Unfixing the Fixpoint: The Theories of the λY-Calculus

Furio Honsell and Marina Lenisa

Dipartimento di Matematica e Informatica, Università di Udine, Italy
furio.honsell@comune.udine.it, marina.lenisa@uniud.it

Dedicated to Samson Abramsky
on the occasion of his 60th birthday

Abstract. We investigate the theories of the *λY-calculus*, *i.e.* simply typed λ-calculus with *fixpoint* combinators. Non-terminating λY-terms exhibit a rich behavior, and one can reflect in λY many results of untyped λ-calculus concerning theories. All theories can be characterized as *contextual theories* à la Morris, w.r.t. a suitable set of *observables*. We focus on theories arising from natural classes of observables, where Y can be approximated, albeit not always initially. In particular, we present the standard theory, induced by *terminating terms*, which features a canonical interpretation of Y as "minimal fixpoint", and another theory, induced by *pure* λ-terms, which features a non-canonical interpretation of Y. The interest of these two theories is that the term model of the λY-calculus w.r.t. the first theory gives a *fully complete model* of the maximal theory of the simply typed λ-calculus, while the term model of the latter theory provides a *fully complete model* for the observational equivalence in unary PCF. Throughout the paper we raise open questions and conjectures.

Introduction

Y, the *fixpoint combinator* lies at the heart of computation, and quite naturally PCF has been a paradigm language for many decades. However, λY-calculus, the purely functional core of PCF, *i.e.* simply typed λ-calculus extended with fixpoint combinators, has not been often studied *per se*. In this paper, we outline a general investigation of the theories of λY, inspired by what has been done in the untyped λ-calculus, see *e.g.* [Bar84, HP09].

From this investigation, we expect to achieve a better understanding of how theories of the λY, and hence of PCF, relate to *e.g. iteration* theories [BE93, PS00], what are the constraints on possible *non initial* interpretations of Y, and which properties of λY-terms can be naturally encoded by non-terminating λ-processes. We think that this research can be quite rewarding given the remarkable results obtained in the semantics of simply typed λ-calculus using games, and other categories, since the fundamental work of Samson Abramsky on full abstraction of PCF, see [AJM00].

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Ultimately, we would be like to answer general questions on the flexibility of notions such as games or continuous functions in modeling adequately the rich computational behavior of syntactical combinators. *E.g.*: "Are all λY-theories modeled by game models? If not, which are they?" Abramsky and Luke Ong [AO93], and one of the authors [HR92], were among the first to realize the constraints imposed by Scott-domains on the semantics of the untyped λ-calculus. Since then a vast literature arose in this area, see *e.g.* [DFH99, CS09, HP09] for the untyped λ-calculus. This paper addresses the above issues for the typed λY-calculus. In particular, we follow a journey around λY-theories analogue to that for the untyped λ-calculus. We start by defining theories contextually, given a set of *observables*. This amounts to reasoning on *term models*, *i.e.* on a model-independent semantics. A straightforward *transfer* result allows us to reflect on the λY-calculus essentially all the complexities of the theories of untyped λ-calculus. But "sometimes less is more" and many more issues arise in the typed setting than one would have expected at the outset. A wealth of intriguing connections appear. For example, interpretations of Y in naturally defined, non standard, theories behave as *sequential* composition in unary PCF, or *if then else* in binary PCF.

Summary. In Section 1, we present the syntax of the λY-calculus. In Section 2, we study general λY-theories and preorders, and we prove a *Transfer Theorem*, providing a correspondence between theories of the untyped λ-calculus and λY-theories. In Section 3, we focus on two special λY-theories related to the maximal theory on typed λ-calculus and to the observational equivalence on unary PCF, respectively. Final remarks, conjectures and open problems appear in Section 4.

1 The λY-Calculus

The λY-calculus is a simply typed lambda-calculus with two base constants $\bot, \top$, and fixpoint combinators Y_σ at each type. The following definitions are standard.

Definition 1 (Syntax).
Types:

$$\sigma ::= \; o \mid \sigma \to \sigma$$

Raw terms:

$$M ::= \; x \mid \bot \mid \top \mid MM \mid \lambda x : \sigma . M \mid Y_\sigma \;,$$

where $\bot, \top$ are constants and $x \in Var$.

Definition 2 (Well-Typed Terms). *The* proof system *for typing terms derives judgements $\Gamma \vdash M : \sigma$, where Γ is a* type environment, i.e. *a finite set of assumptions $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$. The rules of the proof system are the following:*

$$\frac{}{\Gamma \vdash \bot : o} \qquad \frac{}{\Gamma \vdash \top : o} \qquad \frac{}{\Gamma \vdash Y_\sigma : (\sigma \to \sigma) \to \sigma} \qquad \frac{}{\Gamma, x : \sigma \vdash x : \sigma}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x{:}\sigma.M : \sigma \to \tau} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

We denote by Λ_Y (Λ^0_Y) the set of well-typed (closed) λY-terms.
Term contexts are defined as usual. In the sequel, we will denote by $C[\,] : \sigma \to \tau$ a closed context expecting a term of type σ, and producing a term of type τ.

Definition 3 (Reduction/Conversion). *The* reduction *relation between well-typed terms is the least relation generated by the following rules together with the rules for transitive and congruence closure (which we omit):*
(β) $\Gamma \vdash (\lambda x{:}\sigma.M)N \Rightarrow M[N/x] \;:\; \tau$, *where* $\Gamma, x : \sigma \vdash M : \tau$, *and* $\Gamma \vdash N : \sigma$
(η) $\Gamma \vdash \lambda x{:}\sigma.Mx \Rightarrow M \;:\; \sigma \to \tau$, *provided* $x \notin M$
(Y) $\Gamma \vdash Y_\sigma M \Rightarrow M(Y_\sigma M) \;:\; \sigma$, *where* $\Gamma \vdash M \;:\; \sigma \to \sigma$.
Conversion, *denoted by* $=$, *is the symmetric and transitive closure of reduction.*

In the following, we will often omit the environment Γ and/or the type, when they are clear from the context.

2 λY-Theories

We focus on the theories of the λY-calculus, *i.e.* congruence relations on (closed) well-typed terms, which are closed under the conversion relation. We show that all λY-theories admit a contextual characterization. Moreover, we prove a *Transfer Theorem*, giving a correspondence between λ-theories and λY-theories.

