Chapter 7
Higher Order Sliding Mode Control by Keeping a 2-Sliding Constraint

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Abstract. This article investigates a new algorithm for higher order sliding mode control. The proposed control law keeps a constraint in 2-sliding mode such that the finite time stabilization of the chain of integrators is achieved. The proposed switching function has relative degree two with respect to the input and a second order sliding controller is used. The twisting controller is used for achieving finite time convergent 2-sliding mode to the switching manifold. The switching manifold is designed to provide finite time convergence of the integrator chain. The fractional powers in the switching function are carefully designed to prevent the unboundedness or singularity arising because of the switching constraint being kept at zero.

7.1 Introduction

Finite time stability and Sliding Mode Control (SMC) are closely related areas of active research. Since sliding mode control methods require finite time reaching to the sliding manifold, finite time stabilization methods can naturally be applied in SMC. However, SMC has more emphasis on robustness and sliding mode controllers should be robust. The Higher Order Sliding Mode Control (HOSM) is a necessity for achieving robust stabilization or tracking in the systems with outputs having relative degree more that one with respect to the input. The sliding order by definition gives better accuracy ⟦11⟧, ⟦13⟧.

The literature on second order sliding modes is abundant. However, few controllers exist for achieving finite time convergence for an arbitrary order dynamical system. In general this may not be possible but given boundedness of the states

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finite time robust stabilization of an arbitrary chain of integrators is an interesting problem. Finite time stability with non-lipschitzian right hand sides is well documented ⟦2⟧, ⟦8⟧, ⟦15⟧. Homogeneity properties of the vector field are also explored and shown to be quite useful in proving the finite time nature ⟦1⟧, ⟦10⟧. The generalization to systems with dynamics of order higher than 2 has been established in different forms ⟦3⟧, ⟦9⟧. These methods employ fractional powers and provide a continuous control law.

Terminal sliding mode (TSM) was first introduced in ⟦7⟧ as a robust finite time controller for a second order two-link robotic manipulator. The terminal sliding mode has an extraordinary feature of providing finite time convergence of all the system states through a non linear switching function. The more general formulation was presented in ⟦17⟧, where nested non linear switching functions are designed. All switching functions converge to zero in finite time sequentially and as a consequence ultimately all system states converge to zero. The terminal sliding mode can easily be used to achieve finite time stabilization of chain of integrators and thus providing for a higher order sliding mode. However, as mentioned by Levant ⟦12⟧, this form leads to unbounded control for systems of order three and higher. Thus, Filippov solutions are not well defined for this formulation.This has led to development of non-singular terminal sliding mode control. A slightly different switching function has been shown ⟦6⟧, ⟦5⟧ to provide terminal sliding without causing unbounded control. Non singular terminal sliding mode control has been used as second order sliding mode control of uncertain multivariable systems ⟦5⟧.

The discontinuous control i.e., the sliding mode control has also been developed for achieving finite time stabilization of higher order systems ⟦12⟧, ⟦16⟧. The discontinuous feedback has the apparent advantage of robustness over the continuous feedback laws. The most popular HOSM controller is detailed in ⟦12⟧. The idea in ⟦12⟧ for achieving finite time convergence is to keep a properly designed switching constraint in 1-sliding. The novel method proposed in ⟦16⟧ utilizes feedback control with matrix exponentials, but needs the knowledge of initial conditions to compute the gain matrix. The advantage is the reaching time can be easily specified and the designed control law is able to steer the trajectory to the desired sliding set in the specified time. The proposed controller in this article is based on the idea of holding a 2-sliding constraint. The organisation is as follows. The first section provides a basic introduction to HOSM. The second section describes the main proposal and detailed proof that keeping a properly designed constraint in 2-sliding can achieve finite time convergence of a triple integrator. Extension of the idea to the chain of integrators for higher order sliding mode control is described in the third section followed by numerical simulations and an example.

7.1.1 Higher Order Sliding Mode Control

The HOSM is introduced and well defined in terms of a family of real sliding trajectories ⟦11⟧. This article follows the same definitions and assumptions. Let the dynamical system be described by,

$$\dot{x} = f(t,x) + g(t,x)u \tag{7.1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}$. Let $\sigma(x) \in \mathbb{R}$ be some output of the system with stable zero dynamics and the problem is to steer it to zero in finite time. The classical sliding mode approach makes $\dot{\sigma}$ discontinuous using a relay feedback and provides for finite time stabilization. This approach would require $\sigma(x)$ to have relative degree 1 with respect to the input and it makes the input discontinuous. Apparently when the output has relative degree more than one, this approach would not work and HOSM concepts have to be employed. In some situations, discontinuous control might be unacceptable. Then the relative degree can be increased artificially by cascading integrators and shifting the discontinuity in to higher derivatives of input. This approach increases the smoothness of control as well as $\sigma(x)$.
Suppose $\sigma(x)$ has relative degree r with respect to the input (it may have been artificially increased as mentioned). Then the control appears in the r-th derivative of $\sigma(x)$ with the direction derivatives of system functions. Thus,

$$\sigma^{(r)} = \phi(t,x) + \gamma(t,x)u \tag{7.2}$$

where, $\phi(t,x)$ and $\gamma(t,x)$ contains directional derivative(Lie derivatives) of the system functions $f(t,x)$ and $g(t,x)$. It is assumed that

$$|\phi(t,x)| < \Phi,\ 0 < \Gamma_m \leq \gamma(t,x) \leq \Gamma_M \tag{7.3}$$

The problem is to find a control input u which stabilizes (7.2) in finite time.

It can also be stated as finite time stabilization of an integrator chain in the presence of uncertainties in the form of $\phi(t,x)$ and $\gamma(t,x)$. Also, finite time stabilization of an integrator chain without uncertainties is a good starting point for analysis of new controller. The next section describes a new controller for triple integrator without uncertainties and then extends it to the triple integrator with uncertainties.

7.2 Third Order Sliding via Non-singular Terminal Switching Function

This section introduces the idea of keeping a constraint in 2-sliding to obtain third order sliding. At first we consider the triple integrator without uncertainties. We show that finite time convergence of the triple integrator is obtained by holding a switching function in 2-sliding mode. The switching function is the one used in non-singluar terminal sliding mode control. The triple integrator input has relative degree two with respect to this switching function. The twisting controller is the obvious choice to confine the trajectory on the manifold defined by this constraint function.

