograms and hierarchies. The second derives an ultrametric half distance derived from a stratification index. An order relation between dendrograms organizes the hierarchies as a complete lattice. Finally, two adjunctions are defined on dendrograms. Combining erosion and dilation in an adjunction produces openings and closings, from which the classical morphological filters may then be derived [3].

W.G. Kropatsch et al. (Eds.): GbRPR 2013, LNCS 7877, pp. 91–100, 2013.

2 Dendrograms and Hierarchies

The axiomatic definition of dendrograms and hierarchies is due to Benzecri [1]. Let E be a domain with a finite number of elements are called points (for instance the pixels of an image) and $\mathcal{P}(E)$ the family of subsets of E. Let $\mathcal{X}$ be a subset of $\mathcal{P}(E)$, on which we consider an arbitrary order or preorder relation relation $\prec$ (in the present work $\prec$ is the inclusion relation $\subset$ between sets) The union of all sets belonging to $\mathcal{X}$ is called support of $\mathcal{X}$: $\text{supp}(\mathcal{X})$. The subsets of $\mathcal{X}$ may be structured into:

* the summits : $\text{Sum}(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : A \prec B \Rightarrow A = B\}$
* the leaves : $\text{Leav}(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : B \prec A \Rightarrow A = B\}$
* the nodes : $\text{Nod}(\mathcal{X}) = \mathcal{X} - \text{Leav}(\mathcal{X})$
* the predecessors : $\text{Pred}(A) = \{B \in \mathcal{X} \mid A \prec B\}$
* the immediate predecessors :
  $\text{ImPred}(A) = \{B \in \mathcal{X} \mid \{U \mid U \in \mathcal{X}, A \prec U \text{ and } U \prec B\} = (A, B)\}$
* the successors : $\text{Succ}(A) = \{B \in \mathcal{X} \mid B \prec A\}$

Fig.5, at the end of the document, presents a dendrogram in which the letters represent subsets of $\mathcal{P}(E)$; and $A \rightarrow B$ means that $B \prec A$. Then A is the summit ; B is the predecessor of (D, E, F, H, I) and the immediate predecessor of (D, E, F) ; (H, I, E, F, J, G) are leaves ; (J, G) are successors of A and C, and immediate successors of C.

2.1 Dendrograms

We now structure $\mathcal{X}$ as a tree or a dendrogram (also called "partial hierarchy")

Dendrograms : $\mathcal{X}$ is a dendrogram if and only if the set $\text{Pred}(A)$ of the predecessors of A, with the order relation induced by $\prec$ is a total order. The maximal element of this family is a summit, which is the unique summit containing A.

Proposition 1. *The following properties are equivalent:*
1)$\mathcal{X}$ is a dendrogram
2) $U, V, A \in \mathcal{X}$: $A \subset U$ and $A \subset V \Rightarrow U \subset V$ or $V \subset U$
3) $U, V \in \mathcal{X} : U \nsubseteq V$ and $V \nsubseteq U \Rightarrow U \cap V = \varnothing$
4) Any element $A \in \mathcal{X} - \text{Sum}(\mathcal{X})$ possesses a unique immediate predecessor (valid as we suppose E and $\mathcal{X}$ finite)

Proposition 2. *A family $(A_i)_{i \in I}$ of sets in $\mathcal{X}$ with a non empty intersection is completely ordered for $\subset$.*

A dendrogram is said to be connected if it possesses a unique summit. Finite dendrograms are classically represented as a tree : each element $A \in \mathcal{X}$ is a node of the tree, and is linked by an edge with its unique immediate predecessor.

Consider a dendrogram Π verifying : $A \in \text{supp}(\Pi) \Rightarrow \text{Pred}(A) = A$. Such a dendrogram has only one hierarchical level is called *partial partition* (partial partitions have been introduced by C.Ronse in [5]). If $\text{supp}(\Pi) = E$, then it is called partition.

2.2 Hierarchies

Definition 1. *We call hierarchy $\mathcal{H}$ a dendrogram verifying* $\bigcup \text{Leav}(\mathcal{H}) = \text{supp}(\mathcal{H})$

Proposition 3. *A dendrogram $\mathcal{X}$ is a hierarchy if and only any element A of $\mathcal{X}$ is the union of all other elements of $\mathcal{X}$ contained in A:*
$\forall A \in \mathcal{X} : \bigcup \{B \in \mathcal{X} \mid B \subset A \; ; B \neq A\} = \{A, \emptyset\}$

For understanding the difference between dendrograms and hierarchies, we give an example of each used all along the paper.

