Chapter 14
Observers with Discrete-Time Measurements in the Sliding Mode Output-Feedback Stabilization of Nonlinear Systems

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Abstract. The chapter investigates the problem of designing an observer for nonlinear nonaffine systems with discrete-time measurements (continuous-discrete-time systems). The chapter considers the variable-structure control of nonlinear systems when the state vector is not completely available and the output measurements are discrete-time; the use of suitably designed observers is required. The strategy of introducing integrators in the input channel is exploited to enlarge the class of tractable control systems. An observer is proposed and conditions are found under which the convergence to the unique ideal solution is proven for both system and observer. The control problem is solved by forcing a sliding regime for the observer, while satisfying an exponential stability criterion for the observation error state equation.

14.1 Introduction

This chapter deals with nonlinear systems nonaffine in the control law when, due to incomplete state availability, the design of sliding mode control calls for suitable observation procedure.

It is proven in ⟦1⟧ that the control problem has a solution for perfectly known nonlinear nonaffine systems, provided some uniqueness conditions, ⟦2⟧, are satisfied by the coupled state-observer system, and a nonlinear matrix inequality involving the Jacobian matrices of the observer has a solution. This method is "differentiator free", nevertheless in some cases the posed convergence conditions result to be too restrictive.

In ⟦3⟧ nonlinear nonaffine systems are considered and novelties with respect to ⟦1⟧ are presented. Integrators are introduced in the input channel, ⟦4⟧, ⟦5⟧, ⟦6⟧,

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with the aim of strongly simplifying the convexity constraints required to ensure the global convergence of the coupled state-observer system to the unique ideal one.

In the present chapter we consider nonlinear nonaffine control systems. Integrators are introduced in the input channel in order to deal with a larger class of nonlinear nonaffine control systems. A full-order observer is designed and the relevant convergence conditions are found. The sliding motion of the state-observer coupled system on a sliding manifold in the state space of the observer is guaranteed. The analysis of the closed-loop robustness of the proposed scheme is performed with respect to the discrete-time availability of the measurements of the system. Conditions are posed about the considered system and the accessible measurements.

This chapter investigates the problem of designing an observer for nonlinear nonaffine systems with discrete-time measurements (continuous-discrete-time systems). The contribution of the chapter is in the context of output feedback under perfect plant knowledge and with discrete-time measurements.

The use of continuous-discrete observers to estimate the state of nonlinear systems has already been investigated in the literature, [[7]], [[8]], [[9]]. In particular sliding mode observers have been developed in presence of sampled output information, [[10]], [[11]], [[12]].

The practical issues relevant to differentiators, [[13]], [[14]], are not addressed in the chapter.

The chapter is organized as follows. Section 14.2 proposes the statement of the considered variable-structure control problem. Integrators are introduced in the input channel in order to deal with a larger class of nonlinear nonaffine control systems. Conditions are posed about the considered system. In Section 14.3 an observer with continuous time measurements is designed and the relevant convergence conditions are found. In the following Section 14.4 it is considered the case when the state vector is not completely available and the output is accessible via discrete-time measurements: the use of a suitably designed observer is required. Finally a detailed example and simulation results are presented in Section 14.5.

Throughout the chapter a prime denotes transpose and $|\cdot|$ is the Euclidean norm or the induced matrix norm.

14.2 Problem Statement

We consider the nonlinear nonaffine control system

$$\dot{\eta} = \varphi(t, \eta, u) \qquad t \geq 0, \tag{14.1}$$

where $\varphi : [0, +\infty) \times \Omega \times R^m \to R^n$ is a Carathéodory mapping, $\eta \in R^n$ is the state vector, Ω is an open set of R^n, $u \in R^m$ is the available control vector.

The state vector is not completely available and the output vector $\zeta \in R^k$ is expressed by the following equation

$$\zeta = \rho(\eta), \tag{14.2}$$

where ρ is of class C^2.

The output is accessible via discrete-time measurements

$$\zeta_i = \rho\left(\eta\left(t_i\right)\right), \tag{14.3}$$

where t_i, $i = 0, 1, \ldots$, is the sequence of positive real numbers, the sampling instants, defined as $t_{i+1} = t_i + \delta$, $t_0 = 0$, $i = 0, 1, \ldots$, and the constant $\delta > 0$ is the measurement sampling interval.

The sliding manifold is

$$\xi\left(\eta\right) = 0, \tag{14.4}$$

with $\xi\left(\eta\right) \in R^m$.

We assume that $n \geq m$,

$$\xi = \xi\left(\eta\right) : \Omega \rightarrow R^m,$$

ξ is $C^2\left(\Omega\right)$, and the $m \times n$ Jacobian matrix

$$\xi_\eta = \frac{\partial \xi}{\partial \eta}\left(\eta\right) \qquad \text{has maximum rank } m \tag{14.5}$$

for $\eta \in \Omega$.

