An Algebraic Analysis for Binary Intuitionistic L-Fuzzy Relations

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Abstract From the point of view of algebraic logic, this paper presents an algebraic analysis for binary intuitionistic lattice-valued fuzzy relations based on lattice implication algebras, which is a kind of lattice-valued propositional logical algebras. By defining suitable operations, we prove that the set of all binary intuitionistic lattice-valued fuzzy relations is a lattice-valued relation algebra, and some important properties are also obtained. This research shows that the algebraic description is advantageous to studying of structure of intuitionistic fuzzy relations.

Keywords Intuitionistic L-fuzzy relation · L-fuzzy relation · Lattice-valued relation algebra · Lattice implication algebra

1 Introduction

Fuzzy relations, introduced by Zadeh [24] in 1965, permit the gradual assessment of relativity between objects, and this is described with the aid of a membership function valued in the real unit interval [0,1]. Later on, Goguen [9] generalized this concept in 1967 to lattice-valued fuzzy relation (L-fuzzy relation for short) for an arbitrary complete Brouwerian lattice L instead of the unit interval [0,1]. Intuitionistic fuzzy relation, defined by Atanassov in 1984 [1–3], is another kind of generalization for Zadeh's fuzzy relation, give us the possibility to model hesitation and uncertainty by using an additional degree. An intuitionistic L-fuzzy relation (ILFR for short) R between two universes U and V is defined as an intuitionistic L-fuzzy set (ILFS) in $U \times V$, assigns to each element $(x, y) \in U \times V$

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a membership degree $\mu_R(x,y)(\in L)$ and a non-membership degree $\nu_R(x,y)(\in L)$ such that $\mu_R(x,y) \leqslant N(\nu_R(x,y))$, where $N : L \to L$ is an involutive order-reversing operation on the lattice $(L, \leqslant)$. When $L = [0,1]$, the object R is an intuition fuzzy relation (IFR) and the following condition holds: $(\forall(x,y) \in U \times V)(0 \leqslant \mu_R(x,y) + \nu_R(x,y) \leqslant 1)$. For all $(x,y) \in U \times V$, the number $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$ is called the hesitation degree or the intuitionistic index of (x,y) to R. As we know, the structures and properties of operations on intuitionistic fuzzy relations have always been important topics in ILFS community. For that, P. Burillo and H. Bustince discussed the properties of several kinds of operations on ILFRs with different t-norms and t-conorms and characterized the structures of ILFRs in [4, 5]. In [6], G. Deschrijver and E. E. Kerre probed into the triangular compositions of ILFRs. In addition, the applications of ILFRs have also been developed rapidly in recent years, see [11, 13, 23].

As a fundamental conceptual and methodological tool in computer science just like logic, since the mid-1970s, relation algebras have been used intensively in applications of mathematics and computer science, and it also provides an apparatus for an algebraic analysis of ordinary predicate calculus, see e.g., [10, 18]. Following the idea of classical relation algebras, fuzzy relation algebras have been also considered by several researchers in [7, 8, 12]. Taking L to be a complete Heyting algebra, Furusawa developed the fuzzy relational calculus [7], and proved the representation theorems for algebraic formalizations of fuzzy relations in 1998. Furusawa's fuzzy relations algebras are equipped with sup-min composition and a semiscalar multiplication. In [8], Furusawa continued to study Dedekind categories with a cutoff operator, in which the Dedekind formula holds. In [16, 17], Popescu established the notion of MV-relation algebra based on MV-algebras, investigated their basic properties, and given a characterization of the "natural" MV-relation algebras. In order to provide a kind of algebraic model for lattice-valued first-order logic, based on complete lattice implication algebras [22], Pan introduced the notion of lattice-valued fuzzy relation algebra [14]; its basic properties and cylindric filters have also been established. For other categorical description of fuzzy relations, we refer readers to [19–21].

