Chapter 3
First Approach to the Green Functions: The Rayleigh Method

3.1 Introduction

In Sect. 1.3 we have considered a solution method for the scattering problems which was already used by Rayleigh to solve plane wave scattering on periodic gratings. Starting point was the approximation (1.21) of the scattered field by a finite series expansion in terms of any appropriate expansion functions. This approximation was assumed to hold everywhere in the outer region Γ_+. The unknown expansion coefficients in this approximation have been determined afterwards by use of the additional boundary conditions at the scatterer surface $\partial\Gamma$ (see (1.29), for example, if the outer Dirichlet problem is considered). Hereby it was assumed that the primary incident field is the known quantity. But if we look closer on (1.21) and (1.29) we can recognize two different sets of expansion functions. In (1.21), we have the expansion functions $|\,\varphi_{i,\tau}(k_0, \mathbf{x})\rangle$ defined everywhere in the outer region Γ_+. On the other hand, concerning equation (1.29) we used the expansion functions $|\,\varphi_{i,\tau}(k_0, \mathbf{x})\rangle_{\partial\Gamma}$ defined exclusively at the scatterer surface $\partial\Gamma$. The expansion coefficients resulting from the corresponding continuity conditions at the scatterer surface should have their meaning only for the approximation of the scattered field at this surface, as one might expect. Therefore, we have to clarify whether these expansion coefficients can be used also in approximation (1.21) or not. Before going into the details of deriving the Green function related to the outer Dirichlet problem of the Helmholtz and vector-wave equation we will clarify this aspect first.

The Green functions form the decisive link between the differential equation and integral equation formulation of the scattering problems, as we pointed out already in Sect. 2.5. It is exactly this property which will be used in this chapter to approximate the Green functions of the scattering problems by finite series expansions. The procedure is very similar to what is known for the corresponding approximation of Dirac's delta distribution. What does it mean? Let us consider for simplicity the one-dimensional problem with real-valued functions $f(x)$ defined in the interval $x \in [a, b]$. We can expand the function $f(x)$ in terms of some appropriate expansion

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functions $\varphi_i(x)$ according to

$$f^{(N)}(x) = \sum_{i=0}^{N} a_i \cdot \varphi_i(x). \tag{3.1}$$

Let us assume furthermore that these expansion functions form an orthogonal basis in the functional space of square-integrable functions defined on the interval $x \in [a, b]$ like the sine and cosine functions of the conventional Fourier method, for example. Then, if minimizing the mean square error, we can calculate the expansion coefficients of the approximation $f^{(N)}$ from

$$a_i = \int_a^b \varphi_i(x) \cdot f(x)\, dx. \tag{3.2}$$

Due to the assumed orthogonality of the expansion functions these expansion coefficients are final, i.e., they are independent of the truncation parameter N in the finite series (3.1) (see the remarks in Sect. 2.2.2 concerning the best approximation). Inserting (3.2) into (3.1) results in

$$f^{(N)}(x_j) = \sum_{i=0}^{N} \int_a^b \varphi_i(x_j) \cdot \varphi_i(x) \cdot f(x)\, dx \tag{3.3}$$

as the approximation of $f(x)$ in point $x_j \in [a, b]$. Dirac's delta distribution, on the other hand, is defined according to

$$\int_a^b \delta(x - x_j) \cdot f(x)\, dx := f(x_j). \tag{3.4}$$

Replacing $f(x_j)$ on the right hand side of this definition by expression (3.3), and after interchanging integration and summation (this can be done since we restrict our consideration to a finite series expansion) we get finally

$$\delta^{(N)}(x - x_j) = \sum_{i=0}^{N} \varphi_i(x) \cdot \varphi_i(x_j) \tag{3.5}$$

as the corresponding approximation of Dirac's delta distribution. That is the strategy we want to pursue in this chapter to get an approximation of the Green functions related to the scattering problems. Beside the clarification of the interrelation between the expansion coefficients in the approximations of the scattered field at the scatterer surface and in the outer region Γ_+ there is just one additional complication resulting from the assumed non-orthogonality but linearly independence of the relevant expansion functions. But before going into the details of this analysis we will start with the introduction and approximation of the scalar delta distribution at the scatterer surface. This quantity becomes of interest when proving the continuity condition which has to be fulfilled by the Green function related to the outer Dirichlet problem, and in conjunction with the Lippmann-Schwinger equations we will derive at the end of Chap. 5.