The objective is to control the state variables $\eta\left(t\right)$, $t \geq 0$, of the control system (14.1) in order to guarantee that the sliding output

$$\xi\left[\eta\left(t\right)\right] \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

In the following section a solution is proposed, which introduces integrators in the input channel. This procedure, traditionally implemented in order to reduce the chattering phenomenon, allows to consider a larger class of nonlinear nonaffine control systems and results in a strongly simplified convexity condition, ⟦3⟧.

14.2.1 The Introduction of Integrators in the Input Channel

Consider the control system (14.1) and sliding manifold (14.4).

Let us define the following augmented control system

$$\dot{\eta} = \varphi\left(t, \eta, u\right) \quad \dot{u} = v, \quad t \geq 0, \tag{14.6}$$

with control vector $v \in R^m$. We measure $\zeta_1 = \rho_1\left(u\right)$, where $\rho_1 : R^m \rightarrow R^m$ and the Jacobian matrix $\rho_{1u} = \dfrac{\partial \rho_1}{\partial u}\left(u\right)$ has maximum rank m. The output ζ_1 is accessible via discrete-time measurements

$$\zeta_{1\,i} = \rho_1\left(u\left(t_i\right)\right),$$

where t_i, $i = 0, 1, \ldots$, is the sequence of positive real numbers, the sampling instants, defined as $t_{i+1} = t_i + \delta$, $t_0 = 0$, $i = 0, 1, \ldots$, and the constant $\delta > 0$ is the measurement sampling interval.

Assume that φ, ξ are both of class C^2 everywhere. For almost every t, the first time derivative of ξ is given by

$$\dot{\xi} = \xi_\eta(\eta)\,\varphi(t,\eta,u)$$

We introduce a new sliding output

$$s = \dot{\xi} + \Lambda\xi, \tag{14.7}$$

where $\Lambda = \operatorname{diag}(\lambda_j)$, $\lambda_j > 0$, $j = 1, \ldots, m$, is a constant $m \times m$ diagonal matrix.

Let the augmented state vector $x = (\eta', u')' \in R^{n+m}$ and the measured vector $y = (\zeta', \zeta_1')' \in R^{k+m}$, we can write

$$\dot{x} = \begin{bmatrix} \dot{\eta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \varphi(t,\eta,u) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v =$$

$$= A(t,x) + Bv = f(t,x,v) \tag{14.8}$$

$$y = \begin{bmatrix} \zeta \\ \zeta_1 \end{bmatrix} = \begin{bmatrix} \rho(\eta) \\ \rho_1(u) \end{bmatrix} = h(x),$$

where $A(t,x) = \begin{bmatrix} \varphi(t,\eta,u) \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $f(t,x,v) = A(t,x) + Bv$, and $h(x) = \begin{bmatrix} \rho(\eta) \\ \phi(u) \end{bmatrix}$. The control vector is v and the sliding output s is defined by (14.7).

Consider (14.6), (14.2) and (14.7). The following new variable-structure control problem can be defined

$$\dot{x} = f(t,x,v), \quad t \geq 0, \quad \text{state equation}, \tag{14.9}$$

$$\dot{u} = v, \quad \text{control equation}, \tag{14.10}$$

$$y = h(x), \quad \text{output equation}, \tag{14.11}$$

$$y_i = h(x(t_i)), \quad \text{discrete-time measurement equation}, \tag{14.12}$$

$$s(t,x) = 0, \quad \text{sliding manifold}, \tag{14.13}$$

where t_i, $i = 0, 1, \ldots$, is the sequence of positive real numbers, the sampling instants, defined as $t_{i+1} = t_i + \delta$, $t_0 = 0$, $i = 0, 1, \ldots$, and the constant $\delta > 0$ is the measurement sampling interval. The vector field $f(t,x,v)$ is defined by (14.8).

The output function h is such that

$$|h(x)| \leq \psi_1 |x|, \quad \psi_1 > 0, \quad \forall t, x, \quad t \geq 0. \tag{14.14}$$

Assume that the Jacobian matrix

$$\xi_\eta(\eta)\,\varphi_u(t,\eta,u) \quad \text{is everywhere nonsingular.} \tag{14.15}$$

The objective is to control, by the vector v, the state variables $(\eta'(t), u'(t))'$, $t \geq 0$, of the augmented system in order to guarantee the sliding property

$$s(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

14.3 Nonlinear Observer with Continuous Time Measurement

In this section it is considered the case when the output is accessible via continuous-time measurement.

The nonlinear observer with continuous-time measurements, ⟦3⟧, for system (14.9) is defined as

$$\dot{\hat{x}} = f(t,\hat{x},v) + N_1[y(t) - h(\hat{x})]. \tag{14.16}$$

We have $x, \hat{x} \in R^{(n+m)}$, $y \in R^{(k+m)}$, u, v, $s \in R^m$ and $m \leq n$. $N_1 \in R^{(n+m)\times(k+m)}$ is a constant matrix. The function f is defined by (14.8). The functions f, h, s are continuously differentiable in x, with f measurable in t and continuous in (x, v).

