Imperfect Information in Logic and Concurrent Games

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Abstract. This paper builds on a recent definition of concurrent games as event structures and an application giving a concurrent-game model for predicate calculus. An extension to concurrent games with imperfect information, through the introduction of 'access levels' to restrict the allowable strategies, leads to a concurrent-game semantics for a variant of Hintikka and Sandu's Independence-Friendly (IF) logic.

Keywords: Concurrent games, Event structures, IF logic.

1 Introduction

Traditional games and strategies, in which one move is made at a time, have most often been represented by trees. If we are to develop a theory of concurrent, or distributed, games it seems sensible to investigate games and strategies formulated in terms of the concurrent analogue of trees, *viz.* event structures. (Just as transition systems unfold to trees so models such as Petri nets, which give an explicit account of concurrency, unfold to event structures [13]).

Concurrent games as event structures were introduced in [14] as a tentative new basis for the formal semantics of concurrent systems and programming languages. Such games carry an explicit representation of causal dependencies between moves. The concurrent-games model was extended in [5] by winning conditions in order to specify objectives for the players of the game. Games with winning conditions are a useful tool for expressing and solving problems in logic and verification.

The games studied in [5] are of perfect information. They are determined (*i.e.* there is a winning strategy for one of the two players) whenever they are well-founded and satisfy a structural property, called race-freedom, that prevents one player from interfering with the moves available to the other. The paper [5] provides a concurrent-game semantics for the predicate calculus, where nondeterministic winning strategies can be effectively built and deconstructed in a compositional manner.

This paper illustrates how by allowing imperfect information within concurrent games we obtain a compositional game semantics for a variant of Hintikka and Sandu's Independence-Friendly (IF) logic [7]; the concurrent-game semantics in this paper generalises that for the predicate calculus in [5]. A striking mathematical feature of the concurrent-game semantics is the facility with which event

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structures lend themselves to the form of dependence and independence central to IF logic and its variants.

The extension to concurrent games with imperfect information is achieved by adjoining 'access levels.' It was guided originally by the wish to handle games with imperfect information in a way that respects the existing bicategorical structure of concurrent games. There are strong similarities with work by Samson Abramsky and Radha Jagadeesan on an extension of AJM games to handle access control [1].

Related Work. Perhaps the first encounter of logic with imperfect information was in Henkin's generalisation of first-order quantifiers to free up the dependencies between quantified variables [6]. His idea led to other revisions of first-order logic: Hintikka and Sandu's Independence-Friendly (IF) logic [7]; Väänänen's Dependence logic [15] and its 'team semantics;' the latter being a variant of Hodges' compositional semantics of IF logic [8]. Semantics for such logics are often given in terms of games with imperfect information, in which players only have access to a limited, 'visible' part of the history of the games they play. Imperfect information is often captured by requiring that strategies behave in a uniform manner across plays with the same visible history. With concurrent games as event structures we can express imperfect information by specifying the permitted causal dependencies directly.

Within the theory of concurrent computation we see the modal and fixed-point variants of IF logic developed by Bradfield *et al* [4,3] and the alternating-time temporal logic (ATL) of Alur, Henzinger and Kupferman [2]. In modal IF logic a direct link is made between the independence of IF logic and the independence of actions seen in concurrent computation, a correspondence echoed in the semantics of IF logic presented here. The semantics of ATL is given in terms of 'concurrent game structures,' which are essentially Blackwell games [11]. The two players in a Blackwell game play in a series of rounds in which they choose their moves independently. We shall see how to express such rounds via access levels within the broader framework of concurrent games as event structures—see Example 2.

2 Event Structures and Concurrent Games

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy four axioms:

$$\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \text{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Often we are concerned with just the finite configurations of E. We write $\mathcal{C}(E)$ for the *finite* configurations of E.

We say an event structure is *elementary* when the consistency relation consists of all finite subsets of events. Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between plays an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X ; note if $X \in \text{Con}$ then $[X] \in \text{Con}$. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.* $x \subset y$ with nothing in between, and $x \overset{e}{-\!\!\!-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \overset{e}{-\!\!\!-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{-\!\!\!-\!\!\subset} y$ for some y.

Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$ (with both defined) then $e_1 = e_2$. The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Partial maps of event structures compose as partial functions, with identity maps given by identity functions. We say that the map is *total* if the function f is total. A total map of event structures which preserves causal dependency is called *rigid*.

