

# Aggregate and Disaggregate Dynamic Spatial Interaction Approaches to Modeling Coin Diffusion

Marion Le Texier and Geoffrey Caruso

**Abstract** With the 2002 introduction of the euro as a common currency in Europe, the possibility has emerged to assess international mobility using this new tracer, given that every coin bears a specific national side. Using a simple two-country framework, four dynamic modeling strategies were designed in order to simulate the diffusion of coins and to understand how this diffusion is affected by population size, mobility rates and coin exchange processes. Methodological implications are raised with respect to aggregation, synchronicity and stochasticity issues.

Although each model converges to an equilibrium, the time to reach this end stage and the level of coin mixing in each country strongly varies with the modeling strategy. Calibration is undertaken with French data, using mobility rates as adjustment variables. The experiment shows that convergence to a perfect mix of coins can only be obtained if reciprocal exchanges are modeled, with a time horizon around 2064 – while non-reciprocal models indicate an imperfect mix converging in the year 2020 at the latest.

**Keywords** Diffusion • Euro • Spatial interaction • Dynamic system • Individual agent-based model

## Introduction

The Euro is much more than a mere economic instrument. It is a symbol of the *grand mission* of European integration, a tool to ensure democracy and peace on a continent too often ravaged by war (Time, 30th April 2012).

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M. Le Texier (✉)  
University Paris Diderot, Paris, France

University Rouen Normandie, Mont-Saint-Aignan, France  
e-mail: [marion.le-texier@univ-rouen.fr](mailto:marion.le-texier@univ-rouen.fr)

G. Caruso  
University of Luxembourg, Luxembourg, Luxembourg

This is the way Michael Schuman, an American journalist, reports on what Europeans say about the euro and its recent crisis. It seems that the meaning of the common currency for Europeans goes far beyond its economic utility, notably since it facilitates exchanges between residents of the Eurozone.

The euro was put into circulation on January 1, 2002 as a common currency for 11 countries – each euro coin bearing on one side a common EU symbol, and on the other side, a national symbol. At that precise point in time, all coins minted in a given country were located in their countries of origin. Euro coins then started to circulate, conveyed across space and borders by people and along multiple monetary transactions. Assumably, the euro constitutes a wonderful tracer of international mobility and can somehow reveal patterns of European integration. Does a rapid mixing of the different national coins, and a high level of foreign euro coins in a given country, mean stronger integration? Interestingly enough, after 10 years, French coins still represented 66% of total coins in France, while the *Banque Nationale de France* produces only 20% of the coins of the Eurozone (Grasland et al. 2012). Although European integration cannot be reduced to the pace of coins' mixing, analyzing the mechanisms of the euro diffusion process may shed light on particular interaction flows.

Empirical studies based upon surveys of money bags across the EU have revealed intricate patterns of diffusion that go well beyond the simple proximity-based contagion diffusion from borders. A series of models have been developed in the past few years, based upon physical or epidemic processes, to reflect both upon the spatial diffusion of money and the time at which a homogeneous mix of coins across national states can be reached. Models have used approaches as diverse as Brownian movement, Levy flights, or Markov chains.

In this research, we use a simple spatial interaction perspective to the modeling of euro coins since it is well grounded in geographical research to represent flows of people and goods [see Tobler (1970) or Fotheringham and O'Kelly (1989) to name but two]. This model explicitly takes into account the existence, size and characteristics of cities, in addition to distance effects. However, a specificity of coin diffusion is that flows are not directly observed. Rather, it is the mix of coins in different places that can be measured, i.e., the result of a series of consecutive flows and interpersonal interactions occurring between many possible origins and destinations. The question then arises as to whether this diffusion process can be modeled using aggregate behavioral assumptions at the country level, or whether each individual, wearing a money bag, must be modeled across his/her trips and transactions in order to obtain a realistic representation of the diffusion process. The money bags will be described by the share of coins of each particular country of minting that they contain.

This study first recalls basics of dynamic spatial interaction models and discusses existing models of coin diffusion and their assumptions. These modeling strategies raise fundamental issues related to the treatment of time, randomness and agents' aggregation. We then provide a comparative analysis of four models of coin circulation, which are simulated using a simplified geography of two countries. Our models differ in two aspects: (1) the scale of analysis, i.e., aggregate, (place-based)

versus individual-based approaches; (2) the time granularity, i.e., asynchrony versus synchrony of both mobility and exchange processes; and (3) the definition of the exchange process, i.e., the number of coins involved in each transaction. We analyze the time, existence, and characteristics of a steady state in the coins' mixing process through numerical simulation. We then conduct a sensitivity analysis of the time to convergence and proportion of the foreign coin mix to the spatial interaction parameters: population size and mobility rates (in which distance effects and the attractiveness of places are assumed to be embedded).

## Methodology

### *Randomness, Time and Behavior in Spatial Interactions*

The observation of population migration within the United Kingdom at the end of the nineteenth century led several authors to describe empirical spatial interaction laws (Ravenstein 1889), which were then formalized by analogy to the physical law of gravitation (Reilly 1929, 1931; Stewart 1947; Huff 1964; Tobler 1970). Hence the emergence of spatial interaction theory as a field of research examining the location and flow of social and economic processes, emphasizing the role of the attractiveness of places and their relative distances. Spatial interaction models have since evolved in different directions (Roy 2004), including three that are particularly important for this study: probabilistic treatment, cumulative dynamics, and explicit consideration of individual behavior. In the following section, we position our work with respect to these three evolutions.

### Randomness and Stochasticity

Probabilistic treatment was introduced for spatial interaction models by Huff (1964) whose models aimed at identifying theoretical market areas within a set of central places included conditional probability distributions based upon place location. Probabilistic models were then popularized with the emergence of random utility theory and discrete choice models (Wegener 2000) and notably through the use of multinomial Logit models (Nijkamp and Reggiani 1987).

In spatial interaction models, randomness is mainly found within the destination choice stage. For coin diffusion this stage may not represent the main stochastic element. We hypothesize that the mixing of coins results from two sequential stages: (1) a mobility stage, i.e., flows of individuals who choose a destination; and (2) an exchange stage, i.e., flows of money between individuals in a given place. The latter is very likely to be stochastic compared to mobility, which is a reasoned choice made by individuals pursuing a specific objective. Even if we assume a simplified spatial interaction framework, the fact that there is a stochastic element at destination – and a cumulative effect through time due to repetitive movements and repetitive

exchanges – make coin diffusion a complex system. In the present work, we control for the spatial interaction stochasticity by leaving the destination choice aside and considering two cities only. We thus focus on the exchange stochasticity, making our approach closer to epidemic models which consider the probability of an individual to randomly have a contagious interaction at constant rate in time (Andersson and Britton 2000).

## Complex Dynamics

Gravity models also evolved to account for complex system dynamics after the work of Allen and Sanglier (1979) and Wilson (1981). Again transferred from physical concepts, the integration of the ‘arrow of time’ (Prigogine and Stengers 1979; Haken 1977) in the analysis of cities lead to the emergence of principles of self-organization in geographical (Pumain 1998) or economic contexts (Arthur 1988), with particular focus on non-linear dynamics, irreversibilities and path-dependency (O’Sullivan 2004; Guermond 2008). Temporal and cumulative effects have been considered in various fields and at various geographical scales, including transportation and urban economics, urban geography, and interregional migrations (Anas 1983; White 1977, 1978; Allen and Sanglier 1979, 1981; Wilson 1981; Pumain et al. 1989; Weidlich and Haag 1988; Aziz Alaoui and Bertelle 2009).

How and whether or not the system can reach a steady state is analyzed in dynamic systems, and typically entropy is used as a measure of the degree of organization of the system (Wilson 1967, 1970). In the easiest cases, interactions between places lead to stable situations where the aggregate system descriptors do not change, or evolve only slowly. Yet, the resulting structure of the system, even if stable, is often found to be very sensitive to the values of the input parameters (Bretagnolle et al. 2003). In many cases, unstable situations are observed with either internal fluctuations within the system or external perturbations. Non-linear interactions can indeed lead to extremely complicated sets of bifurcations and trajectories, with no possibility to pre-determine any equilibrium point or other attractors (Wilson 1981, 2002; Allen and Sanglier 1981; Allen 1997; Bretagnolle et al. 2003). Moreover, in the case of the evolution of systems of cities through time, if one can find stable mathematical convergence properties, such as scaling laws (Batty 2008), this relative stability and regularity at the macro level might be coupled with a high volatility at the micro level (in terms of cities or firms, for example) (Portugali 2000; Pumain 2003; Pumain et al. 2009).

Furthermore, the causal relationships within a geographical system may create progressively growing contrasts between spatial units; pushing the system away from maximized entropy (Berry 1964). In econophysics, such a behavior has been observed in the distribution of wealth. Successive monetary transactions between agents in a closed system reach a thermodynamic equilibrium with Boltzmann-Gibbs distribution (Chakraborti et al. 2011; Dragulescu and Yakovenko 2001). In population or economic exchange systems, the speed toward convergence has been shown to depend on the degree of mobility (Rappaport 2004), the frequency of interactions (Busnel et al. 2008), as well as on the differences between places at the

original state, as illustrated in the case of economic convergence between countries (Barro and Sala i Martin 1992).

In the case of coin diffusion – which is a mixing process – in time, we can expect a homogenization across the system; however, we also hypothesize that the mixing process should reach an end, at which time the proportions of each coin type in each city match the overall proportions originally produced. Also, there is no apparent reason to think that the diffusion is reversible and would, for example, after reaching a certain mix, return to 100% national coins in each country. There are a few redistribution interventions from National Banks from time to time, but the system is largely bottom-up, driven by millions of individual exchanges. We therefore expect an irreversible process and some form of convergence. Whether some complex trajectories can also be observed because of path dependence is not known a priori. Similarly, some regions may have specific behavior given relative location, lower population ranking, or other regional factors; however, these considerations are beyond the scope of this initial research, in which simplistic geographical structure will be used.