If (14.15) holds for system (14.9) and (14.16), then for every $t \geq 0$, y, $\hat{x}$ there exists a unique solution

$$v_{1*}(t,y,\hat{x}) \tag{14.17}$$

of the equation

$$s_t(t,\hat{x}) + s_x(t,\hat{x})\{f(t,\hat{x},\cdot) + N_1[y - h(\hat{x})]\} = 0,$$

where N_1 is as in (14.16). The mapping v_{1*} is by definition *the observer's equivalent control* corresponding to the output y.

We consider solutions in $[0,+\infty)$ (either in the Filippov or a.e. sense) to (14.9), (14.11), (14.16) corresponding to the observer's equivalent control, i.e. solutions to

$$\dot{x} = f(t,x,v_{1*}(t,h(x),\hat{x})), \tag{14.18}$$

$$\dot{\hat{x}} = f(t,\hat{x},v_{1*}(t,h(x),\hat{x})) + N_1[h(x) - h(\hat{x})], \tag{14.19}$$

the existence of which is guaranteed by previous conditions (14.15) and (14.17).

By specializing the results in ⟦1⟧ and ⟦3⟧ to the coupled state-observer system (14.9) and (14.16), we obtain the following. If there exist matrices $N_1 \in R^{(n+m)\times(k+m)}$, $M_1 \in R^{(n+m)\times(n+m)}$, positive numbers α_1, $\omega_1 \in R^+$ such that the eigenvalues of M_1 are between α_1 and ω_1, and positive number $\varepsilon_1 \in R^+$ such that the following matrix inequality holds

$$M_1(f_x - N_1 h_x) + (f_x - N_1 h_x)' M_1 \leq -\varepsilon_1 I, \tag{14.20}$$

and if $|s_x(t,x)| \leq L$ everywhere, for some positive constant L, then for every $t \geq 0$

$$|s(t,x(t)) - s(t,\hat{x}(t))| \leq L\left(\frac{\omega_1}{\alpha_1}\right)^{\frac{1}{2}} |x(0) - \hat{x}(0)| \exp(c_1 t),$$

where $\omega_1 c_1 = -\varepsilon_1$, for every a.e. solution $(x', \hat{x}')'$ in $[0,+\infty)$ to (14.18) and (14.19).

The observer (14.16) is a nonlinear system with available state vector $\hat{x}$. We assume that the control vector v can be designed to reach in finite time the observer sliding manifold $s(t,\hat{x}) = 0$. Then, from previous inequality, once $s(t,\hat{x}) = 0$, we have that

$$|s(t,x(t))| \leq L\left(\frac{\omega_1}{\alpha_1}\right)^{\frac{1}{2}} |x(0) - \hat{x}(0)| \exp(c_1 t).$$

The previous conditions (14.15) and (14.17) guarantee that the sliding motion of the state-observer coupled system on $s(t,\hat{x}) = 0$ is described by (14.18) and (14.19), independently of the nature (continuous or discontinuous) of the control v suitably designed to enforce $s(t,\hat{x}) = 0$ and actually applied.

14.4 Nonlinear Observer with Discrete-Time Measurement

Consider the system (14.9) and the sliding manifold (14.13). The control objective is to steer to zero the sliding output $s(t,x)$ by the vector v.

The state variables x are not available. The vector $y = h(x)$ is the accessible output, the measurements y_i of which are received at discrete-times t_i, with a fixed sampling period δ, according to (14.12).

The observer with discrete-time measurements for system (14.9) is designed as the following continuous-discrete-time observer defined by the following hybrid system

$$\begin{cases} \dot{\overline{x}}(t) = f(t,\overline{x},v) - Ph(\overline{x}), \quad t \in [t_i, t_{i+1}), \\ \overline{x}(t_i) = \overline{x}(t_i^-) + P\left[y_i - h\left(\overline{x}\left(t_i^-\right)\right)\right], \end{cases} \tag{14.21}$$

where $y_i = h(x_i)$ and $x_i = x(t_i)$, the sampling instants are defined as $t_{i+1} = t_i + \delta$, $t_0 = 0$, $i = 0,1,\ldots$, the constant $\delta > 0$ is the measurement sampling interval, and $P \in R^{(n+m)\times(k+m)}$ is a constant matrix, which will be specified in the sequel. The functions f, h and s are defined by (14.8), (14.11) and (14.13). The functions f, h, s are continuously differentiable in x, with f measurable in t and continuous in (x, v).

Since (14.15) holds for system (14.9) and observer (14.21), then for every $t \geq 0$, y_i, $\overline{x}$ there exists a unique solution

$$v_*(t, y_i, \overline{x}) \tag{14.22}$$

of the equation

$$s_t(t,\overline{x}) + s_x(t,\overline{x})\left[f(t,\overline{x},\cdot) - Ph(\overline{x})\right] = 0,$$

where P is as in (14.21). The mapping v_* is by definition *the observer's equivalent control* corresponding to the measurements y_i of the output y.