7.2.1 Triple Integrator without Uncertainties

Let the triple integrator be $\sigma^{(3)} = u$. The constraint function, as mentioned earlier, has to have relative degree two with respect to the input. Thus, it must be a function of variables σ and $\dot{\sigma}$ such that the input appears in the second time derivative of the function. Consider the following function,

$$\psi(\sigma,\dot{\sigma}) = \sigma + \dot{\sigma}^{\alpha} \tag{7.4}$$

where, α is a ratio of positive odd integers. Specifically $\alpha = p/q$, with $p,q \in \mathbb{N}$ and odd. This function also defines a 2-sliding manifold in σ,$\dot{\sigma}$ and $\ddot{\sigma}$ coordinates as $\mathscr{S} = \{(\sigma,\dot{\sigma},\ddot{\sigma}) | \psi(\sigma,\dot{\sigma}) = \dot{\psi}(\dot{\sigma},\ddot{\sigma}) = 0\}$. We have the following proposition to show that keeping the constraint function (7.4) in a 2-sliding mode achieves finite time stabilization of the triple integrator.

Proposition 7.1. *The control law,*

$$u = -k_1 sign(\sigma + \dot{\sigma}^{\alpha}) - k_2 sign(\dot{\sigma} + \alpha\dot{\sigma}^{\alpha-1}\ddot{\sigma}) \tag{7.5}$$

stabilizes $\sigma^{(3)} = u$ *in finite time with the following sufficient conditions.*

- α *is a ratio of two odd integers with* $1 < \alpha < 1.5$.
- $k_1 > k_2$ *and* $\Xi_3 + \alpha' \Xi_2 \Xi_3^2 + \Xi_2(k_1 - k_2) < -\Xi_3 - \alpha' \Xi_2 \Xi_3^2 + \Xi_2(k_1 + k_2)$.

where, $|\dot{\sigma}| < \frac{1}{\alpha}\Xi_2^{1-\alpha}$, $|\ddot{\sigma}| < \Xi_3$ *and* $\Xi_2,\Xi_3 \in \mathbb{R}$ *are known.*

Proof. Let us define the integrator states as, $\xi_1 = \sigma, \xi_2 = \dot{\sigma}$ and $\xi_3 = \ddot{\sigma}$, then the state equations are

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= u \end{aligned} \tag{7.6}$$

Consider the switching function $\psi(\xi) = \xi_1 + \xi_2^{\alpha}$. The proof is divided in two major parts. First we show that if the constraint $\psi(\xi)$ is held in 2-sliding, i.e., $\psi(\xi) = \dot{\psi}(\xi) = 0$ is kept then the reduced order dynamics of (7.6) is finite time stable. It is equivalent to say that the zero dynamics of the (7.6) with function $\psi(\xi)$ as output is finite time stable. The second part proposes a controller which keeps this constraint in 2-sliding i.e, makes the trajectory reach the 2-sliding manifold in finite time.

The zero dynamics can be easily obtained by equating $\psi(\xi)$ and $\dot{\psi}(\xi)$ to zero. That is,

$$\psi(\xi) = \xi_1 + \xi_2^{\alpha} = 0 \tag{7.7}$$

$$\dot{\psi}(\xi) = \xi_2 + \alpha\xi_2^{\alpha-1}\xi_3 = 0 \tag{7.8}$$

The algebraic relationships are obtained as,

$$\xi_2 = -\xi_1^{\frac{1}{\alpha}} \tag{7.9}$$

$$\xi_3 = -\frac{1}{\alpha}\xi_2^{2-\alpha} \tag{7.10}$$

Note that, (7.8) dictates two algebraic relations namely $\xi_2 = 0$ or $\xi_3 = -\frac{1}{\alpha}\xi_2^{2-\alpha}$. However, $\xi_2 = 0$ is not to be considered for reduced order dynamics for reasons explain later in the proof. Thus, when the constraint ψ is kelp in 2-sliding, the dynamics of (7.6) reduces to one differential equation $\dot{\xi}_1 = -\xi_1^{1/\alpha}$ which is finite time stable with $1 < \alpha < 2$, and the other two states are algebraically related with ξ_1. Here, it is obvious that for finite time stability $\alpha > 1$ is necessary but $\alpha < 2$ is apparent from the algebraic relation (7.10), where the $\xi_2^{2-\alpha}$ converges to zero only with $\alpha < 2$.

We have established that if the constraint $\psi(\xi) = \xi_1 + \xi_2^{\alpha}$ is kept in 2-sliding, then finite time convergence of (7.6) is achieved. Next we show that the proposed controller achieves finite time convergence to the 2-sliding manifold $\mathscr{S}$. To this end, consider the second time derivative of $\psi(\xi)$,

$$\begin{aligned}\ddot{\psi} &= \xi_3 + \alpha(\alpha-1)\xi_2^{\alpha-2}\xi_3^2 + \alpha\xi_2^{\alpha-1}\dot{\xi}_3 &(7.11)\\ &= \phi'(\xi) + \gamma'(\xi)u &(7.12)\end{aligned}$$

where,

$$\phi'(\xi) = \xi_3 + \alpha(\alpha-1)\xi_2^{\alpha-2}\xi_3^2 \tag{7.13}$$

$$\gamma'(\xi) = \alpha\xi_2^{\alpha-1} \tag{7.14}$$

The functions $\phi'(\xi)$ and $\gamma'(\xi)$ are of utmost importance in the subsequent analysis. Some remarks about these functions, follow, which are useful in the proof.

Remark 7.1. The function $\phi'(\xi)$ is not globally bounded so the question can be raised about the existence of Filippov's solution. Note that the equation (7.11) is only a tool to understand the behaviour of ψ and $\dot{\psi}$. The solution of the system is dictated by (7.6). It is obvious that Filippov solutions for ξ are very well defined with the proposed control. The functions ψ and $\dot{\psi}$ are algebraically related with ξ so these are also well defined.

Remark 7.2. The set $\mathscr{S} = \{\xi \in \mathbb{R}^3 | \psi(\xi) = \dot{\psi}(\xi) = 0\}$ is intended to be a positively invariant set, hence, it is necessary to examine the dynamics when the trajectory is within $\mathscr{S}$ i.e., during the sliding mode. Consider (7.11) when $\psi(\xi) \equiv \dot{\psi}(\xi) \equiv 0$,i.e., substituting algebraic relations (7.9)-(7.10) into (7.11),

$$\ddot{\psi} = \alpha\xi_2^{\alpha-1}\left(\frac{\alpha-2}{\alpha^2}\xi_2^{2-3\alpha} + u\right) \tag{7.15}$$

Note that it contains the term $\xi^{2-3\alpha}$ and $2-3\alpha > 0$ is necessary for the right hand side to be bounded while sliding. Thus, we obtain the condition $1 < \alpha < 1.5$.

