Chapter 17
Anomalous Transport from Kubo Formulae

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17.1 Introduction

Anomalies in relativistic field theories of chiral fermions belong to the most intriguing properties of quantum field theory. Comprehensive reviews on anomalies can be found in the textbooks [1–3].

Hydrodynamics is an ancient subject. Even in its relativistic form it appeared that everything relevant to its formulation could be found in [4]. Apart from stability issues that were addressed in the 1960s and 1970s [5–7] leading to a second order formalism there seemed little room for new discoveries. The last years witnessed however an unexpected and profound development of the formulation of relativistic hydrodynamics. The second order contributions have been put on a much more systematic basis applying effective field theory reasoning [8, 9]. The lessons learned from applying the AdS/CFT correspondence [10–12] to the plasma phase of strongly coupled non-abelian gauge theories [13–15] played a major role (see [16] for a recent review).

The presence of chiral anomalies in otherwise conserved currents has profound implications for the formulation of relativistic hydrodynamics. The transport processes related to anomalies have surfaced several times and independently [17–22].

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The axial current was the focus in [23] and the first application of the AdS/CFT correspondence to anomalous hydrodynamics can be found already in [24]. The full impact anomalies have on the formulation of relativistic hydrodynamics was however not fully appreciated until recently.

The renewed interest in the formulation of relativistic hydrodynamics has its origin mostly in the spectacular experimental evidence for collective flow phenomena taking place in the physics of heavy ion collisions at RHIC and LHC. These experiments indicate the creation of a deconfined quark gluon plasma in a strongly coupled regime. In the context of heavy ion collisions it was argued in [25, 26] that the excitation of topologically non-trivial gluon field configurations in the early non-equilibrium stages of a heavy ion collision might lead to an imbalance in the number of left- and right-handed quarks. This situation can be modeled by an axial chemical potential and it was shown that an external magnetic field leads to an electric current parallel to the magnetic field. This chiral magnetic effect leads then to a charge separation perpendicular to the reaction plane in heavy ion collisions. The introduction of an axial chemical potential also allows to define a chiral magnetic conductivity which is simply the factor of proportionality between the magnetic field and the induced electric current. This effect is a direct consequence of the axial anomaly.

The application of the fluid/gravity correspondence to theories including chiral anomalies lead to another surprise: it was found that not only a magnetic field induces a current but that also a vortex in the fluid leads to an induced current [27, 28]. This is the chiral vortical effect. Again it is a consequence of the presence of a chiral anomaly. It was later realized that the chiral magnetic and vortical conductivities are almost completely fixed in the hydrodynamic framework by demanding the existence of an entropy current with positive definite divergence [29]. That this criterion did not fix the anomalous transport coefficients completely was noted in [30] and various terms depending on the temperature instead of the chemical potentials were shown to be allowed as undetermined integration constants. See also [31, 32] for a recent discussion of these anomaly coefficients with applications to heavy ion physics.

In the meanwhile Kubo formulae for the chiral magnetic conductivity [33] and the chiral vortical conductivity [34] had been developed. Up to this point only pure gauge anomalies had been considered to be relevant since the mixed gauge-gravitational anomaly in four dimensions is of higher order in derivatives and was thought not to be able to contribute to hydrodynamics at first order in derivatives. Therefore it came as a surprise that in the application of the Kubo formula for the chiral vortical conductivity to a system of free chiral fermions a purely temperature dependent contribution was found. This contribution was consistent with some the earlier found integration constants and it was shown to arise if and only if the system of chiral fermions features a mixed gauge-gravitational anomaly [35]. In fact these contributions had been found already very early on in [17–20]. The connection to the presence of anomalies was however not made at that time. The gravitational anomaly contribution to the chiral vortical effect was also established in a strongly coupled AdS/CFT approach and precisely the same result as at weak coupling was found [36].

The argument based on a positive definite divergence of the entropy current allows to fix the contributions form pure gauge anomalies uniquely and provides therefore a non-renormalization theorem. No such result is known thus far for the contributions of the gauge-gravitational anomaly.[1]

A gas of weakly coupled Weyl fermions in arbitrary dimensions has been studied in [39] and confirmed that the anomalous conductivities can be obtained directly from the anomaly polynomial under substitution of the field strength with the chemical potential and the first Pontryagin density by the negative of the temperature squared [40]. Recently the anomalous conductivities have also been obtained in effective action approaches [41, 42]. The contribution of the mixed gauge-gravitational anomaly appear in all these approaches as undetermined integrations constants.

We will review here what can be learned from the calculation of the anomalous conductivities via Kubo formulae. The advantage of the usage of Kubo formulae is that they capture all contributions stemming either from pure gauge or from mixed gauge-gravitational anomalies. The disadvantage is that the calculations can be performed only within a particular model and only in the weak or in the gravity dual of the strong coupling regime. Along the way we will explain our point of view on some subtle issues concerning the definition of currents and of chemical potentials when anomalies are present. These subtleties lead indeed to some ambiguous results [43] and [44]. A first step to clarify these issues was done in [45] and a more general exposition of the relevant issues has appeared in [46].

The review is organized as follows. In Sect. 17.2 we will briefly summarize the relevant issues concerning anomalies. We recall how vector like symmetries can always be restored by adding suitable finite counterterms to the effective action [47]. A related but different issue is the fact that currents can be defined either as consistent or as covariant currents. The hydrodynamic constitutive relations depend on what definition of current is used. We discuss subtleties in the definition of the chemical potential in the presence of an anomaly and define our preferred scheme. We discuss the hydrodynamic constitutive relations and derive the Kubo formulae that allow the calculation of the anomalous transport coefficients from two point correlators of an underlying quantum field theory.

In Sect. 17.3 we apply the Kubo formulae to a theory of free Weyl fermions and show that two different contributions arise. They are clearly identifiable as being related to the presence of pure gauge and mixed gauge-gravitational anomalies.

In Sect. 17.4 we define a holographic model that implements the mixed gauge-gravitational anomaly via a mixed gauge-gravitational Chern-Simons term. We calculate the same Kubo formulae as at weak coupling, obtaining the same values for chiral axi-magnetic and chiral vortical conductivities as in the weak coupling model.

We conclude this review with some discussions and outlook to further developments.

[1] See however the very recent attempts to establish non-renormalization theorems in [37] and [38].

17.2 Anomalies and Hydrodynamics

In this section we will review briefly anomalies. We compare the consistent with the covariant form of the anomaly and we introduce the Bardeen counterterm that allows to restore current conservation for vector like symmetries. Then we turn to the question of what we mean when we talk about the chemical potential. Two ways of introducing chemical potential, either through a deformation of the Hamiltonian or by demanding twisted boundary conditions along the thermal circle are shown to be in-equivalent in presence of an anomaly. Equivalence can still be achieved by introduction of a spurious axion field. We explain the implications for holography. The constitutive relations for anomalous currents are introduced in Landau and Laboratory frame. We discuss how they differ if we use the consistent instead of the covariant currents and derive the Kubo formulae for the anomalous conductivities.

17.2.1 Anomalies

Anomalies arise by integrating over chiral fermions in the path integral. They signal a fundamental incompatibility between the symmetries present in the classical theory and the quantum theory.

Unless otherwise stated we will always think of the symmetries as global symmetries. But we still will introduce gauge fields. These gauge fields serve as classical sources coupled to the currents. As a side effect their presence promotes the global symmetry to a local gauge symmetry. It is still justified to think of it as a global symmetry as long as we do not introduce a kinetic Maxwell or Yang-Mills term in the action.

In a theory with chiral fermions we define an effective action depending on these gauge fields by the path integral

$$e^{iW_{eff}[A_\mu]/\hbar} := \int \mathscr{D}\Psi \mathscr{D}\bar{\Psi} e^{iS[\psi,A_\mu]/\hbar}. \tag{17.1}$$

The vector field $A_\mu(x)$ couples to a classically conserved current $J^\mu = \bar{\Psi}\gamma^\mu Q\Psi$. The charge operator Q can be the generator of a Lie group combined with chiral projectors $\mathscr{P}_\pm = \frac{1}{2}(1 \pm \gamma_5)$. General combinations are allowed although in the following we will mostly concentrate on a simple chiral $U(1)$ symmetry for which we can take $Q = \mathscr{P}_+$. The fermions are minimally coupled to the gauge field and the classical action has an underlying gauge symmetry

$$\delta\Psi = -i\lambda(x)Q\Psi, \qquad \delta A_\mu(x) = D_\mu\lambda(x), \tag{17.2}$$

with D_μ denoting the gauge covariant derivative. Assuming that the theory has a classical limit the effective action in terms of the gauge fields allows for an expansion in $\hbar$

$$W_{eff} = W_0 + \hbar W_1 + \hbar^2 W_2 + \cdots. \tag{17.3}$$

We find it convenient to use the language of BRST symmetry by promoting the gauge parameter to a ghost field $c(x)$.[2] The BRST symmetry is generated by

$$sA_\mu = D_\mu c, \qquad sc = -ic^2. \tag{17.4}$$

It is nilpotent $s^2 = 0$. The statement that the theory has an anomaly can now be neatly formalized. Since on gauge fields the BRST symmetry acts just as the gauge symmetry, gauge invariance translates into BRST invariance. An anomaly is present if

$$sW_{\mathit{eff}} = \mathscr{A} \quad \text{and} \quad \mathscr{A} \neq sB. \tag{17.5}$$

Because of the nilpotency of the BRST operator the anomaly has to fulfill the Wess-Zumino consistency condition

$$s\mathscr{A} = 0. \tag{17.6}$$

As indicated in (17.5) this has a possible trivial solution if there exists a local functional $B[A_\mu]$ such that $sB = \mathscr{A}$. An anomaly is present if no such B exists. The anomaly is a quantum effect. If it is of order $\hbar^n$ and if a suitable local functional B exists we could simply redefine the effective action as $\widetilde{W}_{\mathit{eff}} = W_{\mathit{eff}} - B$ and the new effective action would be BRST and therefore gauge invariant. The form and even the necessity to introduce a functional B might depend on the particular regularization scheme chosen. As we will explain in the case of an axial and vector symmetry a suitable B can be found that always allows to restore the vectorlike symmetry, this is the so-called Bardeen counterterm [47]. The necessity to introduce the Bardeen counterterm relies however on the regularization scheme chosen. In schemes that automatically preserve vectorlike symmetries, such as dimensional regularization, the vector symmetries are automatically preserved and no counterterm has to be added. Furthermore the Adler-Bardeen theorem guarantees that chiral anomalies appear only at order $\hbar$. Their presence can therefore by detected in one loop diagrams such as the triangle diagram of three currents.

We have introduced the gauge fields as sources for the currents

$$\frac{\delta}{\delta A_\mu(x)} W_{\mathit{eff}}[A] = \langle J^\mu \rangle. \tag{17.7}$$

For chiral fermions transforming under a general Lie group generated by T^a the chiral anomaly takes the form [1]

$$\begin{aligned} sW_{\mathit{eff}}[A] &= -\int \mathrm{d}^4x\, c^a \left(D_\mu J^\mu\right)^a \\ &= -\frac{\eta}{24\pi^2} \int \mathrm{d}^4x\, c^a \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr}\left[T^a \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma\right)\right]. \end{aligned} \tag{17.8}$$

[2] A recent comprehensive review on BRST symmetry is [48].

Where $\eta = +1$ for left-handed fermions and $\eta = -1$ for right-handed fermions. Differentiating with respect to the ghost field (the gauge parameter) we can derive a local form. To simplify the formulas we specialize this to the case of a single chiral $U(1)$ symmetry taking $T^a = 1$

$$\partial_\mu J^\mu = \frac{\eta}{96\pi^2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{17.9}$$

This is to be understood as an operator equation. Sandwiching it between the vacuum state $|0\rangle$ and further differentiating with respect to the gauge fields we can generate the famous triangle form of the anomaly

$$\left\langle \partial_\mu J^\mu(x) J^\sigma(y) J^\kappa(z) \right\rangle = \frac{1}{12\pi^2}\varepsilon^{\mu\sigma\rho\kappa}\partial^x_\mu \delta(x-y)\partial^x_\rho \delta(x-z). \tag{17.10}$$

The one point function of the divergence of the current is non-conserved only in the background of parallel electric and magnetic fields whereas the non-conservation of the current as an operator becomes apparent in the triangle diagram even in vacuum.

