Chapter 3
Decentralised Variable Structure Control for Time Delay Interconnected Systems

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Abstract. A class of multiple time varying delay interconnected systems with nonlinear disturbances is considered in this Chapter, where both the known and uncertain interconnections involve time delay. A decentralised static output feedback variable structure control is synthesised, which is independent of the time delays, to stabilise the system globally uniformly asymptotically. The stability of the closed loop system is analysed based on the Lyapunov Razumikhin approach. Then, for interconnected systems where each subsystem is square, it is shown that the effects of the uncertain interconnections can be largely rejected by appropriate controllers if the delays are known and the uncertain interconnections are bounded by a class of functions of the outputs and delayed outputs. A case study relating to a river pollution control problem is presented to illustrate the proposed approach.

3.1 Introduction

Interconnected systems exist widely in the real world. Examples include power networks, cellular systems, ecological systems and financial systems. Such systems are often widely distributed in space. A fundamental characteristic of interconnected systems, which holds for both natural and engineered systems, is that they tend to operate in a decentralised manner. For interconnected systems, the presupposition of centrality generally fails to hold due to the lack of centralised information or the lack of a centralised decision making focus. Even with engineered systems, issues such as the economic cost and reliability of communication links, particularly when systems are characterised by geographical separation, limit the appetite

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to develop centralised systems. This has motivated the development of a wide literature in the area of decentralised control for interconnected systems, see, for example, ⟦12, 17, 20, 26, 27⟧.

3.1.1 Interconnected Systems

Interconnected systems are often modelled as dynamical equations composed of interconnections between a collection of lower-dimensional subsystems. A fundamental property of any interconnected system is that a perturbation of one subsystem can affect the other subsystems as well as the overall performance of the network. The purpose of control and monitoring paradigms from the domain of engineering within an interconnected system's architecture is thus to minimise the effect of any perturbation or uncertainty on the overall system behaviour.

Large scale interconnected systems were studied from the engineering perspective as early as the 1970's ⟦24⟧. This early work focussed primarily on linear interconnected systems. The dynamics of large scale natural and engineered interconnected systems are usually highly nonlinear, and thus it is not only the structure of the system which produces complexity but also the nonlinearity of the dynamics. It is clear that although a linear dynamics may approximate the orbit of a nonlinear system locally, it does not permit the existence of the multiple states observed in real networks and does not accommodate global properties of the system. Such global properties can be crucial because they may become significant when the system is perturbed or a subsystem enters a failure state. Increasing requirements on system performance coupled with the ability to model and simulate reality by means of complex, possibly nonlinear, interconnected systems models have motivated increasing contributions to the study of such systems. This interest has been further stimulated by the simultaneous development of nonlinear systems theory and the emergence of powerful mathematical and computational tools which render the formal and constructive study of nonlinear large scale systems increasingly possible.

3.1.2 Decentralised Output Feedback Control

Decentralised output feedback control, where only limited local system state information is available to design any corrective action, has received much attention in the literature and many interesting results have been obtained. Many of these methods are based on Lyapunov approaches or involve adaptive control. In the contributions of Saberi and Khalil ⟦23⟧ and Yan et al. ⟦31⟧, Lyapunov methods are used to form the control scheme and strict structural conditions are imposed on each of the nominal subsystem models. The work also includes some strong limitations on the admissible interconnections. Adaptive control techniques have been employed by Zhou and Wen ⟦34⟧, and Jain and Khorrami ⟦11⟧, but only parametric uncertainty is dealt with due to the limitation of the approach; this is clearly a strong limitation as uncertainty in the possibly nonlinear system structure as well as external perturbations are key factors in any interconnected system of practical significance. The corresponding results can thus only be applied to certain systems with special structure. An appropriate methodology must be able to deal with a broad class of nonlinear subsystems where the subsystems themselves as well as the possibly nonlinear interconnections between them will be uncertain and only limited system state variables will be measurable. So called sliding mode control has been used successfully by many authors in such uncertain, nonlinear scenarios [2, 3, 25, 32]. However, the primary focus in the literature has been on centralised control which is problematic to implement in large-scale interconnected systems using decentralised control.

Sliding mode control schemes for large-scale systems have also been proposed in the literature, see for example [7, 13]. However, in such contributions it is required that the uncertainties and the interconnections have special structure, or else have linear or polynomial bounds. In addition, most methods focus on the so-called state feedback control case where all state information is assumed available to the control design. Much less attention has been paid to the output feedback, or limited information, case. Lee proposed a decentralised output feedback control scheme using sliding mode techniques [13], where not only were the isolated subsystems required to be linear, but the interconnections were restricted to the linear case as well. Also all of the uncertainties and interconnections are required to adopt a specific structure, i.e. satisfy the so-called matching conditions whereby all the perturbations and interconnection effects are assumed to be implicit in the control injection channels. Recent work has made significant contributions to alleviating these constraints and has developed constructive frameworks for the development of output feedback control strategies based on sliding mode techniques [27–29]. This work encompasses nonlinear system representations, uncertainty and unknown perturbations as well as limited available measurements of the system state. A class of nonlinear, large-scale interconnected systems incorporating a broad range of uncertainties has been considered where no statistical information about the uncertainties is imposed.

3.1.3 Time Delay in Interconnected Systems

Interconnections between two or more subsystems in a network are often accompanied by phenomena such as material transfer, energy transfer and information transfer. From a mathematical point of view, transfer phenomena can be represented by delay elements [19]. However, for such a time delay interconnected system, the future evolution frequently depends not only on the present state but also on the past history of the system. The presence of even a small delay may greatly affect the performance of a system; a stable system may become unstable, or chaotic behaviour may result [19]. This has motivated the importance of the study of interconnected systems in the presence of delay [1].