3.2 The Scalar Delta Distribution at the Scatterer Surface

For a moment we keep staying at the scatterer surface $\partial\Gamma$ and consider sufficiently smooth functions $f(\mathbf{x})$ defined on this surface. In close analogy to (3.4) in the above considered one-dimensional case we define the scalar delta distribution $\delta_{\partial\Gamma}(\mathbf{x}' - \mathbf{x})$ at the scatterer surface according to

$$\oint_{\partial\Gamma} \delta_{\partial\Gamma}(\mathbf{x}' - \mathbf{x}) \cdot f(\mathbf{x}')\, dS(\mathbf{x}') \;:=\; f(\mathbf{x}); \quad \mathbf{x} \in \partial\Gamma. \tag{3.6}$$

Now, we look back to the results of Sect. 2.2 and use expansion (2.1) as an approximation of the function $f(\mathbf{x})$ in terms of the linearly independent functions $\varphi_i(\mathbf{x})$ (which must not necessarily be the radiating solutions of Helmholtz's equation):

$$f^{(N)}(\mathbf{x}) \;=\; \sum_{i=0}^{N} b_i^{(N)} \cdot \varphi_i(\mathbf{x}); \quad \mathbf{x} \in \partial\Gamma. \tag{3.7}$$

The expansion coefficients are calculated from Eq. (2.14). Taking the definition (1.34) of the scalar product into account we obtain

$$b_i^{(N)} \;=\; \sum_{j=0}^{N} [A_{\partial\Gamma}^{(g,\varphi)^{-1}}]_{i,j} \cdot \oint_{\partial\Gamma} g_j^*(\mathbf{x}') \cdot f(\mathbf{x}')\, dS(\mathbf{x}') \tag{3.8}$$

for the coefficients. Inserting these coefficients into (3.7), interchanging summation and integration, and comparing the resulting expression with (3.6) where we have again replaced on the right hand side of this latter equation $f(\mathbf{x})$ by its approximation (3.7) we end up with

$$\delta_{\partial\Gamma}^{(N)}(\mathbf{x}' - \mathbf{x}) \;=\; \sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi)^{-1}}\right]_{i,j} \cdot \varphi_i(\mathbf{x}) \cdot g_j^*(\mathbf{x}'); \quad \mathbf{x}, \mathbf{x}' \in \partial\Gamma \tag{3.9}$$

as an appropriate approximation of the scalar surface delta distribution. That is,

$$\oint_{\partial\Gamma} \delta_{\partial\Gamma}^{(N)}(\mathbf{x}' - \mathbf{x}) \cdot f(\mathbf{x}')\, dS(\mathbf{x}') \;:=\; f^{(N)}(\mathbf{x}); \quad \mathbf{x} \in \partial\Gamma \tag{3.10}$$

was used as the corresponding definition of this approximation.