By construction this form of the anomaly fulfills the Wess-Zumino consistency condition and is therefore called the *consistent anomaly*. In analogy we call the current defined by (17.7) the *consistent current*.

For a $U(1)$ symmetry the functional differentiation with respect to the gauge field and the BRST operator s commute,

$$\left[s, \frac{\delta}{\delta A_\mu(x)} \right] = 0. \tag{17.11}$$

An immediate consequence is that the consistent current is not BRST invariant but rather obeys

$$s J^\mu = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu c F_{\rho\lambda} = -\frac{1}{24\pi^2} s K^\mu, \tag{17.12}$$

where we introduced the Chern-Simons current $K^\mu = \varepsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda}$ with $\partial_\mu K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$.

With the help of the Chern-Simons current it is possible to define the so-called *covariant current* (in the case of a $U(1)$ symmetry rather the invariant current)

$$\tilde{J}^\mu = J^\mu + \frac{1}{24\pi^2} K^\mu \tag{17.13}$$

fulfilling

$$s \tilde{J}^\mu = 0. \tag{17.14}$$

The divergence of the covariant current defines the *covariant anomaly*

$$\partial_\mu \tilde{J}^\mu = \frac{1}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{17.15}$$

Notice that the Chern-Simons current cannot be obtained as the variation with respect to the gauge field of any functional. It is therefore not possible to define an effective action whose derivation with respect to the gauge field gives the covariant current.

Let us suppose now that we have one left-handed and one right-handed fermion with the corresponding left- and right-handed anomalies. Instead of the left-right basis it is more convenient to introduce a vector-axial basis by defining the vectorlike current $J^\mu_V = J^\mu_L + J^\mu_R$ and the axial current $J^\mu_A = J^\mu_L - J^\mu_R$. Let $V_\mu(x)$ be the gauge field that couples to the vectorlike current and $A_\mu(x)$ be the gauge field coupling to the axial current. The (consistent) anomalies for the vector and axial current turn out to be

$$\partial_\mu J^\mu_V = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\lambda}F^V_{\mu\nu}F^A_{\rho\lambda}, \tag{17.16}$$

$$\partial_\mu J^\mu_A = \frac{1}{48\pi^2}\varepsilon^{\mu\nu\rho\lambda}\big(F^V_{\mu\nu}F^V_{\rho\lambda} + F^A_{\mu\nu}F^A_{\rho\lambda}\big). \tag{17.17}$$

As long as the vectorlike current corresponds to a global symmetry nothing has gone wrong so far. If we want to identify the vectorlike current with the electric-magnetic current in nature we need to couple it to a dynamical photon gauge field and now the non-conservation of the vector current is worrisome to say the least. The problem arises because implicitly we presumed a regularization scheme that treats left-handed and right-handed fermions on the same footing. As pointed out first by Bardeen this flaw can be repaired by adding a finite counterterm (of order $\hbar$) to the effective action. This is the so-called Bardeen counterterm and has the form

$$B_{ct} = -\frac{1}{12\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\lambda}V_\mu A_\nu F^V_{\rho\lambda}. \tag{17.18}$$

Adding this counterterm to the effective action gives additional contributions of Chern-Simons form to the consistent vector and axial currents. With the particular coefficient chosen it turns out that the anomaly in the vector current is canceled whereas the axial current picks up an additional contribution such that after adding the Bardeen counterterm the anomalies are

$$\partial_\mu J^\mu_V = 0, \tag{17.19}$$

$$\partial_\mu J^\mu_A = \frac{1}{48\pi^2}\varepsilon^{\mu\nu\rho\lambda}\big(3F^V_{\mu\nu}F^V_{\rho\lambda} + F^A_{\mu\nu}F^A_{\rho\lambda}\big). \tag{17.20}$$

This definition of currents is mandatory if we want to identify the vector current with the usual electromagnetic current in nature! It is furthermore worth to point out that both currents are now invariant under the vectorlike $U(1)$ symmetry. The currents are not invariant under axial transformation, but these are anomalous anyway.

Generalizations of the covariant anomaly and the Bardeen counterterm to the non-abelian case can be found e.g. in [1].

There is one more anomaly that will play a major role in this review, the mixed gauge-gravitational anomaly [49–51].[3] So far we have considered only spin one currents and have coupled them to gauge fields. Now we also want to introduce the energy-momentum tensor through its coupling to a fiducial background metric $g_{\mu\nu}$. Just as the gauge fields, the metric serves primarily as the source for insertions of the energy momentum tensor in correlation functions. Just as in the case of vector and axial currents, the mixed gauge-gravitational anomaly is the statement that it is impossible in the quantum theory to preserve at the same time the vanishing of the divergence of the energy-momentum tensor and of chiral (or axial) $U(1)$ currents. It is however possible to add Bardeen counterterms to shift the anomaly always in the sector of the spin one currents and preserve translational (or diffeomorphism) symmetry. If we have a set of left-handed and right-handed chiral fermions transforming under a Lie Group generated by $(T_a)_L$ and $(T_a)_R$ in the background of arbitrary gauge fields and metric, the anomaly is conveniently expressed through the non-conservation of the *covariant* current as

$$\left(D_\mu J^\mu\right)_a = \frac{d_{abc}}{32\pi^2}\varepsilon^{\mu\nu\rho\lambda}F^b_{\mu\nu}F^c_{\rho\lambda} + \frac{b_a}{768\pi^2}\varepsilon^{\mu\nu\rho\lambda}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\rho\lambda}. \tag{17.21}$$

The purely group theoretic factors are

$$d_{abc} = \frac{1}{2}\operatorname{tr}\big(T_a\{T_b,T_c\}\big)_L - \frac{1}{2}\operatorname{tr}\big(T_a\{T_b,T_c\}\big)_R, \tag{17.22}$$

$$b_a = \operatorname{tr}(T_a)_L - \operatorname{tr}(T_a)_R. \tag{17.23}$$

Chiral anomalies are completely absent if and only if $d_{abc}=0$ and b_a=0.

17.2.2 Chemical Potentials for Anomalous Symmetries

Thermodynamics of systems with conserved charges can be described in a grand canonical ensemble where a Lagrange multiplier μ ensures that the partition function fulfills

$$T\frac{\partial\log(Z)}{\partial\mu} = \langle Q\rangle. \tag{17.24}$$

The textbook approach is to consider a deformation of the Hamiltonian

$$H \to H - \mu Q, \tag{17.25}$$

where Q is the charge in question. We can think of this as arising from the coupling of the (fiducial) gauge field A_μ to the current J^μ and giving a vacuum expectation value to $A_0=\mu$. Since the fiducial gauge field leads to local gauge invariance we can remove the μQ coupling in the Hamiltonian by the gauge transformation $A_0 \to A_0 + \partial_0\chi$ with $\chi = -\mu t$.

[3] In $D = 4k+2$ dimensions also purely gravitational anomalies can appear [52].

Fig. 17.1 At finite temperature field theories are defined on the Keldysh-Schwinger contour in the complexified time plane. The initial state at t_i is specified through the boundary conditions on the fields. The endpoint of the contour is at $t_i - i\beta$ where $\beta = 1/T$

Table 17.1 Two formalisms for the chemical potential

Formalism	Hamiltonian	Boundary condition
(A)	$H - \mu Q$	$\Psi(t_i - i\beta) = \pm\Psi(t_i)$
(B)	H	$\Psi(t_i - i\beta) = \pm e^{q\beta\mu}\Psi(t_i)$

In the context of finite temperature field theory such a gauge transformation is however not really allowed. One needs to define the field theory on the Keldysh-Schwinger contour in the complexified time plane as shown in Fig. 17.1. Fields are taken to be periodic or anti-periodic along the imaginary time direction $t = -i\tau$ with period $\beta = 1/T$ where T is the temperature

$$\Psi(t_i - i\beta) = \pm\Psi(t_i), \tag{17.26}$$

with the plus sign for bosons and the minus sign for fermions. The gauge transformation that removes the constant zero component of the gauge field is not periodic along the contour and therefore changes the boundary conditions on the fields. After the gauge transformation with $\chi = -\mu t$ the fields obey the boundary conditions

$$\Psi(t_i - i\beta) = \pm e^{q\mu\beta}\Psi(t_i). \tag{17.27}$$

Demanding these "twisted" boundary conditions is of course completely equivalent to having $A_0 = \mu$. The gauge invariant statement is that a charged field parallel transported around the Keldysh-Schwinger contour picks up a factor of $\exp(q\mu\beta)$. As long as we have honest non-anomalous symmetries under consideration we have therefore two (gauge-)equivalent formalisms of how to introduce the chemical potential summarized in Table 17.1 [53].

One convenient point of view on formalism (B) is the following. In a real time Keldysh-Schwinger setup we demand some initial conditions at initial (real) time $t = t_i$. These initial conditions are given by the boundary conditions in (B). From then on we do the (real) time development with the microscopic Hamiltonian H. In principle there is no need for the Hamiltonian H to preserve the symmetry present at times $t < t_i$. This seems an especially suited approach to situations where the charge in question is not conserved by the real time dynamics. In the case of an anomalous symmetry we can start at $t = t_i$ with a state of certain charge. As long as we have only external gauge fields present the one-point function of the divergence of the current vanishes and the charge is conserved. This is not true on the full theory since even in vacuum the three-point correlators are sensitive to the anomaly. For the formulation of hydrodynamics in external fields the condition that the one-point functions of the currents are conserved as long as there are no parallel electric and magnetic external fields (or a metric that has non-vanishing Pontryagin density) is sufficient.[4]

Let us assume now that Q is an anomalous charge, i.e. its associated current suffers from chiral anomalies. We first consider formalism (B) and ask what happens if we do now the gauge transformation that would bring us to formalism (A). Since the symmetry is anomalous the action transforms as

$$S[A+\partial\chi] = S[A] + \int \mathrm{d}^4x\, \chi\varepsilon^{\mu\nu\rho\lambda}\big(C_1 F_{\mu\nu}F_{\rho\lambda} + C_2 R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda}\big), \qquad (17.28)$$

with the anomaly coefficients C_1 and C_2 depending on the chiral fermion content. It follows that formalisms (A) and (B) are physically inequivalent now, because of the anomaly. However, we would like to still come as close as possible to the formalism of (A) but in a form that is physically equivalent to the formalism (B). To achieve this we proceed by introducing a non-dynamical axion field $\Theta(x)$ and the vertex

$$S_\Theta[A,\Theta] = \int \mathrm{d}^4x\, \Theta\varepsilon^{\mu\nu\rho\lambda}\big(C_1 F_{\mu\nu}F_{\rho\lambda} + C_2 R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda}\big). \qquad (17.29)$$

If we demand now that the "axion" transforms as $\Theta \to \Theta - \chi$ under gauge transformations we see that the action

$$S_{tot}[A,\Theta] = S[A] + S_\Theta[A,\Theta] \qquad (17.30)$$

is gauge invariant. Note that this does not mean that the theory is not anomalous now. We introduce it solely for the purpose to make clear how the action has to be modified such that two field configurations related by a gauge transformation are physically equivalent. It is better to consider Θ as *coupling* and not a field, i.e. we consider it a spurion field. The gauge field configuration that corresponds to formalism (B) is simply $A_0 = 0$. A gauge transformation with $\chi = \mu t$ on the gauge invariant action S_{tot} makes clear that a physically equivalent theory is obtained by choosing the field configuration $A_0 = \mu$ and the time dependent coupling $\Theta = -\mu t$. If we define the current through the variation of the action with respect to the gauge field we get an additional contribution from S_Θ,

$$J^\mu_\Theta = 4C_1\varepsilon^{\mu\nu\rho\lambda}\partial_\nu\Theta F_{\rho\lambda}, \qquad (17.31)$$

[4] If dynamical gauge fields are present, such as the gluon fields in QCD even the one point function of the charge does decay over (real) time due to non-perturbative processes (instantons) or at finite temperature due to thermal sphaleron processes [54]. Even in this case in the limit of large number of colors these processes are suppressed and can e.g. not be seen in holographic models in the supergravity approximation.

and evaluating this for $\Theta = -\mu t$ we get the spatial current

$$J^m_\Theta = 8C_1 \mu B^m. \tag{17.32}$$

We do not consider this to be the chiral magnetic effect! This is only the contribution to the current that comes from the new coupling that we are forced to introduce by going to formalism (A) from (B) in a (gauge-)equivalent way. As we will see in the following chapters the chiral magnetic and vortical effect are on the contrary non-trivial results of dynamical one-loop calculations.

