On the Functor ℓ^2

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Abstract. We study the functor ℓ^2 from the category of partial injections to the category of Hilbert spaces. The former category is finitely accessible, and in both categories homsets are algebraic domains. The functor preserves daggers, monoidal structures, enrichment, and various (co)limits, but has no adjoints. Up to unitaries, its direct image consists precisely of the partial isometries, but its essential image consists of all continuous linear maps between Hilbert spaces.

I am delighted to dedicate this paper to Samson Abramsky, on the occasion of his 60th birthday. Among all the wisdom he has imparted on me is this contradictory gem: "Never solve a problem completely, or noone will have a reason to cite you". My better nature gladly took some time off to let this paper follow his advice.

1 Introduction

The rich theory of Hilbert spaces underpins much of modern functional analysis and therefore quantum physics [24,20], yet important parts of it have resisted categorical treatment. In any categorical analysis of a species of mathematical objects, free objects of that kind play a significant role. The important ℓ^2–construction is in many ways the closest thing there is to a free Hilbert space: if X is a set, then

$$\ell^2(X) = \left\{\varphi\colon X \to \mathbb{C} \;\middle|\; \sum_{x\in X} |\varphi(x)|^2 < \infty\right\}$$

is a Hilbert space, in fact the only one of its dimension up to isomorphism. The ℓ^2–construction can be made into a functor, if we take partial injections as morphisms between the sets X, as first observed by Barr [6]. Outside functional analysis, it also plays a historically important role in the geometry of interaction (which has been noticed by many authors; an incomplete list of references includes [9,1,12,13,17]).

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Explicit categorical properties of the ℓ^2–construction are few and far between in the literature. These notes gather and augment them in a systematic study. Section 2 starts with the category of Hilbert spaces: it is self-dual, has two monoidal structures, and its homsets are algebraic domains, but its enrichment and limit behaviour is wanting. Section 3 discusses the category of partial injections, which is more well-behaved: it is also self-dual, has two monoidal structures, and is enriched over algebraic domains; moreover, it is finitely accessible. Section 4 introduces and studies the functor ℓ^2 itself. It preserves the self-dualities, monoidal structures, and enrichment. It also preserves (co)kernels and finite (co)products, but not general (co)limits. Therefore it has no adjoints, and in that sense does not provide free Hilbert spaces. It is faithful and essentially surjective on objects. Section 5 studies the image of the functor ℓ^2. Up to unitaries, its direct image consists precisely of partial isometries. Remarkably, it is essentially full, that is, its essential image is the whole category of Hilbert spaces.

Choice issues are lurking closely beneath the surface of these results. In fact, $\ell^2(X)$ is not just a Hilbert space; it carries a priviledged orthonormal basis. The functor ℓ^2 is an equivalence between the category of partial injections, and the category of Hilbert spaces with a chosen orthonormal basis and morphisms preserving it. But the latter class of morphisms is too restrictive: all interesting applications of Hilbert spaces require a change of basis. Following the guiding thought "a gentleman does not choose a basis", Section 6 suggests directions for further research.

2 The Codomain

Definition 2.1. We are interested in the category **Hilb**, whose objects are complex Hilbert spaces, and whose morphisms are continuous linear functions.

2.2. The category **Hilb** has a *dagger*, that is, a contravariant involutive functor $\dagger\colon \mathbf{Hilb}^{\mathrm{op}} \to \mathbf{Hilb}$ that acts as the identity on objects. On a morphism $f\colon H \to K$ it is given by the unique adjoint $f^\dagger\colon K \to H$ satisfying $\langle f(x) \,|\, y\rangle = \langle x \,|\, f^\dagger(y)\rangle$. For example, an isomorphism u is unitary when $u^{-1} = u^\dagger$.

2.3. Furthermore, the usual tensor product of Hilbert spaces provides the category **Hilb** with symmetric monoidal structure. The monoidal unit is the 1-dimensional Hilbert space $\mathbb{C}$. In fact, **Hilb** has *dagger symmetric monoidal* structure, *i.e.* $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and all coherence isomorphisms are unitaries.

2.4. Direct sums of Hilbert spaces provide the category **Hilb** with (finite) *dagger biproducts*. That is, $H \oplus K$ is simultaneously a product and a coproduct, the projections are the daggers of the corresponding coprojections, and $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$. Similarly, the 0-dimensional Hilbert space is a *zero object*, *i.e.* simultaneously initial and terminal.

2.5. Let us emphasize that we take continuous linear maps as morphisms between Hilbert spaces, rather than linear contractions. The category of Hilbert spaces with the latter morphisms is rather well-behaved, see *e.g.* [5]. However, it is the former choice of morphisms that is of interest in functional analysis and quantum physics. Unfortunately it also reduces the limit behaviour of the category **Hilb**, as the following lemma shows.

Lemma 2.6. *The category* **Hilb***:*

(i) has (co)equalizers;
(ii) does not have infinite (co)products;
(iii) does not have directed (co)limits.

Proof. Part (i) holds because **Hilb** is enriched over abelian groups and has kernels [16]. For (ii), consider the following counterexample. Define an $\mathbb{N}$-indexed family $H_n = \mathbb{C}$ of objects of **Hilb**. Suppose the family (H_n) had a coproduct H with coprojections $\kappa_n \colon H_n \to H$. Define $f_n \colon H_n \to \mathbb{C}$ by $f_n(z) = n \cdot \|\kappa_n\| \cdot z$. These are bounded maps, since $\|f_n\| = n \cdot \|\kappa_n\|$. Then for all $n \in \mathbb{N}$ the norm of the cotuple $f \colon H \to \mathbb{C}$ of (f_n) must satisfy

$$n \cdot \|\kappa_n\| = \|f_n\| = \|f \circ \kappa_n\| \leq \|f\| \cdot \|\kappa_n\|,$$

so that $n \leq \|f\|$. This contradicts the boundedness and hence continuity of f. Finally, part (iii) follows from (ii) and [23, IX.1.1] □