3.3 The Scalar Green Functions Related to the Helmholtz Equation

3.3.1 The Outer Dirichlet Problem

Now, we will answer the question if the expansion coefficients $a_i^{(N)}$ of approximation (1.21) of the scattered field $u_s(\mathbf{x})$ in the outer region Γ_+ are identical to the expansion coefficients $\alpha_i^{(N)}$ of approximation

$$u_s^{(N)}(\mathbf{x}) \;=\; \sum_{i=0}^{N} \alpha_i^{(N)} \cdot \varphi_i(k_0, \mathbf{x}); \quad \mathbf{x} \in \partial\Gamma \tag{3.11}$$

which holds for the scattered field at the scatterer surface. It is moreover assumed that in both approximations the radiating solutions (2.58) of the scalar Helmholtz equation are used as expansion functions (we would like to recall that in the scalar case considered in the following analysis we can neglect the τ-summation in (1.21)!). As frequently done, we employ again Green's theorem (2.239) but now with the two quantities $\Psi(\mathbf{x}) = u_s(\mathbf{x})$ and $\Phi(\mathbf{x}) = \varphi_i(k_0, \mathbf{x})$. Since u_s as well as φ_i are solutions of the homogeneous Helmholtz equation we get

$$\oint_{\partial\Gamma} \left[u_s(\mathbf{x}) \cdot \frac{\partial \varphi_i(k_0, \mathbf{x})}{\partial \hat{n}_-} \;-\; \varphi_i(k_0, \mathbf{x}) \cdot \frac{\partial u_s(\mathbf{x})}{\partial \hat{n}_-} \right] dS(\mathbf{x}) \;=\; 0. \tag{3.12}$$

$u_s(\mathbf{x})$ in the boundary integral on the right-hand side is next replaced by its approximation (3.11) valid at the scatterer surface. For its normal derivative $\partial u_s(\mathbf{x})/\partial \hat{n}_-$, on the other hand, we have to use approximation (1.21) instead. This is essential since according to the definition (1.7) we have to apply the ∇-operation on $u_s^{(N)}$ first. But this operation must be performed inside Γ_+ and can not be restricted to the scatterer surface. Only then we can apply the scalar multiplication with the normal vector $\hat{n}_-$. Thus, we have

$$\sum_{j=0}^{N} \oint_{\partial\Gamma} \left[\alpha_j^{(N)} \cdot \varphi_j(k_0, \mathbf{x}) \cdot \frac{\partial \varphi_i(k_0, \mathbf{x})}{\partial \hat{n}_-} \right.$$
$$\left. - a_j^{(N)} \cdot \varphi_i(k_0, \mathbf{x}) \cdot \frac{\partial \varphi_j(k_0, \mathbf{x})}{\partial \hat{n}_-} \right] dS(\mathbf{x}) \;=\; 0. \tag{3.13}$$

Furthermore, if employing Green's theorem (2.239) with the two quantities $\Psi(\mathbf{x}) = \varphi_i(k_0, \mathbf{x})$ and $\Phi(\mathbf{x}) = \varphi_j(k_0, \mathbf{x})$ it is easy to show that one gets the relation

$$\oint_{\partial\Gamma} \varphi_j(k_0, \mathbf{x}) \cdot \frac{\partial \varphi_i(k_0, \mathbf{x})}{\partial \hat{n}_-} \, dS(\mathbf{x}) = \oint_{\partial\Gamma} \varphi_i(k_0, \mathbf{x}) \cdot \frac{\partial \varphi_j(k_0, \mathbf{x})}{\partial \hat{n}_-} \, dS(\mathbf{x}). \tag{3.14}$$

Together with (3.13) this results into

$$\sum_{j=0}^{N} \left[\alpha_j^{(N)} - a_j^{(N)} \right] \cdot \oint_{\partial\Gamma} \varphi_j(\mathbf{x}) \cdot \frac{\partial \varphi_i(k_0, \mathbf{x})}{\partial \hat{n}_-} \, dS(\mathbf{x}) = 0 \tag{3.15}$$

which holds for all $i = 0, \ldots, N$. The boundary integral on the right hand side defines the elements $m_{i,j}$ of a matrix $\mathbf{M}$. If this matrix is invertible then we have for the expansion coefficients in the square brackets