Hierarchy : A prototype of hierarchy is a series of nested partitions, where each coarser partition is obtained by merging regions of finer partitions. The leaves are the regions of the finest partition ; their union constitutes the support of the hierarchy.

Dendrogram : A prototype of a dendrogram is constituted by the lake distribution of a topographic surface. Each lake is included in all lakes with a higher level. The leaves are the lakes when they just cover the regional minima. The union of all lakes is larger than the union of the leaves. For each flooding level, the lakes form a partial partition. From one level to a higher level, the extension of a lake may grow, new lakes appear, existing lakes merge.

2.3 Stratification Index and Partial Ultrametric Distances (PUD)

Consider a dendrogram or hierarchy $\mathcal{X}$; $\mathcal{X}$ is a stratified hierarchy, if it is equipped with an index function st from $\mathcal{X}$ into the interval $[0, L]$ of $\mathbb{R}$ which is strictly increasing with the inclusion order:
$\forall A, B \in \mathcal{X} : A \subset B$ and $B \neq A \Rightarrow \text{st}(A) < \text{st}(B)$.
As E is finite, the number of distinct stratification levels is finite. We suppose that for all $A \in \mathcal{X} : \text{st}(A) < L$ and set $\text{st}(\varnothing) = L$.

Each dendrogram $\mathcal{X}$ with a stratification index st induces on the points $p, q \in E$ a partial ultrametric distance $\chi(p, q)$. If no set of $\mathcal{X}$ contains both p and q, then $\chi(p, q) = L$. Otherwise, the family $(A_i)_{i \in I}$ of sets of $\mathcal{X}$ containing both p and q has a non empty intersection, and as established above, is completely ordered for $\subset$. Thus it possesse a smallest element A and $\chi(p, q) = \text{st}(A)$. In particular $\chi(p, p)$ is the stratification index of the smallest set $\mathcal{X}$ of containing p ; if no set of $\mathcal{X}$ contains p, then $\chi(p, p) = \text{st}(\varnothing) = L$; such a point is called "alien" of $\mathcal{X}$.

Properties : χ has the following properties:
$\forall p, q \in E : \chi(p, q) = \chi(q, p)$
$\forall p, q, r \in E : \chi(p, q) \leq \max \{\chi(p, r), \chi(r, q)\}$

Remark: χ is not a distance but an ecart as $\chi(p, q) = 0$ does not necessarily imply $p = q$. We call it partial ultrametric distance.

Properties of the Balls of a Partial Ultrametric Distance
Closed balls, defined as $\text{Ball}(p,\rho) = \{q \in E \mid \chi(p,q) \leq \rho\}$ have strange properties:

* *Each element of a closed ball* $\text{Ball}(p,\rho) = \{q \in E \mid \chi(p,q) \leq \rho\}$ *is centre of this ball.*
* *Two closed balls* $\text{Ball}(p,\rho)$ *and* $\text{Ball}(q,\rho)$ *with the same radius are either disjoint or identical.*
* *The radius of a ball is equal to its diameter.* If A is a set of the dendrogram, p an arbitrary point of A, then A is the ball of center p and of radius equal to the diameter of A (the maximal distance between two points of A.

Inversely, the closed balls of a partial ultrametric distance χ form a dendrogram $\mathcal{X}$.

2.4 Partial Partitions by Thresholding Partial Hierarchies

Consider a partial hierarchy $\mathcal{X}$ with its associated PUD χ. By thresholding the PUD at level λ one obtains a partial binary ultrametric half distance (PBUD):

$$\pi_\lambda(x,y) = \begin{matrix} 1 \text{ if } \chi(x,y) > \lambda \\ 0 \text{ if } \chi(x,y) \leq \lambda \end{matrix} \text{ associated to a partial partition } \Pi_\lambda.$$

Aliens and Singletons

Aliens and singletons of partial partitions. We define aliens and singletons of a partial partition π:

