Chapter 1
Comprehensive Approach to Sliding Mode Design and Analysis in Linear Systems

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Abstract. This chapter considers the design of reduced and integral sliding mode (SM) dynamics for state space systems. The prescribed sliding mode dynamics are selected to have either a desired spectrum or optimal behavior in the linear quadratic regulator (LQR) sense. Due to the operator representation of the system equations, separate treatment of the discrete time (DT) and the continuous time (CT) cases is not needed. Fully decentralized design of the control used to satisfy the reachability problem is possible using the obtained sliding subspaces. For the sake of straightforward analysis of the SM dynamics, a new way to obtain the SM equation, based on singular value decomposition (SVD), is also provided. Algorithms are implemented in MATLAB. Simulations illustrating the usefulness of the developed design method conclude the chapter.

1.1 Introduction

In this section we first review various methods used for the design of linear sliding subspaces for LTI MIMO systems. The next subsection explains the main motivation for this chapter: to enable straightforward synthesis of the sliding subspace, coupled with a rapid analysis of the SM motion, by constructing both new algorithms and

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simplified forms of existing algorithms that are valid for both discrete and continuous LTI systems. The ease of MATLAB implementation is a key requirement.

1.1.1 Previous Approaches

The basic idea of variable structure control with a SM, applied first to CT systems, is well established. Briefly, the control first drives the state to a sliding subspace. Once the state is in the subspace, the control enforces the state to stay in the subspace. Such motion is defined as the reduced order SM. An appropriate design provides beneficial features to the SM motion such as desired dynamics, suppression of disturbances, optimal behavior, robust stability etc. The advance of digital systems motivated the extension of this approach to DT systems, where samples of the state remain in the given sliding subspace.

CT and DT integral, or full order, SM systems were introduced later. In these systems a set of integrators is connected to the controlled system, and the sliding subspace is defined in terms of both the system states and the integrator outputs. The incentive to introduce integral systems was to remove the reaching phase by adjusting the initial values of the output of the integrators. The four basic types of SM considered here are: reduced order CT, reduced order DT, CT integral and DT integral SM.

The two main issues in the design of a control for systems with a SM are: how to determine an appropriate subspace to achieve the desired motion in the SM and how to design the control that guarantees the subspace is first attained and maintained, to ensure a lasting SM motion.

The topic of this chapter is design of the sliding subspace for SM control in LTI MIMO systems where the full state is available. Many papers have been devoted to this issue. They differ in the design aims, the type of SM, and, of course, in the approach to the design. A brief review of these papers follows. Consider first CT controllable systems with a reduced order SM represented by

$$\dot{x}(t) = Ax(t) + Bu(t), x \in \Re^n, u \in \Re^m, A \in \Re^{n\times n}, B \in \Re^{n\times m}, \operatorname{rank} B = m, \quad (1.1)$$

where the design aim is to ensure a given spectrum or optimal behaviour in the SM.

First design of a SM motion in the sliding subspace, $g(t) = Cx(t) = 0$, of order $(n-m)$ was proposed in [[1]]. In that paper a nonsingular transformation of states $\hat{x} = Mx$, $\hat{x} \in \Re^n$ is applied first to obtain the so-called regular form, where nonzero elements of the transformed matrix B are in the last m rows only. The structure of the subspace matrix C in the transformed system ensures (CB) is full rank. For sliding subspace design, the last m equations of the transformed model are dropped, and the last m states denoted as $\hat{x}_2(t)$ are expressed in terms of the first $(n-m)$ states $\hat{x}_1(t)$. This procedure creates the following pair of equation in the SM

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_{11}\hat{x}_1(t) + A_{12}\hat{x}_2(t) \\ \hat{x}_2(t) &= -C_1\hat{x}_1(t) \end{aligned} \quad (1.2)$$

where $\hat{x}_2(t)$ plays the role of a virtual control vector and the matrix C_1 is a feedback matrix. The proper choice of this matrix may fulfill two design goals. The first goal is a given spectrum in the SM and the second aim is to have optimal behavior in the SM in the LQR sense. The first aim is achieved by using a pole placement method to find C_1. In the second aim the importance of the states $\hat{x}_1(t)$ and $\hat{x}_2(t)$ for the system is defined by matrices Q and R. The required value of C_1 is calculated by a classical LQR approach. This regular form approach provides a valid design, but requires many steps to obtain the result.

In the next group of papers, the right eigenvectors are used to obtain a sliding subspace. In [[2]] and [[3]] assignment of right eigenvectors was based on projector matrices. In [[4]] eigenvector construction using Kautsky's algorithm [[5]] was employed to obtain a robust pole assignment with feedback K. Then the matrix C should satisfy two matrix equations: $CB = Z$ and $CV = 0$, where Z is a chosen nonsingular matrix, and the matrix V has as columns $(n-m)$ selected right eigenvectors of the matrix $(A-BK)$.

The main point of interest in papers [[6]], [[7]] and [[8]] is how to obtain the feedback matrix K. Matrix $(A-BK)$ in this approach should have $(n-m)$ given desired eigenvalues, and m arbitrary stable real eigenvalues. The sliding subspace is spanned by the desired right eigenvectors. In [[6]] the arbitrary eigenvalues are all different, while in [[7]] they are all equal to a real value resulting in a closed form expression of C. This idea was exploited in [[8]] for the DT system:

$$x(k+1) = Ax(k) + Bu(k). \tag{1.3}$$

Again state feedback $u = -Kx$ should provide a given spectrum which has $(n-m)$ desired eigenvalues, while the other m eigenvalues are all equal to real number λ which may not be an eigenvalue of A. Then, the sliding subspace matrix is obtained in closed form as $C = K(A-\lambda I_n)^{-1}$.