### **Individual Choice and Aggregation**

Aggregate spatial models directly describe the mechanisms of a system and its evolution using an ad hoc analytical form and a limited series of aggregated state variables (Weidlich 2003). Such a strategy provides indicators regarding system behavior and a set of interpretative clues without the need for simulations (Edwards et al. 2003). The main purpose of an aggregate model is then to understand the evolution of the phenomena overall, rather than its internal dynamics (Sanders 2007). Sometimes hypotheses regarding individual behavior can be integrated at an upper level, as is the case, for example, with the synergetic model of interregional migrations by Weidlich and Haag (1988).

Nevertheless, the form of aggregate models often requires simplification of the system under analysis; in spatial interaction models, this simplification is implemented via the hypothesis that agents are homogeneous and rational (Roy and Less 1981). Moreover the elementary interactions of individual components within a system may show collective dynamics that call into question the capability of equation-based modeling strategies to grasp the macroscopic behavior of many systems in many fields, including chemistry, biology, finance and social sciences (Shnerb et al. 2000; Gilbert and Troitzsch 2011). Interactions at the elementary levels are responsible for the production of intermediate or global processes and dynamics which may be important enough to contrast with macroscopic descriptions and predictions (as in the Lotka-Volterra model). The concept of emergence, relying on the idea that understanding a macro-scale phenomena necessitates the understanding of the most elementary components of the system (Solé et al. 1999), is therefore often preferred by modelers, despite difficulties in adaptation to real systems (Gil-Quijano et al. 2012). Compared to aggregate models, individual-based models (IBM) are also perceived as being easily applicable to empirical

issues because they are more often grounded in a heuristic approach to prediction (Bretagnolle et al. 1999).

An early attempt to integrate individuals in models was conducted by Orcutt (1957) who implemented operating characteristics at the individual and household levels to understand macroeconomic laws. In the particular context of spatial diffusion, the idea dates back to Hagerstrand's model where both a hierarchical model of the society and individuals within it, as well as spatial and temporal constraints upon individuals, are considered (Hägerstrand 1970, 1952). Many models actually account for mechanisms at different scales, notably in geography (Sanders 2007), in order to highlight impacts between scales (White et al. 1997; Quijano et al. 2007). Nevertheless, the complexity of those interrelations may make this analysis impossible; a solution is to consider the system at the most disaggregate level only. In transport research, the disaggregation of models to individuals or households offers a direct way to include economic decision-making explicitly in the choice of activities and trips, in contrast to the ad hoc "physics" of aggregate spatial interaction (Lerman 1979; Kitamura 1984a,b).

It is difficult to form a theory or discover facts about how people behave with coins and the many currency transactions they might have. Money bags can be surveyed at different time intervals, but each exchange of coins cannot be surveyed. The individual behavior which can be represented in a model is therefore simplistic, and randomness may play an important role. Even with this caveat, it is interesting to survey individuals. In this study, individuals have their own money bags, which they carry out when moving out, and whose content changes after a monetary transaction. As stated earlier, we consider only movement between two cities for simplification, therefore mobility choice at the individual level is not considered. Aggregate and disaggregate versions of the exchange stage were developed for the purpose of comparison.

### *Models of Coin Diffusion*

One can identify two approaches to the analysis of coin diffusion in the literature. The first approach considers the circulation of coins to be a process of spatial migration between origins and destinations with the aim of understanding what type of persons and places contribute to circulating coins across borders, and whether barriers can be identified (Berroir et al. 2005). The second approach is epidemiological – the epidemic disease being represented by the presence of foreign coins in each country. Of course, coins are not strictly analogous to diseases or biological vectors since they cannot reproduce themselves; however, the circulation process shares similarities with epidemics in the sense that they both result from movements of people and individual contacts.

Currency circulation has been studied as a proxy of human mobility in many different scientific fields. Historians have been using coin distribution as a proxy for ancestral mobilities for decades already (Bursche 2002; Moasil 2002; Oberländer-

Törnöveanu 2002; Tsotselia 2002; Ujes 2002). Other disciplines began to consider this particular material only recently. Physicists and mathematicians have analyzed dollar bills (Brockmann and Hufnagel 2007; Brockmann 2008, 2010; Brockmann and Theis 2008) or euro coins (van Blokland et al. 2002; Stoyan et al. 2004; Seitz et al. 2009, 2012) using data publicly contributed over the Internet. Geographers have compared static potential models with observed distribution of dollar bills (Tobler 1981), or analyzed border and social effects in terms of “contamination” by foreign euro coins (Berroir et al. 2005; Grasland et al. 2002, 2005a,b, 2012; Grasland and Guérin-Pace 2003; Grasland 2009).

The case of euro coins has attracted interest because it is a unique example of a diffusion process for which the initial state is perfectly known (all coins minted were in their home countries on December 31, 2001). It offers the potential to determine whether the movement of currency follows a particular theoretical law, and whether we can predict the date upon which a perfect mix can be reached. The perfect mix is theoretically the final stage of the diffusion process where the share of each coin type is equal across countries and remains stable. The proportion corresponds to the proportions of the different coins in the total produced since the introduction of the Euro.

From a physical perspective, the diffusion between two places is expected to occur via dispersal from the place where the phenomena is the most concentrated toward the place where it is the least concentrated, as in heat or particle diffusion. Using Markov chains, van Blokland et al. (2002), Stoyan (2002), Stoyan et al. (2004), Seitz et al. (2009, 2012) designed models in which each country is characterized by the proportion of foreign coins, and entropy is measured through time. The proportions evolve according to constant but asymmetric transition probabilities. Those models are aggregated in the sense that they assume that the money bag of each person involved in a transaction is solely characterized by the proportion of foreign coins in his or her country. Using this modeling framework in the case of Germany, the perfect mix of coins was expected by 2012 (Stoyan 2002; Stoyan et al. 2004; Seitz et al. 2009); if losses (in terms of movement of the coins outside of the Eurozone, savings and losses) and new production of coins are accounted for, the perfect mix is expected to be delayed until 2050 (Seitz et al. 2012).

Three key assumptions underlie these models, the implications of which upon the evolution of the system will be tested. First, nation states are used as the relevant elementary entity, which raises the question of being less aggregate by considering individuals rather than territories. Second, the coins available in a given place depends only of the previous stage, meaning that the mobility and the exchange process are asynchronous. Third, the authors consider a uni-directional exchange at destination as only the ‘movers’ get money from the transactions. We consider the effects of these three assumptions by designing and comparing four different modeling strategies for the exchange process.

Another modeling approach has been taken by Grasland and Guérin-Pace (2003), which is not dynamic but includes both population and distance effects, as in gravity models. The authors analyze the effect of the distance decay parameter. Berroir

et al. (2005) use a similar strategy to identify barrier effects. Grasland et al. (2005a) employ a dynamic perspective which is close to the previous Markov models but considers distance between places. The probability for a coin to move toward a particular city is defined according to transition probabilities that decrease with distance from the border. The mix of coins is analyzed with respect to distance. The authors show that even a very active border area may only lead to a small level of penetration of foreign coins as observed at a given point in time since the coins are continuously going forward and backward from the border. When compared to the previous physical models, Grasland et al. (2005a) obtain a better fit to the observed mixing of coins in France and other countries.

In addition to testing the implications of the three assumptions underlying the Markov models discussed above, this model employs a spatial interaction (distance) effect, as in Grasland and Guérin-Pace (2003) and Grasland et al. (2005a). This is conducted in a simpler manner by analyzing sensitivity of model outputs to different cross-border mobility rates.

## Model

### *Definitions and General Assumptions*

#### **Elements of the System**

The model utilizes a simple geographical setting comprised of two places, and undifferentiated interaction flows that represent the movement of people between the two places. In the future, a more complex geographical system including several places with different population and attractiveness attributes (capital cities, tourist attractions,...) would enrich the approach and would facilitate the inclusion of different motives for and intensity of movement (travel, holidays, daily commute, weekend trips, etc.). In the present work, a simpler setting enables the focused examination of the impact of the various modeling assumptions stated above.

The geographical system consists of two places,  $a$  and  $b$ , endowed with total populations  $P_a$  and  $P_b$ . Within each population, a portion is mobile,  $M_a$  and  $M_b$ ; the rest is immobile. The immobile agents are able to participate in transactions, although they do not move out of their home city. Mobility rates for each place are given by  $m_a = \frac{M_a}{P_a}$  and  $m_b = \frac{M_b}{P_b}$ .

Each place produces a certain number of coins (one coin for each individual) with a specific symbol.  $A$  and  $B$  are the total number of coins from each place available in the system.  $A_a^t$ ,  $B_a^t$ ,  $A_b^t$  and  $B_b^t$  correspond to the total number of coins located in a city ( $a, b$ ) at time  $t$ . The initial conditions are such that  $A$  and  $B$  are distributed only among their own total population at the beginning of the system:  $A_a^{t_0} = A$ ;  $A_b^{t_0} = 0$ ;  $B_a^{t_0} = 0$ ;  $B_b^{t_0} = B$ , which depicts the Eurozone at the time euro currency entered circulation (January 1, 2002).



