Chapter 2
Adaptive Sliding Mode Control

Vadim I. Utkin and Alexander S. Poznyak

Abstract. The main obstacles for application of Sliding Mode Control are two interconnected phenomena: chattering and high activity of control action. It is well known that the amplitude of chattering is proportional to the magnitude of a discontinuous control. These two problems can be handled simultaneously if the magnitude is reduced to a minimal admissible level defined by the conditions for the sliding mode to exist. Here an adaptation methodology is discussed for obtaining the minimum possible value of control based on two approaches developed in recent publications:

- *The* σ *- adaptation*, providing adequate adjustments of the magnitude of a discontinuous control within the "reaching phase", that is, when the state trajectories are out of a sliding surface;
- *Dynamic adaptation* or the adaptation within a sliding mode (on a sliding surface), based on the, so-called, *equivalent control* obtained by the direct measurements of the output signals of a first-order low-pass filter containing in the input the discontinuous control with the specially adapted magnitude value.

The application of these methodologies is illustrated by the corresponding adaptive versions of the *Super Twist* controller. It is shown that the adaptation based on the equivalent control application enables reducing of the control action magnitude to the minimum possible value keeping the property of finite-time convergence.

Vadim I. Utkin
Department of Electrical Engineering, The Ohio State University, USA
e-mail: utkin@ece.osu.edu

Alexander S. Poznyak
Dept. Control Automatico, CINVESTAV-IPN, Mexico
e-mail: apoznyak@ctrl.cinvestav.mx

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2.1 Introduction

2.1.1 Brief Survey

The *Sliding Mode Control* is a very popular strategy for control of nonlinear uncertain systems, with a very large frame of applications fields ⟦26⟧, ⟦30⟧. Due to the use of the discontinuous function, its main features are

- the robustness of closed-loop system
- and the finite-time convergence.

However, its design requires the knowledge of the bound on the uncertainties, which could be, from a practical point of view, a hard task: it often follows that this bound is overestimated, which yields excessive gain. Then, the main drawback of the sliding mode control, the well-known *chattering phenomenon* (for its analysis, see ⟦4⟧, ⟦5⟧), is important and could damage actuators and systems. A first way to reduce the chattering is the use of a boundary layer: in this case, many approaches have proposed adequate controller gains tuning ⟦26⟧. A second way to decrease the effect of the chattering phenomenon is the use of higher order sliding mode controller ⟦16⟧, ⟦17⟧, ⟦19⟧, ⟦7⟧, ⟦14⟧, ⟦22⟧. However, in both these control approaches, knowledge of the bound on the uncertainties is required. As the objective is the non-requirement of the uncertainties bound, another way consists in using adaptive sliding mode, the goal being to ensure a dynamic adaptation of the control gain to be as small as possible whereas sufficient to counteract the uncertainties and perturbations.

The basic idea of the *Adaptive Control Approach* consists in designing the systems exhibiting the same dynamic properties under uncertainty conditions based on utilization of current information. It involves modifying the control law used by a controller to cope with the fact that the parameters of the system being controlled are slowly time-varying or uncertain. Even more, adaptive control implies improving dynamic characteristics while properties of a controlled plant or environment are varying ⟦1⟧, ⟦25⟧. Without adaptation the original SMC demonstrates *robustness* properties with respect to parameter variations and disturbances ⟦29⟧. The first attempts to apply the ideas of adaptation in Sliding Mode Control (SMC) were made in the 60's ⟦8⟧, ⟦9⟧ and ⟦10⟧: the control efficiency was improved by changing the position or equation of the discontinuity surfaces without any information on a plant parameters. The design idea might be formulated as follows: if sliding mode exists, then the coefficients of switching plane can be varied to improve the system dynamics. However those early publications did not take into account the main obstacle for SMC application - the *chattering* phenomenon which is inherent in sliding motions (see, for example, ⟦7⟧, ⟦4⟧ and ⟦5⟧). This phenomenon is well-known from literature on power converters and referred to as "*ripple*" ⟦21⟧. Then the efforts of the researchers were oriented to the application of *adaptability principles* to reduce the effect of chattering. Since the amplitude of chattering is proportional to discontinuity magnitude in control, one of possible adaptation methods is related to reducing this magnitude to the minimum admissible value dictated by the conditions for SM to exist. So, in ⟦23⟧ the control gain depended on the distance of system state to a discontinuity surface. The tracks of adaptability can be found in the first publications about variable structure systems with SM (see ⟦29⟧, ⟦30⟧) with the control gain proportional the system state. As recalled previously, this problem is an exciting challenge for applications given that, in many cases, gains are also overestimated, which gives larger control magnitude and larger chattering. In order to adapt the gain, many controllers based on fuzzy tools (see ⟦20⟧, ⟦28⟧) have been published; however, these papers do not guarantee the tracking performances. In ⟦13⟧ gain dynamics directly depends on the tracking error (sliding variable): the control gain is increasing since sliding mode is not established. Once this is the case, gain dynamics equals zero. The main drawback of this approach is the gain over-estimation with respect to uncertainties bound. Furthermore, this approach is not directly applicable, but requires modifications for its application to real systems: thus, the sign function is replaced by a saturation function where the boundary layer width affects accuracy and robustness. Furthermore, no boundary layer width tuning methodology is provided. A method proposed in ⟦15⟧ in order to limit the switching gain must be mentioned. The idea is based on use of equivalent control: once sliding mode occurs, disturbance magnitude is evaluable and allows an adequate tuning of control gain. However, this approach requires the knowledge of uncertainties/perturbations bounds and the use of low-pass filter, which introduces signal magnitude attenuation, delay and transient behavior when disturbances are acting. In ⟦12⟧ a gain-adaptation algorithm is proposed by using sliding mode disturbance observer. The main drawback is that the knowledge of uncertainties bounds is required to design observer-based controller. There exist also adaptive SMC (ASMC) algorithms that allow adjusting dynamically the control gains without knowledge of uncertainties/perturbations bounds. In particular, several adaptive fuzzy SMC algorithms were proposed. However, they do not guarantee the tracking performance or overestimate the switching control gains as in ⟦13⟧. Of course, another efficient tool to suppress chattering is the application of state observers ⟦6⟧, but for this method the plant parameters are assumed to be known.

2.1.2 Objective and the Design Idea

The objective of the current article is to discuss two new methodologies for control of a class of uncertain nonlinear systems:

1) the first one is referred to as *Sigma Adaptation* of Sliding Mode Controllers and deals with models for which uncertainties are bounded, but this bound (which is finite) is not known; this adaptation process with the varying magnitude of the control gain terminates in the moment when the sliding mode starts (see ⟦23⟧ and ⟦27⟧).
2) the second methodology is associated with *Dynamic Adaptation* of Sliding Mode Controllers (under known uncertainty bounds) where the adaptation

process is continued during sliding mode, using the current estimates of the corresponding *equivalent control* (see [[31]]), that leads to minimization of chattering effect.

2.1.3 Main Contribution

The first contribution of this chapter is in developing the adaptation methodology for finding the control gain $k(t)$ providing a minimum value of discontinuity resulting in minimization of the chattering effect.

The second contribution consists in developing the modified version of the suggested methodology for the super-twisting algorithm with a SOSM (Second Order Sliding Mode).

2.2 The σ-Adaptation Method

2.2.1 System Description, Main Assumptions and Restrictions

Consider the nonlinear uncertain system

$$\left.\begin{array}{c} \dot{x}(t) = f(x(t)) + g(x(t))\,u \\ x(0) = x_0,\ t \geq 0 \end{array}\right\} \tag{2.1}$$

where $x(t) \in \mathscr{X} \subset \mathbb{R}^n$ the state vector and $u \in \mathbb{R}$ the control input to be designed. Function $f(x)$ and $g(x)$ are supposed to be smooth uncertain functions and are bounded for all $x \in \mathscr{X}$; furthermore, $f(x)$ contains unmeasured perturbations term and $g(x) \neq 0$ for all $x \in \mathscr{X}$.

The control objective consists in forcing the continuous function $\sigma(x,t)$, named sliding variable, to 0. Supposing that σ admits the relative degree equal to 1 with respect to u, one gets

$$\begin{array}{c} \dot{\sigma}(x,t) = \left(\dfrac{\partial\sigma(x,t)}{\partial x}\right)^{\mathsf{T}} \dot{x} + \dfrac{\partial\sigma(x,t)}{\partial t} = \dfrac{\partial\sigma(x,t)}{\partial t} + \\ \left(\dfrac{\partial\sigma(x,t)}{\partial x}\right)^{\mathsf{T}} f(x) + \left(\dfrac{\partial\sigma(x,t)}{\partial x}\right)^{\mathsf{T}} g(x)\,u = \Psi(x,t) + \Gamma(x,t)\,u \end{array} \tag{2.2}$$

where

$$\Psi(x,t) := \frac{\partial\sigma(x,t)}{\partial t} + \left(\frac{\partial\sigma(x,t)}{\partial x}\right)^{\mathsf{T}} f(x)$$

$$\Gamma(x,t) := \left(\frac{\partial\sigma(x,t)}{\partial x}\right)^{\mathsf{T}} g(x)$$

Functions $\Psi(x,t)$ and $\Gamma(x,t)$ are supposed to be bounded such that for all $x \in \mathscr{X}$ and all $t \geq 0$

$$|\Psi(x,t)| \leq \Psi_M,\ 0 < \Gamma_m \leq \Gamma(x,t) \leq \Gamma_M \tag{2.3}$$

It is assumed that Ψ_M, Γ_m and Γ_M exist but are *not known*. The objective for a designer is to propose a sliding mode controller $u(\sigma,t)$ with the same features as classical SMC, namely, robustness and finite-time convergence but without any information on uncertainties and perturbations (appearing in $f(x)$); this latter is only known to be bounded. Furthermore, this objective allows to ensure a global stability of closed-loop system whereas the classical way (with knowledge of uncertainties bounds) only ensures its semi-global stability.