The paper by [[9]] extends the application of Ackermann's formula, proposed first for SISO systems in [[10]], to MIMO systems.

The integral SM was introduced first for SISO systems, in order to eliminate the reaching phase. An integrator is added to the system, and the switching function encompasses both the state variables and integrator output. The integrator output is adjusted so that motion starts within the sliding subspace. This idea may be applied to MIMO systems, where the number of added integrators equals the number of controls. An explicit formula for CT linear integral MIMO systems is not available, although in [[11]] a recommendation how to obtain this has been given. In [[12]], DT full order SM was considered. The main result consists of a set of formulas defining the switching functions g_k, integrator outputs z_k, and matrix E defining the discrete integrator's outputs for a matrix D satisfying $\mathrm{rank}(DB) = m$

$$\begin{aligned} g_k &= Dx_k - Dx(0) + z_k \\ z_k &= z_{k-1} + Ex_{k-1} \\ E &= -D(A - I + BK) \end{aligned} \tag{1.4}$$

Here K provides a given spectrum to the matrix $(A-BK)$.

The second subsection explains the motivation in more detail and describes the chapter content.

1.1.2 Motivation

As may be seen above, the sliding subspace design for four types of SM were treated completely separately. For a control engineer wishing to implement SM control, identifying a suitable method, understanding the theory, encoding the design, and then checking the design by simulation may not appear straightforward. The main objective of this chapter is to both make application of the SM method more popular in the wider control community by offering simple solutions to effect sliding subspace design in all four cases, and also to enable rapid analysis of the SM system using many of the control system software design tools available for LTI systems. To achieve this goal some innovations are introduced, and some new algorithms are proposed.

First, the design formulas derived for CT and DT SMs show a striking similarity. The operator notation used in ⟦5⟧ is adopted for the first time in this chapter. The application of this notation results in design formulas valid for both CT and DT systems. Last, but not the least it eliminates the need to use brackets and subscripts to represent state variables.

This chapter exploits the similarity of the SM equations to the equations of first integrals. The proposed designs have two stages. In the first stage a linear state feedback is found, such that there exists a first integral of the closed loop system with desired properties. Then the motion of the system is restricted to a subspace by applying a SM control. The subspace equation at the same time defines the first integral of closed loop system. This approach permits the required calculations to be achieved by a few programming statements.

Although the design of the reaching control is not in the scope of this chapter, a sliding subspace is determined so that a fully decentralized reaching control is possible. Each control component effectively annihilates in finite time one switching function by providing a proper sign of the switching function increment, and thus there is no need to verify stability of the reaching and sliding phase.

The last phase of each design is its verification. One of the very important issues in each system is the sensitivity to disturbances. If there are unmatched disturbances in the system, they will affect the motion in the SM and their impact must be assessed. There are many tools to handle this problem, but the model must be in state space form. A novel coordinate transform is introduced, based on SVD, which needs only a couple of MATLAB standard statements to obtain a state space model having as an input additive disturbances.

This model can be used in an iterative design of the system with unmatched additive disturbance. Since the formulas are simple, and simulations are not necessary, a repetitive design procedure can be devised so that the desired spectrum or matrices defining optimal behaviour are systematically modified in order to improve the sensitivity to specific type of disturbances.

The two design aims considered in this chapter are that of achieving a given spectrum or an optimal behavior in the LQR sense in the SM. Four brief algorithms cover the design procedure. The first two algorithms determine the SM subspace which has a given dynamic for both reduced order and integral SM respectively. Two more algorithms determine the sliding subspace for an optimal SM motion for the reduced order and integral SM.

The description of the content of the rest of the chapter is as follows. In Section 1.2 the operator notation taken from [[5]] is used to derive a common model of the reduced order and full order SM. In Section 1.3 the algorithms for achieving a given spectrum in reduced and integral SM are presented as well as the design for optimal behavior in SM. The derivation of the new state space model is the topic of Section 1.4. Section 1.5 contains examples including MATLAB codes and some simulations. The conclusion ends the chapter.

1.2 Common Model of Continuous and Discrete-Time SM Dynamics

Both CT (1.1) and DT (1.3) linear time invariant systems are described by the same equation

$$\delta x = Ax + Bu. \tag{1.5}$$

In CT systems the state $x \in \Re^n$ is a function of continuous time t denoted as $x(t)$, where δ represents the differential operator d/dt. In DT systems, the state x is a function of discrete time k, denoted as x_k while δ denotes the forward shift operator. The control is represented by $u \in \Re^m$, while $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are system matrices. In the following it will be assumed that $\mathrm{rank}(B) = m$, while (A,B) is a controllable pair. The derivations of the SM equations of reduced order and full order follow.

It is assumed that in the reduced order case, a suitable switching type of control places the SM in a subspace defined by the sliding subspace matrix C. SM motion is defined by a pair of equations appended to (1.5): The equation (1.6) says that the state at a moment belongs to a subspace, and (1.7) states that in its further motion it stays in the same subspace:

$$g = Cx = 0, \tag{1.6}$$

$$\delta g = C\delta x = C(Ax + Bu) = 0. \tag{1.7}$$

Here g denotes the so-called vector switching function. The matrix (CB) must have full rank m, and therefore matrix C must also have full rank as well. The solution of (1.7) for u leads to the equivalent control u_{eq} given by

$$u_{eq} = -(CB)^{-1}CAx = -Kx \tag{1.8}$$

where $K \in \Re^{m \times n}$ is referred to as the equivalent feedback matrix. In the equivalent system (1.9), the real switching control is replaced by the equivalent control. Eqs. (1.9) and (1.10) together define a linear system of order $(n-m)$

$$\delta x = (A - BK)x, \tag{1.9}$$

$$Cx = 0. \tag{1.10}$$

To obtain an integral SM, m integrators having outputs represented by the vector σ are connected to the system (1.5). This makes an extended system of order $(n+m)$. The integrator inputs are equal to Ex. As proposed in ⟦12⟧, the SM occurs in the subspace given by

$$g = Dx + \sigma = 0 \tag{1.11}$$

provided DB is a nonsingular matrix.