2.7. Despite the previous lemma, **Hilb** is *conditionally (co)complete*, in the sense that it does have objects that partially obey the universal property of infinite (co)products: for a family H_i of Hilbert spaces,

$$H = \big\{(x_i) \in \prod_i H_i \mid \sum_i \|x_i\|^2 < \infty\big\}.$$

is a well-defined Hilbert space under the inner product $\langle (x_i) \,|\, (y_i)\rangle = \sum_i \langle x_i \,|\, y_i\rangle$ [20]. The evident morphisms $\pi_i \colon H \to H_i$ satisfy $\pi_i \circ \pi_i^\dagger = \mathrm{id}$ and $\pi_i \circ \pi_j^\dagger = 0$ when $i \neq j$. A cone $f_i \colon K \to H_i$ allows a unique well-defined morphism $f \colon K \to H$ satisfying $\pi_i \circ f = f_i$ if and only if $\sum_i \|f_i\|^2 < \infty$. Note, however, that the cone (π_i) itself does not satisfy this condition. In this sense, $\ell^2(X)$ is the conditional coproduct of X many copies of $\mathbb{C}$.

2.8. A similar phenomenon occurs for simpler types of (co)limits. Monomorphisms in **Hilb** are precisely the injective morphisms, and epimorphisms are precisely those morphisms with dense range [15, A.3]. Not every monic epimorphism is an isomorphism. For example, the morphism $f \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $f(\varphi)(n) = \frac{1}{n}\varphi(n)$ is injective, self-adjoint, and hence also has dense image. But it is not surjective, as the vector $\varphi \in \ell^2(\mathbb{N})$ determined by $\varphi(n) = \frac{1}{n}$ is not in its range.

2.9. If $f, g \colon H \to K$ are morphisms in **Hilb**, then so are $f + g$ and zf for $z \in \mathbb{C}$. Because composition respects these operations, **Hilb** is enriched over complex vector spaces. In general, the homsets are not Hilbert spaces themselves [2], so

Hilb is not enriched over itself, and hence not Cartesian closed. At any rate, there is another way to structure the homsets of **Hilb**, which is of more interest here. Say $f \leq g$ when $\ker(f)^\perp \subseteq \ker(g)^\perp$ and $f(x) = g(x)$ for $x \in \ker(f)^\perp$. The following proposition shows that this makes all homsets into *algebraic domains* [4], but that this is not respected by composition. This is closely related to [8, 2.1.4], but **Hilb** is not a restriction category in the sense of that paper: setting $\overline{f}$ to be the projection onto $\ker(f)^\perp$ does not satisfy $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

Proposition 2.10. *All homsets in the category* **Hilb** *are algebraic domains, but composition is not monotone.*

Proof. The least upper bound of a directed family f_i is given by continuous extension to the closure of $\bigcup_i \ker(f_i)^\perp$; this makes all homsets into directed-complete partially ordered sets. If $f \leq \bigvee_i f_i$ always implies $f \leq f_i$ for some i, then $\ker(f)^\perp$ must have been finite-dimensional; thus morphisms f satisfying $\dim(\ker(f)^\perp) < \infty$ are the compact elements. It is now easy to see that any morphism is the directed supremum of compact ones below it, making all homsets into algebraic domains.

Now consider composition. First suppose that $f \leq f'$ and $g \leq g'$. If $x \in \ker(f)$, then clearly $gf(x) = 0$. If $x \in \ker(f)^\perp$, then $f(x) = f'(x)$, so $g'f'(x) = 0$ implies $f(x) \in \ker(g') \subseteq \ker(g)$. Because we may write $\mathrm{dom}(gf) = \ker(f) \oplus \ker(f)^\perp$, we conclude $\ker(gf)^\perp \subseteq \ker(g'f')^\perp$. But unless $f(\ker(gf)^\perp) \subseteq \ker(g)^\perp$, it need not be the case that gf equals $g'f'$ on $\ker(gf)^\perp$. For an explicit counterexample, let

$$f = f' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad g' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $f \leq f'$ and $g \leq g'$. But $\ker(gf)^\perp = \{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right) \mid x \in \mathbb{C}\}^\perp = \{\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right) \mid x \in \mathbb{C}\}$, and $gf\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2x \\ 0 \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 2x \\ x \end{smallmatrix}\right) = g'f'\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right)$, so $gf \not\leq g'f'$. □

3 The Domain

Definition 3.1. A *partial injection* is a partial function that is injective, wherever it is defined. More precisely, it(s graph) is a relation $R \subseteq X \times Y$ such that for each x there is at most one y with $(x, y) \in R$, and for each y there is at most one x with $(y, x) \in R$. Sets and partial injections form a category **PInj** under composition of relations $S \circ R = \{(x, z) \mid \exists y \colon (x, y) \in R, (y, z) \in S\}$.

3.2. Notationally, a partial injection $f \colon X \to Y$ can be conveniently represented as a span $(X \overset{f_1}{\leftarrowtail} F \overset{f_2}{\rightarrowtail} Y)$ of monics in **Set**. Here, f_1 is (the inclusion of) the domain of definition of f, and f_2 is its (injective) action on that domain. Composition in this representation is by pullback. We will also write $\mathrm{Dom}(f) = f_1(F)$ for the domain of definition, and $\mathrm{Im}(f) = f_2(F)$ for the range of f.

If it wasn't already, the span notation immediately makes it clear that **PInj** is a *dagger* category: $(X \overset{f_1}{\leftarrowtail} F \overset{f_2}{\rightarrowtail} Y)^\dagger = (Y \overset{f_2}{\leftarrowtail} F \overset{f_1}{\rightarrowtail} X)$.

3.3. The category **PInj** has two dagger *symmetric monoidal* structures. The first one, that we denote by $\otimes$, acts as the Cartesian product on objects. Because the Cartesian product of injections is again injective, $\otimes$ is well-defined on morphisms of **PInj** as well. The monoidal unit is a singleton set $\mathbf{1}$. Notice that $\otimes$ is not a product, and hence not a coproduct either.