Along the line of the algebraic formalization of fuzzy relations, the present paper aims at investigating the arithmetical properties of mathematical structures formed by binary ILFRs based on the theory of L-fuzzy relation algebras (LRA). The paper is organized as follows. We recall some fundamental notions and properties of lattice implication algebras, Intuitionistic L-fuzzy relations and LRA in Sect. 2.2. Section 2.3 devotes to arithmetical properties of ILFRs. We conclude the paper in Sect. 2.4 with some radical suggestions for further problems.

2 Preliminaries

In this section, we review some basic definitions about intuitionistic L-fuzzy relations, lattice implication algebras, and lattice-valued relation algebras for the purpose of reference and also recall some basic results which will be frequently used in the following, and we will not cite them every time they are used.

Definition 1 [22] A bounded lattice $(L, \vee, \wedge, O, I)$ with order-reversing involution $'$ and a binary operation $\to$ is called a lattice implication algebra if it satisfies the following axioms:
(I_1) $x \to (y \to z) = y \to (x \to z)$,
(I_2) $x \to x = I$,
(I_3) $x \to y = y' \to x'$,
(I_4) $x \to y = y \to x = I \Rightarrow x = y$,
(I_5) $(x \to y) \to y = (y \to x) \to x$,
(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,
for all $x, y, z \in L$.

Example 1 [22] Let $L = [0, 1]$. If for any $a, b \in L$, put

$$a \vee b = \max\{a, b\}, a \wedge b = \min\{a, b\}, a' = 1 - a,$$

$$a \otimes b = \max\{0, a + b - 1\}, a \to b = \min\{1, 1 - a + b\}.$$

Then, $(L, \vee, \wedge, \otimes, \to, ')$ is called the Łukasiewicz algebra. Here, $\to$ is called the Łukasiewicz implication, $\otimes$ is called the Łukasiewicz product, and another operation $\oplus$, $a \oplus b = \min\{1, a + b\}$, is called the Łukasiewicz sum. It is easy to show that $(L, \vee, \wedge, \to, ', 0, 1)$ is also a lattice implication algebra.

Theorem 1 [22] Let $(L, \vee, \wedge, ', \to, 0, 1)$ *be a lattice implication algebra, then* $(L, \vee, \wedge)$ *is a distributive lattice.*

In what follows, we list some well-known properties of lattice implication algebras that will often be used without mention. In a lattice implication algebra L, we define binary operation $\otimes$ *and* $\oplus$ *as follows: for any* $x, y \in L$, $a \otimes b = (a \to b')'$; $a \oplus b = a' \to b$.

Proposition 1 *Let L be a lattice implication algebra, then for any* $x, y, z \in L$, *we have the following:*

(1) $x \to y = I$ *if and only if* $x \leqslant y$;
(2) $x \to y \geqslant x' \vee y$;
(3) $(x \otimes y)' = x' \oplus y', (x \oplus y)' = x' \otimes y'$;
(4) $O \otimes x = O, I \otimes x = x, x \otimes x' = O, O \oplus x = x, I \oplus x = I, x \oplus x' = I$;
(5) $x \otimes y = (x \vee y) \otimes (x \wedge y), x \oplus y = (x \vee y) \oplus (x \wedge y)$;
(6) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z), x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$.

In [14], by generalizing the classical (Boolean) relation algebras, we introduce the notion of lattice-valued relation algebra based on complete lattice implication algebras L.

Definition 2 A lattice-valued relation algebra (**LRA** for short) is an algebra $\mathfrak{L} = (L, \vee, \wedge, ', \rightarrow, O, I, ;, \smile, \triangle)$, where

(1) $(L, \vee, \wedge, ', \rightarrow, O, I)$ is a lattice implication algebra;
(2) $(L, ;, \triangle)$ is a monoid (a semigroup with identity, $\triangle$);
(3) $(x \vee y); z = (x; z) \vee (y; z)$;
(4) $(x \rightarrow y)^{\smile} = x^{\smile} \rightarrow y^{\smile}$;
(5) $x^{\smile\smile} = x$;
(6) $\triangle' \vee \triangle = I$;
(7) $(x; y)^{\smile} = y^{\smile}; x^{\smile}$;
(8) $y' = y' \vee (x^{\smile}; (x; y)')$;
(9) $(a \otimes x); (b \otimes y) \leqslant (a; b) \otimes (x; y)$.