2.2.2 Real and Ideal Sliding Modes

In the sequel, the definitions of *ideal* and *real* sliding mode are recalled.

Definition 2.1. [[2]] We say that an **ideal sliding mode** exists

$$\lim_{\sigma \to +0} \dot{\sigma} < 0 \text{ and } \lim_{\sigma \to -0} \dot{\sigma} > 0$$

In real applications, an 'ideal' sliding mode, as defined in Definition 2.1, cannot be established. Then, it is necessary to introduce the concept of 'real' sliding mode.

Definition 2.2. [[2]], [[16]] If, due to some small positive parameter μ, the state trajectories belong to domain

$$|\sigma(t)| \leq \Delta(\mu),\ \lim_{\mu \to 0} \Delta(\mu) = 0$$

then the motion is called a **real sliding mode**.

As it is common for Sliding Mode Theory [[29]], we will consider the scalar discontinuous control action $u = u(\sigma,t)$ at time t as

$$u(\sigma,t) = -K(t)\,\mathrm{sign}(\sigma)$$

$$\mathrm{sign}(\sigma) := \begin{cases} 1 & if\ \sigma > 0 \\ -1 & if\ \sigma < 0 \\ \in [-1;1] & if\ \sigma = 0 \end{cases} \tag{2.4}$$

Below we consider two different sliding mode control laws:

- the *first* one combines the adaptive schemes formulated in [[13]] and [[15]], while mitigating the deficiencies of the combined gain-adaptation schemes;
- the *second* the adaptive scheme is suggested in [[23]]; it does not require the estimation of the perturbations via equivalent control as in [[15]] and does not overestimate the control gain as in [[13]]. Furthermore, the second adaptive-gain

sliding mode control algorithm requires smaller amount of tuning parameters than the first algorithm, and is developed in the real sliding mode context.

2.2.3 First Adaptive Sliding Mode Control Law

Consider the controller (2.4) with the gain $K(t)$ defined as follows:

- If $|\sigma(x(t),t)| > \varepsilon > 0$, then $K(t)$ is the solution of the following differential equation
$$\dot{K}(t) = \bar{K}_1 |\sigma(x(t),t)| \tag{2.5}$$
with $\bar{K}_1 > 0$ and $K(0) > 0$;
- If $|\sigma(x(t),t)| \leq \varepsilon$, then $K(t)$ reads as
$$\begin{gathered} K(t) = \bar{K}_2 |\eta(t)| + \bar{K}_3 \\ \tau\dot{\eta}(t) + \eta(t) = \operatorname{sign}(\sigma(x(t),t)) \end{gathered} \tag{2.6}$$
with $\bar{K}_2 = K(t^*)$, $\bar{K}_3 > 0$ and $\tau > 0$, where t^* is the largest value of t such that
$$|\sigma(x(t^*-0),t^*-0)| > \varepsilon, \ |\sigma(x(t^*),t^*)| \leq \varepsilon$$

Obviously, this controller is based on the real sliding mode concept. By supposing that
$$|\sigma(x(0),0)| > \varepsilon$$
the adaptive sliding mode control law (2.5) - (2.6) works as follows:

- The gain $K(t)$ is increasing due to the adaptation law (2.5) up to a value large enough to counteract the bounded uncertainty with unknown bounds in (2.1) until the real sliding mode starts. Denote the time instant when the real sliding mode starts for the first time as t_1.
- As sliding mode has started, i.e.,
$$|\sigma(x(t),t)| \leq \varepsilon$$
from $t = t_1$, $K(t)$ follows the gain-adaptation law (2.6). Then, gain $K(t)$ is adapted through (2.6) with $\bar{K}_2 = K(t_1)$. Note that this strategy will allow to decrease the gain and then to adjust it with respect to the current uncertainties and perturbations.
- However, if the varying uncertainty and perturbation exceeds the value $\bar{K}_2 = K(t_1)$, then the real sliding mode will be destroyed and we get that
$$|\sigma(x(t),t)| > \varepsilon$$
- Next, the gain adaptation will happen in accordance with (2.5). The gain $K(t)$ will be increasing until the sliding mode occurs again at the reaching time instant t_2.

- As sliding mode has occurred and $|\sigma(x(t),t)| \leq \varepsilon$ starting from $t = t_2$, $K(t)$ now follows the gain-adaptation law (2.6) with $\bar{K}_2 = K(t_2)$. And so on.

The following theorem confirms the workability of the first adaptive sliding mode control law (2.5) - (2.6).

Theorem 2.1 ([[23]]). *Given the nonlinear uncertain system (2.1) with the sliding variable $\sigma(x(t),t)$, satisfying (2.2), controlled by (2.4), (2.5) and (2.6), there exists a finite time t_F so that a real sliding mode is established for all $t \geq t_F$, i.e.,$|\sigma(x(t),t)| \leq \varepsilon$ for all $t \geq t_F$.*

2.2.4 Second Adaptive Sliding Mode Control Law

The first adaptive sliding mode control law (2.4), (2.5) and (2.6) uses the concept of "equivalent control" [[30]]. It can be obtained by a low-pass filter, although it is not easy to tune its parameter. The controller displayed in this subsection does not estimate the boundary of perturbation and uncertainties. But, there is an eminent price to do that: the new strategy guarantees a real sliding mode only.

Consider the following controller (2.4)

$$u(\sigma,t) = -K(t)\,\mathrm{sign}\,(\sigma(x(t),t))$$

where the gain coefficient $K(t)$ satisfies

$$\dot{K} = \begin{cases} \bar{K}\,|\sigma(x(t),t)|\,\mathrm{sign}\,(|\sigma(x(t),t)| - \varepsilon) \text{ if } K > \mu \\ 0 \text{ if } K \leq \mu \end{cases} \tag{2.7}$$

with $\bar{K} > 0$, $\varepsilon > 0$ and a small enough positive μ. The parameter μ is introduced in order to get only positive values for K.

Once sliding mode with respect to $\sigma(x(t),t)$ is established, the proposed gain-adaptation law (2.7) allows the gain K declining (while $|\sigma(x(t),t)| < \varepsilon$). In other words, the gain K will be kept at the smallest level that allows a given accuracy of σ - stabilization. Of course, as described in the sequel, this adaptation law allows to get an adequate gain with respect to uncertainties/perturbations magnitude.

First, let us prove the following auxiliary result.

Lemma 2.1. *Given the nonlinear uncertain system (2.1) with the sliding variable $\sigma(x(t),t)$ dynamics (2.2) controlled by (2.4) and (2.7), the gain $K(t)$ has an upper-bound, i.e. there exists a positive constant K^* so that $K(t) \leq K^*$ for all $t \geq 0$.*

Proof. Suppose that $|\sigma(x(0),0)| \geq \varepsilon$. From K - dynamics, and given that functions Ψ and Γ are supposed bounded, it follows that K is increasing and there exists a time t_1 (see 2.1) such that

$$K(t_1) = \left|\frac{\Psi(t_1)}{\Gamma(t_1)}\right|$$

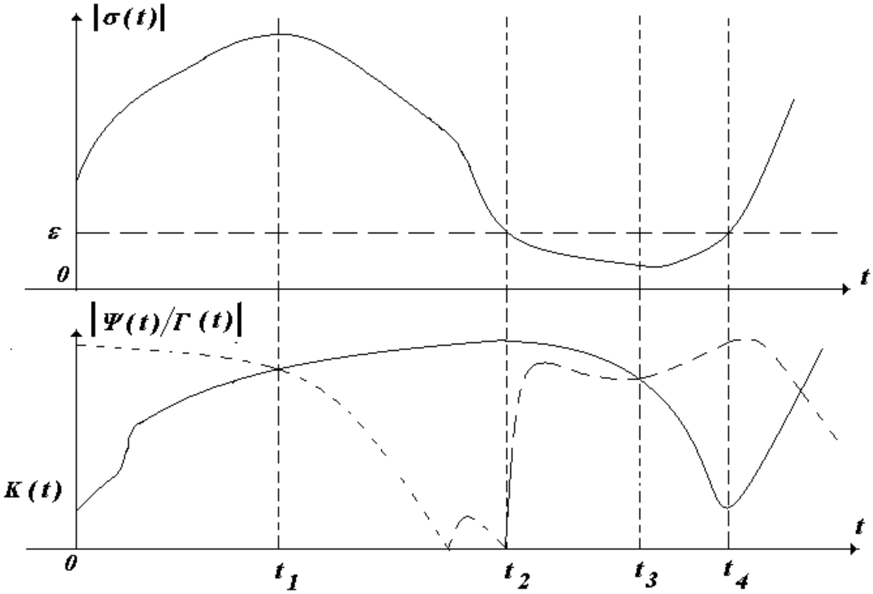

Fig. 2.1 Scheme describing the behaviour of σ (top) and K (bottom) versus time

From $t = t_1$, given K - dynamics, the gain K is large enough to make the sliding variable decreasing. Then, it yields that, in a finite time t_2 (2.1), $|\sigma(x(t),t)| < \varepsilon$. It yields that gain K is decreasing from t_2, the gain K being at a maximum value at $t = t_2$. From K - dynamics, it yields that there exists a time instant $t_3 > t_2$ (2.1) such that

$$K(t_3) = \left|\frac{\Psi(t_3)}{\Gamma(t_3)}\right|$$

From $t = t_3$, the gain K is not large enough to counteract perturbations and uncertainties as it is decreasing. It yields that there exists a time instant $t_4 > t_3$ such that

$$|\sigma(x(t_4),t_4)| \geq \varepsilon$$

The process then restarts from the beginning. By the adaptation law (2.7), the gains $K(t_i)$ remain bounded uniformly on t_i. In fact,

$$K(t_i) = \left|\frac{\Psi(t_i)}{\Gamma(t_i)}\right| \leq \frac{\Psi^{**}}{\Gamma^{**}} := K^{**}$$

and, hence, there always exists a finite constant K^* such that $K^* < K^{**}$, which proves the desired result. ■

