# Chapter 19
# Stability Analysis and Limit Cycles of High Order Sigma-Delta Modulators

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**Abstract.** In this chapter we present an unified approach for study the stability and validation of potential limit cycles of one bit high order Sigma-Delta modulators. The approach is general because it uses the general form of a Sigma-Delta modulator. It is based on a parallel decomposition of the modulator and a direct nonlinear systems analysis. In this representation, the general N-th order modulator is transformed into a decomposition of low order, generally complex modulators, which interact only through the quantizer function. The developed conditions for stability and for validation of potential limit cycles are very easy for implementation and this procedure is very fast.

## 19.1 Introduction

Sigma-Delta modulation has become in recent years an increasingly popular choice for robust and inexpensive analog-to-digital and digital-to-analog conversion [1, 2]. Despite the widespread use of Sigma-Delta modulators theoretical understanding of Sigma-Delta concept is still very limited. This is a consequence of the fact that these systems are nonlinear, due to the presence of a discontinuous nonlinearity - the quantizer. Since the pioneering work of Gray and his co-workers beginning with [3], a number of researchers have contributed to the development of a theory of Sigma-Delta modulation based on the principles of nonlinear dynamics [4, 5, 6]. That work and references there, has succeeded in explaining many fundamentally nonlinear features of this system. The stability of high order interpolative Sigma-Delta ($\Sigma\Delta$) modulators based on nonlinear dynamics has been considered in a couple of papers [7, 8]. The authors present a technique, which in many cases simplifies the analysis. The technique involves a transformation of the state equations of a modulator into a form in which the individual state variables are essentially decoupled and interact

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only within the quantizer function. In [9, 10, 11] and [12] a stability (in the sense of boundness of the states) analysis approach based on decomposition of the general N-th order modulator is presented. This decomposition is considered for all cases of poles of the transfer function of the modulator loop filter. Using this presentation the modulator could be considered as made up of N first order modulators, which interact only through the quantizer function. Based on this decomposition the stability conditions of high order modulators are extracted. They are determined by the stability conditions of each of the first order modulators but shifted with respect to the origin of the quantizer function. Limit cycles are well known phenomena that often appear in practical ΣΔ modulators. For data processing applications it is very important to predict and describe possible limit cycles. Main results concerning the limit cycles for low order Sigma-Delta modulators are presented in [6, 13, 14] and [15]. In [16, 17] authors use state space approach and present a mathematical framework for the description of limit cycles in 1-bit Sigma-Delta modulators for constant inputs. In [18] and [19] an approach for validation of potential limit cycles for high order modulators with constant input signals is presented. The approach is based on the same decomposition of the general N-th order modulator presented in [7, 9, 10, 11] and [12]. The conditions for the existence of limit cycles given in [18] and [19] are easily to be checked and they are basis of a searching procedure for possible limit cycles. In this contribution we do extend both techniques and present unified approach for study the stability and limit cycles of high order sigma-delta modulators. The study is organized as follows. In the next section we describe the parallel decomposition technique for different cases of poles of the loop filter transfer function. Then we present the stability analysis study for first and high order modulators together with an example. In Section 19.5 we present the limit cycle analysis and also give several examples to show the applicability of the presented techniques. The concluding remarks are given in the last section.

## 19.2 Parallel Decomposition of a Sigma Delta Modulator

The structure of a basic ΣΔ modulator is shown in Figure 19.1, and consists of a filter with transfer function $G(z)$ followed by a one-bit quantizer in a feedback loop. The system operates in discrete time.

The input to the loop is a discrete-time sequence $u(n) \in [-1,1]$, which is to appear in quantized form at the output. The discrete-time sequence $x(n)$ is the output

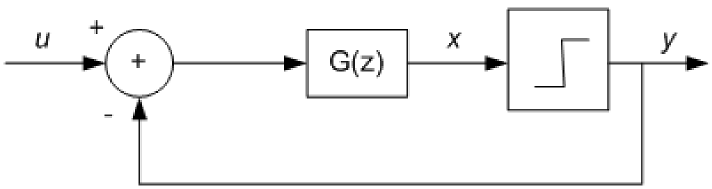


**Fig. 19.1** Basic structure of the sigma-delta modulator

of the filter and the input to the quantizer. Let us consider a N-th order modulator with a loop filter with a transfer function (TF) in the form

$$G(z) = \frac{a_1 z^{-1} + ... + a_N z^{-N}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_N z^{-N}} \tag{19.1}$$

Suppose the transfer function has $N$ real distinct roots of the denominator. Then using partial fraction expansion we get

$$G(z) = \frac{a_1 z^{-1} + ... + a_N z^{-N}}{(1-\lambda_1 z^{-1})...(1-\lambda_N z^{-1})} = \frac{b_1 z^{-1}}{1-\lambda_1 z^{-1}} + ... + \frac{b_N z^{-1}}{1-\lambda_N z^{-1}} \tag{19.2}$$

where the coefficients $b_i, i = 1,2,...,N$ of the fractional components can be found easily using the well known formula $b_i = \frac{(1-\lambda_i z^{-1})}{z^{-1}}\Big|_{z=\lambda_i} G(z)$ .

The corresponding block diagram of the modulator is given in Figure 19.2.

Based on this presentation the state equations of the $\Sigma\Delta$ modulator are

$$\begin{aligned} x_k(n+1) =& \lambda_k x_k(n) + \left[ u(n) - f\left( \sum_{i=1}^{N} b_i x_i(n) \right) \right] = \\ =& \lambda_k x_k(n) + \left[ u(n) - f\left( b_k x_k(n) + \sum_{\substack{i=1 \\ i \neq k}}^{N} b_i x_i(n) \right) \right] \\ k =& 1,2,...,N \end{aligned} \tag{19.3}$$

where $\lambda_1$, $\lambda_2$ ...,$\lambda_N$ are poles (or modes) of the loop filter and the quantizer function $f$ is a sign function. Equation (19.3) also can be rewritten in the form

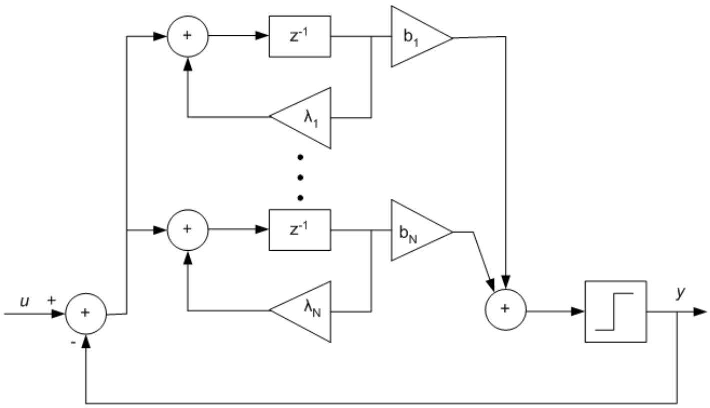


**Fig. 19.2** Block diagram of the modulator using parallel form of the loop filter

$$\begin{aligned} x_k(n+1) =& \lambda_k x_k(n) + \left[u(n) - f\left(\sum_{i=1}^{N} b_i x_i(n)\right)\right] = \\ & \lambda_k x_k(n) + \left[u(n) - f\left(\mathbf{b}^T \mathbf{x}(n)\right)\right] = \\ & \lambda_k x_k(n) + [u(n) - y(n)] \quad k = 1,2,...,N \end{aligned} \tag{19.4}$$

where $\mathbf{b} = (b_1, b_2, ..., b_N)^T$ is the vector of fractional components coefficients and $\mathbf{x} = (x_1, x_2, ..., x_N)^T$ is the state vector. The above presentation indicates that high order modulators could be considered as built up of first order modulators, which interact only through the quantizer function. To simplify the notations, we will drop the indexes and will rewrite equation (19.3) in the following form

$$x_k(n+1) = \lambda x(n) + [u(n) - f(bx(n) + \alpha(n))] = \tag{19.5}$$

where

$$\alpha(n) = \sum_{\substack{i=1 \\ i \neq k}}^{N} b_i x_i(n) \tag{19.6}$$

and

$$y(n) = f\left(\sum_{i=1}^{N} b_i x_i(n)\right) = f\left(\mathbf{b}^T \mathbf{x}(n)\right) = \begin{cases} 1 & \mathbf{b}^T \mathbf{x}(n) \geq 0 \\ -1 & \mathbf{b}^T \mathbf{x}(n) < 0 \end{cases} \tag{19.7}$$

Equation (19.4) describes a first order shifted by $\alpha(n)$ modulator. A detailed analysis of the stability of these modulators will be presented in the next chapter.