What is the Hamiltonian now based on the modified formalism (A)? We have to take of course the new coupling generated by the non-zero Θ. The Hamiltonian now is therefore

$$H - \mu\left(Q + 4C_1 \int \mathrm{d}^3x\, \varepsilon^{0ijk} A_i \partial_j A_k\right), \tag{17.33}$$

where for simplicity we have ignored the contributions from the metric terms.

For explicit computations in Sects. 17.3 and 17.4 we will introduce the chemical potential through the formalism (B) by demanding twisted boundary conditions. It seems the most natural choice since the dynamics is described by the microscopic Hamiltonian H. The modified (A) based on the Hamiltonian (17.33) is however not without merits. It is convenient in holography where it allows vanishing temporal gauge field on the black hole horizon and therefore a non-singular Euclidean black hole geometry.[5]

17.2.2.1 Hydrodynamics and Kubo Formulae

The modern understanding of hydrodynamics is as an effective field theory. The equations of motion are the (anomalous) conservation laws of the energy-momentum tensor and spin one currents. These are supplemented by expression for the energy-momentum tensor and the current which are organized in a derivative expansion, the so-called constitutive relations. Symmetries constrain the possible terms. In the presence of chiral anomalies the constitutive relations for the energy-momentum tensor and the currents in the Landau frame are

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p P^{\mu\nu} - \eta P^{\mu\alpha} P^{\nu\beta} \sigma_{\alpha\beta} - \zeta P^{\mu\nu} \partial_\alpha u^\alpha, \tag{17.34}$$

$$\tilde{J}^\mu_a = \rho_a u^\mu + \Sigma_{ab}\left(E^\mu_b - T P^{\mu\alpha} D_\alpha \frac{\mu_a}{T}\right) + \xi^B_{ab} B^\mu_a + \xi^V_a \omega^\mu. \tag{17.35}$$

It is important to specify that these are the constitutive relations for the *covariant* currents! Here ε is the energy density, p the pressure density, u^μ the local fluid velocity. $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the transverse projector to the fluid velocity.

[5] It is possible to define a generalized formalism to make any choice for the gauge field $A_0 = \nu$, so that one recovers formalism (A) when $\nu = \mu$ and formalism (B) when $\nu = 0$ as particular cases (see [55] for details).

$\sigma_{\mu\nu}$ is the symmetric traceless shear tensor. The non-anomalous transport coefficients are the shear viscosity η, the bulk viscosity ζ and the electric conductivities Σ_{ab}. External electric and magnetic fields are covariantized via $E_a^\mu = F_a^{\mu\nu} u_\nu$ and $B_a^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda} u_\nu F_{a,\rho\lambda}$. The vorticity of the fluid is $\omega^\mu = \varepsilon^{\mu\nu\rho\lambda} u_\nu \partial_\rho u_\lambda$.

The anomalous transport coefficients are the chiral magnetic conductivities ξ_{ab}^B and the chiral vortical conductivities ξ_a^V. At first order in derivatives the notion of fluid velocity is ambiguous and needs to be fixed by prescribing a choice of frame. We remark that the constitutive relations (17.34) and (17.35) are valid in the Landau frame where $T^{\mu\nu} u_\nu = \varepsilon u^\mu$.

To compute the Kubo formulae for the anomalous transport coefficients it turns out that the Landau frame is not the most convenient one. It fixes the definition of the fluid velocity through energy transport. Transport phenomena related to the generation of an energy current are therefore not directly visible, rather they are absorbed in the definition of the fluid velocity. It is therefore more convenient to go to another frame in which we demand that the definition of the fluid velocity is not influenced when switching on an external magnetic field or having a vortex in the fluid. In such a frame the constitutive relations take the form

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} - \eta P^{\mu\alpha} P^{\nu\beta} \sigma_{\alpha\beta} - \zeta P^{\mu\nu} \partial_\alpha u^\alpha \\ &\quad + Q^\mu u^\nu + Q^\nu u^\mu, \end{aligned} \tag{17.36}$$

$$Q^\mu = \sigma_B^\varepsilon B^\mu + \sigma_V^\varepsilon \omega^\mu, \tag{17.37}$$

$$\tilde{J}^\mu = \rho u^\mu + \Sigma \left(E^\mu - T P^{\mu\alpha} D_\alpha \left(\frac{\mu}{T} \right) \right) + \sigma_B B^\mu + \sigma_V \omega^\mu. \tag{17.38}$$

In order to avoid unnecessary clutter in the equations we have specialized now to a single $U(1)$ charge. Notice that now there is a sort of "heat" current present in the constitutive relation for the energy-momentum tensor.

The derivation of Kubo formulae is better based on the usage of the *consistent* currents. Since the covariant and consistent currents are related by adding suitable Chern-Simons currents, the constitutive relations for the consistent current receive additional contribution from the Chern-Simons current

$$J^\mu = \tilde{J}^\mu - \frac{1}{24\pi^2} K^\mu. \tag{17.39}$$

If we were to introduce the chemical potential according to formalism (A) via a background for the temporal gauge field we would get an additional contribution to the consistent current from the Chern-Simons current. In this case it is better to go to the modified formalism (A') that also introduces a spurious axion field and another contribution to the current J_Θ (17.32) has to be added

$$J^\mu = \tilde{J}^\mu - \frac{1}{24\pi^2} K^\mu + J_\Theta^\mu. \tag{17.40}$$

For the derivation of the Kubo formulae it is therefore more convenient to work with formalism (B) in which $A_0 = 0$ and the chemical potential is introduced via the boundary conditions (17.27). Otherwise there arise additional contributions to the two point functions. We will briefly discuss them in the next subsection.

From the microscopic view the constitutive relations should be interpreted as the one-point functions of the operators $T^{\mu\nu}$ and J^μ in a near equilibrium situation, i.e. gradients in the fluid velocity, the temperature or the chemical potentials are assumed to be small. From this point of view Kubo formulae can be derived. In the microscopic theory the one-point function of an operator near equilibrium is given by linear response theory whose basic ingredient are the retarded two-point functions. If we consider a situation with only an external electric field in z-direction and all other sources switched off, i.e. the fluid being at rest $u^\mu = (1,0,0,0)$ and no gradients in temperature or chemical potentials the constitutive relations are simplified to

$$J^z = \Sigma E^z. \tag{17.41}$$

The electric field is $E^z = i\omega A^z$ in terms of the vector potential and using linear response theory the induced current is given through the retarded two-point function by

$$J^z = \langle J^z J^z \rangle A^z. \tag{17.42}$$

Equating the two expressions for the current we find the Kubo formula for the electric conductivity

$$\Sigma = \lim_{\omega\to 0} \frac{-i}{\omega} \langle J^z J^z \rangle. \tag{17.43}$$

This has to be evaluated at zero momentum. The limit in the frequency follows because the constitutive relation are supposed to be valid only to lowest order in the derivative expansion, therefore one needs to isolate the first non-trivial term.

Now we want to find some simple special cases that allow the derivation of Kubo formulae for the anomalous conductivities. A very convenient choice is to go to the restframe $u^\mu = (1,0,0,0)$, switch on a vector potential in the y-direction that depends only on the z direction and at the same time a metric deformation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with the only non-vanishing component h_{0y} depending on z only. To linear order in the background fields the non-vanishing components of the energy-momentum tensor and the current are

$$T^{0x} = -\sigma_B^\varepsilon \partial_z A_y - \sigma_V^\varepsilon \partial_z h_{0y}, \tag{17.44}$$

$$J^x = -\sigma_B \partial_z A_y - \sigma_V \partial_z h_{0y}, \tag{17.45}$$

since in the formalism (B) neither the Chern-Simons term nor the Θ coupling contribute. Going to momentum space and differentiating with respect to the sources A_y and h_{0y} we find therefore the Kubo formulae [26, 34]

$$\boxed{\begin{array}{ll} \sigma_B = \lim_{k_z\to 0} \frac{i}{k_z} \langle J^x J^y \rangle & \sigma_V = \lim_{k_z\to 0} \frac{i}{k_z} \langle J^x T^{0y} \rangle \\[2ex] \sigma_B^\varepsilon = \lim_{k_z\to 0} \frac{i}{k_z} \langle T^{0x} J^y \rangle & \sigma_V^\varepsilon = \lim_{k_z\to 0} \frac{i}{k_z} \langle T^{0x} T^{0y} \rangle \end{array}} \tag{17.46}$$

All these correlators are to be taken at precisely zero frequency. As these formulas are based on linear response theory the correlators should be understood as retarded ones. They have to be evaluated however at zero frequency and therefore the order of the operators can be reversed. From this it follows that the chiral vortical conductivity coincides with the chiral magnetic conductivity for the energy flux $\sigma_V = \sigma_B^\varepsilon$.[6]

We also want to discuss how these transport coefficients are related to the ones in the more commonly used Landau frame. They are connected by a redefinition of the fluid velocity of the form

$$u^\mu \to u^\mu - \frac{1}{\varepsilon + p} Q^\mu, \tag{17.47}$$

to go from (17.36)–(17.38) to (17.34)–(17.35). The corresponding transport coefficients of the Landau frame are therefore

$$\xi_B = \lim_{k_n \to 0} \frac{-i}{2k_n} \sum_{k.l} \varepsilon_{nkl} \left(\langle J^k J^l \rangle - \frac{\rho}{\varepsilon + p} \langle T^{0k} J^l \rangle \right), \tag{17.48}$$

$$\xi_V = \lim_{k_n \to 0} \frac{-i}{2k_n} \sum_{k.l} \varepsilon_{nkl} \left(\langle J^k T^{0l} \rangle - \frac{\rho}{\varepsilon + p} \langle T^{0k} T^{0l} \rangle \right), \tag{17.49}$$

where we have employed a slightly more covariant notation. The generalization to the non-abelian case is straightforward.

It is also worth to compare to the Kubo formulae for the dissipative transport coefficients as the electric conductivity (17.43). In the dissipative cases one first goes to zero momentum and then takes the zero frequency limit. In the anomalous conductivities this is the other way around, one first goes to zero frequency and then takes the zero momentum limit. Another observation is that the dissipative transport coefficients sit in the anti-Hermitean part of the retarded correlators, i.e. the spectral function whereas the anomalous conductivities sit in the Hermitean part. The rate at which an external source f_I does work on a system is given in terms of the spectral function of the operator O^I coupling to that source as

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{2} \omega f_I(-\omega) \rho^{IJ}(\omega) f_J(\omega). \tag{17.50}$$

The anomalous transport phenomena therefore do no work on the system, first they take place at zero frequency and second they are not contained in the spectral function $\rho = \frac{-i}{2}(G_r - G_r^\dagger)$.

[6]Notice that h_{0y} can also be understood as the so-called gravito-magnetic vector potential $\mathbf{A}_g$, which is related to the gravito-magnetic field by $\mathbf{B}_g = \nabla \times \mathbf{A}_g$. This allows to interpret σ_V not only as the generation of a current due to a vortex in the fluid, i.e. the chiral vortical effect, but also as a *chiral gravito-magnetic* conductivity giving rise to a *chiral gravito-magnetic effect*, see [56] for details.