The second dagger symmetric monoidal structure on **PInj**, denoted by $\oplus$, is given by disjoint union on objects. It is easy to see that a disjoint union of injections is again injective, making $\oplus$ well-defined on morphisms of **PInj**. The monoidal unit is the empty set. Notice that $\oplus$ is not a coproduct, and hence not a product either.

Lemma 3.4. *The category* **PInj***:*

(i) has (co)equalizers;
(ii) has a zero object;
(iii) does not have finite (co)products;

Proof. The equalizer of $f, g\colon X \to Y$ is the inclusion of

$$\big\{x \in X \mid x \notin (\mathrm{Dom}(f) \cup \mathrm{Dom}(g)) \vee \big(x \in (\mathrm{Dom}(f) \cap \mathrm{Dom}(g)) \wedge f(x) = g(x)\big)\big\}$$

into X. The empty set is a zero object in **PInj**.

Towards (iii), notice that if $(X \xrightarrow{\kappa_X} X + Y \xleftarrow{\kappa_Y} Y)$ were a coproduct in **PInj**, then one must have $\mathrm{Dom}(\kappa_X) = X$, $\mathrm{Dom}(\kappa_Y) = Y$ and $\mathrm{Im}(\kappa_X) \cap \mathrm{Im}(\kappa_Y) = \emptyset$, because otherwise unique existence of mediating morphisms is violated. Hence any coproduct must contain the disjoint union of X and Y. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be any morphisms. Then a mediating morphism $m\colon X + Y \to Z$ has to satisfy $m(x) = f(x)$ for $x \in \mathrm{Dom}(f)$ and $m(y) = g(y)$ for $y \in \mathrm{Dom}(g)$. But such an m is not unique, unless $\mathrm{Dom}(f) = X$ and $\mathrm{Dom}(g) = Y$. In fact, it is not even a partial injection unless $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$. We conclude that **PInj** does not have binary (co)products. □

3.5. In fact, part (ii) of the previous lemma follows from the existence of directed colimits, which we now work towards. Recall that a category has directed colimits if and only if it has colimits of chains, *i.e.* colimits of well-ordered diagrams [5, Corollary 1.7]. Observe that for a chain $D\colon I \to \mathbf{PInj}$, if $c_i\colon D(i) \to X$ is a cocone on D, then $\mathrm{Dom}(c_i) \subseteq \mathrm{Dom}(D(i \leq j))$ for all $j \geq i$. To see this, notice that $c_i = c_j \circ D(i \leq j)$ since c_i is a cocone, and therefore

$$\mathrm{Dom}(c_i) = \mathrm{Dom}(c_j \circ D(i \leq j)) \subseteq \mathrm{Dom}(D(i \leq j)).$$

This observation suggests that the colimit of a well-ordered diagram in **PInj** should consist of all 'infinite paths'. The following proposition shows that this is indeed a colimit.

Proposition 3.6. *The category* **PInj** *has directed colimits.*

Proof. Let $D\colon I \to \mathbf{PInj}$ be a chain. Define

$$X = \{x \in \coprod_i D(i) \mid \forall_{j \geq i}[x \in \mathrm{Dom}(D(i \leq j))]\}/\sim,$$

where the coproduct is taken in **Set**, and the equivalence relation $\sim$ is generated by $x \sim D(i \leq j)(x)$ for all $i \leq j$ in I and $x \in \mathrm{Dom}(D(i \leq j))$. For $i \in I$, define $c_i\colon D(i) \to X$ by

$$\mathrm{Dom}(c_i) = \{x \in D(i) \mid \forall_{j \geq i}[x \in \mathrm{Dom}(D(i \leq j))]\},$$

and $c_i(x) = [x]$.

First of all, let us show that the c_i form a cocone. One has:

$$\begin{aligned}
&\mathrm{Dom}(c_j \circ D(i \leq j)) \\
&= \{x \in D(i) \mid x \in \mathrm{Dom}(D(i \leq j)) \wedge D(i \leq j)(x) \in \mathrm{Dom}(c_j)\} \\
&= \{x \in D(i) \mid x \in \mathrm{Dom}(D(i \leq j)) \wedge \forall_{k \geq j}[D(i \leq j)(x) \in \mathrm{Dom}(D(j \leq k))]\}.
\end{aligned}$$

The well-orderedness of I implies that

$$\forall_{k \geq i}[P(k)] \Leftrightarrow \forall_{k \geq j}[P(k)] \wedge P(j)$$

for any property P on the objects of I, whence

$$\mathrm{Dom}(c_j \circ D(i \leq j)) = \{x \in D(i) \mid \forall_{k \geq i}[x \in \mathrm{Dom}(D(i \leq k))]\} = \mathrm{Dom}(c_i).$$

Moreover $c_j \circ D(i \leq j)(x) = [D(i \leq j)(x)] = [x] = c_i(x)$ for $x \in \mathrm{Dom}(c_i)$, by definition of the equivalence relation.

Next, we show that c_i is universal. Let $d_i\colon D(i) \to Y$ be any cocone, and define $m\colon X \to Y$ by

$$\mathrm{Dom}(m) = \{[x] \mid x \in \mathrm{Dom}(d_i)\}$$

and $m([x]) = d_i(x)$ for $x \in D(i)$; this is well-defined since d_i is a cocone. Then

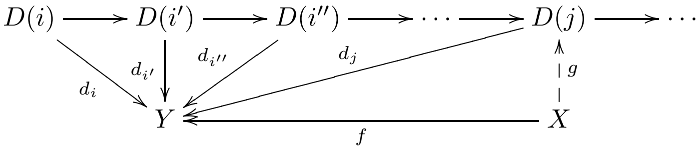
$$\begin{aligned}
\mathrm{dom}(m \circ c_i) &= \{x \in D(i) \mid x \in \mathrm{Dom}(c_i) \wedge m_i(x) \in \mathrm{Dom}(m)\} \\
&= \{x \in D(i) \mid \forall_{j \geq i}[x \in \mathrm{Dom}(D(i \leq j))] \wedge x \in \mathrm{Dom}(d_i)\} \\
&= \mathrm{Dom}(d_i)
\end{aligned}$$

by 3.5, and $m \circ c_i(x) = m([x]) = d_i(x)$ for $x \in D(i)$. Thus $m \circ c_i = d_i$, so m is indeed a mediating morphism.