In the general case the loop filter transfer function can have complex conjugated roots. Without loss of generality we will consider only one pair of complex conjugated roots. In this case (19.2) becomes

$$\begin{aligned} G(z) =& \frac{b_1 z^{-1}}{(1 - \lambda_1 z^{-1})} + ... + G_2(z) = \\ & \frac{b_1 z^{-1}}{(1 - \lambda_1 z^{-1})} + ... \frac{B_{N-1} z^{-1} + B_N z^{-2}}{1 - d_1 z^{-1} - d_2 z^{-2}} \end{aligned} \tag{19.8}$$

The denominator of the last part of (19.8) has a complex conjugated pair of roots. The main idea is to use a complex form of expansion of the last part of $G(z)$. Therefore (19.8) becomes

$$G(z) = \frac{b_1 z^{-1}}{(1 - \lambda_1 z^{-1})} + ... + \frac{b_{N-1} z^{-1}}{(1 - \lambda_{N-1} z^{-1})} + \frac{b_N z^{-1}}{1 - \lambda_N z^{-1}} \tag{19.9}$$

where

$$\begin{aligned} \lambda_{N-1} = \alpha + j\beta, \lambda_N = \alpha - j\beta \\ b_{N-1} = \delta - j\gamma, b_N = \delta + j\gamma \end{aligned} \tag{19.10}$$

i.e. $\lambda_{N-1}$, $\lambda_N$ and $b_{N-1}$, $b_N$ are complex conjugated numbers. Because of this we can use the same parallel presentation given in figure 2. However, the values of the last two blocks are complex. It should be stressed that the output signal of these two blocks is real. In order to make things more clear and without loss of generality we will consider only these blocks. They correspond to a second order ΣΔ modulator with complex conjugated poles of the loop filter transfer function $G(z)$. The block diagram of this modulator is given in figure 19.3.

Here both signals $x_1$ and $x_2$ are complex conjugated, namely

$$\begin{aligned} x_1(k+1) &= m(k+1) + jn(k+1) \\ x_2(k+1) &= m(k+1) - jn(k+1) \end{aligned} \tag{19.11}$$

Because of this the input of the quantizer is real i.e.

$$(\delta - j\gamma)x_1(k) + (\delta + j\gamma)x_2(k) = 2\delta m(k) + 2\gamma n(k) \tag{19.12}$$

As in the case of real poles, the modulator could be considered as two first order modulators interacting only through the quantizer function. The difference now is that the signals connected with both modulators are complex, but the input and output signals ($u$ and $y$) are the "true" signals of the modulator. This model will help us to make analysis simple. We will consider the state of the first order modulators as a point in a complex plane $(m,n)$. Depending on whether the input $2\delta m + 2\gamma n$ of the quantizer is positive or negative the state equation of the second order modulator could be described as follows:

$$\begin{aligned} x_1(k+1) &= (\alpha + j\beta)x_1(k) + [u(k) - 1], 2\delta m(k) + 2\gamma n(k) \geq 0 \\ x_2(k+1) &= (\alpha + j\beta)x_2(k) + [u(k) - 1], 2\delta m(k) + 2\gamma n(k) \geq 0 \end{aligned} \tag{19.13}$$

and

$$\begin{aligned} x_1(k+1) &= (\alpha + j\beta)x_1(k) + [u(k) + 1], 2\delta m(k) + 2\gamma n(k) < 0 \\ x_2(k+1) &= (\alpha + j\beta)x_2(k) + [u(k) + 1], 2\delta m(k) + 2\gamma n(k) < 0 \end{aligned} \tag{19.14}$$

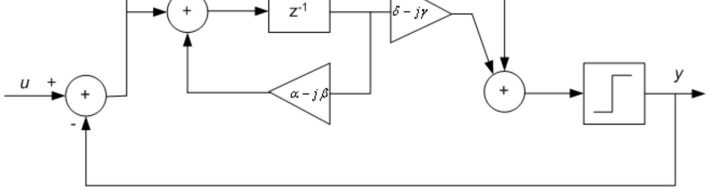


**Fig. 19.3** Block diagram of second order modulator with complex conjugate pair of roots of the loop filter transfer function

where $x_1$ and $x_2$ are given by (19.11). In fact $2\delta m + 2\gamma n$ is a line through the origin in the plane $(m,n)$ and depending on in what half the point $x_1$ is (because $x_1 = m + jn$), the description of the modulator is (19.13) or (19.14).

## 19.3 Stability of Shifted First Order Sigma-Delta Modulators

The shifted first order modulator is described by equation (19.5). Because of the ideal quantizer, the system can be viewed as two linear systems connected at point - $\alpha(n)/b$ and thus the equations describing the dynamics of the first order Sigma-Delta modulator from (19.5) are

$$\begin{aligned} x(n+1) &= \lambda x(n) + [u(n) - 1], x(n) \geq -\alpha(n)/b; b > 0 \\ x(n+1) &= \lambda x(n) + [u(n) + 1], x(n) < -\alpha(n)/b; b > 0 \end{aligned} \tag{19.15}$$

The fixed points of the system are $x' = \frac{u(n)-1}{1-\lambda}, x'' = \frac{u(n)+1}{1-\lambda}$.

In what follows we will consider the input signal $u(n)$ to be from the interval $u(n) \in [-\Delta u, \Delta u], \Delta u > 0$ and because of this the shift $\alpha(n)$ belongs to the interval $[-\Delta\alpha, \Delta\alpha], \Delta\alpha > 0$. The flow diagram of the system is given in Figure 19.4.

### *Stable Mode,* $\lambda < 1$

Depending on the parameters $b, \alpha(n)$ and input signal $u(n)$ the system can have two stable virtual fixed points (the case given in the figure) and a compact region exists between them (in fact this is an invariant set in state space, which has the property that all subsequent states lie in the original set for a certain class of input signals). For another set of parameters one of the virtual fixed points becomes a real fixed point. In each of the cases the system is stable but in the second one, there is

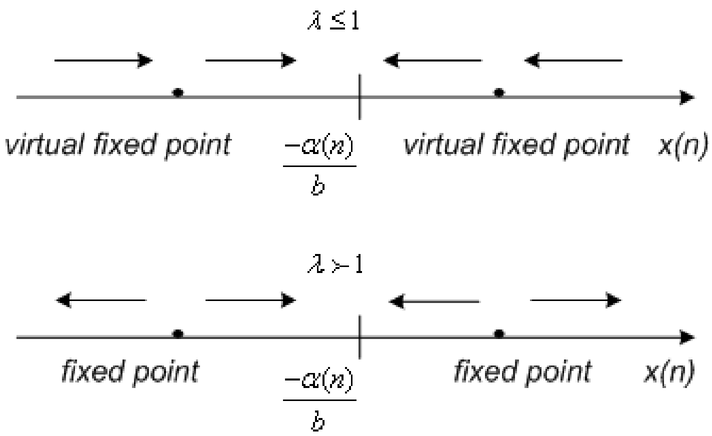


**Fig. 19.4** Flow diagrams of the first order system for the case of $\lambda \leq 1$ and $\lambda > 1$

no compact region. The system moves towards a single attractor at the stable fixed point. Anyway, if the initial condition is between the origin and the real fixed point of system (19.5) the state flow finishes at the equilibrium point (due to asymptotic movement to the single equilibrium point). It should be noted that this is not a desired Sigma-Delta modulator behavior. The Sigma-Delta modulator behavior appear when the first order system has two virtual fixed points and the state of (19.5) jumps between them. Thus the desired bitstream appear at the output of the quantizer.