From the above definition, an **LRA** is an expansion of the corresponding lattice implication algebra with the operations $\smile, ;, \triangle$, where $\smile$ is called the converse operation and ; the relative multiplication operation and $\triangle$ the diagonal element. The class of **LRA**s will be denoted by $\mathfrak{LRA}$.

Let $(L, \vee, \wedge, ', \rightarrow, 0, 1)$ be a complete lattice implication algebra. An intuitionistic L-fuzzy relation R from a universe U to a universe V is an intuitionistic fuzzy set in $U \times V$, i.e. an object having the form $R = \{((x, y), \mu_R(x, y), v_R(x, y))|x \in U, y \in V\}$, where $\mu_R : U \times V \rightarrow L$ and $\mu_R : U \times V \rightarrow L$ satisfy the condition $(\forall (x, y) \in U \times V)(\mu_R(x, y) \leqslant (v_R(x, y))')$. The set of all intuitionistic L-fuzzy relations from U to V will be denoted by $\mathfrak{ILFR}(U \times V)$. We say the intuitionistic L-fuzzy relation R is contained in the intuitionistic L-fuzzy relation S, written $R \subseteq S$ if $\mu_R(x, y) \leqslant \mu_S(x, y)$ and $v_R(x, y) \geqslant v_S(x, y)$ for all $(x, y) \in U \times V$. The zero relation $O_{U \times V}$ and the full relation $I_{U \times V}$ are intuitionistic L-fuzzy relations with $\mu_{O_{U \times V}}(x, y) = O$, $v_{O_{U \times V}}(x, y) = I$ and $\mu_{I_{U \times V}}(x, y) = I$, $v_{I_{U \times V}}(x, y) = O$ for all $(x, y) \in U \times V$, respectively. It is trivial that $\subseteq$ is a partial order on $\mathfrak{ILFR}(U \times V)$ and $O_{U \times V} \subseteq R \subseteq I_{U \times V}$ for all intuitionistic L-fuzzy relations R. The union, intersection complement, and converse of intuitionistic L-fuzzy relations are defined as follows:

$$\begin{aligned} R \cup S &= \{((x, y), \mu_R(x, y) \vee \mu_S(x, y), v_R(x, y) \wedge v_S(x, y))|(x, y) \in U \times V\}; \\ R \cap S &= \{((x, y), \mu_R(x, y) \wedge \mu_S(x, y), v_R(x, y) \vee v_S(x, y))|(x, y) \in U \times V\}; \\ R' &= \{((x, y), v_R(x, y), \mu_R(x, y))|(x, y) \in U \times V\}; \\ R^{\smile} &= \{((x, y), \mu_R(y, x), v_R(y, x))|(x, y) \in U \times V\}. \end{aligned}$$

It is easy to show that $(\mathfrak{ILFR}(U \times V), \cup, \cap, O_{U \times V}, I_{U \times V})$ is a complete lattice. Let $R \in \mathfrak{ILFR}(U \times V)$ and $S \in \mathfrak{ILFR}(V \times W)$, the relative multiplication (or composition) $R; S$ is defined as follows:

$$R; S = \Big((x, z), \bigvee_{y \in V} \big(\mu_R(x, y) \otimes \mu_S(y, z)\big), \bigwedge_{y \in V} \big(v_R(x, y) \oplus v_S(y, z)\big)|(x, z) \in U \times W\Big),$$

then $\bigvee_{y \in V}(\mu_R(x,y) \otimes \mu_S(y,z)) \leqslant (\bigwedge_{y \in V}(\nu_R(x,y) \oplus \nu_S(y,z))'$ for any $(x,z) \in U \times W$. Let $R, S \in \mathfrak{ILFR}(U \times V)$, $R \rightarrow S$ is also an intuitionistic L-fuzzy relation on $U \times V$, where $\mu_{R \rightarrow S}(x,y)$ means the truth degree of the sentence, *if xRy, then xSy*; and $\nu_{R \rightarrow S}(x,y)$ means the truth degree of the sentence, *if xRy does not hold, then xSy does not hold*.