We define the stock of coins available in each place at time  $t$  according to their place of origin in relative terms:  $\alpha_a$  (and, respectively,  $\beta_a, \alpha_b, \beta_b$ ) being the share of  $A$  (and, respectively,  $B$ ) among the total number of coins located in  $a$  or  $b$  at time  $t$  (so that  $\alpha_a^t + \beta_a^t = 1$  and equivalently for  $b$ ):

$$\alpha_a^t = \frac{A_a^t}{A_a^t + B_a^t} \quad (1)$$

$$\beta_a^t = \frac{B_a^t}{A_a^t + B_a^t} \quad (2)$$

$$\alpha_b^t = \frac{A_b^t}{A_b^t + B_b^t} \quad (3)$$

$$\beta_b^t = \frac{B_b^t}{A_b^t + B_b^t} \quad (4)$$

We take particular interest into the value of  $\alpha_b$  and  $\beta_a$  since they indicate the level of foreign coins in a given place. For example ten years after the Euro entered circulation, the share of foreign Euro coins was 35% in France (Grasland et al. 2012) and 75% in the Netherlands (Eurodiffusie 2011).

In this model, the system is closed. Each city produces one coin for each individual and there is no additional production nor loss of coins :

$$A_a^t + B_a^t = P_a \quad (5)$$

$$A_b^t + B_b^t = P_b \quad (6)$$

Last, we assume that every mobile person makes a trip to the other city at each time step. Every mobile person also makes a monetary transaction at destination. Thus, for a mobile agent, a time step consists in four successive actions : (1) a move, (2) a transaction, (3) a move, and (4) a transaction. For an immobile agent, a time step consists only in transactions.

Three additional assumptions underlie the mixing process. First, the direction of the transaction is either one-way or two-ways: in each transaction there is either a giver or a receiver, or both agents participate in the transaction. Such a transaction may represent a situation in which an individual purchases an item in a shop and receives change from the shopkeeper. Second, the exchange of money is either synchronous or asynchronous with the mobility of people: i.e. everybody moves and conducts transaction at the same time or sequentially. In the first case, it is considered that movers can only conduct a transaction with an immobile agent at destination; while in the second case, the people and money available in the city are defined after the moves of mobile agents. Third, each individual may interact with one or with several individuals, thus heterogeneity can be introduced in the quantity of transactions that each individual conducts.

## Convergence Speed, Equilibrium Value and Perfect Mix

The methodology used in this study includes an analysis which examines how the mix of coins changes with time, and whether or not it converges to an equilibrium. The result of interest is the proportion of foreign coins within each city at equilibrium – i.e.,  $\beta_a^*$  and  $\alpha_b^*$ , which we define as:  $\beta_a^* = \lim_{t \rightarrow \infty} \beta_a(t)$  and  $\alpha_b^* = \lim_{t \rightarrow \infty} \alpha_b(t)$ . We thus expect a static equilibrium, from which the properties of the system will remain unchanged over time. The equilibrium point of  $\beta_a$  is such that  $\forall t \in N, \beta_a^{t+1} = \beta_a^t$  (similarly for  $\alpha_b$ ). We denote by  $t^*$  the moment at which these asymptotes are reached. Obviously,  $t^*$  is the same for both places.

Through simulation it is possible to analyze the existence and characteristics of this simplified system's equilibrium, as well as the time required to reach it. This process is presented in the following section. Prior to attempts to reach convergence, and independent of the equilibrium solution, the perfect mix was characterized as a mix in which  $\alpha_a = \alpha_b$  and, accordingly,  $\beta_a = \beta_b$ . Since the shares of  $\alpha$  and  $\beta$  are complementary, we can define the perfect mix using our two variables of interest:  $\beta_a = 1 - \alpha_b$ .

## Set of Models

We design a set of three models, the last one being subdivided into two variants. Models *I* and *II* are aggregated, at a level similar to that of existing models in the literature – i.e., averaged money bags were calculated during the exchange process. Models *IIIa* and *IIIb* are disaggregated – each individual money bag is calculated in the exchange process.

For all models, numerical simulations are conducted (see section “[Theoretical Simulations](#)”) in order to capture the diffusion process and cumulative effects. Some analytical properties can be derived from the first model using a discrete or continuous dynamic formalization (see Appendix). A thorough continuous analysis and comparison with numerical outputs is nevertheless outside the scope of the present work.

## Model I

Model *I* implements an aggregated one-way exchange. This is in line with the models proposed by Stoyan (2002), Stoyan et al. (2004) and Seitz et al. (2009): the diffusion mix results from external trips only and is one-way. The one-way exchange applies hardly to real-life situation since no money is received by local population at destination. The money bag of immobile agents being unchanged at transaction time, the money available for a transaction is directly described by  $(A_i, B_i)$  at destination place for the receiving mobile agents. Mobility and exchange are asynchronous: at each time step  $t$ , mobile agents move then make a transaction

at destination. The mobile agents bring back a proportion of coins  $A$  and  $B$  that corresponds to the proportion in the destination place prior to their arrival.

The stock of money  $A$  (and, respectively,  $B$ ) available in city  $a$  (and, respectively,  $b$ ) at the end of a time step is therefore defined according to:

$$A_a^{t+1} = (1 - m_a)P_a\alpha_a^t + m_aP_a\alpha_b^t \quad (7)$$

$$B_a^{t+1} = (1 - m_a)P_a\beta_a^t + m_aP_a\beta_b^t \quad (8)$$

$$A_b^{t+1} = (1 - m_b)P_b\alpha_b^t + m_bP_b\alpha_a^t = A - A_a^{t+1} \quad (9)$$

$$B_b^{t+1} = (1 - m_b)P_b\beta_b^t + m_bP_b\beta_a^t = B - B_a^{t+1} \quad (10)$$

Each equation is made of two parts: the first refers to the unchanged money bags of the  $(1 - m_a)P_a$  and  $(1 - m_b)P_b$  immobile agents, and the second to the number of coins moving from  $a$  to  $b$  (or  $b$  to  $a$ ), taken by the  $m_aP_a$  and  $m_bP_b$  mobile agents.

## Model II

In daily life, both a buyer and a shopkeeper can receive money from a transaction. We therefore adapt Model *I* to realign with this hypothesis that coin transactions can be two-way. This kind of transactions may represent the money exchanged between a shop-keeper and a buyer, bills and tips, shared expenses, etc. If not every time, at least on average, mobile people and the people they meet can be both givers and receivers in turn. Model *II* therefore implements the idea that all people present in a given city and a given time have the same probability to get a coin from either a mobile agent visiting the city or from an immobile agent living in this same city. This two-way exchange process requires the total currency available in each place to be updated with the currency brought back by all agents, in order to constitute a stock before the next transactions take place.

Since Model *II* is aggregated and two-way, it automatically assumes that movements are synchronous – all mobile agents move together to contribute their coins to the stock, then each returns home with a money bag characterized by this averaged intermediate stock.<sup>1</sup> The averaged intermediate stock represents the average of the destination stock after the arrivals of mobile agents.

In Model *II*, we first define the intra-time update of the stock of coins available in cities after mobility has taken place:

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<sup>1</sup>A two-ways asynchronous model would correspond to a situation where the total currency available in each place would be updated after the moves of agents. Mobile agent would nevertheless be able to conduct a transaction only with immobile agents at destination. Consequently, they would return home with the same money bag as in Model *I*. Besides, if the locals are the receivers, then they would receive the same money as if they were moving to the other city, again resembling Model *I*. A two-ways asynchronous model would thus add little sense at an aggregated level and is not analyzed in this paper.

$$\bar{A}_a^t = (1 - m_a)P_a\alpha_a^t + m_bP_b\alpha_b^t \quad (11)$$

$$\bar{B}_a^t = (1 - m_a)P_a\beta_a^t + m_bP_b\beta_b^t \quad (12)$$

$$\bar{A}_b^t = (1 - m_b)P_b\alpha_b^t + m_aP_a\alpha_a^t \quad (13)$$

$$\bar{B}_b^t = (1 - m_b)P_b\beta_b^t + m_aP_a\beta_a^t \quad (14)$$

Hence, the share of each coins available for transactions in city  $a$  at time  $t$  is denoted by:

$$\bar{\alpha}_a^t = \frac{\bar{A}_a^t}{\bar{A}_a^t + \bar{B}_a^t} \quad (15)$$

$$\bar{\beta}_a^t = \frac{\bar{B}_a^t}{\bar{A}_a^t + \bar{B}_a^t} = 1 - \bar{\alpha}_a^t \quad (16)$$

$$\bar{\alpha}_b^t = \frac{\bar{A}_b^t}{\bar{A}_b^t + \bar{B}_b^t} \quad (17)$$

$$\bar{\beta}_b^t = \frac{\bar{B}_b^t}{\bar{A}_b^t + \bar{B}_b^t} = 1 - \bar{\alpha}_b^t \quad (18)$$

with  $\bar{\alpha}_a^t$  describing the share of money  $A$  in the city  $a$  after every mobile agent moved, and accordingly,  $\bar{\alpha}_b^t$  describing the share of  $A$  in  $b$ ,  $\bar{\beta}_a^t$  the share of  $B$  in  $a$  and  $\bar{\beta}_b^t$  the share of  $B$  in  $b$ .

Given these intermediate proportions (Eqs. 15, 16, 17, and 18), the stocks of coins in each place at the end of the time step (Eqs. 7, 8, 9, and 10) are redefined into:

$$A_a^{t+1} = (1 - m_a)P_a\bar{\alpha}_a^t + m_aP_a\bar{\alpha}_b^t \quad (19)$$

$$B_a^{t+1} = (1 - m_a)P_a\bar{\beta}_a^t + m_aP_a\bar{\beta}_b^t \quad (20)$$

$$A_b^{t+1} = (1 - m_b)P_b\bar{\alpha}_b^t + m_bP_b\bar{\alpha}_a^t \quad (21)$$

$$B_b^{t+1} = (1 - m_b)P_b\bar{\beta}_b^t + m_bP_b\bar{\beta}_a^t \quad (22)$$

### Model III

Model *III* uses a disaggregate approach, with the exchange process defined at the level of individuals. Mobility is still implemented at the inter-city level, since the model includes only two cities and the share of mobile people is known and fixed. Models *IIIa* and *IIIb* are both two-way asynchronous models.

