Chapter 6
A Framework for Coordination in Distributed Stochastic Systems: Perfect State Feedback and Performance Risk Aversion

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Abstract. This research article considers a class of distributed stochastic systems where interconnected systems closely keep track of reference signals issued by a coordinator. Much of the existing literature concentrates on conducting decisions and control synthesis based solely on expected utilities and averaged performance. However, research in psychology and behavioral decision theory suggests that performance risk plays an important role in shaping preferences in decisions under uncertainty. Thus motivated, a new equilibrium concept, called "person-by-person equilibrium" for local best responses is proposed for analyzing signaling effects and mutual influences between an incumbent system, its coordinator and immediate neighbors. Individual member objectives are defined by the multi-attribute utility functions that capture both performance expectation and risk measures to model the satisfaction associated with local best responses with risk-averse attitudes. The problem class and approach of coordination control of distributed stochastic systems proposed here are applicable to and exemplified in military organizations and flexibly autonomous systems.

6.1 Introduction

Control and coordination of distributed stochastic systems offers a framework to analyzing intertemporal strategic interactions between individual agents or controllers, one for each interconnected systems and based on local observations. The importance of evaluating approaches in a dynamic setting and the broad flexibility and adaptability of the decision and control architectures of distributed control with communications has spurred many large-scale applications such as military command and control hierarchies, spacecraft constellations, remotely piloted platform

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formations, teams of humans and autonomous robots, etc. where each member can be in best response to its neighbor actions and yet has no influence on other members to which it has no communication supports.

Despite the broad interest in distributed systems, there remain significant hurdles in applying them to practical problems of interest. Interplay between common team objectives and individual member objectives can yield surprises and complex behaviors. Hence, a form of coordination control that helps balance between cooperative goals and adversarial behavior in addition of fundamentals for team and individual decisions, is necessarily required.

Thus motivated, this research article proposes a new framework and analysis to study risk-averse control of a distributed stochastic system, in particular coordination control with risk-averse attitudes toward performance uncertainty and robustness. The approach of noncooperative game-theoretic decision making and optimization is suited to coordination control, where a distributed stochastic system is distinguished into a coordinator (also known as dominant player) with significant reference signals and incumbent systems (also known as nondominant players) with fringe couplings. To account for uncertainty in inherent design problem and in preference assessment, a multi-attribute utility function that enables incumbent systems' decision makers or controllers to select the best risk-averse strategy for the attribute tradeoffs between performance expectation and risks, is therefore considered. Notice that this dominant/nondominant game structure is also prevalent in both economics [1] and social sciences [2].

The game-theoretic model of mixed player behaviors considered herein is particularly related to the research [3] that has extended the large population linear-quadratic-Gaussian games to include a major player and a large number of minor players. As such, minor players are more sensitive to variations in the behavior of major player than those of individual minor players. To overcome the curse of dimensionality, computational concerns have typically resorted the analysis to the so-called Nash certainty equivalence method, where the key idea is to break the large population game into a family of limiting two-player games. The synthesis of decentralized strategies is obtained via a set of aggregate quantities giving the mean field approximation. In contrast with such existing literature, this appealing research which is the extension of recent accounts [4] and [5] investigates: i) a stochastic dynamic game model of behavior where nondominant players not only keep track closely of the large impact by the dominant player but also monitor rivals from the peers in a less detailed way and ii) a computationally tractable model of payoff uncertainty forecast for which sufficient statistics summarize all payoff relevant information and thus are used in the person-by-person equilibrium strategies by nondominant players.

In summary, the proposed game-theoretic framework is prevalent in distributed stochastic systems with a dominant/fringe coordination structure, capturing the attributes that are important to inherent design problem and preference assessment uncertainties, their tradeoff behavior over these attributes and their risk attitude. The rest of this article is organized as follows. Section 2 introduces a new computationally tractable model for distributed stochastic systems with state-space representations of a dominant coordinator and many nondominant systems. In addition, the preliminary results on sufficient mathematical statistics that summarize all performance measure or utility relevant history and for which the person-by-person equilibrium strategies are optimal for nondominant systems are discussed in great details. Section 3 contains precise problem statements for coordination control analysis and decision optimization for the person-by-person equilibrium or feedback Nash strategy concerned by autonomous agents and incumbent systems. The construction of person-by-person strategies is established in Section 4 while some conclusions and future research directions are drawn in Section 5.

6.2 Problem Formulation

Before going into a formal presentation, it is necessary to consider some conceptual notations in this article. For instance, time t is modeled as continuous and the notation of the time interval is $[t_0, t_f]$. All random variables are defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ which is a triple consisting of a set Ω, a σ-algebra $\mathscr{F}$ and a probability measure $\mathscr{P} : \mathscr{F} \mapsto [0,1]$ and is equiped with a filtration $\{\mathscr{F}_t : t \in [t_0, t_f]\}$. In addition, for a given Hilbert space X with norm $||\cdot||_X$, $1 \le p \le \infty$, a Banach space is defined as follows

$$\mathscr{L}^p_{\mathscr{F}}(t_0, t_f; X) \triangleq \Bigg\{ \phi : [t_0, t_f] \times \Omega \mapsto X \text{ is an } X\text{-valued } \mathscr{F}_t\text{-measurable process} \\ \text{with } E\left\{ \int_{t_0}^{t_f} ||\phi(t,\omega)||_X^p dt \right\} < \infty \Bigg\} \tag{6.1}$$

with norm

$$||\phi(\cdot)||_{\mathscr{F},p} \triangleq \left(E\left\{ \int_{t_0}^{t_f} ||\phi(t,\omega)||_X^p \, dt \right\} \right)^{1/p} . \tag{6.2}$$

Furthermore, the Banach space of X-valued continuous functionals on $[t_0, t_f]$ with the max-norm induced by $||\cdot||_X$ is denoted by $\mathscr{C}(t_0, t_f; X)$. The deterministic version of (6.1) and its associated norm (6.2) is written as $\mathscr{L}^p(t_0, t_f; X)$ and $||\cdot||_p$.

A distributed stochastic system that evolves over $[t_0, t_f]$ captures interactions among a coordinator and finite number of incumbent systems. Each incumbent system that enters the distributed system is assigned a unique positive integer-valued index. The set of indices of incumbent systems is denoted by $\overline{I} \triangleq \{1, 2, \ldots, N\}$ and a typical element by i. The set of immediate neighbors associated with an incumbent system i is denoted by N_i. For concreteness, the heterogeneity of incumbent system i and $i \in \overline{I}$ is distinguished by an individual state that is governed by the stochastic differential equation with the initial-value condition $x_i(t_0) = x_i^0$

$$dx_i(t) = (A_{ii}(t)x_i(t) + B_{ii}(t)u_i(t) + C_{ii}(t)z_i(t) + \sum_{j=1}^{N_i} B_{ij}(t)u_{ij}(t))dt + G_i(t)dw_i(t) \tag{6.3}$$

where continuous-time coefficients $A_{ii} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times n_i})$, $B_{ii} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times m_i})$, $C_{ii} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times q_i})$, $B_{ij} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times r_i})$ and $G_i \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times p_i})$ are deterministic matrix-valued functions. At time t, the recursive state of incumbent system i is denoted by $x_i \in \mathscr{L}^2_{\mathscr{F}_i}(t_0,t_f;\mathbb{R}^{n_i})$ with the initial state $x_i^0 \in \mathbb{R}^{n_i}$ known. The control policies from agent i to that system i are presented by $u_i \in \mathscr{L}^2_{\mathscr{F}_i}(t_0,t_f;\mathbb{R}^{m_i})$ and $z_i \in \mathscr{L}^2_{\mathscr{F}_i}(t_0,t_f;\mathbb{R}^{q_i})$. In addition, the interconnection inputs of that incumbent system i supported by the communication paths from immediate neighbors j and $j \in N_i$ are viewed as the real-valued functions $u_{ij}(t)dt$ of the following random processes

$$du_{ij}(t) = (C_{ij}(t)x_j(t) + D_{ij}(t)u_j(t))dt + dv_j(t)\,, \quad j \in N_i \tag{6.4}$$

where continuous-time coefficients $C_{ij} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{r_i \times n_j})$ and $D_{ij} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{r_i \times m_j})$ are deterministic matrix-valued functions. As the number of incumbent systems grows large, it is unrealistic to believe that binding agents i associated with incumbent systems i and $i \in \overline{I}$ are capable of monitoring the evolution of their immediate neighbors. Instead, it is reasonable to assume that incumbent systems only keep track of actual interactions or signaling references provided by coordinator c and $c \in \overline{I}_c$, where the set of partaking coordinators is predetermined and does not change over time.

In coordination control a coordinator c issues reference signals to two or more incumbent systems i and $i \in \overline{I}$ such that

$$z_{ic}(t)dt = (A_{ic}(t)x_c(t) + B_{ic}(t)u_c(t))dt + G_{ic}(t)dv_c(t) \tag{6.5}$$

but the incumbent systems i do not directly send signals to the coordinator c. In practice, it is further desirable to have decentralized decision making without intensive communication overheads. A potential alternative therefore involves the selection of a crude model of reduced order for the interactions among coordinator c and binding agents i associated with incumbent systems i. The actual reference signals imposed by coordinator c are now approximated by an explicit model-following of the type

$$dz_{ic}(t) = (A_{ic}(t)z_{ic}(t) + B_{ic}(t)u_c(t))dt + G_{ic}(t)dv_c(t)\,, \quad z_{ic}(t_0) = 0 \tag{6.6}$$

whereby continuous-time coefficients $A_{ic} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{q_i \times q_i})$, $B_{ic} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{q_i \times m_c})$ and $G_{ic} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{q_i \times q_c})$ are deterministic matrix-valued function and potentially come from a structural decomposition of a monolithic distributed system with centralized dynamics.

In the state-space representation (6.3) and (6.6) one postulates independent Wiener processes $w_i(t) \triangleq w_i(t,\omega_i) : [t_0,t_f] \times \Omega_i \mapsto \mathbb{R}^{p_i}$ and $v_c(t) \triangleq v_c(t,\omega_c) : [t_0,t_f] \times \Omega_c \mapsto \mathbb{R}^{q_c}$ defined by the underlying filtered probability spaces $(\Omega_i, \mathscr{F}_i, \{\mathscr{F}_i\}_t, \mathscr{P}_i)$ and $(\Omega_c, \mathscr{F}_c, \{\mathscr{F}_c\}_t, \mathscr{P}_c)$ with the correlations of independent increments

$$E\left\{[w_i(\tau_1) - w_i(\tau_2)][w_i(\tau_1) - w_i(\tau_2)]^T\right\} = W_i|\tau_1 - \tau_2|, \quad W_i > 0\,, \quad \tau_1,\tau_2 \in [t_0,t_f]$$
$$E\left\{[v_c(\tau_1) - v_c(\tau_2)][v_c(\tau_1) - v_c(\tau_2)]^T\right\} = V_c|\tau_1 - \tau_2|, \quad V_c > 0\,, \quad \tau_1,\tau_2 \in [t_0,t_f]$$

approximate the inherent design system uncertainty due to variability and lack of knowledge.

Furthermore, the model primitives of the state recursion (6.3) in the absence of links from the immediate neighbors and environmental disturbances are also assumed to be uniformly exponentially stable. For instance, there exist positive constants η_1 and η_2 such that the pointwise matrix norm of the closed-loop state transition matrix associated with incumbent system (6.3) satisfies the inequality

$$||\Phi_i(t,\tau)|| \leq \eta_1 e^{-\eta_2(t-\tau)} \qquad \forall\, t \geq \tau \geq t_0 \ .$$

The pair $(A_{ii}(t),[B_{ii}(t),C_{ii}(t)])$ is pointwise stabilizable if there exist bounded matrix-valued functions $K_{x_i}(t)$ and $K_{z_i}(t)$ so that the closed-loop system $dx_i(t) = (A_{ii}(t)+B_{ii}(t)K_{x_i}(t)+C_{ii}(t)K_{z_i}(t))x_i(t)dt$ is uniformly exponentially stable.

With the local agent dynamics (6.3) considered herein, each agent i associated with incumbent system i only plays a local dynamical game with its immediate neighbors $j \in N_i$. Mutual influence controlled by the control policies from the immediate neighbors of agent i is defined by $u_{-i} \triangleq \{u_{ij} : j \in N_i\}$. Assuming its coalition N_i conveys mutual influence information u_{-i}, agent i selects, at each time instant, a tuple of control policies to optimize its multi-attribute utility function. The tuple of control laws is defined by the control processes u_i and z_i, of which z_i is supposed to follow the prediction process z_{ic} for the reference signals from coordinator c. Thus, the subsequent states of agent i is determined by its current individual states x_i and z_{ic}, its chosen action (u_i,z_i) and the coalition effects u_{-i}. In fact, the selected action (u_i,z_i) will depend on agent i's individual states x_i and z_{ic} as well as the coalition effects u_{-i}.

To further illustrate the applicability of the coordination control framework as proposed here, the classes of admissible control policies associated with (6.3) are defined by $U_i \times Z_i \subset \mathscr{L}^2_{\mathscr{F}_i}(t_0,t_f;\mathbb{R}^{m_i}) \times \mathscr{L}^2_{\mathscr{F}_{mi}}(t_0,t_f;\mathbb{R}^{q_i})$. For any given coalition effects u_{-i}, the 3-tuple $(x_i(\cdot),u_i(\cdot),z_i(\cdot))$ shall be referred to as an admissible 3-tuple if $x_i(\cdot) \in \mathscr{L}^2_{\mathscr{F}_i}(t_0,t_f;\mathbb{R}^{n_i})$ is the solution trajectory of the stochastic differential equation (6.3) when $u_i(\cdot) \in U_i$ and $z_i(\cdot) \in Z_i$.

Next, agent i evaluates its performance and makes control policies that are consistent with its preferences. There are performance tradeoffs among the closeness of local states x_i from desired states ζ_i, the size of local actions u_i and the closeness of interaction enforcements between z_i and z_{ic} on incumbent system i by coordinator c. Henceforth, agent i must carefully balance the three in order to achieve its local performance measure. Mathematically, there assumes existence of an integral-quadratic form (IQF) performance-measure $J_i : U_i \times Z_i \mapsto \mathbb{R}_+$

$$\begin{aligned} J_i(u_i,z_i;u_{-i}) &= [x_i(t_f)-\zeta_i(t_f)]^T Q_i^f [x_i(t_f)-\zeta_i(t_f)] \\ &\quad + \int_{t_0}^{t_f} \{x_i^T(\tau)Q_{ii}(\tau)x_i(\tau) + [x_i(\tau)-\zeta_i(\tau)]^T Q_i [x_i(\tau)-\zeta_i(\tau)]\} d\tau \\ &\quad + \int_{t_0}^{t_f} \{u_i^T(\tau)R_{ii}(\tau)u_i(\tau) + [z_i(\tau)-z_{ic}(\tau)]^T R_{zi}(\tau)[z_i(\tau)-z_{ic}(\tau)]\} d\tau \end{aligned} \qquad (6.7)$$

where the deterministic matrix-valued functions $Q_i^f \in \mathbb{R}^{n_i \times n_i}$, $Q_{ii} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times n_i})$, $Q_i \in \mathscr{C}(t_0,t_f;\mathbb{R}^{n_i \times n_i})$ $R_{ii} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{m_i \times m_i})$ and $R_{zi} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{q_i \times q_i})$ representing design parameters for terminal states, transient states, control efforts and reference mismatches are positive semidefinite with $R_{ii}(t)$ and $R_{zi}(t)$ invertible.