* Singletons are characterized by: $\forall p,q \in E\ ,\ p \neq q, : \pi(p,q) = 1$ and $\pi(p,p) = 0$.
* The support of π is the set of points p verifying : $\pi(p,p) = 0$
* Aliens, which are points outside the support are characterized by: $\forall p \in E\ :\ \pi(p,p) = 1$ implying $\forall q \in E : \pi(p,p) \leq \pi(p,q) \vee \pi(q,r)$ so that $\pi(p,q) = 1$

Aliens and singletons of dendrograms. Consider a PUD χ and its thresholds π_λ at level λ. For increasing values of λ, the partial partitions π_λ obtained by thresholding χ have increasing supports $\text{supp}_\lambda(\chi) = \{p \in E : \chi(p,p) \leq \lambda\}$. Let $p \in E$ be a point verifying $\chi(p,p) = \lambda$ and the partition π_μ obtained by thresholding χ at the level μ. And consider $\nu = \bigwedge_{q \neq p} \chi(p,q)$. Since $\chi(p,p) \leq \chi(p,q) \vee \chi(q,p)$, we have $\nu \geq \lambda$. The status of p will vary in the partitions π_ν for increasing levels ν :

* $\mu < \lambda : \pi_\mu(p,p) = 1$ and p does not belong to the support of π_μ and is an alien
* $\lambda \leq \mu < \nu : \pi_\mu(p,p) = 0$ but for $q \neq p$, $\pi_\mu(p,q) = 1$ and p is a singleton
* $\nu \leq \mu : \pi_\mu(p,p) = 0$ and for there exists $q \neq p$ such that $\pi_\mu(p,q) = 0$ and p is a regular node.

3 The Lattice of Hierarchies

3.1 Order Relation between Hierarchies and Partial Hierarchies

Let $\mathcal{A}$ and $\mathcal{B}$ be two dendrograms with their associated PUD : $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$. The following relation defines an order relation between the hierarchies: $B \leq A \Leftrightarrow \forall p, q \in E \quad \chi_{\mathcal{A}}(p,q) \leq \chi_{\mathcal{B}}(p,q)$

It follows that $\forall p \in E$: $\mathrm{Ball}_{\mathcal{B}}(p,\rho) \subset \mathrm{Ball}_{\mathcal{A}}(p,\rho)$. We say that the hierarchy $\mathcal{A}$ is coarser than the hierarchy $\mathcal{B}$ and that the hierarchy $\mathcal{B}$ is finer than the hierarchy $\mathcal{B}$.

For each $p \notin \mathrm{supp}(\mathcal{A})$: $\chi_{\mathcal{A}}(p,p) = L$ which implies that $\chi_B(p,p) = L$, so that $p \notin \mathrm{supp}(\mathcal{B})$.

The smallest dendrogram has an empty support and contains only aliens, i.e. points p verifying $\forall q \in E$, $\chi(p,q) = L$.

The smallest hierarchy has E as support and contains only singletons $\forall p \neq q \in E$, $\chi(p,q) = L$, and $\forall p \in E$, $\chi(p,q) = 0$. The largest hierarchy is E itself, whose PUD verifies: $\forall p, q \in E : \chi(p,q) = 0$

To binary PUDs $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$ correspond partitions and partial partitions. Their closed balls verify : $\mathrm{Ball}_{\mathcal{B}}(p,0) \subset \mathrm{Ball}_{\mathcal{A}}(p,0)$, the aliens remaining outside the balls. Hence the tiles of the finer partition $\mathcal{B}$ are included in the tiles of the coarser partition $\mathcal{A}$ which is coherent with the usual definition of the order between partitions.

3.2 The Lattice of Dendrograms

Consider a family of dendrograms $(\mathcal{A}_i)_{i \in I}$, the PUD of the dendrogram $\mathcal{A}_i$ being χ_i.

Infimum of Hierarchies. The infimum $\wedge \mathcal{A}_i$ is the largest dendrogram which is smaller than each $\mathcal{A}_i$ and its PUD is the smallest verifying for all elements of the family $\chi_{\wedge \mathcal{A}_i} \geq \chi_i$. As the supremum of PUDs is a PUD, we have $\chi_{\wedge \mathcal{A}_i} = \bigvee_i \chi_i$. And $\mathrm{supp}_\lambda \chi_{\wedge \mathcal{A}_i} = \bigwedge_i \mathrm{supp}_\lambda A_i$.