In contrast to Model *II*, there is no intermediate, aggregate ‘pot of coins’ but a sequence of iterative meetings between two individuals present at the same place. Exchanges are sequentially simulated after randomly pairing individuals in the list

of people present. However, in order to be comparable with the previous aggregate models in terms of quantity of exchanges and the mixing process, two additional assumptions were made:

First, for an individual, the mixing occurs at the destination, but also back home (as in Model *I* and *II*, in which the second stage was included in the update process). The equivalent in disaggregate modeling is to model interpersonal exchanges twice: agents conduct transactions at the destination, and then return home where they conduct transactions again. The list of individuals present at destination is drawn from the list of immobile people of the destination place and the mobile agents from the concurrent city. Back home, the list is made of the population of the city – the immobile agents plus the mobile agents (having returned home).

Second, the quantity of transactions is set similarly to aggregated models in which every exchanger is assumed to have one interaction. Therefore the number of transactions is fixed at half of the population present in a place, i.e.,  $E_a = [(1 - m_a)P_a + m_b P_b]/2$  and  $E_b = [(1 - m_b)P_b + m_a P_a]/2$  for the first round at destination, and  $E'_a = P_a/2$  and  $E'_b = P_b/2$  for the second round at home.

The mix of coins in a given place after completion of a full time step is then measured when every mobile agent is back home:

$$\alpha_a^t = \sum_{i_a}^{P_a} \frac{\alpha_{i_a}^t}{P_a} \quad (23)$$

$$\beta_a^t = \sum_{i_a}^{P_a} \frac{\beta_{i_a}^t}{P_a} \quad (24)$$

$$\alpha_b^t = \sum_{i_b}^{P_b} \frac{\alpha_{i_b}^t}{P_b} \quad (25)$$

$$\beta_b^t = \sum_{i_b}^{P_b} \frac{\beta_{i_b}^t}{P_b} \quad (26)$$

where  $(\alpha_{i_a}^t, \beta_{i_a}^t)$  and  $(\alpha_{i_b}^t, \beta_{i_b}^t)$  represent the money bag mix for each individual  $i$  at time  $t$  after exchange.

The pairing of individuals necessitates a third assumption:  $E_a$ ,  $E_b$ ,  $E'_a$  and  $E'_b$  are sample pairs obtained from a random drawing with replacement. This reflects the ideas that: (1) all exchanges are sequential and the model is thus asynchronous; (2) each agent can make several (or no) transactions regardless of its origin; and (3) at destination, the probability to exchange with a local or a visitor is equal. The latter sounds reasonable, although we can imagine that visitors may concentrate in parts of the city and have more interactions among themselves, as it may be the case in souvenir shops, for instance.

Finally, the exchange between two people itself necessitates several assumptions. We know from Nuno et al. (2005) that individuals have different strategies across transactions, leading to different numbers of coins being exchanged. We also know that the value of coins and the characteristics of individuals influence the number of coins exchanged (Grasland 2009). It is therefore interesting to test the impact of how many coins are exchanged at each transaction onto the convergence level and rate of foreign coin diffusion.

We consider two possibilities. In model *IIIa*, the exchange consists of averaging the money bags of the paired individuals. After exchange, the two money bags are thus similar.

For a pair of individuals  $i = (1,2)$  and  $\tau$ , referring to an intra-time exchange iteration, we have

$$\alpha_1^\tau = \alpha_2^\tau = \alpha_1^{\tau-1} + \alpha_2^{\tau-1}/2 \quad (27)$$

and similarly for  $\beta$  (which is also equal to  $1 - \alpha$ ). Note that when  $\tau = 1$  then  $\alpha^{\tau-1} = \alpha^\tau$  and  $\beta^{\tau-1} = \beta^\tau$ .

In model *IIIb*, a different strategy is used that is expected to lead to the quickest mixing of coins. Instead of averaging the money bags, agents 1 and 2 simply swap money bags:

$$\alpha_1^\tau = \alpha_2^{\tau-1} \quad (28)$$

$$\alpha_2^\tau = \alpha_1^{\tau-1} \quad (29)$$

and similarly for  $\beta$ . This type of transaction could correspond, for instance, to a situation in which two individuals exchange plenty of coins of small values against a few coins of high values.

These models represent only two strategies out of many possibilities. There are intermediate situations in which only a part of each money bag is exchanged, or even more extreme situations where only one coin is exchanged during a transaction. Testing this variety of strategies in its entirety is not feasible for one study. With models *IIIa* and *IIIb*, we aim to show that a disaggregate model of exchange based upon individuals, rather than aggregated exchange, might in itself provide insight into the complexity of the diffusion process, even within a simplified geography.

## Theoretical Simulations

In this section, numerical outputs from models *I*, *II*, *IIIa* and *IIIb* are analyzed and compared based upon the sensitivity of the equilibrium value ( $\beta_a^*$ ) and convergence speed ( $t^*$ ) to changes in population and mobility rates.

## ***Exogenous Parameters***

Sensitivity analysis is based upon a fixed  $P_b = 1,000$  and  $m_b = 0.10$  and an incrementally varying  $P_a$  from 100 to 2,000, and  $m_a$  from 0.01 to 0.20. The benchmark situation is such that both cities have equivalent exogenous parameters, i.e.,  $P_a = P_b = 1,000$  and  $m_a = m_b = 0.10$ .  $m_a$  and  $m_b$  can theoretically be any proportion, but 0.10 is used as a mid-range benchmark. As an indication, 1.1% of the French active population was a cross-border commuter in 2011 (Floch 2011). This would be our lower limit, knowing that other trip purposes contribute to the diffusion of coins.

Population is fixed at the values indicated. These figures were chosen in order to avoid both very small values (because of random effects), and very high values (out of concern for computational limits of this particular research effort). The median population in the Eurozone countries in 2011 was 9.5 million inhabitants. The first and third quartiles were, respectively, 1.9 million and 24 million inhabitants (Eurostat 2012). Sensitivity ranging from 100 to 2,000 was therefore considered to be reasonable.

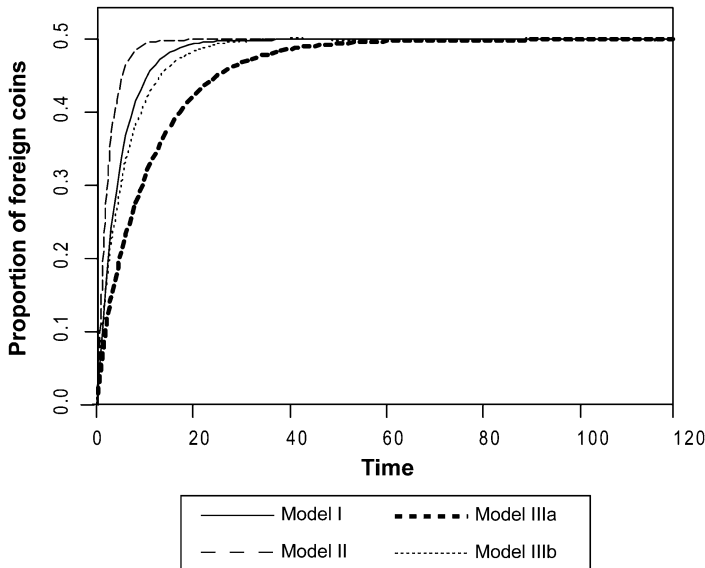
## ***Results***

### **Benchmark: Equal Population and Mobility Ratios**

The benchmark case uses same characteristics for both places, in order to measure the effects of the modeling strategy without external influence (i.e., without altering the values of exogenous parameter inputs  $P_a$ ,  $P_b$ ,  $m_a$  and  $m_b$ ). Following Seitz et al. (2012), if there are no external influences, we expect convergence and perfect mix at equilibrium for the two aggregated models. Also, there is no a priori reason not to reach convergence in the two disaggregated cases, since apart from stochasticity, there is no additional interaction.

All of the models converge, and the convergence process is depicted in Fig. 1. Models *I* and *IIIb* demonstrate very similar behavior. The exchange process is actually very similar in both models, even though the elementary scale is different: in Model *IIIb*, people swap money bags, and they do not mix up their coins. In Model *I*, visitors change their money bags to mimic the locals' money bags. It is surprising, however, to find Model *IIIb* so close to the most basic aggregated model, and so different from Model *IIIa*. This definitely highlights the importance of carefully considering what an aggregate model assumes in terms of individual behavior.

Comparing Models *IIIa* and *IIIb* shows that when more coins are exchanged between individuals, convergence is accelerated. The same idea holds when comparing Model *I* and Model *II*, in which people at the destination also receive money, instead of being givers only.



**Fig. 1** Evolution of coin mixing ( $\beta_a$ ) over time for the four models

Stochasticity in Models *IIIa* and *IIIb* may raise additional questions regarding these results. Therefore Fig. 2a, b report the uncertainty around the convergence state after a Monte Carlo process with 100 replications. The variability of the equilibrium is small in the case of Model *IIIb* and almost nonexistent in Model *IIIa*.

### Sensitivity to Mobility and Population Differentials

We show in Fig. 3a–d the sensitivity of the equilibrium mix ( $\beta_a^*$ ) to changes in mobility rates and population differentials between places. The grids represent incremental variations of the ratio of the two input parameters between the two cities. The darker the square, the higher the proportion of foreign coins in place *a* at equilibrium. The color gradient corresponds to different values for each model.