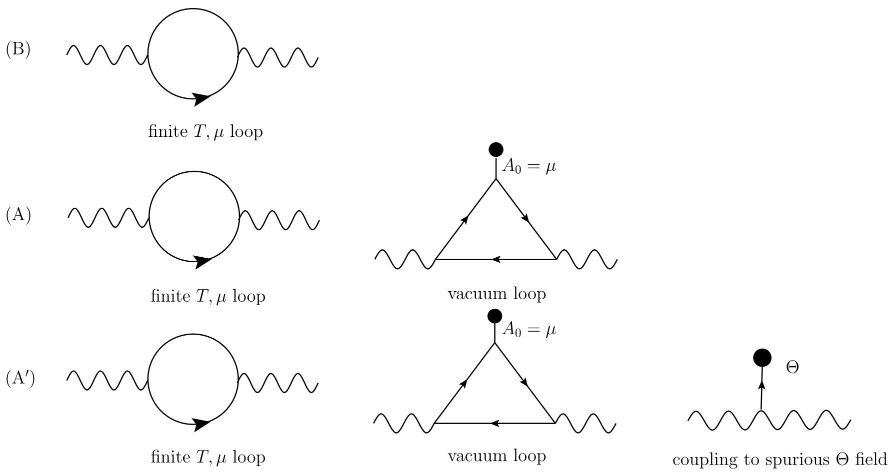

Fig. 17.2 Contributions to the Kubo formula for the chiral magnetic conductivity in the different formalisms for the chemical potential

17.2.3 Contributions to the Kubo Formulae

Now we want to give a detailed analysis of the different Feynman graphs that contribute to the Kubo formulae in the different formalisms for the chemical potentials. The simplest and most economic formalism is certainly the one labeled (B) in which we introduce the chemical potentials via twisted boundary conditions. The Hamiltonian is simply the microscopic Hamiltonian H. Relevant contributions arise only at first order in the momentum and at zero frequency and in this kinematic limit only the Kubo formulae for the chiral magnetic conductivity is affected. In Fig. 17.2 we summarize the different contributions to the Kubo formulae in the three ways to introduce the chemical potential.

The first of the Feynman graphs is the same in all formalisms. It is the genuine finite temperature and finite density one-loop contribution. This graph is finite because the Fermi-Dirac distributions cutoff the UV momentum modes in the loop. In the formalism (A) we need to take into account that there is also a contribution from the triangle graph with the fermions going around the loop in vacuum, i.e. without the Fermi-Dirac distributions in the loop integrals. For a non-anomalous symmetry this graph vanishes simply because on the upper vertex of the triangle sits a field configuration that is a pure gauge. If the symmetry under consideration is however anomalous the triangle diagram picks up just the anomaly. Even pure gauge field configurations become physically distinct from the vacuum and therefore this diagram gives a non-trivial contribution. On the level of the constitutive relations this contribution corresponds to the Chern-Simons current in (17.39). We consider this contribution to be unwanted. After all the anomaly would make even a constant value of the temporal gauge field A_0 observable in vacuum. An example is provided for a putative axial gauge field A^5_μ. If present the absolute value of its temporal component would be observable through the axial anomaly. We can be sure that in nature no such background field is present. The third line (A') introduces also the spurious axion field Θ the only purpose of this field is to cancel the contribution from the triangle graph. This cancellation takes place by construction since (A') is gauge equivalent to (B) in which only the first genuine finite T, μ part contributes. It corresponds to the contribution of the current J^{μ}_{Θ} in (17.40). We further emphasize that these considerations are based on the usage of the consistent currents.

In the interplay between axial and vector currents additional contributions arise from the Bardeen counterterm. It turns out that the triangle or Chern-Simons current contribution to the consistent vector current in the formalism (A) cancels precisely the first one [44, 45]. Our take on this is that a constant temporal component of the axial gauge field $A_0 = \mu_5$ would be observable in nature and can therefore be assumed to be absent. The correct way of evaluating the Kubo formulae for the chiral magnetic effect is therefore the formalism (B) or the gauge equivalent one (A').

At this point the reader might wonder why we introduced yet another formalism (A') which achieves apparently nothing but being equivalent to formalism (B). At least from the perspective of holography there is a good reason for doing so. In holography the strong coupling duals of gauge theories at finite temperature in the plasma phase are represented by five dimensional asymptotically Anti- de Sitter black holes. Finite charge density translates to charged black holes. These black holes have some non-trivial gauge flux along the holographic direction represented by a temporal gauge field configuration of the form $A_0(r)$ where r is the fifth, holographic dimension. It is often claimed that for consistency reasons the gauge field has to vanish on the horizon of the black hole and that its value on the boundary can be identified with the chemical potential

$$A_0(r_{\mathrm{H}}) = 0 \quad \text{and} \quad A_0(r \to \infty) = \mu. \tag{17.51}$$

According to the usual holographic dictionary the gauge field values on the boundary correspond to the sources for currents. A non-vanishing value of the temporal component of the gauge field at the boundary is therefore dual to a coupling that modifies the Hamiltonian of the theory just as in (17.25). Thus with the boundary conditions (17.51) we have the holographic dual of the formalism (A). If anomalies are present they are represented in the holographic dual by five-dimensional Chern-Simons terms of the form $A \wedge F \wedge F$. The two point correlator of the (consistent) currents receives now contributions from the Chern-Simons term that is precisely of the form of the second graph in (A) in Fig. 17.2. As we have argued this is an a priory unwanted contribution. We can however cure that by introducing an additional term in the action of the form (17.29) living only on the boundary of the holographic space-time. In this way we can implement the formalism (A'), cancel the unwanted triangle contribution with the third graph in (A') in Fig. 17.2 and maintain $A_0(r_{\mathrm{H}}) = 0$!

The claim that the temporal component of the gauge field has to vanish at the horizon is of course not unsubstantiated. The reasoning goes as follows. The Euclidean section of the black-hole space time has the topology of a disc in the r, τ

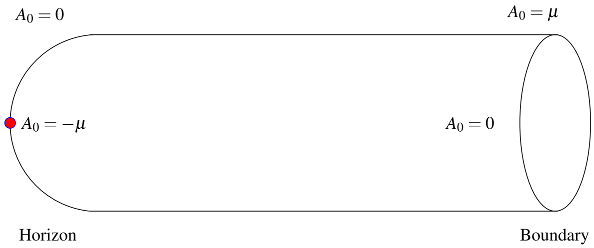

Fig. 17.3 A sketch of the Euclidean black hole topology. A singularity at the horizon arises if we do not choose the temporal component of the gauge field to vanish there. On the other hand allowing the singularity to be present changes the topology to the one of a cylinder and this in turn allows twisted boundary conditions

directions, where τ is the Euclidean time (see Fig. 17.3). This is a periodic variable with period $\beta = 1/T$ where T is the (Hawking) temperature of the black hole and at the same time the temperature in the dual field theory. Using Stoke's law we have

$$\int_{\partial D} A_0 \, \mathrm{d}\tau = \int_D F_{r0} \, \mathrm{d}r \, \mathrm{d}\tau, \tag{17.52}$$

where F_{r0} is the electric field strength in the holographic direction and D is a Disc with origin at $r = r_{\mathrm{H}}$ reaching out to some finite value of r_f. If we shrink this disc to zero size, i.e. let $r_f \to r_{\mathrm{H}}$ the r.h.s. of the last equation vanishes and so must the l.h.s. which approaches the value $\beta A_0(r_{\mathrm{H}})$. This implies that $A_0(r_{\mathrm{H}}) = 0$. If on the other hand we assume that $A_0(r_{\mathrm{H}}) \neq 0$ then the field strength must have a delta type singularity there in order to satisfy Stokes theorem. Strictly speaking the topology of the Euclidean section of the black hole is not anymore that of a disc since now there is a puncture at the horizon. It is therefore more appropriate to think of this as having the topology of a cylinder. Now if we want to implement the formalism (B) in holography we would find the boundary conditions

$$A_0(r_{\mathrm{H}}) = \mu \quad \text{and} \quad A_0(r \to \infty) = 0, \tag{17.53}$$

and precisely such a singularity at the horizon would arise. In addition we would need to impose twisted boundary conditions around the Euclidean time τ for the fields just as in (17.27). Now the presence of the singularity seems to be a good thing: if the space time would still be smooth at the horizon it would be impossible to demand these twisted boundary conditions since the circle in τ shrinks to zero size there. If this is however a singular point of the geometry we can not really shrink the circle to zero size. The topology being rather a cylinder than a disc allows now for the presence of the twisted boundary conditions.

It is also important to note that in all formalisms the potential difference between the boundary and the horizon is given by μ. This has a very nice intuitive interpretation. If we bring a unit test charge from the boundary to the horizon we need the energy $\Delta E = \mu$. In the dual field theory this is just the energy cost of adding one unit of charge to the thermalized system and coincides with the elementary definition of the chemical potential.

From now on we will always only consider the genuine finite T, μ contribution that is the only one that arises in formalism (B).

The rest of this review is devoted to the explicit evaluation of these Kubo formulae in two different systems: free chiral fermions and a holographic model implementing the chiral and gravitational anomalies by suitable five dimensional Chern-Simons terms.

17.3 Weyl Fermions

We will now evaluate the Kubo formulae for the chiral magnetic, chiral vortical and energy flux conductivities (17.46) for a theory of N free chiral fermions Ψ^f transforming under a global symmetry group G generated by matrices $(T_a)^f{}_g$.

We denote the generators in the Cartan subalgebra by H_a. Chemical potentials μ_a can be switched on only in the Cartan subalgebra. Furthermore the presence of the chemical potentials breaks the group G to a subgroup $\widehat{G}$. Only the currents that lie in the unbroken subgroup are conserved (up to anomalies) and participate in the hydrodynamics. The chemical potential for the fermion Ψ^f is given by $\mu^f = \sum_a q_a^f \mu_a$, where we write the Cartan generator $H_a = q_a^f \delta^f{}_g$ in terms of its eigenvalues, the charges q_a^f. The unbroken symmetry group $\widehat{G}$ is generated by those matrices $T_a^f{}_g$ fulfilling

$$T_a^f{}_g \mu^g = \mu^f T_a^f{}_g. \tag{17.54}$$

There is no summation over indices in the last expression. From now on we will assume that all currents $\mathbf{J}_a$ lie in directions indicated in (17.54). We define the chemical potential through the boundary condition on the fermion fields around the thermal circle, i.e. we adopt the formalism (B) discussed in previous section,

$$\Psi^f(\tau - \beta) = -e^{\beta\mu^f} \Psi^f(\tau). \tag{17.55}$$

Therefore the eigenvalues of ∂_τ are $i\tilde{\omega}_n + \mu^f$ for the fermion spiecies f with $\tilde{\omega}_n = \pi T(2n+1)$ the fermionic Matsubara frequencies. A convenient way of expressing the current and the energy-momentum tensor is in terms of Dirac fermions and writing

$$J_a^i = \sum_{f,g=1}^{N} T_a^g{}_f \bar{\Psi}_g \gamma^i \mathscr{P}_+ \Psi^f, \qquad T^{0i} = \frac{i}{2} \sum_{f=1}^{N} \bar{\Psi}_f \big(\gamma^0 \partial^i + \gamma^i \partial^0\big) \mathscr{P}_+ \Psi^f, \tag{17.56}$$

where we used the chiral projector $\mathscr{P}_\pm = \frac{1}{2}(1 \pm \gamma_5)$. The fermion propagator is

$$S(q)^f{}_g = \frac{\delta^f{}_g}{2} \sum_{t=\pm} \Delta_t\big(i\tilde{\omega}^f, \mathbf{q}\big) \mathscr{P}_+ \gamma_\mu \hat{q}_t^\mu, \qquad \Delta_t\big(i\tilde{\omega}^f, q\big) = \frac{1}{i\tilde{\omega}^f - tE_q}, \tag{17.57}$$

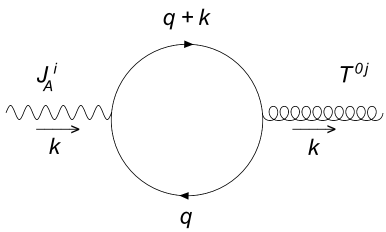

Fig. 17.4 1 loop diagram contributing to the vortical conductivity (17.58)

with $i\tilde{\omega}^f = i\tilde{\omega}_n + \mu^f$, $\hat{q}_t^\mu = (1, t\hat{q})$, $\hat{q} = \frac{\mathbf{q}}{E_q}$ and $E_q = |\mathbf{q}|$. For simplicity in the expressions we consider only left-handed fermions, but one can easily include right-handed fermions as well as they contribute in all our calculations in the same way as the left-handed ones up to a relative minus sign.