2.1 Contextual Characterization of λY-Theories

It is well-known that all theories on the untyped λ-calculus are *contextual*, *i.e.* they admit a contextual characterization à la Morris, see [HR92]. An analogous result holds for λY-theories.

Definition 4 (Contextual λY-Theory). *A λY-theory $\sim$ is* contextual *if there exists a set of terms $\mathcal{Q} \subseteq \Lambda^0_Y$ closed under conversion such that, for all σ and all $M, N \in \Lambda^0_Y$ of type σ,*

$$M \sim N : \sigma \iff \forall C[\,] : \sigma \to \tau.\ (C[M] \in \mathcal{Q} \Leftrightarrow C[N] \in \mathcal{Q})\ .$$

The terms in $\mathcal{Q}$ are called convergent *or* observable *terms.*

The following result holds:

Theorem 1.
(i) If $\emptyset \neq \mathcal{Q} \subsetneq \Lambda^0_Y$ and $\mathcal{Q}$ is closed under conversion, then the contextual theory $\sim_{\mathcal{Q}}$ is non-trivial.
(ii) Every λY-theory is contextual.

Proof.
(i) The proof is standard.
(ii) Let $\sim$ be a λY-theory, define

$$\mathcal{Q} = \{M \mid \exists A, B.\ (A \sim B : \sigma\ \wedge\ M \sim \lambda x : \sigma \to \sigma \to \sigma.xAB)\}\ ,$$

where x does not occur in A or B. Let us denote by $\sim_{\mathcal{Q}}$ the contextual theory induced by $\mathcal{Q}$. If $M \sim N$, it is immediate to show that also $M \sim_{\mathcal{Q}} N$. Vice versa, if $M \not\sim N$, then also $M \not\sim_{\mathcal{Q}} N$, since for $C[\ ] \equiv \lambda x : \sigma \to \sigma \to \sigma.xM[\ ]$ we have $C[M] \in \mathcal{Q}$, while $C[N] \notin \mathcal{Q}$. □

It is useful to introduce also the notion of *contextual preorder* defined by:

Definition 5 (Contextual Preorder). *A preorder $\lesssim$ on closed λ-terms is* contextual *if there exists a set of terms $\mathcal{Q}$ closed under conversion such that, for all σ and all $M, N \in \Lambda^0_Y$ of type σ,*

$$M \lesssim N : \sigma \iff \forall C[\ ] : \sigma \to \tau.\ (C[M] \in \mathcal{Q} \Rightarrow C[N] \in \mathcal{Q})\ .$$

Any preorder $\lesssim$ induces a corresponding theory $\sim = \lesssim \cap (\lesssim)^{-1}$.

Interesting contextual preorders (theories) are those that admit a characterization as *logical relations*, *i.e.* the (pre)equivalence of terms at higher types is determined by the (pre)equivalence at the base type as follows:

Definition 6 (Logical Preorder/Theory). *Let $\lesssim_{\mathcal{Q}}$ ($\sim_{\mathcal{Q}}$) be a contextual preorder (theory) with observables in $\mathcal{Q}$. We say that*
(i) the preorder $\lesssim_{\mathcal{Q}}$ is a logical relation *if, for all $M, N \in \Lambda^0_Y$,*
$M \lesssim_{\mathcal{Q}} N : o \iff \forall C[\ ] : \sigma \to \tau.\ (C[M] \in \mathcal{Q}\ \Rightarrow\ C[N] \in \mathcal{Q})$
$M \lesssim_{\mathcal{Q}} N : \sigma \to \tau \iff \forall P \lesssim_{\mathcal{Q}} Q.\ MP \lesssim_{\mathcal{Q}} NQ : \tau.$
(ii) the theory $\sim_{\mathcal{Q}}$ is a logical relation *if, for all $M, N \in \Lambda^0_Y$,*
$M \sim_{\mathcal{Q}} N : o \iff \forall C[\ ] : \sigma \to \tau.\ (C[M] \in \mathcal{Q}\ \Leftrightarrow\ C[N] \in \mathcal{Q})$
$M \sim_{\mathcal{Q}} N : \sigma \to \tau \iff \forall P \sim_{\mathcal{Q}} Q.\ MP \sim_{\mathcal{Q}} NQ : \tau.$

A natural question to ask is when a preorder (theory) is a logical relation. A sufficient condition is the following:

Definition 7. *Let $\lesssim_{\mathcal{Q}}$ ($\sim_{\mathcal{Q}}$) be a contextual preorder (theory) with observables in $\mathcal{Q}$. Then*
(i) the preorder $\lesssim_{\mathcal{Q}}$ is well-behaved *if, for all $M, N \in \Lambda^0_Y$ of type σ,*

$$M \in \mathcal{Q}\ \wedge\ N \notin \mathcal{Q} \Longrightarrow \exists C[\ ] : \sigma \to o.\ (C[M] \not\lesssim_{\mathcal{Q}} C[N] : o)\ .$$

(ii) the theory $\sim_{\mathcal{Q}}$ is well-behaved *if, for all $M, N \in \Lambda^0_Y$ of type σ,*

$$M \in \mathcal{Q}\ \wedge\ N \notin \mathcal{Q} \Longrightarrow \exists C[\ ] : \sigma \to o.\ (C[M] \not\sim_{\mathcal{Q}} C[N] : o)\ .$$

Proposition 1.
(i) Any well-behaved contextual preorder on the λY-calculus is a logical relation.
(ii) Any well-behaved contextual theory on the λY-calculus is a logical relation.