Figure 3a reveals a strong difference in the behavior of Model *I* compared to the other models. Results from Model *I* are quite counter-intuitive. In fact we expected the differentials of population between places to determine the level of the coins mixing at equilibrium as is the case in Models *II*, *IIIa* and *IIIb*. The proportion of foreign coins at equilibrium should correspond to the ratio of populations, since each city produces one coin per individual: the higher the population of *b* compared to *a*, the higher the share of foreign coins in *a* at equilibrium.

But in Model *I*,  $\beta_a^*$  is only dependent on mobility differentials. Indeed, a closer look at the model shows that the total number of coins is preserved, but not the global share of each coin type in the system – which changes with time in a complex



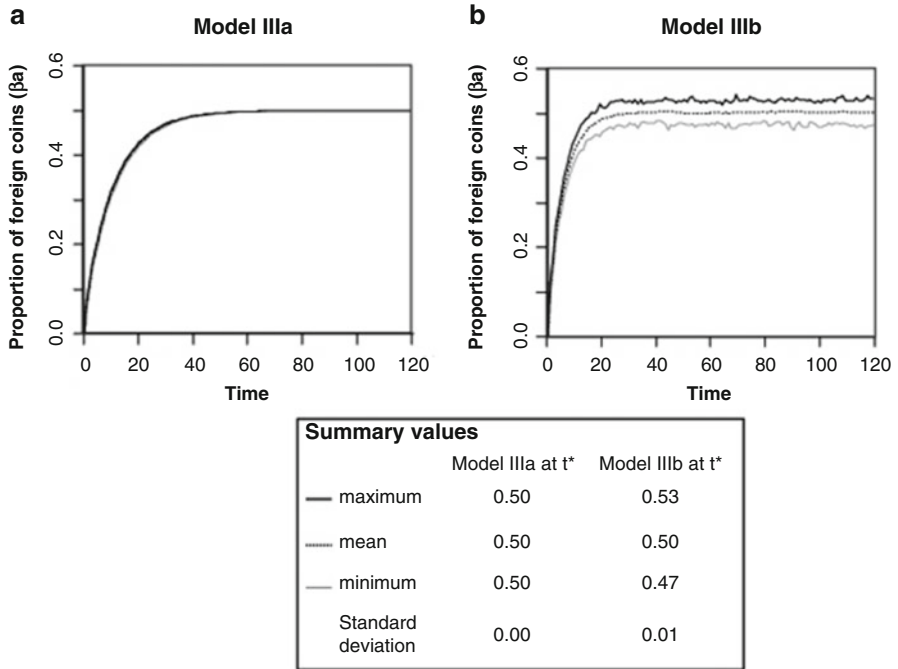


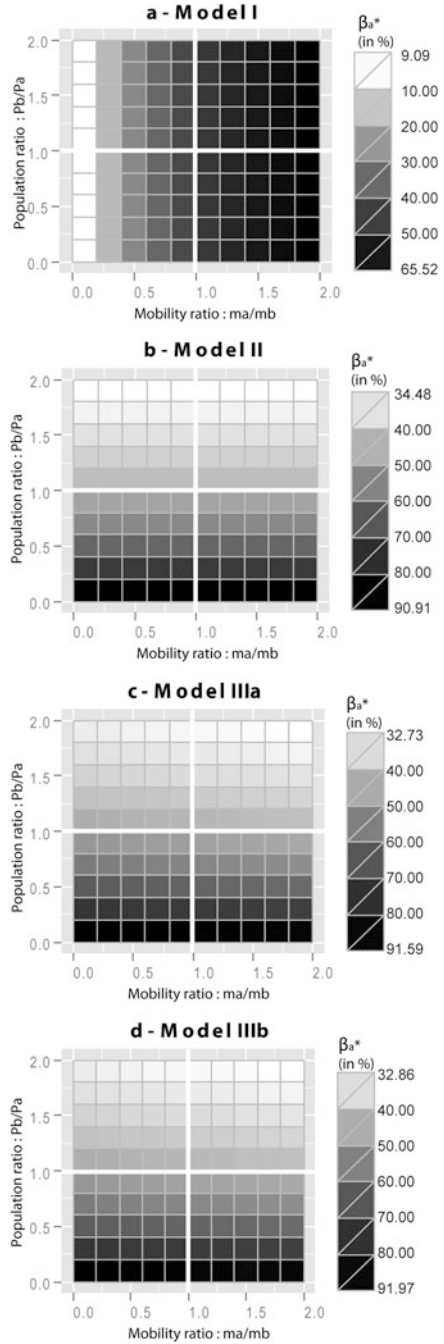
Fig. 2 Evolution of coin mixing ( $\beta_a$ ) over time – Monte Carlo analysis for Models IIIa and IIIb

manner (see section “Calibration: France Versus the Eurozone” and “Appendix”). This transformation in the type of coins is of course unwanted. Model I is however very useful methodologically because it is the closest to Stoyan (2002), Stoyan et al. (2004), Seitz et al. (2009) models who postulate a constant number of coins, but do not control for it. Indeed, the dynamic is not simulated on the stocks but on the share of the origins of coins in Germany and the rest of the Eurozone. When simulating the dynamics of these stocks in Model I, we show that A or B coins are “created” or “deleted” symmetrically revealing that the system is not properly closed.

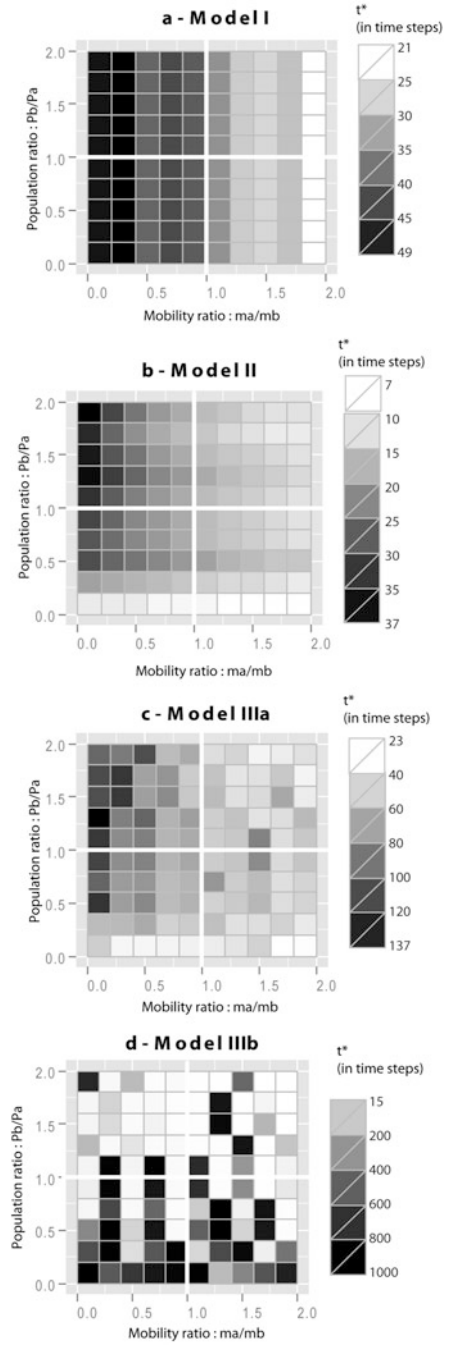
Figure 4a–d show the sensitivity of the time to reach equilibrium, i.e., the sensitivity of  $t^*$  to changes in mobility rates and populations. The darker the square, the more time is needed to reach equilibrium. Again, the color gradient corresponds to different values for each model. Model I again shows a unique pattern. However, there are also strong differences for  $t^*$  between Models II, IIIa and IIIb, whereas these models had similar patterns for  $\beta_a^*$ .

The convergence speed in Model I is not influenced by changes in the population ratio, but is highly dependent upon mobility rates. Quite surprisingly, equivalent mobility rates for cities do not lead to the quickest mixing of coins. On the contrary, the higher is the differential of mobility in favor of city a, the quicker is the convergence speed to the equilibrium value of the share of foreign coins within the

**Fig. 3** Sensitivity of the equilibrium mix  $\beta_a^*$  to changes in the ratio of populations and mobility rates



**Fig. 4** Sensitivity of the time to convergence,  $t^*$ , to changes in the ratio of populations and mobility rates



city (i.e.,  $\beta_a^*$ ). Although the simulation shows that some combinations of mobility rates are associated with lower time to convergence than others, the resulting value of  $t^*$  is not strictly proportional to the ratio of mobility.

In comparing Fig. 4b–d, we observe a certain similarity of the impacts of mobility rates and population differentials in Models *II* and *IIIa*, with a positive effect of mobility superiority and a negative effect of population superiority upon the convergence speed. Model *IIIb* seems to be much sensitive to the interaction of both mobility rates and population differentials than model *IIIa*, which appears itself more sensitive than model *II*.

## Calibration: France Versus the Eurozone

This section presents the calibration of our four models using data for France to determine whether the evolution of the mix can be predicted, and whether or when the perfect mix can be reached. In order to calibrate the model, the proportion of foreign coins in France is observed at 16 points in time via a representative survey of an average of 1,500 individuals and money bags per survey (Grasland et al. 2012). The error (*Err*) of each model is defined as the sum of the squared residuals between the estimated values and the 16 observations.

Before testing the four models, we first develop an empirical model. We test for various functional forms and find that a logistic function<sup>2</sup> best fits the data. Using maximum likelihood estimation we find:

$$\beta_a(t) = 0.3541 * (1 - e^{-t/43.72}) \quad (30)$$

The error amounts to 25.98. The curve is displayed in Fig. 5. The model results in an asymptotic value of the mix at  $\beta_a = 0.3541$ , to be obtained in April 2015. However  $\beta_a$  is far below the expected full mix level 0.75, i.e. the ratio of the two populations. The logistic curve is often used to model diffusion processes (Casetti 1969), although in this case, despite a good fit to empirical data, its usage remains ad hoc to represent mobility and diffusion. This small test emphasizes that some processes come into play which considerably reduce the mixing process – typically additional barrier effects, hoarding, external dissemination of coins, etc., which it is not possible to reflect within the framework of this analysis.