Amongst of some research issues for coordination control which are currently under investigation is how to carry out optimal control synthesis for coordination control of distributed stochastic systems. The approach to handle the problem with a tuple of two or more control laws is to use the noncooperative game-theoretic paradigm. Particularly, an N-tuple policies $\{(u_1^*, z_1^*), (u_2^*, z_2^*), \ldots, (u_N^*, z_N^*)\}$ is said to constitute a person-by-person equilibrium solution for the coordination control problem (6.3) and performance measure (6.7) if

$$J_i^* \triangleq J_i(u_i^*, z_i^*; u_{-i}^*) \le J_i(u_i, z_i; u_{-i}^*), \qquad \forall i \in \bar{I}. \tag{6.8}$$

That is, none of the N agents can deviate unilaterally from the equilibrium policies and gain from doing so. The justification for the restriction to such an equilibrium is that the coalition effects u_{-i}^* sent to agent i does not necessarily support its preference optimization. Therefore, they cannot do better than behave as if they strive for this equilibrium. It is reasonable to conclude that a person-by-person equilibrium of distributed control is identical to the concept of a Nash equilibrium within a noncooperative game-theoretic setting.

Because admissible feedback policy sets for agent i are not discussed, the determination of a person-by-person equilibrium for the distributed stochastic system is still not straightforward. Therefore, a further restriction is imposed next. Given the linear-quadratic properties of the state-space description (6.3) and (6.7), attention is then focused on the search for linear time-varying feedback policies generated from the locally accessible state $x_i(t)$ by

$$u_i(t) = K_{x_i}(t)x_i(t) + p_{x_i}(t) \tag{6.9}$$

$$z_i(t) = K_{z_i}(t)x_i(t) + p_{z_i}(t), \quad t \in [t_0, t_f] \tag{6.10}$$

with $K_{x_i} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{m_i \times n_i})$, $K_{z_i} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{q_i \times n_i})$, $p_{x_i} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{m_i})$ and $p_{z_i} \in \mathscr{C}(t_0,t_f;\mathbb{R}^{q_i})$ admissible feedback policy parameters whose further defining properties will be stated shortly.

For the given (t_0, x_{ai}^0) and subject to the feedback control policies (6.9)-(6.10), agent i forms a local awareness of its state recursion (6.3) as follows

$$dx_{ai}(t) = (A_{ai}(t)x_{ai}(t) + l_{ai}(t))dt + G_{ai}(t)dw_{ai}(t), \quad x_{ai}(t_0) = x_{ai}^0 \tag{6.11}$$

in which the aggregate Wiener process $w_{ai}(t) \triangleq \left[\, w_i^T(t)\ v_c^T(t) \,\right]^T$ has the correlations of independent increments $E\left\{[w_{ai}(\tau_1) - w_{ai}(\tau_2)][w_{ai}(\tau_1) - w_{ai}(\tau_2)]^T\right\} = W_{ai}|\tau_1 - \tau_2|$ for all $\tau_1, \tau_2 \in [t_0, t_f]$ and $W_{ai} > 0$; whereas the augmented state variable x_{ai}, its initial-valued condition x_{ai}^0, the system coefficients and parameters are defined by

$$x_{ai}(t) \triangleq \begin{bmatrix} x_i(t) \\ z_{ic}(t) \end{bmatrix}; \quad x_{ai}^0 \triangleq \begin{bmatrix} x_i^0 \\ 0 \end{bmatrix}; \quad G_{ai}(t) \triangleq \begin{bmatrix} G_i(t) & 0 \\ 0 & G_{ic}(t) \end{bmatrix}; \quad W_{ai} \triangleq \begin{bmatrix} W_i & 0 \\ 0 & V_c \end{bmatrix}$$

$$A_{ai}(t) \triangleq \begin{bmatrix} A_{ii}(t) + B_{ii}(t)K_{x_i}(t) + C_{ii}(t)K_{z_i}(t) & 0 \\ 0 & A_{ic}(t) \end{bmatrix}$$

$$l_{ai}(t) \triangleq \begin{bmatrix} B_{ii}(t)p_{x_i}(t) + C_{ii}(t)p_{z_i}(t) + \sum_{j=1}^{N_i} B_{ij}(t)u_{ij}^*(t) \\ B_{ic}(t)u_c(t) \end{bmatrix}.$$

Moreover, the sample function of the random performance measure (6.7) becomes

$$\begin{aligned} J_i(K_{x_i}, p_{x_i}; K_{z_i}, p_{z_i}) &= x_{ai}^T(t_f)Q_{ai}^f x_{ai}(t_f) + 2x_{ai}^T(t_f)S_{ai}^f + \zeta_i^T(t_f)Q_i^f\zeta_i(t_f) \\ &+ \int_{t_0}^{t_f} [x_{ai}^T(\tau)Q_{ai}(\tau)x_{ai}(\tau) + 2x_{ai}^T(\tau)S_{ai}(\tau) + \zeta_i^T(\tau)Q_i(\tau)\zeta_i(\tau) \\ &+ p_{x_i}^T(\tau)R_{ii}(\tau)p_{x_i}(\tau) + p_{z_i}^T(\tau)R_{zi}(\tau)p_{z_i}(\tau)]d\tau \end{aligned} \quad (6.12)$$

whereby the corresponding weightings are given by

$$Q_{ai}^f \triangleq \begin{bmatrix} Q_i^f & 0 \\ 0 & 0 \end{bmatrix}; \quad S_{ai}^f \triangleq \begin{bmatrix} -Q_i^f\zeta_i(t_f) \\ 0 \end{bmatrix}; \quad S_{ai} \triangleq \begin{bmatrix} K_{x_i}^T R_{ii} p_{x_i} + K_{z_i}^T R_{zi} p_{z_i} - Q_i\zeta_i \\ -R_{zi}p_{z_i} \end{bmatrix}$$

$$Q_{ai} \triangleq \begin{bmatrix} Q_{ii} + Q_i + K_{x_i}^T R_{ii} K_{x_i} + K_{z_i}^T R_{zi} K_{z_i} & -2K_{z_i}^T R_{zi} \\ 0 & R_{zi} \end{bmatrix}.$$

In views of the linear-quadratic structure of the problem (6.11) and (6.12), the performance measure (6.12) is clearly a random variable with chi-squared type. Consequently, agent i adjusts its objective values into an abstract notion of satisfaction. In this research, performance expectations and risks are incorporated into a multi-attribute utility function. Then, the next task involves measuring a finite number of higher-order statistics associated with (6.12) or efficiently computing them to understand their importance. Fortunately, the research on performance assessment uncertainty offers a body of mathematical constructs that provides a starting point for such a knowledge extraction in terms of performance-measure statistics [6] and [7].

Theorem 1 *Performance-Measure Statistics.*
Let the pairs (A_{ii}, B_{ii}) and (A_{ii}, C_{ii}) be uniformly stabilizable on $[t_0, t_f]$ in the incumbent system i and $i \in \overline{I}$ governed by (6.11) and (6.12). Then for the given initial condition (t_0, x_i^0), incumbent agent i obtains the k_i-th cumulant associated with (6.12)

$$\kappa_{k_i}^i = (x_{ai}^0)^T H_i(t_0, k_i)x_{ai}^0 + 2(x_{ai}^0)^T \breve{D}_i(t_0, k_i) + D_i(t_0, k_i), \qquad k_i \in \mathbb{N} \quad (6.13)$$

whereby the supporting variables $\{H_i(s,r)\}_{r=1}^{k_i}$, $\{\breve{D}_i(s,r)\}_{r=1}^{k_i}$ and $\{D_i(s,r)\}_{r=1}^{k_i}$ satisfy the time-backward differential equations (with the dependence of $H_i(s,r)$, $\breve{D}_i(s,r)$ and $D_i(s,r)$ upon the admissible K_{x_i}, K_{z_i}, p_{x_i} and p_{z_i} suppressed)

$$\frac{d}{ds}H_i(s,1) = -A_{ai}^T(s)H_i(s,1) - H_i(s,1)A_{ai}(s) - Q_{ai}(s) \tag{6.14}$$

$$\frac{d}{ds}H_i(s,r) = -A_{ai}^T(s)H_i(s,r) - H_i(s,r)A_{ai}(s) - \sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}H_i(s,v)G_{ai}(s)W_{ai}G_{ai}^T(s)H_i(s,r-v), \quad 2 \le r \le k_i \tag{6.15}$$

$$\frac{d}{ds}\breve{D}_i(s,1) = -A_{ai}^T(s)\breve{D}_i(s,1) - H_i(s,1)l_{ai}(s) - S_{ai}(s) \tag{6.16}$$

$$\frac{d}{ds}\breve{D}_i(s,r) = -A_{ai}^T(s)\breve{D}_i(s,r) - H_i(s,r)l_{ai}(s), \quad 2 \le r \le k_i \tag{6.17}$$

$$\frac{d}{ds}D_i(s,1) = -\mathrm{Tr}\{H_i(s,1)G_{ai}(s)W_{ai}G_{ai}^T(s)\} - 2\breve{D}_i^T(s,1)l_{ai}(s) - p_{x_i}^T(s)R_{ii}(s)p_{x_i}(s) - p_{z_i}^T(s)R_{zi}(s)p_{z_i}(s) - \zeta_i^T(s)Q_i(s)\zeta_i(s) \tag{6.18}$$

$$\frac{d}{ds}D_i(s,r) = -\mathrm{Tr}\{H_i(s,r)G_{ai}(s)W_{ai}G_{ai}^T(s)\} - 2\breve{D}_i^T(s,r)l_{ai}(s), 2 \le r \le k_i \tag{6.19}$$

whereby the terminal-value conditions $H_i(t_f,1) = Q_{ai}^f$, $H_i(t_f,r) = 0$ *for* $2 \le r \le k_i$; $\breve{D}_i(t_f,1) = S_{ai}^f$, $\breve{D}_i(t_f,r) = 0$ *for* $2 \le r \le k_i$; *and* $D_i(t_f,1) = \zeta_i^T(t_f)Q_i^f\zeta_i(t_f)$, $D_i(t_f,r) = 0$ *for* $2 \le r \le k_i$.

Proof. In general, the initial condition (t_0, x_{ai}^0) is parameterized by any arbitrary pair (s, x_{ai}^s). Then, for the given admissible affine inputs p_{x_i} and p_{z_i} in addition with admissible feedback gains K_{x_i} and K_{z_i}, the "running" performance measure is introduced as follows

$$J_i(s,x_{ai}^s) = x_{ai}^T(t_f)Q_{ai}^f x_{ai}(t_f) + 2x_{ai}^T(t_f)S_{ai}^f + \zeta_i^T(t_f)Q_i^f\zeta_i(t_f) + \int_s^{t_f}[x_{ai}^T(\tau)Q_{ai}(\tau)x_{ai}(\tau) + 2x_{ai}^T(\tau)S_{ai}(\tau) + \zeta_i^T(\tau)Q_i(\tau)\zeta_i(\tau) + p_{x_i}^T(\tau)R_{ii}(\tau)p_{x_i}(\tau) + p_{z_i}^T(\tau)R_{zi}(\tau)p_{z_i}(\tau)]d\tau. \tag{6.20}$$

The moment-generating function associated with agent i of (6.20) is defined by

$$\varphi_i(s,x_{ai}^s;\theta_i) \triangleq E\left\{\exp\left(\theta_i J_i\left(s,x_{ai}^s\right)\right)\right\}, \tag{6.21}$$

for some small parameters θ_i in an open interval about 0. Thus, the cumulant-generating function immediately follows

$$\psi_i\left(s,x_{ai}^s;\theta_i\right) \triangleq \ln\left\{\varphi_i\left(s,x_{ai}^s;\theta_i\right)\right\}, \tag{6.22}$$

for some θ_i in some (possibly smaller) open interval about 0 while $\ln\{\cdot\}$ denotes the natural logarithmic transformation.

