Chapter 12
On Discontinuous Observers for Second Order Systems: Properties, Analysis and Design

Jaime A. Moreno

Abstract. Smooth observers are able to converge asymptotically to the actual value of the state, in the case where no measurement noise and no persistently acting perturbations are present. Under the same conditions continuous observers can converge in finite time. However, they are unable to converge if a perturbation/uncertainty is present. In order to achieve finite time and exact convergence in the presence of perturbations, it is necessary to use discontinuous injection terms. In this chapter, some recent developments in this direction for second order systems will be presented and the results will be illustrated by means of simple examples. It will be also shown that by including non globally Lipschitz injection terms the convergence time of the observers can be made independent of the initial condition. The restriction to the two dimensional case is due to the fact that all proofs are done by means of Lyapunov functions, that are only available for planar systems. However, this has as advantage that the treatment is mainly tutorial, and provides on the one side an easy introduction to the topic, and on the other side it presents in the simplest case the main results that are (probably) valid for the general case. We hope to be able to provide a similar treatment of the general case in the near future.

Key words: Sliding Modes, Variable Structure Control, Lyapunov Methods, Discontinuous Observers, Second Order Systems.

12.1 Introduction and Problem Statement

We will consider the class of (second order) systems that are described by the (possibly multivalued or discontinuous) differential equation

Jaime A. Moreno
Eléctrica y Computación, Instituto de Ingeniería,
Universidad Nacional Autónoma de México, 04510 Mexico D.F., Mexico
e-mail: JMorenoP@ii.unam.mx

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$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u) + x_2 + \delta_1(t, x, u) , \\ \dot{x}_2 &= f_2(x_1, x_2, u) + \delta_2(t, x, u, w) , \\ y &= x_1 \end{aligned} \tag{12.1}$$

where $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$ are the states, $u \in \mathbb{R}^m$ is a known input, $w \in \mathbb{R}^r$ represents an unknown input and $y \in \mathbb{R}$ is the measured output. f_1 is a known continuous function and f_2 corresponds to a known possibly discontinuous or multivalued function. δ_1 and δ_2 represent uncertain terms. The measured variables are x_1 and the known input u. It is assumed that system (12.1) has solutions in the sense of Filippov ⟦8⟧.

When the uncertainty $\delta_1(t, x, u) \equiv 0$ in (12.1) the observability map is

$$\mathscr{O}(x, u, w) = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ f_1(x_1, u) + x_2 \end{bmatrix},$$

which is clearly globally invertible for every known and unknown input u and w. In the absence of unknown input w system (12.1) (with $\delta_1(t, x, u) \equiv 0$ and $\delta_2(t, x, u, w = 0) \equiv 0$) is *uniformly observable for every input ⟦10–12⟧*. When there is an unknown input w the system (with $\delta_1(t, x, u) \equiv 0$) is said to be *strongly observable ⟦13, 25⟧*. In both cases it is theoretically possible to determine the unmeasured state x_2 from the measurement of x_1. Note that if the uncertain term $\delta_1(t, x, u) \neq 0$ observability is lost, and it is imposible to determine exactly the state x_2.

Many second order systems are described by equations (12.1). For example (12.1) can represent a mechanical system when $\delta_1(t, x, u) \equiv 0$ and $f_1(x_1, u) \equiv 0$, where x_1 corresponds to the (measured) position and x_2 is the velocity. u can represent a control force (or torque) and w can correspond to uncertain parameters or forces. If there exist Coulomb friction forces or in the presence of back-slash or hysteretic phenomena the functions $f_2(x, u)$ and/or $\delta_2(t, x, u, w)$ are discontinuous or multivalued.

Many other systems, although not represented by (12.1) in original coordinates, can be brought to (12.1) by a (local or global) state diffeomorphism. In particular, it is well known that smooth systems (without uncertainties) and that are uniformly observable for every input *⟦10–12⟧* can be transformed to the form (12.1).

12.1.1 Objectives

Our aim in this chapter is to propose an observer that is able to estimate the unmeasured state x_2 from the measurement of x_1. It is clear from the observability analysis of the previous paragraph that this will be possible in an exact manner only if the perturbation term $\delta_1(t, x, u) \equiv 0$ (we are only considering the case without measurement noise).

Since many existing observer algorithms can be used for this purpose, we will list the distinguishing properties of the proposed observer:

1. It is able to estimate exactly the state x_2 after a *finite time* and *robustly with respect to uncertainties/perturbations*, represented by $\delta_2(t, x, u, w)$ in (12.1), that are persistent. In order to achieve this feature it is necessary to introduce discontinuous functions in the injection terms of the observer. It is important to note that finite time convergence can be achieved without discontinuous injection terms (just with continuous but not locally Lipschitz continuous ones at zero), but only in the absence of uncertainties/perturbations. See subsections 12.2.2, 12.4.1, 12.4.2.
2. The proposed observer is able to converge in a finite time that is *independent of the initial condition* of the plant and of the observer. In order to achieve this property it is required to introduce not globally Lipschitz injection terms. See subsections 12.2.2, 12.4.4.
3. The observer is able to deal with a known function f_1 that is continuous but not necessarily Lipschitz (globally or locally). The function f_2 can be discontinuous, it does not have to be locally or globally Lipschitz in x_1 and it can grow linearly in x_2. See subsections 12.2.2, 12.2.3, 12.4.3.
4. When a bounded uncertainty/perturbation δ_1 is present, the estimation error will be bounded. The same will be true in presence of measurement noise. See subsections 12.2.2, 12.2.3, 12.4.5.
5. The design of the observer proposed in the chapter is in the spirit of the High-Gain (HG) observer: the observer constants are parametrized in terms of a single gain, that has to be set large enough to meet the convergence, robustness and convergence time required. See subsections 12.2.1, 12.2.2, 12.4.
6. All proofs are based on Lyapunov's method. The Lyapunov functions used here are of quadratic type, so that the mathematical machinery required is very similar to what is needed for linear systems. See Section 12.4.
7. The proposed method can be considered as a generalization and improvement of other observer design methods in the literature. See subsections 12.2.3, 12.3.

In order to put in perspective the first two properties, we will in the next subsection illustrate in a simple simulation example the behavior with respect to finite time convergence, robustness to uncertainties/perturbations and the convergence time with increasing initial estimation error for two typical observer design methods: High-Gain Observer ⟦7, 11, 15⟧ and (First Order) Sliding Mode Observer ⟦27⟧.