The category of event structures is rich in useful constructions on processes. In particular, it has products and pullbacks (both forms of synchronised composition) and coproducts (nondeterministic sums). Event structures support a simple form of hiding associated with a factorization system. Let $(E, \leq, \text{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\text{def}} (V, \leq_V, \text{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \text{Con}_V$ iff $X \in \text{Con}$ & $X \subseteq V$. Consider a partial map of event structures $f : E \to E'$. Let $V =_{\text{def}} \{e \in E \mid f(e) \text{ is defined}\}$. Then f clearly factors into the composition $E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$ of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V.

Event Structures with Polarities. Both a game and a strategy in a game are represented in terms of an event structure with polarity, comprising an event structure E together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ (Player) or $-$ (Opponent) to its events; the events correspond to moves. Maps of event structures with polarity are maps of event structures which preserve polarities.

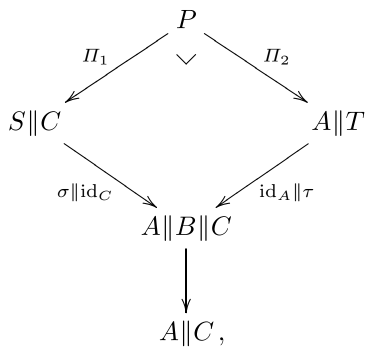
Event structures with polarities support two key operations. The *dual*, $E^\perp$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities. The *simple parallel composition* $E \| E'$ forms the disjoint juxtaposition of E and E', two event structures with polarity; a finite subset of events is consistent if its intersection with each component is consistent.

2.1 Concurrent Games and Strategies

Pre-strategies. Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* and *winning strategy*. Formally, a *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S. A map between pre-strategies $\sigma : S \to A$ and $\tau : T \to A$ in A will be a map $\theta : S \to T$ such that $\sigma = \tau\theta$. When θ is an isomorphism we write $\sigma \cong \tau$.

Let A, B be event structures with polarity. Following Joyal [9], a pre-strategy from A to B is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. We write $\sigma : A \mathrel{\rightarrow\!\!\!\!\!+} B$ to express that σ is a pre-strategy from A to B. Note that a pre-strategy σ in A coincides with a pre-strategy from the empty game $\sigma : \varnothing \mathrel{\rightarrow\!\!\!\!\!+} A$.

Composing Pre-strategies. We can present the composition of pre-strategies via pullbacks. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in the category of event structures as shown below



where the map of event structures $A \| B \| C \to A \| C$ is undefined on B and acts as identity on A and C. The partial map from P to $A \| C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P \downarrow V$, where V is the set of events of P at which the map $P \to A \| C$ is defined, and a total map $P \downarrow V \to A \| C$. The resulting total map gives us the composition $\tau \odot \sigma : P \downarrow V \to A^\perp \| C$ once we reinstate polarities.

Concurrent Copy-Cat. Identities w.r.t. composition are copy-cat strategies. Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$. For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}\mathbb{C}_A$ to comprise the event structure with polarity $A^\perp \| A$ together with the extra causal dependencies generated by $\overline{c} \leq_{\mathbb{C}\mathbb{C}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$. The *copy-cat* pre-strategy $\gamma_A : A \twoheadrightarrow A$ is defined to be the map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ where γ_A is the identity on the common set of events.

Strategies. The main result of [14] is that two conditions on pre-strategies, called *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. Innocence, on the other hand, restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game. Formally:

Receptivity: A pre-strategy σ is *receptive* iff
$\sigma x \overset{a}{\longrightarrow\!\subset}\ \&\ pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\subset}\ \&\ \sigma(s) = a\,.$
Innocence: A pre-strategy σ is *innocent* when it is both
$+$-*innocent:* if $s \rightarrowtriangle s'\ \&\ pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$, and
$-$-*innocent:* if $s \rightarrowtriangle s'\ \&\ pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

Theorem 1 (from [14]). *Let $\sigma : A \twoheadrightarrow B$ be pre-strategy. Copy-cat behaves as identity w.r.t. composition, i.e. $\sigma \circ \gamma_A \cong \sigma$ and $\gamma_B \circ \sigma \cong \sigma$, iff σ is receptive and innocent. Copy-cat pre-strategies $\gamma_A : A \twoheadrightarrow A$ are receptive and innocent.*

Then, a *strategy* is a pre-strategy which is receptive and innocent. In fact, we obtain a bicategory, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \twoheadrightarrow B$ and the 2-cells are maps of (pre-)strategies, defined above. A strategy $\sigma : A \twoheadrightarrow B$ corresponds to a dual strategy $\sigma^\perp : B^\perp \twoheadrightarrow A^\perp$. This duality arises from the correspondence between pre-strategies $\sigma : S \to A^\perp \| B$ and $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$.