Theorem 2.2 ([[23]]). *Given the nonlinear uncertain system (2.1) with the sliding variable $\sigma(x(t),t)$ dynamics (2.2) controlled by (2.4) and (2.7), there exists a finite time t_F so that a real sliding mode is established for all $t \geq t_F$, i.e.,*

$$|\sigma(x(t),t)| < \delta$$

for all $t \geq t_F$ with

$$\delta = \sqrt{\varepsilon^2 + \frac{\Psi_M^2}{\bar{K}\Gamma_m}} \tag{2.8}$$

Proof. The proof is based on the Lyapunov approach and shows that, when $|\sigma(x(t),t)| > \varepsilon$, then control strategy ensures that $|\sigma(x(t),t)| \leq \varepsilon$ in a finite time. Furthermore, it is proven that as soon as σ reaches the domain $|\sigma(x(t),t)| \leq \varepsilon$, it stays in the domain $|\sigma(x(t),t)| \leq \delta$ defined by (2.8) for all consecutive time. Therefore, the proof shows that the real sliding mode is established in finite time in the domain $|\sigma(x(t),t)| \leq \delta$. Consider the following Lyapunov candidate function

$$V = \frac{1}{2}\sigma^2 + \frac{1}{2\gamma}(K - K^*)^2, \ \gamma > 0 \tag{2.9}$$

So one has

$$\dot{V} = \sigma\Psi - \sigma\Gamma K \text{sign}(\sigma) + \frac{1}{\gamma}(K - K^*)|\sigma| \text{sign}(|\sigma| - \varepsilon) \leq$$

$$|\sigma|\Psi_M - |\sigma|\Gamma_m K + \frac{1}{\gamma}(K - K^*)\bar{K}|\sigma| \text{sign}(|\sigma| - \varepsilon) =$$

$$|\sigma|(\Psi_M - \Gamma_m K^*) + (K - K^*)\left(-\Gamma_m|\sigma| + \frac{\bar{K}}{\gamma}|\sigma| \text{sign}(|\sigma| - \varepsilon)\right)$$

By Lemma 2.1 there always exists $K^* > 0$ such that $K(t) - K^* \leq 0$ and hence, for some $\beta_K > 0$ it follows

$$\dot{V} \leq -\beta_\sigma|\sigma| - \beta_K|K - K^*| - \xi$$

$$\xi := |K - K^*|\left(-\Gamma_m|\sigma| + \frac{\bar{K}}{\gamma}|\sigma| \text{sign}(|\sigma| - \varepsilon) - \beta_K\right)$$

$$\beta_\sigma := (\Gamma_m K^* - \Psi_M) > 0$$

which results to

$$\dot{V} \leq -\beta\sqrt{V} - \xi \tag{2.10}$$

for

$$\beta := \sqrt{2}\min\{\beta_\sigma; \beta_K\sqrt{\gamma}\}$$

Case 1: $|\sigma| > \varepsilon$. In this case it is possible to obtain $\xi > 0$ selecting γ

$$\frac{\bar{K}}{\Gamma_m + \beta_K \varepsilon^{-1}} > \gamma$$

which provides

$$-\Gamma_m |\sigma| + \frac{\bar{K}}{\gamma} |\sigma| - \beta_K > 0$$

for all σ satisfying $|\sigma| > \varepsilon$. From (2.10) we get

$$\dot{V} \leq -\beta\sqrt{V} - \xi \leq -\beta\sqrt{V}$$

Therefore, the finite time convergence to a domain $|\sigma| \leq \varepsilon$ is guaranteed from any initial condition corresponding $|\sigma(x(0), 0)| > \varepsilon$.

Case 2: $|\sigma| < \varepsilon$. Function ξ in (2.10) can be negative. It means that $\dot{V}$ would be sign indefinite, and it is not possible to conclude on the closed-loop system stability. Therefore, $|\sigma|$ can increase over ε. As soon as $|\sigma|$ becomes greater than ε , we return to the previous case and V starts decreasing. Apparently, the decrease of V can be achieved via increase of K allowing $|\sigma|$ to increase before it starts decreasing down to $|\sigma| \leq \varepsilon$. Without loss of generality, let us estimate the overshoot when $\sigma(x(0), 0) = \varepsilon^+$ and $K(0) > 0$: considering the 'worst' case (with respect to uncertainties/perturbations), one has

$$\dot{\sigma} = \Psi_M - K\Gamma_m$$

$$\dot{K} = \bar{K}|\sigma| = \bar{K}\sigma$$

leading tot

$$\sigma(x(t), t) = \varepsilon^+ \cos\left(\sqrt{\bar{K}\Gamma_m t}\right) + \frac{\Psi_M - \bar{K}\Gamma_m}{\sqrt{\bar{K}\Gamma_m}} \sin\left(\sqrt{\bar{K}\Gamma_m t}\right)$$

$$K(t) = \varepsilon^+ \sqrt{\frac{\bar{K}}{\Gamma_m}} \sin\left(\sqrt{\bar{K}\Gamma_m t}\right) + \left(K(0) - \frac{\Psi_M}{\Gamma_m}\right) \cos\left(\sqrt{\bar{K}\Gamma_m t}\right) + \frac{\Psi_M}{\Gamma_m}$$

Taking $\varepsilon^+ \to \varepsilon$, for $\delta = \max_t \sigma(x(t), t)$ (maximization is taken over all t within the considered interval), we obtain (2.8). Theorem is proven. ∎

So, the convergence to the domain $|\sigma| \leq \varepsilon$ is in a finite time, but could be sustained in the bigger domain $|\sigma| \leq \delta$. Therefore, the real sliding mode exists in the domain $|\sigma| \leq \delta$.

2.2.5 On the ε - Parameter Tuning

The choice of parameter ε has to be made by an adequate way because a 'bad' tuning could provide either instability and control gain increasing to infinity, or bad accuracy for closed-loop system as described in the sequel.

- If the parameter ε is too small, and due to large gain K and sampling period T_e, system trajectories are such that $|\sigma|$ never stays lower than ε. From K-dynamics ((2.7)) it yields that the gain K is increasing, which induces larger oscillation, and so on.
- If the parameter ε is too large, system trajectories are such that, in spite of large gain K and sampling period T_e, $|\sigma|$ is evolving around ε, it follows that controller accuracy is not as good as possible.

In [[23]] there is suggested to select ε adjusted in time as

$$\varepsilon(t) = 4K(t)T_e$$

2.3 The Dynamic Adaptation Based on the Equivalent Control Method

In the previous section, following to [[23]] and [[27]], the adaptation process with the varying magnitude of the control gain terminates in the moment when the sliding mode starts. In [[15]] the authors tried to continue the adaptation process during sliding mode estimating the corresponding *equivalent control*. However, none of the above algorithms resulted in minimum possible value of the discontinuous control. Finding the solution of this problem under uncertainty conditions is the objective of this section.

2.3.1 Simple Illustrative Example Explaining the Main Idea of the Method

We start with a simple example. It is assumed that for the first-order system

$$\begin{gathered}\dot{x}(t) = a(t) + u \\ u = -k\operatorname{sign}(x(t)),\ k > 0\end{gathered} \tag{2.11}$$

The ranges of a time varying parameter

$$0 < |a(t)| \leq a_+$$

and the upper bound A for its time derivative

$$|\dot{a}(t)| \leq A$$

are known only.

The sliding mode with $x(t) \equiv 0$ exists for all values of unknown parameter $a(t)$ if

$$k > a_+$$

However if parameter $a(t)$ is varying, the gain k can be decreased and, as a result, chattering amplitude can be reduced. The objective of adaptation is decreasing k to the minimal value preserving sliding mode, if parameter a is unknown.

If the condition $k > a_+$ holds, then sliding mode with $x(t) \equiv 0$ occurs and control in (2.11) should be replaced by the, so-called, *equivalent control* u_{eq} [[29]] for which the right-hand side in (2.11) is equal to zero, namely,

$$\dot{x}(t) = 0 = a(t) + u_{eq} \tag{2.12}$$

that leads to

$$k(t)\left[\operatorname{sign}(x(t))\right]_{eq} = a(t) \tag{2.13}$$

If $k < a$, the set $x(t) \equiv 0$ is of zero measure in time and can be disregarded. The function $\left[\operatorname{sign}(x(t))\right]_{eq}$ is an average value, or a slow component of discontinuous function $\operatorname{sign}(x(t))$ switching at high frequency and can be easily obtained by a low pass filter filtering out the high frequency component [[29]]. Of course, the average value is in the range $(-1, 1)$.
Then the design idea of adaptation seems to be evident:

after sliding mode occurs the control parameter $\left[\operatorname{sign}(x(t))\right]_{eq}$ *should be decreased until and becomes close to* 1.

On one hand, the condition $k(t) > a(t)$ should hold. But the chattering amplitude is proportional to $k(t)$. The objective of adaptation process looks now transparent:

the gain $k(t)$ should tend to $a(t)/\alpha$ with $\alpha \in (0,1)$ which is very close to 1.