$$a_j^{(N)} = \alpha_j^{(N)}. \tag{3.16}$$

That is, we can indeed use the coefficients resulting from the application of the continuity condition (1.29) in approximation (1.21). Since every linear combination of the radiating solutions of Helmholtz's equation is a radiating solution itself relation (3.16) holds also if we use the new expansion functions

$$\xi_i(k_0, \mathbf{x}) = \sum_{k=0}^{N} c_{i,k} \cdot \varphi_k(k_0, \mathbf{x}); \quad i = 0, \ldots, N \tag{3.17}$$

instead of the old functions φ_i, and if the resulting matrix $\mathbf{M}$ is again invertible. The invertability of the infinite-dimensional matrix $\mathbf{M}$ (i.e., for the matrix with elements $m_{i,j}$; $i, j = 0, \ldots, N$, and N tends to infinity) can be ensured mathematically only if the radiating solutions form a basis in the functional space $L_2(\partial\Gamma)$. The invertibility of the finite-dimensional matrix, on the other hand, requires only the linearly independence of the expansion functions as it was already discussed in Sect. 2.3.3. But if we have a scatterer geometry whose surface is not of C^2 or Liapounoff type then we can prove the invertibility of the finite-dimensional matrix only by a numerical procedure according to our pragmatic point of view on the convergence behaviour formulated in Sect. 2.3.1. This situation belongs to most of the realistic problems. But it should be also emphasized at this point that the usage of approximation (1.21) for the scattered field everywhere outside the scatterer is not without controversy and strongly related to the problem of the Rayleigh hypothesis we will discuss throughout Chap. 6.

Now we are prepared to approximate the Green function G_{Γ_+} belonging to the outer Dirichlet problem. The cooking recipe for this undertaking is as follows:

First step:

We expand the primary incident field u_{inc} at the scatterer surface according to (2.1) into a series in terms of the functions $\psi_i(k_0, \mathbf{x})$. These could be the regular eigensolutions of Helmholtz's equation, for example, but not necessarily. The corresponding expansion coefficients $b_i^{(N)}$ are then calculated according to (2.14) and (2.15).

Second step:

Utilizing the transformation character (2.18) of the T-matrix (2.19) we accomplish the transition from the expansion functions $\psi_i(k_0, \mathbf{x})$ to the radiating solutions $\varphi_i(k_0, \mathbf{x})$ in the approximation of the primary incident field at the scatterer surface. The new expansion coefficients $a_i^{(N)}$ are calculated according to (2.23) from the old coefficients $b_i^{(N)}$. Due to the identical definitions (2.15) and (2.22) of both matrices $\mathbf{A}_{\partial\Gamma}^{(g,\psi_0)}$ and $\mathbf{B}_{\partial\Gamma}^{(g,\psi_0)}$ which appear in equations (2.14) and (2.19) we get from the continuity condition (1.29) and from the above derived relation (3.16) the following approximation for the scattered field u_s in the outer region Γ_+:

$$u_s^{(N)}(\mathbf{x}) \;=\; -\sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi_0)^{-1}}\right]_{i,j} \cdot < g_j \,|\, u_{inc} >_{\partial\Gamma} \cdot \varphi_i(k_0, \mathbf{x}); \qquad \mathbf{x} \in \Gamma_+. \tag{3.18}$$

Let us write the scalar product $< g_j \,|\, u_{inc} >_{\partial\Gamma}$ in this equation more explicitly. With definition (1.34) we obtain

$$u_s^{(N)}(\mathbf{x}) = -\sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi_0)^{-1}}\right]_{i,j} \cdot \oint_{\partial\Gamma} g_j^*(\mathbf{x}') \cdot u_{inc}(\mathbf{x}')\, dS(\mathbf{x}') \cdot \varphi_i(k_0, \mathbf{x}); \qquad \mathbf{x} \in \Gamma_+,\; \mathbf{x}' \in \partial\Gamma, \tag{3.19}$$

or, if interchanging summation and integration, and after a few rearrangements:

$$u_s^{(N)}(\mathbf{x}) = -\oint_{\partial\Gamma} \sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi_0)^{-1}}\right]_{i,j} \cdot \varphi_i(k_0, \mathbf{x}) \cdot g_j^*(\mathbf{x}') \cdot u_{inc}(\mathbf{x}')\, dS(\mathbf{x}'); \qquad \mathbf{x} \in \Gamma_+,\; \mathbf{x}' \in \partial\Gamma. \tag{3.20}$$