Proof.
(i) Let $\lesssim_{\mathcal{Q}}$ be a well-behaved contextual preorder, and let $\lesssim'_{\mathcal{Q}}$ be the preorder defined by:
$M \lesssim'_{\mathcal{Q}} N : o$ iff $\forall C[\,] : \sigma \to \tau.\ (C[M] \in \mathcal{Q} \ \Rightarrow\ C[N] \in \mathcal{Q})$
$M \lesssim'_{\mathcal{Q}} N : \sigma \to \tau$ iff $\forall P \lesssim'_{\mathcal{Q}} Q.\ MP \lesssim'_{\mathcal{Q}} NQ : \tau$.
We prove that $M \lesssim_{\mathcal{Q}} N \iff M \lesssim'_{\mathcal{Q}} N$.
In order to prove that $M \lesssim'_{\mathcal{Q}} N : \sigma \implies M \lesssim_{\mathcal{Q}} N : \sigma$ (*), we first show that:
(a) $M \lesssim'_{\mathcal{Q}} N : \sigma \implies \forall C[\,] : \sigma \to \tau.\ (C[M] \lesssim'_{\mathcal{Q}} C[N] : \tau)$.
(b) $M \lesssim'_{\mathcal{Q}} N : \sigma \implies (M \in \mathcal{Q} \ \Rightarrow\ N \in \mathcal{Q})$.
To prove item (a) one proceeds by extending the preorder $\lesssim'_{\mathcal{Q}}$ to open terms by substitution with $\lesssim'_{\mathcal{Q}}$-related closed terms as follows. Let M, N open terms with free variables $\boldsymbol{x}$ we define: $M \lesssim'_{\mathcal{Q}} N$ iff $\forall \boldsymbol{P} \lesssim'_{\mathcal{Q}} \boldsymbol{Q}.\ M[\boldsymbol{P}/\boldsymbol{x}] \lesssim'_{\mathcal{Q}} N[\boldsymbol{Q}/\boldsymbol{x}]$. Then we prove the thesis for all possibly open terms, by induction on contexts.
The proof of item (b) requires the hypothesis that the preorder is well-behaved. Namely, assume by contradiction $M \lesssim'_{\mathcal{Q}} N : \sigma$, $M \in \mathcal{Q}$, but $N \not\in \mathcal{Q}$. Then, since the preorder is well-behaved, there exists $C[\,] : \sigma \to o$ such that $C[M] \not\lesssim_{\mathcal{Q}} C[N] : o$. But, by item (a), $C[M] \lesssim'_{\mathcal{Q}} C[N] : o$. Contradiction.
Now we are in the position of proving (*). Assume $M \lesssim'_{\mathcal{Q}} N : \sigma$. Then by (a), for any context $C[\ ] : \sigma \to \tau$, $C[M] \lesssim'_{\mathcal{Q}} C[N] : \tau$, and, by item (b), $C[M] \in \mathcal{Q} \ \Rightarrow\ C[N] \in \mathcal{Q}$.
In order to prove the converse, *i.e.* $M \lesssim_{\mathcal{Q}} N : \sigma \implies M \lesssim'_{\mathcal{Q}} N : \sigma$, we proceed by induction on the type σ. For $\sigma = o$ the thesis is trivial, by definition of $\lesssim_{\mathcal{Q}}$ and $\lesssim'_{\mathcal{Q}}$. For $\sigma = \sigma_1 \to \sigma_2$, let $M \lesssim_{\mathcal{Q}} N : \sigma_1 \to \sigma_2$, if $P \lesssim'_{\mathcal{Q}} N : \sigma_1$, then, by (*), $P \lesssim_{\mathcal{Q}} N : \sigma_1$. Therefore $MP \lesssim_{\mathcal{Q}} NP \lesssim_{\mathcal{Q}} NQ : \sigma_2$, hence by induction hypothesis $MP \lesssim'_{\mathcal{Q}} NP : \sigma_2$, thus $M \lesssim'_{\mathcal{Q}} N : \sigma_1 \to \sigma_2$.
(ii) Similarly to item (i) above. □

2.2 λ-Theories and λY-Theories

The class of λY-theories is rich. Consider, for example, the *unsolvable* λY terms of order 0. As for the untyped λ-calculus, there are no constraints on the equational behavior of such "easy" terms, [Bar84]. At any type τ there are infinitely many non convertible such terms, *e.g.* $Y_{\sigma\to\tau}IM$ for any M, where I denotes the identity of type $(\sigma \to \tau) \to (\sigma \to \tau)$.

The richness of λY-theories is witnessed by the *Transfer Theorem* below, which provides a correspondence between λ-theories, *i.e.* theories on the untyped λ-calculus, and λY-theories. The gist of this translation is an encoding of untyped λ-terms into well-typed λY-terms, whereby untyped terms are transformed into well-typed ones, by suitably inserting terms of the form $Y_\sigma I$ of appropriate types σ, where I denotes the identity of type $\sigma \to \sigma$. A consequence of the Transfer Theorem is that there are $2^{\aleph_0}$ λY-theories.

Definition 8 (Encoding λ-terms into λY-terms). *Let* $in_\sigma : (\sigma \to \sigma) \to \sigma$ *and* $out_\sigma : \sigma \to (\sigma \to \sigma)$ *be the* λY*-terms defined by:*

$$in_\sigma = Y_{(\sigma\to\sigma)\to\sigma} I \quad \textit{and} \quad out_\sigma = Y_{\sigma\to(\sigma\to\sigma)} I\ .$$

Then we define the encoding $\mathcal{E}_\sigma : \Lambda \to \Lambda_Y$*, which, given an untyped term, yields a* λY*-term of type* σ*, as follows:*

$$\mathcal{E}_\sigma(M) = \begin{cases} x : \sigma & \text{if } M \equiv x \\ in_\sigma(\lambda x : \sigma.\mathcal{E}_\sigma(M_1)) : \sigma & \text{if } M \equiv \lambda x.M_1 \\ out_\sigma(\mathcal{E}_\sigma(M_1))\mathcal{E}_\sigma(M_2) : \sigma & \text{if } M \equiv M_1 M_2 \end{cases}$$

Notice that the encoding $\mathcal{E}_\sigma$ is parametric w.r.t. σ.

Theorem 2 (Theory Correspondence). *Let* $\sim_\lambda$ *be a* λ*-theory, and* σ *any type. Then there exists a* λY*-theory* $\sim_{\lambda Y}$ *such that, for all* $M, N \in \Lambda^0$,

$$M \sim_\lambda N \iff \mathcal{E}_\sigma(M) \sim_{\lambda Y} \mathcal{E}_\sigma(N) \ .$$

Proof. (Sketch) Let $\sim_\lambda$ be a λ-theory, take $\sim_{\lambda Y}$ to be the λY-theory induced by the conversion and contextual closure of $\{(\mathcal{E}_\sigma(M), \mathcal{E}_\sigma(N)) \mid M \sim_\lambda N\}$. The argument lies in the fact that $Y_{(\sigma\to\sigma)\to\sigma} I$ and $Y_{\sigma\to(\sigma\to\sigma)} I$ have a completely inactive rôle computationally, see comment in the proof of Theorem 3 below. □

The type σ in the above theorem is generic. This result, albeit extremely simple, indicates that the computational complexity of untyped λ-calculus is immediately captured by the Y combinator. This result should be compared also to the results in [Lai03] for FPC.