As a result, the minimal possible value of discontinuity magnitude is found for the current value of parameter $a(t)$ to reduce the amplitude of chattering. For that purpose select the *adaptation algorithm* in the form

$$\dot{k}(t) = \rho k(t)\operatorname{sign}(\delta(t)) - M\left[k(t) - k^+\right]_+ + M\left[\mu - k(t)\right]_+$$

$$\delta(t) := \left|\left[\operatorname{sign}(x(t))\right]_{eq}\right| - \alpha,\ \alpha \in (0,1) \tag{2.14}$$

$$[z]_+ := \begin{cases} 1 \text{ if } z \geq 0 \\ 0 \text{ if } z < 0 \end{cases},\ M > \rho k^+,\ k^+ > a^+,\ \rho > 0$$

The gain k can vary in the range $[\mu, k^+]$, $\mu > 0$ is a preselected minimal value of k. For the adaptation algorithm (2.14) sliding mode will occur after a finite time interval. Indeed, if it does not exist, then

$$\left|\left[\operatorname{sign}(x(t))\right]_{eq}\right| = 1$$

that leads to $\delta > 0$, and the increasing gain $k(t)$ will reach the value k^+ which is sufficient for enforcing sliding mode for any value of parameter $a(t)$.

Show that in sliding mode the adaptation process (2.14) with

$$\delta(t) = 0 \text{ or } k = \mu$$

is over after a finite time t_f. To do that calculate the time derivative of the Lyapunov function

$$V(\delta) = \delta^2/2$$

First, assume that during the adaptation process $k(t) \in [\mu, k^+]$ which means that

$$|a(t)|/\alpha > \mu \text{ or } (|a(t)| > \alpha\mu)$$

the time derivatives of $\left|[\text{sign}(x(t))]_{eq}\right|$ (2.13) and $|a(t)|$ exist and the terms depending on M in the adaptation algorithm (2.14) are equal to zero. Calculate the time derivative of the Lyapunov function $V(\delta)$ by virtue of (2.13) and (2.14):

$$\begin{aligned}
&\dot{V}(\delta) = \delta\dot{\delta} = \delta\frac{d}{dt}\left|[\text{sign}(x)]_{eq}\right| = \\
&\delta\frac{d}{dt}(|a|/k) = -|a|\delta k^{-2}\dot{k} + \delta k^{-1}\frac{d}{dt}(|a|) = \\
&-|a|\delta k^{-1}\rho\,\text{sign}\left(\delta - M[k-k^+]_+ + M[\mu-k]_+\right) \\
&+\delta k^{-1}\dot{a}\,\text{sign}(a) \leq -|a|\delta k^{-1}\rho\,\text{sign}(\delta) + |\delta|k^{-1}A \\
&\leq -\alpha\mu\rho k^{-1}|\delta| + |\delta|k^{-1}A = -|\delta|k^{-1}(\alpha\mu\rho - A)
\end{aligned} \tag{2.15}$$

and if $\rho > A/\alpha\mu$ it follows

$$\dot{V}(\delta) \leq -\sqrt{2}\frac{(\alpha\mu\rho - A)}{k^+}\sqrt{V(\delta)}$$

It is evident from the solution

$$0 \leq \sqrt{V(\delta(t))} \leq \sqrt{V(\delta(0))} - \frac{(\alpha\mu\rho - A)}{\sqrt{2}k^+}t$$

of the differential inequality (2.15) that $\sqrt{V(\delta(t))} = 0$ at least after

$$t_f = \frac{k^+}{(\alpha\mu\rho - A)}\sqrt{2V(\delta(0))} = \frac{k^+}{(\alpha\mu\rho - A)}|\delta(0)|$$

and, as a result, $\delta(t)$ becomes equal to zero identically after the finite time t_f.

After the adaptation process is over ($t > t_f$) we have

$$\left|[\text{sign}(x(t))]_{eq}\right| = \frac{|a|}{k} = \alpha$$

So, $k = |a|/\alpha$. If in the course of motion $|a(t)|/\alpha < \mu$, then the gain $k(t)$ decreases until $k(t) = \mu$ and, as it follows from (2.14), it will be maintained at this level. Since the gain $a(t)$ is time varying its increase can can result in $|a(t)|/\alpha = \mu$ and $\delta(t) = 0$ at a time t_f. As it follows from the above analysis, for the further motion in the domain $k(t) \in (\mu, k^+]$ with the initial condition $\delta(t_f) = 0$ the time function $\delta(t)$ will be equal to zero with $\alpha = |a(t)|/k(t)$.

Remark 2.1. The function $[\text{sign}(x(t))]_{eq}$ is needed here for the implementation of the adaptation algorithm (2.14). As it was mentioned above, it can be derived by filtering out a high frequency component of the discontinuous function $\text{sign}(x(t))$ by a low pass filter

$$\tau\dot{z} + z = \text{sign}(x(t)),\ z(0) = 0$$

with a small time constant $\tau > 0$ and the output $z(t)$ which is, in fact, an estimate of $[\text{sign}(x(t))]_{eq}$ satisfying

$$\left|z(t) - [\text{sign}(x(t))]_{eq}\right| \leq O(\tau) \underset{\tau\to 0}{\to} 0$$

Then the convergence analysis of (2.14)-(2.15) with $\delta(t) = z(t) - \alpha$ is valid beyond the domain $|\delta(t)| \leq O(\tau)$. This inequality defines the accuracy of adaptation. Note that the switching frequencies of the modern power converters are of order dozens of kHz, and very small time constant τ can be selected to get a high accuracy of adaptation.

Notice also that, as follows from ⟦29⟧,

$$z(t) = \psi(t) + O(\sup|x(t)| + \tau) + O(\sup|x(t)|/\tau)$$

where $\psi(t)$ is the fast rate exponentially decreasing function. The term $\sup|x(t)|$ is inverse proportional to the sliding mode frequency f. It is of order of dozen kHz in the modern switching devices. Therefore it is not a problem to make the term

$$O(\sup|x| + \tau) + O(\sup|x|/\tau)$$

negligible. Of course, this engineering language can be translated into mathematical one, for example as follows: *for any $\varepsilon > 0$ there exists a switching frequency f_0 such that*

$$|z - u_{eq}| < \varepsilon \text{ if } f > f_0$$

implying

$$\text{sign}[\left|[\text{sign}(x(t))]_{eq}\right| - \alpha] = \text{sign}[|z(t)| - \alpha]$$

Certainly, we have described the idea only. The generalization of the adaptive procedure (2.14) for the vector-state models with uncertainties $a = a(t,x)$ constitutes the main result given below.

2.3.2 Main Assumptions

Here we consider an arbitrary order system

$$\begin{gathered}\dot{x}(t) = f(t, x(t)) + b(t, x(t))\, u(t, x(t)) \\ x(t) \in R^n,\ f : R^+ \times R^n \to R^n \\ u : R^+ \times R^n \to R,\ b : R^+ \times R^n \to R^n\end{gathered} \tag{2.16}$$

for which we assume that

A1 the control $u = u(t, x)$ enforces siding mode on some surface

$$\sigma(x) = 0\ (\sigma \in C^1)$$

and is in the following form

$$\begin{gathered}u(t, x) = -k(t)\left(1 + \lambda\sqrt{\|x\|^2 + \varepsilon}\right)\operatorname{sign}(\sigma(x)) \\ \lambda \geq 0,\ \varepsilon > 0, k(t) \in [\mu, k^+], \mu > 0\end{gathered} \tag{2.17}$$

Similarly to the example (2.11) the control gain $k(t)$ is a time varying function governed by the adaptation procedure described below.

A2 the uncertain functions $f(t, x)$ and $b(t, x)$ satisfy the commonly accepted conditions (which are much more general then in (2.3)):

$$\begin{gathered}\|f(t, x)\| \leq f_0 + f_1\|x\| \\ 0 < b_0 \leq \nabla^{\mathsf{T}}\sigma(x)\, b(t, x) \\ \|b(t, x)\| \leq b^+,\ \|\nabla\sigma(x)\| \leq \sigma^+\end{gathered} \tag{2.18}$$

$$\begin{gathered}\Phi(t, x) := \frac{\nabla^{\mathsf{T}}\sigma(x)\, f(t, x)}{\nabla^{\mathsf{T}}\sigma(x)\, b(t, x)} \\ \|\nabla^{\mathsf{T}}\Phi(t, x)\| \leq \Phi_0 + \Phi_1\|x\| \\ \left|\frac{\partial}{\partial t}\Phi(t, x)\right| \leq \varphi_0 + \varphi_1\|x\|\end{gathered} \tag{2.19}$$