Remark 7.3. $\gamma'(\xi)$ is a sign definite positive function. The α is ratio of two odd integers, so $\alpha-1$ has an even number in the numerator. Thus $\xi_2^{\alpha-1}$ is always positive. However, it does not have a lower bound. It is seen that $\gamma'(\xi) \to 0$ as $\xi_2 \to 0$. Also as $\xi_2 \to 0$ the function $\phi'(\xi) \to \infty$.

Remark 7.4. Also, it is crucial to note that on the manifold $\mathscr{S}$ the function $\phi'(\xi)$ is bounded. It is evident from the following limit.

$$\lim_{(\xi_2 \to 0)|\mathscr{S}} \xi_3 + \frac{\alpha(\alpha-1)\xi_2^{\alpha}\xi_3^2}{\xi_2^2} = \frac{\alpha-2}{\alpha}\xi_2^{2-\alpha} = 0 \tag{7.16}$$

Thus, once the trajectories converge to the 2-sliding set $\mathscr{S}$, $\phi'(\xi)$ is bounded.

Consider some identities relating the functions ϕ' and γ'. These identities are useful in the proof.

$$\dot{\gamma}'(\xi) = \alpha(\alpha-1)\xi_2^{\alpha-2}\xi_3 = \alpha'\gamma'(\xi)\xi_2^{-1}\xi_3 \tag{7.17}$$

$$\phi'(\xi) = \xi_3 + \dot{\gamma}'(\xi)\xi_3 \tag{7.18}$$

$$\dot{\psi}(\xi) = \frac{\xi_2}{\alpha'}(\alpha' + \dot{\gamma}'), \quad \alpha' = \alpha - 1 \tag{7.19}$$

The proof of finite convergence will consist of showing that the real trajectory is "majored" by a known fixed trajectory (majorant curve). This majorant curve converges to zero in finite time and thus the real trajectory converges to zero. To this end, consider the $\ddot{\psi}$ again in a more convenient form,

$$\begin{aligned} \ddot{\psi} &= \xi_3 + \dot{\gamma}'(\xi)\xi_3 + \gamma'(\xi)u \quad &(7.20)\\ &= \xi_3 + \gamma'(\xi)\left(\alpha'\xi_2^{-1}\xi_3^2 + u\right) \quad &(7.21) \end{aligned}$$

Next, consider the projection of the trajectory on ψ-$\dot{\psi}$ coordinates. Without loss of generality one can consider the trajectory starting from a point $(0, \dot{\psi}_0)$. The trajectory enters the quadrant $\psi > 0, \dot{\psi} > 0$. We shall show that the real trajectory of the system with proposed control in this quadrant is confined by the axis $\psi = 0$, $\dot{\psi} = 0$ and the trajectory of the equation

$$\ddot{\psi} = \Xi_3 + \alpha'\Xi_2\Xi_3^2 - \Xi_2 k_M \tag{7.22}$$

where, $k_M = k_1 + k_2$,Ξ_2 and Ξ_3 are bounds considered as $|\xi_3| < \Xi_3$,$\gamma'(\xi) < \Xi_2$.

For the justification of this fact let us write the differential inclusion from (7.20) as,

$$\ddot{\psi} \in [-\Xi_3, \Xi_3] + [0, \Xi_2]\left(\frac{\alpha'}{\xi_2}\left[-\Xi_3^2, \Xi_3^2\right] + u\right) \tag{7.23}$$

Table 7.1 Table gain conditions in state-space

Region	ξ_1 ξ_2 ξ_3	Condition for $\dot{\psi} > 0$	Condition for $\phi' < 0$
$\mathscr{O}_1$	+,+,+	Always	Never
$\mathscr{O}_2$	−,+,+	Always	Never
$\mathscr{O}_3$	+,−,+	$\dot{\gamma}' < -\alpha'$	$\dot{\gamma}' < -1$
$\mathscr{O}_4$	+,+,−	$\dot{\gamma}' > -\alpha'$	$\dot{\gamma}' > -1$
$\mathscr{O}_5$	−,+,−	$\dot{\gamma}' > -\alpha'$	$\dot{\gamma}' > -1$

Now, refer to the Table-7.1. The table details the regions in ξ-space where $\dot{\psi}$ can be positive. The third column lists the condition for $\dot{\psi} > 0$ and fourth column lists the condition for $\phi'(\xi)$ as in (7.18) to be negative. Interestingly the condition for $\dot{\psi} > 0$ implies the condition for $\phi'(\xi) < 0$ in the regions $\mathscr{O}_4$ and $\mathscr{O}_5$. For example, consider $\dot{\psi}_0 \in \mathscr{O}_4$. Then, from (7.19) $\dot{\gamma}' > -\alpha'$ is necessary for $\dot{\psi} > 0$. Moreover, in this region $\dot{\gamma}' > -1$ implies $\phi' < 0$. It is trivial that when $\phi'(\xi) < 0$ the real trajectory is obviously confined by trajectory of (7.22). Thus, the regions where $\phi'(\xi) > 0$ are of concern which are $\mathscr{O}_1$, $\mathscr{O}_2$ and $\mathscr{O}_3$. Observe that in these regions $\xi_3 > 0$ and in the quadrant $\psi > 0$, $\dot{\psi} > 0$ we have $\dot{\xi}_3 = -k_M$. Thus, ξ_3 becomes negative in finite time and enters regions $\mathscr{O}_4$ or $\mathscr{O}_5$. In these regions as we have already seen the real trajectory is bounded by the trajectory of (7.22).