Since the models of CT and DT integrators differ in the adopted operator notation, the application of the equivalent control concept leads to two different expressions for CT and DT full order SM. CT integrators connected to system (1.5) are represented by

$$\delta\sigma = Ex, \tag{1.12}$$

while DT integrators are modeled as

$$\delta\sigma = Ex + \sigma \tag{1.13}$$

The introduction of a qualifier q, which is equal to 1 for DT systems and to 0 for CT systems, results in an unique triple of equations describing the SM motion of the extended system

$$\delta x = (A - BK)x, \tag{1.14}$$

$$\sigma + Dx = 0, \tag{1.15}$$

$$K = (DB)^{-1}(D(A - qI) + E). \tag{1.16}$$

An important advantage of the full order system is that the state variables in the SM do not depend on the integrator output. Thus, the equation defining the state variables (1.14) represents at the same time the SM motion. This equation mimics a LTI system with a linear feedback. Therefore, the design of the SM may use numerous methods developed for control design of linear systems. The application of the switching control to obtain a system that behaves as an ordinary linear system with linear feedback may sppear strange. However, recall that the sliding mode system completely rejects matched disturbances and may reduce the effect of unmatched disturbances, as illustrated by simulations.This is a significant advantage when compared to linear feedback.

1.3 The Design of SM Subspace

The first part of this section explains the underlying principle of design: the similarity of the SM equations to the equations of first integrals. It specifies the two design goals considered in this chapter: to obtain a desired dynamics, and to have an optimal behaviour in the LQR sense. The second part deals with reduced order SM design, and the third part with full order SM design.

1.3.1 Design Aims and Philosophy

Notice that the pair of equations (1.9) and (1.10) taken together may have the following interpretation: at the subspace (1.10), the motion of the dynamic system (1.9) is described by a first integral. A first integral is a solution of lower order of a higher order system. This feature is the starting point of the proposed design. The design idea in this chapter is to create first a linear autonomous system having some desired properties by choosing a feedback matrix K. This phase of the design may use the rich body of methods developed for linear systems. Then a first integral is to be found such that the desired property of the autonomous system is maintained. Finally the sliding subspace matrix where first integral motion occurs is calculated.

This idea is applied to the two most common design goals: achieving a desired spectrum or determining optimal behavior in the LQR sense. The determination of a feedback matrix K and matrix C for reduced order SM control, and matrices K, D and E for the case of integral SM control, whereby a desired spectrum is achieved will be presented in the next two subsections.

1.3.2 SM with Given Spectrum for Reduced and Full Order Dynamics

Reduced order design will be considered first. In principle, the obtained method may be extended easily to full order systems by treating full order SM as a reduced order SM of the extended system. However, the structure of the extended system enables a more simple design. Accordingly, separate formulas are used for the reduced and full order cases.

If a given set of $(n-m)$ eigenvalues defines a given spectrum in the SM, then the same eigenvalues must be a subset of the equivalent system spectrum. The remaining m eigenvalues should not appear in the SM. If the initial state belongs to the subspace spanned by the $(n-m)$ right eigenvectors corresponding to the desired $(n-m)$ eigenvalues, this aim will be accomplished. These eigenvectors must be mutually linearly independent to ensure the required system order of $(n-m)$ in the SM. The m eigenvalues that will be eliminated may have arbitrary values, except that complex eigenvalues must appear only in conjugate pairs. That idea was implemented in [[6]]. In the approach adopted here, the required calculation is significantly simplified due to suitable choice of removed eigenvalues.

The first step of the algorithm is the calculation of the feedback matrix K fulfilling two requirements:

a) the matrix $(A-BK)$ is simple, that is, all its eigenvectors are linearly independent;

b) the matrix $(A-BK)$ has m zero eigenvalues and $(n-m)$ given eigenvalues.
The obvious consequence of a) and b) is that $\mathrm{rank}(A-BK)=(n-m)$. The calculation of K is a pole placement problem, having no unique solution in MIMO systems. Kautsky's algorithm [[5]] suits the purpose well since it provides a simple matrix $(A-BK)$. Its only impediment is that multiplicity of eigenvalues can not exceed $\mathrm{rank}(B)$.

The next step is the calculation of the matrix C. It will be shown that this matrix can be obtained as the solution of the following pair of matrix equations:

$$C(A-BK)=0, \tag{1.17}$$

$$CB=I. \tag{1.18}$$

Eq. (1.18) ensures CB is full rank. Since $CB=I$, (1.17) may be replaced by $CA=K$.