We now turn to fitting our aggregate and disaggregate models. This is an attempt to understand how the introduction of mobility and exchange behavior can affect our ability to forecast the diffusion process. In this calibration, populations are exogenously fixed, while mobility rates ( $m_a$  for France and  $m_b$  for the rest of the Eurozone) are adapted in order to minimize the error term. As explained earlier, our interest is indeed to use euro coin diffusion as a marker of international movements.

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<sup>2</sup>Asymptotic regression with lower limit at 0 (2 parameters).

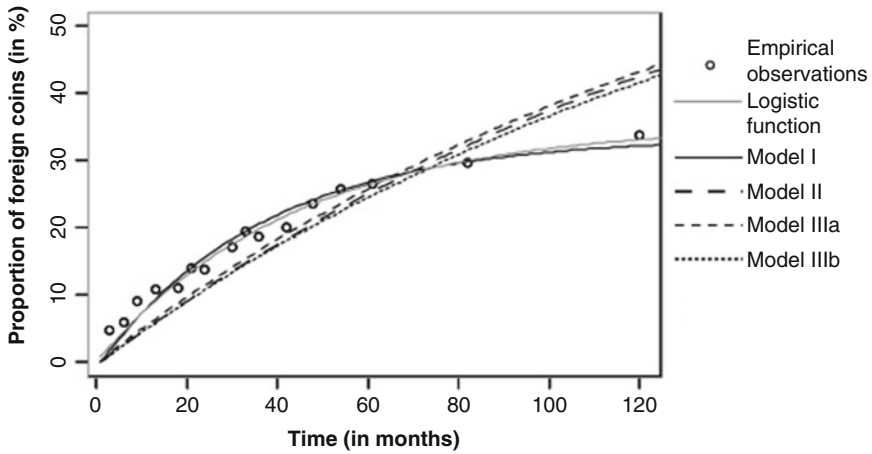


Fig. 5 Fitting of the estimated diffusion curves to empirical observations

Population parameters are taken according to Eurostat (Eurostat 2012). On January 1, 2012 the population of the Eurozone was 333 million inhabitants, as compared to 65 million in France alone (19.5% of the Eurozone total). In order to avoid computational burdens, the number of agents implemented is set at 654 ( $P_a$ ) and 2,674 ( $P_b$ ).

Again, the mobility rates can take any value ranging from 0 to 1. In the absence of global information about international mobility between France and the rest of the Eurozone, there is thus no preconceived starting point for the exploration. First, a crude calibration is conducted in order to narrow down the values of  $m_a$  and  $m_b$  by reducing *Err* by trial and error. Population being fixed, this exploration reveals monotonic behavior. Then, we vary  $m_a$  and  $m_b$  systematically from 0.001 to 0.030 by a 0.001 increment and run each model for this set of parameters.

The models also include a time increment. The unit was defined as a month, which of course impacts the results. However, for a given set of population and migration rates, varying the time increment is similar to changing the number of exchanges. Here one exchange per month was assumed for simplification. The measurement unit of the mobility rate is therefore to be interpreted with these two assumptions in mind.

Simulation results are displayed in Fig. 5 and Table 1.

Figure 5 shows that  $\beta_a$  from Model I follows a curve that is very close to the logistic estimate. Error is slightly increased but much lower than that of the other three models. Model I predicts an equilibrium occurring in May 2020. As with the logistic model, perfect mix is not reached: only 34% of coins in France are foreign coins at equilibrium. Upon closer look at the resulting equilibrium values, we find that they correspond to  $\beta_a^* = m_b P_b^2$  for France ( $\beta_b^* = m_a P_a^2$  for Eurozone) and  $\alpha_b^* = m_a P_a^2$  for France ( $\alpha_a^* = m_b P_b^2$  for Eurozone).

**Table 1** Calibration of the four models to empirical values

	Logistic	Model <i>I</i>	Model <i>II</i>	Model <i>IIIa</i>	Model <i>IIIb</i>
$P_a$		654	654	654	654
$P_b$		2,674	2,674	2,674	2,674
$m_a$		0.009	0.001	0.017	0.003
$m_b$		0.018	0.001	0.001	0.002
<i>Err</i>	0.003	0.004	0.026	0.023	0.025
$\beta_a^*$	0.35	0.34	0.80	0.80	0.80
$t^*$	161	221	861	756	746
Mix date	04/2015	05/2020	09/2073	12/2064	02/2064

Consequently, the numbers of coins  $A^t$  and  $B^t$  differ from the initial amounts across time. After calibration, for example, Model *I* transforms the number of French coins from 654 to 1332 and in parallel reduces the number of Eurozone coins from 2674 to 1996. Total number of coins is constant, but this model does not preserve the initial share. This is problematic, but also striking, in that this is the best model to fit the data. We interpret this as the result of the production of new coins and losses (coins leaving the Eurozone or being hoarded). There are no reported numbers related to losses (European 2003), but we know, for example that new currency production has not being homogeneous in time and space. According to the European Central Bank, 56% of all French euro coins in existence in 2010 were minted prior to the end of 2002; for Luxembourg, this share is 31%.

Model *II*, Model *IIIa* and Model *IIIb* have the same equilibrium value. Indeed, models incorporating a reciprocal exchange process predict that the convergence value will be close to the ratio of population differences, leading to a longer time until equilibrium. Indeed, Model *II* expects an equilibrium in January 2073, i.e. more than 70 years after the introduction of the euro as a common currency. The design of the exchange process does not appear to have great influence on the convergence speed, since Model *IIIa* converges after December 2064 and Model *IIIb* after February 2064. Nevertheless, the fact that the individuals switch the entire contents of their money bags in Model *IIIb* results in an equilibrium value varying from an estimation of 80.1% to 80.7% of total euro coins estimated to be foreign euro coins in France.

Even if Models *II*, *IIIa* and *IIIb* give close *Err* or, equilibrium value and time to convergence, the estimated values of  $m_a$  and  $m_b$  are very different from one modeling strategy to another. The aggregate model (model *II*) uses equivalent mobility rates for the two places, and the mobility rates are very low (only 1% of the global population of place *a* and place *b*), signifying that the number of interactions between France and the rest of the Eurozone is assumed to be very low. Besides, the design of the exchange process impacts the values of the input parameters  $m_a$  and  $m_b$ ; the lower the number of coins exchanged during a transaction, the higher the mobility of the residents of place *a* has to be to compensate for the differences of population. Indeed, model *IIIa* estimates that the mobility of French people (17%) is

much higher than the mobility of people of the rest of the Eurozone entering France (1%), while model *IIIb* allows for few differences between those mobility rates; the mobility in France being set much lower (3%).

## Conclusion

This study aimed at illustrating the results of a particular modeling strategy for analysis using a dynamic spatial interaction model.

Model *I* has been found to be very unique among the models tested: the effects of the input parameters on the equilibrium value and convergence speed differ largely from results in Models *II*, *IIIa* and *IIIb*. The integration or non-integration of substantial reciprocity in the exchange process seems indeed to be the preeminent choice in the design of the models. Consequently, the calibration of Model *I* resulted in very different adjusted parameters and predictions compared to the three other models.

The mechanisms under investigation in Model *I* (in which external trips are the impetus for diffusion, and exchange is only one-way (van Blokland et al. 2002; Stoyan et al. 2004; Seitz et al. 2009, 2012)) and the logistic curve offer very similar calibration to empirical values and time predictions. Since Hägerstrand (1952) the logistic model has often been shown to fit diffusion processes over time, our result may question the benefit of aggregate models to provide modeling benefits within such a simple geographical space. Conversely the disaggregate approaches helped in revealing how aggregate patterns emerge over time from individual behavior.

The system being comprised of two countries, randomness in Models *IIIa* and *IIIb* is not defined at the stage of the destination choice like it has mainly been the case since Huff (1964), but rather is implemented at the moment of the exchange. Models *IIIa* and *IIIb* result in a low sensitivity of the system to the elementary interactions of individuals components. This is due to all mobile agents sharing identical mobility behavior. Low sensitivity to random pairing (equal probability for agents to participate in a transaction) could as well been deduced from the central limit theorem (Roy and Less 1981).

The calibration to empirical values has shown minor differences in the convergence speed of the three models, as expected from Hägerstrand (1952), Shnerb et al. (2000), Sanders (2007), and Quijano et al. (2007). These differences stand in the design of the elementary transactions: mean behavior (model *II*) vs mean behavior with stochasticity (model *IIIa*) and differences in the number of coins exchanged (model *IIIa* and model *IIIb*).

These results are based upon testing of four models, with particular assumptions. Expanded testing with additional models, using a broader range of assumptions and additional sensitivity analyses, will be undertaken in future research to follow up on these results.

In the study of this simple system, we have considered that individuals will participate in one exchange per time step on average, to simplify the comparison with the aggregate models. However, the question of the impact of the number of exchanges per time step is still unanswered, and will be the subject of further research. Controlling the range of variability within the different types of exchanges is expected to bring new understanding of the particular effects of mobility upon the diffusion process.

We should be prudent when using coin diffusion modeling to generalize about international movements. Coins are not equivalently involved in transactions according to the prices of the products, or the payment strategies adopted by individuals (Nuno et al. 2005), or even the characteristics of the people. Empirical surveys (Grasland et al. 2012) have shown for example that the content of money bags depends on individual characteristics: elderly women carry on average more coins than young men, etc. Also, cash represents a substantial parts of payments in Europe (de Meijer 2010), which might not be the case in other regions of the world. In the European context we assume that coin diffusion is a possible proxy for analyzing individual mobility and, subsequently, the internationalization of territories.