Finally, if $m'\colon X \to Y$ satisfies $m \circ c_i = d_i$, then it follows from the above considerations that $\mathrm{Dom}(m') = \mathrm{Dom}(m)$ and $m'(x) = m(x)$ for $x \in \mathrm{Dom}(m)$. Hence m is the unique mediating morphism. □

3.7. Recall that an object X in a category $\mathbf{C}$ is called *finitely presentable* when the hom-functor $\mathbf{C}(X, -)\colon \mathbf{C} \to \mathbf{Set}$ preserves directed colimits. Explicitly, this means that for any directed poset $D\colon I \to \mathbf{C}$, any colimit cocone $d_i\colon D(i) \to Y$

and any morphism $f\colon X \to Y$, there are $j \in I$ and a morphism $g\colon X \to D(j)$ such that $f = d_j \circ g$. Moreover, this morphism g is essentially unique, in the sense that if $f = d_i \circ g = d_i \circ g'$, then $D(i \to i') \circ g = D(i \to i') \circ g'$ for some $i' \in I$.



A category is called *finitely accessible* [5] when it has directed colimits and every object is a directed colimit of finitely presentable objects.

Lemma 3.8. *A set is finitely presentable in* **PInj** *if and only if it is finite.*

Proof. The only thing, in the situation of 3.7 with X finite, is to notice that if a partial injection g is to exist, we must have $\mathrm{Dom}(g) = \mathrm{Dom}(f)$. The rest follows from [5, 1.2.1]. □

Theorem 3.9. *The category* **PInj** *is finitely accessible.*

Proof. It suffices to prove that every set in **PInj** is a directed colimit of finite ones. But that is easy: X is the colimit of the directed diagram consisting of its finite subsets. □

Definition 3.10. An *inverse category* is a category $\mathbf{C}$ in which every morphism $f\colon X \to Y$ allows a unique morphism $f^\dagger\colon Y \to X$ satisfying $f = ff^\dagger f$ and $f^\dagger = f^\dagger f f^\dagger$. Equivalently, it is a dagger category satisfying $f = ff^\dagger f$ and $pq = qp$ for idempotents $p, q\colon X \to X$. The proof of equivalence of these two statements is the same as for inverse semigroups (see [22, Theorem 1.1.3] or [8, Theorem 2.20]). Inverse categories are a special case of *restriction categories* [8].

The category **PInj** is an inverse category under its dagger (see 3.2). The following categorification of the Wagner–Preston theorem [22, Theorem 1.5.1] shows that it is in fact a representative one. See also [8, 3.4].

Proposition 3.11. *Any locally small inverse category* $\mathbf{C}$ *allows a faithful embedding* $F\colon \mathbf{C} \to \mathbf{PInj}$ *that preserves daggers.*

Proof. First suppose that $\mathbf{C}$ is small. Then we may set $F(X) = \coprod_{Z \in \mathbf{C}} \mathbf{C}(X, Z)$. For $f\colon X \to Y$ define $F(f)\colon F(X) \to F(Y)$ by $F(f) = (\_) \circ f^\dagger$ on the domain $\{g \in \mathbf{C}(X,Z) \mid Z \in \mathbf{C}, g = gf^\dagger f\}$; this gives a well-defined partial injection. It is functorial, since clearly $F(\mathrm{id}) = \mathrm{id}$, and

$$\begin{aligned}\mathrm{Dom}(F(gf)) &= \{h\colon X \to Z \mid h = hf^\dagger g^\dagger g f\} \\ &= \{h\colon X \to Z \mid h = hf^\dagger f, hf^\dagger = hf^\dagger g^\dagger g\} = \mathrm{Dom}(F(g) \circ F(f)).\end{aligned}$$

It preserves daggers, because $F(f^\dagger) = (\_) \circ f = F(f)^\dagger$, and

$$\begin{aligned}\mathrm{Dom}(F(f^\dagger)) &= \{h\colon Y \to Z \mid h = hff^\dagger\} \\ &= \{gf^\dagger\colon Y \to Z \mid g = gf^\dagger f\colon X \to Z\} = \mathrm{Im}(F(f)) = \mathrm{Dom}(F(f)^\dagger).\end{aligned}$$

Finally, F is clearly injective on objects. It is also faithful: if $F(f) = F(g)$, then $ff^\dagger = Ff(f) = Fg(f) = fg^\dagger$ and $gf^\dagger = Ff(g) = Fg(g) = gg^\dagger$, whence $fg^\dagger f = f$ and $gf^\dagger g = g$, and so $f = g$.