Supremum of Dendrograms. The supremum $\vee \mathcal{A}_i$ is the smalles dendrogram which is larger than each $\mathcal{A}_i$ and its PUD is the largest verifying for all elements of the family $\chi_{\vee \mathcal{A}_i} \leq \chi_i$. Unfortunately $\bigwedge_i \chi_i$ is not a PUD and $\chi_{\vee \mathcal{A}_i} = \widehat{\bigwedge_i \chi_i}$ is the largest partial ultrametric distance which is lower than $\bigwedge_i \chi_i$. It exists as the set of ultrametric distances lower than $\bigwedge_i \chi_i$ is not empty and contains the largest dendrogram whose PUD verifies $\forall p, q \in E$, $\chi(p,q) = 0$. As this family is closed by supremum it has a largest element.

Illustration

Fig.1 presents two hierarchies HA and HB through their nested partitions. The supremum and infimum of both hierarchies also are represented. The infimum

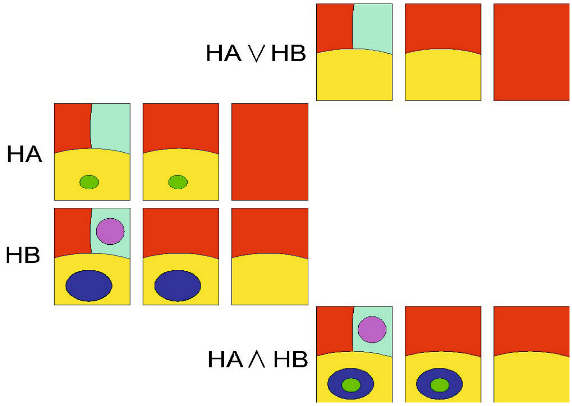

Fig. 1. Two hierarchies HA and HB and their derived supremum and infimum

takes for each threshold the intersection of the corresponding partitions, obtained through intersection of the tiles. The supremum is obtained by keeping only the boundaries existing in each component.

4 Adjunctions on Partial Hierarchies

4.1 Erosion and Dilations by a Structuring Element of Binary Sets

Adjunctions are the mother of morphology. Consider a lattice, A, B two arbitrary elements of the lattice ; the operator δ and ε iff $\delta A < B \Leftrightarrow A < \varepsilon B$ [9]. Then δ and ε are increasing operators, δ a dilation and ε an erosion, $\varepsilon\delta$ a closing and $\delta\varepsilon$ an opening.

Consider the classical erosion and dilation of binary sets by a structuring element. Chosing a point O in the domain E, we associate to each point x the vector $\overrightarrow{Ox}$. Inversely we associate to each affine vector $\overrightarrow{Ox}$ its extremity x. We write $x+b$ for the extremity of the vector $\overrightarrow{Ox}+\overrightarrow{Ob}$ and define $X_b = \bigcup_{x \in X} x+b$, the set X translated by the vector $\overrightarrow{Ob}$ (X_{-b} for the translation $\overrightarrow{bO}$). We have the following equivalence: $B_p \subset X \Leftrightarrow \forall b \in B : p+b \in X \Leftrightarrow \forall b \in B : p \in X_{-b}$ from which we derive two classical and equivalent formulations of the erosion of a set X by a structuring element $B : X \ominus B = \{p \in X \mid B_p \subset X\} = \bigwedge\limits_{b \in B} X_{-b}$. It appears that each of these formulations, which are equivalent for sets lead to two distinct adjunctions in the case of partial hierarchies.

4.2 A First Adjunction Based on the Supremum and Infimum of Translated PUD

Consider a dendrogram $\mathcal{X}$ and its PUD χ. If we translate all elements of $\mathcal{X}$ by a translation $\overrightarrow{bO}$ we get a new hierarchy $\mathcal{X}_{-b}$ with a PUD χ_b defined by