12.1.2 Simulation Example

Consider a simple (mechanical) system described by

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = w(t) \tag{12.2}$$

where x_1 is the measured position, x_2 is the (unmeasured) velocity and w is the unknown applied force. Note that w can represent unmodeled nonlinear and discontinuous phenomena as hysteresis, back-slash or Coulomb friction. The trajectories of the plant, with initial conditions $x_1(0) = 2$ and $x_2(0) = 1$, are shown in Figure

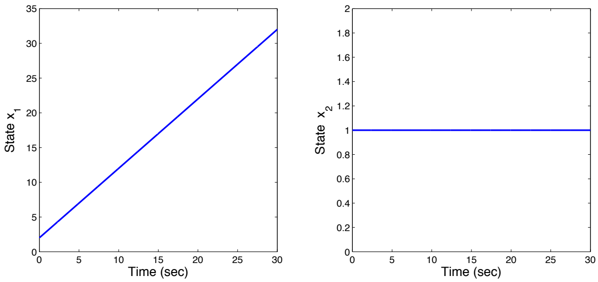

Fig. 12.1 Plant's trajectories with vanishing unknown input $w(t) = 0$

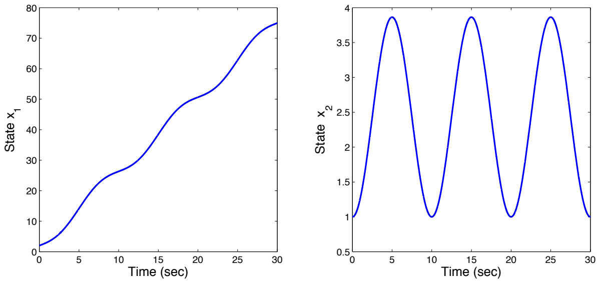

Fig. 12.2 Plant's trajectories with a periodic unknown input $w(t)$

12.1 in the case $w = 0$ and in Figure 12.2 when $w(t) = 0.9\sin(0.2\pi t)$. We will use this conditions for all the following simulations and those in Section 12.3.

12.1.2.1 A Linear Observer

The linear observer

$$\dot{\hat{x}}_1 = -l_1\gamma(\hat{x}_1 - x_1) + \hat{x}_2\,, \quad \dot{\hat{x}}_2 = -l_2\gamma^2(\hat{x}_1 - x_1)$$

with appropriately designed gains $l_1 > 0$, $l_2 > 0$ and $\gamma > 0$ is known to provide an exponentially convergent estimate of the velocity in the absence of unknown input. This can be seen in the simulation in Figure 12.3, where the gains have been selected as $l_1 = l_2 = \gamma = 1$. The initial conditions of the observer have been selected as $\hat{x}(0) = [-2, -1]$.

This is clear from the analysis of the dynamical behavior of the estimation errors $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$, given by

$$\dot{e}_1 = -l_1 e_1 + e_2\,, \quad \dot{e}_2 = -l_2 e_1 - w(t)\,.$$

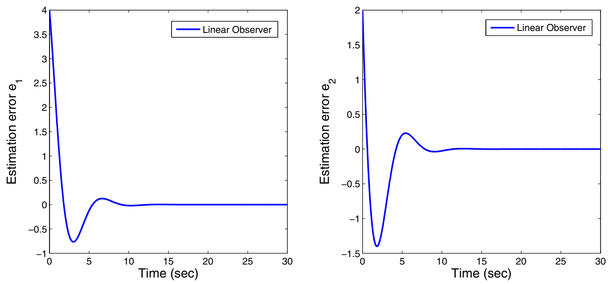

Fig. 12.3 Estimation errors of the Linear Observer without unknown input

However, it is also clear from the last equation that in the presence of a non vanishing unknown input w the estimation error will be unable to converge to zero. This is also illustrated in the simulation in Figure 12.4, with the same gains and periodic unknown input.

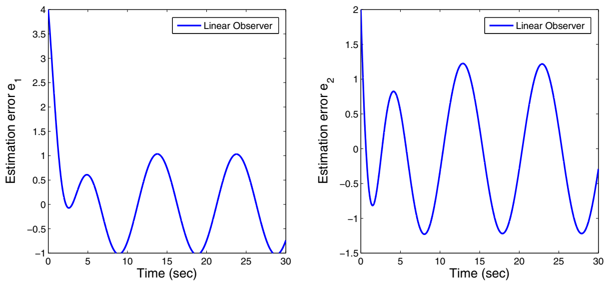

Fig. 12.4 Estimation errors of the Linear Observer with unknown input

In synthesis, the linear observer converges asymptotically (not in finite time) and is not able to converge to the true value of the unmeasured state in the presence of an unknown input. In fact, finite time convergence is impossible for any observer having locally Lipschitz continuous injection terms, and the convergence in the presence of persistent unknown inputs is also impossible for any continuous observer.

12.1.2.2 A Discontinuous First Order Sliding Mode Observer

In order to alleviate the problem, we consider a (First Order) Sliding Modes (SM) Observer [27], that has discontinuous injection terms, and has the form

$$\dot{\hat{x}}_1 = -l_1 \text{sign}(\hat{x}_1 - x_1) + \hat{x}_2, \quad \dot{\hat{x}}_2 = -l_2 \text{sign}(\hat{x}_1 - x_1).$$

However, this observer is also unable to either converge in finite time or to estimate the velocity correctly in the presence of an unknown input. This is illustrated in Figures 12.5 and 12.6, where $l_1 = l_2 = 1$, and the same initial conditions as for the linear observer, i.e. $\hat{x}(0) = [-2, -1]$, have been used.

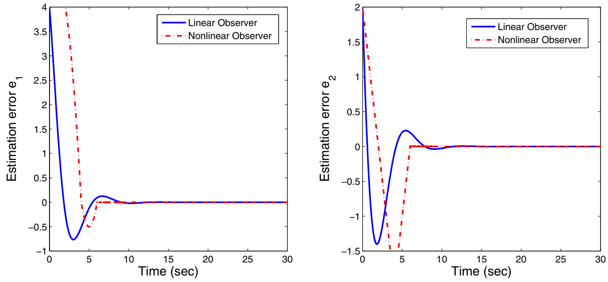

Fig. 12.5 Estimation errors for the Linear and the SM Observers without unknown input

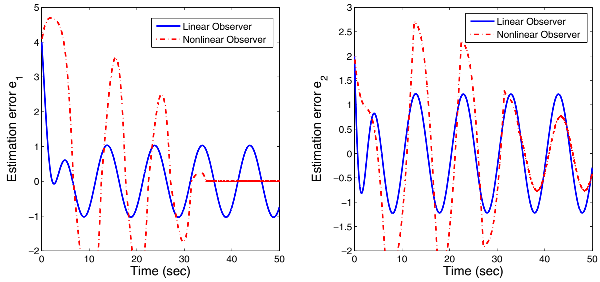

Fig. 12.6 Estimation errors of the linear and the SM Observers with unknown input

Finally, we can observe that for the linear observer (and also for the sliding mode observer) the larger the initial estimation error, the larger the convergence time (see Figure 12.7, where the initial state for the observer has been set to $\hat{x}(0) = 500[-2, -1]$, and compare with Figure 12.3). This means that it is difficult to estimate a priori the time required by the observer to provide a good estimation of the velocity.

12.2 The Proposed Observer: Design Method and Properties

In order to achieve the features for the observer, that have been listed in Subsection 12.1.1, in this section we propose a (discontinuous) observer, named *Generalized Super-Twisting Observer (GSTO)*, for the plant (12.1). We also describe how

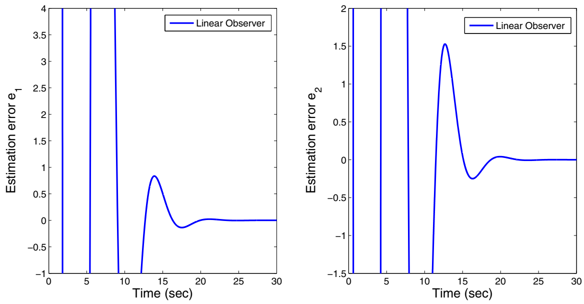

Fig. 12.7 Estimation error of the Linear Observer without UI with very large initial conditions

it is designed and discuss its properties. The proofs of the results will be given in Section 12.4.