However, there exist initial points on $\psi = 0$ axis such that $\ddot{\psi} > 0$. In this case the trajectory is not bounded by (7.22). Note that such initial points exist only in the regions $\mathscr{O}_1$, $\mathscr{O}_2$ and $\mathscr{O}_3$. We have already seen that the trajectory leaves this regions never to enter again. Let the majorant curve intersection point on the axis $\dot{\psi} = 0$ be $(\psi_M, 0)$, then

$$-2(\Xi_3 + \alpha'\Xi_2\Xi_3^2 - \Xi_2 k_M) = \dot{\psi}_0^2 \tag{7.24}$$

Similar analysis can be done to see that the trajectory in the quadrant $\psi > 0, \dot{\psi} < 0$ is confined by the trajectory of

$$\ddot{\psi} = -\Xi_3 - \alpha'\Xi_2\Xi_3^2 - \Xi_2 k_m \tag{7.25}$$

where, $k_m = k_1 - k_2$. Assume that the trajectory intersects the axis $\psi = 0$ at a point $(0, \dot{\psi}_1)$ then,

$$2(\Xi_3 + \alpha'\Xi_2\Xi_3^2 + \Xi_2 k_m) = \dot{\psi}_1^2 \tag{7.26}$$

For the convergence of the majorant trajectory it is sufficient to have $|\dot{\xi}_1|/|\dot{\xi}_0| < 1$. Thus, a sufficient condition can be written as

$$\Xi_3 + \alpha'\Xi_2\Xi_3^2 + \Xi_2 k_m < -\Xi_3 - \alpha'\Xi_2\Xi_3^2 + \Xi_2 k_M \tag{7.27}$$

This completes the proof. ■

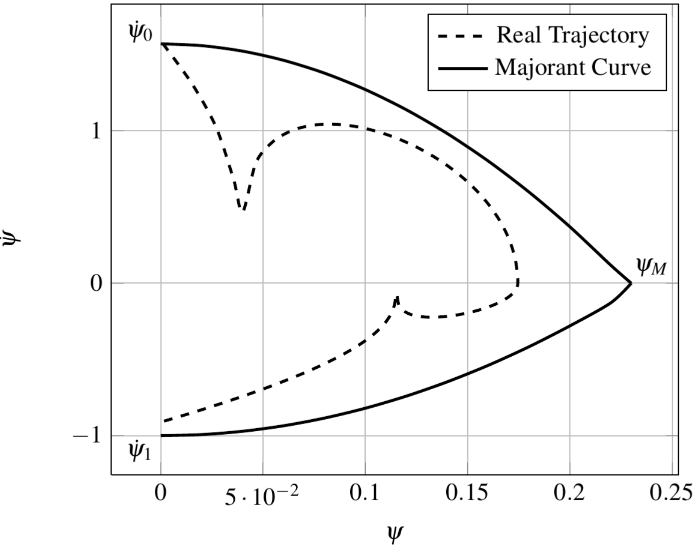

Fig. 7.1 Real trajectory and the Majorant Curve

Thus, it is established that finite time convergence to a 3-sliding set is possible by keeping a switching constraint in 2-sliding mode. Unlike the usual terminal sliding mode, the existence of Filippov solutions is also seen with a condition on the fractional power. It is important to note that the derived gain conditions are too conservative and in practice gains cannot be assigned using these inequalities.

7.2.2 *Uncertain Triple Integrator*

This section considers the triple integrator with uncertainties. The practical application of the 3-sliding algorithm requires stabilization in the presence of uncertain bounded system functions. Thus, it is necessary for the proposed controller to be able to achieve finite time stabilization of,

$$\sigma^{(3)} = \phi(t,x) + \gamma(t,x)u \tag{7.28}$$

with bounds given as,

$$|\phi(t,x)| < \Phi, 0 < \Gamma_m < \gamma(t,x) \leq \Gamma_M \tag{7.29}$$

Proposition 7.2. *The control* (7.5)*, stabilizes the uncertain integrator* (7.28) *in finite time, if the following sufficient conditions hold.*

- *α is a ratio of two odd integers with* $1 < \alpha < 1.5$.

- $\Gamma_m \Xi_3 k_M - \Xi_2 \Xi_3^2 - \Phi > \Gamma_M \Xi_3 k_m + \Xi_3 + \Xi_2 \Xi_3^2 + \Phi$.

Proof. The equation for $\ddot{\psi}$ is changed incorporating these changes as,

$$\ddot{\psi} = \xi_3 + \alpha\alpha' \xi_2^{\alpha-2} \xi_3^2 + \alpha \xi_2^{\alpha-1} \phi(t,x) + \alpha \xi_2^{\alpha-1} \gamma(t,x) u \tag{7.30}$$

Let $\phi'(t,x)$ and $\gamma'(t,x)$ be,

$$\phi'(t,x) = \xi_3 + \alpha\alpha' \xi_2^{\alpha-2} \xi_3^2 + \alpha \xi_2^{\alpha-1} \phi(t,x) \tag{7.31}$$

$$\gamma'(t,x) = \alpha \xi_2^{\alpha-1} \gamma(t,x) \tag{7.32}$$

Exactly the same analysis as in the proof of Proposition 7.1 can be repeated to determine the majorant curve in the ψ-$\dot{\psi}$ plane. The majorant curve in the first quadrant($\psi > 0, \dot{\psi} > 0$) is determined as

$$\ddot{\psi} = \Xi_3 + \Xi_2 \Xi_3^2 + \Phi - \Gamma_m k_M \tag{7.33}$$

In the second quadrant($\psi > 0, \dot{\psi} < 0$) the majorant curve is determined by

$$\ddot{\psi} = -\Xi_3 - \Xi_2 \Xi_3^2 - \Phi - \Gamma_M k_m \tag{7.34}$$

Thus, the sufficient condition in this case can be easily written as,

$$\Gamma_m \Xi_3 k_M - \Xi_2 \Xi_3^2 - \Phi > \Gamma_M \Xi_3 k_m + \Xi_3 + \Xi_2 \Xi_3^2 + \Phi \tag{7.35}$$

Thus, the proposed control produces a 2-sliding mode on the constraint $\psi(\xi)$ and in turn finite time stabilization of the triple integrator. ■

7.3 Extension to Higher Order Sliding

The idea presented in the previous section can also be used to achieve higher order sliding modes. The key idea is to hold a 2-sliding constraint to obtain higher order sliding. Consider the chain of r integrators with uncertainties which might be of the form (7.2).

$$\sigma^{(r)}(x) = \phi(t,x) + \gamma(t,x) u \tag{7.36}$$

Assume that the r-sliding set $\mathscr{S} = \{x \in \mathbb{R}^n | \sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0\}$ is non empty. The objective is to find an input u such that the trajectory of (7.36) is finite time convergent to $\mathscr{S}$. To this end, a switching function $\psi(\sigma, \dot{\sigma}, \cdots, \sigma^{(r-2)})$ is designed such that if ψ and $\dot{\psi}$ are forced to zero the reduced dynamics of (7.36) is finite time convergent.