All coefficients in the right-hand sides of these inequalities are constant and positive. The function $\sigma(x)$ and its time derivative

$$\begin{gathered}\dot{\sigma}(x) = \nabla^{\mathsf{T}}\sigma(x)\, f(t, x) - \\ \nabla^{\mathsf{T}}\sigma(x)\, b(t, x)\, k(t)\left(1 + \lambda\sqrt{\|x\|^2 + \varepsilon}\right)\operatorname{sign}(\sigma(x))\end{gathered} \tag{2.20}$$

should have opposite signs ($\sigma(x)\dot{\sigma}(x) < 0$ if $\sigma(x) \neq 0$) for sliding mode to exist on the surface $\sigma(x) = 0$. The sufficient condition for this follows from (2.18),(2.19) and (2.20):

$$\sigma(x)\dot{\sigma}(x) = \sigma(x)\nabla^{\intercal}\sigma(x) f(t,x) -$$

$$\nabla^{\intercal}\sigma(x) b(t,x) k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)|\sigma(x)|$$

$$\leq [\nabla^{\intercal}\sigma(x) b(t,x)]\,|\sigma(x)| \times$$

$$\left(|\Phi(t,x)| - k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)\right) < 0$$

if

$$|\Phi(t,x)| - k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right) < 0 \tag{2.21}$$

which is always holds when

$$\lambda \geq f_1/f_0,\ \mu > f_0\sigma^+/b_0,\ k(t) \in \left(\mu, k^+\right] \tag{2.22}$$

in view of the relation

$$|\Phi(t,x)| - k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right) \leq$$

$$f_0\frac{\sigma^+(1+\|x\| f_1/f_0)}{b_0} - \mu(1+\lambda\|x\|)$$

To derive the sliding mode equation the function $\mathrm{sign}(\sigma(x))$ should be replaced by the solution of the equation $\dot{\sigma}(x) = 0$ with respect to the term $\mathrm{sign}(\sigma(x))$, called *the equivalent control*:

$$[\mathrm{sign}(\sigma(x))]_{eq} := \begin{cases} \dfrac{\Phi(t,x)}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)} & \text{if } \sigma(x(t)) = 0 \\ \mathrm{sign}(\sigma(x(t))) & \text{if } \sigma(x(t)) \neq 0 \end{cases} \tag{2.23}$$

satisfying (in view of (2.21)) in the sliding mode $(\sigma(x(t)) = 0)$

$$\left|[\mathrm{sign}(\sigma(x))]_{eq}\right| < 1 \tag{2.24}$$

Note that the state-depended magnitude of discontinuity is the conventional tool to minimize chattering. Indeed, in the course of approaching the origin $x = 0$ it is decreasing automatically [15]. Similarly the term

$$\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)$$

may also affect the amplitude of chattering appearing on sliding mode phase. The necessity of this term in (2.17) is related with the considered class of nonlinear functions satisfying

$$\|f(t,x)\| \leq f_0 + f_1\|x\|$$

If $f_1 = 0$ (nonlinear function is bounded satisfying $\|f(t,x)\| \leq f_0$) similarly to the example we may take $\lambda = 0$, and, in this case, the term $\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)$ does not affect a chattering amplitude. It is important, that this methodology is oriented to the worst case - sliding mode should exist for all values of unknown functions or parameters from some range. The method of the paper guarantees the minimal magnitude for their current values of unknown functions and parameters. In general, we may add the term $\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)$ to the gain multiplying the discontinuous function $\operatorname{sign}(\sigma(x))$ to enforce sliding mode when $f(x)$ is unbounded. This term with $\varepsilon = 0$ in the form $(1+\lambda\|x\|)$ can solve this problem as well. However, the adaptation algorithm implies existence of the gradient of this term, but it does not exist for the last case.

Below we will show that in the general case the adaptation of the gain-parameter $k(t)$ only is sufficient to minimize the chattering effect on sliding mode phase since the suggested "learning law" for $k(t)$ variation automatically takes into account the presence of this term.

2.3.3 Adaptation Algorithm in Sliding Mode

2.3.3.1 Description of the Adaptation Procedure

The idea of the *adaptation law* for the control gain $k(t)$ is similar to that for our first-order system in the previous subsection:

$$\dot{k}(t) = \begin{cases} (\gamma_0 + \gamma_1\|x\|)\,k(t)\operatorname{sign}(\delta(t)) \\ -M[k(t)-k^+]_+ + M[\mu - k(t)]_+ \end{cases} \tag{2.25}$$

where

$$\begin{gathered} \delta(t) := \left|[\operatorname{sign}(\sigma(x))]_{eq}\right| - \alpha \\ \alpha \in (0,1),\ \lambda > 0,\ \gamma_0, \gamma_1 > 0 \end{gathered} \tag{2.26}$$

Notice that in (2.23)

$$\frac{|\Phi(t,x)|}{\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)} < k$$

and, moreover,

$$\frac{|\Phi(t,x)|}{\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)} \leq \frac{\sigma^+\left(f_0+f_1\|x\|\right)}{b_0\left(1+\lambda\|x\|\right)} = \tag{2.27}$$
$$\sigma^+\frac{f_0}{b_0}\left(1+\frac{\left(f_1 f_0^{-1}-\lambda\right)\|x\|}{1+\lambda\|x\|}\right) \leq \sigma^+\frac{f_0}{b_0}$$

Select in (2.25)

$$k^+ > \sigma^+\frac{f_0}{b_0} \tag{2.28}$$

If sliding mode does not exist, then

$$\left|[\operatorname{sign}(\sigma(x))]_{eq}\right| = 1$$

and the gain $k(t)$ will be equal to k^+ which results in the occurrence of this motion in the surface $\sigma(x(t)) = 0$.

2.3.3.2 Analysis of the δ-Stability for the Adaptive Version

The following theorem describes the main stability property of the sliding mode controller with the gain adaptation based on the "equivalent control method".

Theorem 2.3 (on the adaptive sliding mode controller). *For the dynamic system (2.16) closed by the control (2.17) with the gain adaptation law (2.25) - (2.26) with the parameters satisfying*

$$\begin{gathered} k^+ > \sigma^+\frac{f_0}{b_0},\ \mu > f_0\sigma^+/b_0,\ 0 < \varepsilon << 1 \\ \gamma_0 > \alpha^{-1}\left[\left(\frac{f_0}{\mu}+b^+\right)\Phi_0+\frac{\varphi_0}{\mu}+f_0+b^+k^+\right] \\ \gamma_1 \geq \alpha^{-1}\left(\frac{f_0}{\mu}+b^+\right)\Phi_1,\ M > \gamma_0 k^+ \end{gathered} \tag{2.29}$$

there exist

$$\theta := \alpha\gamma_0 - \left[\left(\frac{f_0}{\mu}+b^+\right)\Phi_0+\frac{\varphi_0}{\mu}+f_0+b^+k^+\right] > 0 \tag{2.30}$$

and

$$t_f = \theta^{-1} |\delta(0)|$$

(where $\delta(0)$ *is defined by (2.26)) such that for all* $t \geq t_f$ *the condition*

$$\left| [\mathrm{sign}(\sigma(x(t)))]_{eq} \right| = \alpha \tag{2.31}$$

holds. It means that the sliding surface $\sigma(x) = 0$ *is attained in a finite time* t_f*, and for*

$$\alpha = 1 - \varepsilon_0$$

($\varepsilon_0 > 0$ *is a small enough positive number) the suggested adaptation procedure provides* $k(t)$ *tending to a vicinity of the minimum possible value* $k_{\min}(t)$*, that is, as it follows from (2.23), in sliding mode*

$$\begin{gathered} k(t) = \begin{cases} \dfrac{1}{1-\varepsilon_0} k_{\min}(t) & if\ k_{\min}(t) \geq \mu \\ \mu & if\ k_{\min}(t) < \mu \end{cases} \\ k_{\min}(t) := \dfrac{|\Phi(t, x(t))|}{1 + \lambda \sqrt{\|x(t)\|^2 + \varepsilon}} \end{gathered} \tag{2.32}$$

Proof. Consider the Lyapunov function candidate as

$$V(\delta) := \frac{1}{2}\delta^2 \tag{2.33}$$

and it is assumed that during adaptation process $k(t) \in [\mu, k^+]$ which means that

$$\frac{|\Phi(t,x)|}{k(t)\left(1 + \lambda \sqrt{\|x\|^2 + \varepsilon}\right)} > \alpha$$

$$|\Phi(t,x)| > \alpha\mu(1 + \lambda \|x\|)$$

taking into account that

$$[\mathrm{sign}(\sigma(x))]_{eq} \neq 0$$

and the time derivative of $\left| [\mathrm{sign}(\sigma(x))]_{eq} \right|$ exists. If in the course of adaptation process the above inequality does not hold, the gain $k(t)$ will decrease and after the $k(t) = \mu$ it will remain constant. Note that the similar consideration was given for the simple example in the previous subsection. Calculate the time derivative of (2.9) (it exists with any $\varepsilon > 0$, while it does not exist with $\varepsilon = 0$):

$$\dot{V}(\delta) = \delta\dot{\delta} = \delta \frac{d}{dt}\left[\frac{|\boldsymbol{\Phi}(t,x)|}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)} - \alpha\right]$$

$$= \delta\left[\frac{\nabla^{\intercal}\boldsymbol{\Phi}(t,x)(f+bu) + \frac{\partial}{\partial t}\boldsymbol{\Phi}(t,x)}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)}\operatorname{sign}(\boldsymbol{\Phi}(t,x)) - \right. \tag{2.34}$$

$$\left. |\boldsymbol{\Phi}(t,x)| \frac{\dot{k}(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right) + k(t)\lambda\frac{x^{\intercal}(f+bu)}{\sqrt{\|x\|^2+\varepsilon}}}{k^2(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)^2}\right]$$