12.2.1 The Generalized Super-Twisting Observer (GSTO)

When the plant is given in the form (12.1), the proposed GSTO has the form

$$\begin{aligned}\dot{\hat{x}}_1 &= -l_1\gamma\phi_1(e_1) + f_1(\hat{x}_1, u) + \hat{x}_2\,,\\ \dot{\hat{x}}_2 &= -l_2\gamma^2\phi_2(e_1) + f_2(\hat{x}_1, \hat{x}_2, u)\,,\end{aligned} \tag{12.3}$$

where $e_1 = \hat{x}_1 - x_1$, and $e_2 = \hat{x}_2 - x_2$ are the state estimation errors. $l_1 > 0$ and $l_2 > 0$ are positive, $\gamma > 0$ is an observer gain that has to be selected large enough to assure the convergence of the observer. The injection nonlinearities ϕ_1 and ϕ_2 are of the form

$$\phi_1(e_1) = \mu_1 |e_1|^{\frac{1}{2}} \operatorname{sign}(e_1) + \mu_2 |e_1|^q \operatorname{sign}(e_1)\ ,\ \mu_1\ ,\ \mu_2 \geq 0\,, \tag{12.4}$$

$$\phi_2(e_1) = \frac{\mu_1^2}{2}\operatorname{sign}(e_1) + \mu_1\mu_2\left(q+\frac{1}{2}\right)|e_1|^{q-\frac{1}{2}}\operatorname{sign}(e_1) + \mu_2^2|e_1|^{2q-1}\operatorname{sign}(e_1)\ , \tag{12.5}$$

where μ_1 and μ_2 are non negative constants, not both zero, and $q \geq \frac{1}{2}$ is a real number. Note that ϕ_1 and ϕ_2 are related, since $\phi_2(e_1) = \phi_1'(e_1)\phi_1(e_1)$, that they are both monotonically increasing functions of e_1 and ϕ_1 is continuous while ϕ_2 is discontinuous at $e_1 = 0$. Solutions of the observer (12.3) are understood in the sense of Filippov [[8]]. The state estimation errors (i.e. the estimation error vector $e = [e_1, e_2]^T$) satisfy the differential equation

$$\begin{aligned}\dot{e}_1 &= -l_1\gamma\phi_1(e_1) + e_2 + \rho_1(t, e, x, u)\\ \dot{e}_2 &= -l_2\gamma^2\phi_2(e_1) + \rho_2(t, e, x, u, w)\,,\end{aligned} \tag{12.6}$$

where

$$\rho_1(t, e_1, x, u) = f_1(x_1 + e_1, u) - f_1(x_1, u) - \delta_1(t, x, u) \quad (12.7)$$
$$\rho_2(t, e, x, u, w) = f_2(x_1 + e_1, x_2 + e_2, u) - f_2(x_1, x_2, u) - \delta_2(t, x, u, w) \,. \quad (12.8)$$

Each of the perturbation terms ρ_1 and ρ_2 has two components:

- $\rho_{1f} = f_1(x_1 + e_1, u) - f_1(x_1, u)$, $\rho_{2f} = f_2(x_1 + e_1, x_2 + e_2, u) - f_2(x_1, x_2, u)$ are due to the known terms of the dynamics. Note that (in the absence of noise) the term $\rho_{1f} = f_1(x_1 + e_1, u) - f_1(x_1, u)$ can be eliminated if one uses $f_1(y, u)$ instead of $f_1(\hat{x}_1, u)$ in the observer (12.3).
- $\rho_{1\delta} = -\delta_1$, $\rho_{2\delta} = -\delta_2$ due to the uncertain/perturbation terms δ_1 and δ_2.

Each term has a different influence on the behavior of the observer, and this will be discussed below.

If the dynamics of the plant is given by

$$\begin{aligned} \dot{z}_1 &= F_1(t, z_1, z_2, u, w) \,, \\ \dot{z}_2 &= F_2(t, z_1, z_2, u, w) \,, \\ y &= H(t, z_1, z_2, u) \end{aligned} \quad (12.9)$$

where $z_1 \in \mathbb{R}$, $z_2 \in \mathbb{R}$ are the states, and it can be transformed into the form (12.1) by a global diffeomorphism $x = \Phi(z)$, an observer in original coordinates can be obtained from (12.3) as

$$\begin{aligned} \frac{d}{dt}\hat{z} &= F(t, \hat{z}_1, \hat{z}_2, u, 0) - \left(\frac{\partial \Phi(\hat{z})}{\partial z}\right)^{-1} \begin{bmatrix} l_1 \gamma \phi_1(\hat{y} - y) \\ l_2 \gamma^2 \phi_2(\hat{y} - y) \end{bmatrix} \,, \\ \hat{y} &= H(t, \hat{z}_1, \hat{z}_2, u) \,. \end{aligned} \quad (12.10)$$

12.2.2 Observer Design

In this subsection we will discuss how to design the gains l_1, l_2, γ and q of the observer, so that, in the absence of perturbation $\delta_1 = 0$ (and measurement noise), the estimation error e converges in finite time to the origin, and robustly with respect to a perturbation δ_2, when δ_2 and the terms ρ_{1f} and ρ_{2f} satisfy some growth conditions (to be specified later). Moreover, the effect of the gains in the convergence time will be discussed. When a perturbation δ_1 is present, we know from the observability properties that it is impossible to obtain convergence to zero of the estimation error. In this case we show that "practical" stability is achieved.

We impose the following growth conditions on the perturbation terms (when $\delta_1 = 0$):

Property 12.1. We assume that there exist a real number $0 \leq r$, two real numbers $\frac{1}{2} \leq s_1 \leq s_2$ and non negative (real) constants α_0, α_1, α_2, β_1 and β_2 such that

$$|\rho_1| \leq \beta_1 |e_1|^{s_1} + \beta_2 |e_1|^{s_2}, \qquad (12.11)$$
$$|\rho_2| \leq \alpha_0 + \alpha_1 |e_1|^r + \alpha_2 |e_2|.$$

The next Theorem provides a procedure to design the observer (See the proof in Section 12.4):

Theorem 12.1. *Assume that* $\delta_1 = 0$. *Suppose further that the perturbation terms satisfy Property 12.1. Select the parameter* q *such that*

$$q \geq \max\left\{1, s_2, \frac{r+1}{2}\right\}.$$

Select $l_1 > 0$ *and* $l_2 > 0$ *arbitrarily, what implies that the matrix*

$$A_l = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$$

is Hurwitz. Set $\mu_1 > 0$ *and* $\mu_2 > 0$. *Under these conditions there exists a value* $\gamma_0 > 0$ *such that for every* $\gamma > \gamma_0$ *the state estimation error* $e(t)$ *converges to zero in finite time, for every initial condition and robustly with respect to the perturbations satisfying* (12.11). *Moreover, if* $q > 1$ *the convergence time is upper bounded by a constant, independent of the initial estimation error. Furthermore, if the perturbation* δ_1 *is a signal uniformly bounded for all the time, the estimation error* $e(t)$ *will be ultimately and uniformly bounded* ⟦*15 . Sect. 9.2*⟧*, i.e. there exists a positive constant* b *and a finite time* T *such that* $\|e(t)\| \leq b$ *for all* $t > T$.