Finally, further research efforts will attempt to: (1) increase the realism/complexity of the geography of the system by introducing new places (in a 1-D and/or 2-D space) and by differentiating the attractiveness of the different places; and (2) articulate assumptions on macroscopic mobility with sociological parameters of individual behavior. These assumptions may be best investigated by preliminary social network analysis and methods based in time-geography in order to accurately model mobility decisions (scheduling of activities and destination choices).

## Appendix

Mathematical resolutions of Model *I* are presented in the following subsections.

### *Discrete Formalism*

From Eqs. (1) and (3) on the one hand and (5) and (6) on the other hand, within Eq. 7, we find that:

$$\begin{aligned} A_a^{t+1} &= (1 - m_a)P_a\alpha_a^t + m_aP_a\alpha_b^t \\ A_a^{t+1} &= (1 - m_a)P_a\frac{A_a^t}{P_a} + m_aP_a\frac{A_b^t}{P_b} \\ A_a^{t+1} &= \frac{(1 - m_a)P_aA_a^t}{P_a} + \frac{m_aP_aA_b^t}{P_b} \end{aligned}$$



$$\begin{aligned}
A_a^{t+1} &= (1 - m_a)A_a^t + \frac{m_a P_a}{P_b}(P_b - B_b^t) \\
A_a^{t+1} &= (1 - m_a)A_a^t + \frac{m_a P_a}{P_b}(P_b - (P_b - B_a^t)) \\
A_a^{t+1} &= (1 - m_a)A_a^t + \frac{m_a P_a}{P_b}B_a^t \\
A_a^{t+1} &= (1 - m_a)A_a^t + \frac{m_a P_a}{P_b}(P_a - A_a^t) \\
A_a^{t+1} &= (1 - m_a)A_a^t + \frac{m_a P_a}{P_b}(P_a - A_a^t)
\end{aligned} \tag{31}$$

Knowing that for  $A_a$ , we can proceed similarly with others:

$$B_a^{t+1} = (1 - m_a)B_a^t + \frac{m_a P_a}{P_b}(P_b - B_a^t) \tag{32}$$

$$A_b^{t+1} = (1 - m_b)A_b^t + \frac{m_b P_b}{P_a}(P_a - A_b^t) \tag{33}$$

$$B_b^{t+1} = (1 - m_b)B_b^t + \frac{m_b P_b}{P_a}(P_b - B_b^t) \tag{34}$$

Knowing that  $A_a^{t+1} = (1 - m_a)A_a^t + \frac{m_a P_a}{P_b}(P_a - A_a^t)$ , if we define  $x = (1 - m_a)$ ,  $y = \frac{m_a P_a}{P_b}$ ,  $a = x - y$  and  $b = yP_a$ , we have:

$$aA_a^t + b \tag{35}$$

We know that arithmetico-geometric series may be solved by the following theorem:  $\forall n \geq n_0, U_{n+1} = aU_n + b$ , having  $r = \frac{b}{1 - a}$ , thus  $\forall n \geq n_0, U_n = a^{n-n_0}(U_{n_0} - r) + r$ , with  $U$  being the arithmetic series and  $r$  being the common difference. Consequently, the change in the stocks of coins of each city varies according to the populations in the two cities and mobility rates of the city where coins are observed.

### ***Continuous Formalism***

Trying to define the variation of  $A_a, A_b, B_a$  et  $B_b$  under the formalism of deterministic ordinary differential equations of first order, we find that:

$$A_a^{t+1} - A_a^t = -m_a \left( 1 + \frac{P_a}{P_b} \right) A_a^t + \frac{m_a P_a^2}{P_b}$$

Consequently, with  $d$  being the derivative, the equilibrium point which verifies  $\frac{dA_a}{dt} = 0$  is given by the following expression:

$$\frac{dA_a}{dt} = \frac{P_a^2}{P_b + P_a}$$

Similarly, the equilibrium points which verify  $\frac{dB_a}{dt} = 0$ ,  $\frac{dA_b}{dt} = 0$  and  $\frac{dB_b}{dt} = 0$  are given by the expressions:

$$B_a = \frac{P_a P_b}{P_b + P_a}$$

$$A_b = \frac{P_a P_b}{P_b + P_a}$$

$$B_b = \frac{P_b^2}{P_b + P_a}$$

The variation between two consecutive states are dependent upon both the mobility rates in the place of minting and the populations differences between the two places. The value of the equilibrium points of the redistribution of coins between cities is only dependent upon the population shares, i.e., upon the number of coins originally produced.

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## References

- Allen, P., & Sanglier, M. (1979). A dynamic model of growth in a central place system. *Geographical Analysis*, 11, 256–272.
- Allen, P., & Sanglier, M. (1981). Urban evolution, self organisation and decision-making. *Environment and Planning*, 13, 168–183.
- Allen, P. M. (1997). *Cities and regions as self-organizing systems: Models of complexity*. London: Taylor and Francis.
- Anas, A. (1983). Discrete choice theory, information theory and the multinomial logit and gravity models. *Transportation Research B*, 17, 13–23.
- Andersson, H., & Britton, T. (2000). *Stochastic epidemic models and their statistical analysis*. New York: Springer.
- Arthur, W. B. (1988). Urban systems and historical path dependence. In J. Ausubel & R. Herman (Eds.), *Cities and their vital systems: Infrastructure, past, present and future* (pp. 85–97). Washington, DC: National Academy Press.
- Aziz-Alaoui, M., & Bertelle, C. (Eds.) (2009). *From system complexity to emergent properties*. Berlin: Springer.

- Barro, R. J., & Sala-i-Martin, X. X. (1992). Convergence. *Journal of Political Economy*, 100, 223–251.
- Batty, M. (2008). The size, scale, and shape of cities. *Science*, 319(5864), 769–771.
- Berry, B. J. L. (1964). Cities as systems within systems of cities. *Papers in Regional Science*, 13, 147–163.
- Berroir, S., Grasland, C., Guérin-Pace, F., & Hamez, G. (2005). La diffusion spatiale des pièces euro en Belgique et en France. *Revue belge de Géographie (Belgeo)*, 4, 345–358.
- Bretagnolle, A., Daudé, E., & Pumain, D. (2003). From theory to modelling: Urban systems as complex systems. Cybergeog: European Journal of Geography, Dossiers, 13e Colloque Européen de Géographie Théorique et Quantitative, Lucca, Italie, 8–11 Septembre 2003. <http://cybergeog.revues.org/2420>
- Bretagnolle, A., Mathian, H., Pumain, D., & Rozenblat, C., Dossiers, 11th European Colloquium on Quantitative and Theoretical Geography, Durham Castle, UK, September 3–7 1999. Long-term dynamics of European towns and cities: towards a spatial model of urban growth. Cybergeog: Revue européenne de géographie [on line], mis en ligne le 29 Mars 2000. <http://cybergeog.revues.org/566>
- Brockmann, D. (2008). Money circulation science – fractional dynamics in human mobility. In K. Rainer, G. Radons, & I. M. Sokolov (Eds.), *Anomalous transport: Foundations and applications* (pp. 459–484). Weinheim: Wiley-VCH.
- Brockmann, D. (2010). Following the money. *Physics World*, 23, 31–34.
- Brockmann, D., & Hufnagel, L. (2007). The scaling law of human travel – a message from George. In: B. Blasius (Ed.), *Nonlinear modeling in ecology, epidemiology and genetics* (pp. 109–127). Hackensack: World Scientific.
- Brockmann, D., & Theis, F. (2008). Money circulation, trackable items, and the emergence of universal human mobility patterns. *IEEE Pervasive Computing*, 7, 28–35.
- Bursche, A. (2002). Circulation of Roman coinage in northern Europe in late antiquity. *Histoire & Mesure*, XVII(3), 121–141.
- Busnel, Y., Bertier, M., & Kermarrec, A. M. (2008). On the impact of the mobility on convergence speed of population protocols. Research Report, Institute National de Recherche en Informatique et Application (INRIA), No 6580, July 2008.
- Casetti, E. (1969). Why do diffusion processes conform to logistic trends? *Geographical Analysis*, 1(1), 101–105.
- Chakraborti, A., Muni Toke, I., Patriarca, M., & Abergel, F. (2011). Econophysics review: II Agent-based models. *Quantitative Finance*, 11(7), 1013–1041.
- de Meijer, C. R. W. (2010). The single euro cash area. *European Payments Council Newsletter*, 8(10), 1–9.
- Dragulescu, A., & Yakovenko, V. M. (2001). Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States. *Physica A*, 299, 213–221.
- Edwards, M., Huet, S., Goreaud, F., & Deffuant, G. (2003). Comparing an individual-based model of behaviour diffusion with its mean field aggregate approximation. *Journal of Artificial Societies and Social Simulation*, 6(4), <http://jasss.soc.surrey.ac.uk/6/4/9.html>.
- Eurodiffusie. (2011). Volg de verspreiding van de euromunten door Europa! <http://www.eurodiffusie.nl/>. Accessed Feb 2011.
- European, C. (2003). The introduction of euro banknotes and coins – one year after. *Official Journal of the European Union*. [COM(2002) 747 final - Official Journal C 36 of 15.02.2003].
- Eurostat. (2012). Statistic database. <http://epp.eurostat.ec.europa.eu/>. Accessed Sept 2012.
- Floch, J.-M. (2011). Vivre en deca de la frontière, travailler au-delà. *Insee Première*, 1337, 1–4.
- Fotheringham, A., & O’Kelly, M. E. (1989). *Spatial interaction models: Formulations and applications*. Dordrecht: Kluwer.
- Gil-Quijano, J., Louail, T., & Hutzler, G. (2012). From biological to urban cells: Lessons from three multilevel agent-based models. In N. Desai, A. Liu, & M. Winikoff (Eds.), *Principles and practice of multi-agent systems* (pp. 620–635). Berlin/Heidelberg: Springer.
- Gilbert, N., Troitzsch, K. G. (2011). *Simulation for the social scientist*, 2nd ed. Buckingham: Open University Press.