$\chi_b(p,q) = \chi(p-b, q-b)$. To the eroded hierarchy $\mathcal{X} \ominus B = \bigwedge_{b \in B} \mathcal{X}_{-b}$ corresponds the PUD $\bigvee_{b \in B} \chi_{-b}$ defined by $\bigvee_{b \in B} \chi_{-b}(p,q) = \bigvee_{b \in B} \chi(p-b, q-b \mid b \in B)$. The adjunction dilation is then $\mathcal{X} \oplus B = \bigvee_{b \in B} \mathcal{X}_b$ with the associated PUD $\overbrace{\bigwedge_{b \in B} \chi_b}$. For showing that the first $\mathcal{X} \ominus B = \bigwedge_{b \in B} \mathcal{X}_b$ is an erosion and the second $\mathcal{X} \oplus B = \bigvee_{b \in B} \mathcal{X}_b$ a dilation, we have to show that they form an adjunction: for any two hierarchies $\mathcal{X}, \mathcal{Y} \in \mathcal{X}(E) : \mathcal{X} \oplus B < \mathcal{Y} \Leftrightarrow \mathcal{X} < \mathcal{Y} \ominus B$.

We will prove the adjunction through the PUD χ and ζ associated to the hierarchies $\mathcal{X}$ and $\mathcal{Y}$: $\mathcal{X} < \mathcal{Y} \ominus B \Leftrightarrow \chi > \bigvee_{b \in B} \zeta_{-b} \Leftrightarrow \forall b \in B : \chi > \zeta_{-b} \Leftrightarrow \forall b \in B : \chi_b > \zeta \Leftrightarrow \bigwedge_{b \in B} \chi_b > \zeta$

Remains to establish : $\bigwedge_{b \in B} \chi_b > \zeta \Leftrightarrow \overbrace{\bigwedge_{b \in B} \chi_b} > \zeta$:

* $\overbrace{\bigwedge_{b \in B} \chi_b} > \zeta \Rightarrow \bigwedge_{b \in B} \chi_b > \zeta$ since $\overbrace{\bigwedge_{b \in B} \chi_b}$ is the largest ultrametric ecart below $\bigwedge_{b \in B} \chi_b$

* Suppose now $\bigwedge_{b \in B} \chi_b > \zeta$. Since ζ is an ultrametric ecart below $\bigwedge_{b \in B} \chi_b$, it is smaller or equal to the largest ultrametric ecart below $\bigwedge_{b \in B} \chi_b$, that is $\overbrace{\bigwedge_{b \in B} \chi_b}$

This completes the proof :

$$\mathcal{X} < \mathcal{Y} \ominus B \Leftrightarrow \chi > \bigvee_{b \in B} \zeta_{-b} \Leftrightarrow \bigwedge_{b \in B} \chi_b > \zeta \Leftrightarrow \overbrace{\bigwedge_{b \in B} \chi_b} > \zeta \Leftrightarrow \mathcal{X} \oplus B < \mathcal{Y}$$

The expression of the PUD is

$$\chi \ominus B(p,q) = \left[\bigvee_{b \in B} \chi_{-b}\right](p,q) = \bigvee \{\chi(p-b, q-b) \mid b \in B\}$$

$$\chi \oplus B(p,q) = \left[\overbrace{\bigwedge_{b \in B} \chi_b}\right](p,q) = \overbrace{\bigwedge \{\chi(p+b, q+b) \mid b \in B\}}$$

If there exists a $b \in B$, such that $\chi(p-b, p-b) = \lambda$, then $\chi \ominus B(p,p) \geq \lambda$; in other words, aliens of E for χ are dilated by the structuring element B.

We illustrate in figures 2 and 3 the erosion and the opening of a one dimensional hierarchy by a structuring element made of three pixels.

4.3 Adjunction on Hierarchies and Partial Hierarchies, Defined on a Tile by Tile Basis

Adjunctions on Partial Partition. The second formulation of the erosion for sets $\{p \in X \mid B_p \subset X\}$ will now be adapted to a partial partition with its PUD χ. Two points p and q belong to the same tile of the partition eroded by a structuring element B, if they are centers of disks entirely included in the same tile of the initial partition (see fig.4), which is the case if and only if all pairs