Throughout this paper, unless otherwise stated, L always denotes a complete lattice implication algebra. For more details of lattice implication algebras, we refer readers to [15, 22].

3 The Arithmetical Properties of $\mathfrak{ILFR}(U \times U)$

In this section, we study the arithmetical properties of intuitionistic L-fuzzy relations on the universe U.

Theorem 2 *Let U be any non-empty set. The union, intersection, complement, converse, and composition of intuitionistic L-fuzzy relations in U are defined as that in Sect. 2.2. We also define the operations $\rightarrow, \oplus, \otimes$ on $\mathfrak{ILFR}(U \times U)$, for any $R, S \in \mathfrak{ILFR}(U \times U)$,*

$$R \rightarrow S = ((x,y), \nu_R(x,y) \oplus \mu_S(x,y), \mu_R(x,y) \otimes \nu_S(x,y) | (x,y) \in U \times U),$$
$$R \oplus S = ((x,y), \mu_R(x,y) \oplus \mu_S(x,y), \nu_R(x,y) \otimes \nu_S(x,y) | (x,y) \in U \times U),$$
$$R \otimes S = ((x,y), \mu_R(x,y) \otimes \mu_S(x,y), \nu_R(x,y) \oplus \nu_S(x,y) | (x,y) \in U \times U),$$

and define the identity relation $\triangle_U$ is an intuitionistic L-fuzzy relation on U such that for any $(x,y) \in U \times U$,

$$\mu_{\triangle_U}(x,y) = \begin{cases} I & \text{if } x = y, \\ O, & \text{otherwise.} \end{cases} \quad \text{and} \quad \nu_{\triangle_U}(x,y) = \begin{cases} O & \text{if } x = y, \\ I, & \text{otherwise.} \end{cases}$$

Then, $(\mathfrak{ILFR}(U \times U), \cup, \cap, ', \rightarrow, O_U, I_U, ;, \smile, \triangle_U)$ is an ***LRA***.

Proof The zero relation and the full relation on U are denoted by O_U and I_U, respectively. As we have discussed above, $(\mathfrak{ILFR}(U \times U), \cup, \cap, O_U, I_U)$ is a bounded lattice. For any $R, S, T \in \mathfrak{ILFR}(U \times U)$, since

$$\begin{aligned} \mu_{R \rightarrow (S \rightarrow T)}(x,y) &= \nu_R(x,y) \oplus \mu_{S \rightarrow T}(x,y) = \nu_R(x,y) \oplus (\nu_S(x,y) \oplus \mu_T(x,y)) \\ &= \nu_S(x,y) \oplus (\nu_R(x,y) \oplus \mu_T(x,y)) = \nu_S(x,y) \oplus \mu_{R \rightarrow T}(x,y) \\ &= \mu_{S \rightarrow (R \rightarrow T)}(x,y), \end{aligned}$$

and $\nu_{R \rightarrow (S \rightarrow T)}(x,y) = \nu_{S \rightarrow (R \rightarrow T)}(x,y)$ can also be proved similarly, I_1 holds. In this way, we can validate $I_2, \ldots, I_5$ and L_1, L_2. Hence, $(\mathfrak{ILFR}(U \times U), \cup, \cap, ', \rightarrow, O_U, I_U)$ is a lattice implication algebra. The associativity $(R;S);T = R;(S;T)$ and the unitary law $R;\triangle_U = \triangle_U;R = R$ and the zero law $R;O_U = O_U;R = O_U$ are obvious. So far, we have proved (1), (2) in definition 2. (4), (5), and (6) are obvious. The rest is to prove (3), (7), (8), and (9). Note that L is an infinite distributive lattice, hence,

$$\begin{aligned} v_{(R\cup S);T}(x,z) &= \bigwedge_{y\in U}\big((v_R(x,y)\wedge v_S(x,y))\oplus v_T(y,z)\big)\\ &= \bigwedge_{y\in U}\big((v_R(x,y)\oplus v_T(y,z))\wedge(v_S(x,y)\oplus v_T(y,z))\big)\\ &= \bigwedge_{y\in U}\big((v_R(x,y)\oplus v_T(y,z))\big)\wedge\bigwedge_{y\in U}\big((v_S(x,y)\oplus v_T(y,z))\big)\\ &= v_{R;T}(x,z)\wedge v_{S;T}(x,z) = v_{(R;T)\cup(S;T)}(x,z). \end{aligned}$$