We consider solutions in $[0,+\infty)$ (either in the Filippov or a.e. sense) to (14.9), (14.11), (14.21) corresponding to the observer's equivalent control, i.e. solutions to

$$\dot{x} = f\left(t, x, v_*\left(t, y_i, \overline{x}\right)\right), \tag{14.23}$$

$$\dot{\overline{x}} = f\left(t, \overline{x}, v_*\left(t, y_i, \overline{x}\right)\right) - Ph\left(\overline{x}\right), \tag{14.24}$$

the existence of which is guaranteed by previous conditions (14.15) and (14.22).

Assumption 14.1. *The observer (14.21) is a perfectly known nonlinear system with available state vector* $\overline{x}$. *The control vector* v *can be designed to reach in finite time the observer sliding manifold* $s\left(t, \overline{x}\right) = 0$.

We assume that the sliding output is designed such that the state of the observer on the sliding manifold of the observer is exponentially stable.

Assumption 14.2. *For every a.e. solution* $\overline{x}$ *in* $[0,+\infty)$ *to (14.24), on* $s\left(t, \overline{x}\right) = 0$, *we assume that there exist a matrix* $M \in R^{(n+m)\times(n+m)}$, *two positive numbers* α, $\varepsilon \in R^+$ *such that the eigenvalues of* M *are between* α *and* ε, *and a positive number* $\beta \in R^+$ *such that the first time derivative of the Lyapunov function* $V\left(\overline{x}\right) = \overline{x}' M \overline{x}$ *satisfies*

$$\dot{V} \leq -\beta V \leq -\beta\alpha \left|\overline{x}\right|^2 .$$

Consider the coupled state-observer system (14.9) and (14.21), the following theorem can be stated.

Theorem 14.1. *Consider the system (14.9) and the observer (14.21), for which the previously posed conditions, particularly Assumption 14.1 and 14.2, hold.*
Assume that it is possible to find a symmetric matrix $Q \in R^{(n+m)\times(n+m)}$ *such that the eigenvalues of* Q *are between two positive numbers* μ *and* κ, *and such that the eigenvalues of the symmetric part of* $Q\left(f_x - Ph_x\right)$ *are less or equal* $-\nu$ *everywhere, being* ν *a positive number.*
Assume moreover that $\left|s_x\left(t, x\right)\right| \leq D$ *everywhere, for some constant* D.

Provided μ, κ, *and* ν *are such that* $c_1 = \left(\frac{\nu}{\kappa} - 2\frac{\kappa^2}{\nu\mu}\psi^2\right) > 0$ *and* $c_2 = \left(\beta - 2\frac{\kappa^2}{\nu\alpha}\psi^2\right) > 0$ *with* $\psi = |P|\,\psi_1$, ψ_1, P, α, *and* β *defined by (14.14), (14.21), and Assumption 14.2, then for every* $t \geq 0$

$$\left|s\left(t, x(t)\right)\right| \leq D\left[\frac{\kappa}{\mu}\left|x(0) - \overline{x}(0)\right|^2 + \frac{\varepsilon}{\mu}\left|\overline{x}(0)\right|^2\right]^{\frac{1}{2}} \exp\left(-ct\right), \tag{14.25}$$

where $2c = \min\left(c_1, c_2\right)$, *for every a.e. solution* $\left(x', \overline{x}'\right)'$ *in* $[0,+\infty)$ *to (14.23) and (14.24), such that* $s\left(0, \overline{x}(0)\right) = 0$.

Proof.

Set

$$W(t) = V_1(t) + V_2(t) \\ = [x(t) - \overline{x}(t)]' Q [x(t) - \overline{x}(t)] + \overline{x}'(t) M \overline{x}(t), \quad t \geq 0.$$

Then a.e. for $t \geq 0$,

$$\dot{W}(t) = 2[x(t) - \overline{x}(t)]' Q \\ \{f(t, x, v_*) - f(t, \overline{x}, v_*) + Ph(x) - P[h(x) - h(\overline{x})]\} + \dot{V}_2(t),$$

where

$$v_*(t) = v_*(t, y_i, \overline{x}).$$

Therefore $\dot{W}(t) =$

$$= 2[x(t) - \overline{x}(t)]' Q \left[\int_0^1 (f_x - Ph_x)\, da\right] [x(t) - \overline{x}(t)] \\ + 2[x(t) - \overline{x}(t)]' QPh(x) + \dot{V}_2(t),$$

where f_x and h_x are evaluated at $(t, \overline{\alpha}(a,t), v_*)$ and $\overline{\alpha}(a,t) = ax(t) + (1-a)\overline{x}(t)$.

We have $\dot{W}(t) =$

$$= \int_0^1 (x - \overline{x})' \left[Q(f_x - Ph_x) + (f_x - Ph_x)' Q'\right] (x - \overline{x})\, da \\ + 2(x - \overline{x})' QPh(x) + \dot{V}_2(t),$$

from which, recalling that the symmetric part of a square matrix A is by definition $\frac{A+A'}{2}$,

$$\dot{W}(t) \leq -2\int_0^1 \nu (x - \overline{x})' (x - \overline{x})\, da + 2(x - \overline{x})' QPh(x) + \dot{V}_2(t)$$

$$\leq -2\frac{\nu}{\kappa} V_1(t) + 2(x - \overline{x})' QPh(x) + \dot{V}_2(t);$$

then

$$\dot{W}(t) \leq -2bV_1(t) + 2(x - \overline{x})' QPh(x) + \dot{V}_2(t),$$

with $b = \frac{\nu}{\kappa}$.