It should be noted that time delay is another important factor which makes the study of interconnected systems complex [21]. Mahmoud and Bingulac [18] considered a class of interconnected systems where delay does not appear in the interconnection terms. Although time delay interconnected systems have been considered, and many results have been achieved ⟦1, 8⟧, most of the existing results are based on the fact that the system states are available. The associated decentralised output feedback results for time-delayed interconnected systems are few ⟦10,15,33⟧. An output feedback decentralised control scheme is given in ⟦16⟧ where discrete interconnected systems are considered. A class of nonlinear interconnected systems with triangular structure is considered in ⟦10⟧, and an interconnected system composed of a set of single input single output subsystems with dead zone input is considered in ⟦33⟧. In both ⟦10⟧ and ⟦33⟧, the control schemes are based on dynamical output feedback which increases the computation greatly due to the associated closed-loop system possessing possibly double the order of the actual plant. A decentralised model reference adaptive control scheme is proposed in ⟦15⟧ where the considered interconnections are linear and matched.

Some work considers systems of particular structure, such as the work of Hua and Guan ⟦9⟧ where a triangular structure is assumed. In all of the existing output feedback control strategies for interconnected time delay systems, the nominal isolated subsystems are required to be linear, and the bounds on the disturbances are functions of the outputs and/or largely linear ⟦9, 16, 33⟧. A class of interconnected systems with time delay is considered in ⟦8⟧ where a model following problem is considered and state feedback is employed. Building on a strong track record of work in the area of control of delay systems ⟦29⟧ and interconnected systems ⟦27,28⟧, recent work has sought to develop a global decentralised static output feedback robust control scheme for interconnected systems ⟦30⟧ where it is assumed that all the time delays are known.

3.1.4 Contribution

In this Chapter, a variable structure control is synthesised to stabilise a class of time delay interconnected systems with nonlinear disturbances. The bounds on the uncertainties are nonlinear and time delayed. Both the isolated subsystems and the interconnections involve multiple time varying delays. A decentralised variable structure control scheme using only output information is proposed firstly which is independent of time delay. Based on the Lyapunov Razumikhin approach, sufficient conditions are derived such that the closed-loop systems formed by the designed control and the considered interconnected systems are globally uniformly asymptotically stable. Then, for interconnected systems composed of a set of square subsystems, it is shown that the effects of the nonlinear interconnections can be largely rejected if their bounds are nonlinear functions of only the outputs and delayed outputs, and the delays are available for design. The limitation that the rate of change of the time delay is less than one, is not required. A compensator, which increases the required computation levels for large-scale interconnected systems, is not required either. A case study on the river pollution problem is presented to demonstrate the work.

3.2 Preliminaries

This section will provide the required notation and some basic results which will be used later in this Chapter.

3.2.1 Notation

In this Chapter, $\mathscr{R}^+$ denotes the nonnegative set of real numbers $\{t \mid t \geq 0\}$. The symbol $\mathscr{C}_{[a,b]}$ represents the set of $\mathscr{R}^n$-valued continuous function on $[a,b]$ and I_n denotes the unit matrix with dimension n. For a matrix A, the expression $A > 0$ ($A < 0$) means that A is symmetric positive (negative) definite and $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) represents its maximum (minimum) eigenvalue. The symbol diag$\{A_1, A_2, \cdots, A_n\}$ represents diagonal/block-diagonal matrix with diagonal entries $A_1, A_2, \cdots, A_n$. For vectors $x = (x_1, x_2, \ldots, x_{n_1})^T \in \mathscr{R}^{n_1}$ and $y = (y_1, y_2, \ldots, y_{n_2})^T \in \mathscr{R}^{n_2}$, the expression $f(x,y)$ denotes a function $f(x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$ defined on $\mathscr{R}^{n_1+n_2}$. Finally, $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

3.2.2 Basic Results

Definition 3.1. (see, [[6]]) A continuous function $\alpha : [0,a) \mapsto [0,\infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. Further, it is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

Consider a time-delay system

$$\dot{x}(t) = f(t, x(t - d(t)) \tag{3.1}$$

with initial condition

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0]$$

where $f : \mathscr{R}^+ \times \mathscr{C}_{[-\overline{d},0]} \mapsto \mathscr{R}^n$ takes $\mathscr{R}\times$ (bounded sets of $\mathscr{C}_{[-\overline{d},0]}$) into bounded sets in $\mathscr{R}^n$; $d(t)$ is the time-varying delay and $\overline{d} := \sup_{t\in\mathscr{R}^+}\{d(t)\} < \infty$.

Lemma 3.1. *If there exist class $\mathscr{K}_\infty$ functions $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, a class $\mathscr{K}$ function $\zeta_3(\cdot)$ and a continuous function $V_1(\cdot) : [-\overline{d}, \infty] \times \mathscr{R}^n \mapsto \mathscr{R}^+$ satisfying*

$$\zeta_1(\|x\|) \leq V_1(t,x) \leq \zeta_2(\|x\|), \qquad t \in \mathscr{R}^+, \quad x \in \mathscr{R}^n$$

such that the time derivative of V_1 along the solution of system (3.1) satisfies

$$\dot{V}_1(t,x) \leq -\zeta_3(\|x\|) \tag{3.2}$$

whenever

$$V_1(t+d, x(t+d)) \leq V_1(t, x(t))$$

for any $d \in [-\overline{d}, 0]$, *then the system (3.1) is uniformly stable. If in addition,* $\zeta_3(\tau) > 0$ *for* $\tau > 0$ *and there exists a continuous nondecreasing function* $\xi(\tau) > \tau$ *for* $\tau > 0$ *such that (3.2) is strengthened to*

$$\dot{V}_1(t,x) \leq -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t+d, x(t+d)) \leq \xi(V_1(t, x(t))) \tag{3.3}$$

for $d \in [-\overline{d}, 0]$ *, then the system (3.1) is uniformly asymptotically stable. Further, if in addition* $\lim_{\tau\to\infty} \zeta_1(\tau) = \infty$, *then, the system (3.1) is globally uniformly asymptotically stable.*

Proof. See pages 14-15 in [6].

Lemma 3.1 is the well known Razumikhin Theorem [6]. From Lemma 3.1, the following result can be obtained.