Theorem 7.1. *Let $\psi_0 = \sigma$ and further the switching functions defined as,*

$$\psi_i = \psi_{i-1} + \dot{\psi}_{i-1}^{\alpha_i}, \; i = 1, \ldots, r-2 \tag{7.37}$$

where, $r \geq 4$ *and each* α_i *is a ratio of odd integers with* $1 < \alpha_i < \frac{r-i+1}{r-i}$.

$$u = -k_1 sgn(\psi_{r-2}) - k_2 sgn(\dot{\psi}_{r-2}),\, k_1 > k_2 \tag{7.38}$$

The controller with sufficiently large gains k_1 *and* k_2 *makes the trajectories of* (7.36) *reach the r-sliding set* $\mathscr{S}$ *in finite time.*

Proof. The control law (7.5) first establishes a 2-sliding mode on ψ_{r-2}, i.e., the trajectory reaches the set $\{\psi_{r-2} = \dot{\psi}_{r-2} = 0\}$. This provides finite time convergence of ψ_{r-3} and $\dot{\psi}_{r-3}$. In turn the trajectory is successively transferred to the set $\mathscr{N} = \{\psi_0 = \psi_1 = \cdots = \psi_{r-2} = 0\}$. As seen earlier it is clear that if the trajectory reaches within $\mathscr{N}$ and stays in $\mathscr{N}$ then the reduced order dynamics are such that $\sigma, \dot{\sigma}, \cdots, \sigma^{(r-1)}$ converges to the origin in finite time.

The second time derivative of ψ_i can be obtained exactly as (7.14). A similar argument holds for existence of Filippov solutions and the evaluation of $\ddot{\psi}_i$ on the constraint manifolds is required which is obtained as (details are given in the appendix),

$$\begin{aligned}\ddot{\psi}_i = \prod_{j=1}^{r-2} \alpha_j \dot{\psi}_{i-1}^{\alpha_i} \Big(\beta_i \dot{\psi}_{i-1}^{3-2\alpha_i} + \dot{\psi}_{i-2}^{\alpha_{i-1}-1} \Big(\beta_{i-1} \dot{\psi}_{i-2}^{4-3\alpha_{i-1}} \\ + \dot{\psi}_{i-3}^{\alpha_{i-2}-1} \Big(\beta_{i-2} \dot{\psi}_{i-3}^{5-4\alpha_{i-2}} + \ldots + u\Big)\Big)\Big)\end{aligned} \tag{7.39}$$

It is easy to see that for the right hand side of (7.39) to be well defined, the following inequalities are necessary.

$$\alpha_i < \frac{3}{2},\; \alpha_{i-1} < \frac{4}{3},\; \ldots,\; \alpha_1 < \frac{r}{r-1} \tag{7.40}$$

The (7.40) can be written collectively as

$$\alpha_i < \frac{r-i+1}{r-i} \tag{7.41}$$

As we have noted in previous sections with k_1, k_2 sufficiently large, ψ_{r-2} is attractive in finite time. Since, (7.37) define nested surfaces, ultimately the r-sliding set $\mathscr{S}$ is reached in finite time. ■

This proves to be another algorithm for stabilizing an uncertain chain of integrators in finite time. It can be noted that as this algorithm involves keeping the constraint in 2-sliding, it provides for extra accuracy compared to algorithms keeping a 1-sliding constraint. The following section considers some simulation examples to illustrate the proposed algorithm.

7.4 Simulation Examples

All simulations are carried out using MATLAB®'s ODE45 program with all tolerances set to 10^{-4}.

Example 7.1. This numerical simulation shows stabilization of an uncertain triple integrator with the proposed controller. Consider the perturbed triple integrator

$$\sigma^{(3)} = 1.5sin(2t) + (1.5 + 0.5sin(t) + 0.5cos(3t))\,u \tag{7.42}$$

The switching function $\psi(\sigma, \dot{\sigma})$ is taken as,

$$\psi(\sigma, \dot{\sigma}) = \sigma + \dot{\sigma}^{\frac{7}{5}} \tag{7.43}$$

According to the condition derived in Proposition 7.1, we have $1 < \frac{7}{5} < 1.5$ and the controller used for simulation is

$$u = -8sgn(\psi) - 6sgn(\dot{\psi}) \tag{7.44}$$

The Figure-7.3 shows that the switching function and its derivative reaches zero at about 1 unit of time. Figure-7.2 shows σ,$\dot{\sigma}$ and $\ddot{\sigma}$ reaching zero in finite time as desired.

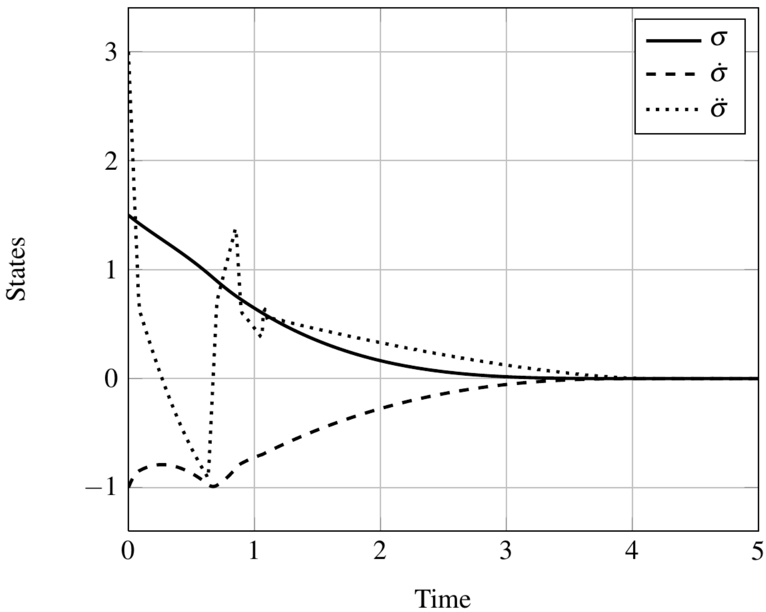

Fig. 7.2 Switching function and its derivatives in 3-sliding in Example-1

Example 7.2. This numerical simulation shows stabilization of pure fourth order integrator with the proposed control. Consider the fourth order integrator chain as $\sigma^{(4)} = u$. The switching functions are defined as

$$\psi_0 = \sigma \tag{7.45}$$

$$\psi_1 = \psi_0 + \dot{\psi}_0^{\frac{9}{7}} \tag{7.46}$$

$$\psi_2 = \psi_1 + \dot{\psi}_1^{\frac{7}{5}} \tag{7.47}$$

Since $\frac{7}{5} < \frac{3}{2}$, and $\frac{9}{7} < \frac{4}{3}$ these are proper choice of fraction powers according to the Proposition 7.2. The control law is determined as $u = -13 sign(\psi_2) - 11 sign(\dot{\psi}_2)$ and the initial conditions are $(-0.1\ \ 0.7\ \ 0.8\ \ 0.3)^T$. Figure-7.4 shows the 4-sliding trajectories of σ,$\dot{\sigma}$,$\ddot{\sigma}$, $\sigma^{(3)}$.