From previous inequality, by standard computations, we obtain

$$\dot{W}(t) \leq -bV_1(t) + \frac{1}{b} h'(x) P' QPh(x) + \dot{V}_2(t),$$

then

$$\dot{W}(t) \leq -bV_1(t) + \frac{\kappa}{b} |Ph(x)|^2 + \dot{V}_2(t)$$

and by (14.14)

$$\dot{W}(t) \leq -bV_1(t) + \frac{\kappa}{b} \psi^2 |(x - \overline{x}) + \overline{x}|^2 + \dot{V}_2(t),$$

where $\psi = |P|\, \psi_1$.

This can be rewritten as

$$\dot{W}(t) \leq -bV_1(t) + 2\frac{\kappa}{b\mu}\psi^2 V_1(t) + 2\frac{\kappa}{b}\psi^2 |\overline{x}|^2 + \dot{V}_2(t)$$

and since Assumption 14.2 holds

$$\dot{W}(t) \leq -\left(b - 2\frac{\kappa}{b\mu}\psi^2\right) V_1(t) - \left(\beta - 2\frac{\kappa}{b\alpha}\psi^2\right) V_2(t).$$

If μ, κ, and ν are such that

$$c_1 = \left(\frac{\nu}{\kappa} - 2\frac{\kappa^2}{\nu\mu}\psi^2\right) > 0$$

and

$$c_2 = \left(\beta - 2\frac{\kappa^2}{\nu\alpha}\psi^2\right) > 0$$

with $\psi = |P|\,\psi_1$, ψ_1, P, α, and β defined by (14.14), (14.21), and Assumption 14.2, we obtain that

$$\dot{W}(t) \leq -2cW(t),$$

where $2c = \min\left[\left(\frac{\nu}{\kappa} - 2\frac{\kappa^2}{\nu\mu}\psi^2\right), \left(\beta - 2\frac{\kappa^2}{\nu\alpha}\psi^2\right)\right]$.

If we set

$$W_1(t) = W(t)\exp(2ct),$$

we have that $\dot{W}_1(t) \leq 0$, thus giving

$$W(t) = W(0)\exp(-2ct).$$

Then

$$\mu|x - \overline{x}|^2 + \alpha|\overline{x}|^2 \leq W(t)$$

$$\leq \left[\kappa|x(0) - \overline{x}(0)|^2 + \varepsilon|\overline{x}(0)|^2\right]\exp(-2ct).$$

We can conclude

$$|x - \overline{x}| \leq \left[\frac{\kappa}{\mu}|x(0) - \overline{x}(0)|^2 + \frac{\varepsilon}{\mu}|\overline{x}(0)|^2\right]^{\frac{1}{2}}\exp(-ct),$$

$$|s(t,x) - s(t,\overline{x})| \leq D|x - \overline{x}|$$

and therefore (14.25) since it holds $s(t,\overline{x}) = 0$.

□

14.5 Example

We consider the following variable-structure control system

$$\begin{aligned}\dot{\eta} &= \varphi(t,\eta,u) = \\ &= \begin{bmatrix} \left(3+\sin^2(\omega t)\right)(\eta_2-\eta_1) - 4\left(\eta_1+\eta_1^3\right) \\ \left(2-\sin^2(\omega t)\right)\eta_1 - 4\left(\eta_2+\eta_2^3\right) + \rho_1(u) \end{bmatrix}, \\ \dot{u} &= v, \end{aligned}$$

where $\eta \in R^2$, $u \in R$ and $\rho_1(u) = \sqrt{3}(u-1) + \sqrt{u^2+3}$; $\omega = 2\pi$; the output equation has the form $\zeta = \rho(\eta) = \eta_1$, $\zeta \in R$; the sliding manifold is designed as $\xi(\eta) = \eta_1 - \eta_2 = 0$, $\xi \in R$; it is trivial to verify that the corresponding zero-dynamics is asymptotically stable.

Let the augmented state vector $x = (\eta', u')'$; we consider the variable-structure control system

$$\begin{aligned}\dot{x} &= f(t,x,v) = A(t,x) + Bv = \\ &= \begin{bmatrix} \left(3+\sin^2(\omega t)\right)(x_2-x_1) - 4\left(x_1+x_1^3\right) \\ \left(2-\sin^2(\omega t)\right)x_1 - 4\left(x_2+x_2^3\right) + \rho_1(x_3) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v, \end{aligned} \tag{14.26}$$

where $x \in R^3$ and $\rho_1(x_3) = \sqrt{3}(x_3-1) + \sqrt{x_3^2+3}$.
The sliding manifold is designed as

$$s(x) = \dot{\xi} + \Lambda\xi = A_1(t,x) - A_2(t,x) + \Lambda(x_1 - x_2) = 0, \tag{14.27}$$

where $s \in R$, $A_1(t,x)$ and $A_2(t,x)$ are respectively the first and second element of the vector field $A(t,x)$ in (14.26), and $\Lambda \in R$ is chosen $\Lambda = 10$. On $s(t,x) = 0$ the system (14.26) is stable.
The output vector $y \in R^2$ is

$$y = \begin{bmatrix} \zeta \\ \zeta_1 \end{bmatrix} = h(x) = \begin{bmatrix} x_1 \\ \rho_1(x_3) \end{bmatrix}. \tag{14.28}$$