Lemma 3.2. *Consider system (3.1). If there exists a function* $V_0(x) = x^T P x$ *with* $P > 0$ *such that for* $d \in [-\overline{d}, 0]$, *the time derivative of* V_0 *along the solution of system (3.1) satisfies*

$$\dot{V}_0(x) \leq -q_1\|x\|^2 \quad \text{if} \quad V_0(x(t+d)) \leq q_2 V_0(x(t)) \tag{3.4}$$

for some $q_1 > 0$ *and* $q_2 > 1$, *then system (3.1) is globally uniformly asymptotically stable.*

Proof. See Lemma 1 of Appendix 1 in [30].

Lemma 3.3. *Assume the matrix/vector functions* $H_{ij}(t, x_j) \in \mathscr{R}^{n_i \times m_j}$ *with* n_i *and* m_j *positive integral numbers, and* $x = col(x_1, x_2, \cdots, x_n)$ *where* $x_i \in \mathscr{R}^{n_i}$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ij}(t, x_j) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ji}(t, x_i)$$

Proof. From the fact that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij}(t, x_j) = \sum_{j=1}^{n} \sum_{i=1}^{n} H_{ij}(t, x_j)$$

it follows that

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ij}(t, x_j) \\
&= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij}(t, x_j) - H_{11}(t, x_1) - H_{22}(t, x_2) - \cdots - H_{nn}(t, x_n) \\
&= \textstyle\sum_{j=1}^{n} \sum_{i=1}^{n} H_{ij}(t, x_j) - \sum_{j=1}^{n} H_{jj}(t, x_j) \\
&= \textstyle\sum_{j=1}^{n} \left(\sum_{i=1}^{n} H_{ij}(t, x_j) - H_{jj}(t, x_j)\right) \\
&= \textstyle\sum_{j=1}^{n} \sum_{\substack{i=1 \\ i\neq j}}^{n} H_{ij}(t, x_j) \\
&= \textstyle\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ji}(t, x_i)
\end{aligned}$$

Hence the conclusion follows.

The results presented in this section will be used in the later analysis.

3.3 System Description and Basic Assumptions

In this section, the systems considered in this chapter will be presented and basic assumptions will be imposed.

3.3.1 Interconnected System Description

Consider a time-varying delayed interconnected system composed of n n_i-th order subsystems

$$\dot{x}_i = A_i x_i + B_i \left(u_i + G_i\left(t, x_i, x_{id_i}\right)\right) + \sum_{\substack{j=1\\ j\neq i}}^{n} \left(E_{ij} x_{jd_j} + F_{ij} x_j + \Phi_{ij}(t, x_j, x_{jd_j})\right) \quad (3.5)$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, n, \quad (3.6)$$

where $x := col(x_1, \ldots, x_n)$, $x_i \in \mathscr{R}^{n_i}$, $u_i \in \mathscr{R}^{m_i}$ and $y_i \in \mathscr{R}^{p_i}$ are the state variables, inputs and outputs of the i-th subsystem respectively. The triple (A_i, B_i, C_i) and $E_{ij}, F_{ij} \in \mathscr{R}^{n_i \times n_j}$ with $i \neq j$ represent constant matrices of appropriate dimensions with B_i and C_i of full rank. The functions $G_i(\cdot)$ are matched nonlinear uncertainties in the i-th subsystem. The terms

$$\sum_{\substack{j=1\\ j\neq i}}^{n} (E_{ij} x_{jd_j} + F_{ij} x_j) \qquad \text{and} \qquad \sum_{\substack{j=1\\ j\neq i}}^{n} \Phi_{ij}(t, x_j, x_{jd_j})$$

are, respectively, the known and uncertain interconnections of the i-th subsystem; $x_{id_i} := x_i(t - d_i)$ are the delayed states, and the symbols $d_i := d_i(t)$ denote the time-varying delays which are assumed to be known, nonnegative and bounded in $\mathscr{R}^+$, that is

$$\overline{d}_i := \sup_{t \in \mathscr{R}^+} \{d_i(t)\} < \infty, \qquad i = 1, 2, \ldots, n$$

The initial conditions associated with the time delays are given by

$$x_i(t) = \phi_i(t), \qquad t \in [-\overline{d}_i, 0]$$

where $\phi_i(\cdot)$ are continuous in $[-\overline{d}_i, 0]$ for $i = 1, 2, \ldots, n$. It is assumed that all the nonlinear functions are smooth enough such that the unforced interconnected system has a unique continuous solution.

Definition 3.2. Consider system (3.5)–(3.6). The systems

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_i\big(u_i + G_i\big(t, x_i, x_{id_i}\big)\big)\\ y_i &= C_i x_i, \qquad i = 1, 2, \ldots, n,\end{aligned}$$

are called the i-th isolated subsystems of the system (3.5)–(3.6), and the systems

$$\dot{x}_i = A_i x_i + B_i u_i \tag{3.7}$$
$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, n, \tag{3.8}$$

are said to be the i-th nominal isolated subsystems of the system (3.5)–(3.6).

3.3.2 Assumptions

For the interconnected system (3.5)–(3.6), it is required to impose the following conditions.

Assumption 3.1. There exist known continuous functions $\rho_i(\cdot)$ and $\varpi_i(\cdot)$ and constants α_{ij} and β_{ij} such that for $i \neq j,\ i, j = 1, 2, \ldots, n$

$$\|G_i(t, x, x_{id_i})\| \le \rho_i(t, y_i) + \varpi_i(t, y_i)\|x_{id_i}\| \tag{3.9}$$
$$\|\Phi_{ij}(t, x_j, x_{jd_j})\| \le \alpha_{ij}\|x_j\| + \beta_{ij}\|x_{jd_j}\| \tag{3.10}$$

Remark 1. Assumption 3.1 is a limitation on the uncertainties that can be tolerated by the system. It is not required that the interconnections are described or bounded by functions of the system outputs. Unlike ⟦22, 33⟧, time delays are involved in the interconnections; and the result obtained in this chapter will be global.