12.2.3 Discussion of the Observer and Its Properties

It is important to note that from the inequalities (12.11), due to the parameter α_0, the perturbation ρ_2 does not have to vanish at the equilibrium point, i.e. when $e = 0$, and despite of this the estimation error can converge to zero. This situation appears for example when a persistent perturbation, due to an external unknown input w, is acting on the system (see (12.8)). Convergence under non vanishing unknown perturbations is imposible for continuous systems. The GSTO is able to achieve this property due to the discontinuous term in the injection function ϕ_2 (see (12.5)). This is a distinguishing feature of the GSTO, since a continuous observer, as the well-known High-Gain Observer (HGO) ⟦3, 7, 11, 15⟧ cannot achieve this property.

For different values of the parameters (μ_1, μ_2, q) some important particular cases are recovered:

(HG) The linear (or High Gain) Observer is recovered when $(\mu_1, \mu_2, q) = (0, 1, 1)$, so that $\phi_1(x_1) = x_1$, $\phi_2(x_1) = x_1$, and its properties can be derived in the same form as for Theorem 12.1. However, to avoid confusion in the redaction they are not included in the Theorem (see the related results in ⟦23⟧). The GST observer has much stronger properties, as described in the listing in paragraph 12.1.1.

(STA) The classical Super-Twisting Algorithm (STA), originally proposed in [13], is obtained by setting $(\mu_1, \mu_2, q) = (1, 0, q)$, so that $\phi_1(x_1) = |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1)$, $\phi_2(x_1) = \frac{1}{2}\operatorname{sign}(x_1)$. In this case $\phi_2(x_1)$ is a discontinuous function. The algorithm has been used for observation in mechanical systems by [6]. A comparison of (some of) the properties of the ST and the GST Observers is done in Section 12.3.

(H) A Homogeneous Algorithm is obtained if $\phi_1(x_1) = |x_1|^q \operatorname{sign}(x_1)$, $\phi_2(x_1) = q|x_1|^{2q-1} \operatorname{sign}(x_1)$, for $q \geq \frac{1}{2}$. In this case system (12.6) without perturbations is homogeneous [4, 18]. When $q = \frac{1}{2}$ the previous ST algorithm is recovered. For $\frac{1}{2} < q < 1$ the algorithm is continuous but not locally Lipschitz, and it is able to converge in finite time. However, it is not able to converge to zero when a non vanishing perturbation δ_2 is present. When $q > 1$ the algorithm is smooth, but not globally Lipschitz, and although it converges only asymptotically its convergence time is uniform in the initial conditions. An algorithm combining both terms (with $q < 1$ and $q > 1$) can be obtained in the same framework as the GSTO (see [23]), and it combines both convergence properties. This is in the spirit of the observers designed in a recursive manner by [1]. The GSTO is not recursive, and it provides the whole set of all possible gains. The structure of the injection terms is different, and so is also the Lyapunov function used for the proof. Finally, the insensitivity properties of the GSTO when there is a persistent perturbation δ_2 cannot be achieved by these continuous algorithms.

(UD) The Uniform differentiator introduced in [5] is recovered when $q = \frac{3}{2}$.

(GSTA) For $q = 1$ the Generalized Super-Twisting Algorithm (GSTA) proposed in [21] is obtained.

The design method in Theorem 12.1 resembles the standard procedure for High-Gain observers (HGO) [3, 11, 15] in which a gain has to be designed high enough to assure the convergence. The design method presented here differs from the one that can be derived from [23] (and originally proposed in [21] for the case $q = 1$) since in [21, 23] the design of the gains l_1 and l_2 requires the solution of a Riccati Algebraic Equation when there are perturbations. Here this is much simpler, since only the gain γ has to be set large enough.

The value of the gain γ required to assure the convergence depends on the growth conditions of the perturbation terms (12.11) and on the selected gains l_1 and l_2 (see (12.17) for an expression of the gain γ_0 and subsection 12.4.3). Instead of calculating this gain explicitly, what can be a difficult task, it is possible to tune the observer by increasing γ until its performance is acceptable. Note that increasing γ results also in a smaller convergence time, as can be seen from the convergence time estimations provided by (12.19) and (12.20). Moreover, the larger γ is selected the smaller will be the effect of the perturbation δ_1 on the estimation error (see subsection 12.4.5), but the estimation error cannot be better than a certain minimal bound, depending on the size of the perturbation δ_1 (see equation 12.25). This is coherent with the observability analysis for the system (see Section 12.1), that indicates that x_2 can be estimated at the best within an error of the size of the pertubation δ_1. However, as for HGO, a large gain γ can produce a large peaking in the initial transient of the observer, what is an undesirable effect (see subsection 12.4.2). Furthermore, in the presence of measurement noise, a large gain γ will amplify the effect of noise in the estimation error.

12.3 Simulation Example (Continued).

In order to illustrate some of the properties of the observer proposed in the previous section, we will perform a simulation study using the example presented in subsection 12.1.2. In particular we want to show the effect of the two terms in the injection nonlinearity ϕ_1 (12.4) (and the corresponding ones in ϕ_2 (12.5)), that are obtained setting $\mu_1 = 0$ or $\mu_2 = 0$.

12.3.1 Super-Twisting Observer

In this subsection we design an observer, derived from the GSTO (12.3) by setting $\mu_2 = 0$. One obtains the well-known Super-Twisting Algorithm (STA), that has been proposed by Levant [[14]] as a differentiator and also for control in [[9, 13]]. In [[6]] this algorithm has been used as an observer for mechanical systems, that correspond to (12.1) without the known nonlinearities f_1 and f_2. The observer is given by (with $\gamma = 1$)

$$\begin{aligned}\dot{\hat{x}}_1 &= -l_1 \left|e_1\right|^{\frac{1}{2}} \operatorname{sign}(e_1) + \hat{x}_2\,,\\ \dot{\hat{x}}_2 &= -l_2 \operatorname{sign}(e_1)\,.\end{aligned}$$

The effect of the discontinuous term in ϕ_2 is twofold:

- Convergence in finite time to zero of the estimation error. This can be seen in the simulation in Figure 12.8, where the unknown input is $w = 0$, and the same initial conditions for plant and observer as in the linear case were used.
- More importantly: The convergence in finite time to zero error is kept despite of a non vanishing unknown input $w(t) = 0.9\sin(0.2\pi t)$. This can be appreciated in Figure 12.9. This is a distinguishing feature of this observer, and it is clearly due to the discontinuity in ϕ_2.

The Super-Twisting Observer (STO) has, however, some disadvantages:

1. The convergence time grows very fast (and unboundedly) with the size of the initial estimation error. This can be observed in Figure 12.10, where the initial condition of linear and ST Observers is $\hat{x}(0) = 500[-2, -1]$, i.e. 500 times its value for Figure 12.8. One notes here that the convergence time of the STO grows faster than that for the Linear Observer.
2. When the bound of the perturbation δ_2 is larger than the gain l_2, then the STO can diverge. It is not possible to assure the desirable property that a bounded perturbation produces a bounded estimation error (see [[20, 23]] for more details).

3. The STO is not able to assure global convergence in the presence of known terms f_1 and/or f_2 in the plant's model (12.1).

The two last features are not illustrated in the simulations.