### *Unstable Mode,* $\lambda > 1$

The stability in this case is connected with existence of a compact region between the unstable fixed points (not virtual). It is important to point out that $b > 0$. Otherwise the dynamic of the system is described by

$$\begin{aligned} x(n+1) &= \lambda x(n) + [u(n) - 1], x(n) < -\alpha(n)/b; b < 0 \\ x(n+1) &= \lambda x(n) + [u(n) + 1], x(n) \geq -\alpha(n)/b; b < 0 \end{aligned} \tag{19.16}$$

and it is easy to observe that the above system is always unstable, because at least one of the fixed points is virtual. Let's consider the map (19.5), given by (19.15) depicted in Figure 19.5. For a compact region (CR), to exists the fixed point should not be virtual i.e. $-\frac{\alpha(n)}{b} < \frac{u(n)-1}{(1-\lambda)}$ and $-\frac{\alpha(n)}{b} > \frac{u(n)+1}{(1-\lambda)}$

This should be true for the worst case i.e. $-\frac{\alpha(n)}{b} < \frac{\Delta u-1}{(1-\lambda)}$ and $-\frac{\alpha(n)}{b} > \frac{-\Delta u+1}{(1-\lambda)}$.
Taking into account that $(1-\lambda) < 0$ and $b > 0$ we get

$$\frac{b}{\lambda-1}\Delta u - \frac{b}{(\lambda-1)} < \alpha(n) < -\frac{b}{\lambda-1}\Delta u + \frac{b}{(\lambda-1)} \tag{19.17}$$

The second condition for the existence of a compact region is that it has to be included into the region between the fixed points i.e. the stable region. The maximum jump of the variable $x(n)$ from the Negative Half Line (NHL), with respect to $-\alpha(n)/b$, to the Positive Half Line (PHL), with respect to $-\alpha(n)/b$, is $[-\alpha(n)/b]\lambda + [u(n)+1]$ and the maximum jump from PHL to NHL is $[-\alpha(n)/b]\lambda + [u(n)-1]$. Hence in the worst case

$$-\frac{\alpha(n)}{b}\lambda + [\Delta u + 1] < \frac{\Delta u - 1}{(1-\lambda)}, \quad -\frac{\alpha(n)}{b}\lambda + [-\Delta u - 1] > \frac{-\Delta u + 1}{(1-\lambda)}$$

Solving the above inequalities with respect to $\alpha(n)$ we find that a compact region can only exist if $b > 0$ and

$$\begin{aligned} & b > 0 \\ & \frac{b}{\lambda-1}\Delta u - \frac{b(2-\lambda)}{\lambda(\lambda-1)} < \alpha(n) < -\frac{b}{\lambda-1}\Delta u + \frac{b(2-\lambda)}{\lambda(\lambda-1)} \end{aligned} \tag{19.18}$$

Because the above should be valid for all $y$ and for $y = 0$ as well then $(2-\lambda)/\lambda > 0$ or $\lambda < 2$, i.e. $1 < \lambda < 2$. Due to this $(2-\lambda)/\lambda < 1$ and hence if (19.18) is

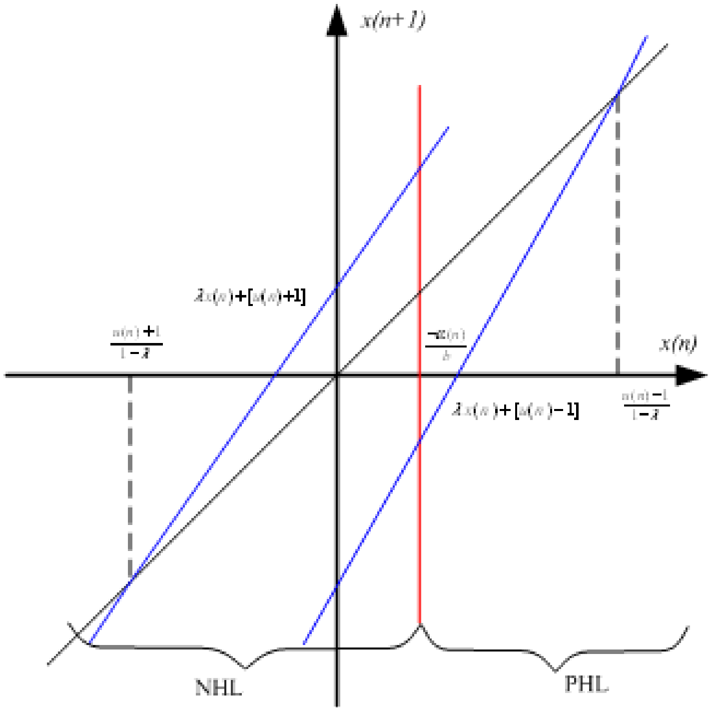


**Fig. 19.5** Map (19.5) given by (19.15) for the case of $\lambda > 1$

satisfied then (19.17) will be satisfied as well. Considering again these two conditions, the maximal shift of the input signal $\Delta u$, which ensures that the compact region is included into the region between the fixed points i.e. the stable region is given by

$$\Delta u < -\frac{\Delta\alpha(\lambda - 1)}{b} + \frac{2 - \lambda}{\lambda} \tag{19.19}$$

Note that condition (19.18) is a sufficient but not necessary condition. It has been derived for the worst case and if satisfied, the first order modulator is stable for the range of input signal given by (19.19). However, if (19.18) is not satisfied the modulator could be stable for certain input signal.

## 19.4 Stability of High Order Sigma-Delta Modulators

### *Stability of High Order Sigma-Delta Modulators with Real Poles*

Taking into account the parallel presentation given in Section 19.2, the stability of the high order Sigma-Delta modulator depends on the stability of each of the first order modulators. If all modes $\lambda_k$, are stable, i.e. $\lambda_k < 1$ then the corresponding high order Sigma-Delta modulator is stable in the sense of boundness of the states. If there exists even one unstable mode $\lambda_k$, i.e. $1 < \lambda_k < 2$, the stability conditions for shifted modulators given above should be applied. In this case the shift $\alpha_k(n)$ depends on the values of the other variables $x_i(n)$ i.e.

$$\lambda_k(n) = \sum_{\substack{i=1 \\ i \neq k}}^{N} b_i x_i(n) \tag{19.20}$$

From (19.18), we have

$$\begin{aligned} &\sum_{\substack{i=1 \\ i \neq k}}^{N} b_i x_i(n) < -\frac{b_k}{\lambda_k - 1}\Delta u + \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k - 1)} \\ &\sum_{\substack{i=1 \\ i \neq k}}^{N} b_i x_i(n) > \frac{b_k}{\lambda_k - 1}\Delta u - \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k - 1)}, \\ &k = 1,2,...,N \end{aligned} \tag{19.21}$$

The above should still be true when $x_k$ makes the maximal "jumps" into the PHL or into the NHL. Without loss of generality we will consider the first $p$ modes $\lambda_k$ of the high order Sigma-Delta modulator to correspond to $1 < \lambda_k < 2, k = 1,2,\dots,p$ whereas the remaining $N-p$ modes correspond to $\lambda_k < 1, k = p+1,\dots,N$. In this case only the first $p$ coefficients $b_k$ must be positive and the remaining $N-p$ coefficients could have any real value. The maximal "jumps" of the state variables corresponding to the first $p$ modes in the PHL and the NHL are $\frac{u(n)-1}{1-\lambda_k}$ and $\frac{u(n)+1}{1-\lambda_k}$, respectively (the fixed points of the system with respect to $x_k, k = 1,2,\dots,p$). Similarly, the maximal "jumps" of the state variables corresponding to the last $N-p$ modes in the PHL and the NHL, are $\frac{u(n)+1}{1-\lambda_k}$ and $\frac{u(n)-1}{1-\lambda_k}$, respectively (the virtual or real fixed points of the system with respect to $x_k, k = p+1,\dots,N$). Therefore from (19.21) for the worst case with respect to the input signal one can obtain

$$\begin{aligned} &\sum_{\substack{i=1 \\ i \neq k}}^{p} b_i \frac{-\Delta u - 1}{1-\lambda_i} + \sum_{i=p+1}^{N} |b_i| \frac{\Delta u + 1}{1-\lambda_i} < -\frac{b_k}{\lambda_k - 1}\Delta u + \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k - 1)} \\ &\sum_{\substack{i=1 \\ i \neq k}}^{p} b_i \frac{\Delta u + 1}{1-\lambda_i} + \sum_{i=p+1}^{N} |b_i| \frac{-\Delta u - 1}{1-\lambda_i} > \frac{b_k}{\lambda_k - 1}\Delta u - \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k - 1)} \\ &k = 1,2,...,p \end{aligned} \tag{19.22}$$

Note that we apply (19.22) only for the shifts connected to the first $p$ modulators. The other $N-p$ first order modulators are stable, because for their corresponding $\lambda_k, \lambda_k \leq 1, k = p+1,\dots,N$. If there exists a region $[-\Delta u, \Delta u] \subseteq [-1,1]$, such that

$u \in [-\Delta u, \Delta u]$ and for this region conditions (19.22) are satisfied, then the Sigma-Delta modulator will be stable for all input signals from this region. Taking into account equation (19.22) we get

$$\left[\sum_{i=1}^{p} \frac{b_i}{\lambda_i - 1} - \sum_{i=p+1}^{N} \frac{|b_i|}{\lambda_i - 1}\right] \Delta u < \sum_{i=p+1}^{N} \frac{|b_i|}{\lambda_i - 1} - \sum_{\substack{i=1 \\ i \neq k}}^{p} \frac{b_i}{\lambda_i - 1} + \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k) - 1}$$
$$k = 1, 2, ..., p \tag{19.23}$$