For notational simplicity, it is convenient to define $\varpi_i\left(s,x_{ai}^s;\theta_i\right) \triangleq \exp\left\{\theta_i J_i\left(s,x_{ai}^s\right)\right\}$ and $\varphi_i\left(s,x_{ai}^s;\theta_i\right) \triangleq E\left\{\varpi_i\left(s,x_{ai}^s;\theta_i\right)\right\}$ together with the time derivative of

$$\frac{d}{ds}\varphi_i(s,x_{ai}^s;\theta_i) = -\theta_i\Big\{(x_{ai}^s)^T Q_{ai}(s)x_{ai}^s + 2(x_{ai}^s)^T S_{ai}(s) + \zeta_i^T(s)Q_i(s)\zeta_i(s) + p_{x_i}^T(s)R_{ii}(s)p_{x_i}(s) + p_{z_i}^T(s)R_{zi}(s)p_{z_i}(s)\Big\}\varphi_i(s,x_{ai}^s;\theta_i). \quad (6.23)$$

Using the standard Ito's formula, it yields

$$\begin{aligned} d\varphi_i(s,x_{ai}^s;\theta_i) &= E\{d\varpi_i(s,x_{ai}^s;\theta_i)\}, \\ &= \varphi_{i,s}(s,x_{ai}^s;\theta_i)\,ds + \varphi_{i,x_{ai}^s}(s,x_{ai}^s;\theta_i)\,[A_{ai}(s)x_{ai}^s + l_{ai}(s)]\,ds \\ &\quad + \frac{1}{2}\mathrm{Tr}\{\varphi_{i,x_{ai}^s x_{ai}^s}(s,x_{ai}^s;\theta_i)\,G_{ai}(s)\,W_{ai}G_{ai}^T(s)\}ds. \end{aligned}$$

Furthermore, the moment-generating function of (6.20) can also be expressed by

$$\varphi_i(s,x_{ai}^s;\theta_i) \triangleq \rho_i(s;\theta_i)\exp\left\{(x_{ai}^s)^T\Upsilon_i(s;\theta_i)x_{ai}^s + 2(x_{ai}^s)^T\eta_i(s;\theta_i)\right\} \quad (6.24)$$

whereby all the supporting entities are going to be determined in the sequel. In particular, the partial derivatives of (6.24) results in

$$\begin{aligned} \frac{d}{ds}\varphi_i(s,x_{ai}^s;\theta_i) &= \Bigg\{\frac{\frac{d}{ds}\rho_i(s;\theta_i)}{\rho_i(s,\theta_i)} + (x_{ai}^s)^T\frac{d}{ds}\Upsilon_i(s;\theta_i)x_{ai}^s + 2(x_{ai}^s)^T\frac{d}{ds}\eta_i(s;\theta_i) \\ &\quad + (x_{ai}^s)^T A_{ai}^T(s)\Upsilon_i(s;\theta_i)x_{ai}^s + (x_{ai}^s)^T\Upsilon_i(s;\theta_i)A_{ai}(s)x_{ai}^s + 2(x_{ai}^s)^T A_{ai}^T(s)\eta_i(s;\theta_i) \\ &\quad + 2(x_{ai}^s)^T\Upsilon_i(s;\theta_i)l_{ai}(s) + 2\eta_i^T(s;\theta_i)l_{ai}(s) + \mathrm{Tr}\{\Upsilon_i(s;\theta_i)G_{ai}(s)W_{ai}G_{ai}^T(s)\} \\ &\quad + 2(x_{ai}^s)^T\Upsilon_i(s;\theta_i)G_{ai}(s)W_{ai}G_{ai}^T(s)\Upsilon_i(s;\theta_i)x_{ai}^s\Bigg\}\varphi_i(s,x_{ai}^s;\theta_i). \end{aligned} \quad (6.25)$$

Equating the expression (6.23) with that of (6.25) and having both linear and quadratic terms independent of x_{ai}^s yield the following results, wherein $\upsilon_i(s;\theta_i) \triangleq \ln\{\rho_i(s;\theta_i)\}$

$$\begin{aligned} \frac{d}{ds}\Upsilon_i(s;\theta_i) &= -A_{ai}^T(s)\Upsilon_i(s;\theta_i) - \Upsilon_i(s;\theta_i)A_{ai}(s) \\ &\quad - 2\Upsilon_i(s;\theta_i)G_{ai}(s)W_{ai}G_{ai}^T(s)\Upsilon_i(s;\theta_i) - \theta_i Q_{ai}(s) \end{aligned} \quad (6.26)$$

$$\frac{d}{ds}\eta_i(s;\theta_i) = -A_{ai}^T(s)\eta_i(s;\theta_i) - \Upsilon_i(s;\theta_i)l_{ai}(s) - \theta_i S_{ai}(s) \quad (6.27)$$

$$\begin{aligned} \frac{d}{ds}\upsilon_i(s;\theta_i) &= -\mathrm{Tr}\left\{\Upsilon_i(s;\theta_i)G_{ai}(s)\,W_{ai}G_{ai}^T(s)\right\} - 2\eta_i^T(s;\theta_i)l_{ai}(s) - \theta_i\zeta_i^T(s)Q_i(s)\zeta_i(s) \\ &\quad - \theta_i p_{x_i}^T(s)R_{ii}(s)p_{x_i}(s) - \theta_i p_{z_i}^T(s)R_{zi}(s)p_{z_i}(s) \end{aligned} \quad (6.28)$$

At the final time $s = t_f$, it follows that

$$\begin{aligned} \varphi_i(t_f,x_{ai}(t_f);\theta_i) &= \rho_i(t_f;\theta_i)\exp\left\{x_{ai}^T(t_f)\Upsilon_i(t_f;\theta_i)x_{ai}(t_f) + 2x_{ai}^T(t_f)\eta_i(t_f;\theta_i)\right\} \\ &= E\left\{\exp\left\{\theta_i[x_{ai}^T(t_f)Q_{ai}^f x_{ai}(t_f) + 2x_{ai}^T(t_f)S_{ai}^f + \zeta_i^T(t_f)Q_i^f\zeta_i(t_f)]\right\}\right\} \end{aligned}$$

which in turn yields the terminal-value conditions as $\Upsilon_i(t_f;\theta_i) = \theta_i Q_{ai}^f$; $\eta_i(t_f;\theta_i) = \theta_i S_{ai}^f$; and $\upsilon_i(t_f;\theta_i) = \theta_i\zeta_i^T(t_f)Q_i^f\zeta_i(t_f)$.

As it turns out, all the higher-order characteristic distributions associated with performance uncertainty are very well captured in higher-order performance-measure statistics associated with the chi-squared random performance measure (6.20). In views of the expression (6.24) and the definition of (6.22), the cumulant-generating function or second-order characteristic function of (6.20) is rewritten as follows

$$\psi_i\,(s, x_{ai}^s; \theta_i) = (x_{ai}^s)^T \Upsilon_i(s;\theta_i) x_{ai}^s + 2(x_{ai}^s)^T \eta_i(s;\theta_i) + \upsilon_i(s;\theta_i)\,. \tag{6.29}$$

Subsequently, higher-order statistics of the random performance measure (6.20) that depict the performance uncertainty can now be determined by a Maclaurin series expansion of the cumulant-generating function (6.29); e.g.,

$$\psi_i\,(s, x_{ai}^s; \theta_i) = \sum_{r=1}^{\infty} \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \psi_i(s, x_{ai}^s; \theta_i) \bigg|_{\theta_i = 0} \frac{\theta_i^r}{r!}\,, \tag{6.30}$$

from which all $\kappa_r \triangleq \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \psi_i(s, x_{ai}^s; \theta_i)\Big|_{\theta_i=0}$ are known as the mathematical statistics of the chi-squared random performance measure (6.20). Moreover, the series expansion coefficients are computed by using the cumulant-generating function (6.29)

$$\begin{aligned} \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \psi_i(s, x_{ai}^s; \theta_i) \bigg|_{\theta_i=0} &= (x_{ai}^s)^T \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \Upsilon_i(s;\theta_i) \bigg|_{\theta_i=0} x_{ai}^s \\ &\quad + 2(x_{ai}^s)^T \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \eta_i(s;\theta_i) \bigg|_{\theta_i=0} + \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \upsilon_i(s;\theta_i) \bigg|_{\theta_i=0}\,. \end{aligned} \tag{6.31}$$

In view of the definition (6.30), the rth performance-measure statistic is given by

$$\begin{aligned} \kappa_r = (x_{ai}^s)^T & \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \Upsilon_i(s;\theta_i) \bigg|_{\theta_i=0} x_{ai}^s \\ &\quad + 2(x_{ai}^s)^T \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \eta_i(s;\theta_i) \bigg|_{\theta_i=0} + \frac{\partial^{(r)}}{\partial \theta_i^{(r)}} \upsilon_i(s;\theta_i) \bigg|_{\theta_i=0} \end{aligned} \tag{6.32}$$

for any finite $1 \le r < \infty$. For notational convenience, the change of notations

$$H_i(s,r) \triangleq \frac{\partial^{(r)} \Upsilon_i(s;\theta_i)}{\partial \theta_i^{(r)}} \bigg|_{\theta_i=0} ; \breve{D}_i(s,r) \triangleq \frac{\partial^{(r)} \eta_i(s;\theta_i)}{\partial \theta_i^{(r)}} \bigg|_{\theta_i=0} ; D_i(s,r) \triangleq \frac{\partial^{(r)} \upsilon_i(s;\theta_i)}{\partial \theta_i^{(r)}} \bigg|_{\theta_i=0}$$

is introduced. What remains is to show that the solutions $H_i(s,r)$, $\breve{D}_i(s,r)$ and $D_i(s,r)$ for $1 \le r \le k_i$ and $k_i \in \mathbb{N}$ indeed satisfy the time-backward matrix, vector and scalar-valued differential equations (6.14)-(6.19). Notice that these differential equations

(6.14)-(6.19) are readily obtained by successively taking derivatives with respect to θ_i of the cumulant-supporting equations (6.26)-(6.28) under the assumption of (A_{ii}, B_{ii}) and (A_{ii}, C_{ii}) uniformly stabilizable on the interval $[t_0, t_f]$. □

Furthermore, some attractive properties of the solutions to the cumulant-generating equations (6.14)-(6.19), for which the problem of coordination control with risk-averse performance of the class of distributed stochastic systems considered here is therefore well-posed, are presented as follows.

Theorem 2 *Existence of Solutions for Performance-Measure Statistics.*
Let the pairs $(A_{ii}(\cdot), B_{ii}(\cdot))$ and $(A_{ii}(\cdot), C_{ii}(\cdot))$ be uniformly stabilizable. Then, for any given $k_i \in \mathbb{N}$, the cumulant-generating equations (6.14)-(6.19) admit unique and bounded solutions $\{H_i(\cdot,r)\}_{r=1}^{k_i}$, $\{\breve{D}_i(\cdot,r)\}_{r=1}^{k_i}$ and $\{D_i(\cdot,r)\}_{r=1}^{k_i}$ on $[t_0, t_f]$.

Proof. Under the assumption of stabilizability, there always exist some feedback parameters $K_{x_i}(\cdot)$ and $K_{z_i}(\cdot)$ such that the continuous-time aggregate state matrix $A_{ai}(\cdot)$ is exponentially stable on $[t_0, t_f]$. According to the results in [8], the state transition matrix $\Phi_{ai}(t,t_0)$, associated with the continuous-time composite state matrix $A_{ai}(\cdot)$, has the following properties

$$\frac{d}{dt}\Phi_{ai}(t,t_0) = A_{ai}(t)\Phi_{ai}(t,t_0)\,, \qquad \Phi_{ai}(t_0,t_0) = I\,,$$

$$\lim_{t_f \to \infty} ||\Phi_{ai}(t_f,\tau)|| = 0\,, \qquad \lim_{t_f \to \infty} \int_{t_0}^{t_f} ||\Phi_{ai}(t_f,\tau)||^2 d\tau < \infty\,.$$

By the matrix variation of constant formula, the unique solutions to the time-backward matrix differential equations (6.14)-(6.19) together with the terminal-value conditions are then written as follows

$$H_i(s,1) = \Phi_{ai}^T(t_f,s)Q_{ai}^f\Phi_{ai}(t_f,s) + \int_s^{t_f} \Phi_{ai}^T(\tau,s)Q_{ai}(\tau)\Phi_{ai}(\tau,s)d\tau$$

$$H_i(s,r) = \int_s^{t_f} \Phi_{ai}^T(\tau,s)\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}H_i(\tau,v)G_{ai}(\tau)W_{ai}G_{ai}^T(\tau)H_i(\tau,r-v)\Phi_{ai}(\tau,s)d\tau$$

$$\breve{D}_i(s,1) = -\Phi_{ai}^T(t_f,s)Q_i^f\zeta_i(t_f) + \int_s^{t_f} \Phi_{ai}^T(\tau,s)\{H_i(\tau,1)l_{ai}(\tau) + S_{ai}(\tau)\}d\tau$$

$$\breve{D}_i(s,r) = \int_s^{t_f} \Phi_{ai}^T(\tau,s)H_i(\tau,r)l_{ai}(\tau)d\tau\,, \quad 2 \le r \le k_i$$

$$\begin{aligned} D_i(s,1) = \zeta_i^T(t_f)Q_i^f\zeta_i(t_f) + \int_s^{t_f} &\{\mathrm{Tr}\{H_i(\tau,1)G_{ai}(\tau)W_{ai}G_{ai}^T(\tau)\} + 2\breve{D}_i^T(\tau,1)l_{ai}(\tau) \\ &+ p_{x_i}^T(\tau)R_{ii}(\tau)p_{x_i}(\tau) + p_{z_i}^T(\tau)R_{zi}(\tau)p_{z_i}(\tau) + \zeta_i^T(\tau)Q_i(\tau)\zeta_i(\tau)\}d\tau \end{aligned}$$

$$D_i(s,r) = \int_s^{t_f} \{\mathrm{Tr}\{H_i(\tau,r)G_{ai}(\tau)W_{ai}G_{ai}^T(\tau)\} + 2\breve{D}_i^T(\tau,r)l_{ai}(\tau)\}d\tau, \quad 2 \le r \le k_i\,.$$

As long as the growth rates of the integrals are not faster than those of exponentially decreasing $\Phi_{ai}(\cdot,\cdot)$ and $\Phi_{ai}^T(\cdot,\cdot)$ factors, it is therefore concluded that there exist upper bounds on the non-negative and monotically increasing solutions $H_i(\cdot,r)$, $\breve{D}_i(\cdot,r)$ and $D_i(\cdot,r)$ for any time interval $[t_0,t_f]$.

6.3 Problem Statements

The problem of adapting to performance uncertainty is now addressed by leveraging increased insight into the roles played by performance-measure statistics (6.13). It is interesting to note that all the performance-measure statistics (6.13) are functions of time-backward evolutions and do not depend on intermediate recursive state values $x_{ai}(t)$ governed by the state-space representation (6.11)-(6.12) for incumbent agent i at each point of time $t \in [t_0,t_f]$. Henceforth, these time-backward evolutions (6.14)-(6.19) of which the admissible decision variables K_{x_i}, K_{z_i}, p_{x_i} and p_{z_i} from the 2-tuple person-by-person equilibrium strategy (6.9)-(6.10) are embedded, are therefore considered as the new dynamical equations with the associated state variables $H_i(\cdot,r)$, $\breve{D}_i(\cdot,r)$ and $D_i(\cdot,r)$, not the traditional system states $x_{ai}(\cdot)$.