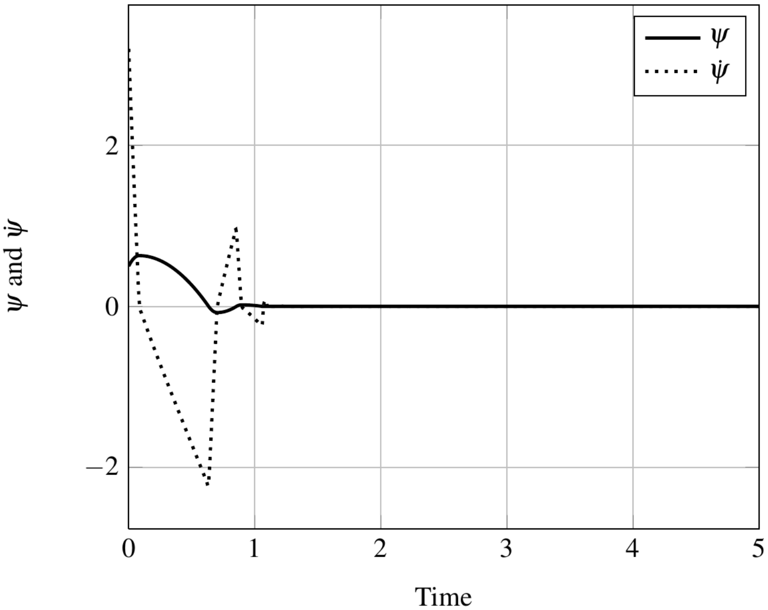

Fig. 7.3 Switching function and its derivative for Example-1

Example 7.3. To realize the proposed algorithm with uncertain functions, we consider a non linear DC motor model. DC motors have been used widely for motion control in industries. A wide variety of DC motors are available for specific applications. This example considers a series excited DC motor model. Motors are usually non linear devices but frequently linearized for control design. However, there are several applications where a linear model that ignores the friction nonlinearity is not adequate. For example, for orientation angle control of large telescopes. Since speeds are very small in these applications the friction force is dominant and cannot be ignored. One such non linear model is considered here.

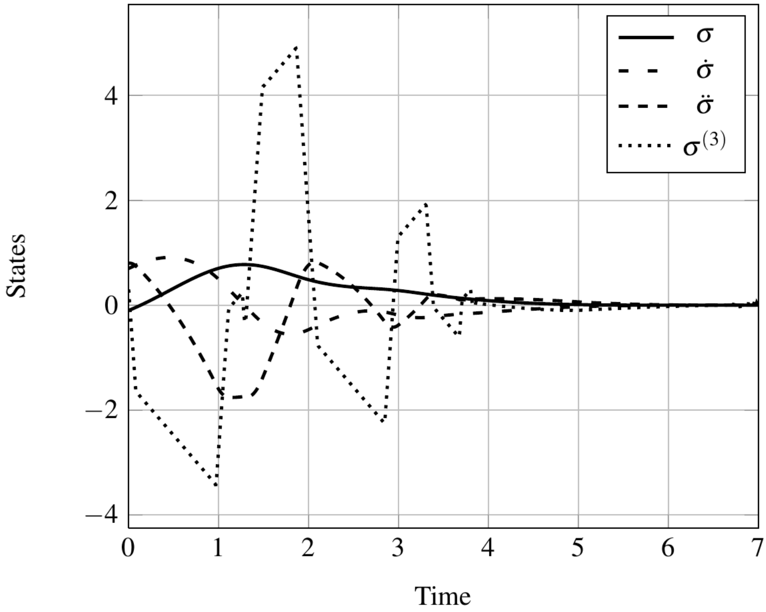

Fig. 7.4 Switching function and its derivatives in 4-sliding

The model is described by [[4]],

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -m_1x_2 + m_2x_3 - m_6 sgn(x_2)\\ &\quad - m_7e^{m_8|x_3|}sgn(x_2)\\ \dot{x}_3 &= -m_4x_2 - m_3x_3 + m_5u\end{aligned} \tag{7.48}$$

where, $x_1 = \theta$ (angular position), $x_2 = \omega$ (angular speed), $x_3 = i_a$ (motor current), $u = V_a$ (armature voltage). k_i are motor parameters given as $m_1 = 0.0110, m_2 = 16.16, m_3 = 50.66, m_4 = 1.31, m_5 = 15.8, m_6 = 19.27, m_7 = 10.11, m_8 = 0.0051$.

The problem is to design a control to track a given position i.e., $\theta_d(t)$ is given and the motor angle should track the desired angle at all times. Defining $\sigma = x_1 - x_{1d}$, it is clear that it has relative degree three with the input. Thus, the proposed method in Section 7.2 is applicable here. Please note that the *signum* function is $\dot{x}_2$ equation is just a notational aid to represent the friction force reversal when the direction of the rotation reverses. Thus, it does not affect the relative degree in any way.

The switching function is designed as $\psi = \sigma + \dot{\sigma}^{\frac{7}{5}}$ and $\dot{\psi} = \dot{\sigma} + \frac{7}{5}\dot{\sigma}^{2/5}\ddot{\sigma}$ is obtained. The gains for the control law are determined as $k_1 = 17$ and $k_2 = 13$. Figure-7.5,7.6 show the simulations with $\theta(0) = 0.35^R$ and all other initial conditions zero. The desired position is given as $\theta_d = \pi/2^R$.