More detailed considerations of the above inequality shows that in order to ensure a consistent solution of (19.23) with respect to $\Delta u$

$$\sum_{\substack{i=1 \\ i \neq k}}^{p} \frac{b_i}{\lambda_i - 1} - \sum_{i=p+1}^{N} \frac{|b_i|}{\lambda_i - 1} - \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k - 1)} < 0, \quad k = 1, 2, .., p \tag{19.24}$$

Hence the maximal shift of input signal $\Delta u$ ensuring the stability is given by

$$\Delta u < \frac{\sum_{i=p+1}^{N} \frac{|b_i|}{\lambda_i - 1} - \sum_{\substack{i=1 \\ i \neq k}}^{p} \frac{b_i}{\lambda_i - 1} + \frac{b_k(2-\lambda_k)}{\lambda_k(\lambda_k - 1)}}{\sum_{i=1}^{p} \frac{b_i}{\lambda_i - 1} - \sum_{i=p+1}^{N} \frac{|b_i|}{\lambda_i - 1}}, \quad k = 1, 2, ..., p \tag{19.25}$$

Note that inequalities (19.25) should be valid simultaneously for each $k, k = 1, 2, \ldots, p$. Therefore, together with $b_k > 0, k = 1, 2, \ldots, p$, equation (19.24) gives the sufficient conditions for the stability of the Sigma-Delta modulator, namely

$$\frac{(2-\lambda_k)}{\lambda_k} \frac{b_k}{(\lambda_k - 1)} > \sum_{\substack{i=1 \\ i \neq k}}^{p} \frac{b_i}{\lambda_i - 1} - \sum_{i=p+1}^{N} \frac{|b_i|}{\lambda_i - 1}, \quad k = 1, 2, ..., p \tag{19.26}$$

For the poles outside the unit circle, $k = 1, 2, \ldots, p$, we have that $(2 - \lambda_k)/\lambda_k < 1$. This implies that the inequality, Eq. (19.26), can only hold for one value of $k$. Hence, Eq. (19.26) provides a sufficient condition for stability when $p = 1$ i.e. there is at most one unstable mode, and this sufficient condition cannot hold when there is more than one pole outside the unit circle. It is clear now that in the case of repeated poles $(\lambda_1, \ldots, \lambda_m = \lambda)$ of the loop transfer function, the Sigma-Delta modulator is stable only when the corresponding modes are stable i.e. $\lambda \leq 1$. Let us consider more precisely the case of identical poles. Without losing the generality we will consider that the pole $\lambda_1$ is repeated with order 2 i.e. $\lambda_1 = \lambda_2 = \lambda$. In this case (2) becomes

$$G(z) = \frac{b_1 z^{-1}}{1 - \lambda z^{-1}} + \frac{b_2 z^{-2}}{(1 - \lambda z^{-1})^2} + ... + \frac{b_N z^{-1}}{1 - \lambda_N z^{-1}} \tag{19.27}$$

And the state equations may be given as

$$\begin{aligned} x_1(n+1) &= \lambda x_1(n) + u(n) - sgn[b_1 x_1(n) + \sum_{i=2}^{N} b_i x_i(n)] \\ x_2(n+1) &= x_1(n) + \lambda x_2(n) \\ x_k(n+1) &= \lambda_k x_k(n) + u(n) - sgn[b_k x_k(n) + \sum_{\substack{i=1 \\ i \neq k}}^{N} b_i x_i(n)] \end{aligned} \tag{19.28}$$

$k = 3,..,N$

If $\lambda$ is an unstable mode, *i.e.* $1 < \lambda < 2$ then the corresponding first and second modulators should be stable in the sense of boundedness of the states. The first one can satisfy the conditions given by (19.18). The second one in fact is a linear system described by

$$x_2(n+1) = \lambda x_2(n) + x_1(n) \tag{19.29}$$

where the state variable $x_1$ could be considered as an input signal for this system. If $1 < \lambda < 2$ then all possible symbolic sequences represent admissible periodic orbits of $x_1$. Because of this, depending on the initial conditions a certain periodic orbit of $x_1$ could influence the instability in $x_2$.

## *Stability of High Order Sigma-Delta Modulators with Complex Poles*

In the particular case of two complex conjugated poles given in Figure 19.3, the dynamics of the Sigma-Delta Modulator is described by (19.13) or (19.14). The analysis of the behavior of both first order "complex" modulators is similar to the analysis of the first order "real" modulators, given in Section 19.3. Here we always should keep in mind that both modulators work cooperative, because their signals are conjugated. These modulators do not exist in the real Sigma-Delta modulator. They are introduced (like in the "real" case as well) to help us to carry out the analysis of the behavior of the whole system.

## *Stable Mode,* $|\lambda_1| = |\lambda_2| < 1$

In this case both modulators have two stable equilibrium points (in every half plane):

first modulator: $\frac{u-1}{1-\lambda_1}$ and $\frac{u+1}{1-\lambda_1}$ i.e. $\frac{(u-1)[(1-\alpha)+j\beta]}{(1-\alpha)^2+\beta^2}$ and $\frac{(u+1)[(1-\alpha)+j\beta]}{(1-\alpha)^2+\beta^2}$

second modulator: $\frac{u-1}{1-\lambda_2}$ and $\frac{u+1}{1-\lambda_2}$ i.e. $\frac{(u-1)[(1-\alpha)-j\beta]}{(1-\alpha)^2+\beta^2}$ and $\frac{(u+1)[(1-\alpha)-j\beta]}{(1-\alpha)^2+\beta^2}$

These fixed points could be virtual or real. Taking into account equations (19.12), (19.13) and (19.14), the fixed points of both modulators are ”virtual” when $2\delta(1-\alpha)+2\gamma\beta>0$ and ”non-virtual” when $2\delta(1-\alpha)+2\gamma\beta<0$. Both complex modulators are stable and the second order modulator is stable as well. As was mentioned in section 19.3, the Sigma-Delta modulator behavior appears when the first order system has two virtual fixed points and the states of (19.13), (19.14) jump between them. Thus the desired bitstream appears at the output of the quantizer. According to [12], in the general case, when the last two first order modulators are ”complex”, i.e. correspond to a stable complex conjugated pair of roots; condition (19.26) has the form

$$\frac{(2-\lambda_1)}{\lambda_1}\frac{b_1}{(\lambda_1-1)} > -\sum_{i=2}^{N-2}\frac{|b_i|}{\lambda_i-1}+\frac{2|\delta(1-\alpha)+\gamma\beta|}{(1-\alpha)^2+\beta^2} \tag{19.30}$$

and the maximal range of input signal $\Delta u$ ensuring the stability is expressed by

$$\Delta u < \frac{\sum_{i=2}^{N-2}\frac{|b_i|}{\lambda_i-1}+\frac{2|\delta(1-\alpha)+\gamma\beta|}{(1-\alpha)^2+\beta^2}+\frac{b_1(2-\lambda_1)}{\lambda_1(\lambda_1-1)}}{\frac{b_1}{\lambda_1-1}-\sum_{i=2}^{N-2}\frac{|b_i|}{\lambda_i-1}+\frac{2|\delta(1-\alpha)+\gamma\beta|}{(1-\alpha)^2+\beta^2}} \tag{19.31}$$

## Unstable Mode, $|\lambda_1| = |\lambda_2| > 1$

In this case both modulators have two unstable fixed points (in every half plane). Depending on parameters, these points could be ”non-virtual” or ”virtual”. In the case of virtual fixed points, both ”complex” modulators are unstable and the whole system is unstable. In the case of real fixed points, the possibility for Sigma-Delta modulator behavior is connected with the existence of a compact region in the complex plane.