To properly develop the problem statements within the concept of the person-by-person equilibrium strategy for agent i and $i \in \overline{I}$, the new dynamics (6.14)-(6.19) based upon the performance-measure statistics of (6.13) is rewritten in accordance of the following matrix partitions, for $1 \le r \le k_i$ and $k_i \in \mathbb{N}$

$$H_i(\cdot,r) \triangleq \begin{bmatrix} H_{i,r}^{11}(\cdot) & H_{i,r}^{12}(\cdot) \\ H_{i,r}^{21}(\cdot) & H_{i,r}^{22}(\cdot) \end{bmatrix}, \qquad \breve{D}_i(\cdot,r) \triangleq \begin{bmatrix} \breve{D}_{i,r}^{11}(\cdot) \\ \breve{D}_{i,r}^{21}(\cdot) \end{bmatrix}.$$

For notational simplicity, it is now useful to denote the right members of the dynamics (6.14)-(6.19) as the mappings

$$\begin{aligned}
\mathscr{F}_i^r &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \times \mathbb{R}^{m_i\times n_i} \times \mathbb{R}^{q_i\times n_i} \mapsto \mathbb{R}^{n_i\times n_i} \\
\mathscr{F}_i^{k_i+r} &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \times \mathbb{R}^{m_i\times n_i} \times \mathbb{R}^{q_i\times n_i} \mapsto \mathbb{R}^{n_i\times n_i} \\
\mathscr{F}_i^{2k_i+r} &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \times \mathbb{R}^{m_i\times n_i} \times \mathbb{R}^{q_i\times n_i} \mapsto \mathbb{R}^{n_i\times n_i} \\
\mathscr{F}_i^{3k_i+r} &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \mapsto \mathbb{R}^{n_i\times n_i} \\
\breve{\mathscr{G}}_i^r &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{m_i\times n_i} \times \mathbb{R}^{q_i\times n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \mapsto \mathbb{R}^{n_i} \\
\breve{\mathscr{G}}_i^{k_i+r} &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \mapsto \mathbb{R}^{n_i} \\
\mathscr{G}_i^r &: [t_0,t_f] \times (\mathbb{R}^{n_i\times n_i})^{4k_i} \times (\mathbb{R}^{n_i})^{2k_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \mapsto \mathbb{R}
\end{aligned}$$

with the rules of action

$$\begin{aligned}
\mathscr{F}_i^1(s,\mathscr{H}_i,K_{x_i},K_{z_i}) \triangleq &-[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]^T\mathscr{H}_i^1(s) \\
&-\mathscr{H}_i^1(s)[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)] \\
&-Q_{ii}(s)-Q_i(s)-K_{x_i}^T(s)R_{ii}(s)K_{x_i}(s)-K_{z_i}^T(s)R_{zi}(s)K_{z_i}(s)
\end{aligned}$$

$$\begin{aligned}\mathscr{F}_i^r(s,\mathscr{H}_i,K_{x_i},K_{z_i}) \triangleq& -[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]^T\mathscr{H}_i^r(s)\\ &-\mathscr{H}_i^r(s)[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^v(s)G_i(s)W_iG_i^T(s)\mathscr{H}_i^{r-v}(s)\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{k_i+v}(s)G_{ic}(s)V_cG_{ic}^T(s)\mathscr{H}_i^{2k_i+r-v}(s)\end{aligned}$$

$$\begin{aligned}\mathscr{F}_i^{k_i+1}(s,\mathscr{H}_i,K_{x_i},K_{z_i}) \triangleq& -[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]^T\mathscr{H}_i^{k_i+1}(s)\\ &-\mathscr{H}_i^{k_i+1}(s)A_{ic}(s)+2K_{z_i}^T(s)R_{zi}(s)\end{aligned}$$

$$\begin{aligned}\mathscr{F}_i^{k_i+r}(s,\mathscr{H}_i,K_{x_i},K_{z_i}) \triangleq& -[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]^T\mathscr{H}_i^{k_i+r}(s)\\ &-\mathscr{H}_i^{k_i+r}(s)A_{ic}(s)-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^v(s)G_i(s)W_iG_i^T(s)\mathscr{H}_i^{k_i+r-v}(s)\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{k_i+v}(s)G_{ic}(s)V_cG_{ic}^T(s)\mathscr{H}_i^{3k_i+r-v}(s)\end{aligned}$$

$$\begin{aligned}\mathscr{F}_i^{2k_i+1}(s,\mathscr{H}_i,K_{x_i},K_{z_i}) \triangleq& -A_{ic}^T(s)\mathscr{H}_i^{2k_i+1}(s)\\ &-\mathscr{H}_i^{2k_i+1}(s)[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]\end{aligned}$$

$$\begin{aligned}\mathscr{F}_i^{2k_i+r}(s,\mathscr{H}_i,K_{x_i},K_{z_i}) \triangleq& -A_{ic}^T(s)\mathscr{H}_i^{2k_i+r}(s)\\ &-\mathscr{H}_i^{2k_i+r}(s)[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{2k_i+v}(s)G_i(s)W_iG_i^T(s)\mathscr{H}_i^{r-v}(s)\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{3k_i+v}(s)G_{ic}(s)V_cG_{ic}^T(s)\mathscr{H}_i^{2k_i+r-v}(s)]\end{aligned}$$

$$\mathscr{F}_i^{3k_i+1}(s,\mathscr{H}_i) \triangleq -A_{ic}^T(s)\mathscr{H}_i^{3k_i+1}(s)-\mathscr{H}_i^{3k_i+1}(s)A_{ic}(s)-R_{zi}(s)$$

$$\begin{aligned}\mathscr{F}_i^{3k_i+r}(s,\mathscr{H}_i) \triangleq& -A_{ic}^T(s)\mathscr{H}_i^{3k_i+r}(s)-\mathscr{H}_i^{3k_i+r}(s)A_{ic}(s)-R_{zi}(s)\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{2k_i+v}(s)G_i(s)W_iG_i^T(s)\mathscr{H}_i^{k_i+r-v}(s)\\ &-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{3k_i+v}(s)G_{ic}(s)V_cG_{ic}^T(s)\mathscr{H}_i^{3k_i+r-v}(s)\end{aligned}$$

$$\breve{\mathcal{G}}_i^1(s,\mathcal{H}_i,\breve{\mathcal{D}}_i,K_{x_i},K_{z_i},p_{x_i},p_{z_i}) \triangleq -[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]^T\breve{\mathcal{D}}_i^1(s)$$
$$-\mathcal{H}_i^1(s)[B_{ii}(s)p_{x_i}(s)+C_{ii}(s)p_{z_i}(s)+\sum_{j=1}^{N_i}B_{ij}(s)u_{ij}^*(s)]$$
$$-\mathcal{H}_i^{k_i+1}(s)B_{ic}(s)u_{ic}(s)-K_{x_i}^T(s)R_{ii}(s)p_{x_i}(s)-K_{z_i}^T R_{zi}(s)p_{z_i}(s)+Q_i(s)\zeta_i(s)$$

$$\breve{\mathcal{G}}_i^r(s,\mathcal{H}_i,\breve{\mathcal{D}}_i,K_{x_i},K_{z_i},p_{x_i},p_{z_i}) \triangleq -[A_{ii}(s)+B_{ii}(s)K_{x_i}(s)+C_{ii}(s)K_{z_i}(s)]^T\breve{\mathcal{D}}_i^r(s)$$
$$-\mathcal{H}_i^r(s)[B_{ii}(s)p_{x_i}(s)+C_{ii}(s)p_{z_i}(s)+\sum_{j=1}^{N_i}B_{ij}(s)u_{ij}^*(s)]-\mathcal{H}_i^{k_i+r}(s)B_{ic}(s)u_{ic}(s)$$

$$\breve{\mathcal{G}}_i^{k_i+1}(s,\mathcal{H}_i,\breve{\mathcal{D}}_i,p_{x_i},p_{z_i}) \triangleq -A_{ic}^T(s)\breve{\mathcal{D}}_i^{k_i+1}(s)-\mathcal{H}_i^{3k_i+1}(s)B_{ic}(s)u_{ic}(s)+R_{zi}(s)p_{z_i}(s)$$
$$-\mathcal{H}_i^{2k_i+1}(s)[B_{ii}(s)p_{x_i}(s)+C_{ii}(s)p_{z_i}(s)+\sum_{j=1}^{N_i}B_{ij}(s)u_{ij}^*(s)]$$

$$\breve{\mathcal{G}}_i^{k_i+r}(s,\mathcal{H}_i,\breve{\mathcal{D}}_i,p_{x_i},p_{z_i}) \triangleq -A_{ic}^T(s)\breve{\mathcal{D}}_i^{k_i+r}(s)-\mathcal{H}_i^{3k_i+r}(s)B_{ic}(s)u_{ic}(s)$$
$$-\mathcal{H}_i^{2k_i+r}(s)[B_{ii}(s)p_{x_i}(s)+C_{ii}(s)p_{z_i}(s)+\sum_{j=1}^{N_i}B_{ij}(s)u_{ij}^*(s)]$$

$$\mathcal{G}_i^1(s,\mathcal{H}_i,\breve{\mathcal{D}}_i,p_{x_i},p_{z_i}) \triangleq -\mathrm{Tr}\{\mathcal{H}_i^1(s)G_i(s)W_iG_i^T(s)+\mathcal{H}_i^{3k_i+1}(s)G_{ic}(s)V_cG_{ic}^T(s)\}$$
$$-2(\breve{\mathcal{D}}_i^1)^T(s)[B_{ii}(s)p_{x_i}(s)+C_{ii}(s)p_{z_i}(s)+\sum_{j=1}^{N_i}B_{ij}(s)u_{ij}^*(s)]-2(\breve{\mathcal{D}}_i^{k_i+1})^T(s)B_{ic}(s)u_{ic}(s)$$
$$-\zeta_i^T(s)Q_i(s)\zeta_i(s)-p_{x_i}^T(s)R_{ii}(s)p_{x_i}(s)-p_{z_i}^T(s)R_{zi}(s)p_{z_i}(s)$$

$$\mathcal{G}_i^r(s,\mathcal{H}_i,\breve{\mathcal{D}}_i,p_{x_i},p_{z_i}) \triangleq -\mathrm{Tr}\{\mathcal{H}_i^r(s)G_i(s)W_iG_i^T(s)+\mathcal{H}_i^{3k_i+r}(s)G_{ic}(s)V_cG_{ic}^T(s)\}$$
$$-2(\breve{\mathcal{D}}_i^r)^T(s)[B_{ii}(s)p_{x_i}(s)+C_{ii}(s)p_{z_i}(s)+\sum_{j=1}^{N_i}B_{ij}(s)u_{ij}^*(s)]-2(\breve{\mathcal{D}}_i^{k_i+r})^T(s)B_{ic}(s)u_{ic}(s)$$

whereby the components of $4k_i$-tuple $\mathcal{H}_i$, $2k_i$-tuple $\breve{\mathcal{D}}_i$ and k_i-tuple $\mathcal{D}_i$ variables are defined by

$$\mathcal{H}_i \triangleq (\mathcal{H}_i^1,\ldots,\mathcal{H}_i^{k_i},\mathcal{H}_i^{k_i+1},\ldots,\mathcal{H}_i^{2k_i},\mathcal{H}_i^{2k_i+1},\ldots,\mathcal{H}_i^{3k_i},\mathcal{H}_i^{3k_i+1},\ldots,\mathcal{H}_i^{4k_i})$$
$$\equiv (H_{i,1}^{11},\ldots,H_{i,k_i}^{11},H_{i,1}^{12},\ldots,H_{i,k_i}^{12},H_{i,1}^{21},\ldots,H_{i,k_i}^{21},H_{i,1}^{22},\ldots,H_{i,k_i}^{22}),$$
$$\breve{\mathcal{D}}_i \triangleq (\breve{\mathcal{D}}_i^1,\ldots,\breve{\mathcal{D}}_i^{k_i},\breve{\mathcal{D}}_i^{k_i+1},\ldots,\breve{\mathcal{D}}_i^{2k_i}) \equiv (\breve{D}_{i,1}^{11},\ldots,\breve{D}_{i,k_i}^{11},\breve{D}_{i,1}^{21},\ldots,\breve{D}_{i,k_i}^{21})$$
$$\mathcal{D}_i \triangleq (\mathcal{D}_i^1,\ldots,\mathcal{D}_i^{k_i}) \equiv (D_i^1,\ldots,D_i^{k_i}).$$

Hence, the product system of dynamical equations, whose mapping are
$\mathscr{F}_i^1 \times \cdots \times \mathscr{F}_i^{4k_i} : [t_0,t_f] \times (\mathbb{R}^{n_i \times n_i})^{4k_i} \times \mathbb{R}^{m_i \times n_i} \times \mathbb{R}^{q_i \times n_i} \mapsto (\mathbb{R}^{n_i \times n_i})^{4k_i}$
$\breve{\mathscr{G}}_i^1 \times \cdots \times \breve{\mathscr{G}}_i^{2k_i} : [t_0,t_f] \times (\mathbb{R}^{n_i \times n_i})^{4k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{m_i \times n_i} \times \mathbb{R}^{q_i \times n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \mapsto (\mathbb{R}^{n_i})^{2k_i}$
$\mathscr{G}_i^1 \times \cdots \times \mathscr{G}_i^{k_i} : [t_0,t_f] \times (\mathbb{R}^{n_i \times n_i})^{4k_i} \times (\mathbb{R}^{n_i})^{2k_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \mapsto \mathbb{R}^{k_i}$

in coordination control of the problem class with performance risk aversion, becomes

$$\frac{d}{ds}\mathscr{H}_i(s) = \mathscr{F}_i(s,\mathscr{H}_i(s),K_{x_i}(s),K_{z_i}(s)), \qquad \mathscr{H}_i(t_f) = \mathscr{H}_i^f, \tag{6.33}$$

$$\frac{d}{ds}\breve{\mathscr{D}}_i(s) = \breve{\mathscr{G}}_i(s,\mathscr{H}_i(s),\breve{\mathscr{D}}_i(s),K_{x_i}(s),K_{z_i}(s),p_{x_i}(s),p_{z_i}(s)), \quad \breve{\mathscr{D}}_i(t_f) = \breve{\mathscr{D}}_i^f, \tag{6.34}$$

$$\frac{d}{ds}\breve{\mathscr{D}}_i(s) = \breve{\mathscr{G}}_i(s,\mathscr{H}_i(s),\breve{\mathscr{D}}_i(s),p_{x_i}(s),p_{z_i}(s)), \quad \breve{\mathscr{D}}_i(t_f) = \breve{\mathscr{D}}_i^f, \tag{6.35}$$

whereby $\mathscr{F}_i \triangleq \mathscr{F}_i^1 \times \cdots \times \mathscr{F}_i^{4k_i}$, $\breve{\mathscr{G}}_i \triangleq \breve{\mathscr{G}}_i^1 \times \cdots \times \breve{\mathscr{G}}_i^{2k_i}$ and $\mathscr{G}_i \triangleq \mathscr{G}_i^1 \times \cdots \times \mathscr{G}_i^{k_i}$, in addition with the product system of the terminal-value conditions

$$\mathscr{H}_i^f \triangleq Q_i^f \times \underbrace{0 \times \cdots \times 0}_{(k_i-1)\text{-times}} \times \underbrace{0 \times \cdots \times 0}_{k_i\text{-times}} \times \underbrace{0 \times \cdots \times 0}_{k_i\text{-times}} \times \underbrace{0 \times \cdots \times 0}_{k_i\text{-times}}$$

$$\breve{\mathscr{D}}_i^f \triangleq -Q_i^f \zeta_i(t_f) \times \underbrace{0 \times \cdots \times 0}_{(k_i-1)\text{-times}} \times \underbrace{0 \times \cdots \times 0}_{k_i\text{-times}}$$

$$\mathscr{D}_i^f \triangleq \zeta_i^T(t_f) Q_i^f \zeta_i(t_f) \times \underbrace{0 \times \cdots \times 0}_{(k_i-1)\text{-times}} .$$

Once immediate neighbors $j \in N_i$ of agent i fix the corresponding person-by-person equilibrium strategies u_j^* and thus the signaling or coordination effects u_{-i}^*, agent i then obtains an optimal stochastic control problem with risk-averse performance considerations. The construction of agent i's person-by-person policy now involves the 4-tuple $(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i})$. Furthermore, the solutions of the equations (6.33)-(6.35) also depend on the admissible feedback gains K_{x_i} and K_{z_i}, in addition with the affine inputs p_{x_i} and p_{z_i}. In the sequel and elsewhere, when this dependence is needed to be clear, then the notations $\mathscr{H}_i(s,K_{x_i},K_{z_i};u_{-i}^*)$, $\breve{\mathscr{D}}_i(s,K_{x_i},K_{z_i},p_{x_i},p_{z_i};u_{-i}^*)$ and $\mathscr{D}_i(s,K_{x_i},K_{z_i},p_{x_i},p_{z_i};u_{-i}^*)$ should be used to denote the solution trajectories of the dynamics (6.33)-(6.35) with the admissible 5-tuple $(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}; u_{-i}^*)$.

