Nothing Can Be Fixed

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Abstract. We establish the existence of zero elements in certain partially ordered monoids and use them to prove the existence of least fixed points in domain theory. This algebraic stance is the magic underlying Pataraia's constructive proof of the fixed point theorem.

To Samson, on the 30th Anniversary of His 30th Birthday

In the Michaelmas Term of 2000, Samson gave a seminar on Pataraia's constructive proof of the fixed point theorem. Because of his remarkable lucidity that day, every detail of the presentation has stayed with me for over twelve years, and as a result, I now have something new to report: there are times when algebraic zeroes can yield fixed points, hence our title. Alternate interpretations, such as "nothing can stay the same" (like one's age) or "things are broken and cannot be repaired," are probably coincidental.

Definition 1. A *partially ordered monoid* is a monoid $(M, \cdot, 1)$ with a partial order $\leq$ such that

$$a \leq b \;\&\; x \leq y \Longrightarrow ax \leq by$$

for all $a, b, x, y \in M$. It is *directed complete* when all of its directed sets have suprema and is said to have a *zero* when there is an element e with $e = ex = xe$ for all x. Zero elements are unique when they exist.

Theorem 1. *If M is a directed complete monoid with $1 \leq x$ for all $x \in M$, then M has a zero.*

Proof. Let $x, y \in M$. Using $1 \leq x$ and $1 \leq y$, multiply the first on the right by y and the second on the left by x to get $x, y \leq xy$. Then M is a directed set. Let $e = \bigsqcup M$. We then have $e \leq ex, xe$ but also $ex, xe \leq e$ since e is above every element of M. Then $e = ex = xe$ is the zero element of M. □

The last result enables an algebraic proof of the fixed point theorem in domain theory:

B. Coecke, L. Ong, and P. Panangaden (Eds.): Abramsky Festschrift, LNCS 7860, pp. 195–196, 2013.

Theorem 2. *A monotone map $f : D \to D$ on a dcpo D with least element $\perp$ has a least fixed point.*

Proof. Let $S = \{x \in D : x \sqsubseteq f(x)\}$. Notice that $S \neq \emptyset$ since $\perp \in S$ and is a dcpo by the monotonicity of f. Let $P(S)$ denote the set of monotone maps above the identity which take S into S. Then $P(S)$ is a directed complete monoid under composition with least element 1. By Theorem 1, it has a zero e. Since $f \in P(S)$, we then have $f \circ e = e$, which implies that $f(e(x)) = e(x)$ for all $x \in S$ and thus that f has a fixed point.

To prove f has a least fixed point, let M denote the set of $A \subseteq D$ closed under directed suprema in D with $\perp \in A$ and $f(A) \subseteq A$. Under the operation of intersection and order of reverse inclusion, M is a directed complete monoid with identity D. By Theorem 1, it has a zero e, and as seen above, f has a fixed point $\text{fix}(f) \in e$. Given any $x = f(x) \in D$, the set $\{a : a \sqsubseteq x\} \in M$ and thus contains e. Then $\text{fix}(f) \sqsubseteq x$. □

We applied Theorem 1 twice in the proof of Theorem 2 only to make the point that both the existence of the fixed point as well as its leastness can be handled with Theorem 1. Alternatively, the same effect can be achieved by taking S in the first part of Theorem 2 to be $e \in M$.

References

1. Escardo, M.H.: Joins in the frame of nuclei. Applied Categorical Structures 11, 117–124 (2003)
2. Pataraia, D.: A constructive proof of Tarski's fixed-point theorem for dcpo's. Presented in the 65th Peripatetic Seminar on Sheaves and Logic, Aarhus, Denmark (November 1997)