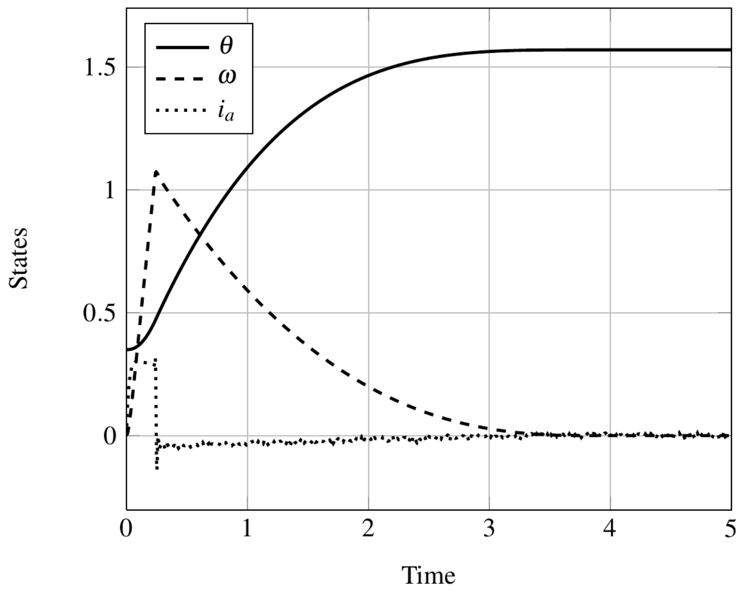

Fig. 7.5 Angle, speed and armature current with 3-sliding

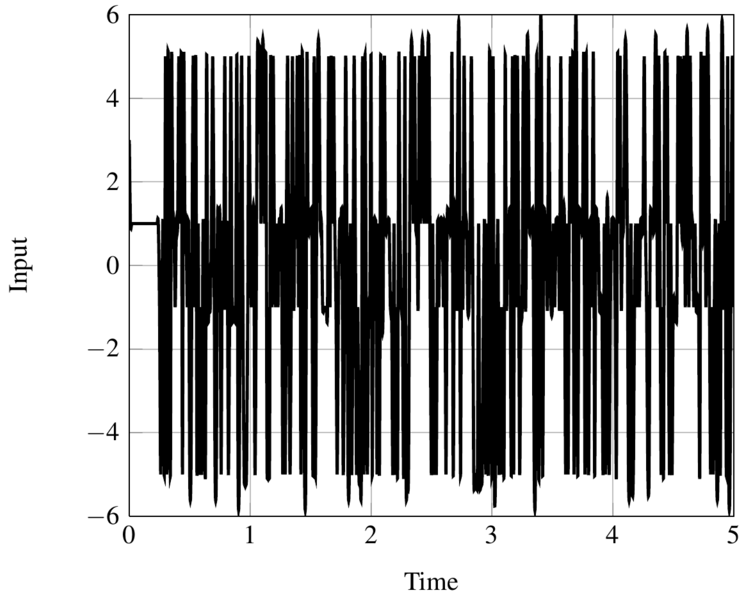

Fig. 7.6 Armature Voltage with 3-sliding

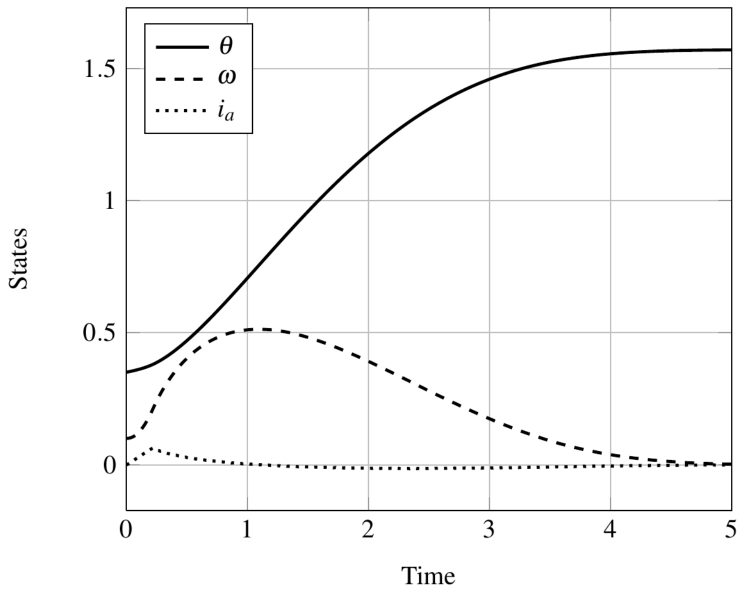

Fig. 7.7 State Trajectories with 4-sliding control

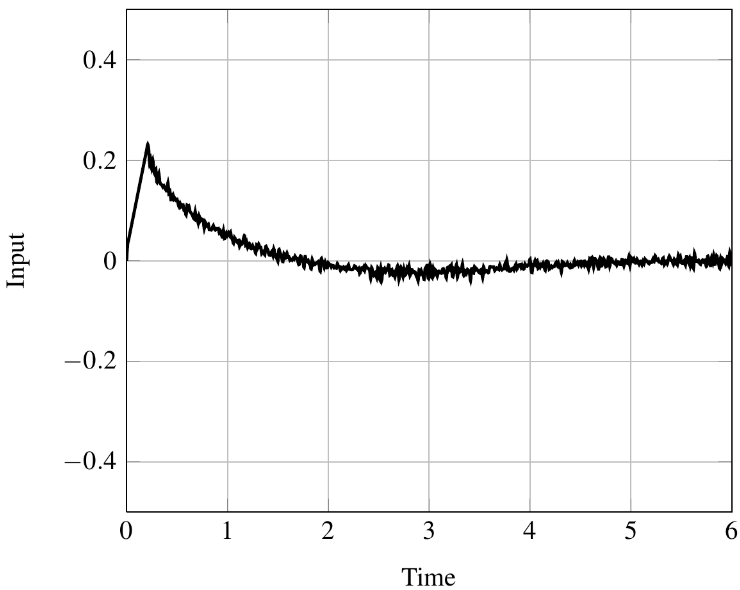

Fig. 7.8 Armature Voltage with 4-sliding control

It is observed that having relative degree three output, the control is discontinuous and if smooth control is desired then 4-th or higher order sliding mode control has to be designed. According to the proposed method, a nested switching function can be designed as

$$\psi_0 = x_1 - \theta_d \tag{7.49}$$

$$\psi_1 = \psi_0 + \dot{\psi}_0^{\frac{9}{7}} \tag{7.50}$$

$$\psi_2 = \psi_1 + \dot{\psi}_1^{\frac{7}{5}} \tag{7.51}$$

The controller is given as $u = -3sign(\psi_2) - 2sign(\dot{\psi}_2)$. Figure-7.7,7.8 shows trajectories and input with 4-sliding controller. Comparatively smooth input can be recognised.