For the given terminal data $(t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f)$, the theoretical framework for risk-averse control of the distributed stochastic system with possibly noncooperative u_{-i}^*, is then analyzed by a class of admissible feedback policies employed by agent i.

Definition 1 *Admissible Feedback Policies.*
Let compact subsets $\overline{K}^{x_i} \subset \mathbb{R}^{m_i \times n_i}$, $\overline{K}^{z_i} \subset \mathbb{R}^{q_i \times n_i}$, $\overline{P}^{x_i} \subset \mathbb{R}^{m_i}$ *and* $\overline{P}^{z_i} \subset \mathbb{R}^{q_i}$ *be the sets of allowable feedback form values available at agent* i *and* $i \in \overline{I}$. *For the*

given $k_i \in \mathbb{N}$ and sequence $\mu_i = \{\mu_r^i \geq 0\}_{r=1}^{k_i}$ with $\mu_1^i > 0$, the set of feedback gains $\mathscr{K}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $\mathscr{K}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $\mathscr{P}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$ and $\mathscr{P}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$ are assumed to be the classes of $\mathscr{C}(t_0,t_f;\mathbb{R}^{m_i \times n_i})$, $\mathscr{C}(t_0,t_f;\mathbb{R}^{q_i \times n_i})$, $\mathscr{C}(t_0,t_f;\mathbb{R}^{m_i})$ and $\mathscr{C}(t_0,t_f;\mathbb{R}^{q_i})$ with values $K_{x_i}(\cdot) \in \overline{K}^{x_i}$, $K_{z_i}(\cdot) \in \overline{K}^{z_i}$, $p_{x_i}(\cdot) \in \overline{P}^{x_i}$ and $p_{z_i}(\cdot) \in \overline{P}^{z_i}$, for which the solutions to the dynamic equations (6.33)-(6.35) with the terminal-value conditions $\mathscr{H}_i(t_f) = \mathscr{H}_i^f$, $\breve{\mathscr{D}}_i(t_f) = \breve{\mathscr{D}}_i^f$ and $\mathscr{D}_i(t_f) = \mathscr{D}_i^f$ exist on the interval of optimization $[t_0,t_f]$.

To determine agent i's the person-by-person equilibrium strategy with risk bearing so as to minimize its performance vulnerability of (6.12) against all the sample-path realizations from uncertain environments w_{ai} and noncooperative coordination u^*_{-i} from immediate neighbors j and $j \in N_i$, performance risks are henceforth interpreted as worries and fears about certain undesirable characteristics of performance distributions of (6.12) and thus are proposed to manage through a finite set of selective weights. This custom set of design freedoms representing particular uncertainty aversions is hence different from the ones with aversion to risk captured in risk-sensitive optimal control [9] and [10].

On $\mathscr{K}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $\mathscr{K}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $\mathscr{P}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$ and $\mathscr{P}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, the performance index with risk-value considerations in risk-averse decision making is subsequently defined as follows.

Definition 2 *Risk-Value Aware Performance Index.*
Let incumbent agent i and $i \in \overline{I}$ select $k_i \in \mathbb{N}$ and the sequence of scalar coefficients $\mu_i = \{\mu_r^i \geq 0\}_{r=1}^{k_i}$ with $\mu_1^i > 0$. Then for the given initial condition (t_0,x_i^0), the risk-value aware performance index, $\phi_i^0 : \{t_0\} \times (\mathbb{R}^{n_i \times n_i})^{k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{k_i} \mapsto \mathbb{R}^+$ pertaining to risk-averse decision making of agent i over $[t_0,t_f]$ is defined by

$$\phi_i^0(t_0,\mathscr{H}_i(t_0),\breve{\mathscr{D}}_i(t_0),\mathscr{D}_i(t_0)) \triangleq \underbrace{\mu_1^i \kappa_1^i}_{\text{Value Measure}} + \underbrace{\mu_2^i \kappa_2^i + \cdots + \mu_{k_i}^i \kappa_{k_i}^i}_{\text{Risk Measures}}$$
$$= \sum_{r=1}^{k_i} \mu_r^i [(x_i^0)^T \mathscr{H}_i^r(t_0) x_i^0 + 2(x_i^0)^T \breve{\mathscr{D}}_i^r(t_0) + \mathscr{D}_i^r(t_0)], \quad (6.36)$$

wherein the additional design freedom by means of μ_r^i's utilized by agent i with risk-averse attitudes are sufficient to meet and exceed different levels of performance-based reliability requirements, for instance, mean (i.e., the average of performance measure), variance (i.e., the dispersion of values of performance measure around its mean), skewness (i.e., the anti-symmetry of the density of performance measure), kurtosis (i.e., the heaviness in the density tails of performance measure), etc., pertaining to closed-loop performance variations and uncertainties while the supporting solutions $\{\mathscr{H}_i^r(s)\}_{r=1}^{k_i}$, $\{\breve{\mathscr{D}}_i^r(s)\}_{r=1}^{k_i}$ and $\{\mathscr{D}_i^r(s)\}_{r=1}^{k_i}$ evaluated at $s = t_0$ satisfy the dynamical equations (6.33)-(6.35).

To specifically indicate the dependence of the risk-value aware performance index (6.36) expressed in Mayer form on $(K_{x_i},K_{z_i},p_{x_i},p_{z_i})$ and the signaling effects u^*_{-i} issued by all immediate neighbors j from N_i, the multi-attribute utility function or performance index (6.36) for agent i is now rewritten explicitly as $\phi_i^0(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}; u_{-i}^*)$.

Definition 3 *Nash Equilibrium Solution.*
An N-tuple of policies $\{(K_{x_1}^*, K_{z_1}^*, p_{x_1}^*, p_{z_1}^*), \ldots, (K_{x_N}^*, K_{z_N}^*, p_{x_N}^*, p_{z_N}^*)\}$ *is said to constitute a Nash equilibrium solution for the distributed N-agent stochastic game if, for all* $i \in \overline{N}$*, the Nash inequality condition holds*

$$\phi_i^0(K_{x_1}^*, K_{z_1}^*, p_{x_1}^*, p_{z_1}^*; u_{-i}^*) \leq \phi_i^0(K_{x_1}, K_{z_1}, p_{x_1}, p_{z_1}; u_{-i}^*) \,. \tag{6.37}$$

For the sake of time consistency and subgame perfection, a Nash equilibrium solution is required to have an additional property that its restriction on the interval $[t_0, \tau]$ is also a Nash solution to the truncated version of the original problem, defined on $[t_0, \tau]$. With such a restriction so defined, the Nash equilibrium solution is now termed as a feedback Nash equilibrium solution, which is now free of any informational nonuniqueness, and thus whose derivation allows a dynamic programming type argument.

Definition 4 *Feedback Nash Equilibrium.*
Let $(K_{x_i}^*, K_{z_i}^*, p_{x_i}^*, p_{z_i}^*)$ *constitute a feedback Nash strategy for agent i such that*

$$\phi_i^0(K_{x_i}^*, K_{z_i}^*, p_{x_i}^*, p_{z_i}^*; u_{-i}^*) \leq \phi_i^0(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}; u_{-i}^*) \,, \qquad i \in \overline{I} \tag{6.38}$$

for admissible $K_{x_i} \in \mathscr{K}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, $K_{z_i} \in \mathscr{K}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, $p_{x_i} \in \mathscr{P}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$ *and* $p_{z_i} \in \mathscr{P}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, *upon which the solutions to the dynamical systems (6.33)-(6.35) exist on* $[t_0, t_f]$.

Then, $\{(K_{x_1}^*, K_{z_1}^*, p_{x_1}^*, p_{z_1}^*), \ldots, (K_{x_N}^*, K_{z_N}^*, p_{x_N}^*, p_{z_N}^*)\}$ *when restricted to the interval* $[t_0, \tau]$ *is still a N-tuple feedback Nash equilibrium solution for the multiperson Nash decision problem with the appropriate terminal-value condition* $(\tau, \mathscr{H}_i^*(\tau), \breve{\mathscr{D}}_i^*(\tau), \mathscr{D}_i^*(\tau))$ *for all* $\tau \in [t_0, t_f]$.

In conformity with the rigorous formulation of dynamic programming, the following development is important. Let the terminal time t_f and 3-tuple states $(\mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f)$, the other end condition involved the initial time t_0 and 3-tuple states $(\mathscr{H}_i^0, \breve{\mathscr{D}}_i^0, \mathscr{D}_i^0)$ be specified by a target set requirement.

Definition 5 *Target Sets.*
$(t_0, \mathscr{H}_i^0, \breve{\mathscr{D}}_i^0, \mathscr{D}_i^0) \in \mathscr{M}_i$, *where the* target set $\mathscr{M}_i$ *is a closed subset of* $[t_0, t_f] \times (\mathbb{R}^{n_i \times n_i})^{k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{k_i}$.

Now, the decision optimization residing at incumbent agent i is to minimize the risk-value aware performance index (6.36) over admissible feedback strategies composed by $K_{x_i} \equiv K_{x_i}(\cdot) \in \mathscr{K}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, $K_{z_i} \equiv K_{z_i}(\cdot) \in \mathscr{K}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, $p_{x_i} \equiv p_{x_i}(\cdot) \in \mathscr{P}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$ and $p_{z_i} \equiv p_{z_i}(\cdot) \in \mathscr{P}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$ while subject to interconnection links from all immediate neighbors with corresponding feedback Nash policies u_{-i}^*.

Definition 6 *Optimization of Mayer Problem.*
Given the sequence of scalars $\mu_i = \{\mu_r^i \geq 0\}_{r=1}^{k^i}$ with $\mu_1^i > 0$, the decision optimization over $[t_0, t_f]$ at agent i and $i \in \overline{I}$ is given by

$$\min_{K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}} \phi_i^0(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}; u_{-i}^*) \tag{6.39}$$

subject to the dynamical equations (6.33)-(6.35) on $[t_0, t_f]$.

Notice that the optimization considered here is in Mayer form and can be solved by applying an adaptation of the Mayer form verification results as given in [11]. To embed this optimization facing agent *i* into a larger problem, the terminal time and states $(t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f)$ are parameterized as $(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$, whereby $\mathscr{Y}_i \triangleq \mathscr{H}_i(\varepsilon)$, $\breve{\mathscr{Z}}_i \triangleq \breve{\mathscr{D}}_i(\varepsilon)$ and $\mathscr{Z}_i \triangleq \mathscr{D}_i(\varepsilon)$. Thus, the value function for this optimization problem is now depending on the parameterization of terminal-value conditions.

Definition 7 *Value Function.*
Suppose $(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) \in [t_0, t_f] \times (\mathbb{R}^{n_i \times n_i})^{k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{k_i}$ is given and fixed for agent i and $i \in \overline{I}$. Then, the value function $\mathscr{V}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ is defined by

$$\mathscr{V}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) \triangleq \inf_{K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}} \phi_i^0(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}; u_{-i}^*)\,.$$

For convention, $\mathscr{V}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) \triangleq \infty$ when $\mathscr{K}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \times \mathscr{K}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \times \mathscr{P}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \times \mathscr{P}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$ is empty. Next, some candidates for the value function are constructed with the help of the concept of reachable set.

Definition 8 *Reachable Sets.*
Associate with agent i and $i \in \overline{I}$ a reachable set defined by $\mathscr{Q}_i \triangleq \Big\{(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) \in [t_0, t_f] \times (\mathbb{R}^{n_i \times n_i})^{k_i} \times (\mathbb{R}^{n_i})^{k_i} \times \mathbb{R}^{k_i}$ such that the Cartesian product $\mathscr{K}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \times \mathscr{K}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \times \mathscr{P}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \times \mathscr{P}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i} \neq \emptyset \Big\}$.

Moreover, it can be shown that the value function associated with agent *i* is satisfying a partial differential equation at interior points of $\mathscr{Q}_i$, at which it is differentiable.

Theorem 3 *Hamilton-Jacobi-Bellman (HJB) Equation-Mayer Problem.*
Let $(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ be any interior point of the reachable set $\mathscr{Q}_i$, at which the value function $\mathscr{V}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ is differentiable. If there exists a feedback Nash strategy which is supported by $K_{x_i}^*(\cdot) \in \mathscr{K}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, $K_{z_i}^*(\cdot) \in \mathscr{K}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, $p_{x_i}^*(\cdot) \in \mathscr{P}^{x_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$ and $p_{z_i}^*(\cdot) \in \mathscr{P}^{z_i}_{t_f, \mathscr{H}_i^f, \breve{\mathscr{D}}_i^f, \mathscr{D}_i^f; \mu_i}$, then the differential equation

$$
\begin{aligned}
0 = \min_{K_{x_i}\in\overline{K}^{x_i}, K_{z_i}\in\overline{K}^{z_i}, p_{x_i}\in\overline{P}^{x_i}, p_{z_i}\in\overline{P}^{z_i}} &\left\{ \frac{\partial}{\partial\varepsilon}\mathscr{V}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,\mathscr{Z}_i) \right. \\
&+ \frac{\partial}{\partial\,\mathrm{vec}(\mathscr{Y}_i)}\mathscr{V}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,\mathscr{Z}_i)\mathrm{vec}(\mathscr{F}_i(\varepsilon,\mathscr{Y}_i,K_{x_i},K_{z_i})) \\
&+ \frac{\partial}{\partial\,\mathrm{vec}(\breve{\mathscr{Z}}_i)}\mathscr{V}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,\mathscr{Z}_i)\mathrm{vec}(\breve{\mathscr{G}}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,K_{x_i},K_{z_i},p_{x_i},p_{z_i}) \\
&\left. + \frac{\partial}{\partial\,\mathrm{vec}(\mathscr{Z}_i)}\mathscr{V}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,\mathscr{Z}_i)\mathrm{vec}(\mathscr{G}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,p_{x_i},p_{z_i}) \right\}
\end{aligned} \tag{6.40}
$$

is satisfied whereby $\mathscr{V}_i(t_0,\mathscr{Y}_i(t_0),\breve{\mathscr{Z}}_i(t_0),\mathscr{Z}_i(t_0)) = \phi_i^0(\mathscr{H}_i(t_0),\breve{\mathscr{D}}_i(t_0),\mathscr{D}_i(t_0))$.