Deterministic Games and Strategies. There is the important subcategory of *deterministic* strategies. An event structure with polarity S is deterministic iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = -\ \&\ \exists s \in X.\ s' \leq s\}$. Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic. Deterministic strategies are necessarily mono, and so can be identified with certain subfamilies of configurations of the game, and in fact coincide with the receptive ingenuous strategies of Mimram and Melliès [12]. While deterministic strategies do compose, a copy-cat strategy γ_A can fail to be deterministic. However, γ_A is deterministic iff there is no immediate conflict between +ve and −ve events, a condition we call ‘race-free:’

$$x \overset{a}{\longrightarrow\!\subset}\ \&\ x \overset{a'}{\longrightarrow\!\subset}\ \&\ pol(a) \neq pol(a') \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(Race-free)}$$

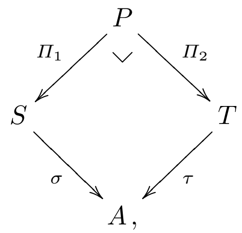
We obtain a sub-bicategory by restricting objects to race-free games and strategies to deterministic ones. Via the presentation of deterministic strategies as subfamilies of configurations, the sub-bicategory of deterministic games and strategies is equivalent to a mathematically simpler order-enriched category.

3 Winning Strategies and Determinacy

A *concurrent game with winning conditions* [5] comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player. We define the *losing conditions* to be $\mathcal{C}^\infty(A) \setminus W$.

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$—a configuration x is +-maximal if whenever $x \stackrel{s}{\longrightarrow\!\!\!\!\subset}$ then the event s has −ve polarity.

Equivalently, a strategy for Player is winning if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose that $\sigma : S \to A$ is a strategy in a game (A, W). A counter-strategy is a strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. We can view σ as a strategy $\sigma : \varnothing \mathrel{\longrightarrow\!\!\!\!+} A$ and τ as a strategy $\tau : A \mathrel{\longrightarrow\!\!\!\!+} \varnothing$. Their composition $\tau \odot \sigma : \varnothing \mathrel{\longrightarrow\!\!\!\!+} \varnothing$ is not very informative; rather it is the set of configurations in $\mathcal{C}^\infty(A)$ their full interaction induces what decides which player wins. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,



to obtain the event structure P resulting from the interaction of σ and τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^\infty(P)$. A maximal configuration z images to a configuration $\sigma \Pi_1 z = \tau \Pi_2 z$ in $\mathcal{C}^\infty(A)$. Define the set of *results* of playing σ against τ to be

$$\langle \sigma, \tau \rangle =_{\mathrm{def}} \{\sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P)\}\,.$$

It can be shown [5], that a strategy σ is a winning for Player iff all the results of the interaction $\langle \sigma, \tau \rangle$ lie within the winning configurations W, for any (deterministic) counter-strategy $\tau : T \to A^\perp$ of Opponent.

Operations. There is a *dual*, $G^\perp$, of a game with winning conditions $G = (A, W_G)$, defined as $G^\perp = (A^\perp, \mathcal{C}^\infty(A) \setminus W_G)$, which reverses the role of Player and Opponent, and consequently that of winning and losing conditions. Moreover, the *parallel composition* of two games with winning conditions $G = (A, W_G)$, $H = (B, W_H)$ is $G \parr H =_{\text{def}} (A \| B,\ W_{G \parr H})$ where, for $x \in \mathcal{C}^\infty(A\|B)$, $x \in W_{G\parr H}$ iff $x_1 \in W_G$ or $x_2 \in W_H$ —a configuration x of $A\|B$ comprises the disjoint union of a configuration x_1 of A and a configuration x_2 of B. To win in $G \parr H$ is to win in either game. The unit of $\parr$ is $(\varnothing, \varnothing)$. Defining $G \otimes H =_{\text{def}} (G^\perp \| H^\perp)^\perp$ we obtain a game where to win is to win in both games G and H. The unit of $\otimes$ is $(\varnothing, \{\varnothing\})$. Defining $G \multimap H =_{\text{def}} G^\perp \parr H$, a win in $G \multimap H$ is a win in H conditional on a win in G: For $x \in \mathcal{C}^\infty(A^\perp \| B)$, $x \in W_{G \multimap H}$ iff $x_1 \in W_G \implies x_2 \in W_H$.