The weighting functions $g_j(\mathbf{x})$ are not yet specified, and we will keep this situation to allow for a certain degree of flexibility in the ongoing analysis. But if they are specified, then, with expression (3.20) we have already found an approximate solution of the outer Dirichlet problem! If we choose the same set of functions as weighting and expansion functions, for example, the primary incident field at the scatterer surface is approximated in terms of the best approximation discussed in Sect. 2.2 Replacing $u_{inc}(\mathbf{x}')$ in the boundary integral on the right hand side of (3.20) by the approximation (1.28) we obtain once again the relation (1.42) between the expansion coefficients of the scattered and primary incident field. If the primary incident field is given by the plane wave (2.102), for example, and if we use in approximation (1.28) the regular eigensolutions of the Helmholtz equation, then the expansion coefficients b_i of the plane wave are given by (2.108). But deriving the Green function of the outer Dirichlet problem (or, better, its approximation) from approximation (3.20) requires some additional steps.

Third step:

We use Green's theorem (2.239) with the two functions $\Psi(\mathbf{x}) = u_s(\mathbf{x})$ and $\Phi(\mathbf{x}) = G_{\Gamma_+}(\mathbf{x}, \mathbf{x}')$. u_s is a solution of the homogeneous Helmholtz equation whereas G_{Γ_+} is a solution of the inhomogeneous Helmholtz equation. Taking the boundary conditions (1.10) and (2.280) as well as the radiation condition at S_∞ into account provides

$$u_s(\mathbf{x}) \;=\; \oint_{\partial\Gamma} \frac{\partial G_{\Gamma_+}(\mathbf{x}', \mathbf{x})}{\partial \hat{n}'_-} \cdot u_{inc}(\mathbf{x}')\, dS(\mathbf{x}') \tag{3.21}$$

as a representation of the scattered field in Γ_+.

$$G_{\partial\Gamma}(\mathbf{x}, \mathbf{x}') \;:=\; \frac{\partial G_{\Gamma_+}(\mathbf{x}', \mathbf{x})}{\partial \hat{n}'_-} \;=\; \hat{n}'_- \cdot \nabla_{x'}\, G_{\Gamma_+}(\mathbf{x}', \mathbf{x}) \qquad \mathbf{x} \in \Gamma_+,\; \mathbf{x}' \in \partial\Gamma \tag{3.22}$$

is the definition of the surface Green function $G_{\partial\Gamma}$ belonging to the Green function G_{Γ_+}. Please, note that one argument of the surface Green function is always located at the scatterer surface. The other argument can be located everywhere in G_{Γ_+}. With this surface Green function we can reformulate Eq. (3.21) into

$$u_s(\mathbf{x}) \;=\; \oint_{\partial\Gamma} G_{\partial\Gamma}(\mathbf{x}, \mathbf{x}') \cdot u_{inc}(\mathbf{x}')\, dS(\mathbf{x}'). \tag{3.23}$$

Comparing this equation with (3.20) provides

$$G^{(N)}_{\partial\Gamma}(\mathbf{x}, \mathbf{x}') \;=\; -\sum_{i,j=0}^{N} \left[A^{(g,\varphi_0)^{-1}}_{\partial\Gamma} \right]_{i,j} \cdot \varphi_i(k_0, \mathbf{x}) \cdot g^*_j(\mathbf{x}') \;; \qquad \mathbf{x}' \in \partial\Gamma,\; \mathbf{x} \in \Gamma_+. \tag{3.24}$$

as an appropriate approximation of the surface Green function. The corresponding approximation of the Green function G_{Γ_+} is obtained by two additional steps.