The existence of solution will be considered first. The number of independent scalar equations represented by matrix equation (1.17) is at most $m(n-m)$, due to the reduced rank of $(A-BK)$, while the number of independent scalar equations represented by the matrix equation (1.18) is m^2. The total number of independent scalar equations does not exceed $m(n-m)+m^2=nm$. Since the number of unknown elements of the matrix C is nm, there exists a solution for C. To obtain the solution, (1.17) and (1.18) are rewritten as

$$C\begin{bmatrix}A & B\end{bmatrix}=\begin{bmatrix}K & I_m\end{bmatrix}. \tag{1.19}$$

The solution of this equation comes out by help of application of matrix pseudo-inverse denoted by the superscript +:

$$C=\begin{bmatrix}K & I_m\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}^{+}. \tag{1.20}$$

Next, it is shown that this solution provides the desired spectrum. Eq. (1.9) represents an autonomous system with eigenvalues defined by the requirement b). Eq. (1.10) indicates that the rows of C are spanned by m left eigenvectors corresponding to zero eigenvalues. Therefore the components of the solution space of $(A-BK)$ corresponding to zero eigenvalues are not present in the SM motion. It follows that the SM dynamics is defined by the remaining $(n-m)$ eigenvalues that constitute the desired spectrum.

The adoption of the condition $CB=I$ has two benefits. In integral SM this guarantees that the influence of unmatched disturbances will not increase in the SM [[13]]. The other advantage is related to reaching control design. Although this design is not the topic of this chapter, the significance of the adopted condition should be pointed out. Basically, the prevalent reasoning in the design of the reaching control is to ensure that the amplitudes of the scalar switching functions decrease all the time with a nonzero speed until all scalar switching functions became equal to zero. In CT systems this is achieved by maintaining opposing signs of the scalar switching function and its derivative. In DT systems the sliding subspace may be attained in one step, therefore the future value of the switching functions should be zero. Thus, for both systems the design is based on δg. Combining equations (1.5), (1.17) and (1.18) one obtains the following equation in the reaching stage

$$\delta g = Kx + u = -u_{eq} + u. \tag{1.21}$$

This expression allows one switching function and one component of the control to be paired. This is a useful feature in the overall control design, since the coordination of controls is an important issue in reaching phase design. However, some other design aim may impose a broader restriction as $CB = Z$, as in [[2]]. Then Z may be put into (1.20) instead of I.

The dynamics in integral SM depends on the matrices E and D. A very important advantage of integral systems is that there are more free parameters in design, since the number of available parameters in the matrices D and E is two times larger than in reduced order systems. Since DB must be nonsingular, the condition $DB = I_m$ is set. From this equation, one obtains D using the pseudo-inverse as $D = B^+$. Such a choice of D provides the equation of the reaching mode in the same form as (1.21). Eqs. (1.14) and (1.16) indicate that the SM eigenvalues are determined by the matrix $(A - BK)$. The needed matrix K may be found by using a pole placement procedure. From equation (1.16), one obtains

$$E = (K - B^+(A - qI_n)). \tag{1.22}$$

It should be mentioned that the expression for the matrix E obtained in [[11]] for discrete systems differs from (1.22) in the value of the matrix D. In [[12]] D is taken as B^{T}, and thus there is no advantage of easy reaching control design.

1.3.3 SM with Optimal Behavior for Reduced and Full Order Dynamics

As already explained, two different formulas for integral and reduced order SM will be used, and reduced order design will be considered first. Let the feedback matrix K in (1.9) be obtained by using LQR design. The closed loop system is then an asymptotically stable system. The matrix $Q \geq 0$ defines priorities regarding the states in the system dynamic, while the matrix $R > 0$ affects the amplitude of control Kx. If the value of control amplitude is not crucial, and system (1.5) is minimum phase, the cheap control concept with zero matrix R may be applied. However, MATLAB does not currently have a routine for this case. Therefore, it is practical to assign a nonzero value to the matrix R. A reasonable compromise is to take R in a sense 'smaller' than Q. The resulting matrix $(A - BK)$ is fully represented by its right eigenvectors which form the columns of the matrix V, and the corresponding eigenvalues form the diagonal elements of the diagonal matrix D_g. If all the eigenvalues are different, this matrix is simple. If this condition is not fulfilled for a given choice of R and Q, modifying these matrices may produce a simple matrix.

Trajectories starting in a subspace spanned by any subset of $(n-m)$ right eigenvectors of the closed-loop V_s will be optimal for given A, B, Q and R. If the SM of the system (1.5) is made in such a subspace, all its trajectories will be optimal in the same sense, as well. The obvious choice to construct C is to use the method given in [[8]].

The matrix C defining this subspace must satisfy the equation

$$CV_s = 0 \tag{1.23}$$

where V_s is a matrix that has as columns the selected $(n-m)$ right eigenvectors. The matrix equation (1.23) is equivalent to $m(n-m)$ independent scalar equations. A unique solution may be obtained by adding the equation (1.18). The matrix C is then defined by the matrix equation

$$C\begin{bmatrix} V_s & B \end{bmatrix} = \begin{bmatrix} 0_{m,n-m} & I_m \end{bmatrix}. \tag{1.24}$$

A reasonable way to choose particular right eigenvectors in V_s is based on their corresponding eigenvalues. The question is which eigenvalues/eigenvectors should be dropped. Some obvious options are to delete the dominant eigenvalues often met in LQR design to eliminate overshoot, or to eliminate some real eigenvalues close to zero to improve the stability margin. The calculation of C is eased by creating the selection matrix S, obtained from D_g by replacing desired $(n-m)$ eigenvalues by 1, and other m eigenvalues with 0. The matrix product VS then has as its columns $(n-m)$ selected eigenvectors and m zero columns. Using this matrix, (1.24) may be rewritten as

$$C\begin{bmatrix} VS & B \end{bmatrix} = \begin{bmatrix} 0_{m,n} & I_m \end{bmatrix}. \tag{1.25}$$

The unique solution of this system is

$$C = \begin{bmatrix} 0_{m,n} & I_m \end{bmatrix}\begin{bmatrix} VS & B \end{bmatrix}^{+}. \tag{1.26}$$

The paper by [[1]] presented a way to obtain an optimal motion in the SM with $R = 0$ using the regular form of the system (1.1). The approach applied here is different: it only guarantees that the behavior in the SM is not worse than the behavior of an LQR optimal system. Its advantage is the possibility to remove undesired components from the solution space. Also, the condition $CB = I$, convenient for the design of switching control, is not built into the regular form approach.