Again following Joyal, a (winning) strategy from G to H, two games with winning conditions, is a (winning) strategy in $G \multimap H$. We compose strategies as before. The composition of winning strategies is winning. However, for a general game (A, W) the copy-cat strategy need not be winning. A necessary and sufficient condition for copy-cat to be winning is given in [5]—see **(Cwins)** of Section 4 for its precise statement. The condition is assured for games which are race-free. We can refine the bicategories studied in [14] to bicategories of concurrent games with winning conditions [5].

Determinacy for Well-Founded, Race-Free Concurrent Games. A game with winning conditions is said to be *determined* when either Player or Opponent has a winning strategy. Not all games are determined.

Example 1. Consider the event structure A with two inconsistent events $\oplus$ and $\ominus$ with the obvious polarities and winning conditions $W = \{\{\oplus\}\}$. In the game (A, W) no strategy for either player wins against all other counter-strategies of the other player. In particular, let σ be the unique map of event structures that contains $\oplus$ and τ a particular counter-strategy for Opponent:

Then, neither $\langle\sigma, \tau\rangle \subseteq W$ nor $\langle\sigma, \tau\rangle \subseteq L$ since $\{\{\oplus\}, \{\ominus\}\} \subseteq \langle\sigma, \tau\rangle$. ∎

Note that G is not *race-free*. Being race-free is not in itself sufficient to ensure a game is determined. However, with respect to the class of well-founded games, *i.e.* games where all configurations in $\mathcal{C}^\infty(A)$ are finite, we have the following:

Theorem 2 (from [5]). *Let A be a well-founded event structure with polarity. The game (A, W) determined for all winning conditions W iff A is race-free.*

It is tempting to believe that a nondeterministic winning strategy always has a winning deterministic sub-strategy. This is not so and determinacy does not hold for well-founded race-free games if we restrict to deterministic strategies.

Nondeterministic (winning) strategies are also useful if one wants to define a partial-order concurrent-game semantics for classical logics—as shown next.

3.1 Application: Concurrent Games for the Predicate Calculus

The syntax for predicate calculus: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\ \phi \mid \forall x.\ \phi$$

where R ranges over basic relation symbols and $x, x_1, x_2, \cdots, x_k$ over variables.

A model M for the predicate calculus comprises a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M. We can then define, by structural induction, the truth of a formula of predicate logic w.r.t. an assignment of values in V_M to the variables of the formula. We write $\rho \vDash_M \phi$ iff formula ϕ is true in M w.r.t. environment ρ; we take an environment to be a function from variables to values.

W.r.t. a model M and an environment ρ, we can denote a formula ϕ by $[\![\phi]\!]_M \rho$, a concurrent game with winning conditions, so that $\rho \vDash_M \phi$ iff there is a winning strategy in $[\![\phi]\!]_M \rho$ (for Player). The denotation as a game is defined as:

$$[\![R(x_1, \cdots, x_k)]\!]_M \rho = \begin{cases} (\varnothing, \{\varnothing\}) & \text{if } \rho \vDash_M R(x_1, \cdots, x_k)\,, \\ (\varnothing, \varnothing) & \text{otherwise.} \end{cases}$$

$$[\![\phi \wedge \psi]\!]_M \rho = [\![\phi]\!]_M \rho \otimes [\![\psi]\!]_M \rho \quad [\![\phi \vee \psi]\!]_M \rho = [\![\phi]\!]_M \rho \parr [\![\psi]\!]_M \rho \quad [\![\neg\phi]\!]_M \rho = ([\![\phi]\!]_M \rho)^\perp$$

$$[\![\exists x.\ \phi]\!]_M \rho = \bigoplus_{v \in V_M} [\![\phi]\!]_M \rho[v/x] \qquad [\![\forall x.\ \phi]\!]_M \rho = \bigominus_{v \in V_M} [\![\phi]\!]_M \rho[v/x]\,.$$