Now suppose **C** is locally small. Consider the diagram of small inverse subcategories **D** of **C**. It clearly has is a cocone to **C**. If $G_{\mathbf{D}} \colon \mathbf{D} \to \mathbf{E}$ is another one, there is a unique mediating functor $M \colon \mathbf{C} \to \mathbf{E}$ as follows. For an object X of **C**, let $\mathbf{D}'$ be the full subcategory of **C** with only one object X, and set $M(X) = G_{\mathbf{D}'}(X)$. For a morphism $f \colon X \to Y$ of **C**, let $\mathbf{D}''$ be the full subcategory of **C** on the objects X, Y, and set $M(f) = G_{\mathbf{D}''}(f)$. This gives a well-defined functor. So **C** is the colimit in **Cat** of its small inverse subcategories. By the above, any small inverse subcategory **C** embeds into **PInj**. It follows that **C** itself embeds into **PInj**. □

3.12. Like any inverse category, the homsets of **PInj** carry a natural partial order: $f \leq g$ when $f = gf^\dagger f$. Concretely, $f \leq g$ means $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)$ and $f(x) = g(x)$ for $x \in \mathrm{Dom}(f)$. It is easy to see that this makes homsets into directed-complete partially ordered sets, with $\mathrm{Dom}(\bigvee_i f_i) = \bigcup_i \mathrm{Dom}(f_i)$ for a directed family of morphisms $f_i \colon X \to Y$. In fact, as in Proposition 2.10, homsets are algebraic domains: any partial injection is the supremum of compact ones below it, which are those partial injections with finite domain. Moreover, composition respects these operations. Thus **PInj** is enriched in algebraic domains. This is a satisfying reflection of Theorem 3.9 on the level of homsets.

4 The Functor

Definition 4.1. There is a functor $\ell^2 \colon \mathbf{PInj} \to \mathbf{Hilb}$, acting on a set X as

$$\ell^2(X) = \{\varphi \colon X \to \mathbb{C} \mid \sum_{x \in X} |\varphi(x)|^2 < \infty\}.$$

This vector space becomes a well-defined Hilbert space under the inner product $\langle \varphi \,|\, \psi \rangle = \sum_{x \in X} \overline{\varphi(x)} \psi(x)$. The action on morphisms sends a partial injection $(X \leftarrow f_1 \prec F \succ f_2 \rightarrow Y)$ to the linear function $\ell^2 f \colon \ell^2(X) \to \ell^2(Y)$ determined informally by $\ell^2 f = (\_) \circ f^\dagger$. Explicitly,

$$(\ell^2 f)(\varphi)(y) = \sum_{x \in f_2^{-1}(y)} \varphi(f_1(x)).$$

4.2. In verifying that $\ell^2 f$ is indeed a well-defined morphism of **Hilb**, it is essential that f is a (partial) injection.

$$\begin{aligned} \sum_{y \in Y} \left|(\ell^2 f)(\varphi)(y)\right|^2 &= \sum_{y \in Y} \Big| \sum_{x \in f_2^{-1}(y)} \varphi(f_1(x))\Big|^2 \leq \sum_{y \in Y} \sum_{x \in f_2^{-1}(y)} |\varphi(f_1(x))|^2 \\ &= \sum_{x \in F} |\varphi(f_1(x))|^2 \leq \sum_{x \in X} |\varphi(x)|^2 < \infty. \end{aligned}$$

That this breaks down for functions f in general, instead of (partial) injections, was first noticed in [6], and further studied in [13]. That is, ℓ^2 is well-defined on the category of sets and partial injections; on the category of finite sets and functions; but not on the category of sets and functions; nor on the category of finite sets and relations. Functoriality of ℓ^2 is easy to verify.

4.3. The following calculation shows that the ℓ^2 functor preserves daggers. For a partial injection $(X \overset{f_1}{\twoheadleftarrow} F \overset{f_2}{\rightarrowtail} Y)$, $\varphi \in \ell^2(X)$ and $\psi \in \ell^2(Y)$:

$$\begin{aligned}\langle (\ell^2 f)(\varphi) \,|\, \psi\rangle_{\ell^2(Y)} &= \sum_{y\in Y} \overline{(\ell^2 f)(\varphi)(y)} \cdot \psi(y) = \sum_{y\in Y} \sum_{x \in f_2^{-1}(y)} \overline{\varphi(f_1(x))} \cdot \psi(y) \\ &= \sum_{x\in F} \overline{\varphi(f_1(x))} \cdot \psi(f_2(x)) = \sum_{x\in X} \sum_{x' \in f_1^{-1}(x)} \overline{\varphi(x)} \cdot \psi(f_2(x')) \\ &= \sum_{x\in X} \overline{\varphi(x)} \cdot \Big(\sum_{x'\in f_1^{-1}(x)} \psi(f_2(x'))\Big) = \langle \varphi \,|\, \ell^2(f^\dagger)(\psi)\rangle_{\ell^2(X)}.\end{aligned}$$

4.4. The functor ℓ^2 preserves the tensor product $\otimes$, *i.e.* it is symmetric (strong) monoidal. There is a canonical isomorphism $\mathbb{C} \cong \ell^2(\mathbf{1})$. The required natural morphisms $\ell^2(X) \otimes \ell^2(Y) \to \ell^2(X \otimes Y)$ are given by mapping (φ, ψ) to the function $(x, y) \mapsto \varphi(x)\psi(y)$. That there are inverses is seen when one realizes that $\ell^2(X \otimes Y)$ is the Cauchy-completion of the set of functions $X \times Y \to \mathbb{C}$ with finite support. The required coherence diagrams follow easily.

4.5. Also, the ℓ^2 functor is symmetric (strong) monoidal with respect to $\oplus$. There is a canonical isomorphism between the 0-dimensional Hilbert space and the set $\ell^2(\emptyset)$ consisting only of the empty function. The natural morphisms $\ell^2(X) \oplus \ell^2(Y) \to \ell^2(X \oplus Y)$ map (φ, ψ) to the cotuple $[\varphi, \psi] \colon X \oplus Y \to \mathbb{C}$. One sees that these are isomorphisms by recalling that $\ell^2(X \oplus Y)$ is the closure of the span of the Kronecker functions δ_x and δ_y for $x \in X$ and $y \in Y$, on which the inverse acts as the appropriate coprojection. Coherence properties readily follow.

4.6. From the description of the structure of homsets in **PInj** and **Hilb** as algebraic domains in 3.12 and 2.9, respectively, it is clear that the functor ℓ^2 preserves this enrichment: $\ell^2(\bigvee_i f_i) = \bigvee_i \ell^2 f_i$ if $f_i \colon X \to Y$ is a directed family of morphisms in **PInj**. See also [18, Theorem 13].