$\mu_{(R\cup S);T}(x,z) = \mu_{(R;T)\cup(S;T)}(x,z)$ is similar. Thus, (3) holds. In this way, we can obtain (7). In order to prove (8), we only need to prove $R^{\smile};(R;S)'\leqslant S'$, or $\big(R^{\smile};(R;S)'\big)'\geqslant S$. As

$$\begin{aligned} \mu_{(R^{\smile};(R;S)')'}(x,y) &= \bigwedge_{z\in U}\Big((\mu_{R^{\smile}}(x,z))'\oplus\bigvee_{w\in U}\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\\ &= \bigwedge_{z\in U}\Big((\mu_R(z,x))'\oplus\bigvee_{w\in U}\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\\ &= \bigwedge_{z\in U}\bigvee_{w\in U}\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big); \end{aligned}$$

and for any $z\in U$,

$$\begin{aligned} &\bigvee_{w\in U}\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\\ &= \bigvee_{w\in U,w\neq x}\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\bigvee\\ &\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,x)\otimes\mu_S(x,y)\big)\Big)\\ &= \bigvee_{w\in U,w\neq x}\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\bigvee\\ &\big(\mu_R(z,x)\to(\mu_R(z,x)\to(\mu_S(x,y))')'\big)\\ &= \bigvee_{w\in U,w\neq x}\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\bigvee\\ &\big((\mu_S(x,y)\to(\mu_R(z,x))')\to(\mu_R(z,x))'\big)\\ &= \bigvee_{w\in U,w\neq x}\Big((\mu_R(z,x))'\oplus\big(\mu_R(z,w)\otimes\mu_S(w,y)\big)\Big)\bigvee\big(\mu_S(x,y)\vee(\mu_R(z,x))'\big)\\ &\geqslant \mu_S(x,y). \end{aligned}$$

Similarly, we can prove $v_{(R^{\smile};(R;S)')'}(x,y)\leqslant v_S(x,y)$. Thus, (8) holds. For (9), we only validate the part of the degree of membership, the other part is similar.

$$
\begin{aligned}
\mu_{(S\otimes R);(T\otimes U)}(x,y) &= \bigvee_{z\in U}\Big(\mu_{S\otimes R}(x,z)\otimes\mu_{T\otimes U}(z,y)\Big)\\
&= \bigvee_{z\in U}\Big(\mu_S(x,z)\otimes\mu_R(x,z)\otimes\mu_T(z,y)\otimes\mu_U(z,y)\Big)\\
&= \bigvee_{z\in U}\Big(\mu_S(x,z)\otimes\mu_T(z,y)\otimes\mu_R(x,z)\otimes\mu_U(z,y)\Big)\\
&\leqslant \bigvee_{z\in U}\big(\mu_S(x,z)\otimes\mu_T(z,y)\big)\otimes\bigvee_{z\in U}\big(\mu_R(x,z)\otimes\mu_U(z,y)\big)\\
&= \mu_{(S;T)\otimes(R;U)}(x,y).
\end{aligned}
$$

Thus, (9) holds. To sum up, $(\mathfrak{ILFR}(U\times U),\cup,\cap,{}',\rightarrow,O_U,I_U,;,{}^{\smile},\triangle_U)$ is an **LRA**.