We will address in detail the computation of the vortical conductivities and sketch only the calculation of the magnetic conductivities since the latter one is a trivial extension of the calculation of the chiral magnetic conductivity in [33]. Then we show the results for the other conductivity coefficients.

17.3.1 Chiral Vortical Conductivity

The vortical conductivity is defined from the retarded correlation function of the current $J_a^i(x)$ and the energy momentum tensor or energy current $T^{0j}(x')$ (17.56), i.e.

$$G_a^V(x - x') = \frac{1}{2}\varepsilon_{ijn}\, i\theta(t - t')\langle[J_a^i(x), T^{0j}(x')]\rangle. \tag{17.58}$$

Going to Fourier space, one can evaluate this quantity as

$$G_a^V(k) = \frac{1}{4}\sum_{f=1}^{N} T_a{}^f{}_f \frac{1}{\beta}\sum_{\tilde{\omega}^f}\int \frac{d^3q}{(2\pi)^3}\varepsilon_{ijn}\,\mathrm{tr}\big[S^f{}_f(q)\gamma^i S^f{}_f(q+k)\big(\gamma^0 q^j + \gamma^j i\tilde{\omega}^f\big)\big], \tag{17.59}$$

which corresponds to the one loop diagram of Fig. 17.4. The vertex of the two quarks with the graviton is $\sim\delta^f{}_g$, and therefore we find only contributions from the diagonal part of the group $\widehat{G}$. The metric we use through this section is the usual one in field theory computations, $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We can split G_a^V into two contributions, i.e.

$$G_a^V(k) = G_{a,(0j)}^V(k) + G_{a,(j0)}^V(k), \tag{17.60}$$

which correspond to the terms $\gamma^0 q^j$ and $\gamma^j i\tilde{\omega}^f$ in (17.59) respectively. We will focus first on the computation of $G_{a,(0j)}^V$. After computation of the Dirac trace in (17.59), this term writes

$$G^V_{a,(0j)}(k) = \frac{1}{8}\sum_{f=1}^{N} T_a^f{}_f \frac{1}{\beta}\sum_{\tilde{\omega}^f}\int \frac{d^3q}{(2\pi)^3} q^j \sum_{t,u=\pm}\bigg[\varepsilon_{ijn}\bigg(t\frac{q^i}{E_q} + u\frac{k^i+q^i}{E_{q+k}}\bigg) + i\frac{tu}{E_q E_{q+k}}(q_j k_n - q_n k_j)\bigg]\Delta_t\big(i\tilde{\omega}^f, \mathbf{k}\big)\Delta_u\big(i\tilde{\omega}^f + i\omega_n, \mathbf{q}+\mathbf{k}\big). \tag{17.61}$$

At this point one can make a few simplifications. Note that due to the antisymmetric tensor ε_{ijn}, the two terms proportional to q^i inside the bracket in (17.61) vanish. Regarding the term $\varepsilon_{ijn} q^j k^i$, it leads to a contribution $\sim\varepsilon_{ijn} k^j k^i$ after integration in d^3q, which is zero. Then the only term which remains is the one not involving ε_{ijn}. We can now perform the sum over fermionic Matsubara frequencies. One has

$$\frac{1}{\beta}\sum_{\tilde{\omega}^f}\Delta_t\big(i\tilde{\omega}^f, \mathbf{q}\big)\Delta_u\big(i\tilde{\omega}^f + i\omega_n, \mathbf{q}+\mathbf{p}\big) = \frac{tn(E_q - t\mu^f) - un(E_{q+k} - u\mu^f) + \frac{1}{2}(u-t)}{i\omega_n + tE_q - uE_{q+k}}, \tag{17.62}$$

where $n(x) = 1/(e^{\beta x}+1)$ is the Fermi-Dirac distribution function. In (17.62) we have considered that $\omega_n = 2\pi T n$ is a bosonic Matsubara frequency. This result is also obtained in Ref. [33]. After doing the analytic continuation, which amounts to replacing $i\omega_n$ by $k_0 + i\varepsilon$ in (17.62), one gets

$$G^V_{a,(0j)}(k) = -\frac{i}{8}\sum_{f=1}^{N} T_a^f{}_f \int \frac{d^3q}{(2\pi)^3}\frac{\mathbf{q}^2 k_n - (\mathbf{q}\cdot\mathbf{k})q_n}{E_q E_{q+k}} \times \sum_{t,u=\pm}\frac{un(E_q - t\mu^f) - tn(E_{q+k} - u\mu^f) + \frac{1}{2}(t-u)}{k_0 + i\varepsilon + tE_q - uE_{q+k}}. \tag{17.63}$$

The term proportional to $\sim\frac{1}{2}(t-u)$ corresponds to the vacuum contribution, and it is ultraviolet divergent. By removing this term the finite temperature and chemical potential behavior is not affected, and the result becomes ultraviolet finite because the Fermi-Dirac distribution function exponentially suppresses high momenta. By making both the change of variable $\mathbf{q} \to -\mathbf{q} - \mathbf{k}$ and the interchange $u \to -t$ and $t \to -u$ in the part of the integrand involving the term $-tn(E_{q+k} - u\mu^f)$, one can express the vacuum substracted contribution of (17.63) as

$$\widehat{G}^V_{a,(0j)}(k) = \frac{i}{8}k_n \sum_{f=1}^{N} T_a^f{}_f \int \frac{d^3q}{(2\pi)^3}\frac{1}{E_q E_{q+k}}\bigg(\mathbf{q}^2 - \frac{(\mathbf{q}\cdot\mathbf{k})^2}{\mathbf{k}^2}\bigg) \times \sum_{t,u=\pm} u\frac{n(E_q - \mu^f) + n(E_q + \mu^f)}{k_0 + i\varepsilon + tE_q + uE_{q+k}}, \tag{17.64}$$

where we have used that $n(E_q - t\mu^f) + n(E_q + t\mu^f) = n(E_q - \mu^f) + n(E_q + \mu^f)$ since $t = \pm 1$. The result has to be proportional to k_n, so to reach this expression we have replaced q_n by $(\mathbf{q}\cdot\mathbf{k})k_n/\mathbf{k}^2$ in (17.63). At this point one can perform the sum over u by using $\sum_{u=\pm} u/(a_1 + ua_2) = -2a_2/(a_1^2 - a_2^2)$, and the integration over angles by considering $\mathbf{q}\cdot\mathbf{k} = E_q E_k x$ and $E_{q+k}^2 = E_q^2 + E_k^2 + 2E_q E_k x$, where $x := \cos(\theta)$ and θ is the angle between $\mathbf{q}$ and $\mathbf{k}$. Then one gets the final result

$$\begin{aligned}\widehat{G}^V_{a,(0j)}(k) &= \frac{i}{16\pi^2}\frac{k_n}{k^2}\big(k^2 - k_0^2\big)\sum_{f=1}^{N} T_a^f{}_f \int_0^\infty \mathrm{d}q\, q f^V(q)\\ &\quad\times\left[1 + \frac{1}{8qk}\sum_{t=\pm}\big[k_0^2 - k^2 + 4q(q + tk_0)\big]\log\left(\frac{\Omega_t^2 - (q+k)^2}{\Omega_t^2 - (q-k)^2}\right)\right],\end{aligned} \tag{17.65}$$

where $\Omega_t = k_0 + i\varepsilon + tE_q$, and

$$f^V(q) = n\big(E_q - \mu^f\big) + n\big(E_q + \mu^f\big). \tag{17.66}$$

The steps to compute $G^V_{a,(j0)}$ in (17.60) are similar. In this case the Dirac trace leads to a different tensor structure, in which the only contribution comes from the trace involving γ_5. The sum over fermionic Matsubara frequencies involves an extra $i\tilde{\omega}^f$, i.e.

$$\begin{aligned}&\frac{1}{\beta}\sum_{\tilde{\omega}^f} i\tilde{\omega}^f \Delta_t\big(i\tilde{\omega}^f, \mathbf{q}\big)\Delta_u\big(i\tilde{\omega}^f + i\omega_n, \mathbf{q}+\mathbf{k}\big)\\ &\quad= \frac{1}{i\omega_n + tE_q - uE_{q+k}}\Big[E_q n\big(E_q - t\mu^f\big) - (E_{q+k} - ui\omega_n)n\big(E_{q+k} - u\mu^f\big)\\ &\qquad - \frac{1}{2}(E_q - E_{q+k} + ui\omega_n)\Big].\end{aligned} \tag{17.67}$$

The last term inside the bracket in the r.h.s. of (17.67) corresponds to the vacuum contribution which we choose to remove, as it leads to an ultraviolet divergent contribution after integration in d^3q. Making similar steps as for $\widehat{G}^V_{a,(0j)}$, i.e. performing the sum over u and integrating over angles, one gets the final result

$$\begin{aligned}\widehat{G}^V_{a,(j0)}(k) &= -\frac{i}{32\pi^2}\frac{k_n}{k^3}\sum_{f=1}^{N} T_a^f{}_f \int_0^\infty \mathrm{d}q \sum_{t=\pm} f_t^V(q, k_0)\\ &\quad\times\left[4tqkk_0 - \big(k^2 - k_0^2\big)(2q + tk_0)\log\left(\frac{\Omega_t^2 - (q+k)^2}{\Omega_t^2 - (q-k)^2}\right)\right],\end{aligned} \tag{17.68}$$

where

$$f_t^V(q, k_0) = qf^V(q) + tk_0 n\big(E_q + t\mu^f\big). \tag{17.69}$$

The result for the vacuum substracted contribution of the retarded correlation function of the current and the energy momentum tensor, $\widehat{G}^V_a(k)$, writes as a sum of (17.65) and (17.68), according to (17.60). From these expressions one can compute the zero frequency, zero momentum, limit. Since

$$\lim_{k\to 0}\lim_{k_0\to 0}\sum_{t=\pm}\log\left(\frac{\Omega_t^2-(q+k)^2}{\Omega_t^2-(q-k)^2}\right)=\frac{2k}{q}, \tag{17.70}$$

the relevant integrals are

$$\int_0^\infty \mathrm{d}q\, q f^V(q)=\int_0^\infty \mathrm{d}q\, f_t^V(q,k_0=0)=\frac{(\mu^f)^2}{2}+\frac{\pi^2}{6}T^2. \tag{17.71}$$

Finally it follows from (17.65) and (17.68) that the zero frequency, zero momentum, vortical conductivity writes

$$\begin{aligned}(\sigma_V)_a &= \frac{1}{8\pi^2}\sum_{f=1}^{N} T_a{}^f{}_f\left[\left(\mu^f\right)^2+\frac{\pi^2}{3}T^2\right]\\ &= \frac{1}{16\pi^2}\left[\sum_{b,c}\mathrm{tr}\big(T_a\{H_b,H_c\}\big)\mu_b\mu_c+\frac{2\pi^2}{3}T^2\,\mathrm{tr}(T_a)\right].\end{aligned} \tag{17.72}$$

Both $\widehat{G}^V_{a,(0j)}$ and $\widehat{G}^V_{a,(j0)}$ lead to the same contribution in $(\sigma_V)_a$. Equation (17.72) was first derived in [35], and it constitutes our main result in this section. The term involving the chemical potentials is induced by the chiral anomaly. More interesting is the term $\sim T^2$ which is proportional to the gravitational anomaly coefficient b_a [49–52]. This means that a non-zero value of this term has to be attributed to the presence of a gravitational anomaly. The Matsubara frequencies $\tilde{\omega}_n=\pi T(2n+1)$ generate a dependence on πT in the final result as compared to the chemical potentials, and then no factors of π show up for the term $\sim T^2$ in (17.72). Right-handed fermions contribute in the same way but with a relative minus sign. Therefore the $\sim T^2$ term appears only when the current in (17.58) has an axial component. The correlator with a vector current does not have this gravitational anomaly contribution.