4.7. The functor ℓ^2 preserves (co)kernels and finite (co)products (because **PInj** has very few of the latter). But it follows from Lemma 2.6(iii) and Proposition 3.6 that ℓ^2 cannot preserve arbitrary (co)limits. For an explicit counterexample to preservation of equalizers, take $X = \{0, 1\}$, $Y = \{a\}$, and let $f, g \colon X \to Y$ be the partial injections $f = \{(0, a)\}$ and $g = \{(1, a)\}$. Their equaliser in **PInj** is $\emptyset$.

But

$$\begin{aligned}\mathrm{eq}(\ell^2(f), \ell^2(g)) &= \{\varphi \in \ell^2(X) \mid \ell^2(f)(\varphi) = \ell^2(g)(\varphi)\} \\ &= \Big\{\varphi \in \ell^2(X) \mid \forall_{y \in Y}.\ \sum_{u \in f_2^{-1}(y)} \varphi(f_1(u)) = \sum_{v \in g_2^{-1}(y)} \varphi(g_1(v))\Big\} \\ &= \{\varphi\colon \{0,1\} \to \mathbb{C} \mid \varphi(0) = \varphi(1)\} \cong \mathbb{C}.\end{aligned}$$

Hence $\mathrm{eq}(\ell^2(f), \ell^2(g)) \cong \mathbb{C} \not\cong \{\emptyset\} = \ell^2(\mathrm{eq}(f, g))$.

Corollary 4.8. *The functor* $\ell^2\colon \mathbf{PInj} \to \mathbf{Hilb}$ *has no adjoints.*

Proof. If ℓ^2 had an adjoint, it would preserve (co)limits, contradicting 4.7. □

4.9. The functor ℓ^2 is clearly faithful. It is also essentially surjective on objects: every Hilbert space H has an orthonormal basis X, so $H \cong \ell^2(X)$. It cannot be full because of 4.8, but it does reflect isomorphisms: if $\ell^2 f$ is invertible, so is f.

4.10. If X is a set, $\ell^2(X)$ is not just a Hilbert space; it comes equipped with a chosen orthonormal basis (given by the Kronecker functions $\delta_x \in \ell^2(X)$ for $x \in X$). Hence we could think of ℓ^2 as a functor to a category of Hilbert spaces H with a priviledged orthonormal basis $X \subseteq H$. If we choose as morphisms $(H, X) \to (K, Y)$ those continuous linear $f\colon H \to K$ satisfying $f(X) \subseteq Y$ and $ff^\dagger f = f$, then the functor ℓ^2 in fact becomes (half of) an equivalence of categories [3, 4.3].

4.11. Lemma 4.8 showed that $\ell^2(X)$ is not the free Hilbert space on X, at least not in the categorically accepted meaning. It also makes precise the intuition that 'choosing bases is unnatural': the functor $\ell^2\colon \mathbf{PInj} \to \mathbf{Hilb}$ cannot have a (functorial) converse, even though one can choose an orthonormal basis for every Hilbert space.

It is perhaps also worth mentioning that ℓ^2 is not a fibration in the technical sense of the word, not even a nonsplit or noncloven one, as the reader might perhaps think; Cartesian liftings in general do not exist because 'choosing bases is unnatural'.

5 The Image

5.1. The choice of morphisms in 4.10 is quite strong, and does not capture all morphisms of interest to quantum physics. From that point of view, one would at least like to relax to *partial isometries*: morphisms i of Hilbert spaces that satisfy $ii^\dagger i = i$. Equivalently, the restriction of i to the orthogonal complement of its kernel is an isometry. The following proposition proves that, up to isomorphisms, the direct image of the functor ℓ^2 consists precisely of partial isometries.

Definition 5.2. For a category $\mathbf{C}$, denote by $\mathbf{C}_{\cong}$ the groupoid with the same objects as $\mathbf{C}$ whose morphisms are the isomorphisms of $\mathbf{C}$.

The category $\mathbf{Hilb}_{\cong}$ is a groupoid, and hence has a dagger. It carries two dagger symmetric monoidal structures: $\oplus$ and $\otimes$. Because having (co)limits only depends on a skeleton of the specifying diagram, $\mathbf{Hilb}_{\cong}$ does not have (co)equalizers, nor (finite) (co)products, but does have directed (co)limits.

Proposition 5.3. *A morphism in* **Hilb** *is a partial isometry if and only if it is of the form* $v \circ \ell^2 f \circ u$ *for morphisms* f *in* **PInj** *and unitaries* u, v *in* $\mathbf{Hilb}_{\cong}$.

Proof. Clearly a map of the form $v \circ \ell^2 f \circ u$ is a partial isometry. Conversely, suppose that $i \colon H \to K$ is a partial isometry. Choose an orthonormal basis $X \subseteq H$ for its initial space $\ker(i)^\perp$, and choose an orthonormal basis $X' \subseteq H$ for $\ker(i)$, giving a unitary morphism $u \colon H \to \ell^2(X \oplus X')$. Let $Y = i(X) \subseteq K$. Then Y will be an orthonormal basis for the final space $\ker(i^\dagger)^\perp$ because i acts isometrically on X. Choose an orthonormal basis $Y' \subseteq K$ for $\ker(i^\dagger)$, giving a unitary $v \colon \ell^2(Y \oplus Y') \to K$. Now, if we define $f = (X \oplus X' \longleftarrow\!\!< X \rightarrowtail^{i} Y \oplus Y')$, then $i = v \circ \ell^2 f \circ u$. □

5.4. However, partial isometries are not closed under composition. To see this, consider the partial isometries $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \colon \mathbb{C} \to \mathbb{C}^2$ and $\begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \colon \mathbb{C}^2 \to \mathbb{C}$ for a fixed real number θ. Their composition is $\begin{pmatrix} \sin(\theta) \end{pmatrix} \colon \mathbb{C} \to \mathbb{C}$, which is not a partial isometry unless θ is a multiple of $\pi/2$. There are other compositions that do make partial isometries into a category [19], but these are not of interest here. Instead, we shall extend the previous proposition to highlight one of the most remarkable features of the functor ℓ^2.