To summarize the results on stability of high order Sigma-Delta modulators from this section, we have the following: 1. Any Sigma-Delta modulator comprised entirely of parallel sections with poles inside the unit circle is inherently stable. 2. Any Sigma-Delta modulator with only real poles is guaranteed to be stable if (19.26) holds and (19.25) provides the maximum input for stability. Equation (19.26) also implies that the sufficient conditions for stability are violated if at least 2 real poles are outside the unit circle. 3. Any Sigma-Delta modulator comprised entirely of parallel sections with poles inside the unit circle and one complex conjugate pair inside the unit circle is inherently stable. 4. Any Sigma-Delta modulator comprised entirely of parallel sections with some real poles outside the unit circle and one complex conjugate pair inside the unit circle is guaranteed to be stable if (19.30) holds, and (19.31) provides the maximum input for stability. Equation (19.30) also implies that the sufficient conditions for stability are violated if at least 2 real poles are outside the unit circle. It should be emphasized, that present theoretical study includes only the cases considered above; real poles not equal to 1, or complex poles inside the unit circle. To demonstrate the applicability of the presented conditions we consider a $\Sigma\Delta$ modulator with the following loop filter transfer function

$$G(z) = \frac{b_1 z^{-1}}{1-\lambda_1 z^{-1}} + \frac{2r\cos\theta z^{-1} - r^2 z^{-2}}{1-2r\cos\theta z^{-1} + r^2 z^{-2}}, \quad \lambda_1 > 1$$

In this case $\lambda_2 = \alpha + j\beta, \quad \lambda_3 = \alpha - j\beta; \quad b_2 = \delta - j\gamma, \quad b_3 = \delta + j\gamma$ where $\alpha = r\cos\theta, \quad \beta = r\sin\theta, \quad \delta = r\cos\theta; \quad \gamma = r\frac{\cos 2\theta}{2\sin\theta}$, Then the stability condition becomes

$$\frac{(2-\lambda_1)}{\lambda_1}\frac{b_1}{(\lambda_1 - 1)} > \frac{2r\cos\theta - r^2}{1-2r\cos\theta + r^2} \tag{19.32}$$

Let's consider two different modulators with the following set of parameters: $r = 0.9, \theta = 15°, \lambda_1 = 1.05, b_1 = 0.5$ and $r = 0.9, \theta = 15°, \lambda_1 = 1.05, b_1 = 1$. One can simulate numerically the behavior of both modulators and could observe that the first modulator is unstable, whereas the second one is stable for a certain range of input signal (given by (19.31)) because stability condition (19.32) is satisfied.

## 19.5 Analysis of Limit Cycles in High Order Sigma-Delta Modulators

In what follows, the following case will be considered:

1. The input signal $u = u(n)$ is constant from interval [-1, 1]

$$u = const., u \in [-1, 1] \tag{19.33}$$

2. The poles of the loop filter $\lambda_1, \lambda_2, \ldots, \lambda_n$ are in the unit circle

$$(\forall_{k=1}^{N} : |\lambda_k| < 1) \tag{19.34}$$

One of the important observations in [4] is that the case of pairs of complex conjugated poles can be considered on similar way with the help of presentation given in Figure 19.2. However, in this case the corresponding signals and coefficients are complex conjugated. As it has been stressed the contribution of the state variables corresponding to every pair of complex conjugated poles, to the input of the quantizer is real. In the next investigations we are going to skip also the case of real repeated roots. This assumption is practical, because it is very difficult to have this case due to unavoidable noise in every sigma-delta modulator realization. Without loss of generality we will consider the case with real distinct poles. Thus the discrete time sequence for state variables $x_1, x_2, \ldots, x_N$ is given by:

$$
\begin{aligned}
x_k(1) =& \lambda_k x_k(0) + [u(0) - y(0)], \\
x_k(2) =& \lambda_k x_k(1) + [u(1) - y(1)] = \lambda_k^2 x_k(0) + \\
& + [u(0) - y(0)]\lambda_k^1 + [u(1) - y(1)] \\
& \ldots\ldots \\
x_k(n) =& \lambda_k x_k(n-1) + [u(n-1) - y(n-1)] = \\
& \lambda_k^n x_k(0) + [u(0) - y(0)]\lambda_k^{n-1} + [u(1) - y(1)]\lambda_k^{n-2} + \ldots \\
& [u(n-2) - y(n-2)]\lambda_k^1 + [u(n-1) - y(n-1)] = \\
& \lambda_k^n x_k(0) + \sum_{i=0}^{n-1} [u(i) - y(i)]\lambda_k^{n-i-1} \\
k =& 1,2,\ldots,N
\end{aligned}
\tag{19.35}
$$

The limit cycles correspond to periodic solutions in time domain. The periodic solutions can be observed at the output of the modulator as repetitive sequences of 1's and -1's. Let's consider a periodic sequence $y(0), y(1), \ldots, y(M-1)$ with length $M$ at the output of the modulator. In this case $y(M) = y(0), y(M+1) = y(1), \ldots, y(2M-1) = y(M-1)$, etc. Every periodic output sequence corresponds to a periodic sequence in the states i.e. every state variable $x_k$ is periodic. This can be observed easily if we write the state variable $x_k$ after $L$ periods.

$$
x_k(L.M) = \lambda_k^{L.M} x_k(0) + \sum_{i=0}^{LM-1} [u(i) - y(i)]\lambda_k^{LM-i-1} \qquad k = 1,2,\ldots,N
\tag{19.36}
$$

Taking into account that every $[u(i) - y(i)]$ is the same after each $M$ samples, (19.36) can be rewritten as

$$
\begin{aligned}
x_k(L.M) =& \lambda_k^{L.M} x_k(0) + \sum_{i=0}^{LM-1} [u(i) - y(i)]\lambda_k^{LM-i-1} \\
& \lambda_k^{L.M} x_k(0) + \sum_{p=0}^{L-1} \lambda_k^{p.M} \left( \sum_{i=0}^{M-1} [u(i) - y(i)]\lambda_k^{M-i-1} \right) = \\
& \lambda_k^{L.M} x_k(0) + \frac{1 - \lambda_k^{LM}}{1 - \lambda_k^M} \left( \sum_{i=0}^{M-1} [u(i) - y(i)]\lambda_k^{M-i-1} \right) \\
k =& 1,2,\ldots,N
\end{aligned}
\tag{19.37}
$$

The above is correct, because $\sum_{p=0}^{L-1} \lambda_k^{p.M}$ is a partial sum of the first $L$ terms of a geometric series with value $\frac{1-\lambda_k^{LM}}{1-\lambda_k^M}$. If $|\lambda_k| < 1$, for every $L$ that is large enough (after enough time) $x_k(L.M) = \frac{1}{1-\lambda_k^M} \left( \sum_{i=0}^{M-1} [u(i) - y(i)]\lambda_k^{M-i-1} \right)$, i.e. $x_k(L.M)$ does not depend on $L$. This means repetition of the value of state $x_k$ after every

$M$ instances, i.e. the states are periodic. If $|\lambda_k| > 1$, from (19.37) follows that the boundness of the states is ensured if

$$x_k(0) = \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u(i)-y(i)]\lambda_k^{M-i-1}\right) \tag{19.38}$$

and thus $x_k(L.M) = x_k(0)$. This means that the initial condition with respect to $x_k$, should be taken in accordance with (19.38), in order to ensure stability of the solution. This fits with the results in concerning the stability of high order modulators when $\lambda_k > 1$. If $\lambda_k = 1$, $x_k(L.M) = x_k(0)$ for every $L$ and every $x_k(0)$, because at the periodic orbit $\sum_{i=0}^{M-1}[u - y(i)] = 0$ for constant input signal $u$. This actually means that periodicity with respect to $x_k$ is ensured. In the case of complex pair of poles the results are similar, but the initial conditions connected with the complex conjugated pair of poles are also complex conjugated. We should stress again that the contribution of the state variables corresponding to these poles, to the input of the quantizer is real. The obtained results have been derived without matching the time sequence of the states $x_k(0), x_k(1), \ldots, x_k(M-1), k = 1,2,\ldots N$ with the time sequence of the output signal $y(0), y(1), \ldots, y(M-1)$ in the framework of one period. In fact to have a valid output sequence $y(0), y(1), \ldots, y(M-1)$ condition (19.7) should be satisfied. Thus,

$$\begin{aligned}
&\left(\sum_{k=1}^{N} b_k x_k(n)\right) = (\mathbf{b}^T\mathbf{x}(n)) \geq 0, \quad if \quad y(n) = 1\\
&\left(\sum_{k=1}^{N} b_k x_k(n)\right) = (\mathbf{b}^T\mathbf{x}(n)) < 0, \quad if \quad y(n) = -1\\
&n = 1,2,\ldots,M
\end{aligned} \tag{19.39}$$

or

$$\begin{aligned}
&\sum_{k=1}^{N} b_k\lambda_k^n x_k(0) \geq -\sum_{k=1}^{N} b_k\left(\sum_{i=0}^{n-1}[u(i)-y(i)]\lambda_k^{n-i-1}\right), \quad if \quad y(n) = 1\\
&\sum_{k=1}^{N} b_k\lambda_k^n x_k(0) < -\sum_{k=1}^{N} b_k\left(\sum_{i=0}^{n-1}[u(i)-y(i)]\lambda_k^{n-i-1}\right), \quad if \quad y(n) = -1\\
&n = 1,2,\ldots,M
\end{aligned} \tag{19.40}$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{N} b_k\lambda_k^n x_k(0) \geq -\sum_{i=1}^{n-1}\left([u(i)-y(i)]\sum_{k=1}^{N} b_k\lambda_k^{n-i-1}\right), \quad if \quad y(n) = 1\\
&\sum_{k=1}^{N} b_k\lambda_k^n x_k(0) < -\sum_{i=1}^{n-1}\left([u(i)-y(i)]\sum_{k=1}^{N} b_k\lambda_k^{n-i-1}\right), \quad if \quad y(n) = -1\\
&n = 1,2,\ldots,M
\end{aligned} \tag{19.41}$$