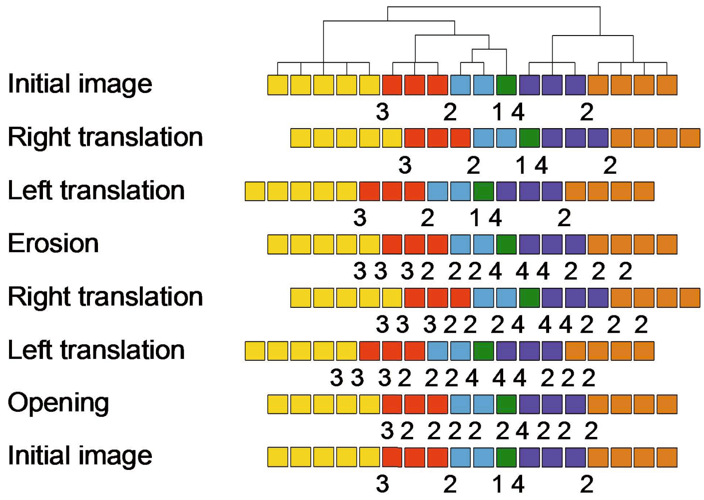

Fig. 2. Erosion and opening by a segment of 3 pixels: intermediate steps

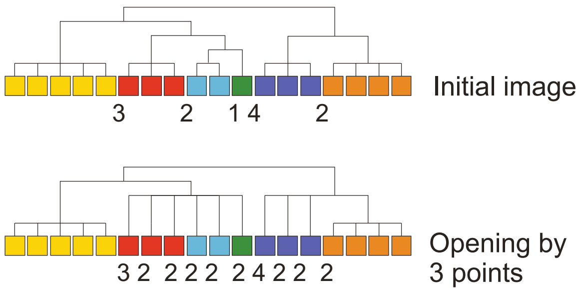

Fig. 3. Dendrogram of an initial image and its opening by a segment of 3 points

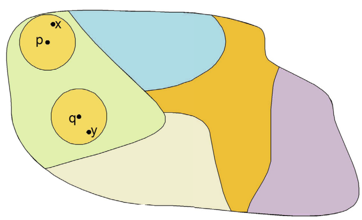

Fig. 4. The points p and q belong to the same tile of the partition eroded by a disk, as they are centers of disks entirely included in the same tile of the initial partition

$x, y \in B_p \cup B_q$ belong to the same tile of the partition, i.e. $\chi(x, y) = 0$. Hence the PUD $\varepsilon\chi$ of the eroded hierarchy is $\delta\chi(p, q) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_q\}$.

Consider now a point p such that B_p is not included in any tile of the partition χ. For each B_p, there exists $s, t \in B_p$ such that $\chi(s, t) = 1$, hence $\delta\chi(p, p) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_p\} = 1$, showing that p is an alien in the eroded partition χ. On the other hand if there exists a tile of the partition containing B_p and the erosion of this tile is reduced to a singleton p, then $\delta\chi(p, p) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_p\} = 0$. In other terms, this erosion of partial partitions adjusts the support of the partial partition by including aliens when necessary, and is identical with the erosion defined by Ronse in [7] (but distinct from the adjunction defined by J.Serra for partitions [10], where each tile of a partition is eroded and dilated separately, empty spaces being filled with singletons).

Adjunctions on Dendrograms. The expression established for partial partitions is still valid for arbitrary dendrograms:
$\delta\chi(p, q) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_q\}$.
It may be reformulated as the supremum of three terms:
$\delta\chi(p, q) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_q\} = \bigvee \{\chi(p + b_1, p + b_2) \mid b_1, b_2 \in B\}$
$\vee \bigvee \{\chi(p + b_1, q + b_2) \mid b_1, b_2 \in B\} \vee \bigvee \{\chi(q + b_1, q + b_2) \mid b_1, b_2 \in B\}$.
The first and last terms are dominated by the central term. Indeed, for each couple $b_1, b_2 \in B : \chi(p + b_1, p + b_2) \leq \chi(p + b_1, q + b_2) \vee \chi(q + b_2, p + b_2)$.

We obtain like that a simpler expression for this dilation : $\delta\chi = \bigvee \big\{\chi(p + b_1, q + b_2) \mid b_1, b_2 \in B\big\}$, PUD of the erosion of the hierarchy $\varepsilon\mathcal{X}$. The adjunct dilation $\delta\mathcal{X}(x, y)$ of the dendrogram is defined by the erosion of its PUD
$\varepsilon\chi(x, y) = \overbrace{\bigwedge \{\chi(x - b_1, y - b_2) \mid b_1, b_2 \in B\}}$.

The couple $(\varepsilon\mathcal{X}, \delta\mathcal{X})$ forms an adjunction for the partial hierarchies.