Proof. By what have been shown in the recent results by the first author [7], the proof for the result herein is readily proven.

Finally, the following result gives the sufficient condition used to verify a feedback Nash strategy for incumbent agent i and $i \in \overline{I}$.

Theorem 4 *Verification Theorem.*
Let $\mathscr{W}_i(\varepsilon,\mathscr{Y}_i,\breve{\mathscr{Z}}_i,\mathscr{Z}_i)$ *be continuously differentiable solution of the HJB equation (6.40) for agent i and* $i \in \overline{I}$*, which satisfies the boundary condition*

$$
\mathscr{W}_i(t_0,\mathscr{H}_i(t_0),\breve{\mathscr{D}}_i(t_0),\mathscr{D}_i(t_0)) = \phi_i^0(t_0,\mathscr{H}_i(t_0),\breve{\mathscr{D}}_i(t_0),\mathscr{D}_i(t_0)) \ . \tag{6.41}
$$

Let $(t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f)$ *be a 4-tuple point in* $\mathscr{Q}_i$*; let* $K_{x_i} \in \mathscr{K}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $K_{z_i} \in \mathscr{K}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $p_{x_i} \in \mathscr{P}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $p_{z_i} \in \mathscr{P}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$*; and let* $\mathscr{H}_i(\cdot)$, $\breve{\mathscr{D}}_i(\cdot)$ *and* $\mathscr{D}_i(\cdot)$ *be the corresponding solutions of the equations of motion (6.33)-(6.35). Then,* $\mathscr{W}_i(s,\mathscr{H}_i(s),\breve{\mathscr{D}}_i(s),\mathscr{D}_i(s))$ *is time-backward increasing function of s.*

If $K^*_{x_i} \in \mathscr{K}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $K^*_{z_i} \in \mathscr{K}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$, $p^*_{x_i} \in \mathscr{P}^{x_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$ *and* $p^*_{z_i} \in \mathscr{P}^{z_i}_{t_f,\mathscr{H}_i^f,\breve{\mathscr{D}}_i^f,\mathscr{D}_i^f;\mu_i}$ *defining a person-by-person equilibrium or feedback Nash strategy for agent i and* $i \in \overline{I}$ *with the corresponding solutions* $\mathscr{H}_i^*(\cdot)$, $\breve{\mathscr{D}}_i^*(\cdot)$ *and* $\mathscr{D}_i^*(\cdot)$ *of the dynamical equations (6.33)-(6.35) such that, for* $s \in [t_0,t_f]$

$$
\begin{aligned}
0 = \frac{\partial}{\partial\varepsilon}&\mathscr{W}_i(s,\mathscr{H}_i^*(s),\breve{\mathscr{D}}_i^*(s),\mathscr{D}_i^*(s)) + \frac{\partial}{\partial\,\mathrm{vec}(\mathscr{Y}_i)}\mathscr{W}_i(s,\mathscr{H}_i^*(s),\breve{\mathscr{D}}_i^*(s),\mathscr{D}_i^*(s)) \\
&\cdot\mathrm{vec}(\mathscr{F}_i(s,\mathscr{Y}_i^*(s),K^*_{x_i}(s),K^*_{z_i}(s))) + \frac{\partial}{\partial\,\mathrm{vec}(\breve{\mathscr{Z}}_i)}\mathscr{W}_i(s,\mathscr{H}_i^*(s),\breve{\mathscr{D}}_i^*(s),\mathscr{D}_i^*(s)) \\
&\quad\cdot\mathrm{vec}(\breve{\mathscr{G}}_i(s,\mathscr{H}_i^*(s),\breve{\mathscr{D}}_i^*(s),K^*_{x_i}(s),K^*_{z_i}(s),p^*_{x_i}(s),p^*_{z_i}(s)) \\
+ \frac{\partial}{\partial\,\mathrm{vec}(\mathscr{Z}_i)}&\mathscr{W}_i(s,\mathscr{H}_i^*(s),\breve{\mathscr{D}}_i^*(s),\mathscr{D}_i^*(s))\mathrm{vec}(\mathscr{G}_i(s,\mathscr{H}_i^*(s),\breve{\mathscr{D}}_i^*(s),p^*_{x_i}(s),p^*_{z_i}(s))
\end{aligned} \tag{6.42}
$$

then $(K^*_{x_i}, K^*_{z_i}, p^*_{x_i}, p^*_{z_i})$ *results in a feedback Nash strategy or person-by-person equilibrium for agent i in* $\mathscr{K}^{x_i}_{t_f,\mathscr{H}^f_i,\breve{\mathscr{D}}^f_i,\mathscr{D}^f_i;\mu_i} \times \mathscr{K}^{z_i}_{t_f,\mathscr{H}^f_i,\breve{\mathscr{D}}^f_i,\mathscr{D}^f_i;\mu_i} \times \mathscr{P}^{x_i}_{t_f,\mathscr{H}^f_i,\breve{\mathscr{D}}^f_i,\mathscr{D}^f_i;\mu_i} \times \mathscr{P}^{z_i}_{t_f,\mathscr{H}^f_i,\breve{\mathscr{D}}^f_i,\mathscr{D}^f_i;\mu_i}$*. Furthermore, it follows that*

$$\mathscr{W}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) = \mathscr{V}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)\,, \tag{6.43}$$

whereby $\mathscr{V}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ *is the value function associated with incumbent agent i.*

Proof. With the aid of the recent development [7], the proof then follows for the verification theorem herein.

6.4 Distributed Person-by-Person Equilibrium Strategies

Reflecting on the Mayer-form optimization problem of the person-by-person equilibrium strategy concerned by incumbent agent i and $i \in \bar{I}$, the technical approach is to apply an adaptation of the Mayer-form verification theorem of dynamic programming as given in [11]. In the framework of dynamic programming, it is often required to denote the terminal time and states of a family of optimization problems as $(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ rather than $(t_f, \mathscr{H}^f_i, \breve{\mathscr{D}}^f_i, \mathscr{D}^f_i)$. Stating precisely, for $\varepsilon \in [t_0, t_f]$ and $1 \le r \le k_i$, the states of the performance robustness (6.33)-(6.35) defined on the interval $[t_0, \varepsilon]$ have the terminal values denoted by $\mathscr{H}_i(\varepsilon) \equiv \mathscr{Y}_i$, $\breve{\mathscr{D}}_i(\varepsilon) \equiv \breve{\mathscr{Z}}_i$ and $\mathscr{D}_i(\varepsilon) \equiv \mathscr{Z}_i$.

Since the performance index (6.36) is quadratic affine in terms of arbitrarily fixed x^0_i, the resulting insight suggests a solution to the adapted HJB equation (6.40) is of the form as follows. It is assumed that $(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ is any interior point of the reachable set $\mathscr{Q}_i$ at which the real-valued function

$$\begin{aligned}\mathscr{W}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) &= (x^0_i)^T \sum_{r=1}^{k_i} \mu^i_r (\mathscr{Y}^r_i + \mathscr{E}^r_i(\varepsilon)) x^0_i \\ &\quad + 2(x^0_i)^T \sum_{r=1}^{k_i} \mu^i_r (\breve{\mathscr{Z}}^r_i + \breve{\mathscr{T}}^r_i(\varepsilon)) + \sum_{r=1}^{k_i} \mu^i_r (\mathscr{Z}^r_i + \mathscr{T}^r_i(\varepsilon))\end{aligned} \tag{6.44}$$

is differentiable. The parametric functions of time $\mathscr{E}^r_i \in \mathscr{C}^1(t_0, t_f; \mathbb{R}^{n_i \times n_i})$, $\breve{\mathscr{T}}^r_i \in \mathscr{C}^1(t_0, t_f; \mathbb{R}^n_i)$ and $\mathscr{T}^r_i \in \mathscr{C}^1(t_0, t_f; \mathbb{R})$ are yet to be determined. Furthermore, the time derivative of $\mathscr{W}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i)$ can be shown to be

$$\begin{aligned}\frac{d}{d\varepsilon}\mathscr{W}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) &= (x^0_i)^T \sum_{r=1}^{k_i} \mu^i_r (\mathscr{F}^r_i(\varepsilon, \mathscr{Y}_i, K_{x_i}, K_{z_i}) + \frac{d}{d\varepsilon}\mathscr{E}^r_i(\varepsilon)) x^0_i \\ &\quad + 2(x^0_i)^T \sum_{r=1}^{k_i} \mu^i_r (\breve{\mathscr{G}}^r_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i}) + \frac{d}{d\varepsilon}\breve{\mathscr{T}}^r_i(\varepsilon)) \\ &\quad + \sum_{r=1}^{k_i} \mu^i_r (\mathscr{G}^r_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, p_{x_i}, p_{z_i}) + \frac{d}{d\varepsilon}\mathscr{T}^r_i(\varepsilon))\,.\end{aligned} \tag{6.45}$$

The substitution of this hypothesized solution (6.44) into the HJB equation (6.40) and making use of (6.45) results in

$$0 \equiv \min_{K_{x_i} \in \overline{K}^{x_i}, K_{z_i} \in \overline{K}^{z_i}, p_{x_i} \in \overline{P}^{x_i}, p_{z_i} \in \overline{P}^{z_i}} \left\{ (x_i^0)^T \sum_{r=1}^{k_i} \mu_r^i \frac{d}{d\varepsilon} \mathscr{E}_i^r(\varepsilon) x_i^0 + 2(x_i^0)^T \sum_{r=1}^{k_i} \mu_r^i \frac{d}{d\varepsilon} \breve{\mathscr{T}}_i^r(\varepsilon) \right.$$
$$+ \sum_{r=1}^{k_i} \mu_r^i \frac{d}{d\varepsilon} \mathscr{T}_i^r(\varepsilon) + 2(x_i^0)^T \sum_{r=1}^{k_i} \mu_r^i \breve{\mathscr{G}}_i^r(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i})$$
$$\left. + (x_i^0)^T \sum_{r=1}^{k_i} \mu_r^i \mathscr{F}_i^r(\varepsilon, \mathscr{Y}_i, K_{x_i}, K_{z_i}) x_i^0 + \sum_{r=1}^{k_i} \mu_r^i \mathscr{G}_i^r(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, p_{x_i}, p_{z_i}) \right\}. \quad (6.46)$$

Differentiating the expression within the bracket of (6.46) with respect to K_{x_i}, K_{z_i}, p_{x_i} and p_{z_i} yields the necessary conditions for an extremum of (6.40) on $[t_0, \varepsilon]$,

$$0 \equiv -2[B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_r^i \mathscr{Y}_i^r + \mu_i^1 R_{ii}(\varepsilon) K_{x_i}] x_i^0 (x_i^0)^T$$
$$- 2[B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r + \mu_i^1 R_{ii}(\varepsilon) p_{x_i}] (x_i^0)^T$$
$$0 \equiv -2[C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r + \mu_i^1 R_{zi}(\varepsilon) K_{z_i}] x_i^0 (x_i^0)^T$$
$$- 2[C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r + \mu_i^1 R_{zi}(\varepsilon) p_{z_i}] (x_i^0)^T$$
$$0 \equiv -2[B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r + \mu_i^1 R_{ii}(\varepsilon) K_{x_i}] x_i^0$$
$$- 2[B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r + \mu_i^1 R_{ii}(\varepsilon) p_{x_i}]$$
$$0 \equiv -2[C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r + \mu_i^1 R_{zi}(\varepsilon) K_{z_i}] x_i^0$$
$$- 2[C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r + \mu_i^1 R_{zi}(\varepsilon) p_{z_i}]$$

Because all $x_i^0 (x_i^0)^T$, $(x_i^0)^T$ and x_i^0 have arbitrary ranks of one, it must be true that

$$K_{x_i} = -(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r, \quad (6.47)$$

$$K_{z_i} = -(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r , \tag{6.48}$$

$$p_{x_i} = -(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r , \tag{6.49}$$

$$p_{z_i} = -(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r . \tag{6.50}$$

Replacing these results (6.47)-(6.50) into the right member of the HJB equation (6.40) yields the value of the minimum

$$\begin{aligned}
&(x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r \frac{d}{d\varepsilon} \mathscr{E}_i^r(\varepsilon) x_i^0 + 2(x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r \frac{d}{d\varepsilon} \breve{\mathscr{T}}_i^r(\varepsilon) + \sum_{r=1}^{k_i} \mu_i^r \frac{d}{d\varepsilon} \mathscr{T}_i^r(\varepsilon) \\
&\quad + (x_i^0)^T \Big\{ - \big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s \\
&\quad - C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s \big]^T \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r \\
&\quad - \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r \big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s \\
&\quad - C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s \big] - \mu_i^1 Q_{ii}(\varepsilon) - \mu_i^1 Q_i(\varepsilon) \\
&\quad - \mu_i^1 \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} R_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r \\
&\quad - \mu_i^1 \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} R_{zi}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r \\
&\quad - \sum_{r=2}^{k_i} \mu_r^i \sum_{v=1}^{r-1} \frac{2r!}{v!(r-v)!} \mathscr{Y}_i^v G_i(\varepsilon) W_i G_i^T(\varepsilon) \mathscr{Y}_i^{r-v} \\
&\quad - \sum_{r=2}^{k_i} \mu_r^i \sum_{v=1}^{r-1} \frac{2r!}{v!(r-v)!} \mathscr{Y}_i^{k_i+v} G_{ic}(\varepsilon) V_c G_{ic}^T(\varepsilon) \mathscr{Y}_i^{2k_i+r-v} \Big\} x_i^0 \\
&\quad + 2(x_i^0)^T \Big\{ - \big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s
\end{aligned}$$

$$
\begin{aligned}
&- C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s \big]^T \sum_{r=1}^{k_i} \mu_i^r \mathscr{Z}_i^r \\
&\quad - \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r \big[- B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \breve{\mathscr{Z}}_i^s \\
&\quad - C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \breve{\mathscr{Z}}_i^s + \sum_{j=1}^{N_i} B_{ij}(\varepsilon) u_{ij}^*(\varepsilon) \big] \\
&\quad - \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^{k_i+r} B_{ic}(\varepsilon) u_{ic}(\varepsilon) + \mu_i^1 Q_i(\varepsilon) \zeta_i(\varepsilon) \\
&- \mu_i^1 \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} R_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r \\
&- \mu_i^1 \sum_{s=1}^{k_i} \mu_i^s \mathscr{Y}_i^s C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} R_{zi}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{Z}}_i^r \Big\} \\
&\quad - \mathrm{Tr}\Big\{ \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^r G_i(\varepsilon) W_i G_i^T(\varepsilon) + \sum_{r=1}^{k_i} \mu_i^r \mathscr{Y}_i^{3k_i+r} G_{ic}(\varepsilon) V_c G_{ic}^T(\varepsilon) \Big\} \\
&\quad - 2 \sum_{r=1}^{k_i} \mu_i^r (\breve{\mathscr{Z}}_i^r)^T \big[- B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \breve{\mathscr{Z}}_i^s \\
&\quad - C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \breve{\mathscr{Z}}_i^s + \sum_{j=1}^{N_i} B_{ij}(\varepsilon) u_{ij}^*(\varepsilon) \big] \\
&\quad - 2 \sum_{r=1}^{k_i} \mu_i^r (\breve{\mathscr{Z}}_i^{k_i+r})^T B_{ic}(\varepsilon) u_{ic}(\varepsilon) - \mu_i^1 \zeta_i^T(\varepsilon) Q_i(\varepsilon) \zeta_i(\varepsilon) \\
&- \mu_i^1 \sum_{r=1}^{k_i} \mu_i^r (\breve{\mathscr{Z}}_i^r)^T B_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} R_{ii}(\varepsilon)(\mu_i^1 R_{ii}(\varepsilon))^{-1} B_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \breve{\mathscr{Z}}_i^s \\
&- \mu_i^1 \sum_{r=1}^{k_i} \mu_i^r (\breve{\mathscr{Z}}_i^r)^T C_{ii}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} R_{zi}(\varepsilon)(\mu_i^1 R_{zi}(\varepsilon))^{-1} C_{ii}^T(\varepsilon) \sum_{s=1}^{k_i} \mu_i^s \breve{\mathscr{Z}}_i^s . \qquad (6.51)
\end{aligned}
$$