We use $\rho[v/x]$ to mean the environment ρ updated to assign value v to variable x. The game $(\varnothing, \{\varnothing\})$, the unit w.r.t. $\otimes$, is the game used to denote true and the game $(\varnothing, \varnothing)$, the unit w.r.t. $\parr$, to denote false. Denotations of conjunctions and disjunctions are denoted by the operations of $\otimes$ and $\parr$ on games, while negations denote dual games. Universal and existential quantifiers denote *prefixed sums* of games, operations which we now describe in the following paragraph.

The game $\bigoplus_{v \in V} (A_v, W_v)$ has underlying event structure with polarity the sum $\sum_{v \in V} \oplus.A_v$ where the winning conditions of a component are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. In $\sum_{v \in V} \oplus.A_v$ a configuration is winning iff it is the image of a winning configuration in a component under the injection to the sum. Note in particular that the empty configuration of $\bigoplus_{v \in V} G_v$ is not winning—Player must make a move in order to win. The game $\bigominus_{v \in V} G_v$ is defined dually, as $(\bigoplus_{v \in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move. Writing $G_v = (A_v, W_v)$, the underlying event structure of $\bigominus_{v \in V} G_v$ is the sum $\sum_{v \in V} \ominus.A_v$ with a configuration winning iff it is empty or the image under the injection of a winning configuration in a prefixed component.

It is easy to check by structural induction that for any formula ϕ the game $[\![\phi]\!]_M \rho$ is well-founded and race-free, so determined by Theorem 2. With the help of techniques to build and deconstruct strategies we can establish:

Theorem 3 (from [5]). *For all formulae ϕ and environments ρ, we have that $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M \rho$ has a winning strategy, for Player.*

4 Concurrent Games with Imperfect Information

We show how to extend concurrent games by imperfect information to form a bicategory, which in the case of deterministic strategies specializes to an order-enriched bicategory.

We first introduce the framework of games with imperfect information through a simple example.

Consider the game "rock, scissors, paper" in which the two participants Player and Opponent independently sign one of r ("rock"), s ("scissors"), or p ("paper"). The participant with the dominant sign w.r.t. the relation

$$r \textit{ beats } s, \;\; s \textit{ beats } p \text{ and } p \textit{ beats } r$$

wins. We could represent this game by RSP, the event structure with polarity

with the three mutually inconsistent signings of Player in parallel with the three mutually inconsistent signings of Opponent. Without neutral configurations, a reasonable choice is to take the *losing* configurations (for Player) to be

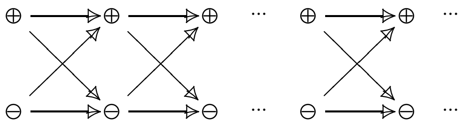
$$\{s_1, r_2\}, \;\; \{p_1, s_2\}, \;\; \{r_1, p_2\}$$

and all other configurations as winning for Player. In this case there is a winning strategy for Player, *viz.* await the move of Opponent and then beat it with a dominant move. But this strategy cheats. In "rock, scissors, paper" participants are intended to make their moves *independently.* The problem with the game RSP as it stands is that it is a game of *perfect information* in the sense that all moves are visible to both participants. This permits the winning strategy above with its unwanted dependencies on moves which should be unseen by Player. In order to model "rock, scissors, paper" more adequately we can use a concurrent game with *imperfect information* where some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out.

To extend concurrent games with imperfect information while respecting the bicategorical structure of games we assume a fixed preorder of *levels* $(\Lambda, \preceq)$. The levels are to be thought of as levels of access, or permission [1]. Moves in games and strategies are to respect levels: moves will be assigned levels in such a way that a move is only permitted to causally depend on moves at equal or lower levels; it is as if from a level only moves of equal or lower level can be seen.