- Grasland, C. (2009). Spatial analysis of social facts. In: F. Bavaud, & C. Mager (Eds.), *Handbook of theoretical and quantitative geography* (pp. 117–174). Lausanne: FGSE.
- Grasland, C., & Guérin-Pace, F., 5–9 septembre 2003. Euroluca: A simulation model of euro coins diffusion. In *Proceedings of the 13th European Colloquium on Theoretical and Quantitative Geography* (pp. 24), Lucca.
- Grasland, C., Guérin-Pace, F., Le Texier, M., & Garnier, B. (2012). Diffusion of foreign euro coins in France, 2002–2012. *Population and Societies*, 488, 1–4.
- Grasland, C., Guérin-Pace, F., & Nuno, J.-C. (2005a). Interaction spatiale et réseaux sociaux. modélisation des déterminants physiques, sociaux et géographiques de la diffusion des pièces euro entre pays de la zone euro. In: 2eme Colloque “Systèmes Complexes en SHS”. Paris.
- Grasland, C., Guérin-Pace, F., Terrier, C., 2005b. La diffusion spatiale, sociale et temporelle des pièces euros étrangères : un problème complexe. Actes des journées de Méthodologie Statistique.
- Grasland, C., Guérin-Pace, F., & Tostain, A. (2002). La circulation des euros, reflet de la mobilité des hommes. *Population et Sociétés*, 384, 4.
- Guermond, Y. (2008). From classic models to incremental models. In Y. Guermond (Ed.), *The modeling process in geography: From determinism to complexity* (GIS series, pp. 15–38). London: ISTE and WILEY.
- Haken, H. (1977). *Synergetics, an introduction*. New York: Springer.
- Hägerstrand, T. (1952). *The propagation of innovation waves. B(4)*. Lund: Royal University of Lund.
- Hägerstrand, T. (1970). What about people in regional science? *Regional Science Association Papers*, 24, 7–21.
- Huff, D. (1964). Defining and estimating a trading area. *Journal of Marketing*, 28, 34–38.
- Kitamura, R. (1984a). A model of daily time allocation to discretionary out-of-home activities and trips. *Transportation Research B*, 18, 255–266.
- Kitamura, R. (1984b). Incorporating trip chaining into analysis of destination choice. *Transportation Research B*, 18, 67–81.
- Lerman, S. (1979). The use of disaggregate choice models in semi-Markov process models of trip chaining behaviour. *Transportation Science*, 13, 273–291.
- Moisil, D. (2002). The Danube limes and the barbaricum (248–498 a.d.). A study in coin circulation. *Histoire & Mesure*, 17(3), 79–120.
- Nijkamp, P., & Reggiani, A. (1987). Spatial interaction and discrete choice: Staties and dynamics. In J. Hauer, H. Timmermans, & N. Wrigley (Eds.), *Contemporary developments in quantitative geography*. Dordrecht: Reidel.
- Nuno, J., Grasland, C., Blasco, F., Guérin-Pace, F., & Olarra, J. (2005). How many coins are you carrying in your pocket? *Physica A*, 354, 432–436.
- Oberländer-Törnoveanu, E. (2002). La monnaie byzantine des vi-viiième siècles au-delà de la frontière du bas-danube. entre politique, économie et diffusion culturelle. *Histoire & Mesure*, 17(3), 155–196.
- Orcutt, G. (1957). A new type of socio-economic system. *Review of Economics and Statistics*, 58, 773–797.
- O’Sullivan, D. (2004). Complexity science and human geography. *Transactions of the Institute of British Geographers*, 29(3), 282–295.
- Portugali, J. (2000). *Self-organization and the city*. Berlin: Springer.
- Prigogine, I., & Stengers, I. (1979). *La nouvelle alliance*. Paris: Gallimard.
- Pumain, D. (1998). Les modèles d’auto-organisation et le changement urbain. *Cahiers de Géographie de Québec*, 42(117), 349–366.
- Pumain, D. (2003). Une approche de la complexité en géographie. *Géocarrefour*, 78(1), 25–31.
- Pumain, D., Paulus, F., & Vacchiani-Marcuzzo, C. (2009). Innovation cycles and urban dynamics. In D. Lane, D. Pumain, S. Van der Leeuw, & G. West (Eds.), *Complexity perspectives on innovation and social change* (Methodos series, Vol. 7, pp. 237–260). Berlin: Springer.
- Pumain, D., Sanders, L., & Saint-Julien, T. (1989). *Villes et auto-organisation*. Paris: Economica.

- Quijano, J. G., Piron, M., & Drogoul, A. (2007). Vers une simulation multi-agent de groupes d'individus pour modéliser les mobilités résidentielles intra-urbaines. *Revue internationale de Géomatique/European Journal of GIS and Spatial Analysis*, 17(2), 161–181.
- Rappaport, J. (2004). A simple model of city crowdedness. Research Working Paper RWP 04-12, Federal Reserve Bank of Kansas City.
- Ravenstein, E. (1889). The laws of migration. *Journal of the Royal Statistical Society of London*, 52(2), 241–305.
- Reilly, W. J. (1929). *Methods for the study of retail relationships* (No. 2944). Austin: University of Texas Bulletin.
- Reilly, W. (1931). *The law of retail gravitation*. New York: Knickerbocker Press.
- Roy, J. R. (2004). *Spatial interaction modelling: A regional science context*. Berlin/New-York: Springer.
- Roy, J. R., & Less, P. (1981). On appropriate microstate descriptions in entropy modelling. *Transportation Research B*, 15, 85–96.
- Sanders, L. (2007). Objets géographiques et simulation agent, entre thématique et méthodologie. *Revue internationale de Géomatique/European Journal of GIS and Spatial Analysis*, 17(2), 135–160.
- Seitz, F., Stoyan, D., & Tödter, K.-H. (2009). *Coin migration within the euro area* (Discussion Paper Series 1: Economic Studies 27). Frankfurt: Deutsche Bundesbank.
- Seitz, F., Stoyan, D., & Tödter, K.-H. (2012). Coin migration and seigniorage within the euro area. *Jahrbücher f. Nationalökonomie u. Statistik*, 232(1), 84–92. Deutsche Bundesbank.
- Shnerb, N. M., Louzoun, Y., Bettelheim, E., & Solomon, S. (2000). The importance of being discrete: Life always wins on the surface. In *PNAS (Proceedings of the National Academy of Sciences of the United States of America)* (Vol. 97).
- Solé, R. V., Gamarra, J. G. P., Ginovart, M., & Lopez, D. (1999). Controlling chaos in ecology: From deterministic to individual-based models. *Bulletin of Mathematical Biology*, 61, 1187–1207.
- Stewart, J. (1947). Empirical mathematical rules concerning the distribution and equilibrium of population. *Geographical Review*, 37, 461–486.
- Stoyan, D. (2002). Statistical analyses of euro coin mixing. *Mathematical Spectrum*, 35, 50–55.
- Stoyan, D., Stoyan, H., & Dödge, G. (2004). Statistical analyses and modelling of the mixing process of euros coins in Germany and Europe. *Australian & New Zealand Journal of Statistics*, 46, 67–77.
- Tobler, W. (1970). A computer movie simulating urban growth in the Detroit region. *Economic Geography*, 46, 234–240.
- Tobler, W. R. (1981). A model of geographical movement. *Geographical Analysis*, 13(1), 1–20.
- Tsotselia, M. (2002). Recent Sasanian coin findings on the territory of Georgia. *Histoire & Mesure*, 17(3), 143–153.
- Ujes, D. (2002). Coins of the Macedonian kingdom in the interior of Balkans. Their inflow and use in the territory of the Scordisci. *Histoire & Mesure*, 17(3), 7–41.
- van Blokland, P., Boot, L., Hiremath, K., Hochstenbach, M., Koole, G., Pop, S., Quant, M., & Wirosoetisno, D. (2002). The euro diffusion project. *Proceedings of the 42nd European Study Group with Industry*.
- Wegener, M. (2000). Spatial models and GIS. In A. S. Fotheringham & M. Wegener (Eds.), *Spatial models and GIS: New potential and new models* (pp. 3–20). London: Taylor and Francis.
- Weidlich, W. (2003). Socio-dynamics – a systematic approach to mathematical modelling in the social sciences. *Chaos, Solitons and Fractals*, 18, 431–437.
- Weidlich, W., & Haag, G. (1988). The migratory equations of motion. In W. Weidlich & G. Haag (Eds.), *Interregional migration* (pp. 21–32). Berlin: Springer.
- White, R., Engelen, G., & Uljee, I. (1997). The use of constrained cellular automata for high-resolution modelling of urban land use dynamics. *Environment and Planning B*, 24, 323–343.
- White, R. W. (1977). Dynamical central place theory. *Geographical Analysis*, 9, 226–243.
- White, R. W. (1978). The simulation of central place dynamics: Two sector systems and the rank size rule. *Geographical Analysis*, 10, 201–208.

- Wilson, A. (1967). A statistical theory of spatial distribution models. *Transportation Research, 1*, 253–269.
- Wilson, A. (1970). *Entropy in urban and regional modelling*. London: Pion.
- Wilson, A. (1981). *Catastrophe theory and bifurcation: Application to urban and regional system*. London: Croom Helm.
- Wilson, A. G. (2002). Complex spatial systems: Challenges for modellers. *Mathematical and Computer Modelling, 36*(3), 379–387.