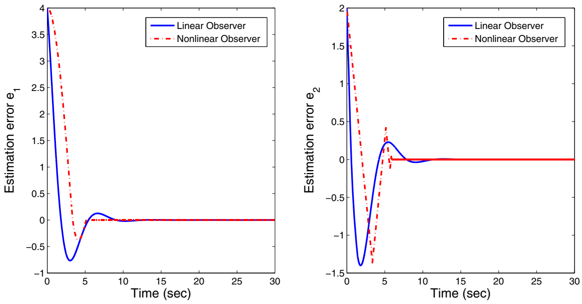

Fig. 12.8 Estimation error for the Linear and the Super-Twisting Observers without unknown input

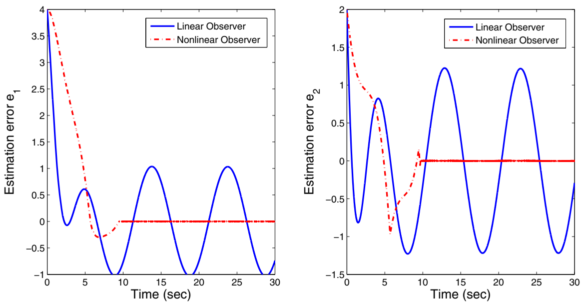

Fig. 12.9 Estimation error for the Linear and the Super-Twisting Observers with unknown input

12.3.2 Generalized Super-Twisting Observers

All these drawbacks of the STO can be solved by the GSTO (12.3) proposed here. For the simulation we have used the Generalized Super-Twisting Observer (GSTO) (12.3) with $q = \frac{3}{2}$, $\mu_1 = \mu_2 = 1$ and $\gamma = 1$. Figures 12.11 and 12.12 illustrate two features: i) The GSTO converges to zero in finite time, with or without unknown input. ii) The convergence time is basically the same for very large initial estimation error conditions. This nice feature of the GSTO is due to the introduction of a nonlinear term with a power q larger than one.

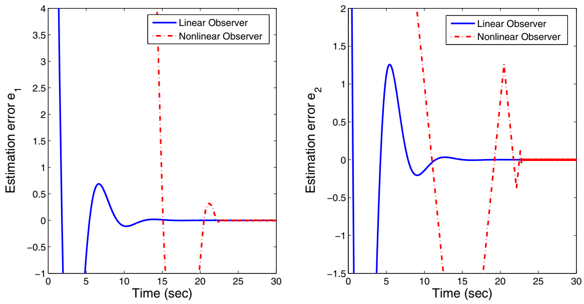

Fig. 12.10 Estimation error for the linear and the Super-Twisting Observer without UI with large initial conditions

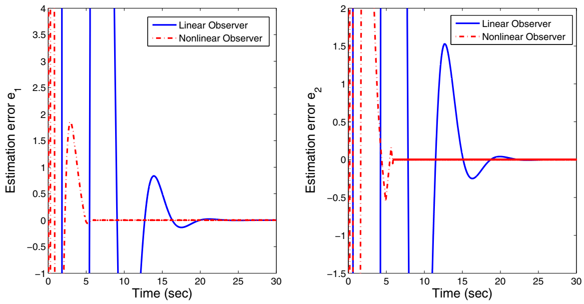

Fig. 12.11 Estimation error for the Linear and the Generalized Super-Twisting Observers without unknown input and large initial conditions, i.e. $\hat{x}(0) = 10[-2, -1]$

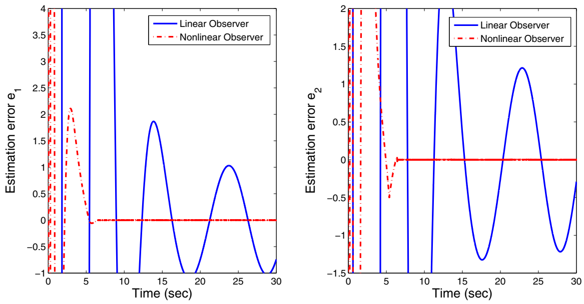

Fig. 12.12 Estimation error for the Linear and the Generalized Super-Twisting Observer with UI with very large initial conditions, i.e. $\hat{x}(0) = 500[-2, -1]$

12.4 Proofs of the Main Results

In this section we provide the proofs of the results presented previously. In particular, we provide a proof for Theorem 12.1. We proceed in several steps.

12.4.1 *The Convergence Proof Using a Quadratic Lyapunov Function*

In ⟦19–21, 23⟧ a quadratic Lyapunov function (LF), that is continuous but not Lipschitz continuous, has been introduced for the analysis of the convergence and robustness properties of Super-Twisting-like algorithms. This Lyapunov function is quadratic not in the state vector, but in a vector

$$\varepsilon^T = \varphi^T(e) = \left[\phi_1(e_1)\ ,\ e_2\right]\ , \tag{12.12}$$

where $\varphi(e)$ is a homeomorphism (i.e. it is continuous and bijective, with a continuous inverse). To take the derivative of the LF it is necessary to calculate the time derivative of ε, that is given by (where it exists)

$$\dot{\varepsilon} = \phi_1'(e_1)\begin{bmatrix} -l_1\gamma\phi_1(e_1) + e_2 + \rho_1(t,e) \\ -l_2\gamma^2\phi_1(e_1) + \frac{\rho_2(t,e)}{\phi_1'(e_1)} \end{bmatrix} = \phi_1'(e_1)\left\{(A_0 - \Gamma L_0 C_0)\varepsilon + \tilde{\rho}\right\}\ ,$$

with

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ L_0 = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},\ C_0 = \left[1,\ 0\right],\ \Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^2 \end{bmatrix}$$

and

$$\tilde{\rho}(t,\varepsilon,\cdot) = \begin{bmatrix} \rho_1(t,e,\cdot) \\ \left(\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2|e_1|^{q-\frac{1}{2}}}\right)\rho_2(t,e,\cdot) \end{bmatrix}_{e=\varphi^{-1}(\varepsilon)}\ ,$$

where we have used the error equation (12.6). Note that the characteristic polynomial of the matrix $(A_0 - \Gamma L_0 C_0)$ is

$$p(s) = \det\left(s\mathbb{I} - (A_0 - \Gamma L_0 C_0)\right) = s^2 + \gamma l_1 s + \gamma^2 l_2 = (s - \gamma\lambda_1)(s - \gamma\lambda_2)$$

where λ_1, λ_2 are the eigenvalues of the (Hurwitz) matrix $A_l = (A_0 - L_0 C_0)$, i.e matrix $(A_0 - \Gamma L_0 C_0)$ with $\gamma = 1$. This shows that the eigenvalues of $(A_0 - \Gamma L_0 C_0)$ are $\gamma\lambda_1$, $\gamma\lambda_2$, multiples of the eigenvalues of $(A_0 - L_0 C_0)$.