Assumption 3.2. There exist matrices K_i, D_i and $P_i > 0$ such that for $i = 1, 2, \ldots, n$

$$-Q_i := (A_i - B_i K_i C_i)^T P_i + P_i(A_i - B_i K_i C_i) < 0 \tag{3.11}$$
$$B_i^T P_i = D_i C_i \tag{3.12}$$

Remark 2. Assumption 3.2 describes a structural property associated with the triple (A_i, B_i, C_i) which is the standard Constrained Lyapunov Problem (CLP) ⟦5⟧. A similar limitation has been imposed by many authors (see e.g, ⟦5, 31⟧). Necessary and sufficient conditions for solving the CLP can be found in ⟦4, 5⟧.

3.3.3 Problem Statement

In this chapter, it is assumed that all the isolated subsystems (3.7) and (3.8) are output feedback stabilisable. The objective is to design a variable structure control law of the form

$$u_i = u_i(t, y_i), \qquad i = 1, 2, \ldots, n \tag{3.13}$$

such that the associated closed-loop system formed by applying the control law in (3.13) to the interconnected system (3.5)–(3.6), is globally uniformly asymptotically stable even in the presence of the uncertainties and time delays. Since the control elements u_i in (3.13) are only dependent on the time t and output y_i, and are independent of time delay, they are called a memoryless decentralised static output feedback control. Then, for interconnected systems with square subsystems, delay dependent decentralised output feedback control elements

$$u_i = u_i(t, y_i, y_{id_i}), \qquad i = 1, 2, \ldots, n$$

are proposed such that the effects of the uncertain interconnections are largely rejected.

3.4 Decentralised Delay Independent Control

In this section, a decentralised output feedback controller which is independent of the time delay will be designed for the interconnected systems (3.5)–(3.6).

3.4.1 Designed Control

Consider the control

$$u_i = -K_i y_i - \frac{1}{2\varepsilon_i} D_i y_i \varpi_i^2(t, y_i) + u_i^a(t, y_i), \qquad i = 1, 2, \ldots, n \tag{3.14}$$

where $K_i \in \mathscr{R}^{m_i \times p_i}$ are design parameters satisfying Assumption 3.2, $\varepsilon_i > 0$ are constant and $u_i^a(\cdot)$ are defined by

$$u_i^a(\cdot) := \begin{cases} -\frac{D_i y_i}{\|D_i y_i\|} \rho_i(t, y_i), & D_i y_i \neq 0 \\ 0, & D_i y_i = 0 \end{cases} \tag{3.15}$$

where D_i satisfy (3.12). Since the structure of the control u_i in (3.14) are variable due to $u_i^a(\cdot)$ in (3.15), they are called a variable structure control. Clearly it is decentralised because u_i is only dependent on time t and local output information y_i. Thus u_i in (3.14) are called decentralised output feedback variable structure controllers.

3.4.2 Main Result

The following result can now be presented:

Theorem 3.1. *Assume that Assumptions 3.1–3.2 hold. Then, the closed-loop system formed by applying the control (3.14)–(3.15) into system (3.5)–(3.6) is globally* *uniformly asymptotically stable if* $W^T + W > 0$ *where the matrix* $W = [w_{ij}]_{2n\times 2n}$ *is defined by*

$$w_{ij} = \begin{cases} \lambda_{\min}(Q_i) - q\lambda_{\max}(P_i), & 1 \le i = j \le n \\ \lambda_{\min}(P_i) - \varepsilon_i, & n+1 \le i = j \le 2n \\ -2\|P_iF_{ij}\| - 2\alpha_{ij}\|P_i\|, & i \ne j \quad and \quad 1 \le i,j \le n \\ -2\|P_iE_{i(j-n)}\| - 2\beta_{i(j-n)}\|P_i\|, & 1 \le i \le n,\ j > n \ and\ j-n \ne i \\ -2\|P_{i-n}E_{(i-n)j}\| - 2\beta_{(i-n)j}\|P_{i-n}\|, & i > n,\ 1 \le j \le n \ and\ i-n \ne j \\ 0, & otherwise \end{cases}$$

for constants $q > 1$ *and* $\varepsilon_i > 0$*, where* α_{ij} *and* β_{ij} *are defined in (3.10) for* $i,j = 1,2,\dots,n,\ i \ne j$.

Proof. Applying the control (3.14)–(3.15) into system (3.5)–(3.6), it follows that the closed-loop system is described by

$$\dot{x}_i = A_ix_i + B_i\left(-K_iC_ix_i - \frac{1}{2\varepsilon_i}D_iy_i\varpi_i^2(t,y_i) + u_i^a(t,y_i) + G_i\big(t,x_i,x_{id_i}\big)\right) + \sum_{\substack{j=1\\ j\ne i}}^{n}\Big(E_{ij}x_{jd_j} + F_{ij}x_j + \Phi_{ij}(t,x_j,x_{jd_j})\Big) \tag{3.16}$$

where $u_i^a(\cdot)$ are given by (3.15) for $i = 1,2,\dots,n$. For system (3.16), consider the Lyapunov function candidate

$$V(x(t)) = \sum_{i=1}^{n} x_i^T(t)P_ix_i(t) \tag{3.17}$$

where $P_i > 0$ satisfy Assumption 3.2 for $i = 1,2,\dots,n$. Then, the time derivative of $V(\cdot)$ along the trajectories of system (3.16) is given by

$$\begin{aligned} \dot{V} = & -\sum_{i=1}^{n}x_i^TQ_ix_i + 2\sum_{i=1}^{n}x_i^TP_iB_i\left(-\frac{1}{2\varepsilon_i}D_iy_i\varpi_i^2(t,y_i) + u_i^a(t,y_i)\right) \\ & +2\sum_{i=1}^{n}x_i^TP_iB_iG_i(t,x_i,x_{id_i}) + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\ne i}}^{n}x_i^TP_iE_{ij}x_{jd_j} \\ & +2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\ne i}}^{n}x_i^TP_iF_{ij}x_j + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\ne i}}^{n}x_i^TP_i\Phi_{ij}(t,x_j,x_{jd_j}) \end{aligned} \tag{3.18}$$