The state vector x is not completely available. The vector $y = h(x)$ is the accessible output, the measurements y_i of which are received at discrete-times t_i, with a fixed sampling period δ, according to (14.12).

The observer with discrete-time measurements for system (14.26) is designed as the following continuous-discrete-time observer defined by the following hybrid system

$$\begin{cases} \dot{\overline{x}}(t) = f(t,\overline{x},v) - Ph(\overline{x}), \quad t \in [t_i, t_{i+1}), \\ \overline{x}(t_i) = \overline{x}\left(t_i^-\right) + P\left[y_i - h\left(\overline{x}\left(t_i^-\right)\right)\right], \end{cases} \tag{14.29}$$

where $y_i = h(x_i)$ and $x_i = x(t_i)$, the sampling instants are defined as $t_{i+1} = t_i + \delta$, $t_0 = 0$, $i = 0, 1, \ldots$, the constant $\delta = 0.2\,sec$ is the measurement sampling interval, and $P \in R^{3\times 2}$ is a constant matrix, which will be specified in the sequel.

In particular the first equation of the observer (14.29) for system (14.26)–(14.27) takes the form

$$\begin{aligned} \dot{\overline{x}} &= f(t,\overline{x},v) - Ph(\overline{x}) = \\ &= \begin{bmatrix} \left(3+\sin^2(\omega t)\right)(\overline{x}_2 - \overline{x}_1) - 4\left(\overline{x}_1 + \overline{x}_1^3\right) \\ \left(2-\sin^2(\omega t)\right)\overline{x}_1 - 4\left(\overline{x}_2 + \overline{x}_2^3\right) + \rho_1(\overline{x}_3) \\ v \end{bmatrix} \\ &\quad - P\begin{bmatrix} \overline{x}_1 \\ \rho_1(\overline{x}_3) \end{bmatrix}, \end{aligned} \tag{14.30}$$

where $P \in R^{3\times 2}$; $x_1(0) = 1.5$, $x_2(0) = -1.5$ and $x_3(0) = -1.5$.

We have the two jacobian matrices $f_x(t,x) =$

$$\begin{bmatrix} -\left(3+\sin^2(\omega t)\right) - 4\left(1+3x_1^2\right) & \left(3+\sin^2(\omega t)\right) & 0 \\ \left(2-\sin^2(\omega t)\right) & -4\left(1+3x_2^2\right) & \left(\sqrt{3}+\frac{x_3}{\sqrt{x_3^2+3}}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$h_x(t,x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \left(\sqrt{3}+\frac{x_3}{\sqrt{x_3^2+3}}\right) \end{bmatrix}.$$

Let us choose $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 8 & 0 \\ 5 & 1 \\ 0 & 1 \end{bmatrix}$. It is easy to verify that the symmetric part of $\theta = Q(f_x - Ph_x)$, that is the matrix $\dfrac{\theta+\theta'}{2} =$

$$\begin{bmatrix} -\left(15+\sin^2(\omega t)+3x_1^2\right) & 0 & 0 \\ 0 & -4\left(1+3x_2^2\right) & 0 \\ 0 & 0 & -\left(\sqrt{3}+\frac{x_3}{\sqrt{x_3^2+3}}\right) \end{bmatrix},$$

is globally negative definite, independently of u, v and on the chosen sliding manifold. The conditions of Theorem 1 holds.

Let us consider the sliding output $s(t,\overline{x}) = A_1(t,\overline{x}) - A_2(t,\overline{x}) + \Lambda(\overline{x}_1 - \overline{x}_2)$, $\Lambda = 10$, and its first time derivative, which can be expressed as

$$\dot{s}(t) = \boldsymbol{\Phi}(t,\overline{x},y) - \left(\sqrt{3}+\frac{\overline{x}_3}{\sqrt{\overline{x}_3^2+3}}\right)v, \tag{14.31}$$