Application of integral SM results in a system that mimics a linear LQR system, and also suppresses matched disturbances. In integral SM, the integrator outputs σ need not be optimized. The feedback matrix K in (1.9) or (1.16) may be obtained by the LQR technique. The calculation of D and E is the same as in the design for desired spectrum.

1.4 SM State Space Equations

A dynamical system of n-th order in the reduced order SM with $\mathrm{rank}(C) = m$ behaves as a dynamical system of $n-m$th order. If the design is performed using a state space model of the plant, the motion is described by the equivalent system dynamic equation of n-th order and the sliding subspace algebraic equation of m-th order. This is not a state space model in the strict sense, and numerous tools available in control software cannot be applied for its analysis.

The first way to obtain a state space model for a system of reduced order was proposed in [[1]] and described in the Introduction. The controlled system state space model is first transformed into regular form. The action of any unmatched disturbances was not considered in this approach. Since their presence may deteriorate system performance, it is very important to include them as an input in the system model.

In further derivations of a new way to obtain such a state space model, a matrix denoted by 0 has all its elements equal to zero, and I is the unity matrix. The matrix dimensions are explicitly stated only if necessary. The reduced SM model will be treated first.

Consider a system given in operator notation having additive disturbances

$$\delta x = Ax + Bu + Gf. \tag{1.27}$$

The system output is

$$y = Hx. \tag{1.28}$$

$G \in \Re^{n\times r}$ is the constant disturbance matrix and $f \in \Re^{r}$ is the disturbance vector. $H \in \Re^{q\times n}$ is the output matrix. Sliding occurs at $g = Cx = 0$, where $g \in \Re^{m}$ is the vector switching function. In the sliding subspace, the equivalent system equation is

$$\delta x = P(Ax + Gf), \tag{1.29}$$

where P denotes the projector matrix:

$$P = I - B(CB)^{-1}C. \tag{1.30}$$

It is easy to verify that the matrix C satisfies the equation $CP = 0$. Since C is of full rank, $\mathrm{rank}(P) = n - m$. Represent P with its SVD:

$$P = USV^{\mathrm{T}}. \tag{1.31}$$

Since P is a quadratic matrix of rank equal to $(n-m)$, $S \in \Re^{n\times n}$ has the following form:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}, S_1 \in \Re^{(n-m)\times(n-m)} \tag{1.32}$$

The elements of the diagonal matrix S_1 are $(n-m)$ nonzero singular values of P, and thus S_1 is invertible. Since S has the last m rows equal to zero, and the last m columns equal to zero, for any matrix T of appropriate dimensions, the following two properties hold:

(A) product ST has its last m rows equal to zero;

(B) product TS has the last m columns equal to zero.

Matrices U and V are square and unitary, that is their inverse is equal to their transpose. Therefore, U^{T} qualifies as a transform matrix. Introduce new state variables

$$\begin{bmatrix} z \\ w \end{bmatrix} = U^{\mathrm{T}}x, \text{ i.e} x = U \begin{bmatrix} z \\ w \end{bmatrix}, \text{ where} z \in \Re^{n-m}, w \in \Re^{m}. \tag{1.33}$$

The equation of the equivalent system after some matrix manipulations becomes

$$\begin{bmatrix} \delta z \\ \delta w \end{bmatrix} = SV^{\mathrm{T}}AU \begin{bmatrix} z \\ w \end{bmatrix} + SV^{\mathrm{T}}Gf. \tag{1.34}$$

Due to the property (A) matrices $SV^{\mathrm{T}}AU$ and $SV^{\mathrm{T}}G$ may be broken into blocs as follows

$$\begin{bmatrix} \delta z \\ \delta w \end{bmatrix} = \begin{bmatrix} A_z & A_w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} + \begin{bmatrix} G_z \\ 0 \end{bmatrix} f. \tag{1.35}$$

Since $\delta w = 0$, in the SM the value of the vector w is constant. Now consider the sliding subspace equation:

$$Cx = CU \begin{bmatrix} z \\ w \end{bmatrix} = 0. \tag{1.36}$$

To find CU, start from $CP = CUSV^{\mathrm{T}}$. Since V is invertible, this reduces to $CUS = 0$. Break also C and U into blocs to obtain

$$CUS = \begin{bmatrix} C_z & C_w \end{bmatrix} \begin{bmatrix} U_{zz} & U_{zw} \\ U_{wz} & U_{ww} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (C_z U_{zz} + C_w U_{wz}) S_1 & 0 \end{bmatrix} = 0. \tag{1.37}$$

Since S_1 is an invertible matrix, $C_z U_{zz} + C_w U_{wz} = 0$. Now consider the sliding subspace equation (1.36)

$$\begin{bmatrix} C_z & C_w \end{bmatrix} \begin{bmatrix} U_{zz} & U_{zw} \\ U_{wz} & U_{ww} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} C_z U_{zz} + C_w U_{wz} & C_z U_{zw} + C_w U_{ww} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = 0. \tag{1.38}$$