5.5. The example in 5.4 shows that any linear function $\mathbb{C} \to \mathbb{C}$ between -1 and 1 is a composition of partial isometries. Note that the projections $\pi_i \colon \mathbb{C}^m \to \mathbb{C}$ and coprojections $\pi_i^\dagger \colon \mathbb{C} \to \mathbb{C}^n$ are partial isometries, as are the weighted diagonal $\Delta/\sqrt{n} \colon \mathbb{C} \to \mathbb{C}^n$ given by $\Delta(x) = (x, \ldots, x)$ and the weighted codiagonal $\Delta^\dagger/\sqrt{m} \colon \mathbb{C}^m \to \mathbb{C}$ given by $\Delta^\dagger(x_1, \ldots, x_m) = \sum_i x_i$. Moreover, it is easy to see that if f and g are (compositions of) partial isometries, then so is $f \oplus g$. Finally, any linear map $f \colon \mathbb{C}^m \to \mathbb{C}^n$ has a matrix expansion, and can hence be written in terms of biproduct structure as $f = \Delta^\dagger \circ (\bigoplus_{i=1}^m \bigoplus_{j=1}^n \pi_j^\dagger \circ \pi_j \circ f \circ \pi_i^\dagger \circ \pi_i) \circ \Delta$. Thus any $f \colon \mathbb{C}^m \to \mathbb{C}^n$ with $\|f\| \leq 1/\sqrt{mn}$ is a composition of partial isometries.

5.6. The *essential image* of a functor $F \colon \mathbf{C} \to \mathbf{D}$ is the smallest subcategory of $\mathbf{D}$ that contains all morphisms $F(f)$ for f in $\mathbf{C}$, and that is closed under composition with isomorphisms of $\mathbf{D}$.

It follows from 5.5 that the essential image of the functor ℓ^2 contains at least all morphisms of **Hilb** of finite rank. For infinite rank that strategy fails because Δ is then no longer a valid morphism (see 2.7). Nevertheless, Theorem 5.11 below will prove that the essential image of ℓ^2 is all of **Hilb**. In preparation we accommodate an intermezzo on *polar decomposition.*

A morphism $p \colon H \to H$ in **Hilb** is *nonnegative* when $\langle px \,|\, x \rangle \geq 0$ for all $x \in H$, and *positive* when $\langle px \,|\, x \rangle > 0$. Nonnegative maps are precisely those of the form $p = f^\dagger f$ for some morphism f.

Proposition 5.7. *For every morphism $f\colon H \to K$ between Hilbert spaces, there exist a unique nonnegative map $p\colon H \to H$ and partial isometry $i\colon H \to K$ satisfying $f = ip$ and* $\ker(p) = \ker(i)$.

Proof. See [14, problem 134]. □

5.8. The previous proposition stated the usual formulation of polar decomposition, but the unicity condition $\ker(p) = \ker(i)$ is something of a red herring. It should be understood as saying that both i and p are uniquely determined on the orthogonal complement of $\ker(f) = \ker(p) = \ker(i)$. On each point of $\ker(f)$, one of i and p must be zero, but the other's behaviour has no restrictions apart from being a partial isometry or positive map, respectively. Dropping the unicity condition, we may take p to be a positive map, by altering i to be zero on $\ker(f)$, and p to be nonzero on $\ker(f)$. More precisely, define $p' = p$ on $\ker(f)^{\perp}$ and $p' = \mathrm{id}$ on $\ker(f)$; since $\ker(f)$ is a closed subspace, $H \cong \ker(f) \oplus \ker(f)^{\perp}$, and this gives a well-defined positive operator $p'\colon H \to H$. Similarly, setting $i' = i$ on $\ker(f)^{\perp}$ and $i' = 0$ on $\ker(f)$ gives a well-defined partial isometry $i'\colon H \to K$, satisfying $f = i'p'$.

Lemma 5.9. *Positive operators on Hilbert spaces are isomorphisms.*

Proof. Let $p\colon H \to H$ be a positive operator in **Hilb**. Since p is self-adjoint, the spectral theorem [24] guarantees the existence of a measure space (S, Σ, μ), a unitary $u\colon H \to L^2(S, \Sigma, \mu)$, and a measurable function $f\colon S \to \mathbb{C}$ whose range is the spectrum $\sigma(p)$ of p, such that $p = u^{\dagger} \circ m \circ u$, where m is the multiplication operator induced by f. Because p is positive, f must take values in $\mathbb{R}^{>0}$. This makes the function $f^{-1}\colon S \to \mathbb{C}$ given by $s \mapsto f(s)^{-1}$ a well-defined measurable function. Let m^{-1} be the multiplication operator induced by f^{-1}. Then $u^{\dagger} \circ m^{-1} \circ u$ is the inverse of p. □

Definition 5.10. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *essentially full* when for each morphism g in $\mathbf{D}$ there exist f in $\mathbf{C}$ and u, v in $\mathbf{C}_{\cong}$ such that $g = v \circ Ff \circ u$.

It follows that the essential image of such a functor is all of $\mathbf{D}$.