A Λ-game (G, l_G) comprises a game $G = (A, W)$ with winning conditions together with a *level function* $l_G : A \to \Lambda$ such that $a \leq_A a' \implies l_G(a) \preceq l_G(a')$ for all events $a, a' \in A$. A Λ-strategy in the Λ-game (G, l_G) is a winning strategy $\sigma : S \to A$ for which $s \leq_S s' \implies l_G \sigma(s) \preceq l_G \sigma(s')$ for all $s, s' \in S$. For example, for "rock, scissors, paper" we can take Λ to be the discrete preorder consisting of levels 1 and 2 unrelated to each other under $\leq$. To make RSP into a suitable Λ-game the level function l_G takes +ve events in RSP to level 1 and −ve events to level 2. The (winning) strategy above, where Player awaits the move of Opponent then beats it with a dominant move, is now disallowed as it is not a Λ-strategy—it introduces causal dependencies which do not respect levels. If instead we took Λ to be the unique preorder on a single level the Λ-strategies would coincide with all the strategies.

Example 2. Through levels we can restrict play to a series of rounds in the way of *Blackwell games* [11] and *concurrent game structures* [2]. An appropriate choice of Λ is the infinite elementary event structure:



Consider A, a race-free concurrent game, for which there is a (necessarily unique) polarity-preserving rigid map from A to Λ—this map becomes the level function. The existence of such a map ensures moves in A occur in rounds comprising a choice of move for Opponent and a choice of move for Player made concurrently. ▮

The introduction of levels meshes smoothly with the bicategorical structure of concurrent games. For a Λ-game (G, l_G), define its dual $(G, l_G)^\perp$ to be $(G^\perp, l_{G^\perp})$ where $l_{G^\perp}(\overline{a}) = l_G(a)$, for a an event of G. Similarly, for Λ-games (G, l_G) and (H, l_H), define their parallel composition $(G, l_G) \parr (H, l_H)$ to comprise $G \parr H$ with levels those inherited from the components. A strategy between Λ-games from (G, l_G) to (H, l_H) is a strategy in $(G, l_G)^\perp \parr (H, l_H)$.

As mentioned earlier, in general a copycat strategy is not necessarily winning. Each event structure with polarity A possesses a 'Scott order' on its configurations $\mathcal{C}^\infty(A)$: $x' \sqsubseteq x$ iff $x' \supseteq^- x \cap x' \subseteq^+ x$, where we use the inclusions $x \subseteq^- y$ iff $x \subseteq y$ & $pol_A(y \setminus x) \subseteq \{-\}$ and $x \subseteq^+ y$ iff $x \subseteq y$ & $pol_A(y \setminus x) \subseteq \{+\}$, for $x, y \in \mathcal{C}^\infty(A)$. The 'Scott-order' is in fact a partial order. It is helpful in expressing a necessary and sufficient condition for copy-cat to be winning w.r.t. a game (G, l_G):

$$\begin{aligned}&\text{if } x' \sqsubseteq x \ \&\ x' \text{ is } +\text{-maximal} \ \&\ x \text{ is } -\text{-maximal},\\ &\text{then} \quad x \in W \implies x' \in W, \ \text{ for all } x, x' \in \mathcal{C}^\infty(A)\,.\end{aligned} \qquad \textbf{(Cwins)}$$

The condition **(Cwins)** is automatically satisfied when A is race-free. We can now state:

Theorem 4. *Let (G, l_G) be a Λ-game.*
(i) If G satisfies **(Cwins)***, then the copy-cat strategy on G is a Λ-strategy.*
(ii) The composition of Λ-strategies is a Λ-strategy.

5 Λ-IF: A Parametrized Logic of Independence

We present a variant of Hintikka and Sandu's Independence-Friendly (IF) logic and propose a semantics in terms of concurrent games with imperfect information. Our logic is parametrized by a preorder that states the possible dependencies between variables. Assume a preorder $(\Lambda, \leq)$. The syntax for our logic, denoted by Λ-IF, is essentially that of the predicate calculus, but with levels in Λ associated with quantifiers: formulae, where $\lambda \in \Lambda$, are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists^{\lambda} x.\ \phi \mid \forall^{\lambda} x.\ \phi$$

where R ranges over basic relation symbols and $x, x_1, x_2, \cdots, x_k$ over variables.

Assume M, a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M. W.r.t. a model M and an environment ρ, we denote each closed formula ϕ of Λ-IF logic by a Λ-game, following very closely the definitions for predicate calculus. The differences are the assignment of levels to events and that the order on Λ has to be respected by the (modified) prefixed sums which quantified formulae denote.