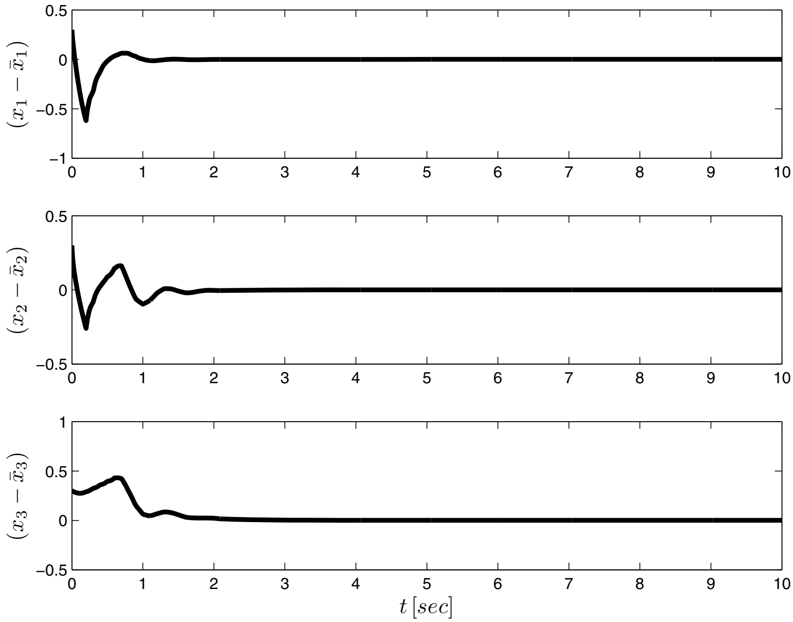

Fig. 14.1 The observation error vector $(x-\overline{x})$ converges to zero exponentially

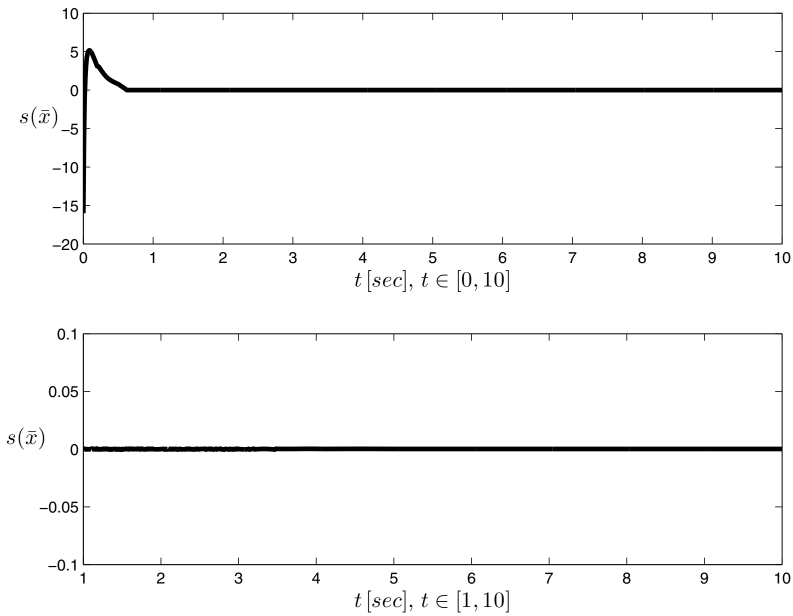

Fig. 14.2 The sliding output $s\,(\overline{x})$ converges to zero in finite time