Similar to the by now classical proof method for High-Gain Observer ⟦3, 11, 15⟧ we introduce here a further change of variables

$$\xi = \theta\Gamma^{-1}\varepsilon = \begin{bmatrix} \frac{\theta}{\gamma}\varepsilon_1 \\ \frac{\theta}{\gamma^2}\varepsilon_2 \end{bmatrix},$$

where $\theta > 0$ is an arbitrary positive constant, and we obtain (since $\Gamma^{-1}A_0\Gamma = \gamma A_0$ and $C_0\Gamma = \gamma C_0$)

$$\dot{\xi} = \theta\Gamma^{-1}\phi_1'(e_1)\left\{(A_0 - \Gamma L_0 C_0)\frac{1}{\theta}\Gamma\xi + \tilde{\rho}\right\} = \phi_1'(e_1)\left\{\gamma(A_0 - L_0C_0)\xi + \theta\Gamma^{-1}\tilde{\rho}\right\}.$$

Using a quadratic Lyapunov function (see [[23]])

$$V(\xi) = \xi^T P \xi$$

where $P = P^T > 0$ is the unique, symmetric and positive definite solution of the Algebraic Lyapunov Equation

$$(A_0 - L_0C_0)^T P + P(A_0 - L_0C_0) = -Q,$$

for $Q = Q^T > 0$, an arbitrary positive definite and symmetric matrix. The derivative of V along the solutions of the error equation (almost everywhere) is given by

$$\begin{aligned}\dot{V} &= \phi_1'(e_1)\left\{\gamma\xi^T\left[(A_0 - L_0C_0)^T P + P(A_0 - L_0C_0)\right]\xi + 2\xi^T P\theta\Gamma^{-1}\tilde{\rho}\right\} \\ &= \phi_1'(e_1)\left\{-\gamma\xi^T Q\xi + 2\xi^T P\theta\Gamma^{-1}\tilde{\rho}\right\} \\ &\leq \phi_1'(e_1)\left\{-\gamma\lambda_{\min}\{Q\}\|\xi\|^2 + 2\|\xi\|\|P\|\left\|\theta\Gamma^{-1}\tilde{\rho}\right\|\right\}\end{aligned} \tag{12.13}$$

where $\lambda_{\min}\{Q\}$ is the minimal eigenvalue of Q, $\|\xi\|$ is the Euclidean norm of ξ and $\|P\| = \lambda_{\max}\{P\}$ is the induced (Euclidean) norm of matrix P. Recall that $\phi_1'(e_1) \geq 0$ since $\phi_1(e_1)$ is monotone increasing.

Here we will consider the case that $\delta_1 = 0$, and we will assume here that there exist some constants k_1, k_2 such that the perturbation terms satisfy the following restrictions

$$|\tilde{\rho}_1(t,e_1)| = |\rho_1(t,e_1)| \leq k_1|\phi_1(e_1)| = k_1\left(\mu_1 + \mu_2|e_1|^{q-\frac{1}{2}}\right)|e_1|^{\frac{1}{2}} \tag{12.14}$$

and

$$|\tilde{\rho}_2(t,e)| = \left(\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2|e_1|^{q-\frac{1}{2}}}\right)|\rho_2(t,e)| \leq k_2\left(\phi_1^2(e_1) + e_2^2\right)^{\frac{1}{2}}. \tag{12.15}$$

Below, in subsection 12.4.3, it will be proved that (12.14-12.15) follow from the Property 12.1. Using the relations

$$\frac{\gamma}{\theta}\xi_1 = \varepsilon_1 = \phi_1(e_1), \quad \frac{\gamma^2}{\theta}\xi_2 = \varepsilon_2 = e_2. \tag{12.16}$$

we obtain the inequalities

$$\left\|\theta\Gamma^{-1}\tilde{\rho}\right\|^2 = \frac{\theta^2}{\gamma^2}\tilde{\rho}_1^2 + \frac{\theta^2}{\gamma^4}\tilde{\rho}_2^2 \le \left(\frac{\theta^2}{\gamma^2}k_1^2 + \frac{\theta^2}{\gamma^4}k_2^2\right)\phi_1^2(e_1) + \frac{\theta^2}{\gamma^4}k_2^2 e_2^2$$
$$= \left(k_1^2 + \frac{1}{\gamma^2}k_2^2\right)\xi_1^2 + k_2^2\xi_2^2 \le k^2\|\xi\|^2$$

for

$$k^2 \ge \max\left\{k_1^2 + \frac{1}{\gamma^2}k_2^2, k_2^2\right\}.$$

This implies that

$$\dot{V} \le -\phi_1'(e_1)(\gamma\lambda_{\min}\{Q\} - 2k\lambda_{\max}\{P\})\|\xi\|^2$$

so that $\dot{V}$ is negative definite for a sufficiently large gain γ, i.e. for

$$\gamma > \gamma_0 \triangleq 2k\frac{\lambda_{\max}\{P\}}{\lambda_{\min}\{Q\}}. \tag{12.17}$$

This can always be achieved, since P and Q are independent of γ and k decreases with γ.

Recall the standard inequality for quadratic forms

$$\lambda_{\min}\{P\}\|\xi\|_2^2 \le \xi^T P\xi \le \lambda_{\max}\{P\}\|\xi\|_2^2 ,$$

where

$$\|\xi\|_2^2 = \xi_1^2 + \xi_2^2 = \frac{\theta^2}{\gamma^2}\phi_1^2(e_1) + \frac{\theta^2}{\gamma^4}e_2^2$$
$$= \frac{\theta^2}{\gamma^2}\left(\mu_1^2|e_1| + 2\mu_1\mu_2|e_1|^{q+\frac{1}{2}} + \mu_2^2|e_1|^{2q}\right) + \frac{\theta^2}{\gamma^4}e_2^2$$

is the Euclidean norm of ξ. Note that the inequality

$$|e_1|^{\frac{1}{2}} \le \frac{1}{\mu_1}|\phi_1(e_1)| \le \frac{\gamma}{\mu_1\theta}\|\xi\| \le \frac{\gamma}{\mu_1\theta\lambda_{\min}^{\frac{1}{2}}\{P\}}V^{\frac{1}{2}}(\xi) \tag{12.18}$$

is satisfied for $\mu_1 > 0$, and therefore

$$-\frac{1}{|e_1|^{\frac{1}{2}}} \le -\frac{\mu_1\theta}{\gamma\|\xi\|} \le -\frac{\mu_1\theta\lambda_{\min}^{\frac{1}{2}}\{P\}}{\gamma}V^{-\frac{1}{2}}(\xi) .$$

Since

$$\phi_1'(e_1) = \frac{1}{2}\mu_1\frac{1}{|e_1|^{\frac{1}{2}}} + q\mu_2|e_1|^{q-1}$$

it follows that

$$
\begin{aligned}
\dot{V} &\leq -\left(\gamma\lambda_{\min}\{Q\} - 2k\|P\|\right)\left(\frac{1}{2}\mu_1\frac{1}{|e_1|^{\frac{1}{2}}} + q\mu_2|e_1|^{q-1}\right)\|\xi\|^2 \\
&\leq -\left(\gamma - \frac{2k\|P\|}{\lambda_{\min}\{Q\}}\right)\lambda_{\min}\{Q\}\left(\frac{1}{2}\mu_1^2\frac{\theta}{\gamma}\|\xi\| + q\mu_2|e_1|^{q-1}\|\xi\|^2\right) \\
&\leq -\mu_1^2\frac{\theta(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{2\gamma\lambda_{\max}^{\frac{1}{2}}\{P\}}V^{\frac{1}{2}}(\xi) - \mu_2\frac{q(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}|e_1|^{q-1}V(\xi),
\end{aligned}
$$

where we have used the definition of γ_0 in (12.17), and therefore $V(\xi(t))$ is monotonically decreasing, and the origin is asymptotically stable.