For each agent i, it is necessary to exhibit $\{\mathscr{E}_i^r(\cdot)\}_{r=1}^{k_i}$, $\{\breve{\mathscr{T}}_i^r(\cdot)\}_{r=1}^{k_i}$ and $\{\mathscr{T}_i^r(\cdot)\}_{r=1}^{k_i}$ which render the left side of the HJB equation (6.40) equal to zero for $\varepsilon \in [t_0, t_f]$, when $\{\mathscr{Y}_i^r\}_{r=1}^{k_i}$, $\{\breve{\mathscr{Z}}_i^r\}_{r=1}^{k_i}$ and $\{\mathscr{Z}_i^r\}_{r=1}^{k_i}$ are evaluated along the solution trajectories of the dynamical equations (6.33)-(6.35). With a careful examination of the expression (6.51), it reveals that

$$\frac{d}{d\varepsilon}\mathscr{E}_i^1(\varepsilon) = [A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]^T\mathscr{H}_i^1(\varepsilon)$$
$$+ \mathscr{H}_i^1(\varepsilon)\big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big] + Q_{ii}(\varepsilon) + Q_i(\varepsilon)$$
$$+ \sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}\mathscr{H}_i^r(\varepsilon)$$
$$+ \sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}\mathscr{H}_i^r(\varepsilon) \quad (6.52)$$

$$\frac{d}{d\varepsilon}\mathscr{E}_i^r(\varepsilon) = [A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]^T\mathscr{H}_i^r(\varepsilon) + \mathscr{H}_i^r(\varepsilon)\big[A_{ii}(\varepsilon)$$
$$- B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon) - C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]$$
$$+ \sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^v(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon)\mathscr{H}_i^{r-v}(\varepsilon)$$
$$+ \sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{k_i+v}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\mathscr{H}_i^{2k_i+r-v}(\varepsilon), \quad 2 \le r \le k_i \quad (6.53)$$

$$\frac{d}{d\varepsilon}\breve{\mathscr{T}}_i^1(\varepsilon) = \big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]^T\mathscr{D}_i^1(\varepsilon)$$
$$+ \mathscr{H}_i^1(\varepsilon)\big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i}B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\big]$$

$$
\begin{aligned}
&+ \mathscr{H}_i^{k_i+1}(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon) - Q_i(\varepsilon)\zeta_i(\varepsilon) \\
&+ \sum_{s=1}^{k_i} \frac{\mu_i^s}{\mu_i^1} \mathscr{H}_i^s(\varepsilon)B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \frac{\mu_i^r}{\mu_i^1} \breve{\mathscr{D}}_i^r(\varepsilon) \\
&+ \sum_{s=1}^{k_i} \frac{\mu_i^s}{\mu_i^1} \mathscr{H}_i^s(\varepsilon)C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \frac{\mu_i^r}{\mu_i^1} \breve{\mathscr{D}}_i^r(\varepsilon)
\end{aligned} \tag{6.54}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\breve{\mathscr{T}}_i^r(\varepsilon) = &\Big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\Big]^T \mathscr{D}_i^r(\varepsilon) \\
&+ \mathscr{H}_i^r(\varepsilon)\Big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i} B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\Big] \\
&+ \mathscr{H}_i^{k_i+r}(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon), \qquad 2 \le r \le k_i
\end{aligned} \tag{6.55}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{T}_i^1(\varepsilon) = &\operatorname{Tr}\{\mathscr{H}_i^1(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon) + \mathscr{H}_i^{3k_i+1}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\} \\
&+ 2(\breve{\mathscr{D}}_i^1)^T(\varepsilon)\Big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i} B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\Big] \\
&+ 2(\breve{\mathscr{D}}_i^{k_i+1})^T(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon) + \zeta_i^T(\varepsilon)Q_i(\varepsilon)\zeta_i(\varepsilon) \\
&+ \sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}(\breve{\mathscr{D}}_i^r)^T(\varepsilon)B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&+ \sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}(\breve{\mathscr{D}}_i^r)^T(\varepsilon)C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon)
\end{aligned} \tag{6.56}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{T}_i^r(\varepsilon) = &\operatorname{Tr}\{\mathscr{H}_i^r(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon) + \mathscr{H}_i^{3k_i+r}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\} \\
&+ 2(\breve{\mathscr{D}}_i^r)^T(\varepsilon)\Big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i} B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\Big] \\
&+ 2(\breve{\mathscr{D}}_i^{k_i+r})^T(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon), \quad 2 \le r \le k_i
\end{aligned} \tag{6.57}
$$

will work. Furthermore, the boundary condition associated with the verification theorem requires that

$$(x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r (\mathscr{H}_i^r(t_0) + \mathscr{E}_i^r(t_0)) x_i^0 + 2(x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r (\breve{\mathscr{D}}_i^r(t_0) + \breve{\mathscr{T}}_i^r(t_0))$$
$$+ \sum_{r=1}^{k_i} \mu_i^r (\mathscr{D}_i^r(t_0) + \mathscr{T}_i^r(t_0))$$
$$= (x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r \mathscr{H}_i^r(t_0) x_i^0 + 2(x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{D}}_i^r(t_0) + \sum_{r=1}^{k_i} \mu_i^r \mathscr{D}_i^r(t_0).$$

Thus, matching the boundary condition yields the initial value conditions $\mathscr{E}_i^r(t_0) = 0$, $\breve{\mathscr{T}}_i^r(t_0) = 0$ and $\mathscr{T}_i^r(t_0) = 0$ for the forward-in-time differential equations (6.52)-(6.57).

Applying the 4-tuple $(K_{x_i}, K_{z_i}, p_{x_i}, p_{z_i})$ in (6.47)-(6.50) that is defining the person-by-person equilibrium for each agent i and $i \in \bar{I}$ along the solution trajectories of the backward-in-time differential equations (6.33)-(6.35), these equations become the backward-in-time Riccati-type differential equations

$$\frac{d}{d\varepsilon}\mathscr{H}_i^1(\varepsilon) = -\Big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\Big]^T\mathscr{H}_i^1(\varepsilon)$$
$$- \mathscr{H}_i^1(\varepsilon)\Big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)$$
$$- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\Big] - Q_{ii}(\varepsilon) - Q_i(\varepsilon)$$
$$- \sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}\mathscr{H}_i^r(\varepsilon)$$
$$- \sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}\mathscr{H}_i^r(\varepsilon) \qquad (6.58)$$

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\mathscr{H}_i^r(\varepsilon) = -\big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon) \\
&\quad - C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]^T\mathscr{H}_i^r(\varepsilon) - \mathscr{H}_i^r(\varepsilon)\big[A_{ii}(\varepsilon) \\
&\quad - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon) - C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big] \\
&\quad - \sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^v(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon)\mathscr{H}_i^{r-v}(\varepsilon) \\
&\quad - \sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_i^{k_i+v}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\mathscr{H}_i^{2k_i+r-v}(\varepsilon), \quad 2 \le r \le k_i
\end{aligned} \tag{6.59}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\breve{\mathscr{D}}_i^1(\varepsilon) = &-\big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]^T\mathscr{D}_i^1(\varepsilon) \\
&- \mathscr{H}_i^1(\varepsilon)\big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i}B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\big] \\
&- \mathscr{H}_i^{k_i+1}(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon) + Q_i(\varepsilon)\zeta_i(\varepsilon) \\
&- \sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}\breve{\mathscr{D}}_i^r(\varepsilon) \\
&- \sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}\breve{\mathscr{D}}_i^r(\varepsilon)
\end{aligned} \tag{6.60}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\breve{\mathscr{D}}_i^r(\varepsilon) = &-\big[A_{ii}(\varepsilon) - B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\mathscr{H}_i^s(\varepsilon)\big]^T\mathscr{D}_i^r(\varepsilon) \\
&- \mathscr{H}_i^r(\varepsilon)\big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i}B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\big] \\
&- \mathscr{H}_i^{k_i+r}(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon), \qquad 2 \le r \le k_i
\end{aligned}
\tag{6.61}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{D}_i^1(\varepsilon) = &-\mathrm{Tr}\{\mathscr{H}_i^1(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon) + \mathscr{H}_i^{3k_i+1}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\} \\
&- 2(\breve{\mathscr{D}}_i^1)^T(\varepsilon)\big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i}B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\big] \\
&- 2(\breve{\mathscr{D}}_i^{k_i+1})^T(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon) - \zeta_i^T(\varepsilon)Q_i(\varepsilon)\zeta_i(\varepsilon) \\
&- \sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}(\breve{\mathscr{D}}_i^r)^T(\varepsilon)B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- \sum_{r=1}^{k_i}\frac{\mu_i^r}{\mu_i^1}(\breve{\mathscr{D}}_i^r)^T(\varepsilon)C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon)
\end{aligned}
\tag{6.62}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{D}_i^r(\varepsilon) = &-\mathrm{Tr}\{\mathscr{H}_i^r(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon) + \mathscr{H}_i^{3k_i+r}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\} \\
&- 2(\breve{\mathscr{D}}_i^r)^T(\varepsilon)\big[-B_{ii}(\varepsilon)R_{ii}^{-1}(\varepsilon)B_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) \\
&- C_{ii}(\varepsilon)R_{zi}^{-1}(\varepsilon)C_{ii}^T(\varepsilon)\sum_{s=1}^{k_i}\frac{\mu_i^s}{\mu_i^1}\breve{\mathscr{D}}_i^s(\varepsilon) + \sum_{j=1}^{N_i}B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)\big] \\
&- 2(\breve{\mathscr{D}}_i^{k_i+r})^T(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon), \quad 2 \le r \le k_i
\end{aligned}
\tag{6.63}
$$

where the terminal-value conditions $\mathscr{H}_i^1(t_f) = Q_i^f$, $\mathscr{H}_i^r(t_f) = 0$ for $2 \le r \le k_i$; $\breve{\mathscr{D}}_i^r(t_f) = -Q_i^f\zeta_i(t_f)$, $\breve{\mathscr{D}}_i^r(t_f) = 0$ for $2 \le r \le k_i$; and $\mathscr{D}_i^r(t_f) = \zeta_i(t_f)Q_i^f\zeta_i(t_f)$, $\mathscr{D}_i^r(t_f)$ for $2 \le r \le k_i$. Thus, whenever the coupled backward-in-time differential equations (6.58)-(6.63) admit the matrix-valued solutions $\{\mathscr{H}_i^r(\cdot)\}_{r=1}^{k_i}$, vector-valued solutions $\{\breve{\mathscr{D}}_i^r(\cdot)\}_{r=1}^{k_i}$ and scalar-valued solutions $\{\mathscr{D}_i^r(\cdot)\}_{r=1}^{k_i}$, then the existence of the matrix-valued solutions $\{\mathscr{E}_i^r(\cdot)\}_{r=1}^{k_i}$, vector-valued solutions $\{\breve{\mathscr{T}}_i^r(\cdot)\}_{r=1}^{k_i}$ and scalar-valued solutions $\{\mathscr{T}_i^r(\cdot)\}_{r=1}^{k_i}$ satisfying the coupled forward-in-time differential equations (6.52)-(6.57) are assured. By comparing the time-forward differential equations (6.52)-(6.57) to those of time-backward differential equations (6.58)-(6.63), one may recognize that these sets of differential equations are related to one another by

$$\frac{d}{d\varepsilon}\mathscr{E}_i^r(\varepsilon) = -\frac{d}{d\varepsilon}\mathscr{H}_i^r(\varepsilon); \quad \frac{d}{d\varepsilon}\breve{\mathscr{T}}_i^r(\varepsilon) = -\frac{d}{d\varepsilon}\breve{\mathscr{D}}_i^r(\varepsilon); \quad \frac{d}{d\varepsilon}\mathscr{T}_i^r(\varepsilon) = -\frac{d}{d\varepsilon}\mathscr{D}_i^r(\varepsilon).$$

Enforcing the initial-value conditions of $\mathscr{E}_i^r(t_0) = 0$, $\breve{\mathscr{T}}_i^r(t_0) = 0$ and $\mathscr{T}_i^r(t_0) = 0$ uniquely implies the following results

$$\mathscr{E}_i^r(\varepsilon) = \mathscr{H}_i^r(t_0) - \mathscr{H}_i^r(\varepsilon); \quad \breve{\mathscr{T}}_i^r(\varepsilon) = \breve{\mathscr{D}}_i^r(t_0) - \breve{\mathscr{D}}_i^r(\varepsilon); \quad \mathscr{T}_i^r(\varepsilon) = \mathscr{D}_i^r(t_0) - \mathscr{D}_i^r(\varepsilon)$$

for all $\varepsilon \in [t_0, t_f]$ and yields a value function

$$\mathscr{W}_i(\varepsilon, \mathscr{Y}_i, \breve{\mathscr{Z}}_i, \mathscr{Z}_i) = (x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r \mathscr{H}_i^r(t_0) x_i^0 + 2(x_i^0)^T \sum_{r=1}^{k_i} \mu_i^r \breve{\mathscr{D}}_i^r(t_0) + \sum_{r=1}^{k_i} \mu_i^r \mathscr{D}_i^r(t_0)$$

for which the sufficient condition (6.42) of the verification theorem is satisfied. Therefore, the extremal person-by-person equilibrium policy (6.47)-(6.50) minimizing (6.36) become optimal

$$K_{x_i}^*(\varepsilon) = -R_{ii}^{-1}(\varepsilon) B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i^r \mathscr{H}_i^{*r}(\varepsilon), \tag{6.64}$$

$$K_{z_i}^*(\varepsilon) = -R_{zi}^{-1}(\varepsilon) C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i^r \mathscr{H}_i^{*r}(\varepsilon), \tag{6.65}$$

$$p_{x_i}^*(\varepsilon) = -R_{ii}^{-1}(\varepsilon) B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i^r \breve{\mathscr{D}}_i^{*r}(\varepsilon), \tag{6.66}$$

$$p_{z_i}^*(\varepsilon) = -R_{zi}^{-1}(\varepsilon) C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i^r \breve{\mathscr{D}}_i^{*r}(\varepsilon), \tag{6.67}$$

whereby $\hat{\mu}_i^r = \frac{\mu_i^r}{\mu_i^1}$.