As a corollary of Theorem 2 above, given the results in *e.g.* [Bar84], we have that:

Corollary 1. *There are* $2^{\aleph_0}$ λY*-theories.*

However, all the theories deriving from Theorem 2 are included in the theory which equates all non-normalizable constant-free terms. A different argument based on the "easy" nature of unsolvables of order 0 is necessary in order to show that:

Theorem 3. *There are* $2^{\aleph_0}$ *maximal* λY*-theories.*

Proof. $Y_{(o\to o)\to(o\to o)} I$ plays the rôle of $(\lambda x.xx)(\lambda x.xx)$ in untyped λ-calculus. Because of its computationally inactive rôle, it can be "anything it should not be", [BB79]. For example, $Y_{(o\to o)\to(o\to o)} I$ can encode the characteristic function of any subset of Church numerals. Any theory extending two such theories would therefore equate $\lambda xy.x$ and $\lambda xy.y$, and hence it would be inconsistent. □

2.3 Approximable Theories

The traditional understanding of Y is that of an *initial* or *least* fixed point. The pragmatics underpinning this concept is to explain away the Y combinator, by approximating its action with an *iterated* application of M on a *kick off* term N_0. Canonically, the kick-off term is $\bot$.

But this is only the "tip of the iceberg". We call this "iceberg" *approximable theories.* These are the theories of λY which support a form of generalized initiality.

Definition 9 (Approximable Theory). *A contextual λY-theory $\sim_{\mathcal{Q}}$, with $\mathcal{Q}$ the set of convergent terms, is* approximable *if for all σ there exists $N_\sigma \in \Lambda^0_Y$ such that*

$$\forall M : \sigma.\ \forall C[\ ] : \sigma \to \tau.\ \exists n.\ C[Y_\sigma M] \sim_{\mathcal{Q}} C[M^n(N_0)]$$

It goes without saying that approximable theories feature an Approximation Theorem. It should be interesting to study this in the context of *iteration* theories [BE93, PS00].

The standard argument, often used in connection with finite models of PCF, that the Y combinator can be dropped, is clearly due to the fact that the theories are uniformly approximable.

3 Canonical and Non-canonical Interpretations of Fixpoint Combinators

Canonical theories of the λY-calculus arise from the interpretation of Y_σ combinators as minimal fixpoint combinators, as in the standard Scott model. Different interpretations of the fixpoint combinators give rise to different (non-canonical) theories.

In this section, we focus on a canonical theory, $\sim_{YC}$, and on a non-canonical one, $\sim_{YN}$. The interest of these two theories lies in the fact that the first is connected to the maximal theory of the simply typed λ-calculus, and it provides a *fully-complete* interpretation of it, while the latter gives a *fully complete* interpretation of the observational equivalence on unary PCF.

3.1 A Canonical λY-Theory

The canonical λY-theory $\sim_{YC}$ on which we focus on can be defined as the contextual theory obtained by taking as convergent those terms that are convertible to a normal form without $\perp$. This is the paramount example of an approximable theory.

Definition 10 (Canonical λY-Theory).
(i) Let $\lesssim_{YC}$ be the preorder defined by, for all $M, N \in \Lambda^0_Y$,

$$M \lesssim_{YC} N : \sigma \text{ iff } \forall C[\ ] : \sigma \to \tau.\ (C[M] \in \mathcal{Q}_{YC} \Rightarrow C[N] \in \mathcal{Q}_{YC})\ ,$$

where $\mathcal{Q}_{YC} = \{M \in \Lambda^0_Y \mid \exists M' \text{ normal form}.\ (\perp \notin M' \ \wedge\ M = M')\}$.
(ii) Let $\sim_{YC}$ be the theory induced by $\lesssim_{YC}$, i.e. $\sim_{YC} = \lesssim_{YC} \cap (\lesssim_{YC})^{-1}$.

Canonical preorder and theory are logical relations, which admit a very simple characterization at the base type:

Proposition 2.
(i) $M \lesssim_{YC} N : o \iff (M \in \mathcal{Q}_{YC} \ \Rightarrow\ N \in \mathcal{Q}_{YC})$
$M \lesssim_{YC} N : o \iff \forall P \lesssim_{YC} Q.\ (MP \lesssim_{YC} NQ)$.
(ii) $M \sim_{YC} N : o \iff (M \in \mathcal{Q}_{YC} \ \Leftrightarrow\ N \in \mathcal{Q}_{YC})$
$M \sim_{YC} N : o \iff \forall P \sim_{YC} Q.\ (MP \sim_{YC} NQ)$.

Proof.
(i) First one can easily show that $M \lesssim_{YC} N : o \iff (M \in \mathcal{Q}_{YC} \Rightarrow N \in \mathcal{Q}_{YC})$. Then the thesis follows from the fact that $\lesssim_{YC}$ is well-behaved. Namely, let M, N be terms of type σ such that $M \in \mathcal{Q}_{YC}$ but $N \notin \mathcal{Q}_{YC}$. Then, if $\sigma = o$, the discriminating context is $(\lambda x : o.x)[\ ]$, otherwise, for $\sigma = \sigma_1 \to \ldots \to \sigma_n \to o$, $C[\ ]$ is $[\ ]\Pi_1 \ldots \Pi_n$, where $\Pi_1, \ldots, \Pi_n$ are suitable projections "extracting" the discriminating subterms.
(ii) Analogous to the above proof. □

The following properties are satisfied by the preorder $\lesssim_{YC}$ and the corresponding theory $\sim_{YC}$:

Lemma 1.
(i) $\bot \lesssim_{YC} \top : o$.
(ii) At any type there are only finitely many equivalence classes w.r.t. $\sim_{YC}$.
(iii) For any type σ *and any term* $M : \sigma \to \sigma$,

$$Y_\sigma M \sim_{YC} M^{p(\sigma)} \bot_\sigma \ ,$$

where

- $\bot_\sigma$, *for* $\sigma = \sigma_1 \to \ldots \to \sigma_n \to o$, *denotes* $\lambda \boldsymbol{x} : \boldsymbol{\sigma}.\bot$,
- $p(\sigma)$ *is any number greater than the number of* $\sim_{YC}$*-equivalence classes at type* σ, e.g. $p(o) = 2$ *and* $p(\sigma \to \tau) = p(\tau)^{p(\sigma)}$.