Corollary 1 *Let U be any non-empty set. For any $R,S\in\mathfrak{ILFR}(U\times U)$, the following identities are true:*

(1) $I_U^{\smile}=I_U,\triangle_U^{\smile}=\triangle_U,O_U^{\smile}=O_U$;
(2) $(R')^{\smile}=(R^{\smile})'$;
(3) $(R\vee S)^{\smile}=R^{\smile}\vee S^{\smile}$, $(R\wedge S)^{\smile}=R^{\smile}\wedge S^{\smile}$;
(4) $(R\oplus S)^{\smile}=R^{\smile}\oplus S^{\smile}$, $(R\otimes S)^{\smile}=R^{\smile}\otimes S^{\smile}$.

According to Definition 2, the following conclusions can be obtained. Here, we only prove (5); for more details, please refer to [14].

Proposition 2 *Let U be any non-empty set. For any $R,S,T\in\mathfrak{ILFR}(U\times U)$, the following conclusions are true:*

(1) $R;(S\vee T)=(R;S)\vee(R;T)$;
(2) $S'=S'\vee((S;R^{\smile})';R)$;
(3) $R;S\leqslant T'\Leftrightarrow R^{\smile};T\leqslant S'\Leftrightarrow T;S^{\smile}\leqslant R'$;
(4) $R\leqslant I_U;R;I_U$, $I_U=I_U;I_U$;
(5) $R;(S\oplus T)\leqslant(R;S)\oplus(I_U;T)$, $(R\oplus S);T\leqslant(R;T)\oplus(S;I_U)$;
(6) $I_U;(R\rightarrow S);I_U\leqslant(I_U;R';I_U)'\rightarrow(I_U;S;I_U)$;
(7) $(I_U;R';I_U)'\leqslant I_U;R;I_U$;
(8) $R^{\smile}\leqslant S^{\smile}$ if and only if $R\leqslant S$.

Proof (5) We only prove the first inequality, the second can be proved similarly. Since

$$
\begin{aligned}
\big(R^{\smile};(R;S)'\big)\otimes\big(I_U;(I_U;T)'\big)\geqslant(R^{\smile}\otimes I_U);\big((R;S)'\otimes(I_U;T)'\big)\\
=R^{\smile};\big((R;S)'\otimes(I_U;T)'\big),
\end{aligned}
$$

and note that $R^{\smile};(R;S)'\leqslant S'$ and $I_U;(I_U;T)'=I_U^{\smile};(I_U;T)'\leqslant T'$, thus

$$R^{\smile};\big((R;S)'\otimes(I_U;T)'\big)\leqslant S'\otimes T'=(S\oplus T)',$$

this is equivalent to $R^{\smile}; (S \oplus T) \leqslant \big((R;S)' \otimes (I_U;T)'\big)' = (R;S) \oplus (I_U;T)$.

Corollary 2 *Let U be any non-empty set. For any $R, S \in \mathfrak{ILFR}(U \times U)$, then,*

(1) *$R = R^{\smile}$, or R and $R^{\smile}$ are incomparable, in notation, $R \| R^{\smile}$;*
(2) *if $S \leqslant \triangle_U$, then $R \geqslant R;S$ and $R \geqslant S;R$; if $S \geqslant \triangle_U$, then $R \leqslant R;S$ and $R \leqslant S;R$.*

In the following, for any $R, S \in \mathfrak{ILFR}(U \times U)$, we let $\Box_U = \triangle'_U$, $R \dagger S = (R';S')'$, where $\Box_U$ is called the diversity element in $\mathfrak{ILFR}(U \times U)$, $\dagger$ is called relative addition. By Theorem 2, Corollary 1, and Proposition 2, the following conclusions can be obtained, the proofs are omitted.

Proposition 3 *Let U be any non-empty set. For any $R, S, T \in \mathfrak{ILFR}(U \times U)$, then,*