In the case of a complex pair of poles $\lambda_i, \lambda_{i+1}$ the result has the same form. It should be noted that the left and right parts of inequalities (19.41) are real. The strategy for searching the limit cycles that correspond to a given output sequence of 1's and -1's with arbitrary length $M$ is based on finding the appropriate initial conditions $x_k(0), k = 1,2,\ldots,N$ with respect to state variables that ensure periodicity after the first period. Afterward the validity of the corresponding output sequences has to be checked. To simplify conditions (19.41) for validation of a given limit cycle connected with the corresponding vector of initial conditions $x_k(0), k = 1,2,\ldots,N$ obtained by (19.38), we are going, substitute (19.38) in all conditions (19.41). Thus taking into account (19.7) we get

$$\begin{aligned}
&\sum_{k=1}^{N} b_k \lambda_k^n \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u-y(i)]\lambda_k^{M-i-1}\right) \geq -\sum_{i=0}^{n-1}\left([u-y(i)]\sum_{k=1}^{N} b_k \lambda_k^{n-i-1}\right),\\
&if \quad y(n)=1\\
&\sum_{k=1}^{N} b_k \lambda_k^n \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u-y(i)]\lambda_k^{M-i-1}\right) < -\sum_{i=0}^{n-1}\left([u-y(i)]\sum_{k=1}^{N} b_k \lambda_k^{n-i-1}\right),\\
&if \quad y(n)=-1\\
&n=1,2,...,M
\end{aligned} \tag{19.42}$$

Conditions (19.42) can be combined in one, multiplying both sides by $y(n)$ that is either 1 or -1.

$$\begin{aligned}
&y(n).\sum_{k=1}^{N} b_k \lambda_k^n \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u-y(i)]\lambda_k^{M-i-1}\right) \geq\\
&-y(n).\sum_{i=0}^{n-1}\left([u-y(i)]\sum_{k=1}^{N} b_k \lambda_k^{n-i-1}\right)\\
&n=1,2,...,M
\end{aligned} \tag{19.43}$$

The above inequalities (19.43) can be developed further as follows

$$\begin{aligned}
&y(n).\sum_{k=1}^{N} b_k \lambda_k^n \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u-y(i)]\lambda_k^{M-i-1}\right) \geq\\
&-y(n).\sum_{k=1}^{N} b_k \sum_{i=0}^{n-1}[u-y(i)]\lambda_k^{n-i-1}\\
&n=1,2,...,M
\end{aligned} \tag{19.44}$$

or

$$
\begin{aligned}
& y(n).\sum_{k=1}^{N} b_k\lambda_k^n \frac{u}{1-\lambda_k^M}(1+\lambda_k+\lambda_k^2+...+\lambda_k^{M-1})- \\
& y(n).\sum_{k=1}^{N} b_k\lambda_k^n \frac{u}{1-\lambda_k^M}(y(0)\lambda_k^{M-1}+y(1)\lambda_k^{M-2}+...+y(M-2)\lambda_k+y(M-1)) \geq \\
& -y(n)\sum_{k=1}^{N} b_k u.(1+\lambda_k+\lambda_k^2+...+\lambda_k^{n-1})+ \\
& +y(n).\sum_{k=1}^{N} b_k(y(0)\lambda_k^{n-1}+y(1)\lambda_k^{n-2}+...+y(n-2)\lambda_k+y(n-1)) \\
& n=1,2,...,M
\end{aligned}
\tag{19.45}
$$

Taking into account that the value of the sum $\left(1+\lambda_k+\lambda_k^2+...+\lambda_k^{M-1}\right)$ is $\frac{1-\lambda_k^M}{1-\lambda_k}$ and the value of the sum $\left(1+\lambda_k+\lambda_k^2+...+\lambda_k^{n-1}\right)$ is $\frac{1-\lambda_k^n}{1-\lambda_k}$, inequalities (19.45) become

$$
\begin{aligned}
& y(n).\sum_{k=1}^{N} b_k\lambda_k^n \frac{u}{1-\lambda_k} + y(n)\sum_{k=1}^{N} b_k u\frac{1-\lambda_k^n}{1-\lambda_k} \geq \\
& y(n).\sum_{k=1}^{N} b_k\lambda_k^n \frac{u}{1-\lambda_k^M}(y(0)\lambda_k^{M-1}+y(1)\lambda_k^{M-2}+...+y(M-2)\lambda_k+y(M-1))+ \\
& +y(n).\sum_{k=1}^{N} b_k(y(0)\lambda_k^{n-1}+y(1)\lambda_k^{n-2}+...+y(n-2)\lambda_k+y(n-1))+ \\
& n=1,2,...,M
\end{aligned}
\tag{19.46}
$$

Thus we get

$$
\begin{aligned}
& y(n).u.\sum_{k=1}^{N} b_k\frac{1}{1-\lambda_k} \geq \\
& y(n).\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k^M}(y(0)\lambda_k^{n+M-1}+y(1)\lambda_k^{n+M-2}+...+y(M-2)\lambda_k^{n+1}+y(M-1)\lambda_k^n)+ \\
& +y(n).\sum_{k=1}^{N} b_k(y(0)\lambda_k^{n-1}+y(1)\lambda_k^{n-2}+...+y(n-2)\lambda_k+y(n-1))+ \\
& n=1,2,...,M
\end{aligned}
\tag{19.47}
$$

Further investigations on (19.47) leads to

$$
\begin{aligned}
&y(n).u.\sum_{k=1}^{N} b_k \frac{1}{1-\lambda_k} \geq \\
&y(n).\sum_{k=1}^{N} b_k \Big[\Big(\frac{\lambda_k^{n+M-1}}{1-\lambda_k^M} + \lambda_k^{n-1}\Big)y(0) + \Big(\frac{\lambda_k^{n+M-2}}{1-\lambda_k^M} + \lambda_k^{n-2}\Big)y(1) + \\
&... + \Big(\frac{\lambda_k^{M}}{1-\lambda_k^M} + 1\Big)y(n-1) + \frac{\lambda_k^{M-1}}{1-\lambda_k^M}y(n) + ... + \frac{\lambda_k^{n+1}}{1-\lambda_k^M}y(M-2) + \frac{\lambda_k^{n}}{1-\lambda_k^M}y(M-1)\Big] \\
&n = 1,2,...,M
\end{aligned}
$$

and hence

$$
\begin{aligned}
&y(n).u.\sum_{k=1}^{N} b_k \frac{1}{1-\lambda_k} \geq \\
&y(n).\sum_{k=1}^{N} b_k \Big[\Big(\frac{\lambda_k^{n-1}}{1-\lambda_k^M}\Big)y(0) + \Big(\frac{\lambda_k^{n-2}}{1-\lambda_k^M}\Big)y(1) + \\
&... + \Big(\frac{1}{1-\lambda_k^M}\Big)y(n-1) + \frac{\lambda_k^{M-1}}{1-\lambda_k^M}y(n) + \\
&... + \frac{\lambda_k^{n+1}}{1-\lambda_k^M}y(M-2) + \frac{\lambda_k^{n}}{1-\lambda_k^M}y(M-1)\Big] \\
&n = 1,2,...,M
\end{aligned}
$$

Therefore, with respect to the output bitstream sequence $y(0), y(1), \ldots, y(M-1)$ the inequalities that have to be satisfied are $M$ linear inequalities in form

$$
\begin{aligned}
&y(n).u.\sum_{k=1}^{N} b_k \frac{1}{1-\lambda_k} \geq \left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{n-1}}{1-\lambda_k^M}\right)y(0) + \\
&+\left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{n-2}}{1-\lambda_k^M}\right)y(1) + ... + \left(y(n).\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k^M}\right)y(n-1) + \\
&+\left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M}\right)y(n) + ... + \left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{n+1}}{1-\lambda_k^M}\right)y(M-2) + \\
&+\left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{n}}{1-\lambda_k^M}\right)y(M-1) \\
&n = 1,2,...,M
\end{aligned}
\tag{19.48}
$$

Inequalities (19.48) have a geometrical interpretation. In the $M$ dimensional space of the output sequences $y(0), y(1), \ldots, y(M-1)$ with length $M$, every output bitstream of 1's and -1's is a vertex of the $M$ dimensional hypercube in this space. Such a vertex represents a possible limit cycle if it is on the corresponding side of all $M$ hyperplanes, given by (19.48) that are equivalent to this vertex.