Proof.
(i) Immediate form Proposition 2.
(ii) Clearly, at type o there are only two equivalence classes, $[\bot]_{\sim_{YC}}$ and $[\top]_{\sim_{YC}}$. Hence, by the characterization of $\sim_{YC}$ given in Proposition 4, there are only finitely many equivalence classes at any type $\sigma \to \tau$.
(iii) Since $\bot \lesssim_{YC} Y_\sigma M : \sigma$, then $M^{p(\sigma)}\bot_\sigma \lesssim_{YC} M^{p(\sigma)}(Y_\sigma M) = Y_\sigma M$, hence $M^{p(\sigma)}\bot_\sigma \lesssim_{YC} Y_\sigma M$. In order to prove the converse, *i.e.* $Y_\sigma M \lesssim_{YC} M^{p(\sigma)}\bot_\sigma$, one proceeds by showing that if there exists a context $C[\ ]$ such that $C[Y_\sigma M] \Rightarrow^* P$, for some P normal and $\bot$-free, and the number of reductions of $Y_\sigma M$ in the chain are less than n, then there exists P' such that both $C[M^n(Y_\sigma M)] \Rightarrow^* P'$, for some P' normal and such that $\bot \notin P'$, without any reductions of $Y_\sigma M$. Hence we have also that $C[M^n(\bot_\sigma)] \Rightarrow^* P'$ and we can replace $M^n(\bot_\sigma)$ for $Y_\sigma M$ in $C[\ ]$. Because of item (i) n can be chosen uniformly for all terms of type σ. □

As a consequence of item (iii) of the above lemma, we have:

Theorem 4. *The* λY*-theory* $\sim_{YC}$ *is approximable.*

The term model determined by the theory $\sim_{YC}$ on the λY-calculus is *sequential*, in the sense that each equivalence class at type $\boldsymbol{\sigma} \to \tau$ behaves either as a constant function in all arguments or it is strict in at least one argument, *i.e.* when this argument is $\bot$, the result of the application is $\bot$.

Theorem 5. *The term model of the λY-calculus induced by the theory $\sim_{YC}$ is sequential.*

Proof. First of all, notice that:
(a) $Y_\sigma I \sim_{YC} \perp_\sigma$, for all σ.
(b) Normal forms are strict in the head variable.
(c) Since the theory is approximable, every term $Y_\sigma M$ can be replaced by $M^k(Y_{(\sigma\to\sigma)\to\sigma} IM)$ for some k, getting a $\sim_{YN}$-equivalent term.
Given (a)–(c), the thesis follows easily by induction on terms. □

Relating the Canonical Theory to the Simply Typed λ-Calculus. The λY-theory $\sim_{YC}$ is related to the maximal theory of the simply typed λ-calculus with constants $\perp, \top$ at the base type o, defined by:

$$M \sim_\lambda N : \sigma \text{ iff } \forall C[\,] : \sigma \to \tau.\ (C[M] \in \mathcal{Q}_\lambda \Leftrightarrow C[N] \in \mathcal{Q}_\lambda)\ ,$$

where $\mathcal{Q}_\lambda = \{M \in \Lambda^0 \mid M \text{ of type } \sigma\ \wedge\ M \neq \perp_\sigma\}$.

In the following, we show that the term model of the λY-calculus w.r.t. $\sim_{YC}$ is fully complete for the theory $\sim_\lambda$ of the simply typed λ-calculus.

Cleary, any term of the simply typed λ-calculus can be viewed as a term of the λY-calculus via a trivial emdedding $\mathcal{I}$. Vice versa, one can define a mapping $\mathcal{L} : \Lambda_Y \to \Lambda$, by encoding Y_σ combinators as follows:

$$\mathcal{L}(Y_\sigma) = \lambda x : \sigma \to \sigma . x^{p(\sigma)}(\perp_\sigma)\ .$$

The above is justified by item (iii) of Lemma 1.

Then, it is easy to check that:

Proposition 3.
(i) $M \sim_\lambda N \iff \mathcal{L}(M) \sim_{YC} \mathcal{L}(N)$.
(ii) $M \sim_{YC} N \iff \mathcal{I}(M) \sim_\lambda \mathcal{I}(N)$.

Hence, we have:

Theorem 6. *The term model of the λY-calculus w.r.t. $\sim_{YC}$ is fully complete w.r.t. the maximal theory $\sim_\lambda$ of the simply typed λ-calculus.*

3.2 A Non-canonical λY-Theory

In this section we focus on a non-canonical λY-theory $\sim_{NY}$, which exhibits a number of intriguing connections with many results in the literature on models of unary PCF. It can be defined as the contextual theory obtained by taking as convergent terms those that are convertible to a term in which $\perp$ does not appear. Its counterpart in the context of the untyped λ-calculus is the theory discussed in [HR92].

Definition 11 (Non-canonical λY-theory).
(i) Let $\lesssim_{YN}$ be the preorder defined by, for all $M, N \in \Lambda^0_Y$,

$$M \lesssim_{YN} N : \sigma \quad \textit{iff} \quad \forall C[\,] : \sigma \to \tau.(C[M] \in \mathcal{Q}_{YN} \Rightarrow C[N] \in \mathcal{Q}_{YN}) \ ,$$

where $\mathcal{Q}_{YN} = \{M \in \Lambda^0_Y \mid \exists M'. \ (\bot \notin M' \ \wedge \ M = M')\}$.
(ii) Let $\sim_{YN}$ be the theory induced by $\lesssim_{YN}$, i.e. $\sim_{YN} = \lesssim_{YN} \cap (\lesssim_{YN})^{-1}$.

Non canonical preorder and theory are logical relations, admitting the following characterization:

Proposition 4.
(i) $M \lesssim_{YN} N : o \iff (M \in \mathcal{Q}_{YN} \ \Rightarrow \ N \in \mathcal{Q}_{YN})$
$M \lesssim_{YN} N : o \iff \forall P \lesssim_{YN} Q. \ (MP \lesssim_{YN} NQ)$.
(ii) $M \sim_{YN} N : o \iff (M \in \mathcal{Q}_{YN} \ \Leftrightarrow \ N \in \mathcal{Q}_{YN})$
$M \sim_{YN} N : o \iff \forall P \sim_{YN} Q. \ (MP \sim_{YN} NQ)$.