The prefixed game $\oplus^{\lambda}.(A, W, l)$ comprises the event structure with polarity $\oplus.A$ in which all the events of $a \in A$ where $\lambda \leq l(a)$ are made to causally depend on a fresh +ve event $\oplus$, itself assigned level λ. Its winning conditions are those configurations $x \in \mathcal{C}^{\infty}(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. The game $\bigoplus^{\lambda}_{v \in V}(A_v, W_v, l_v)$ has underlying event structure with polarity the sum $\sum_{v \in V} \oplus^{\lambda}.A_v$, maintains the same levels as its components, with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. The game $\bigominus^{\lambda}_{v \in V} G_v$ is defined dually, as $(\bigoplus^{\lambda}_{v \in V} G_v^{\perp})^{\perp}$.

True denotes the Λ-game the unit w.r.t. $\otimes$ and false denotes the unit w.r.t. $\parr$. Denotations of conjunctions and disjunctions are given by the operations of $\otimes$ and $\parr$ on Λ-games, while negations denote dual games. W.r.t. an environment ρ and a model M, quantifiers are denoted by *prefixed sums* of Λ-games:

$$[\![\exists^{\lambda} x.\ \phi]\!]^{\Lambda}_{M} \rho = \bigoplus^{\lambda}_{v \in V_M} [\![\phi]\!]^{\Lambda}_{M} \rho[v/x] \qquad [\![\forall^{\lambda} x.\ \phi]\!]^{\Lambda}_{M} \rho = \bigominus^{\lambda}_{v \in V_M} [\![\phi]\!]^{\Lambda}_{M} \rho[v/x] \, .$$

Definition 1. *For all Λ-IF formulae ϕ, environments ρ, and models M, we say that: (i) ϕ is* true *in M w.r.t. ρ, written $\rho \vDash^{\Lambda}_{M} \phi$, iff Player has a winning strategy in the Λ-game $[\![\phi]\!]^{\Lambda}_{M}\rho$; (ii) ϕ is* false *in M w.r.t. ρ, written $\rho \overline{\vDash}^{\Lambda}_{M} \phi$, iff Opponent has a winning strategy in $[\![\phi]\!]^{\Lambda}_{M}\rho$ (note that $\rho \overline{\vDash}^{\Lambda}_{M} \phi$ is equivalent to $\rho \vDash^{\Lambda}_{M} \neg\phi$ but different from $\rho \nvDash^{\Lambda}_{M} \phi$); and (iii) ϕ is* undetermined *in M w.r.t. ρ, otherwise.*

6 Λ-IF Logic vs. IF Logic

The language of Λ-IF formulae is, essentially, that of IF logic where two Λ-IF variables are incomparable w.r.t. $(\Lambda, \leq)$ if they are independent within IF logic. Some similarities and differences between IF logic and Λ-IF logic are illustrated in the following examples.

Example 3. Let Λ be the poset with two incomparable elements 1 and 2 , *i.e.* neither $1 \leq 2$ nor $2 \leq 1$, and consider the formula: $\phi = \forall^1 x.\ \exists^2 y.\ x = y$, whose semantics gives rise to a Λ-game (G, l_G) played on an event structure A whose set of events is (isomorphic to) $V_M + V_M$, with the discrete partial order as causal dependency, consistency $X \in \mathrm{Con}$ if the restriction of X to either copy of V_M is a singleton set of events, and polarity $pol((1,v)) = -$ and $pol((2,v)) = +$. The winning conditions are $W = \{\varnothing, \{\{(2,v)\} \mid v \in V_M\}, \{\{(1,v),(2,v)\} \mid v \in V_M\}\}$. Finally, the level function l_G sends $(1,v)$ to 1 and $(2,v)$ to 2, for all $v \in V_M$. It can be checked that whenever V_M has at least two distinct elements neither player has a winning strategy. Then, ϕ is *undetermined* in the model M. ∎

Example 4. Now, consider the same formula as in Example 3 but with the partial order Λ' containing two elements 1, 2 with $1 \leq 2$. Its interpretation yields a Λ'-game (G', l'_G) played on the event structure with polarity A' which differs from A in that we now have $(1,v) \leq (2,v)$ for all $v \in V_M$. The winning condition and the level function are unchanged, but this game now has a winning strategy: the identity map of event structures $\mathrm{id}_A : A \to A$ is receptive, innocent, and winning. Therefore, ϕ is true as a Λ'-formula, but undetermined as a Λ-formula. ∎