17.3.2 Chiral Magnetic Conductivity

The chiral magnetic conductivity in the case of a vector and an axial $U(1)$ symmetry was computed at weak coupling in [33]. The corresponding Kubo formula involves the two point function of the current, see first expression in (17.46). Following the same method, we have computed it for the unbroken (non-abelian) symmetry group $\widehat{G}$. The relevant Green function is [35]

$$G^B_{ab}(k) = \frac{1}{2}\sum_{f,g} T_a^{f}{}_{g} T_b^{g}{}_{f} \frac{1}{\beta}\sum_{\tilde{\omega}^f}\int \frac{d^3q}{(2\pi)^3}\varepsilon_{ijn}\,\mathrm{tr}\big[S^f{}_f(q)\gamma^i S^f{}_f(q+k)\gamma^j\big]. \tag{17.73}$$

The evaluation of this expression is exactly as in [33] so we skip the details. The zero frequency, zero momentum, limit of the magnetic conductivity is

$$(\sigma_B)_{ab} = \frac{1}{4\pi^2}\sum_{f,g=1}^{N} T_a^{f}{}_{g} T_b^{g}{}_{f}\,\mu^f = \frac{1}{8\pi^2}\sum_c \mathrm{tr}\big(T_a\{T_b, H_c\}\big)\mu_c. \tag{17.74}$$

In the second equality of (17.74) we have made use of (17.54). No contribution proportional to the gravitational anomaly coefficient is found in this case.

17.3.3 *Conductivities for the Energy Flux*

We will include for completeness the result of the chiral magnetic and vortical conductivities for the energy flux, corresponding to the last two expressions in (17.46).

The chiral magnetic conductivity for energy flux, σ^ε_B, follows from the correlation function of the energy momentum tensor and the current, and so it computes in the same way as the vortical conductivity in Sect. 17.3.1. From an evaluation of the corresponding Feynman diagram one finds that the result is the same as (17.59). Then one concludes that

$$\big(\sigma^\varepsilon_B\big)_a = (\sigma_V)_a, \tag{17.75}$$

where $(\sigma_V)_a$ is given by (17.72). Although these coefficients are equal, they describe different transport phenomena. Whereas $(\sigma^\varepsilon_B)_a$ describes the generation of an energy flux due to an external magnetic field $\mathbf{B}_a$, $(\sigma_V)_a$ describes the generation of the current $\mathbf{J}_a$ due to an external field that sources the energy-momentum tensor T^{0i}.

Finally the chiral vortical conductivity for the energy flux, σ^ε_V, follows from the correlation function of two energy momentum tensors. There are three contributions out of the four possible terms. One of these terms involves a sum over fermionic Matsubara frequencies of the form

$$\begin{aligned}
&\frac{1}{\beta}\sum_{\tilde{\omega}^f}\big(i\tilde{\omega}^f\big)^2 \Delta_t\big(i\tilde{\omega}^f, \mathbf{q}\big)\Delta_u\big(i\tilde{\omega}^f + i\omega_n, \mathbf{q}+\mathbf{k}\big)\\
&\quad = \mathscr{F}(i\omega_n, E_q, E_{q+k}, t, u) + \frac{1}{i\omega_n + tE_q - uE_{q+k}}\\
&\qquad \times \big[tE_q^2 n\big(E_q - t\mu^f\big) - u(E_{q+k} - ui\omega_n)^2 n\big(E_{q+k} - u\mu^f\big)\big],
\end{aligned} \tag{17.76}$$

where $\mathscr{F}$ corresponds to the ultraviolet divergent vacuum contribution which we choose to remove. The zero frequency, zero momentum, limit of the chiral vortical conductivity for the energy flux writes

$$\begin{aligned}\sigma_V^\varepsilon &= \frac{1}{12\pi^2}\sum_{f=1}^{N}\left[\left(\mu^f\right)^3 + \pi^2 T^2 \mu^f\right] \\ &= \frac{1}{24\pi^2}\left[\sum_{a,b,c}\operatorname{tr}\left(H_a\{H_b, H_c\}\right)\mu_a\mu_b\mu_c + 2\pi^2 T^2 \sum_a \operatorname{tr}(H_a)\mu_a\right]. \end{aligned} \tag{17.77}$$

This coefficient describes the generation of an energy flux due to a vortex (or a gravito-magnetic field). The correlators (17.75) and (17.77) enter the chiral magnetic and vortical conductivities in the Landau frame, respectively, as defined in [27–29], see (17.48)–(17.49). We have also checked that to lowest order in ω and k one has $\langle T^{0z}T^{0z}\rangle = p$, where p is the pressure of a free gas of massless fermions, and $\langle T^{0z}J^z\rangle = 0$ [34].

17.3.4 Summary and Specialization to the Group $U(1)_V \times U(1)_A$

The results for the different conductivities are neatly summarized as

$$(\sigma_B)_{ab} = \frac{1}{4\pi^2} d_{abc}\mu^c, \tag{17.78}$$

$$(\sigma_V)_a = \left(\sigma_B^\varepsilon\right)_a = \frac{1}{8\pi^2} d_{abc}\mu^b\mu^c + \frac{T^2}{24} b_a, \tag{17.79}$$

$$\sigma_V^\varepsilon = \frac{1}{12\pi^2} d_{abc}\mu^a\mu^b\mu^c + \frac{T^2}{12} b_a\mu^a. \tag{17.80}$$

The axial and mixed gauge-gravitational anomaly coefficients are defined by

$$d_{abc} = \frac{1}{2}\left[\operatorname{tr}\left(T_a\{T_b, T_c\}\right)_L - \operatorname{tr}\left(T_a\{T_b, T_c\}\right)_R\right], \tag{17.81}$$

$$b_a = \operatorname{tr}(T_a)_L - \operatorname{tr}(T_a)_R, \tag{17.82}$$

where the subscripts L, R stand for the contributions of left-handed and right-handed fermions. The result shows that these conductivities are non-zero if and only if the theory features anomalies.

For phenomenological reasons it is interesting to specialize these results to the symmetry group $U(1)_V \times U(1)_A$, i.e. one vector and one axial current with chemical potentials $\mu_L = \mu + \mu_A$, $\mu_R = \mu - \mu_A$, charges $q^L_{V,A} = (1, 1)$ and $q^R_{V,A} = (1, -1)$ for one left-handed and one right-handed fermion. We find (for a vector magnetic field)

$$(\sigma_B)_{VV} = \frac{\mu_A}{2\pi^2}, \qquad (\sigma_B)_{AV} = \frac{\mu}{2\pi^2}, \tag{17.83}$$

$$(\sigma_V)_V = \left(\sigma_B^\varepsilon\right)_V = \frac{\mu\mu_A}{2\pi^2}, \qquad (\sigma_V)_A = \left(\sigma_B^\varepsilon\right)_A = \frac{\mu^2+\mu_A^2}{4\pi^2} + \frac{T^2}{12}, \tag{17.84}$$

$$\sigma_V^\varepsilon = \frac{\mu_A}{6\pi^2}\left(3\mu^2+\mu_A^2\right) + \frac{\mu_A}{6}T^2. \tag{17.85}$$

Here $(\sigma_B)_{VV}$ is the chiral magnetic conductivity [33], $(\sigma_B)_{AV}$ describes the generation of an axial current due to a vector magnetic field [57], $(\sigma_V)_V$ is the vector vortical conductivity, $(\sigma_V)_A$ is the axial vortical conductivity, and σ_V^ε is the vortical conductivity for the energy flux. The vector and axial magnetic conductivities for energy flux $(\sigma_B^\varepsilon)_V$ and $(\sigma_B^\varepsilon)_A$ coincide with the chiral vortical conductivities.

17.4 Holographic Model

In this section for simplicity we will consider a holographic system which realizes a single chiral $U(1)$ symmetry with a gauge and mixed gauge-gravitational anomaly [36]. As we saw in the previous section in a more realistic model $U(1)_V \times U(1)_A$ the transport coefficients receive contribution from the gravitational part only in the axial sector. For a study of such a system with a pure gauge anomaly using Kubo formulae, see [45].

17.4.1 Notation and Holographic Anomalies

Let us fix some conventions we will use in the Gravity Theory. We choose the five dimensional metric to be of signature $(-,+,+,+,+)$. Five dimensional indices are denoted with upper case Latin letters. The epsilon tensor has to be distinguished from the epsilon symbol by $\varepsilon_{ABCDE} = \sqrt{-g}\,\varepsilon(ABCDE)$. The symbol is defined by $\varepsilon(rtxyz) = +1$. We assume the metric can be decomposed in ADM like way and define an outward pointing normal vector to the holographic boundary of an asymptotically AdS space $n_A \propto g^{AB}\frac{\partial r}{\partial x^B}$ with unit norm $n_A n^A = 1$. So that the induced metric takes the form

$$h_{AB} = g_{AB} - n_A n_B. \tag{17.86}$$

In general a foliation of the space-time M with timelike surfaces defined through $r(x) = \text{const}$ can be written as

$$\mathrm{d}s^2 = \left(N^2 + N_A N^A\right)\mathrm{d}r^2 + 2N_A\,\mathrm{d}x^A\,\mathrm{d}r + h_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B. \tag{17.87}$$

The Christoffel symbols, Riemann tensor and extrinsic curvature are given by

$$\Gamma^M_{NP} = \frac{1}{2}g^{MK}(\partial_N g_{KP} + \partial_P g_{KM} - \partial_K g_{NP}), \tag{17.88}$$

$$R^M{}_{NPQ} = \partial_P\Gamma^M_{NQ} - \partial_Q\Gamma^M_{NP} + \Gamma^M_{PK}\Gamma^K_{NQ} - \Gamma^M_{QK}\Gamma^K_{NP}, \tag{17.89}$$

$$K_{AV} = h_A^C \nabla_C n_V = \frac{1}{2} \pounds_n h_{AB}, \tag{17.90}$$

where $\pounds_n$ denotes the Lie derivative in direction of n_A. Finally we can define our model. The action is given by

$$\begin{aligned} S = \frac{1}{16\pi G} \int_M \mathrm{d}^5 x \sqrt{-g} \bigg[R + 2\Lambda - \frac{1}{4} F_{MN} F^{MN} + \varepsilon^{MNPQR} A_M \\ \times \left(\frac{\kappa}{3} F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right) \bigg] + S_{GH} + S_{CSK}, \end{aligned} \tag{17.91}$$

$$S_{GH} = \frac{1}{8\pi G} \int_{\partial M} \mathrm{d}^4 x \sqrt{-h}\, K, \tag{17.92}$$

$$S_{CSK} = -\frac{1}{2\pi G} \int_{\partial M} \mathrm{d}^4 x \sqrt{-h}\, \lambda n_M \varepsilon^{MNPQR} A_N K_{PL} D_Q K^L_R, \tag{17.93}$$

where S_{GH} is the usual Gibbons-Hawking boundary term and D_A is the induced covariant derivative on the four dimensional surface. The second boundary term S_{CSK} is introduced to reproduce the gravitational anomaly at general hypersurface.