Applying here the estimates (2.18), (2.19) and using (2.17), we derive

$$\dot{V}(\delta) \le \delta\left[\frac{\|\nabla^{\intercal}\boldsymbol{\Phi}(t,x)\|\,\|f\| + \left|\frac{\partial}{\partial t}\boldsymbol{\Phi}(t,x)\right|}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)} + \right.$$

$$\|\nabla^{\intercal}\boldsymbol{\Phi}(t,x)\|\,\|b\| - |\boldsymbol{\Phi}(t,x)|\frac{(\gamma_0+\gamma_1\|x\|)\operatorname{sign}(\delta(t))}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)}$$

$$\left. + \ |\boldsymbol{\Phi}(t,x)|\frac{\lambda x^{\intercal}(f+bu)}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)\sqrt{\|x\|^2+\varepsilon}}\right]$$

which (in view of (2.24)) implies

$$\dot{V}(\delta) \le \delta\left[\frac{(\boldsymbol{\Phi}_0+\boldsymbol{\Phi}_1\|x\|)(f_0+f_1\|x\|)+(\varphi_0+\varphi_1\|x\|)}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)}\right.$$

$$+\|\nabla^{\intercal}\boldsymbol{\Phi}(t,x)\|\,\|b\| - |\boldsymbol{\Phi}(t,x)|\frac{(\gamma_0+\gamma_1\|x\|)\operatorname{sign}(\delta(t))}{k(t)\left(1+\lambda\sqrt{\|x\|^2+\varepsilon}\right)}$$

$$+ |\Phi(t,x)| \frac{\lambda x^{\mathsf{T}} (f + bu)}{k(t) \left(1 + \lambda \sqrt{\|x\|^2 + \varepsilon}\right)^3} \Bigg] \leq$$

$$|\delta| \frac{f_0 (\Phi_0 + \Phi_1 \|x\|) \left(1 + \frac{f_1}{f_0} \|x\|\right) + \varphi_0 \left(1 + \frac{\varphi_1}{\varphi_0} \|x\|\right)}{\mu (1 + \lambda \|x\|)}$$

$$+ \ |\delta| (\Phi_0 + \Phi_1 \|x\|) b^+ - |\delta| \alpha (\gamma_0 + \gamma_1 \|x\|)$$

$$+ |\delta| \frac{\lambda \|x\| \left(f_0 + f_1 \|x\| + b^+ k^+ \left(1 + \lambda \sqrt{\|x\|^2 + \varepsilon}\right)\right)}{\left(1 + \lambda \sqrt{\|x\|^2 + \varepsilon}\right)^2}$$

Selecting $\lambda \geq \max \left\{ \frac{f_1}{f_0}, \frac{\varphi_1}{\varphi_0} \right\}$ we get

$$\dot{V}(\delta) \leq |\delta| \left[\left(\frac{f_0}{\mu} + b^+ \right) (\Phi_0 + \Phi_1 \|x\|) + \frac{\varphi_0}{\mu} \right]$$

$$- |\delta| \alpha (\gamma_0 + \gamma_1 \|x\|) + |\delta| \left[f_0 \frac{1 + \frac{f_1}{f_0} \|x\|}{1 + \lambda \|x\|} + b^+ k^+ \right]$$

$$\leq |\delta| \left[\left(\frac{f_0}{\mu} + b^+ \right) \Phi_0 + \frac{\varphi_0}{\mu} + f_0 + b^+ k^+ - \alpha \gamma_0 \right.$$

$$\left. + \|x\| \left(\left(\frac{f_0}{\mu} + b^+ \right) \Phi_1 - \alpha \gamma_1 \right) \right]$$

Taking γ_0 and γ_1 satisfying (2.29) we finally get

$$\dot{V}(\delta) \leq - |\delta| \theta = - \theta \sqrt{2V(\delta)}$$

where θ is given by (2.30). Then similarly to the primitive example it can be shown that the adaptation process will be over after time instant

$$t_f = \theta^{-1} |\delta(0)|$$

and if in the course of motion $k(t)$ decreases and becomes equal to μ, then it will be maintained at this level.

2.4 Adaptive Super-Twist Control

In this section we consider the application of the presented adaptation concept for, the adaptive version of the, so-called, Super-Twist Control.

2.4.1 Main Properties of the Standard Super Twist without Adaptation

Consider the simple two dimensional nonlinear system containing discontinuous nonlinearity in the right-hand side of the second component:

$$\begin{cases} \dot{x}(t) = y(t) - \bar{\alpha}\sqrt{|x(t)|}\operatorname{sign}(x(t)) \\ \dot{y}(t) = \phi(t) + u(t) \\ u(t) := -\bar{\beta}\operatorname{sign}(x(t)) \end{cases} \tag{2.35}$$

referred below to as the "super-twist" controller ⟦16⟧, ⟦17⟧ and ⟦19⟧.

Remark 2.2. Notice that if $y(0) = 0$, then (2.35) can be represented as

$$\dot{x}(t) = -\bar{\alpha}\sqrt{|x(t)|}\operatorname{sign}(x(t)) - \bar{\beta}\int_{s=0}^{t}\operatorname{sign}(x(s))ds$$

which is exactly a *PI*-controller (with the *P*-part modulation) with respect to the $\operatorname{sign}(x)$ - term. Recall that standard *PI*-controllers contain the same Proportional and Integration terms (PI terms), but with respect to the state variable x.

In (2.35) it is supposed that

$$\bar{\alpha} > 0 \text{ and } |\phi(t)| \leq \phi_0 < \bar{\beta} \tag{2.36}$$

Of course, similarly ⟦16⟧, ⟦17⟧, ⟦19⟧ it is assumed that the super twisting algorithm is applied for a system of an arbitrary order with scalar control; x and y are state variable of the controller only, while the function $\phi(t)$ depends on both x, y and the system state.

2.4.1.1 Convergence Analysis of a Standard Super - Twist Controller without Adaptation

a) The both state variable are sign-varying, therefore the initial conditions can be selected as follows

$$x(0) = -x_0, x_0 > 0, y(0) = 0$$

In our case $\dot{x}(0) > 0$ and, hence, $x(t)$ is increasing. Denote

$$t_1^* := \inf\{t > 0 : x(t_1^*) = 0, x(t) < 0 \text{ for } t \in [0, t_1^*)\}$$

Next, compare two ODE's:

$$\dot{x}(t) = y(t) - \bar{\alpha}\frac{x(t)}{\sqrt{|x(t)|}},\ x(0) = -x_0 < 0 \tag{2.37}$$

where

$$y(t) = \int_{\tau=0}^{t} \left[\bar{\beta} + \phi(\tau)\right] d\tau$$

satisfying

$$\begin{array}{c} y(t) \geq \int_{\tau=0}^{t} \left[\bar{\beta} - \phi_0\right] d\tau = m \cdot t,\ m := \bar{\beta} - \phi_0 > 0 \\ y(t) \leq \int_{\tau=0}^{t} \left[\bar{\beta} + \phi_0\right] d\tau = Mt,\ M := \bar{\beta} + \phi_0 \end{array} \tag{2.38}$$

and

$$\dot{z}(t) = m \cdot t - \bar{\alpha}\frac{z(t)}{\sqrt{x_0}},\ z(0) = x(0) = -x_0 < 0 \tag{2.39}$$

Obviously that the ODE (2.37) is equivalent to (2.35). Since $|x(t)|$ is decreasing it follows that

$$\frac{1}{\sqrt{|x(t)|}} > \frac{1}{\sqrt{x_0}} := k_0 \text{ for } t > 0$$

For any $t \in [0, t_1^*)$ we have $x(t) < 0$ and $z(t) < 0$ which implies $\dot{x}(t) > \dot{z}(t)$ and, as the result, $x(t) > z(t)$. So that

$$t_1^* > t' := \inf\{t > 0 : z(t) = 0\}$$

The solution to (2.39) is

$$z(t) = \frac{m}{\bar{\alpha}k_0}\left(t - \frac{1}{\bar{\alpha}k_0}\right) + \left[\frac{m}{(\bar{\alpha}k_0)^2} - x_0\right] e^{-\bar{\alpha}k_0 t}$$

and one can conclude that

$$t' = (k_0)^{-1} q\left(\frac{1}{\bar{\alpha}}\right) = \sqrt{x_0} q\left(\frac{1}{\bar{\alpha}}\right) < t_1^*$$

for a large enough $\bar{\alpha}$. Here t' is the solution of the transcendent algebraic equation $z(t') = 0$ and $q(s) \to 0$ when $s \to 0$. By (2.38) it follows that $y(t) \leq Mt$ and hence

$$y(t_1^*) \leq Mt_1^* \leq Mt' = M\sqrt{x_0} q\left(\frac{1}{\bar{\alpha}}\right)$$

b) For $t > t_1^*$ we already have that $x(t) > 0$ and $y(t)$ is a decaying function since

$$\dot{y}(t) = \phi(t) - \bar{\beta}\operatorname{sign}(x(t)) = \phi(t) - \bar{\beta} \leq 0$$

that implies

$$y(t) \leq y(t_1^*) - m \cdot t \tag{2.40}$$

For the instant t_1^{**}

$$t_1^{**} := \inf\{t > t_1^* : y(t) = 0\}$$

we have

$$\begin{aligned} t_1^{**} &\leq t_1^* + y(t_1^*)/m = t_1^* + \frac{M}{m}\sqrt{x_0}q\left(\frac{1}{\bar{\alpha}}\right) \\ &= \left(1 + \frac{M}{m}\right)\sqrt{x_0}q\left(\frac{1}{\bar{\alpha}}\right) \end{aligned} \tag{2.41}$$

So, by (2.35) and (2.40)

$$\begin{aligned} x(t_1^{**}) &= \int_{\tau=t_1^*}^{t_1^{**}} \left[y(\tau) - \bar{\alpha}\sqrt{|x(\tau)|}\operatorname{sign}(x(\tau))\right] d\tau \\ &= \int_{\tau=t_1^*}^{t_1^{**}} \left[y(\tau) - \bar{\alpha}\sqrt{|x(\tau)|}\right] d\tau \leq \int_{\tau=t_1^*}^{t_1^{**}} y(t_1^*) d\tau = \\ & y(t_1^*)\left(t_1^{**} - t_1^*\right) \leq y^2(t_1^*)/m \leq \gamma x_0 \end{aligned} \tag{2.42}$$

where

$$\gamma := \frac{M^2}{m}\left[q\left(\frac{1}{\bar{\alpha}}\right)\right]^2 \tag{2.43}$$

Selecting $\bar{\alpha}$ large enough we may conclude that $\gamma \in (0,1)$ and

$$x(t_1^{**}) \leq \gamma x_0$$

Here $x(t_1^{**})$ is an initial value of (2.35) for the second interval $\Delta t_2 := t_2^{**} - t_1^{**}$ where

$$t_2^{**} := \inf\{t > t_1^{**} : y(t) = 0\}$$

Similarly to (2.42)

$$|x(t_2^{**})| \leq \gamma x(t_1^{**}) \leq \gamma^2 x_0 \tag{2.44}$$

c) Iterating this process we may conclude that

$$|x(t_i^{**})| \leq \gamma \left|x(t_{i-1}^{**})\right| \leq \cdots \leq \gamma^i x_0 \tag{2.45}$$

and

$$\begin{aligned} \Delta t_i := t_i^{**} - t_{i-1}^{**} &\leq \sqrt{\left|x(t_{i-1}^{**})\right|}\left(1 + \frac{M}{m}\right) q\left(\frac{1}{\bar{\alpha}}\right) \\ &\leq \gamma^{i/2}\sqrt{x_0}\left(1 + \frac{M}{m}\right) q\left(\frac{1}{\bar{\alpha}}\right) \end{aligned} \tag{2.46}$$

Last two inequalities permit to formulate the following result.