In the previous two examples the two Λ-IF specifications are semantically equivalent to their corresponding IF logic counterparts. However:

Example 5. Take the Λ-IF formula $\phi = (\forall^1 x.\ \exists^2 y.\ x = y) \vee (\exists^1 x.\ \forall^2 y.\ x \neq y)$, where neither $1 \leq 2$ nor $2 \leq 1$. Even though the two sub-games under $\vee$ are undetermined in the general case, the concurrent game induced by ϕ has a winning strategy (for Player): the copy-cat strategy for $[\![\forall^1 x.\ \exists^2 y.\ x = y]\!]^{\Lambda}_{M}\rho$. ∎

6.1 On the Expressivity of Λ-IF

Although IF and Λ-IF are semantically different logics, a translation from IF logic to Λ-IF logic is possible provided the dependencies of variables in IF logic can be described as a partial order (or even as a preorder), as is the case for IF-formulae without signaling [10]. In this case a translation from such IF-formulae to Λ-IF-formulae can be given directly. It also follows from a (fairly direct) translation of Henkin logic to Λ-IF. Henkin logic is propositional logic extended with Henkin quantifiers [6]; a Henkin (or branching) quantifier comprises a finite partial order of quantified variables—the partial order assigning the dependency between variables, in the same manner as levels $(\Lambda, \leq)$. For example, in the Henkin formula

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2)\,.$$

the variable y_1 may depend on x_1, but not on x_2, or y_2. This formula has meaning given in terms of Skolem functions by

$$\exists f. \exists g. \forall x_1. \forall x_2.\ \phi(x_1, x_2, f(x_1), g(x_2))$$

and has semantics given by the following game with imperfect information: in a Henkin quantifier $\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2)$ Player chooses y_2 independently of the choice of x_1 by Opponent, who chooses x_2 independently of the choice of y_1 by Player. This behaviour is the same as given by the Λ-IF formula

$$\forall^a x_1.\exists^b y_1.\forall^c x_2 \exists^d y_2.\ \phi(x_1, x_2, y_1, y_2)$$

with $a \leq b$, $c \leq d$, $a\ co\ d$, and $b\ co\ c$ (where $\lambda\ co\ \lambda'$ iff neither $\lambda \nleq \lambda'$ nor $\lambda' \nleq \lambda$).

Being able to encode the Henkin quantifier has two consequences. Firstly, the ability to encode IF logic from the interpretation of IF-logic in Henkin logic. Secondly, that Λ-IF inherits from Henkin logic the expressive power of the existential fragment of second-order logic.

Example 6. We illustrate the translation from IF formulae to Henkin formulae to Λ-IF formulae. Take the IF formula $\forall x.\exists y/x.\ x = y$. Its translation to Henkin logic is

$$\begin{pmatrix} \forall x \exists y_1 \\ \forall x_2 \exists y \end{pmatrix} x = y$$

and to Λ-IF logic is (given the translation above)

$$\forall^a x.\exists^b y_1.\forall^c x_2.\exists^d y.\ x = y$$

with $a \leq b$, $c \leq d$, $a\ co\ d$, and $b\ co\ c$; and eliminating unnecessary quantifiers into

$$\forall^a x.\exists^d y.\ x = y$$

with $a\ co\ d$. As described in Example 3 such a formula is undetermined in any model M with more than two distinct elements. ∎

Remark. Note that, in fact, a formula with a Henkin quantifier

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2)$$

can be translated to the Λ-IF formula

$$\forall^a x_1.\exists^a y_1.\forall^d x_2 \exists^d y_2.\ \phi(x_1, x_2, y_1, y_2)$$

with $a\ co\ d$, which has *only two* incomparable levels.

7 Conclusion

Although strongly related to IF, the logic Λ-IF has a different evaluation game: as illustrated by Example 5, the formula $\psi \vee \neg\psi$ is always a tautology within Λ-IF (as copy-cat is winning there), whereas it is not in IF when ψ is undetermined. There is the possibility of giving a proof theory for Λ-IF since it satisfies the axiom rule, which is not the case for IF logic.

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