In extended form the inequalities (19.48) could be rewritten as follows. For $n = 1$

$$y(1).u.\sum_{k=1}^{N} b_k \frac{1}{1-\lambda_k} \geq \left(y(1).\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k^M}\right) y(0) + \left(y(1).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M}\right) y(1) +$$
$$... + \left(y(1).\sum_{k=1}^{N} \frac{b_k\lambda_k^2}{1-\lambda_k^M}\right) y(M-2) + \left(y(1).\sum_{k=1}^{N} \frac{b_k\lambda_k}{1-\lambda_k^M}\right) y(M-1)$$

For $n = 2$

$$y(1).u.\sum_{k=1}^{N} b_k \frac{1}{1-\lambda_k} \geq \left(y(2).\sum_{k=1}^{N} \frac{b_k\lambda_k}{1-\lambda_k^M}\right) y(0) + \left(y(2).\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k^M}\right) y(1) +$$
$$+ \left(y(2).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M}\right) y(2) + ... +$$
$$+ \left(y(2).\sum_{k=1}^{N} \frac{b_k\lambda_k^3}{1-\lambda_k^M}\right) y(M-2) + \left(y(2).\sum_{k=1}^{N} \frac{b_k\lambda_k^2}{1-\lambda_k^M}\right) y(M-1)$$
$$n = 1,2,...,M$$

For $n = M, y(M) = y(0)$, because the limit cycle is with length $M$

$$y(0).u.\sum_{k=1}^{N} b_k \frac{1}{1-\lambda_k} \geq \left(y(0).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M}\right) y(0) +$$
$$+ \left(y(0).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-2}}{1-\lambda_k^M}\right) y(1) + ... + \left(y(0).\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k^M}\right) y(M-1)$$

Taking into account that at the limit cycle $y(0) = y(M), y(1) = y(M+1), \ldots, y(M-1) = y(2M-1), y(M) = y(2M) = y(0)$ or $y(p) = y(p+M)$ for $p = 1,2,\ldots,M-1$, we can rewrite conditions (19.48) in more general form:

$$y(n).u.\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k} \geq \left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M}\right) y(n) +$$
$$+ \left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-2}}{1-\lambda_k^M}\right) y(n+1) + \left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k^{M-3}}{1-\lambda_k^M}\right) y(n+2) + ... +$$
$$\left(y(n).\sum_{k=1}^{N} \frac{b_k\lambda_k}{1-\lambda_k^M}\right) y(n+M-2) + \left(y(n).\sum_{k=1}^{N} \frac{b_k}{1-\lambda_k^M}\right) y(n+M-1)$$
$$n = 1,2,...,M \tag{19.49}$$

This result simplify conditions (19.41) for validation of a given limit cycle connected with an output bitstream $y(0), y(1), \ldots, y(M-1)$ with length $L$, because directly incorporates the values of the bitsteram sequence, the constant input signal $u$ and the parameters of parallel presentation of the loop filer of the sigma-delta modulator considered. It should be stressed that the coefficients $\sum_{k=1}^{N}\frac{b_k}{1-\lambda_k}$, $\sum_{k=1}^{N}\frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M}$, $\sum_{k=1}^{N}\frac{b_k\lambda_k^{M-2}}{1-\lambda_k^M},\ldots,$ $\sum_{k=1}^{N}\frac{b_k}{1-\lambda_k^M}$ are common for all inequalities and thus the conditions (19.49) could be checked very easy.

For better understanding the validation formulas (19.49) we are going to present a particular case for verification of limit cycles with length $M=4$ for a sigma-delta modulator with a third order loop filter $N=3$. In this case formulas (19.49) become

$$y(1).u.\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k}\geq\left(y(1).\sum_{k=1}^{3}\frac{b_k\lambda_k^3}{1-\lambda_k^4}\right)y(1)+\left(y(1).\sum_{k=1}^{3}\frac{b_k\lambda_k^2}{1-\lambda_k^4}\right)y(2)+$$
$$+\left(y(1).\sum_{k=1}^{3}\frac{b_k\lambda_k}{1-\lambda_k^4}\right)y(3)+\left(y(1).\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k^4}\right)y(0)$$
$$y(2).u.\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k}\geq\left(y(2).\sum_{k=1}^{3}\frac{b_k\lambda_k^3}{1-\lambda_k^4}\right)y(2)+\left(y(1).\sum_{k=1}^{3}\frac{b_k\lambda_k^2}{1-\lambda_k^4}\right)y(3)+$$
$$+\left(y(2).\sum_{k=1}^{3}\frac{b_k\lambda_k}{1-\lambda_k^4}\right)y(0)+\left(y(1).\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k^4}\right)y(1)$$
$$y(3).u.\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k}\geq\left(y(3).\sum_{k=1}^{3}\frac{b_k\lambda_k^3}{1-\lambda_k^4}\right)y(3)+\left(y(3).\sum_{k=1}^{3}\frac{b_k\lambda_k^2}{1-\lambda_k^4}\right)y(0)+$$
$$+\left(y(3).\sum_{k=1}^{3}\frac{b_k\lambda_k}{1-\lambda_k^4}\right)y(1)+\left(y(3).\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k^4}\right)y(2)$$
$$y(0).u.\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k}\geq\left(y(0).\sum_{k=1}^{3}\frac{b_k\lambda_k^3}{1-\lambda_k^4}\right)y(0)+\left(y(0).\sum_{k=1}^{3}\frac{b_k\lambda_k^2}{1-\lambda_k^4}\right)y(1)+$$
$$+\left(y(0).\sum_{k=1}^{3}\frac{b_k\lambda_k}{1-\lambda_k^4}\right)y(2)+\left(y(0).\sum_{k=1}^{3}\frac{b_k}{1-\lambda_k^4}\right)y(3)$$

Based on considerations here, given periodic output sequence of 1's and -1's with arbitrary length $M$, corresponds to a limit cycle if the inequalities (19.49) are satisfied. The application of the approach considered consists of checking inequalities (19.49) for every possible output sequence of 1's and -1's with length $M$. The number of these sequences is $2^M$. Developed conditions (19.49) are $M$ inequalities for every output sequence. Because the coefficients $\sum_{k=1}^{N}\frac{b_k}{1-\lambda_k},\sum_{k=1}^{N}\frac{b_k\lambda_k^{M-1}}{1-\lambda_k^M},\sum_{k=1}^{N}\frac{b_k\lambda_k^{M-2}}{1-\lambda_k^M},\ldots,\sum_{k=1}^{N}\frac{b_k}{1-\lambda_k^M}$ are common for all inequalities, conditions (19.49) are checked very fast and easy. This result accelerates the validation check for the limit cycles in the general case considered in this section. To demonstrate applicability of the new conditions (19.49), we consider a second order sigma-delta modulator with the following loop filter transfer function [4]

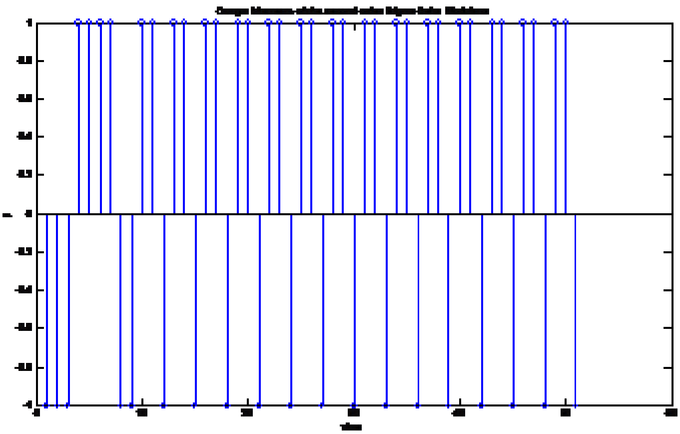

**Fig. 19.6** Output bitstream of the second order Sigma-Delta modulator with loop filter transfer function given by (19.50)

$$G(z) = \frac{2r\cos\theta z^{-1} - r^2 z^{-2}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} \tag{19.50}$$

In this case $\lambda_1 = \alpha + j\beta$, $\lambda_2 = \alpha - j\beta$, $b_1 = \delta - j\gamma$, $b_2 = \delta + j\gamma$, where $\alpha = r.\cos\theta$, $\beta = r.\sin\theta$, $\delta = r.\cos\theta$, $\gamma = r.(\cos 2\theta)/(2\sin\theta)$. When $r = 0.9$ and $\theta = 30°$, $\lambda_1 = 0.7794 + j0.4500$, $\lambda_2 = 0.7794 - j0.4500$, $b_1 = 0.7794 - j0.4500$, $b_2 = 0.7794 + j0.4500$ and the initial conditions that lead to a periodic output sequence $y(0) = 1$, $y(1) = 1$, $y(2) = -1$ without transient are $x_1(0) = 0.779 - j0.123$, $x_2(0) = 0.779 + j0.123$ [18], [19]. When the constant input is $u = 0.4$, the periodic output sequence with length $M = 3$ is $y(0) = 1, y(1) = 1, y(2) = -1$, and can be detected in Figure 19.6.