Proposition 2.1. *If*

$$|\phi(t)| \leq \phi_0 < \beta$$

then for any initial value $x(0)$ from bounded domain there exists large enough $\bar{\alpha} > 0$, such that the super-twist procedure (2.35) has a finite time convergence or reaching time proceeding ***the second order sliding mode****, and the following properties holds:*

1)

$$|x(t)| \simeq q\left(\gamma^{t/2}\right) \underset{t\to\infty}{\to} 0$$

$$\gamma := \frac{M^2}{m}\left[q\left(\frac{1}{\bar{\alpha}}\right)\right]^2 \in (0,1)$$

2) The reaching time

$$t_{reach} := \inf_{\bar{t}\geq 0}\{\bar{t} : x(t) = 0 \, \mathit{for\ all}\ t \geq \bar{t}\}$$

is estimated by

$$\begin{aligned} t_{reach} &\leq \sum_{i=1}^{\infty}\Delta t_i \leq \sqrt{x_0}\left(1+\frac{M}{m}\right)q\left(\frac{1}{\bar{\alpha}}\right)\sum_{i=1}^{\infty}\gamma^{i/2} \\ &\leq \frac{\sqrt{\gamma}}{1-\sqrt{\gamma}}\sqrt{x_0}\left(1+\frac{M}{m}\right)q\left(\frac{1}{\bar{\alpha}}\right) \end{aligned} \tag{2.47}$$

The important comments can be done:

- the reaching time tends to zero with gain $\bar{\alpha} \to \infty$;
- a finite-time convergence takes place for any small $m := \bar{\beta} - \phi_0 > 0$;
- the sufficient convergence conditions, derived in previous publications(see, for example, [[18]], [[24]]) led to upper estimate of admissible disturbance less than $0.5\bar{\beta}$. Note that the system is not even asymptotically stable for $\phi_0 \geq \bar{\beta}$. As it follows from (2.35) in this case $y(t)$ is constant or diverging, if the disturbance ϕ is such that $|\phi(t)| \geq \bar{\beta}$, and has sign opposite to control $u(t)$.
- the upper bound (2.47) for the reaching time t_{reach} is proportional to the root of the initial state, namely, $\sqrt{|x_0|}$ (since, in our case $y_0 = 0$) and inverse-proportinal to the parameter $\bar{\alpha}$, i.e., $q\left(\frac{1}{\bar{\alpha}}\right)$, which coincides with the estimates in [[18]] proportional also to ($y_0 = 0$)

$$\sqrt{|x_0|} + |y_0| = \sqrt{|x_0|}$$

2.4.1.2 Simulations of a Super-Twist Control without Adaptation

The figure 2.2 illustrates the dynamics of the super-twist controller with

$$\phi(t) = \phi_0 \sin(\omega t)$$

the following parameters:

$$\bar{\alpha} = 1.2,\ \bar{\beta} = 0.6,\ \phi_0 = 1,\ \omega = 0.09 \text{ and } x(0) = \begin{bmatrix} -0.3 \ 0.25 \end{bmatrix}^{\mathsf{T}}$$

One can see a finite-time convergence to zero (approximately in 3.5 sec.) of the first state variable $x(t)$ and the corresponding discontinuous control of the amplitude $\bar{\beta}$.

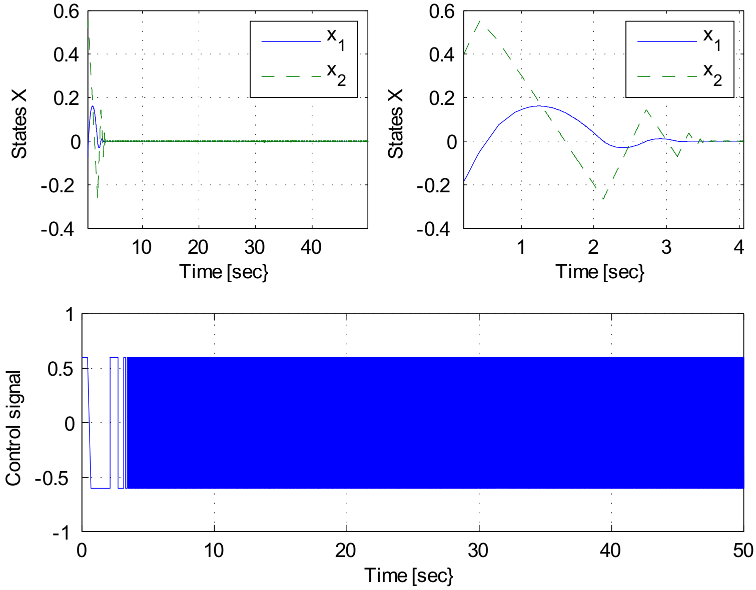

Fig. 2.2 The states and the control signal for the super-twist controller without adaptation of the gain parameter $\bar{\beta}$

2.4.2 *Super-Twist Control with Adaptation*

Denoting

$$x_1 = x,\ \ x_2 = y,\ k := \bar{\beta}$$

the system (2.35) can be represented as

$$\begin{cases} \dot{x}_1 = x_2 - \bar{\alpha}\sqrt{|x_1|}\text{sign}(x_1) \\ \dot{x}_2 = \phi(t) + u \\ u := -k\text{sign}(x_1) \end{cases}$$

or, in the vector format (2.16)

$$\dot{x} = f(t,x) + b(t,x)u$$

with

$$f(t,x) := \begin{pmatrix} x_2 - \bar{\alpha}\sqrt{|x_1|}\mathsf{sign}(x_1) \\ \phi(t) \end{pmatrix},\ b(t,x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Taking

$$\sigma(x) = x_1$$

and permitting for the gain parameter to be time-varying, i.e.,

$$k(t) = \beta(t)$$

we may apply the adaptation procedure (2.25)-(2.26) in spite of the fact that $\left\|\frac{\partial}{\partial x} f(t,x)\right\|^2$ is unbounded since in this case it does not participate directly in the construction of $[\operatorname{sign}(\sigma(x(t)))]_{eq}$. Below we will consider this with more details.

2.4.2.1 The σ-Adaptation Method

Here, following to [[23]], we apply the adaptation law given by

$$u(t) = -k(t)\operatorname{sign}(x_1(t))$$
$$\dot{k}(t) = \begin{cases} k(t)\,|\sigma(x(t))|\operatorname{sign}(|\sigma(x(t)| - \varepsilon) & \text{if } k(t) > \bar{\mu} \\ \bar{\mu} & \text{if } k(t) \le \bar{\mu} \end{cases} \tag{2.48}$$

referred to as "$\sigma-$*adaptation*". In (2.48)

$$k(0) = 2.8, \varepsilon = 0.0003 \text{ and } \bar{\mu} = 0.04$$

The specific feature of this procedure is that the adaptation process practically stops after the reaching time t_{reach} when

$$\sigma(x(t)) = x_1(t) = 0$$

for any $t \ge t_{reach}$, and, as the result, the gain parameter $k(t) = \beta(t)$, defining the size of the discontinuous control (or a chattering amplitude) may be still too far from the disturbance level $|\phi(t)| \le \phi_0$ which is minimal possible one guarantying the finite-time convergence. This effect is clearly seen in the Figure 2.3: the reaching time $t_{reach} \simeq 1\,\mathrm{sec}$., but the gain parameter (the chattering amplitude) remains around the initial level 2.8 (in fact, 3.5) which is too high comparing with $\phi_0 = 1$. So, the adaptation period is to short to decrease significantly the gain parameter $k(t) = \bar{\beta}(t)$, and in sliding mode regime there is no adaptation.