(1) $R\dagger(S\dagger T) = (R\dagger S)\dagger T$;
(2) $(R\dagger S)^{\smile} = S^{\smile}\dagger R^{\smile}$;
(3) $R^{\smile}; (R'\dagger S) \leqslant S$;
(4) $\Box_U^{\smile} = \Box_U$;
(5) $R\dagger(S \wedge T) = (R\dagger S) \wedge (R\dagger T)$, $(R \wedge S)\dagger T = (R\dagger T) \wedge (S\dagger T)$;
(6) $R; (S\dagger T) \leqslant (R;S)\dagger T$;
(7) $(R\dagger S); T \leqslant R\dagger(S;T)$;
(8) $\triangle_U \leqslant R\dagger(R')^{\smile}$, $R;(R')^{\smile} \leqslant \Box_U$;
(9) $\triangle_U \leqslant R\dagger S$ *if and only if* $\triangle_U \leqslant S\dagger R$ *if and only if* $\triangle_U \leqslant R^{\smile}\dagger S^{\smile}$ *if and only if* $\triangle_U \leqslant S^{\smile}\dagger R^{\smile}$;
(10) $R\dagger\Box_U = \Box_U\dagger R = R$, $R\dagger I_U = I_U\dagger R = I_U$;

Theorem 3 *Let U be any non-empty set and $R, S, P \in \mathfrak{ILFR}(U \times U)$. Then, $(S^{\smile};R')' = \bigvee_{S;T \leqslant R} T$. In particular, $P^{\smile} = \bigvee_{P';T \leqslant \Box_U} T$.*

Proof For the first equality, on the one hand, by Definition 2, we have $S;(S^{\smile};R')' \leqslant R$; on the other hand, if $S;T \leqslant R$, by Proposition 2, this is equivalent to $S^{\smile};R' \leqslant T'$, that is $T \leqslant (S^{\smile};R')'$. Hence, $(S^{\smile};R')' = \bigvee_{S;T \leqslant R} T$. Let $S = P', R = \Box_U$, we can obtain the second equality.

In what follows, we prove the main properties of the ILFRs using the calculus of the algebraic theory. Let $R \in \mathfrak{ILFR}(U \times U)$. We say that R is reflexive if $R \geqslant \triangle_U$, antireflexive if $R \leqslant \Box_U$, symmetrical if $R = R^{\smile}$, and transitive if $R \geqslant R;R$.

Theorem 4 *Let U be any non-empty set and $R, S \in \mathfrak{ILFR}(U \times U)$. Then,*

(1) R is reflexive if and only if R' is antireflexive;
(2) *R is symmetrical if and only if R' is symmetrical;*
(3) *R is transitive if and only if $R^{\smile}$ is transitive.*

Proof

(1) R is reflexive if and only if $R \geqslant \triangle_U$ if and only if $R' \leqslant (\triangle_U) = \square_U$ if and only if R' is antireflexive.
(2) By Corollary III.1 (2), we have $(R')^{\smile} = (R^{\smile})'$, so R is symmetrical if and only if $R = R^{\smile}$ if and only if $R' = (R^{\smile})' = (R')^{\smile}$ if and only if R' is symmetrical.
(3) If R is transitive, then $R \geqslant R; R$, by Proposition III.1(8), we have $R^{\smile} \geqslant (R; R)^{\smile} = R^{\smile}; R^{\smile}$; hence, $R^{\smile}$ is transitive, and vice versa.

Corollary 3 *Let U be any non-empty set and* $R \in \mathfrak{ILFR}(U \times U)$. *The following conclusions hold:*

(1) *if* R *is reflexive, then so are* $R; R$ *and* $R^{\smile}$;
(2) *if* R *is symmetrical, then so are* $R; R$ *and* $R^{\smile}$;
(3) *if* R *is transitive, then so are* $R; R$ *and* $R^{\smile}$;
(4) *if* R *is antireflexive, then* $R \geqslant R \dagger R$.

4 Conclusion

In this paper, we present an algebraic analysis for binary intuitionistic lattice-valued fuzzy relations based on lattice implication algebras. For the theory and application of intuitionistic L-fuzzy relations, the algebraic description shows its advantages. More importantly, by the algebraization of the set of intuitionistic L-fuzzy relations, we can obtain a denotational semantics of intuitionistic L-fuzzy theory and hence a mathematical theory to reason about notions like correctness. Consequently, one may prove such properties using the calculus of the algebraic theory, the results, and methods of applications of fuzzy theory may be described by simple terms in this language.

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