Taking into consideration $C_z U_{zz} + C_w U_{wz} = 0$ it follows that

$$\begin{bmatrix} 0 & C_z U_{zw} + C_w U_{ww} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = (C_z U_{zw} + C_w U_{ww}) w = 0. \tag{1.39}$$

Matrix C has full rank, and U is nonsingular, therefore CU has also full rank. It follows that the quadratic matrix $(C_zU_{zw} + C_wU_{ww})$ is nonsingular, and the above equation reduces to $w = 0$. Therefore in the SM the constant value of w is zero. Hence, the SM is defined by

$$\delta z = A_z z + G_z f, \tag{1.40}$$

$$w = 0. \tag{1.41}$$

In these equations, the coordinates w and z are completely decoupled. Eq. (1.41) defines the sliding subspace, and (1.40) is a state space equation of $(n-m)$ order, describing the SM motion under the influence of additive disturbances f seen as an input. The output of the system (1.41) is

$$y = Hx = HU \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} H_z & H_w \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} H_z & H_w \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix} = H_z z. \tag{1.42}$$

Now consider the full order SM motion of system (1.27). The extended system equations with additive disturbances are

$$\delta x = Ax + Bu + Gf, \tag{1.43}$$

$$\delta \sigma = Ex + q\sigma. \tag{1.44}$$

The sliding subspace is defined by the equation $\delta\sigma = 0$. If D guarantees that DB is full rank, the equivalent control is

$$u_{eq} = -(DB)^{-1}((D(A-qI)+E)x + DGf). \tag{1.45}$$

The equation of the SM is

$$\delta x = A_{sm}x + G_{sm}f, \tag{1.46}$$

where $A_{sm} = A - B(DB)^{-1}(D(A-qI)+E)$ and $G_{sm} = G - B(DB)^{-1}DG$. Since the output in the SM is $y = Hx$, $H_{sm} = H$.

This completes the construction of the state space model.

1.5 Examples and Simulations

This section contains four design examples and two simulations. The examples illustrate design of the four basic types of SM given in Section 1.3 and generation of SM state space models. The aim of the first simulation is to compare the trajectories of a standard LQR design, and SM design in a system supplied with matched disturbance, with a trajectory having a reaching phase and a sliding phase. The second simulation uses integral SM. Since there is no reaching phase, the trajectory is obtained using a SM state space model. The simulations use designs performed in corresponding examples of the presented theory. MATLAB implementations are supplied to help the reader to apply the results of this chapter in practice. All examples and simulations use the same linearized CT model of an aircraft given in [[14]] to illustrate the application of the proposed design. The system matrices are:

$$A=\begin{bmatrix}0 & 0 & 1.132 & 0 & -1\\ 0 & -0.0638 & -0.1712 & 0 & 0.0705\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0.0468 & 0 & -0.8556 & -1.013\\ 0 & -0.2908 & 0 & 1.0532 & -0.6059\end{bmatrix},\ B=\begin{bmatrix}0 & 0 & 0\\ -0.12 & 1 & 0\\ 0 & 0 & 0\\ 4.419 & 0 & -1.656\\ 1.575 & 0 & -0.0732\end{bmatrix}.$$

The data for the DT version of the same model with $T = 0.1$ s, is obtained using the MATLAB command `[Ad,Bd]=c2d(A,B,T)`.

Reduced order CT SM design: A matrix C with desired poles $P = \begin{bmatrix}-1.2+1.6\imath & -1.2-1.6\imath\end{bmatrix}$ is obtained using a single statement according to (1.20)

```
C=[place(A,B,[P zeros(1,rank(B))]) eye(rank(B))]*pinv([A B])
```

$$C=\begin{bmatrix}-0.6332 & 0 & 0.5038 & -0.0318 & 0.7243\\ -0.076 & 1 & 0.0605 & -0.0038 & 0.0869\\ -2.5386 & 0 & 0.3831 & -0.6851 & 1.9222\end{bmatrix}.$$

The state space model for reduced order CT SM: System matrices are A and B. Unmatched disturbance and output matrices are respectively $G = \begin{bmatrix}1 & 0 & 0 & 0 & -1\end{bmatrix}^{\mathrm{T}}$ and $H = \begin{bmatrix}1 & 1 & 0 & 0 & 0\end{bmatrix}$. The following commands generate the reduced order SM model:

```
[n,m]=size(B)
[n,r]=size(G)
[q,n]=size(H)
P=eye(n)-B*(B*C)^-1*C                          %Projector matrix
[U,S,V]=svd(P)                                 %SVD of P
SYSeq=ss(P*A,P*G,H,zeros(q,r))                 %Equivalent system
SYStr=ss2ss(SYSeq,U')                          %Transformed system
SYSsm=modred(SYStr,[n-m+1:n],'truncate')       %Trunc. of m variab.
[Asm,Gsm,Hsm,Dsm]=ssdata(SYSsm)                %System matrices:
```

$$A_{sm}=\begin{bmatrix}-1.015 & -1.967\\ 1.319 & -1.385\end{bmatrix},\ G_{sm}=\begin{bmatrix}-1.56\\ -1.12\end{bmatrix},\ H_{sm}=\begin{bmatrix}-0.281 & -0.5\end{bmatrix},\ D_{sm}=0.$$

Full order DT SM design: Let the desired poles be

$$P_d=[e^{\mathrm{T}(-1.2-1.6\imath)}\ e^{\mathrm{T}(-1.2+1.6\imath)}\ e^{-5\mathrm{T}}\ e^{-7\mathrm{T}}\ e^{-10\mathrm{T}}].$$