Fourth step:

From (1.8), (2.271), (1.286), (3.23), and by assuming that the source $\rho(\mathbf{x})$ of the primary incident field is located somewhere in the outer region Γ_+ we obtain

$$\int_{\Gamma_+} G_{\Gamma_+}(\mathbf{x}, \mathbf{x}') \cdot \rho(\mathbf{x}')\, dV(\mathbf{x}') = \int_{\Gamma_+} G_0(\mathbf{x}, \mathbf{x}') \cdot \rho(\mathbf{x}')\, dV(\mathbf{x}') + \oint_{\partial\Gamma} G_{\partial\Gamma}(\mathbf{x}, \mathbf{x}') \cdot u_{inc}(\mathbf{x}')\, dS(\mathbf{x}'). \tag{3.25}$$

Next, we use again (2.271) to replace $u_{inc}(\mathbf{x}')$ on the right-hand side of this expression. Comparing the integrands on both sides of the resulting equation provides

$$G_{\Gamma_+}(\mathbf{x},\mathbf{x}') \;=\; G_0(\mathbf{x},\mathbf{x}') \;+\; \oint_{\partial\Gamma} G_{\partial\Gamma}(\mathbf{x},\bar{\mathbf{x}}) \cdot G_0(\bar{\mathbf{x}},\mathbf{x}')\, dS(\bar{\mathbf{x}}) \tag{3.26}$$

as the relation between the Green function G_{Γ_+} of the outer Dirichlet problem and its surface Green function $G_{\partial\Gamma}$. This relation turns out to be very important for all our ongoing discussions and can be considered as Huygens' principle formulated solely in terms of Green functions. Of course, it is usually not allowed to infer the equality of integrands from the equality of the integrals. The way we used above to derive (3.26) is therefore only a way of plausibility although it is in agreement with the linear superposition of the primary incident and scattered field to the total field. Another possibility to derive (3.26) which avoids this problem is offered with Green's theorem (2.239) employed with the two quantities $\Psi(\mathbf{x}) = G_{\Gamma_+}(\mathbf{x},\mathbf{x}'')$ and $\Phi(\mathbf{x}) = G_0(\mathbf{x},\mathbf{x}')$. We get

$$G_{\Gamma_+}(\mathbf{x}',\mathbf{x}'') \;=\; G_0(\mathbf{x}'',\mathbf{x}') \;+\; \oint_{\partial\Gamma} G_{\partial\Gamma}(\mathbf{x}'',\mathbf{x}) \cdot G_0(\mathbf{x},\mathbf{x}')\, dS(\mathbf{x}). \tag{3.27}$$

From this expression (3.26) follows immediately if taking the symmetry relations (2.245) and (2.284) into account.

Fifth step:

We replace the surface Green function on the right-hand side of (3.26) by its approximation (3.24) and obtain

$$\begin{aligned} G^{(N)}_{\Gamma_+}(\mathbf{x},\mathbf{x}') &= G_0(\mathbf{x},\mathbf{x}') \\ &\quad - \sum_{i,j=0}^{N} \left[A^{(g,\varphi_0)^{-1}}_{\partial\Gamma}\right]_{i,j} \cdot \varphi_i(k_0,\mathbf{x}) \cdot \tilde{g}^*_j(\mathbf{x}')\;;\quad \mathbf{x},\mathbf{x}' \in \Gamma_+ \end{aligned} \tag{3.28}$$

with $\tilde{g}^*_j(\mathbf{x}')$ therein given by

$$\tilde{g}^*_j(\mathbf{x}') \;=\; \oint_{\partial\Gamma} g^*_j(\bar{\mathbf{x}}) \cdot G_0(\bar{\mathbf{x}},\mathbf{x}')\, dS(\bar{\mathbf{x}}). \tag{3.29}$$