Theorem 5.11. *The functor $\ell^2\colon \mathbf{PInj} \to \mathbf{Hilb}$ is essentially full.*

Proof. Let g be a morphism in **Hilb**. By Proposition 5.7 and 5.8, we can write $g = pi$ for a positive morphism p and a partial isometry i. Use Proposition 5.3 to decompose $i = v' \circ \ell^2 f \circ u$ for f in **PInj** and unitaries v', u. Finally, Lemma 5.9 shows that $v = p \circ v'$ in $\mathbf{Hilb}_{\cong}$ satisfies $g = v \circ \ell^2 f \circ u$. □

5.12. Writing $\mathbf{2}$ for the ordinal $2 = (0 \leq 1)$ regarded as a category, the category $\mathbf{C}^{\mathbf{2}}$ is the *arrow category* of $\mathbf{C}$: its objects are morphisms of $\mathbf{C}$, and its morphisms are pairs of morphisms of $\mathbf{C}$ making the square commute. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is essentially full if and only if $F^{\mathbf{2}}\colon \mathbf{C}^{\mathbf{2}} \to \mathbf{D}^{\mathbf{2}}$ is essentially surjective on objects. From this point of view Definition 5.10 is quite natural. Nonetheless we might consider weakening it to take $u = \mathrm{id}$ or $v = \mathrm{id}$. But this would break the previous theorem. For example, if $g\colon \ell^2(X) \to \ell^2(Y)$ is a morphism in **Hilb**, there need not be $f: X \to Y$ in **PInj** and v in $\mathbf{Hilb}_{\cong}$ with $g = v \circ \ell^2 f$. For a counterexample, take $X = Y = \{a, b\}$, and $g(a) = g(b) = a$; if $g = v \circ \ell^2 f$, then $(v \circ \ell^2 f)(a) = (v \circ \ell^2 f)(b)$, so $(\ell^2 f)(a) = (\ell^2 f)(b)$, so $f(a) = f(b)$, whence f cannot be a partial injection. Similarly, because of the dagger, if $g: \ell^2(X) \to \ell^2(Z)$ is a morphism in **Hilb**, there need not be $f: Y \to Z$ in **PInj** and u in $\mathbf{Hilb}_{\cong}$ with $g = \ell^2 f \circ u$.

6 The Future

6.1. Theorem 5.11 naturally raises a coherence question: is there any regularity to the isomorphisms u and v that enable us to write an arbitrary morphism of **Hilb** in the form $v \circ \ell^2 f \circ u$? How do they behave under composition? Curiously enough, essentially full functors do not seem to have been studied in the categorical literature at all. The results in this article suggest such a study.

It would be very interesting to reconstruct **Hilb** (up to equivalence) from $\mathbf{Hilb}_{\cong}$ and **PInj** via the ℓ^2 functor. The objects are easily recovered, because they are the same as those of $\mathbf{Hilb}_{\cong}$. Theorem 5.11 also lets us recover the homsets and identities, as soon as we can identify when two morphisms in **Hilb** of the form $v \circ \ell^2 f \circ u$ are equal. The main problem is how to recover composition, which requires a way to turn $\ell^2 g \circ v \circ \ell^2 f$ into $w \circ \ell^2 h \circ u$. (Note that turning $\ell^2 g \circ v$ into $w \circ \ell^2 h$ would be sufficient, because we could then use functoriality of ℓ^2 and composition in **PInj**. But 5.12 obstructs this; the isomorphism v in the middle is crucial.) This will likely lead into bicategorical territory.

6.2. The ℓ^2–construction has a continuous counterpart, that turns a measure space (X, μ) into a Hilbert space $L^2(X, \mu)$ of square integrable complex functions on X. The L^2–construction is quite fundamental and well-studied, but surprisingly enough functorial aspects seem not to have been considered before. One possibility is to mimic Definition 4.1, and endow the category of measure spaces with essential injections $(X, \mu) \to (Y, \nu)$ as morphisms, *i.e.* subsets $R \subseteq X \times Y$ such that $\nu(\{y \mid xRy\}) = 0$ for all $x \in X$ and $\mu(\{x \mid xRy\}) = 0$ for all $y \in Y$.

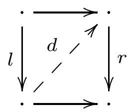
The importance of L^2–spaces lies in the following formulation of the spectral theorem: every normal operator $f: H \to H$ is of the form $f = u^{-1} \circ g \circ u$ for a unitary $u: H \to L^2(X, \mu)$ and an operator g induced by multiplication with a measurable function $X \to \mathbb{C}$. This perspective warrants choosing complex measurable functions as (endo)morphisms on measure spaces, with multiplication for composition. With 5.8 in mind, we could even restrict to a groupoid of positive maps. A solution to 6.1 could then be regarded as reconstructing quantum mechanics (as embodied by **Hilb**) from its continuous, quantitative aspects (encoded by the L^2 functor), and its discrete, qualitative aspects (encoded by the ℓ^2 functor).

At any rate, the continuous cousin L^2 of ℓ^2 poses an interesting research topic.

6.3. Letting $\mathcal{L}$ be the class of positive morphisms, and $\mathcal{R}$ the class of partial isometries in **Hilb**:

1. every morphism f can be factored as $f = rl$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$;

2. every commutative square as below with $l \in \mathcal{L}$ and $r \in \mathcal{R}$ allows a unique diagonal fill-in d making both triangles commute.



The second property follows immediately from Lemma 5.9. The established notion of *orthogonal factorization system* additionally demands that (3) both $\mathcal{L}$ and $\mathcal{R}$ are closed under composition, and (4) all isomorphisms are in both $\mathcal{L}$ and $\mathcal{R}$. But (3) is not satisfied by 5.4, and the map $-1\colon H \to H$ is a counterexample to (4).

Write **3** for the ordinal $3 = (0 \leq 1 \leq 2)$, regarded as a category. Then objects of $\mathbf{C}^\mathbf{3}$ are composable pairs of morphisms. Recall that a *functorial factorization* is a functor $F\colon \mathbf{C}^\mathbf{2} \to \mathbf{C}^\mathbf{3}$ that splits the composition functor. Lemma 5.9 ensures that polar decomposition at least provides a functorial factorization system. It is usual to require extra conditions on top of a functorial factorization, such as in a *natural weak factorization system*. For details we refer to [11]. It leads too far afield here, but polar decomposition does not satisfy the axioms of a natural weak factorization system.

In short, polar decomposition unquestionably provides a notion of factorization. But it does not fit existing categorical notions, despite the fact that factorization has been a topic of quite intense study in category theory [10,7,11,21,25]. This is an interesting topic for further investigation.

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