12.4.2 About the Convergence Velocity of the Error

From the differential inequality satisfied by the LF, it is possible to estimate the convergence velocity of the state estimation errors. We will do this explicitly for two (simple) cases.

12.4.2.1 The Case When $\mu_1 \neq 0$ and q Is Arbitrary

From the differential inequality satisfied by the Lyapunov function it follows that

$$
\dot{V} \leq -\mu_1^2\frac{\theta(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{2\gamma\lambda_{\max}^{\frac{1}{2}}\{P\}}V^{\frac{1}{2}}(\xi).
$$

Since the solution of the differential equation

$$
\dot{v} = -\gamma_1 v^{\frac{1}{2}}, \quad v(0) = v_0 \geq 0
$$

is given by

$$
v(t) = \left(v_0^{\frac{1}{2}} - \frac{1}{2}\gamma_1 t\right)^2 \text{ if } \gamma_1 > 0,
$$

it follows from the comparison principle that

$$
V(t) \leq \left(V^{\frac{1}{2}}(\xi_0) - \frac{1}{2}\mu_1^2\frac{\theta(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{2\gamma\lambda_{\max}^{\frac{1}{2}}\{P\}}t\right)^2,
$$

before the finite convergence time. This implies that

$$
\lambda_{\min}\{P\}\|\xi(t)\|^2 \leq \xi^T(t)P\xi(t) \leq \left(\left(\xi_0^T P\xi_0\right)^{\frac{1}{2}} - \frac{1}{2}\mu_1^2\frac{\theta(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{2\gamma\lambda_{\max}^{\frac{1}{2}}\{P\}}t\right)^2,
$$

and therefore

$$\|\xi(t)\| \leq \frac{1}{\lambda_{\min}^{\frac{1}{2}}\{P\}}\left(\xi_0^T P \xi_0\right)^{\frac{1}{2}} - \frac{1}{2}\mu_1^2 \frac{\theta(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{2\gamma\lambda_{\min}^{\frac{1}{2}}\{P\}\lambda_{\max}^{\frac{1}{2}}\{P\}} t \, .$$

In original coordinates (see (12.16)), and noting that (for $\gamma \geq 1$)

$$\frac{\theta}{\gamma^2}\|\varepsilon(t)\| \leq \left\| \begin{bmatrix} \frac{\theta}{\gamma}\phi_1(e_1(t)) \\ \frac{\theta}{\gamma^2}e_2(t) \end{bmatrix} \right\| = \|\xi(t)\| \leq \frac{\theta}{\gamma}\|\varepsilon(t)\| ,$$

one obtains that

$$\|\varepsilon(t)\| \leq c_P \gamma \|\varepsilon_0\| - \mu_1^2 \frac{\gamma(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{4\lambda_{\min}^{\frac{1}{2}}\{P\}\lambda_{\max}^{\frac{1}{2}}\{P\}} t \, , \; c_P = \sqrt{\frac{\lambda_{\max}\{P\}}{\lambda_{\min}\{P\}}} \, ,$$

where c_P is the condition number of matrix P. The finite convergence time can be estimated by

$$T(\varepsilon_0) \leq \frac{4\lambda_{\max}\{P\}}{\mu_1^2(\gamma-\gamma_0)\lambda_{\min}\{Q\}}\|\varepsilon_0\| \, . \tag{12.19}$$

We notice that the convergence time can be made as small as desired by increasing the gain γ. However, the initial deviation term, given by $c_P\gamma$ grows also with the gain γ. This corresponds to the peaking phenomenon, well-known for High-Gain Observers [[15]].

12.4.2.2 The Case When $\mu_1 \neq 0$ and $q = 1$

The Lyapunov function satisfies the differential inequality

$$\dot{V} \leq -\mu_1^2 \frac{\theta(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{2\gamma\lambda_{\max}^{\frac{1}{2}}\{P\}} V^{\frac{1}{2}}(\xi) - \mu_2 \frac{(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} V(\xi) \, .$$

From the solution of the Differential Equation (See e.g. [[23]])

$$\dot{v} = -\gamma_1 v^{\frac{1}{2}} - \gamma_2 v \, , \quad v(0) = v_0 \geq 0,$$

given by

$$v^{\frac{1}{2}}(t) = \exp\left(-\frac{1}{2}\gamma_2 t\right) v_0^{\frac{1}{2}} - \frac{\gamma_1}{\gamma_2}\exp\left(-\frac{1}{2}\gamma_2 t\right)\left[\exp\left(\frac{1}{2}\gamma_2 t\right) - 1\right] ,$$

and the comparison principle [[15]] it follows that

$$V^{\frac{1}{2}}(\xi(t)) \leq \exp\left(-\frac{1}{2}\gamma_2 t\right)\left[V^{\frac{1}{2}}(\xi_0) + \frac{\gamma_1}{\gamma_2}\right] - \frac{\gamma_1}{\gamma_2} \, ,$$

or

$$V^{\frac{1}{2}}(\xi(t)) \leq \exp\left(-\frac{1}{2}\mu_2 \frac{(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} t\right) \left[V^{\frac{1}{2}}(\xi_0) + \frac{\mu_1^2 \theta \lambda_{\max}^{\frac{1}{2}}\{P\}}{2\mu_2\gamma}\right] - \frac{\mu_1^2 \theta \lambda_{\max}^{\frac{1}{2}}\{P\}}{2\mu_2\gamma} .$$

This implies that

$$\|\xi(t)\| \leq c_P \left\{\exp\left(-\frac{1}{2}\mu_2 \frac{(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} t\right) \left[\|\xi_0\| + \frac{\mu_1^2\theta}{2\mu_2\gamma}\right] - \frac{\mu_1^2\theta}{2\mu_2\gamma}\right\} .$$

In original coordinates (see (12.16)) results (for $\gamma \geq 1$)

$$\|\varepsilon(t)\| \leq \gamma c_P \left\{\exp\left(-\frac{1}{2}\mu_2 \frac{(\gamma-\gamma_0)\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} t\right) \left[\|\varepsilon_0\| + \frac{\mu_1^2}{2\mu_2}\right] - \frac{\mu_1^2}{2\mu_2}\right\} .$$

When $\mu_1 > 0$ the Finite Convergence time can be estimated as

$$T(\varepsilon_0) \leq \frac{2\lambda_{\max}\{P\}}{\mu_2(\gamma-\gamma_0)\lambda_{\min}\{Q\}} \ln\left(\frac{2\mu_2}{\mu_1^2}\|\varepsilon_0\| + 1\right) . \tag{12.20}$$

It is clear that this time can be made arbitrarily small by selecting a gain γ sufficiently large. However, the initial bound (for $t = 0$), given by

$$\|\varepsilon(0)\| \leq \gamma \frac{\lambda_{\max}^{\frac{1}{2}}\{P\}}{\lambda_{\min}^{\frac{1}{2}}\{P\}} \|\varepsilon_0\| ,$$

also grows with the gain γ, which corresponds to the Peaking Phenomenon.