Based on $DB = I_m$ and (1.20), the following statements give D_d and E_d:

```
Dd=pinv(Bd)
Ed=place(Ad,Bd,Pd)-pinv(Bd)*(Ad-eye(size(Ad)))
```

$$D_d = \begin{bmatrix} -0.3785 & 0.1076 & -0.0303 & -0.7205 & 7.4516 \\ -0.0461 & 10.0397 & -0.0039 & -0.0829 & 0.8585 \\ -1.0064 & 0.2988 & -0.3973 & -8.1575 & 19.4651 \end{bmatrix},$$

$$E_d = \begin{bmatrix} -3.4769 & -0.075 & 0.0898 & -0.5184 & 3.7368 \\ 0.2787 & 6.2641 & 0.5838 & 0.2326 & 0.1806 \\ -13.3457 & -0.2739 & -5.2685 & -5.0862 & 10.7894 \end{bmatrix}.$$

Reduced order CT optimal SM design: The chosen optimization matrices are $R = I_3$, $Q = I_5$. The statement `K=lqr(A,B,Q,R)` gives

$$K = \begin{bmatrix} -0.3004 & -0.0965 & 0.4406 & 0.8111 & 0.6052 \\ 0.2756 & 1.0507 & 0.7903 & 0.1981 & -0.5369 \\ -0.9131 & -0.2837 & -2.679 & -0.8635 & 1.5199 \end{bmatrix}. \tag{1.47}$$

Matrices V and D_g are obtained with `[V,Dg]=eigs(A-B*K)`, where D_g is a diagonal matrix: $\text{diag}(D_g) = (-4.4786, -1.8769, -1.0004, -0.5621 - 0.4812\imath, -0.5621 + 0.4812\imath)$. There are no multiple poles and matrix $(A - BK)$ is simple. The three values nearest to the origin are discarded. The selection of matrices S and VS are then

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, VS = \begin{bmatrix} -0.0592 & 0.0416 & 0 & 0 & 0 \\ 0.028 & -0.0251 & 0 & 0 & 0 \\ 0.2174 & -0.4298 & 0 & 0 & 0 \\ -0.9733 & 0.8036 & 0 & 0 & 0 \\ -0.0191 & -0.4089 & 0 & 0 & 0 \end{bmatrix}.$$

The following statements, based on (1.26), give the matrix C

```
[n,m]=size(B)
C=[zeros(m,n) eye(m)]*pinv([V*S B])
```

$$C = \begin{bmatrix} -3.8119 & 0 & -1.118 & -0.032 & 0.7248 \\ -0.0555 & 1 & -0.1537 & -0.0038 & 0.087 \\ -1.2012 & 0 & -3.2446 & -0.6894 & 1.9341 \end{bmatrix}. \tag{1.48}$$

The matrix of the corresponding equivalent system $(A - B(CB)^{-1}CA)$ has eigenvalues (-4.4789,-1.8695,0,0,0), which confirms the design procedure.

Full order DT optimal SM design: Since MATLAB calculates a DT optimal feedback matrix based on the continuous-time system model, to avoid solving a discrete Riccati equation the following is used `Kd=lqrd(A,B,Q,R,0.1)`

$$K_d = \begin{bmatrix} -0.2936 & -0.0958 & 0.1956 & 0.6245 & 0.5381 \\ 0.2481 & 0.9981 & 0.7221 & 0.1825 & -0.4901 \\ -0.8712 & -0.2846 & -0.5302 & -0.779 & 1.4928 \end{bmatrix}.$$

By further application of (1.20)

```
Dd=pinv(Bd)
Ed=Kd-pinv(Bd)*(Ad-eye(size(Ad)))
```