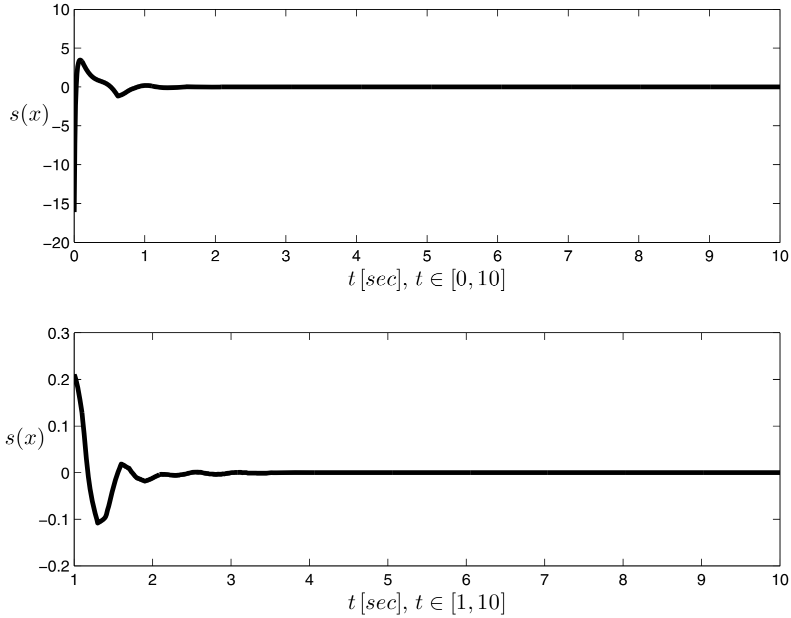

Fig. 14.3 The sliding output $s(x)$ converges to zero exponentially

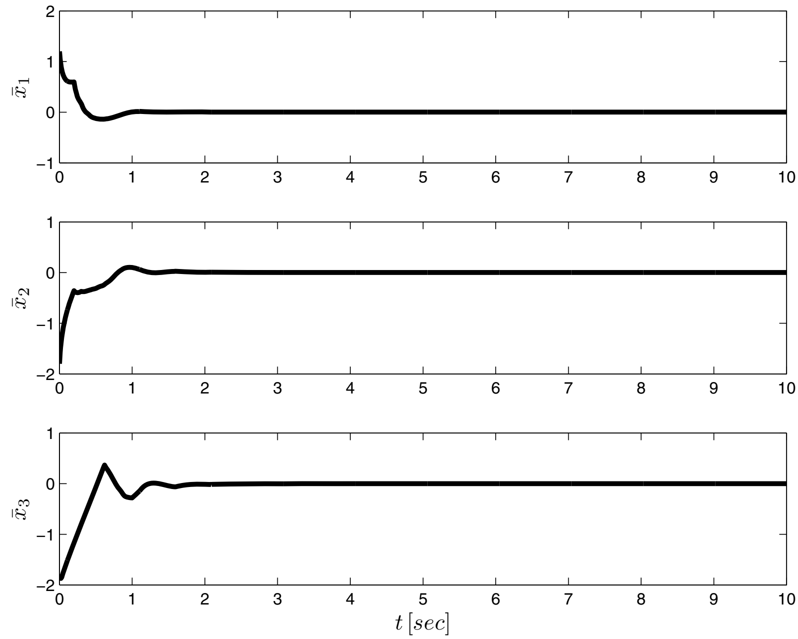

Fig. 14.4 On $s(\overline{x}) = 0$ the observer's state vector $\overline{x}$ converges to zero

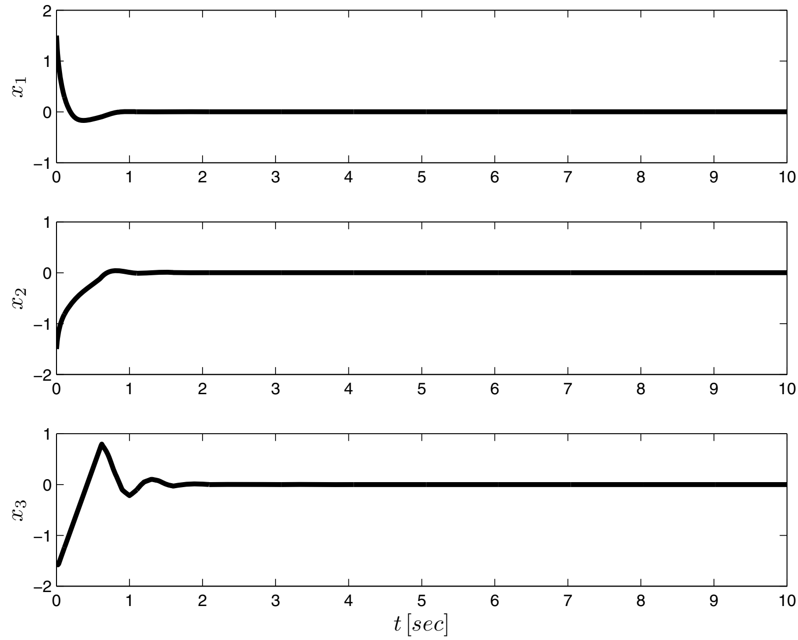

Fig. 14.5 The system's state vector x converges to zero

where the term $\Phi(t,\bar{x},y)$ is known and in the second term the control v is modulated by a known function with constant sign. It is applied the control law $v = -\tilde{K}(t,\bar{x})\,\mathrm{sign}[s(t,\bar{x})]$, where $\tilde{K}(t,\bar{x})$ is chosen to be able to dominate the drift terms in (14.31) and therefore to guarantee $s\dot{s} \leq -\varepsilon^2 |s|$, $\varepsilon \neq 0$, according to standard first order sliding mode technique.

According to the proposed method, the observation error $(x-\bar{x})$ converges to zero exponentially, Figure 14.1. The controller relies on the availability of the vector $\bar{x}$ from the nonlinear observer (14.30) with discrete-time measurement. The discontinuous control law steers to zero in finite time the sliding output $s(t,\bar{x})$, Figure 14.2. According to Theorem 1, once $s(t,\bar{x}) = 0$, the sliding output $s(t,x)$ converges to zero exponentially, Figure 14.3.

On $s(t,\bar{x}) = 0$ the observer (14.30) is stable, Figure 14.4, as well as the system (14.26), the state of which converges to zero, Figure 14.5.

14.6 Conclusions

The contribution of the chapter is in the context of output feedback under perfect plant knowledge and with discrete-time measurements.

A class of nonlinear nonaffine systems is considered when the state vector is not completely available and the output is accessible via discrete-time measurements; the use of suitably designed observers is required.

The proposed methodology introduces integrators in the input channel and combines sliding mode and Luenberger-like observers.

The procedure considers an augmented state and a new control, which is the first time derivative of the original one. The strategy attains chattering reduction, while ruling out possible ambiguous behaviors.

A full-order observer is proposed and conditions are found under which the convergence to the unique ideal solution is proven for both system and observer despite the discrete-time measurements.

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