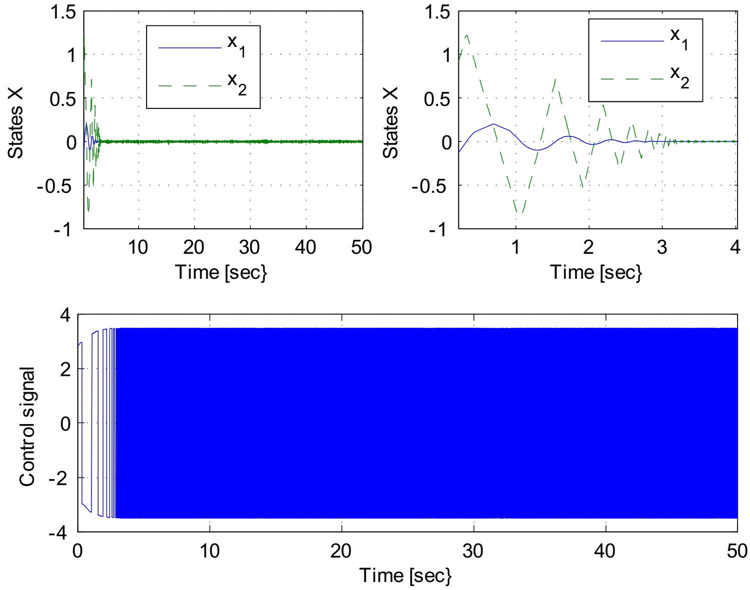

Fig. 2.3 The states and the control signal (with the zooms in the right column) for the super-twist controller with σ-adaptation of the gain parameter $\bar{\beta}$

2.4.2.2 Adaptation Based on the "Equivalent Control"

Theoretical Analysis

The adaptation procedure (2.25)-(2.26) suggested here is applied to minimize the magnitude of discontinuous input $\bar{\beta}\text{sign}(x(t))$ in (2.35). In sliding mode $y(t) \equiv 0$ therefore

$$[\text{sign}\,(\sigma\,(x\,(t)))]_{eq} = \phi(t)/\bar{\beta}(t) = \phi(t)/k(t) \tag{2.49}$$

and the algorithm (2.25)-(2.26) with

$$\lambda = \gamma_1 = 0$$

can be used directly for this case if time derivative of $|\phi(t)|$ is bounded, namely, if

$$\frac{d}{dt}|\phi(t)| \leq L \tag{2.50}$$

Indeed, following to (2.26) and (2.49) for $V(\delta) = \delta^2/2$ we have

$$\dot{V}(\delta\,(t)) = \delta\,(t)\,\dot{\delta}\,(t) = \delta\,(t)\,\frac{d}{dt}\left(\left|[\text{sign}\,(\sigma\,(x))]_{eq}\right|\right)$$

If $|\phi(t)|$ is differentiable then

$$\frac{d}{dt}\left(\left|[\mathrm{sign}\,(\sigma\,(x))]_{eq}\right|\right) = \frac{1}{k}\frac{d}{dt}\,|\phi| - \frac{|\phi|}{k^2}\dot{k}$$

and

$$\dot{V}\,(\delta\,(t)) = \delta\,(t)\left(\frac{1}{k}\frac{d}{dt}\,|\phi| - \frac{|\phi|}{k^2}\dot{k}\right) \tag{2.51}$$

Substitution

$$\dot{k}\,(t) = \gamma_0 k\,(t)\,\mathrm{sign}\,(\delta\,(t)) \tag{2.52}$$

in (2.51) and using (2.50) imply

$$\dot{V}\,(\delta\,(t)) = \delta\,(t)\left[\frac{1}{k}\frac{d}{dt}\,|\phi| -\right.$$

$$\left.\frac{|\phi|}{k^2}k\gamma_0\mathrm{sign}\,(\delta\,(t))\right] \leq \frac{|\delta\,(t)|}{k}\,(L - \phi_0\gamma_0)$$

Taking $\gamma_0 > L/\phi_0$ and denoting

$$\varkappa := \phi_0\gamma_0 - L > 0$$

from the last inequality we get

$$\dot{V}\,(\delta\,(t)) \leq -\,|\delta\,(t)|\,\varkappa/k^+ = -\varkappa/k^+\sqrt{2V\,(\delta\,(t))}$$

which proofs the finite convergence before $t_f = |\delta\,(0)|\,k^+/\varkappa$. So, the following statement can be formulated.

Theorem 2.4 (on adaptive super-twist [[31]]). *The system (2.35) with disturbances $\phi(t)$ having a bounded derivative (fulfilling (2.50)), and with the parameter $\bar{\beta}(t) = k(t)$ adapted on-line according to the adaptation law*

$$\dot{k}\,(t) = \begin{cases} \gamma_0 k(t)\mathrm{sign}\,(\delta\,(t)) - M\,[k(t) - k^+]_+ + M\,[\mu - k(t)]_+ \\ \qquad\qquad \mathit{if}\; 0 < \mu \leq k\,(t) \leq k^+ \\ \\ \qquad\qquad 0 \;\mathit{otherwise} \end{cases}$$

where $\gamma_0 > L/\mu$ converges in the finite time

$$t_f = |\delta\,(0)|\,k^+/\,(\mu\gamma_0 - L)$$

to the sliding mode regime

$$\sigma(x) = x_1 = 0$$

maintaining within the relation

$$\phi(t)/k(t) = \alpha = 1 - \varepsilon_0$$

for small enough $\varepsilon_0 > 0$.

Numerical Illustration

To demonstrate the properties of the adaptation procedure (2.25)-(2.26), simulation was performed for the case $\sigma(x) = x_1$ with the following parameters:

$$\gamma_0 = 2,\ \phi_0 = 1,\ \mu = 0.04$$

$$\alpha = 0.95,\ k^+ = 10\ ,\ k(0) = 2.8$$

In the simulations, the filter (given in Remark 1) with $\tau = 0.5$ was used to calculate the function $\delta(t)$. We obtained the following dynamics (see the figure):

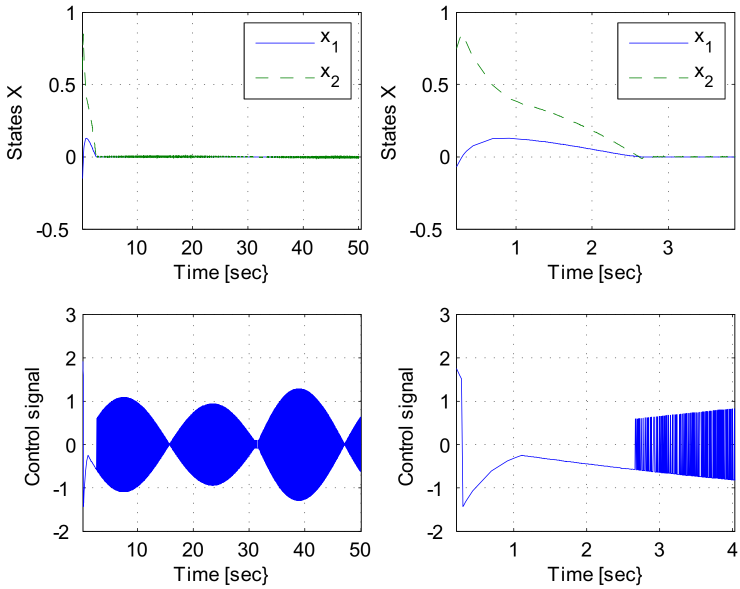

Fig. 2.4 The states and the control signal (with the zooms in the right column) for the super-twist controller with adaptation of the gain parameter $\bar{\beta}$ based on the “equivalent control” signal

Here is clearly seen from Fig.2.5 and Fig.2.4 that gain parameter $k(t) = \beta(t)$, defining the chattering amplitude in the sliding mode (after the reaching time $t_{reach} \simeq$ 1 sec .), continues to decrease attaining after 1.3 sec .the level 0.1 and after follows the amplitude of the external perturbation signal $\phi(t)$. Notice that the simulation with lower frequency demonstrates perfect adaptation process.

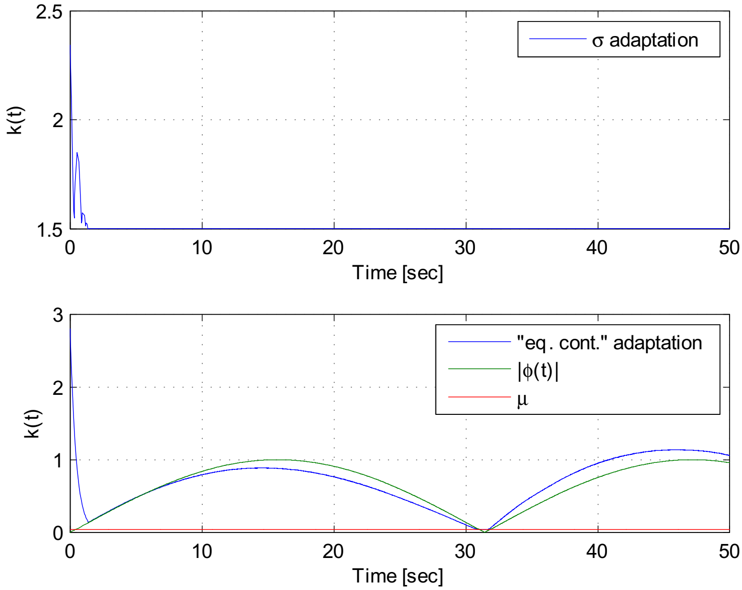

Fig. 2.5 The gain parameter $k(t)$ for σ-adaptation and the adaptation process based on "equivalent control method"

Remark 2.3. The main source of chattering of the super-twisting controller (2.35) is the square root, and not the relay term. A large $\bar{\alpha}$ is really bad selection. If the disturbance is close to the magnitude of the relay function, convergence takes place only for high enough value of $\bar{\alpha}$. So, the magnitude of the relay should be increased to decrease $\bar{\alpha}$. The open problem is to find the trade off to minimize chattering.

2.4.3 Conclusions

In this work an adaptation methodology is developed to find the control gain of a sliding-mode control providing a minimum value of discontinuity resulting in minimization of the chattering effect. The application of this methodology to the super - twist control enables reducing of the control action magnitude to minimum possible value along with a finite-time convergence. The numerical examples clearly illustrate the positive effect of the gain coefficient adaptation being applied to the SOSM controllers (in particularly, to the super-twist controller).

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