7.5 Conclusion

A new algorithm to obtain higher order sliding mode control is presented in this paper. The proposed algorithm keeps a properly designed switching constraint in 2-sliding. The constraint design utilizes a form of non-singular terminal sliding mode switching function, and keeping it in 2-sliding is achieved using twisting controller. It has been shown that Filippov solutions exist with properly chosen fractional powers. The proposed controller is simulated numerically for third and fourth order sliding modes. Also the proposed method is applied to a DC motor with friction in an angle tracking problem, where fourth order sliding is used to avoid chattering in the controller.

7.6 Appendix

7.6.1 The Switching Constraints and its Derivatives

The nested switching functions are listed as defined in (7.37),

$$\begin{aligned}
\psi_0 &= \sigma \\
\psi_1 &= \psi_0 + \dot{\psi}_0^{\alpha_1} \\
\psi_2 &= \psi_1 + \dot{\psi}_1^{\alpha_2} \\
\psi_3 &= \psi_2 + \dot{\alpha}_2^{\alpha_3} \\
&\vdots
\end{aligned} \tag{7.52}$$

The time derivatives of the functions (7.52) can also be listed accordingly

$$\begin{aligned}\dot{\psi}_1 &= \dot{\psi}_0 + \alpha_1 \dot{\psi}^{\alpha_1 - 1} \ddot{\psi}_0 \\ \dot{\psi}_2 &= \dot{\psi}_1 + \alpha_2 \dot{\psi}^{\alpha_2 - 1} \ddot{\psi}_1 \\ \dot{\psi}_3 &= \dot{\psi}_2 + \alpha_3 \dot{\psi}^{\alpha_3 - 1} \ddot{\psi}_2 \\ &\vdots\end{aligned} \tag{7.53}$$

Since we are interested in the form of $\ddot{\psi}_i$, while $\psi_i = \dot{\psi}_i = 0$, it is necessary to evaluate the identities provided by (7.52) and (7.53).

$$\dot{\psi}_0 = -\psi_0^{\frac{1}{\alpha_1}} \tag{7.54}$$

$$\dot{\psi}_1 = -\psi_1^{\frac{1}{\alpha_1}} \tag{7.55}$$

$$\dot{\psi}_2 = -\psi_2^{\frac{1}{\alpha_1}} \tag{7.56}$$

$$\vdots \tag{7.57}$$

and from first time derivatives,

$$\ddot{\psi}_0 = -\frac{1}{\alpha_1}\dot{\psi}_0^{2-\alpha_1} \tag{7.58}$$

$$\ddot{\psi}_1 = -\frac{1}{\alpha_2}\dot{\psi}_1^{2-\alpha_2} \tag{7.59}$$

$$\ddot{\psi}_2 = -\frac{1}{\alpha_3}\dot{\psi}_2^{2-\alpha_3} \tag{7.60}$$

$$\vdots \tag{7.61}$$

Now, if desired sliding order is $r = 3$ then using (7.54) and (7.58),

$$\ddot{\psi}_1 = \alpha_1 \psi_0^{\alpha_1 - 1}\left(\frac{\alpha_1 - 2}{\alpha_1^2}\psi_0^{3-2\alpha_1} + \psi_0^{(3)}\right) \tag{7.62}$$

$$= \alpha_1 \psi_0^{\alpha_1 - 1}\left(\beta_1 \psi_0^{3-2\alpha_1} + \psi_0^{(3)}\right) \tag{7.63}$$

For $r = 4$, using (7.55) and (7.59)

$$\begin{aligned}\ddot{\psi}_2 = \alpha_1 \alpha_2 \dot{\psi}_1^{\alpha_2 - 1}\Bigg(&\frac{\alpha_2 - 2}{\alpha_1 \alpha_2^2}\dot{\psi}_1^{3-2\alpha_2} \\ &+ \dot{\psi}_0^{\alpha_1 - 1}\left(\left(\frac{(\alpha_1 - 2)^2}{\alpha_1^2} - \alpha_1 + 1\right)\dot{\psi}_0^{4-3\alpha_1} + \psi_0^{(4)}\right)\Bigg)\end{aligned} \tag{7.64}$$

Denoting real number terms of α_1 and α_2 by β_1 and β_2 respectively

$$\ddot{\psi}_2 = \alpha_1 \alpha_2 \dot{\psi}_1^{\alpha_2 - 1} \left(\beta_2 \dot{\psi}_1^{3-2\alpha_2} + \dot{\psi}_0^{\alpha_1 - 1} \left(\beta_1 \dot{\psi}_0^{4-3\alpha_1} + \psi_0^{(4)} \right) \right) \tag{7.65}$$

For $r = 5$, using (7.56) and (7.60),

$$\ddot{\psi}_3 = \alpha_1 \alpha_2 \alpha_3 \dot{\psi}_2^{\alpha_3 - 1} \left(\beta_3 \dot{\psi}_2^{3-2\alpha_3} + \dot{\psi}_1^{\alpha_2 - 1} \left(\beta_2 \dot{\psi}_1^{4-3\alpha_2} + \dot{\psi}_0^{\alpha_1 - 1} \left(\beta_1 \dot{\psi}^{5-4\alpha_1} + \psi_0^{(5)} \right) \right) \right) \tag{7.66}$$

Thus, this recursion leads to the r^{th}-order equation as,

$$\ddot{\psi}_i = \prod_{j=1}^{r-2} \alpha_j \dot{\psi}_{i-1}^{\alpha_i} \left(\beta_i \dot{\psi}_{i-1}^{3-2\alpha_i} + \dot{\psi}_{i-2}^{\alpha_{i-1}-1} \left(\beta_{i-1} \dot{\psi}_{i-2}^{4-3\alpha_{i-1}} + \dot{\psi}_{i-3}^{\alpha_{i-2}-1} \left(\beta_{i-2} \dot{\psi}_{i-3}^{5-4\alpha_{i-2}} + \ldots + u \right) \right) \right) \tag{7.67}$$

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