From (3.9), (3.12) and Young's inequality, it follows that for any $\varepsilon_i > 0$

$$\begin{aligned} x_i^T P_i B_i G_i(t,x_i,x_{id_i}) &= (D_i y_i)^T G_i(t,x_i,x_{id_i}) \\ &\le \|D_i y_i\| \rho_i(t,y_i) + \|D_i y_i\| \varpi_i(t,y_i) \|x_{id_i}\| \\ &\le \|D_i y_i\| \rho_i(t,y_i) + \frac{1}{2\varepsilon_i} \|D_i y_i\|^2 \varpi_i^2(t,y_i) + \frac{\varepsilon_i}{2} \|x_{id_i}\|^2 \end{aligned} \quad (3.19)$$

From (3.12) and the definition of $u_i^a(\cdot)$ in (3.15), it follows that

i) if $D_i y_i = 0$, then $u_i^a(\cdot) = 0$, and thus

$$x_i^T P_i B_i u_i^a(t,y_i) + \|D_i y_i\| \rho_i(t,y_i) = \|D_i y_i\| \rho_i(t,y_i) = 0$$

ii) if $D_i y_i \neq 0$, from the definition of $u_i^a(\cdot)$ in (3.15),

$$\begin{aligned} & x_i^T P_i B_i u_i^a(t,y_i) + \|D_i y_i\| \rho_i(t,y_i) \\ & \le -(D_i y_i)^T \frac{D_i y_i}{\|D_i y_i\|} \rho(t,y_i) + \|D_i y_i\| \rho_i(t,y_i) \\ & = 0 \end{aligned}$$

Thus, from i) and ii) above,

$$x_i^T P_i B_i u_i^a(t,y_i) + \|D_i y_i\| \rho_i(t,y_i) \le 0, \qquad i = 1,2,\cdots,n \quad (3.20)$$

Further, from (3.12),

$$\begin{aligned} & -\frac{1}{2\varepsilon_i} x_i^T P_i B_i D_i y_i \varpi_i^2(t,y_i) + \frac{1}{2\varepsilon_i} \|D_i y_i\|^2 \varpi_i^2(t,y_i) \\ &= -\frac{1}{2\varepsilon_i} x_i^T C_i^T D_i^T D_i y_i \varpi_i^2(t,y_i) + \frac{1}{2\varepsilon_i} \|D_i y_i\|^2 \varpi_i^2(t,y_i) \\ &= -\frac{1}{2\varepsilon_i} (D_i y_i)^T D_i y_i \varpi_i^2(t,y_i) + \frac{1}{2\varepsilon_i} \|D_i y_i\|^2 \varpi_i^2(t,y_i) = 0 \end{aligned} \quad (3.21)$$

Therefore, from (3.19), (3.20) and (3.21)

$$\begin{aligned} 2\sum_{i=1}^{n} x_i^T P_i B_i \left(-\frac{1}{2\varepsilon_i} D_i y_i \varpi_i^2(t,y_i) + u_i^a(t,y_i) \right) + 2\sum_{i=1}^{n} x_i^T P_i B_i G_i(t,x_i,x_{id_i}) \\ \le \sum_{i=1}^{n} \varepsilon_i \|x_{id_i}\|^2 \end{aligned} \quad (3.22)$$

From (3.10),

$$\begin{aligned} x_i^T P_i \Phi_{ij}(t,x_j,x_{jd_j}) &\le \|x_i\| \, \|P_i\| \left(\alpha_{ij} \|x_j\| + \beta_{ij} \|x_{jd_j}\| \right) \\ &= \alpha_{ij} \|P_i\| \, \|x_i\| \, \|x_j\| + \beta_{ij} \|P_i\| \, \|x_i\| \|x_{jd_j}\| \end{aligned} \quad (3.23)$$

Applying (3.22) and (3.23) to equation (3.18) yields

$$\dot{V} \leq -\sum_{i=1}^{n} x_i^T Q_i x_i + \sum_{i=1}^{n} \varepsilon_i \|x_{id_i}\|^2 + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i E_{ij} x_{jd_j} + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i F_{ij} x_j$$
$$+2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \Big(\alpha_{ij}\|P_i\|\,\|x_i\|\,\|x_j\| + \beta_{ij}\|P_i\|\,\|x_i\|\|x_{jd_j}\|\Big) \qquad (3.24)$$

From the definition of $V(\cdot)$ in (3.17), it is clear that

$$V(x_{1d_1}, x_{2d_2}, \ldots, x_{nd_n}) \leq qV(x_1, x_2, \ldots, x_n), \qquad (q > 1)$$

implies that

$$q\sum_{i=1}^{n} \lambda_{\max}(P_i)\|x_i\|^2 - \sum_{i=1}^{n} \lambda_{\min}(P_i)\|x_{id_i}\|^2 \geq q\sum_{i=1}^{n} x_i^T P_i x_i - \sum_{i=1}^{n} x_{id_i}^T P_i x_{id_i} \geq 0 \qquad (3.25)$$

Therefore, from (3.25) and (3.24), it follows that when $V(x_{1d_1}, \ldots, x_{nd_n}) \leq qV(x_1, \ldots, x_n)$,

$$\dot{V} \leq -\sum_{i=1}^{n} x_i^T Q_i x_i + \sum_{i=1}^{n} \varepsilon_i \|x_{id_i}\|^2 + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i E_{ij} x_{jd_j} + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i F_{ij} x_j$$
$$+2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \Big(\alpha_{ij}\|P_i\|\,\|x_i\|\,\|x_j\| + \beta_{ij}\|P_i\|\,\|x_i\|\|x_{jd_j}\|\Big)$$
$$+q\sum_{i=1}^{n} \lambda_{\max}(P_i)\|x_i\|^2 - \sum_{i=1}^{n} \lambda_{\min}(P_i)\|x_{id_i}\|^2$$

$$\leq -\sum_{i=1}^{n} \Big(\lambda_{\min}(Q_i) - q\lambda_{\max}(P_i)\Big)\|x_i\|^2 - \sum_{i=1}^{n} \Big(\lambda_{\min}(P_i) - \varepsilon_i\Big)\|x_{id_i}\|^2$$
$$+2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \Big(\|P_i F_{ij}\| + \alpha_{ij}\|P_i\|\Big)\|x_i\|\,\|x_j\|$$
$$+2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \Big(\|P_i E_{ij}\| + \beta_{ij}\|P_i\|\Big)\|x_i\|\,\|x_{jd_j}\|$$
$$= -\frac{1}{2} Y(W^T + W)Y^T$$
$$\leq -\frac{1}{2}\lambda_{\min}(W^T + W)(\|x\|^2 + \|x_d\|^2)$$
$$\leq -\frac{1}{2}\lambda_{\min}(W^T + W)\|x\|^2$$

where $Y := \left[\, \|x_1\| \;\cdots\; \|x_n\| \; \|x_{1d_1}\| \;\cdots\; \|x_{nd_n}\| \,\right]$. From $W^T + W > 0$, it follows that $\lambda_{\min}(W^T + W) > 0$. Hence the conclusion follows from Lemma 3.2.