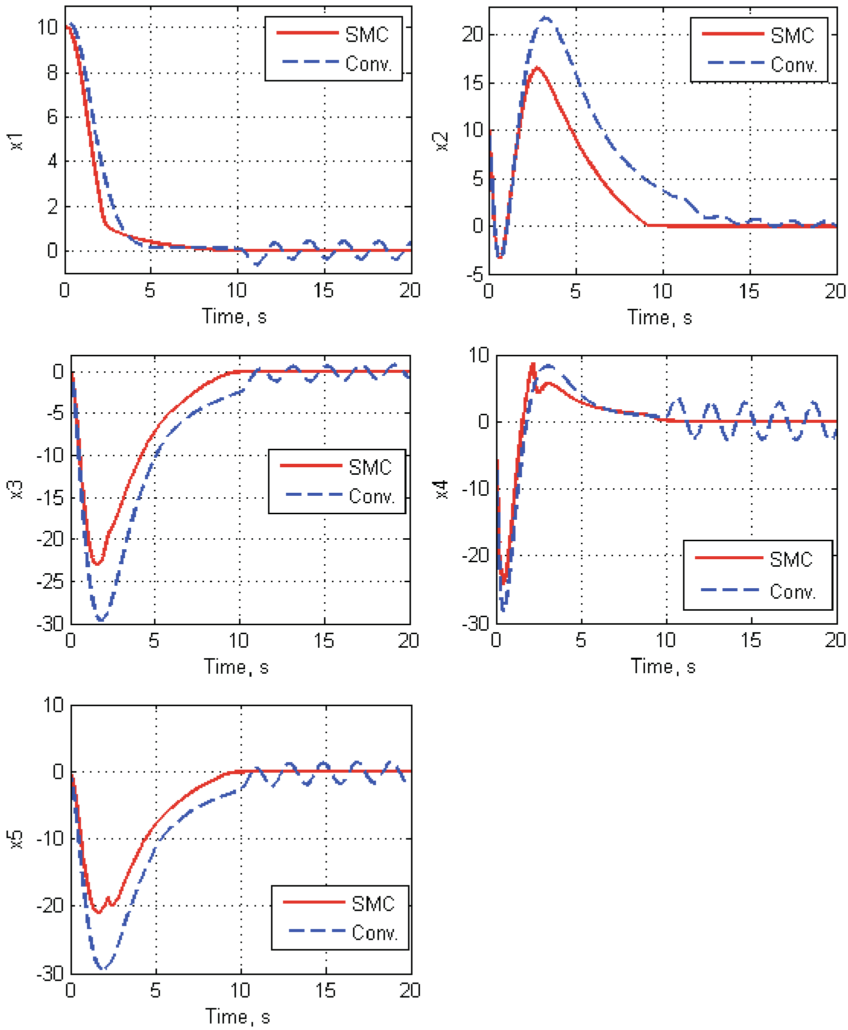

Fig. 1.1 Comparative study of conventional LQR control (Conv.) and CT SM LQR control (SMC). Initial state: $x(0) = [10, 10, 0, 0, 0]^{\mathrm{T}}$. Disturbance $f(t) = 5h(t-10)\sin(\pi t)$ is applied only at the first input. Control is $u = u_{LQR} - [6\mathrm{sgn}(g_1), \mathrm{sgn}(g_2), \mathrm{sgn}(g_3)]^{\mathrm{T}}$.

$$D_d = \begin{bmatrix} -0.3785 & 0.1075 & -0.0304 & -0.7224 & 7.4277 \\ -0.0459 & 10.0447 & -0.0039 & -0.0831 & 0.8557 \\ -1.0074 & 0.2996 & -0.3997 & -8.2068 & 19.5076 \end{bmatrix},$$

$$E_d = \begin{bmatrix} -0.2936 & 0.1165 & 0.2383 & -0.1611 & 0.9072 \\ 0.2481 & 1.0873 & 0.8985 & 0.0969 & -0.5164 \\ -0.8712 & 0.3101 & -2.4162 & -3.3601 & 1.8656 \end{bmatrix}.$$

Reduced order CT optimal SM simulation: Trajectories of a conventional (1.47) (Fig.1.1, dashed lines) and SM version (1.48) (Fig.1.1, solid lines) of an LQR optimal system show that, due to the elimination of slow solution space components, the trajectories supplied with SM control generally reach steady state in a shorter time. Moreover, in these systems outer disturbances are completely suppressed in the sliding mode phase, while the states of systems with conventional control oscillate around the origin.

Full order CT optimal SM simulation: This simulation, Fig.1.2, has only the sliding mode part of the trajectories, since the reaching phase in these systems may be eliminated. Moreover SIMULINK is not used, as in the above simulation, but the SM state space model derived in Section 1.4. The disturbances are matched.

System matrices are A, B, along with $H = [2,0,0,0,0]$, $G = [0\ -0.36\ 0\ 13.257\ 4.725]^{\mathrm{T}}$, $Q = I_5$, $R = 0.2I_3$. The disturbance is $f(t) = 5\sin(5t)$, the initial state $x_0 = [2,0,0,0,0]^{\mathrm{T}}$. The program follows:

```
[n,r]=size(G)
[k,n]=size(H)
[n,m]=size(B)
Kopt=lqr(A,B,Q,R)
SYSopt=ss(A-B*Kopt,G,zeros(k,r))     %Conventional optimal system
D=pinv(B)                            %Parameters of integral SM
E=Kopt-D*A
q=0                                  %Continuous system qualifier
Asm=A-B*(D*B)^-1*(D*(A-q*eye(n))+E) %State space model matrices
Gsm=G-B*(D*B)^-1*D*G
SYSsm=ss(Asm,Gsm,H,zeros(k,r))
```

The response in Fig.1.2 is obtained using:

```
[u,t]=gensig('sin',1,8,0.001)
x0=[2 0 0 0 0]'
lsim(SYSopt,SYSsm,5*u,t,x0)
```

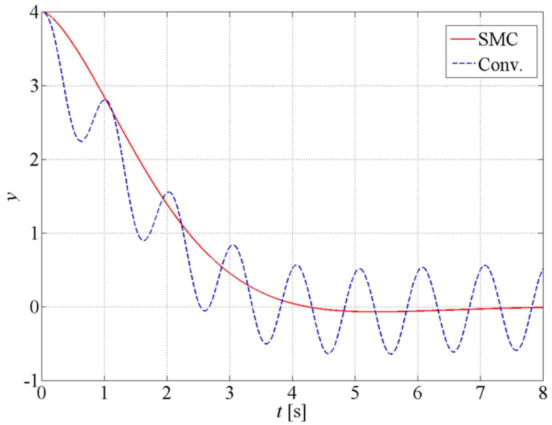

Fig. 1.2 Output trajectories of conventional LQR (dashed line) and integral SM optimal system (solid line). SM system rejects completely a strong disturbance.

1.6 Conclusions

This chapter presents four simple algorithms fulfilling the following goals: the desired spectrum is achieved in both reduced order and integral SM, approximately optimal behavior is determined in reduced order SM and optimal behavior in integral SM. Due to the convenient operator representation of the controlled system, these algorithms work both for CT and DT systems. The switching control design is easy, since scalar controls and scalar switching functions are paired, and thus the issue of coordination of controls in the reaching phase is resolved. A new and simple way to obtain the state space equation of the SM system using SVD is provided. The challenging problem is to extend the proposed approach based on the similarity of SM equations and first integral equations to other issues such as robustness and attenuation of unmatched disturbances, descriptor systems and possibly to some classes of nonlinear systems.

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