In this case the inequalities (19.49) have the form

$$y(1).u.\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k} \geq \left(y(1).\sum_{k=1}^{2}\frac{b_k\lambda_k^2}{1-\lambda_k^3}\right)y(1) + \left(y(1).\sum_{k=1}^{2}\frac{b_k\lambda_k}{1-\lambda_k^3}\right)y(2) + \left(y(1).\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3}\right)y(0)$$
$$y(2).u.\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k} \geq \left(y(2).\sum_{k=1}^{2}\frac{b_k\lambda_k^2}{1-\lambda_k^3}\right)y(2) + \left(y(2).\sum_{k=1}^{2}\frac{b_k\lambda_k}{1-\lambda_k^3}\right)y(0) + \left(y(2).\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3}\right)y(1)$$
$$y(3).u.\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k} \geq \left(y(3).\sum_{k=1}^{2}\frac{b_k\lambda_k^2}{1-\lambda_k^3}\right)y(0) + \left(y(3).\sum_{k=1}^{2}\frac{b_k\lambda_k}{1-\lambda_k^3}\right)y(1) + \left(y(3).\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3}\right)y(2)$$

where $y(3) = y(0)$.

The coefficients $\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k} = 2.9816$, $\sum_{k=1}^{2}\frac{B_k\lambda_k^2}{1-\lambda_k^3} = 0.4775$, $\sum_{k=1}^{2}\frac{b_k\lambda_2}{1-\lambda_k^3} = 1.0578$, $\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3} = 1.4463$ are common for the above 3 inequalities that are satisfied for the output bitstream sequence $y(0) = 1, y(1) = 1, y(2) = -1$.

$$y(1).u.\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k}-\left[\left(y(1).\sum_{k=1}^{2}\frac{b_k\lambda_k^2}{1-\lambda_k^3}\right)y(1)+\left(y(1).\sum_{k=1}^{2}\frac{b_k\lambda_k}{1-\lambda_k^3}\right)y(2)+\left(y(1).\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3}\right)y(0)\right]$$
$$=0.3267\geq 0$$
$$y(2).u.\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k}-\left[\left(y(2).\sum_{k=1}^{2}\frac{b_k\lambda_k^2}{1-\lambda_k^3}\right)y(2)+\left(y(2).\sum_{k=1}^{2}\frac{b_k\lambda_k}{1-\lambda_k^3}\right)y(0)+\left(y(2).\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3}\right)y(1)\right]$$
$$=0.8340\geq 0$$
$$y(3).u.\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k}-\left[\left(y(3).\sum_{k=1}^{2}\frac{b_k\lambda_k^2}{1-\lambda_k^3}\right)y(0)+\left(y(3).\sum_{k=1}^{2}\frac{b_k\lambda_k}{1-\lambda_k^3}\right)y(1)+\left(y(3).\sum_{k=1}^{2}\frac{b_k}{1-\lambda_k^3}\right)y(2)\right]$$
$$=1.1037\geq 0$$

## 19.6 Conclusions

In this chapter we present an unified approach for study the stability and validation of potential limit cycles of one bit high order Sigma-Delta modulators. The approach is general because it uses the general form of a Sigma-Delta modulator. It is based on a parallel decomposition of the modulator and a direct nonlinear systems analysis. In this representation, the general $N-th$ order modulator is transformed into a decomposition of low order, generally complex modulators, which interact only through the quantizer function. The developed conditions for stability and for validation of potential limit cycles are very easy for implementation and this procedure is very fast. The reported results can be elaborated further for some particular cases, and investigating the possibilities to skip the check of some output bitstream sequences and thus to accelerate extra the limit cycle validation procedure. Furthermore it is an open problem how to use the developed conditions for Sigma-Delta modulators design, i.e. to design a Sigma-Delta modulator working on a desired limit cycle.

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## References

1. Candy, J.C., Temes, G.C.: Oversampling Delta-Sigma Data Converters. IEEE Press, New York (1992)
2. Norsworthy, S.R., Schreier, R., Temes, G.C.: Delta-Sigma Data Converters. IEEE Press, New York (1997)
3. Gray, R.M.: Oversampled sigma-delta modulation. IEEE Trans. Commun. 35, 481–489 (1987)
4. Feely, O., Chua, L.O.: The effect of integrator leak in Sigma-delta modulation. IEEE Trans. Circuits and Systems 38, 1293–1305 (1991)
5. Feely, O., Chua, L.O.: Nonlinear dynamics of a class of analog-to-digital converters. Int. J. of Bifurcation and Chaos 2, 325–340 (1992)
6. Feely, O.: A tutorial introduction to non-linear dynamics and chaos and their application to sigma-delta modulators. Int. J. Circuit Theory and Applications 25, 347–367 (1997)
7. Steiner, P., Yang, W.: A framework for analysis of high-order sigma-delta modulators. IEEE Transactions on Circuits and Systems II: CAS II 44, 1–10 (1997)

8. Steiner, P., Yang, W.: Stability of high order sigma-delta modulators. In: International Symposium on Circuits and Systems, ISCAS 1996, vol. 3, pp. 52–55 (1996)
9. Mladenov, V., Hegt, H., Roermund, A.V.: Stability analysis of high order sigma-delta modulators. In: Proc. of the 15th European Conference on Circuit Theory and Design ECCTD 2001, pp. I-313 – I-316. Helsinki University of Technology, Finland (2001)
10. Mladenov, V., Hegt, H., van Roermund, A.: On the Stability of High Order Sigma-Delta Modulators. In: Proceedings of the 8th IEEE International Conference on Electronics, Circuits and Systems, ICECS 2001, Malta, pp. 1383–1386 (2001)
11. Mladenov, V., Hegt, H., van Roermund, A.: On the Stability Analysis of High Order Sigma-Delta Modulators. An International Journal on Analog Integrated Circuits and Signal Processing 36(1-2), 47–55 (2003)
12. Mladenov, V., Hegt, H., van Roermund, A.: On the Stability Analysis of Sigma-Delta Modulators. In: Proceedings of the 16th European Conference on Circuit Theory and Design, ECCTD 2003, Cracow, Poland, September 1-4, pp. I-97– I-100 (2003)
13. Friedman, V.: The structure of the limit cycles in sigma delta modulation. IEEE Transactions on Communications 36, 972–979 (1988)
14. Mann, S., Taylor, D.: Limit cycle behavior in the double-loop bandpass sigma delta a/d converter. IEEE Transactions on Circuits and Systems-II 46, 1086–1089 (1999)
15. Hyun, D., Fischer, G.: Limit cycles and pattern noise in single-stage single-bit delta-sigma modulators. IEEE Trans. Circuits and Systems I 49, 646–656 (2002)
16. Reefman, D., Reiss, J., Janssen, E., Sandler, M.: Description of limit cycles in sigma delta modulators. IEEE Trans. on CAS 52(6), 1211–1223 (2005)
17. Reiss, J.D., Sandler, M.B.: Detection and removal of limit cycles in sigma delta modulators. IEEE Transactions on Circuits and Systems I: Regular Papers 55(10), 3119–3130 (2008)
18. Mladenov, V.: A Method for Searching the Limit Cycles of High Order Sigma-Delta Modulators. In: Proceedings of the 19th European Conference on Circuit Theory and Design, ECCTD 2009, Antalya, Turkey, August 23-27, pp. 543–546 (2009)
19. Mladenov, V.: A Method for Validation the Limit Cycles of High Order Sigma-Delta Modulators. In: Proceedings of the 3rd International Workshop on Nonlinear Dynamics and Synchjronization, INDS 2011, Klagenfurt, Austria, July 25-27, pp. 234–238 (2011)