Let us study now the gauge symmetries of our model. We note that the action is diffeomorphism invariant, but they do depend explicitly on the gauge connection A_M. Under gauge transformations $\delta A_M = \nabla_M \xi$ they are therefore invariant only up to a boundary term. We have

$$\begin{aligned} \delta S = \frac{1}{16\pi G} \int_{\partial M} \mathrm{d}^4 x \sqrt{-h}\, \xi \varepsilon^{MNPQR} \left(\frac{\kappa}{3} n_M F_{NP} F_{QR} + \lambda n_M R^A{}_{BNP} R^B{}_{AQR} \right) \\ - \frac{\lambda}{4\pi G} \int_{\partial M} \mathrm{d}^4 x \sqrt{-h}\, n_M \varepsilon^{MNPQR} D_N \xi K_{PL} D_Q K^L_R. \end{aligned} \tag{17.94}$$

Now without loss of generality we can choose the gauge $N = 1$ and $N_A = 0$ which defines the so called Gaussian normal coordinates, and the metric takes the form $\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$. After doing the decomposition in terms of induced surface and orthogonal fields, all the terms depending on the extrinsic curvature cancel thanks to the contributions from S_{CSK}! The gauge variation of the action depends only on the intrinsic four dimensional curvature of the boundary and is given by

$$\delta S = \frac{1}{16\pi G} \int_{\partial M} \mathrm{d}^4 x \sqrt{-h} \varepsilon^{mnkl} \left(\frac{\kappa}{3} \widehat{F}_{mn} \widehat{F}_{kl} + \lambda \widehat{R}^i{}_{jmn} \widehat{R}^j{}_{ikl} \right). \tag{17.95}$$

This has to be interpreted as the anomalous variation of the effective quantum action of the dual field theory. As consequence of the discussion in Sect. 17.2.1 we can recognize the form of the consistent anomaly and use (17.9) to fix κ for a single fermion transforming under a $U(1)_L$ symmetry. Similarly we can fix λ by matching to the gravitational anomaly of a single left-handed fermion (17.21) and find

$$-\frac{\kappa}{48\pi G} = \frac{1}{96\pi^2}, \qquad -\frac{\lambda}{16\pi G} = \frac{1}{768\pi^2}. \tag{17.96}$$

The bulk equations of motion are

$$G_{MN} - \Lambda g_{MN} = \frac{1}{2}F_{ML}F_N{}^L - \frac{1}{8}F^2 g_{MN} + 2\lambda\varepsilon_{LPQR(M}\nabla_B\big(F^{PL}R^B{}_{N)}{}^{QR}\big), \tag{17.97}$$

$$\nabla_N F^{NM} = -\varepsilon^{MNPQR}\big(\kappa F_{NP}F_{QR} + \lambda R^A{}_{BNP}R^B{}_{AQR}\big). \tag{17.98}$$

A remarkable fact is that the mixed Chern-Simons term does not introduce new singularities into the on-shell action for any asymptotically *AdS* solution, i.e. no new counterterm is needed to renormalize the theory. See [36] for a detailed discussion of the renormalization of the model and Appendix 1 to see the counterterms.

17.4.2 Applying Kubo Formulae and Linear Response

In order to compute the conductivities under study using the Kubo formulae (17.46), we will use tools of linear response theory. To do so we introduce metric and gauge fluctuations over a charged black hole background and use the AdS/CFT dictionary to compute the retarded propagators [58, 59]. Therefore we split the backgrounds and fluctuations as,

$$g_{MN} = g^{(0)}_{MN} + \varepsilon h_{MN}, \tag{17.99}$$

$$A_M = A^{(0)}_M + \varepsilon a_M. \tag{17.100}$$

After the insertion of these fluctuations and background fields in the action and expanding up to second order in ε we can read the on-shell boundary second order action which is needed to get the desired propagators [60],

$$\delta S^{(2)}_{ren} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\big\{\Phi^I_{-k}\mathscr{A}_{IJ}\Phi'^J_k + \Phi^I_{-k}\mathscr{B}_{IJ}\Phi^J_k\big\}\big|_{r\to\infty}, \tag{17.101}$$

where prime means derivative with respect to the radial coordinate, Φ^I_k is a vector constructed with the Fourier transformed components of a_M and h_{MN},

$$\Phi^I\big(r, x^\mu\big) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\Phi^I_k(r)e^{-i\omega t + i\mathbf{kx}}, \tag{17.102}$$

and $\mathscr{A}$ and $\mathscr{B}$ are two matrices extracted from the boundary action and that we will show below.

For a coupled system the holographic computation of the correlators consists in finding a maximal set of linearly independent solutions that satisfy infalling boundary conditions on the horizon and that source a single operator at the AdS boundary [58–61]. To do so we can construct a matrix of solutions $F^I{}_J(k,r)$ such that each of its columns corresponds to one of the independent solutions and normalize it to the unit matrix at the boundary. Therefore, given a set of boundary values for the perturbations, φ_k^I, the bulk solutions are

$$\Phi_k^I(r) = F^I{}_J(k,r)\varphi_k^J. \tag{17.103}$$

Finally using this decomposition we obtain the matrix of retarded Green's functions

$$G_{IJ}(k) = -2\lim_{r\to\infty}\big(\mathscr{A}_{IM}\big(F^M{}_J(k,r)\big)' + \mathscr{B}_{IJ}\big). \tag{17.104}$$

The system of equations (17.97)–(17.98) admit the following exact background AdS Reissner-Nordström black-brane solution

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\big(-f(r)\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2\big) + \frac{L^2}{r^2 f(r)}\,\mathrm{d}r^2, \tag{17.105}$$

$$A^{(0)} = \phi(r)\,\mathrm{d}t = \left(\nu - \frac{\mu r_{\mathrm{H}}^2}{r^2}\right)\mathrm{d}t, \tag{17.106}$$

where the horizon of the black hole is located at $r = r_{\mathrm{H}}$ and the blackening factor of the metric is

$$f(r) = 1 - \frac{ML^2}{r^4} + \frac{Q^2L^2}{r^6}. \tag{17.107}$$

The parameters M and Q of the RN black hole are related to the chemical potential μ and the horizon r_{H} by[7]

$$M = \frac{r_{\mathrm{H}}^4}{L^2} + \frac{Q^2}{r_{\mathrm{H}}^2}, \quad Q = \frac{\mu r_{\mathrm{H}}^2}{\sqrt{3}}. \tag{17.108}$$

The Hawking temperature is given in terms of these black hole parameters as

$$T = \frac{r_{\mathrm{H}}^2}{4\pi L^2} f'(r_{\mathrm{H}}) = \frac{(2r_{\mathrm{H}}^2 M - 3Q^2)}{2\pi r_{\mathrm{H}}^5}. \tag{17.109}$$

The pressure of the gauge theory is $P = \frac{M}{16\pi G L^3}$ and its energy density is $\varepsilon = 3P$ due to the underlying conformal symmetry.

[7] The chemical potential is introduced as the energy needed to introduce an unit of charge from the boundary to behind the horizon $A(\infty) - A(r_{\mathrm{H}})$ which corresponds to the prescription (B) in Table 17.1. Observe that we have left the source value $A(\infty) = \nu$ as an arbitrary constant for reasons we will explain later.

To study the effect of anomalies we just turned on the shear sector (transverse momentum fluctuations) a_α and h_t^α and set without loss of generality the momentum k in the y-direction at zero frequency, so $\alpha = x, z$. Since we are interested in the hydrodynamical regime ($k, \omega \ll T$), it is just necessary to find solutions up to first order in momentum. So that we expand the fields in terms of the dimensionless momentum $p = k/4\pi T$ such as

$$h_t^\alpha(r) = h_t^{(0)\alpha}(r) + p h_t^{(1)\alpha}(r), \tag{17.110}$$

$$B_\alpha(r) = B_\alpha^{(0)}(r) + p B_\alpha^{(1)}(r), \tag{17.111}$$

with the gauge field redefined as $B_\alpha = a_\alpha/\mu$. For convenience we redefine new parameters and radial coordinate

$$\bar{\lambda} = \frac{4\mu\lambda L}{r_{\mathrm{H}}^2}; \qquad \bar{\kappa} = \frac{4\mu\kappa L^3}{r_{\mathrm{H}}^2}; \qquad a = \frac{\mu^2 L^2}{3r_{\mathrm{H}}^2}; \qquad u = \frac{r_{\mathrm{H}}^2}{r^2}. \tag{17.112}$$

In this new radial coordinate the horizon sits at $u = 1$ and the AdS boundary at $u = 0$. At zero frequency the system of differential equations consists on four second order equations.[8] The relevant physical boundary conditions on fields are: $h_t^\alpha(0) = \widetilde{H}^\alpha$, $B_\alpha(0) = \widetilde{B}_\alpha$; where the 'tilde' parameters are the sources of the boundary operators. The second condition compatible with the ingoing one at the horizon is regularity for the gauge field and vanishing for the metric fluctuation [34].

After solving the system perturbatively (see [36] for solutions), we can go back to the formula (17.104) and compute the corresponding holographic Green's functions. If we consider the vector of fields to be

$$\Phi_k^\top(u) = \big(B_x(u), h_t^x(u), B_z(u), h_t^z(u)\big), \tag{17.113}$$

the $\mathscr{A}$ and $\mathscr{B}$ matrices for that setup take the following form

$$\mathscr{A} = \frac{r_{\mathrm{H}}^4}{16\pi G L^5}\,\mathrm{Diag}\left(-3af, \frac{1}{u}, -3af, \frac{1}{u}\right), \tag{17.114}$$

$$\mathscr{B}_{AdS+\partial} = \frac{r_{\mathrm{H}}^4}{16\pi G L^5}\begin{pmatrix} 0 & -3a & \frac{4\kappa i k\mu^2\phi L^5}{3r_{\mathrm{H}}^4} & 0 \\ 0 & -\frac{3}{u^2} & 0 & 0 \\ \frac{-4\kappa i k\mu^2\phi L^5}{3r_{\mathrm{H}}^4} & 0 & 0 & -3a \\ 0 & 0 & 0 & -\frac{3}{u^2} \end{pmatrix}, \tag{17.115}$$

[8] The complete system of equations depending on frequency and momentum is showed in Appendix 2. The system consists of six dynamical equations and two constraints.

$$\mathscr{B}_{CT} = \frac{r_{\rm H}^4}{16\pi G L^5}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{3}{u^2\sqrt{f}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{u^2\sqrt{f}} \end{pmatrix}, \tag{17.116}$$

where $\mathscr{B} = \mathscr{B}_{AdS+\partial} + \mathscr{B}_{CT}$.[9] Notice that there is no contribution to the matrices coming from the Chern-Simons gravity part, because the corresponding contributions vanish at the boundary. These matrices and the perturbative solutions are the ingredients to compute the matrix of propagators. Undoing the vector field redefinition introduced in (17.111) the non-vanishing retarded correlation functions at zero frequency are then

$$G_{x,tx} = G_{z,tz} = \frac{\sqrt{3}Q}{4\pi G L^3}, \tag{17.117}$$

$$G_{x,z} = -G_{z,x} = \frac{i\sqrt{3}kQ\kappa}{2\pi G r_{\rm H}^2} + \frac{ik\nu\kappa}{6\pi G}, \tag{17.118}$$

$$G_{x,tz} = G_{tx,z} = -G_{z,tx} = -G_{tz,x} = \frac{3ikQ^2\kappa}{4\pi G r_{\rm H}^4} + \frac{2ik\lambda\pi T^2}{G}, \tag{17.119}$$

$$G_{tx,tx} = G_{tz,tz} = \frac{M}{16\pi G L^3}, \tag{17.120}$$

$$G_{tx,tz} = -G_{tz,tx} = +\frac{i\sqrt{3}kQ^3\kappa}{2\pi G r_{\rm H}^6} + \frac{4\pi i\sqrt{3}kQT^2\lambda}{G r_{\rm H}^2}. \tag{17.121}$$

We can do an important remark observing (17.118). Remember that we left the boundary value of the background gauge field (17.106) arbitrary as a constant ν. But as the $U(1)$ symmetry is anomalous in the Field Theory side, physical quantities have to be sensitive to the source A_0,[10] indeed as we can check they are. In particular if we choose the value $\nu = \mu$ which corresponds to formalism (A) in Table 17.1, we need to include the counterterm (17.29) in order to get the same propagator as at weak coupling. In fact in [44, 45] it has been shown that in the case of a propagator between two vector currents, choosing this specific value for ν the propagator would be zero, giving us in consequence a zero value for the chiral magnetic conductivity. Hence to be consistent with the scheme we are working at, let us just consider ν as a source in the field theory. Therefore the real propagator is the one with $\nu = 0$ because as is well known we have to set all sources to zero after taking the second

[9] $\mathscr{B}_{CT}$ is coming from the counterterms of the theory.