The goals in this research investigation have been methodological. A noncooperative game-theoretic methodology for coordination control of distributed stochastic systems is successfully sought for theory building in contexts in which signaling effects are issued by a coordinator and distributed person-by-person equilibrium strategies by autonomous agents i and $i \in \overline{I}$ are placed toward performance robustness. At this point, it makes sense to integrate all of the contending results into the following unified theorem.

Theorem 5 *Person-by-Person Equilibrium Strategies.*
Consider a distributed stochastic system governed by (6.3)-(6.7) whose pairs (A_{ii}, B_{ii}) *and* (A_{ii}, C_{ii}) *are uniformly stabilizable on* $[t_0, t_f]$. *A N-tuple* $\{(u_1^*, z_1^*), \ldots, (u_N^*, z_N^*)\}$ *of control policies constitutes a feedback Nash equilibrium for the class of distributed stochastic system considered here. Furthermore, 2-tuple* (u_i^*, z_i^*) *imposing a person-by-person equilibrium strategy for the corresponding agent i and* $i \in \overline{I}$ *is implemented forwardly in time by*

$$u_i^*(t) = K_{x_i}^*(t) x_i(t) + p_{x_i}^*(t) \tag{6.68}$$

$$z_i^*(t) = K_{z_i}^*(t) x_i(t) + p_{z_i}^*(t), \quad t = t_f + t_0 - \varepsilon, \quad \varepsilon \in [t_0, t_f], \tag{6.69}$$

which strives to optimize the risk-value aware performance index (6.36) composed by a preferential set of mathematical statistics of the chi-squared cost random variable (6.7). The construction of the person-by-person equilibrium for each agent i is determined backwardly in time; e.g.,

$$K_{x_i}^*(\varepsilon) = -R_{ii}^{-1}(\varepsilon) B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i{}^r \mathscr{H}_i^{*r}(\varepsilon), \tag{6.70}$$

$$K_{z_i}^*(\varepsilon) = -R_{zi}^{-1}(\varepsilon) C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i{}^r \mathscr{H}_i^{*r}(\varepsilon), \tag{6.71}$$

$$p_{x_i}^*(\varepsilon) = -R_{ii}^{-1}(\varepsilon) B_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i{}^r \breve{\mathscr{D}}_i^{*r}(\varepsilon), \tag{6.72}$$

$$p_{z_i}^*(\varepsilon) = -R_{zi}^{-1}(\varepsilon) C_{ii}^T(\varepsilon) \sum_{r=1}^{k_i} \hat{\mu}_i{}^r \breve{\mathscr{D}}_i^{*r}(\varepsilon), \tag{6.73}$$

wherein the normalized preferences $\hat{\mu}_i^r \triangleq \mu_i^r / \mu_i^1$*'s are mutually chosen by each incumbent agent i for risk-averse coordinations toward co-design of individual performance robustness. The optimal set of supporting solutions* $\{\mathscr{H}_{i*}^r(\varepsilon)\}_{r=1}^{k_i}$, $\{\mathscr{H}_{i*}^{k_i+r}(\varepsilon)\}_{r=1}^{k_i}$, $\{\mathscr{H}_{i*}^{2k_i+r}(\varepsilon)\}_{r=1}^{k_i}$, $\{\mathscr{H}_{i*}^{3k_i+r}(\varepsilon)\}_{r=1}^{k_i}$ *and* $\{\breve{\mathscr{D}}_{i*}^r(\varepsilon)\}_{r=1}^{k_i}$ *satisfy the time-backward and matrix-valued differential equations*

$$\begin{aligned} \frac{d}{d\varepsilon} \mathscr{H}_{i*}^1(\varepsilon) = &-[A_{ii}(\varepsilon) + B_{ii}(\varepsilon) K_{x_i}^*(\varepsilon) + C_{ii}(\varepsilon) K_{z_i}^*(\varepsilon)]^T \mathscr{H}_{i*}^1(\varepsilon) \\ &- \mathscr{H}_{i*}^1(\varepsilon)[A_{ii}(\varepsilon) + B_{ii}(\varepsilon) K_{x_i}^*(\varepsilon) + C_{ii}(\varepsilon) K_{z_i}^*(\varepsilon)] \\ &- Q_{ii}(\varepsilon) - Q_i(\varepsilon) - K_{x_i}^{*T}(\varepsilon) R_{ii}(\varepsilon) K_{x_i}^*(\varepsilon) - K_{z_i}^{*T}(\varepsilon) R_{zi}(\varepsilon) K_{z_i}^*(\varepsilon) \end{aligned} \tag{6.74}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{r}(\varepsilon) = -[A_{ii}(\varepsilon)+B_{ii}(\varepsilon)K_{x_i}^{*}(\varepsilon)+C_{ii}(\varepsilon)K_{z_i}^{*}(\varepsilon)]^T\mathscr{H}_{i*}^{r}(\varepsilon)$$
$$-\mathscr{H}_{i*}^{r}(\varepsilon)[A_{ii}(\varepsilon)+B_{ii}(\varepsilon)K_{x_i}^{*}(\varepsilon)+C_{ii}(\varepsilon)K_{z_i}^{*}(\varepsilon)]$$
$$-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{v}(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon)\mathscr{H}_{i*}^{r-v}(\varepsilon)$$
$$-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{k_i+v}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{2k_i+r-v}(\varepsilon)\,,\quad 2\le r\le k_i \tag{6.75}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{k_i+1}(\varepsilon) = -[A_{ii}(\varepsilon)+B_{ii}(\varepsilon)K_{x_i}^{*}(\varepsilon)+C_{ii}(s)K_{z_i}^{*}(\varepsilon)]^T\mathscr{H}_{i*}^{k_i+1}(\varepsilon)$$
$$-\mathscr{H}_{i*}^{k_i+1}(\varepsilon)A_{ic}(\varepsilon)+2K_{z_i}^{*T}(\varepsilon)R_{zi}(\varepsilon) \tag{6.76}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{k_i+r}(\varepsilon) = -[A_{ii}(\varepsilon)+B_{ii}(\varepsilon)K_{x_i}^{*}(\varepsilon)+C_{ii}(\varepsilon)K_{z_i}^{*}(\varepsilon)]^T\mathscr{H}_{i*}^{k_i+r}(\varepsilon)$$
$$-\mathscr{H}_{i*}^{k_i+r}(\varepsilon)A_{ic}(\varepsilon)-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{v}(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon)\mathscr{H}_{i*}^{k_i+r-v}(\varepsilon)$$
$$-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{k_i+v}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{3k_i+r-v}(\varepsilon)\,,\quad 2\le r\le k_i \tag{6.77}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{2k_i+1}(\varepsilon) = -A_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{2k_i+1}(\varepsilon)$$
$$-\mathscr{H}_{i*}^{2k_i+1}(\varepsilon)[A_{ii}(\varepsilon)+B_{ii}(\varepsilon)K_{x_i}^{*}(\varepsilon)+C_{ii}(\varepsilon)K_{z_i}^{*}(\varepsilon)] \tag{6.78}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{2k_i+r}(\varepsilon) = -\mathscr{H}_{i*}^{2k_i+r}(\varepsilon)[A_{ii}(\varepsilon)+B_{ii}(\varepsilon)K_{x_i}^{*}(\varepsilon)+C_{ii}(\varepsilon)K_{z_i}^{*}(\varepsilon)]$$
$$-A_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{2k_i+r}(\varepsilon)-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{2k_i+v}(\varepsilon)G_i(\varepsilon)W_iG_i^T(\varepsilon)\mathscr{H}_{i*}^{r-v}(\varepsilon)$$
$$-\sum_{v=1}^{r-1}\frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{3k_i+v}(\varepsilon)G_{ic}(\varepsilon)V_cG_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{2k_i+r-v}(\varepsilon)]\,,\quad 2\le r\le k_i \tag{6.79}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{3k_i+1}(\varepsilon) = -A_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{3k_i+1}(\varepsilon)-\mathscr{H}_{i*}^{3k_i+1}(\varepsilon)A_{ic}(\varepsilon)-R_{zi}(\varepsilon) \tag{6.80}$$

$$\frac{d}{d\varepsilon}\mathscr{H}_{i*}^{3k_i+r}(\varepsilon) = -A_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{3k_i+r}(\varepsilon) - \mathscr{H}_{i*}^{3k_i+r}(\varepsilon)A_{ic}(\varepsilon) - R_{zi}(\varepsilon)$$
$$- \sum_{v=1}^{r-1} \frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{2k_i+v}(\varepsilon)G_i(\varepsilon)W_i G_i^T(\varepsilon)\mathscr{H}_{i*}^{k_i+r-v}(\varepsilon)$$
$$- \sum_{v=1}^{r-1} \frac{2r!}{v!(r-v)!}\mathscr{H}_{i*}^{3k_i+v}(\varepsilon)G_{ic}(\varepsilon)V_c G_{ic}^T(\varepsilon)\mathscr{H}_{i*}^{3k_i+r-v}(\varepsilon), \quad 2 \le r \le k_i, \quad (6.81)$$

in addition with the backward-in-time and vector-valued differential equations

$$\frac{d}{d\varepsilon}\breve{\mathscr{D}}_{i*}^1(\varepsilon) = -[A_{ii}(\varepsilon) + B_{ii}(\varepsilon)K_{x_i}^*(\varepsilon) + C_{ii}(s)K_{z_i}^*(\varepsilon)]^T \breve{\mathscr{D}}_{i*}^1(\varepsilon)$$
$$- \mathscr{H}_{i*}^1(\varepsilon)[B_{ii}(\varepsilon)p_{x_i}^*(\varepsilon) + C_{ii}(\varepsilon)p_{z_i}^*(\varepsilon) + \sum_{j=1}^{N_i} B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)] + Q_i(\varepsilon)\zeta_i(\varepsilon)$$
$$- \mathscr{H}_{i*}^{k_i+1}(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon) - K_{x_i}^{*T}(\varepsilon)R_{ii}(\varepsilon)p_{x_i}^*(\varepsilon) - K_{z_i}^{*T}R_{zi}(\varepsilon)p_{z_i}^*(\varepsilon) \quad (6.82)$$

$$\frac{d}{d\varepsilon}\breve{\mathscr{D}}_{i*}^r(\varepsilon) = -[A_{ii}(\varepsilon) + B_{ii}(\varepsilon)K_{x_i}^*(\varepsilon) + C_{ii}(s)K_{z_i}^*(\varepsilon)]^T \breve{\mathscr{D}}_{i*}^r(\varepsilon)$$
$$- \mathscr{H}_{i*}^r(\varepsilon)[B_{ii}(\varepsilon)p_{x_i}^*(\varepsilon) + C_{ii}(\varepsilon)p_{z_i}^*(\varepsilon) + \sum_{j=1}^{N_i} B_{ij}(\varepsilon)u_{ij}^*(\varepsilon)]$$
$$- \mathscr{H}_{i*}^{k_i+r}(\varepsilon)B_{ic}(\varepsilon)u_{ic}(\varepsilon), \qquad 2 \le r \le k_i \quad (6.83)$$

whereby the terminal-value conditions $\mathscr{H}_{i*}^1(t_f) = Q_i^f$, $\mathscr{H}_{i*}^r(t_f) = 0$ *for* $2 \le r \le k_i$; $\mathscr{H}_{i*}^{k_i+r}(t_f) = 0$ *for* $1 \le r \le k_i$; $\mathscr{H}_{i*}^{2k_i+r}(t_f) = 0$ *for* $1 \le r \le k_i$; $\mathscr{H}_{i*}^{3k_i+r}(t_f) = 0$ *for* $1 \le r \le k_i$; *and* $\breve{\mathscr{D}}_{i*}^1(t_f) = -Q_i^f \zeta_i(t_f)$, $\breve{\mathscr{D}}_{i*}^r(t_f) = 0$ *for* $2 \le r \le k_i$.

6.5 Conclusions

The noncooperative game-theoretic framework presented here offers a contribution to coordination control science's existing portfolio of methodologies. This portfolio contains a coordinator which directs two or more interconnected stochastic systems. Thinking about risk-averse attitudes toward performance uncertainty and distribution suggests new ideas for extending existing theories of distributed control and multiperson decision analysis. In this sense, the present research article offers a theoretic lens and a novel approach that direct attention towards mathematical statistics of the chi-squared random performance measures concerned by incumbent agents of the class of distributed stochastic systems herein and thus provide new insights into complex dynamics of performance robustness and reliability. To account for mutual influence from immediate neighbors that give rise to interaction complexity such as potential noncooperation, each incumbent system or self-directed agent autonomously focuses on the search for a person-by-person equilibrium which is in turn locally supported by perfect state observations. Further discussions show that the person-by-person equilibrium is equivalent to the concept of feedback Nash strategy. Research issues discussed include adjusting the risk-averse attitudes against which performance statistics are measured is now termed as risk-value aware performance indices. The process of adjustment for performance risk aversion imposes some computational requirements as needed by the construction of the states of the person-by-person equilibrium.

Future work will be another attainments of distributed control with explicit communications and partial information patterns, wherein research issues are: (i) when is periodic communication the optimal communication policy for the considered cost function? (ii) in what control and decision architectures, are partial local state information shared? and (iii) how to distribute global objectives of distributed and/or hierarchical stochastic systems over constituent systems and/or different layers?

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