Remark 3. Consider (3.10) in Assumption 3.1. The bounds on the uncertain interconnections in system (3.5) are dependent on the systems states, and thus they cannot be employed in the control design since static output feedback is used in this chapter. The effects of such interconnections have been reflected through α_{ij} and β_{ij} in the matrix W.

3.5 Decentralised Control Synthesised for Square Case

Consider interconnected systems where all of the subsystems are square (each subsystem has the same number of outputs as the number of inputs). In this case, it is possible to design decentralised controllers such that the effect of the uncertain interconnections can be largely rejected if delay is available for design. This problem is far from trivial because the uncertain interconnections in each subsystem involve all subsystems' output information while the decentralised control is only allowed to use local output information.

3.5.1 Controller Design

The following assumption is imposed on the system (3.5)–(3.6).

Assumption 3.3. It is assumed that $m_i = p_i$ and the time delays d_i are known. The uncertainties $G_i(\cdot)$ satisfy (3.9) and the uncertainties $\Phi_{ij}(\cdot)$ satisfy

$$\|\Phi_{ij}(t, x_j, x_{jd_j})\| \le \xi_{ij}(t, y_j, y_{jd_j})\|y_j\| \tag{3.26}$$

where functions $\xi_{ij}(\cdot)$ are known nonnegative and continuous for $i \neq j$ and $i, j = 1, 2, \ldots, n$.

Since both B_i and C_i are of full rank, it follows that under Assumption 3.2 the matrix D_i is nonsingular in square case. Then, consider the following control law

$$u_i = -K_i y_i - \frac{1}{2\varepsilon_i} D_i y_i \varpi_i^2(t, y_i) + u_i^a(\cdot) + u_i^b(\cdot) \tag{3.27}$$

where K_i and $u_i^a(\cdot)$ are the same as in (3.14), and $u_i^b(\cdot)$ is defined by

$$u_i^b(\cdot) = -D_i^{-T} y_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\varepsilon_{ji}} (\lambda_{\max}(P_j))^2 \xi_{ji}^2(t, y_i, y_{id_i}) \tag{3.28}$$

where $\varepsilon_{ji} > 0$ $(j \neq i)$ are constants for $i, j = 1, 2, \ldots, n$. It is obvious that the delays are employed in the control design for u_i^b and thus it is required to be known.The result in section 3.5.2 will show that the controllers in (3.27) can largely reject the effects of the uncertain interconnections.

3.5.2 Main Result

Theorem 3.2. *Under Assumptions 3.2 and 3.3, the closed-loop system formed by applying control (3.27) with $u_i^b(\cdot)$ defined in (3.28) into system (3.5)–(3.6) is globally uniformly asymptotically stable if $\Gamma^T + \Gamma > 0$ where the matrix*

$$\Gamma := \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

is defined by

$$\Gamma_{11} := \begin{bmatrix} \Pi_1^a & -2P_1F_{12} & \cdots & -2P_1F_{1n} \\ -2P_2F_{21} & \Pi_2^a & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2P_{n-1}F_{(n-1)n} \\ -2P_nF_{n1} & \cdots & -2P_nF_{n(n-1)} & \Pi_n^a \end{bmatrix}$$

$$\Gamma_{12} := \begin{bmatrix} 0 & -2P_1E_{12} & \cdots & -2P_1E_{1n} \\ -2P_2E_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2P_{n-1}E_{(n-1)n} \\ -2P_nE_{n1} & \cdots & -2P_nE_{n(n-1)} & 0 \end{bmatrix}$$

$$\Gamma_{21} := \begin{bmatrix} 0 & -2P_2E_{21} & \cdots & -2P_nE_{n1} \\ -2P_1E_{12} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2P_{n-1}E_{(n-1)n} \\ -2P_nE_{n1} & \cdots & -2P_nE_{n(n-1)} & 0 \end{bmatrix}$$

and

$$\Gamma_{22} := diag\left\{\Pi_1^b, \Pi_2^b, \cdots, \Pi_n^b\right\}$$

where $\Pi_i^a := Q_i - (q\lambda_{\max}(P_i) + \sum_{\substack{j=1 \\ j \neq i}}^n \varepsilon_{ij})I_{n_i}$ and $\Pi_i^b := (\lambda_{\min}(P_i) - \varepsilon_i)I_{n_i}$ for $i = 1, \ldots, n$ and $q > 1$.

Proof. Consider the uncertain interconnection terms $\sum_{\substack{j=1\\j\neq i}}^{n} \Phi_{ij}(t,x_j,x_{jd_j})$. From the condition (3.26) and Young's inequality ($ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ for $\varepsilon > 0$),

$$\begin{aligned} &2x_i^T P_i \Phi_{ij}(t,x_j,x_{jd_j}) \\ &\le 2\lambda_{\max}(P_i)\|x_i\|\xi_{ij}(t,y_j,y_{jd_j})\|y_j\| \\ &\le \varepsilon_{ij}\|x_i\|^2 + \frac{1}{\varepsilon_{ij}}(\lambda_{\max}(P_i))^2\,\xi_{ij}^2(t,y_j,y_{jd_j})\|y_j\|^2 \end{aligned} \tag{3.29}$$

for constant scalars $\varepsilon_{ij} > 0$. Then, from inequalities (3.29)