4.4 Ordering the Adjunctions on Partial Hierarchies or Partitions

Both adjunctions established above are ordered as:

- $\bigvee \{\chi(p + b, q + b) \mid b \in B\} \leq \bigvee \{\chi(p + b_1, q + b_2) \mid b_1, b_2 \in B\}$, showing that the partial hierarchy $\varepsilon\mathcal{X}$ is coarser than the partial hierarchy $\mathcal{X} \ominus B$
- $\overbrace{\bigwedge \{\chi(p - b_1, q - b_2) \mid b_1, b_2 \in B\}} \leq \overbrace{\bigwedge \{\chi(p - b, q - b) \mid b \in B\}}$ showing that the partial hierarchy $\delta\mathcal{X}$ is finer than the partial hierarchy $\mathcal{X} \oplus B$

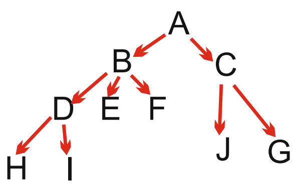

Fig. 5. A dendrogram

If the origin belongs to the structuring element we have the following order relations between the partial hierarchies $\varepsilon\mathcal{X} \leq \mathcal{X} \ominus B \leq \mathcal{X} \leq \mathcal{X} \oplus B \leq \delta\mathcal{X}$.

5 Conclusion

We have established two adjunctions for partial hierarchies, also valid for hierarchies, partitions and partial partitions through the associated PUD. The adjustment of the supports is treated automatically thanks to the introduction of aliens. Aliens and singletons have distinct definitions and distinct fates in the transformations. The morphological corpus can now be derived from these adjunctions. Iterating erosions or dilations increase their sizes. An erosion followed by its adjunct dilation produces an opening γ, a dilation followed by its adjunct erosion a closing φ. The classical filters $\gamma\varphi$, $\varphi\gamma$, $\gamma\varphi\gamma$, $\varphi\gamma\varphi$ may then be derived. Alternate sequential filters may be obtained by concatenating alternatively openings and closings of increasing sizes. One may also imagine geodesic dilations of hierarchies, by iterating an elementary geodesic dilation of a hierarchy under a hierarchy : $\delta\mathcal{X} \wedge \mathcal{Y}$ and $\varepsilon\mathcal{X} \vee \mathcal{Y}$.

References

1. Benzécri, J.P.: L'analyse des données 1. La taxinomie, ch. 3, pp. 119–153. Dunod (1973)
2. Heijmans, H.: Connected morphological operators for binary images. Computer Vision and Image Understanding 73(1), 99–120 (1999)
3. Matheron, G.: Filters and lattices. In: Serra, J. (ed.) Mathematical Morphology Volume II: theoretical advances. Academic Press, London (1988)
4. Meyer, F.: An overview of morphological segmentation. International Journal of Pattern Recognition and Artificial Intelligence 17(7), 1089–1118 (2001)
5. Ronse, C.: Partial partitions, partial connections and connective segmentation. J. Math. Imaging Vis. 32(2), 97–125 (2008)
6. Ronse, C.: Reconstructing masks from markers in non-distributive lattices. Appl. Algebra Eng. Commun. Comput. 19, 51–85 (2008)
7. Ronse, C.: Adjunctions on the lattices of partitions and of partial partitions. Appl. Algebra Eng., Commun. Comput. 21(5), 343–396 (2010)
8. Salembier, P., Garrido, L.: Connected operators based on region-tree pruning. In: Goutsias, J., Vincent, L., Bloomberg, D.S. (eds.) Mathematical Morphology and its Applications to Image and Signal Processing, vol. 18, pp. 169–178. Springer, US (2002)
9. Serra, J. (ed.): Image Analysis and Mathematical Morphology. II: Theoretical Advances. Academic Press, London (1988)
10. Serra, J.: Morphological operators for the segmentation of colour images. In: Bilodeau, M.S.M., Meyer, F. (eds.) Space, structure, and randomness. Contributions in honor of Georges Matheron in the fields of geostatistics, random sets, and mathematical morphology. Lecture Notes in Statistics, vol. 183, pp. 223–255, xviii, 395 p. Springer, New York (2005)
11. Serra, J.: A lattice approach to image segmentation. J. Math. Imaging Vis. 24, 83–130 (2006)