[10] In principle A_0 could be gauged away for the symmetric case and in consequence observables should not depend on its value. For example look at [45] to see how in presence of a $U(1)_V \times U(1)_A$ symmetry with only the $U(1)_V$ conserved, propagators do not depend on the specific value of the zero component of the vector gauge source V_0.

functional derivative of the effective action. Finally using the Kubo formulae (17.46) we recover the vortical and axial-magnetic conductivities

$$\sigma_B = -\frac{\sqrt{3}Q\kappa}{2\pi G r_{\mathrm{H}}^2} = \frac{\mu}{4\pi^2}, \tag{17.122}$$

$$\sigma_V = \sigma_B^\varepsilon = -\frac{3Q^2\kappa}{4\pi G \bar{r}_{\mathrm{H}}^4} - \frac{2\lambda\pi T^2}{G} = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}, \tag{17.123}$$

$$\sigma_V^\varepsilon = -\frac{\sqrt{3}Q^3\kappa}{2\pi G r_{\mathrm{H}}^6} - \frac{4\pi\sqrt{3}QT^2\lambda}{G r_{\mathrm{H}}^2} = \frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}. \tag{17.124}$$

All these expressions coincide with the results in Sect. 17.3, (17.78), (17.79) and (17.80) if we specialize to $d_{abc} = 1$ and $b_a = 1$. They are in perfect agreement with the literature [27–29, 34] except for the contribution coming from the gravitational anomaly which is manifest by the presence of the extra λT^2. All the numerical coefficients coincide precisely with the ones obtained at weak coupling; this we take as a strong hint that the anomalous conductivities are indeed completely determined by the anomalies and are not renormalized beyond one loop. Evidence for non-renormalization comes also from [62] where a holographic renormalization group running of the conductivities showed the same result at any value of the holographic cut-off. We also point out that the T^3 term that appears as undetermined integration constant in the hydrodynamic considerations in [63] should make its appearance in σ_V^ε. We do not find any such term which is consistent with the argument that this term is absent due to CPT invariance.

It is also interesting to write down the magnetic and vortical conductivities using (17.48) and (17.49) as they appear in the Landau frame to compare with the Son and Surowka form [29]

$$\xi_B = -\frac{\sqrt{3}Q(ML^2 + 3r_{\mathrm{H}}^4)\kappa}{8\pi GML^2 r_{\mathrm{H}}^2} + \frac{\sqrt{3}Q\lambda\pi T^2}{GM} = \frac{1}{4\pi^2}\left(\mu - \frac{1}{2}\frac{n(\mu^2 + \frac{\pi^2T^2}{3})}{\varepsilon + P}\right), \tag{17.125}$$

$$\begin{aligned}\xi_V &= -\frac{3Q^2\kappa}{4\pi GML^2} - \frac{2\pi\lambda T^2(r_{\mathrm{H}}^6 - 2L^2Q^2)}{GML^2 r_{\mathrm{H}}^2} \\ &= \frac{\mu^2}{8\pi^2}\left(1 - \frac{2}{3}\frac{n\mu}{\varepsilon + P}\right) + \frac{T^2}{24}\left(1 - \frac{2n\mu}{\varepsilon + P}\right).\end{aligned} \tag{17.126}$$

These expressions agree with the literature except for the λT^2 term. A last comment can be done, the shear viscosity entropy ratio is not modified by the presence of the gravitational anomaly. We know that $\eta \propto \lim_{\omega\to 0}\frac{1}{\omega}\langle T^{xy}T^{xy}\rangle_{k=0}$, so we should solve the system at $k = 0$ for the fluctuations h_y^i but the anomalous coefficients always appear with a momentum k (see Appendix 2), therefore if we switch off the momentum, the system looks precisely as the theory without anomalies. In [64] it has been shown that the black hole entropy doesn't depend on the extra Chern-Simons term.[11]

17.5 Conclusion and Outlook

In the presence of external sources for the energy momentum tensor and the currents, the anomaly is responsible for a non conservation of the latter. This is conveniently expressed through [52]

$$D_\mu J^\mu_a = \varepsilon^{\mu\nu\rho\lambda}\left(\frac{d_{abc}}{32\pi^2}F^b_{\mu\nu}F^c_{\rho\lambda} + \frac{b_a}{768\pi^2}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\rho\lambda}\right), \tag{17.127}$$

where the axial and mixed gauge-gravitational anomaly coefficients, d_{abc} and b_a, are given by (17.22) and (17.23) respectively.

We have discussed in Sect. 17.2 the constitutive relations and derived the Kubo formulae that allow the calculation of transport coefficients at first order in the hydrodynamic expansion. We explained also subtleties in the definition of the chemical potential in the presence of anomalies. The explicit evaluation of these Kubo formulae in quantum field theory has been performed in Sect. 17.3 for the chiral magnetic, chiral vortical and energy flux conductivities of a relativistic fluid at weak coupling, and we found contributions proportional to the anomaly coefficients d_{abc} and b_a. Non-zero values of these coefficients are a necessary and sufficient condition for the presence of anomalies [52]. Therefore the non-vanishing values of the transport coefficients have to be attributed to the presence of chiral and gravitational anomalies.

In order to perform the analysis at strong coupling via AdS/CFT methods, we have defined in Sect. 17.4 a holographic model implementing both type of anomalies via gauge and mixed gauge-gravitational Chern-Simons terms. We have computed the anomalous magnetic and vortical conductivities from a charged black hole background and have found a non-vanishing vortical conductivity proportional to $\sim T^2$. These terms are characteristic for the contribution of the gravitational anomaly and they even appear in an uncharged fluid. The T^2 behavior had appeared already previously in neutrino physics [17–20]. In [30] similar terms in the vortical conductivities have been argued for, but just in terms of undetermined integration constants without any relation to the gravitational anomaly. Very recently a generalization of the results (17.122)–(17.124) to any even space-time dimension as a polynomial in μ and T [39] has been proposed. Finally, the consequences of this anomaly in hydrodynamics have been studied using a group theoretic approach, which seems to suggest that their effects could be present even at $T = 0$ [66]. The numerical values of the anomalous conductivities at strong coupling are in perfect agreement with weak coupling calculations, and this suggests the existence of a non-renormalization theorem including the contributions from the gravitational anomaly.

[11] For a four dimensional holographic model with gravitational Chern-Simons term and a scalar field this has also been shown in [65].

There are important phenomenological consequences of the present study to heavy ion physics. In [67] enhanced production of high spin hadrons (especially Ω^- baryons) perpendicular to the reaction plane in heavy ion collisions has been proposed as an observational signature for the chiral separation effect. Three sources of chiral separation have been identified: the anomaly in vacuum, the magnetic and the vortical conductivities of the axial current J^μ_A. Of these the contribution of the vortical effect was judged to be subleading by a relative factor of 10^{-4}. The T^2 term in (17.123) leads however to a significant enhancement. If we take μ to be the baryon chemical potential $\mu \approx 10$ MeV, neglect μ_A as in [67] and take a typical RHIC temperature of $T = 350$ MeV, we see that the temperature enhances the axial chiral vortical conductivity by a factor of the order of 10^4. We expect the enhancement at the LHC to be even higher due to the higher temperature.

In this review we have presented the computation of the transport coefficients, and in particular their gravitational anomaly contributions, via Kubo formulae. It would be interesting to calculate directly the constitutive relations of the hydrodynamics of anomalous currents via the fluid/gravity correspondence within the holographic model of Sect. 17.4, [27, 28, 68]. This approach will allow us to compute transport coefficients at higher orders [69, 70]. This study is currently in progress [71].

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Appendix 1: Boundary Counterterms

The result one gets for the counterterm coming from the regularization of the boundary action of the holographic model in Sect. 17.4 is

$$S_{ct} = -\frac{3}{8\pi G}\int_{\partial M} \mathrm{d}^4x\sqrt{-h}\left[1 + \frac{1}{2}P - \frac{1}{12}\left(P^i_j P^j_i - P^2 - \frac{1}{4}\widehat{F}_{ij}\widehat{F}^{ij}\right)\log e^{-2\rho}\right], \tag{17.128}$$

where hat on the fields means the induced field on the cut-off surface and

$$P = \frac{1}{6}\widehat{R}, \qquad P^i_j = \frac{1}{2}\big[\widehat{R}^i_j - P\delta^i_j\big]. \tag{17.129}$$

As a remarkable fact there is no contribution in the counterterm coming from the gauge-gravitational Chern-Simons term. This has also been derived in [72] in a similar model that does however not contain S_{CSK}.

Appendix 2: Equations of Motion for the Shear Sector

These are the complete linearized set of six dynamical equations of motion,

$$0 = B_\alpha''(u) + \frac{f'(u)}{f(u)} B_\alpha'(u) + \frac{b^2}{uf(u)^2}\big(\omega^2 - f(u)k^2\big) B_\alpha(u) - \frac{h_t^{\alpha'}(u)}{f(u)}$$
$$+ ik\varepsilon_{\alpha\beta}\left(\frac{3}{uf(u)}\bar{\lambda}\left(\frac{2}{3a}\big(f(u)-1\big)+u^3\right)h_t^{\beta'}(u) + \bar{\kappa}\frac{B_\beta(u)}{f(u)}\right), \tag{17.130}$$

$$0 = h_t^{\alpha''}(u) - \frac{h_t^{\alpha'}(u)}{u} - \frac{b^2}{uf(u)}\big(k^2 h_t^\alpha(u) + h_y^\alpha(u)\omega k\big) - 3auB_\alpha'(u)$$
$$\times i\bar{\lambda}k\varepsilon_{\alpha\beta}\Big[\big(24au^3 - 6(1-f(u))\big)\frac{B_\beta(u)}{u} + \big(9au^3 - 6(1-f(u))\big)B_\beta'(u)$$
$$+ 2u\big(uh_t^{\beta'}(u)\big)' - \frac{2ub^2}{f(u)}\big(h_y^\beta(u)\omega k + h_t^\beta(u)k^2\big)\Big], \tag{17.131}$$

$$0 = h_y^{\alpha''}(u) + \frac{(f/u)'}{f/u}h_y^{\alpha'}(u) + \frac{b^2}{uf(u)^2}\big(\omega^2 h_y^\alpha(u) + \omega k h_t^\alpha(u)\big) + 2uik\bar{\lambda}\varepsilon_{\alpha\beta}\Big[uh_y^{\beta''}(u)$$
$$+ \big(9f(u) - 6 + 3au^3\big)\frac{h_y^{\beta'}(u)}{f(u)} + \frac{b^2}{f(u)^2}\big(\omega k h_t^\beta(u) + w^2 h_y^\beta(u)\big)\Big], \tag{17.132}$$

and two constraints for the fluctuations at $\omega, k \neq 0$

$$0 = \omega\big(h_t^{\alpha'}(u) - 3auB_\alpha(u)\big) + f(u)kh_y^{\alpha'}(u) + ik\bar{\lambda}\varepsilon_{\alpha\beta}\big[2u^2\big(\omega h_t^{\beta'} + f(u)kh_y^{\beta'}(u)\big)$$
$$+ \big(9au^3 - 6(1-f(u))\big)B_\beta(u)\big]. \tag{17.133}$$

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