Proof.
(i) First one can easily show that $M \lesssim_{YN} N : o \iff (M \in \mathcal{Q}_{YN} \ \Rightarrow \ N \in \mathcal{Q}_{YN})$. Then the thesis follows from the fact that $\lesssim_{YN}$ is well-behaved. Namely, let M, N be terms of type σ such that $M \in \mathcal{Q}_{YN}$ but $N \notin \mathcal{Q}_{YN}$. Then, if $\sigma = o$, the discriminating context is $(\lambda x : o.x)[\,]$, otherwise, for $\sigma = \sigma_1 \to \ldots \to \sigma_n \to o$, $C[\,]$ is $[\,](Y_{\sigma_1} I) \ldots (Y_{\sigma_n} I)$.
(ii) Analogous to the above proof. □

The following properties are satisfied by the preorder and the theory $\sim_{YN}$:

Lemma 2.
(i) $\bot \lesssim_{YN} \top : o$ and $Y_o I \sim_{YN} \top : o$.
(ii) At any type there are only finitely many equivalence classes w.r.t. $\sim_{YN}$.
(iii) For any type $\sigma \to \tau$ and any term $N : \sigma$,

$$(Y_{\sigma \to \tau} I)N \sim_{YN} \begin{cases} Y_\tau I : \tau & \textit{if } Y_\sigma I \lesssim_{YN} N \\ \bot : \tau & \textit{otherwise} \ . \end{cases}$$

(iv) For any type σ and any term $M : \sigma \to \sigma$,

$$Y_{(\sigma \to \sigma) \to \sigma} IM \lesssim_{YN} Y_\sigma M \ .$$

(v) For any type σ and any term $M : \sigma \to \sigma$,

$$Y_\sigma M \sim_{YI} M^{p(\sigma)} M_0 \ ,$$

where

- *$M_0 \equiv Y_{(\sigma \to \sigma) \to \sigma} IM : \sigma$,*
- *$p(\sigma)$ is any number greater than the number of $\sim_{YI}$-equivalence classes at type σ.*

Proof.
(i) Immediate, from Proposition 4.
(ii) Clearly, at type o there are only two equivalence classes, $[\bot]_{\sim_{YN}}$ and $[\top]_{\sim_{YN}}$. Hence, using the characterization of $\sim_{YN}$ given in Proposition 4, there are only finitely many equivalence classes at any type $\sigma \to \tau$.
(iii) First one shows that for all σ, τ, $\quad Y_{\sigma\to\tau}I(Y_\sigma I) \sim_{YN} Y_\tau I \quad (*)$. This is immediate, observing that, for any context $C[\ ]$, both terms are "inactive", *i.e.* either they reduce via the fixpoint reduction rule without involving the context, or if they appear in a redex involving the context, then they play a "passive" role as argument. Hence $\bot \in^* C[Y_{\sigma\to\tau}I(Y_\sigma I)]$ iff $\bot \in^* C[Y_\tau I]$, where by $\bot \in^* M$ we denote the fact that for all M' such that $M = M'$, $\bot \in M'$.
Now assume $Y_\sigma I \lesssim_{YN} M$, *i.e.* $\forall C[\ ].\ \bot \in^* C[M] \ \Rightarrow\ \bot \in^* C[Y_\sigma I]$.
We prove that $\forall C[\ ].\ \bot \in^* C[Y_{\sigma\to\tau}IM] \ \Leftrightarrow\ \bot \in^* C[Y_\tau I]$.
($\Rightarrow$) Assume $\bot \in^* C[Y_{\sigma\to\tau}IM]$. Then $\bot \in^* C'[Y_\sigma I]$, where $C'[\ ] = C[Y_{\sigma\to\tau}I[\ ]]$. Then, by $(*)$, $\bot \in^* C[Y_\tau I]$.
($\Leftarrow$) If $\bot \in^* C[Y_\tau I]$, then since $Y_\tau I$ and $Y_{\sigma\to\tau}IM$ are both inactive, then also $\bot \in^* C[Y_{\sigma\to\tau}IM]$.
Now assume $Y_\sigma I \not\lesssim_{YN} M$, *i.e.* there exists $C[\ ]$ such that $\bot \not\in^* C[Y_\sigma I]$ but $\bot \in^* C[M]$. Then we show that $\bot \in^* C[Y_{\sigma\to\tau}IM]$. Namely, $Y_\sigma I$ is inactive, and hence there exists $C'[\ ]$ such that $C[\ ] = C'[\ ]$, $\bot \not\in C'[\ ]$, but $\bot \in^* C'[M]$. Hence $\bot \in M$, and $\bot \in^* C'[Y_{\sigma\to\tau}IM]$.
(iv) The proof follows by an argument similar to the ones used above.
(v) By item (iv), for all k, $M^k\bot \lesssim_{YN} Y_\sigma M$. Hence $M^{p(\sigma)}\bot \lesssim_{YN} Y_\sigma M$. In order to prove the converse, *i.e.* $Y_\sigma M \lesssim_{YN} M^{p(\sigma)}\bot$, one proceeds by showing that if there exists a context $C[\ ]$ such that $C[Y_\sigma M] \Rightarrow^* P$, $\bot \not\in P$, and the number of reductions of $Y_\sigma M$ in the chain are less than n, then there exists P' such that $C[M^n(Y_\sigma IM)] \Rightarrow^* P'$, $\bot \not\in P'$, without any reduction of $Y_\sigma M$, and hence also $C[M^n(Y_{(\sigma\to\sigma)\to\sigma}IM)] \Rightarrow^* P'$. Hence we can replace $M^n(Y_{(\sigma\to\sigma)\to\sigma}IM)$ for $Y_\sigma M$ in $C[\ ]$. Because of item (i) n can be chosen uniformly for all terms of type σ. □

As a consequence of item (v) of the above lemma, we have that:

Theorem 7. *The λY-theory $\sim_{YN}$ is approximable.*

Moreover, we have:

Theorem 8. *The term model of the λY-calculus induced by the theory $\sim_{YN}$ is sequential.*

Proof. First of all, notice that:
(a) The terms $Y_\sigma I$ are strict in all arguments, namely:

$$Y_{\sigma_1\to\ldots\to\sigma_n\to o}IM_1\ldots M_n = \begin{cases} \top & \text{if } \forall i.\ Y_{\sigma_i}I \leq_{YN} M_i \\ \bot & \text{otherwise .} \end{cases}$$

(b) Normal forms are strict in the head variable.