12.4.3 About the Restrictions on the Perturbations

Here we show that (12.14-12.15) follow from the Property 12.1. To show (12.14) it suffices to consider the case

$$|\rho_1| \leq \beta_0 |e_1|^s, \quad \frac{1}{2} \leq s \leq q.$$

It is clear that there exists a constant k_1 such that

$$|\tilde{\rho}_1(t, e_1)| = |\rho_1(t, e_1)| \leq k_1 |\phi_1(e_1)| = k_1 \left(\mu_1 + \mu_2 |e_1|^{q-\frac{1}{2}}\right) |e_1|^{\frac{1}{2}} .$$

To show (12.15) suppose that

$$|\rho_2| \leq \alpha_0 + \alpha_1 |e_1|^r + \alpha_2 |e_2| , \quad 0 \leq r \leq 2q-1, q \geq 1 .$$

We will show that there exists a constant $k_2 > 0$ such that

$$|\tilde{\rho}_2| \leq \left(\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}}} \right) (\alpha_0 + \alpha_1 |e_1|^r + \alpha_2 |e_2|) \leq k_2 \left(\phi_1^2 (e_1) + e_2^2 \right)^{\frac{1}{2}} . \tag{12.21}$$

It is clear that the previous inequality follows if the following three are satisfied:

$$\left(\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}}} \right) \alpha_0 \leq k_{21} \left(\phi_1^2 (e_1) + e_2^2 \right)^{\frac{1}{2}} \tag{12.22}$$

$$\left(\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}}} \right) \alpha_1 |e_1|^r \leq k_{22} \left(\phi_1^2 (e_1) + e_2^2 \right)^{\frac{1}{2}} \tag{12.23}$$

$$\left(\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}}} \right) \alpha_2 |e_2| \leq k_{23} \left(\phi_1^2 (e_1) + e_2^2 \right)^{\frac{1}{2}} . \tag{12.24}$$

The inequality (12.22) is equivalent to inequality (12.23) for $r = 0$. So we prove (12.23), which is equivalent to

$$4\alpha_1^2 |e_1|^{2r+1} \leq k_{22}^2 \left(\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}} \right)^2 \left(\phi_1^2 (e_1) + e_2^2 \right) .$$

Extracting the two terms with the highest and the lowest power of e_1 in the right hand side of the previous inequality one obtains that

$$k_{22}^2 \left(\mu_1^4 |e_1| + 4q^2 \mu_2^4 |e_1|^{4q-1} \right) \leq k_{22}^2 \left(\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}} \right)^2 \left(\phi_1^2 (e_1) + e_2^2 \right) ,$$

and therefore (12.23) follows if

$$4\alpha_1^2 |e_1|^{2r+1} \leq k_{22}^2 \left(\mu_1^4 |e_1| + 4q^2 \mu_2^4 |e_1|^{4q-1} \right) .$$

Clearly there exists a constant k_{22} if $1 \leq 2r + 1 \leq 4q - 1$, or equivalently if $0 \leq r$ and $r \leq 2q - 1$. So both (12.23) and (12.22) are satisfied.

Now we show that (12.24) is fulfilled. It follows from the simple observation that for $q \geq 1$ the function $1/\phi_1'(e_1)$ is bounded by a constant, i.e.

$$\frac{2|e_1|^{\frac{1}{2}}}{\mu_1 + 2q\mu_2 |e_1|^{q-\frac{1}{2}}} \leq M .$$

We can conclude that (12.21) is satisfied.

12.4.4 On the Convergence Uniform in the Initial Conditions

When $q > 1$ it is affirmed in the Theorem 12.1 that there is a constant value $T > 0$ so that all the trajectories will converge to zero within a time lesser than T, i.e. for every initial condition. The previous paragraphs show that this property does not follow from the quadratic Lyapunov function, which is a well-known fact, as it is discussed in detail in [[23]]. In that reference a non-quadratic LF has been proposed to show the uniformity in the initial conditions property. A similar procedure can be used in our case, but the details are too long to be presented here. We refer the reader to references [[5, 23]] for those details.

12.4.5 *The Effect of a Non Vanishing Perturbation* δ_1

So far we have considered only the case when the perturbation $\delta_1 = 0$. If we take into account this term in the derivative of the LF (see (12.13)) we obtain

$$\begin{aligned}\dot{V} &= \phi_1'(e_1)\left\{-\gamma\xi^T Q\xi + 2\xi^T P\theta\Gamma^{-1}\left(\tilde{\rho}_0 + \begin{bmatrix}-1\\0\end{bmatrix}\delta_1\right)\right\}\\ &\leq \phi_1'(e_1)\left\{-\gamma\lambda_{\min}\{Q\}\|\xi\|^2 + 2\|\xi\|\|P\|\left(\left\|\theta\Gamma^{-1}\tilde{\rho}_0\right\| + \frac{\theta}{\gamma}|\delta_1|\right)\right\}\\ &\leq -\phi_1'(e_1)\left\{(\gamma-\gamma_0)\lambda_{\min}\{Q\}\|\xi\| - 2\lambda_{\max}\{P\}\frac{\theta}{\gamma}|\delta_1|\right\}\|\xi\|\end{aligned}$$

where $\tilde{\rho}_0$ represents $\tilde{\rho}$ without the term δ_1, and we have assumed that $\tilde{\rho}_0$ satisfies Property 12.1. If the gain γ is set larger than the corresponding γ_0 in (12.17) it is clear that $\dot{V} < 0$ outside a ball containing the origin, i.e. for

$$\|\xi\| > \frac{2\lambda_{\max}\{P\}\theta|\delta_1|}{\gamma(\gamma-\gamma_0)\lambda_{\min}\{Q\}}.$$

Using standard arguments [[15]] it follows that the trajectories are ultimately uniformly bounded, if δ_1 is bounded. Moreover, in original coordinates

$$\|\varepsilon(t)\| > \frac{2\lambda_{\max}\{P\}}{\lambda_{\min}\{Q\}}\frac{\gamma}{(\gamma-\gamma_0)}|\delta_1|, \tag{12.25}$$

which implies that the final bound has an infimum value that can be approached the larger the gain γ is selected. A similar proof (see also [[23]]) can be used to show the boundedness of the estimation error when the perturbation δ_2 is bounded, but its bound is larger than the one used to set the gain γ, or when there is measurement noise.

12.5 Conclusions

We have presented in this chapter a unified method to design a class of discontinuous observers for second order systems. It generalizes and improves several other known methods, as for example the High-Gain Observer, the Super-Twisting Observer and the Uniform Differentiator, enhancing their properties. We have restricted the treatment to the two dimensional case for two reasons: i) We present all proofs in a unified Lyapunov framework, which is at the moment only available for planar systems. ii) We provide a tutorial presentation that allows an easy introduction to the topic and also presents the main results in the simplest case.

Much work is still necessary to complete the program. In particular a discussion of the effect of measurement noise is crucial for estimation, that has not been included here. It is clear that increasing the gain γ will improve the performance of the observer with respect to convergence velocity and reduction of the effect of the perturbations (unknown input), but it will also increase the effect of noise, and viceversa. So a clear trade-off between estimation error due to noise and to perturbations/unknown inputs is to be considered. For High-Gain Observers (used as differentiators) this has been done recently in ⟦26⟧, where a method to optimize the gain γ has been presented. For the GSTO there are some preliminary results ⟦2⟧.

It is also clear that the extension of the results for higher order systems is an important step, that is part of ongoing research. Applications of the observers are manifold. In ⟦22⟧ they are applied for a class of chemical reactors, output feedback control is presented in a Lyapunov framework in ⟦24⟧. We hope to be able to provide a similar treatment of the general case in the near future.

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