$$\begin{aligned} &2\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} x_i^T P_i \Phi_{ij}(t,x_j,x_{jd_j}) \\ &\le \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} \varepsilon_{ij}\|x_i\|^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\varepsilon_{ij}}(\lambda_{\max}(P_i))^2\,\xi_{ij}^2(t,y_j,y_{jd_j})\|y_j\|^2 \\ &= \sum_{i=1}^{n}\Bigg(\sum_{\substack{j=1\\j\neq i}}^{n}\varepsilon_{ij}\Bigg)\|x_i\|^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\varepsilon_{ji}}(\lambda_{\max}(P_j))^2\xi_{ji}^2(t,y_i,y_{id_i})\|y_i\|^2 \end{aligned} \tag{3.30}$$

where Lemma 3.3 is used to obtain the last equality. From the definition of $u_i^b(\cdot)$ in (3.28),

$$\begin{aligned} &x_i^T P_i B_i u_i^b(\cdot) + \sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\varepsilon_{ji}}(\lambda_{\max}(P_j))^2\xi_{ji}^2(t,y_i,y_{id_i})\|y_i\|^2 \\ &\le -x_i^T C_i^T D_i^T D_i^{-T} y_i \sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\varepsilon_{ji}}(\lambda_{\max}(P_j))^2\xi_{ji}^2(t,y_i,y_{id_i}) \\ &\quad + \sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\varepsilon_{ji}}(\lambda_{\max}(P_j))^2\xi_{ji}^2(t,y_i,y_{id_i})\|y_i\|^2 \\ &= 0 \end{aligned} \tag{3.31}$$

Therefore, from (3.30) and (3.31)

$$2\sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t,y_i) + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} x_i^T P_i \Phi_{ij}(t,x_j,x_{jd_j}) \le \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\varepsilon_{ij}\|x_i\|^2 \tag{3.32}$$

Consider the same Lyapunov function as given in (3.17). Following the analysis and proof in Theorem 1, it is straightfoward to see that when $V(x_{1d_1},\ldots,x_{nd_n}) \le qV(x_1,\ldots,x_n)$,

$$
\begin{aligned}
\dot{V} \le & -\sum_{i=1}^{n} x_i^T Q_i x_i + \sum_{i=1}^{n} \varepsilon_i \|x_{id_i}\|^2 + 2\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i E_{ij} x_{jd_j} + 2\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i F_{ij} x_j \\
& + \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_{ij} \|x_i\|^2 + q \sum_{i=1}^{n} \lambda_{\max}(P_i) \|x_i\|^2 - \sum_{i=1}^{n} \lambda_{\min}(P_i) \|x_{id_i}\|^2 \\
\le & -\sum_{i=1}^{n} x_i^T \Big(Q_i - \Big(q\lambda_{\max}(P_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_{ij} \Big) I_{n_i} \Big) x_i - \sum_{i=1}^{n} \Big(\lambda_{\min}(P_i) - \varepsilon_i \Big) x_{id_i}^T x_{id_i} \\
& + 2\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i E_{ij} x_{jd_j} + 2\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} x_i^T P_i F_{ij} x_j \\
= & -\frac{1}{2} Z^T (\Gamma^T + \Gamma) Z \le -\frac{1}{2} \lambda_{\min}(\Gamma^T + \Gamma) \|Z\|^2 \le -\frac{1}{2} \lambda_{\min}(\Gamma^T + \Gamma) \|x\|^2
\end{aligned}
$$

where $Z := col(x_1, \cdots, x_n, x_{1d_1}, \ldots, x_{nd_n})$.

Hence, the conclusion follows from $\Gamma + \Gamma^T > 0$.

Remark 4. From the proof of Theorem 3.2, it is clear to see that the only terms resulting from the uncertain interconnections, are $\sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_{ij}$ in the matrix Γ. Compared the matrix W in Theorem 3.1 and the matrix Γ in Theorem 3.2, it is straightforward to see that the effects of the uncertain interconnections have been largely rejected by the control (3.27) because the terms $\sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_{ij}$ appeared in the matrix Γ, can be very small if the parameters ε_{ij} are chosen to be small enough although small ε_{ij} usually result in high gain control.

3.6 Case Study—River Pollution Control Problem

Consider a two-reach model of a river pollution control problem [[14]]. It is assumed that the concentration of biochemical oxygen demand (BOD) for the first subsystem is perturbed by a time delay. Then, the system can be described by (See, [[30]])

$$
\dot{x}_1 = \underbrace{\begin{bmatrix} -1.32\delta & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}_{B_1} \Big(u_1 + \underbrace{(-13.2(1-\delta)) y_{1d_1}}_{G_1(\cdot)} \Big) + \Phi_{12}(\cdot) \tag{3.33}
$$

$$
\begin{aligned}
\dot{x}_2 = & \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}_{B_2} (u_2 + G_2(\cdot)) + \underbrace{\begin{bmatrix} 0.9\delta & 0 \\ 0 & 0 \end{bmatrix}}_{E_{21}} x_{1d_1} \\
& + \underbrace{\begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}}_{F_{21}} x_1 + \underbrace{\begin{bmatrix} -0.9\delta y_1 \\ 0 \end{bmatrix}}_{\Phi_{21}(\cdot)}
\end{aligned} \tag{3.34}
$$

$$y_1 = \underbrace{[1 \quad 0]}_{C_1} x_1, \qquad y_2 = \underbrace{[1 \quad 0]}_{C_2} x_2 \tag{3.35}$$

where $x_1 := col(x_{11}, x_{12})$ and $x_2 := col(x_{21}, x_{22})$. The variables x_{i1} and x_{i2} represent the concentration of the BOD and the concentration of dissolved oxygen respectively, and the control u_i are the BOD of the effluent discharge into the river for $i = 1, 2$. The constant $\delta \in [0, 1]$ is the retarded coefficient. The uncertainties $G_2(\cdot)$ and $\Phi_{12}(\cdot)$ are added to illustrate the results obtained.

By direct computation, it is obtained that the matrix $W + W^T$ defined in Theorem 3.1 is not positive definite even if it is assumed that both $G_2(\cdot)$ and $\Phi_{12}(\cdot)$ are zero. Therefore, for the system (3.33)–(3.35), the result in Theorem 3.1 does not hold.