(c) Finally, since the theory is approximable, every term $Y_\sigma M$ can be replaced by $M^k(Y_{(\sigma\to\sigma)\to\sigma} IM)$ for some k, getting a $\sim_{YN}$-equivalent term.
Given (a)–(c), the thesis follows easily by induction on terms. □

Relating the Non-canonical Theory to Unary PCF. Interestingly, one can show that the λY-theory $\sim_{YN}$ captures exactly the behavioral equivalence of unary PCF, providing a fully complete model for it.

More precisely, we can define a mapping from unary PCF terms into λY-terms and vice versa, preserving the correspondence between theories.

We recall that unary PCF is a simply typed λ-calculus over a single base type o, containing two constants $\bot, \top$, and with a "sequential composition" operation $\wedge$ of type $o \to (o \to o)$. The conversion relation of unary PCF is generated by the $\beta\eta$-conversion together with the equations $\bot \wedge M = M \wedge \bot = M$ and $\top \wedge M = M \wedge \top = M$. We denote by Λ_{UP} (Λ^0_{UP}) the set of well-typed (closed) terms of unary PCF. The *behavioral equivalence* on unary PCF is the contextual theory induced by the set $\mathcal{Q}_{UP} = \{M \in \Lambda^0_{UP} \mid M \text{ of type } \tau \Rightarrow M = \bot_\tau\}$, *i.e.*:

$$M \sim_{UP} N : \sigma \;\text{ iff }\; \forall C[\,] : \sigma \to \tau.\ (C[M] = \bot_\tau \Leftrightarrow\ C[N] = \bot_\tau)\ .$$

Alternatively,
$M \sim_{UP} N : o \iff (M = \bot \Leftrightarrow\ N = \bot)$
$M \sim_{UP} N : \sigma \to \tau \iff \forall P \sim_{UP} Q : \sigma.\ (MP \sim_{UP} NQ)$.
Correspondingly, one can define a preorder $\lesssim_{UP}$.

The observational equivalence $\sim_{UP}$ over unary PCF corresponds to the theory $\sim_{YN}$ on the λY-calculus, in the sense that one can define a bijective correspondence between equivalence classes of PCF terms w.r.t. $\sim_{UP}$ and equivalence classes of λY-terms w.r.t. $\sim_{YN}$.

Definition 12.
(i) Let $\mathcal{T} : \Lambda_{UP} \to \Lambda_Y$ be the (type-respecting) mapping inductively defined by:
$\mathcal{T}(M) = M$ *if* $M \in \mathit{Var}$ *or* $M \in \{\bot, \top\}$
$\mathcal{T}(\lambda x : \sigma.M) = \lambda x : \sigma.\mathcal{T}(M)$
$\mathcal{T}(\wedge) = Y_{o\to(o\to o)} I$
$\mathcal{T}(MN) = \mathcal{T}(M)\mathcal{T}(N)$.
(ii) Let $\mathcal{S} : \Lambda_Y \to \Lambda_{UP}$ be the (type-respecting) mapping inductively defined by:
$\mathcal{S}(M) = M$ *if* $M \in \mathit{Var}$ *or* $M \in \{\bot, \top\}$
$\mathcal{S}(\lambda x : \sigma.M) = \lambda x : \sigma.\mathcal{S}(M)$

$$\mathcal{S}(Y_\sigma I) = \begin{cases} \top & \text{if } \sigma \equiv o \\ \lambda x : \sigma'.\lambda \boldsymbol{z} : \boldsymbol{\tau}.(x\mathcal{S}(Y_{\sigma_1} I)\dots\mathcal{S}(Y_{\sigma_n} I) \wedge \mathcal{S}(Y_{\boldsymbol{\tau}} I)\boldsymbol{z}) & \text{if } \sigma = \sigma' \to \tau \end{cases}$$

where $\sigma' = \sigma_1 \to \dots \to \sigma_n \to o$ *and* $\tau = \tau_1 \to \dots \to \tau_m \to o$.
$\mathcal{S}(Y_\sigma) = \lambda x : \sigma \to \sigma.x^{p(\sigma)}\mathcal{S}(Y_{(\sigma\to\sigma)\to\sigma} Ix)$
where $p(\sigma)$ is greater than the number of $\sim_{YI}$-equivalence classes at type σ,

$$\mathcal{S}(MN) = \begin{cases} \mathcal{S}(Y_\sigma I)\mathcal{S}(P_1)\dots\mathcal{S}(P_n) & \text{if } MN \equiv Y_\sigma IP_1\dots P_n \\ \mathcal{S}(M)\mathcal{S}(N) & \text{otherwise}\ . \end{cases}$$

Then we have:

Proposition 5.
(i) For any PCF-term M of type σ, $M \sim_{UP} \mathcal{S}(\mathcal{T}(M)) : \sigma$.
(ii) For any λY-term M of type σ, $M \sim_{YN} \mathcal{T}(\mathcal{S}(M)) : \sigma$.
(iii) For all PCF-terms M, N of type σ,

$$M \sim_{UP} N : \sigma \iff \mathcal{T}(M) \sim_{YI} \mathcal{T}(N) : \sigma \ .$$

(iv) For all λY-terms M, N of type σ,

$$M \sim_{YN} N : \sigma \iff \mathcal{S}(M) \sim_{UP} \mathcal{S}(N) : \sigma \ .$$

Proof.
(i) First of all, we extend the equivalence $\sim_{UP}$ to open terms as follows. Let M, N be terms of type $\boldsymbol{\sigma} \to \tau$ with free variables $x_1, \ldots, x_n$ of type $\sigma_1, \ldots, \sigma_n$, respectively. Then we define

$$M \sim_{UP} N \ \text{ iff } \ \forall \boldsymbol{P} \sim_{UP} \boldsymbol{Q} : \boldsymbol{\sigma}.\ M[\boldsymbol{P}/\boldsymbol{x}] \sim_{UP} N[\boldsymbol{P}/\boldsymbol{x}] : \tau \ .$$