Next a decentralised controller based on the result given in section 3.5 will be proposed. Rewrite the system (3.33)–(3.35) as follows

$$\dot{x}_1 = \underbrace{\begin{bmatrix} -1.32\delta & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}_{B_1} \Big(u_1 + \underbrace{(-13.2(1-\delta)) y_{1d_1}}_{G_1(\cdot)} \Big) + \Phi_{12}(\cdot) \tag{3.36}$$

$$\begin{aligned} \dot{x}_2 = & \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}_{B_2} (u_2 + G_2(\cdot)) + \underbrace{\begin{bmatrix} 0.9\delta & 0 \\ 0 & 0 \end{bmatrix}}_{E_{21}} x_{1d_1} \\ & + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0.9 \end{bmatrix}}_{F_{21}} x_1 + \underbrace{\begin{bmatrix} (1-\delta)0.9 y_1 \\ 0 \end{bmatrix}}_{\Phi_{21}(\cdot)} \end{aligned} \tag{3.37}$$

$$y_1 = \underbrace{[1 \quad 0]}_{C_1} x_1, \qquad y_2 = \underbrace{[1 \quad 0]}_{C_2} x_2 \tag{3.38}$$

It is assumed that

$$|G_2(\cdot)| \le \underbrace{1 + \sin y_2}_{\rho_2} + \underbrace{|y_2|}_{\varpi_2} \|x_{2d_2}\|, \qquad \|\Phi_{12}\| \le \underbrace{|y_2 y_{2d_2}| \sin^2 t}_{\xi_{12}} |y_2| \tag{3.39}$$

Let $\rho_1 = 0$, $\varpi_1(\cdot) = 13.2(1-\delta)$ and $\xi_{21} = 0.9(1-\delta)$. It is clear to see that the Assumption 3.3 hold. Then let $K_1 = 20$, $K_2 = 30$ and

$$Q_1 = \begin{bmatrix} 4.5280 & 0.3200 \\ 0.3200 & 2.4000 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 8.6400 & 0.3200 \\ 0.3200 & 2.4000 \end{bmatrix}$$

The solutions to the Lyapunov equations in (3.11) are $P_1 = P_2 = I_2$ and the equations (3.12) are satisfied with $D_1 = D_2 = 0.1$. Comparing system (3.36)–(3.37) with the system (3.5)–(3.6), it is straightforward to see that $E_{12} = F_{12} = 0$. Let $\varepsilon_1 = \varepsilon_2 = 0.5$ and $\varepsilon_{12} = \varepsilon_{21} = 0.1$. By direct computation,

$$\Gamma_{11} = \begin{bmatrix} 3.4180 & 0.3200 & 0 & 0 \\ 0.3200 & 1.2900 & 0 & 0 \\ 0 & 0 & 7.5300 & 0.3200 \\ 0 & -1.8000 & 0.3200 & 1.2900 \end{bmatrix}$$

$$\Gamma_{22} = \begin{bmatrix} 0.5000 & 0 & 0 & 0 \\ 0 & 0.5000 & 0 & 0 \\ 0 & 0 & 0.5000 & 0 \\ 0 & 0 & 0 & 0.5000 \end{bmatrix}$$

$$\Gamma_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.3600 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma_{21} = \begin{bmatrix} 0 & 0 & -0.3600 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the matrix $\Gamma^T + \Gamma$ where Γ is defined in Theorem 3.2, is positive definite. Clearly the controllers (3.27)–(3.28) are well defined, and, from Theorem 3.2, they stabilise the system (3.33)-(3.35) globally asymptotically.

For simulation purposes, choose $\sigma = 0.20$ and assume the delays are chosen as $d_1(t) = 3 - 2\sin(t)$ and $d_2(t) = 2 - \cos t$, and the delay related initial conditions are chosen as $\phi_1(t) = col(2\cos t, 1)$ and $\phi_2(t) = col(0, 1 - \sin(t))$. The simulation results shown in Figure 3.1 are as expected.

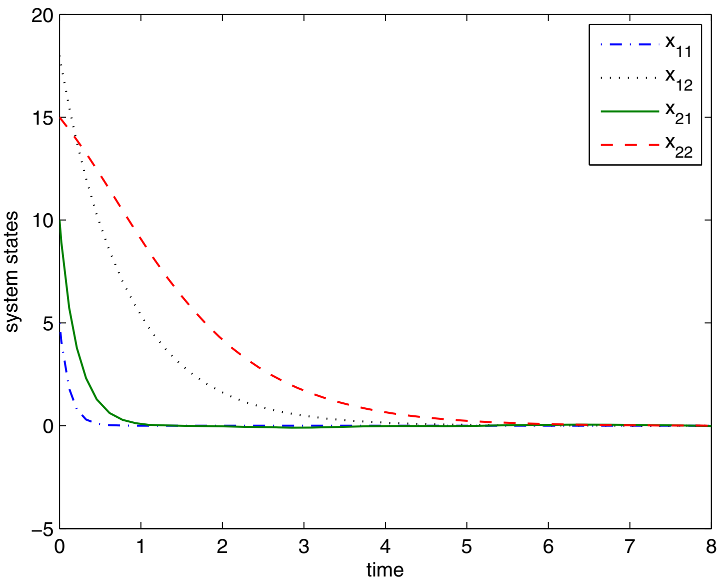

Fig. 3.1 The time responses of the state variables of system (3.33)–(3.34)

3.7 Conclusions

This Chapter has presented two control strategies based on different classes of uncertain interconnections. The proposed control schemes are decentralised and based only on output information, which is convenient for real implementation. The differences between the inaccessible bounds which cannot be used in the control design, and the accessible bounds on the uncertain interconnections which can be employed in the control design, are shown. The proposed approach can be used to accommodate mismatched uncertain interconnections if the delay is known and the bounds on the uncertain interconnections are a class of functions of the outputs and delayed outputs. The limitation on the rate of change of the time varying delay is not required, as is required using the Lyapunov-Krasovskii approach. The case study shows the practicability of the proposed approach.

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