Then the proof of item (i) proceeds by induction on the (possibly) open term M.
(ii) The proof is similar to the proof of the above item, using the extension of $\sim_{YN}$ to open terms.
(iii) The proof follows from the fact that $\forall M \in \Lambda_{UP}.\ (M = \bot : \tau \ \Leftrightarrow \ \bot \in^* \mathcal{T}(M))$. This latter fact is proved by induction on M.
(iv) The proof follows from the fact that $\forall M \in \Lambda_Y.\ \bot \in^* M \ \Leftrightarrow \ \bot \in^* \mathcal{S}(M) = \bot : \tau$. This latter fact is proved by induction on M. □

A consequence of Proposition 5 is the following:

Theorem 9. *The term model of the λY-calculus w.r.t. $\sim_{YN}$ is fully complete for the observational equivalence on unary PCF.*

4 Final Remarks, Conjectures, Open Problems

Infinitary Böhm Trees. Coalgebraic versions of λ-calculus and infinitary Böhm trees are closely related to λY. More results are needed here, involving the λY analogue of the *lazy* λ-calculus [AO93] equating all unsolvable λY terms of order n, for each n.

Categorical Formalization. It would be interesting to cast the results in this paper in a categorical setting.

More Non-standard Approximable Theories. Clearly, given a model of finitary PCF we are quite freed in interpreting the Y combinator. For instance, one can start iterations from the *maximal* element, if it exists. Or simply fix the fixpoint combinator to yield, on any given combinator, an appropriate value, chosen at will. A case in point would be to take YI always to be I. For each such choice the “game” is to find the contextual characterization which uses the most insightful observables.

Binary PCF. A very intriguing example derives from the universal model of binary PCF, because it yields a novel perspective on the *if then else* combinator. The construction generalizes the steps we followed for the unary PCF, the role of the sequential composition being replaced by that of *if then else*. The main surprise lies in the natural contextual theory capturing this choice of the fixed point. The set of observable terms amounts to the set of terms which can be reduced to a term of the λIY-calculus, *i.e.* terms where all abstracted variables *do* occur. We assume at least three constants of type o: $\bot, \mathbf{tt}, \mathbf{ff}$.

Conjecture 1.
(i) $\bot_o \lesssim_{YI} \mathbf{tt}, \mathbf{ff} : \mathbf{o}$ and $Y_o I \sim_{YI} \mathbf{tt}$.
(ii) At any type there are only finitely many equivalence classes w.r.t. $\sim_{YI}$.
(iii) For any type $\sigma \to \tau$ and any term $N : \sigma$,

$$(Y_{\sigma\to\tau} I)N \sim_{YI} \begin{cases} Y_\tau I & \text{if } Y_\sigma I \lesssim_{YI} N \\ \lambda x_1 : \tau_1 \dots x_n : \tau_n.\mathbf{ff} & \text{if } \lambda x_1 : \sigma_1 \dots x_n : \sigma_n.\mathbf{ff} \lesssim_{YI} N \\ \lambda x_1 : \tau_1 \dots x_n : \tau_n.\bot & \text{otherwise .} \end{cases}$$

(v) For any type σ there exists a natural number $p(\sigma)$ such that, for any term $M : \sigma \to \sigma$,

$$Y_\sigma M \sim_{YI} M^{p(\sigma)}(Y_{(\sigma\to\sigma)\to\sigma} IM) .$$

Models of $\boldsymbol{\lambda Y}$-Theories. By Proposition 3, each model of the maximal theory $\sim_\lambda$ on the simply typed λ-calculus is a model of the theory $\sim_{YC}$ of the λY-calculus, and vice versa. As a consequence, the PER model of [AL01] provides a fully complete model of the λY-theory $\sim_{YC}$.

Similarly, by Proposition 5, each model of the theory $\sim_{UP}$ on unary PCF is a model of the λY-theory $\sim_{NC}$. Models of unary PCF have been studied *e.g.* in [Lai03, BLP03]. In particular, in [Lai03] it is shown that any *standard order-extensional model* of unary PCF is fully complete either for unary PCF or for unary PCF extended with *parallel or*. More precisely, any standard order-extensional model of unary PCF, which is *sequential*, is fully complete for unary PCF, while non-sequential models are fully complete for the extended language. *E.g.* the standard Scott model is fully complete for unary PCF with parallel or, while the bidomain model of [Lai03] is fully complete for unary PCF.

It is interesting to notice that, in the context of games, we can recover both kinds of models.

Namely, the game model of unary PCF built over the Sierpinski game, being sequential, is fully complete. On the other hand, one can build a non-sequential game model by changing the definition of tensor product, as in [HL13]. In the standard notion of tensor product of games, see *e.g.* [AJM00], on the game $A\otimes B$, at each step, the player who has the turn can move exactly in one of the two components, A or B. In [HL13], an alternative notion of tensor product, *i.e.* $A \vee B$, has been considered, where at each step the player who has the turn can either move in A, or in B, or in *both* components. A form of parallelism is then recovered in the game model. This construction is based on Conway's *selective sum*, while tensor of traditional game semantics resembles of Conway's *disjunctive* sum, [Con01].

In [HL13], it has been shown that the game $A \vee B$, together with a non-standard definition of strategy composition, gives rise to a tensor product in a category of *coalgebraic games*. This category turns out to be linear, *i.e.* symmetric monoidal closed together with a symmetric monoidal comonad. An analogous construction can be carried out *e.g.* in the category of [AJM00]-games. Working in this category, one could build a non-sequential model of unary PCF over the Sierpinski game $\mathcal{O}$. Parallel or $\vee : o \to (o \to o)$ can then be interpreted by the strategy on $!\mathcal{O} \vee !\mathcal{O} \longrightarrow \mathcal{O}$, where Opponent opens in the right-hand $\mathcal{O}$-component, and Player answers with a *pair* of moves asking *both* arguments; then if Opponent answers in at least one argument (*i.e.* at least one argument is different from $\perp$), Player provides the final answer in the right-hand component. In this way, the theory of standard Scott model is recovered in the context of games.

Open Questions. We conclude with a few open questions:

- Which λY-theories are approximable?
- Are bidomain models complete w.r.t. λY?
- Are game models *complete* w.r.t. λY?

Fixing an answer to such questions would help also to fix ideas on unfixing fixpoints.

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