Next, let us replace in this last expression the free-space Green function G_0 by the expansion (2.278) of $G_0^<$ thus providing

$$\tilde{g}^*_j(\mathbf{x}') \;=\; (ik_0) \sum_{k=0}^{N} \left[B^{(g,\psi_0)}_{\partial\Gamma}\right]_{j,k} \cdot \tilde{\varphi}_k(k_0,\mathbf{x}') \tag{3.30}$$

with matrix elements $\left[B_{\partial\Gamma}^{(g,\psi_0)}\right]_{j,k}$ defined in (2.22). The usage of $G_0^<$ instead of G_0 in (3.29) is allowed only if **the source point $\mathbf{x}'$ of the primary incident field is located somewhere outside the smallest spherical surface circumscribing the scatterer!** But, as we will see later, this assumption provides no restriction for the plane wave scattering problems. With this replacement we get from (3.28) the final approximation

$$G_{\Gamma_+}^{(N)}(\mathbf{x},\mathbf{x}') = G_0(\mathbf{x},\mathbf{x}') \; - \; (ik_0) \cdot \sum_{i,k=0}^{N} [T_{\partial\Gamma}]_{i,k} \cdot \varphi_i(k_0,\mathbf{x}) \cdot \tilde{\varphi}_k(k_0,\mathbf{x}') \; ; \mathbf{x},\mathbf{x}' \in \Gamma_+ \tag{3.31}$$

of the Green function G_{Γ_+} related to the outer Dirichlet problem. Since the second term on the right-hand side of (3.31) represents the scattering part of the Green function, from which one can calculate the scattered field, we will denote it with $G_s(\mathbf{x},\mathbf{x}')$, i.e., we write

$$G_{\Gamma_+}^{(N)}(\mathbf{x},\mathbf{x}') \;=\; G_0(\mathbf{x},\mathbf{x}') \;+\; G_s^{(N)}(\mathbf{x},\mathbf{x}') \tag{3.32}$$

with G_s given by

$$G_s^{(N)}(\mathbf{x},\mathbf{x}') \;=\; -\,(ik_0) \sum_{i,k=0}^{N} [T_{\partial\Gamma}]_{i,k} \cdot \varphi_i(k_0,\mathbf{x}) \cdot \tilde{\varphi}_k(k_0,\mathbf{x}'). \tag{3.33}$$

This is a remarkable result since the matrix elements

$$[T_{\partial\Gamma}]_{i,k} \;=\; \sum_{j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi_0)^{-1}}\right]_{i,j} \cdot \left[B_{\partial\Gamma}^{(g,\psi_0)}\right]_{j,k} \tag{3.34}$$

are nothing but the elements of the transformation matrix (2.19). We can state moreover that $G_{\Gamma_+}^{(N)}(\mathbf{x},\mathbf{x}')$ solves the defining equation (2.279) subject to the radiation condition with respect to $\mathbf{x}$. But what happens with the additional boundary condition (2.280)? Looking back at Huygens' principle (3.26) one can infer that this condition will be fulfilled if

$$G_{\partial\Gamma}(\mathbf{x},\bar{\mathbf{x}}) \;=\; -\delta_{\partial\Gamma}(\bar{\mathbf{x}}-\mathbf{x}) \tag{3.35}$$

holds for every $\mathbf{x} \in \partial\Gamma$. Comparing (3.9) with (3.24) reveals that this is indeed true for the respective approximations, i.e., that

$$G_{\partial\Gamma}^{(N)}(\mathbf{x},\bar{\mathbf{x}}) \;=\; -\delta_{\partial\Gamma}^{(N)}(\bar{\mathbf{x}}-\mathbf{x}) \tag{3.36}$$

holds if the expansion functions in (3.9) are the radiating solutions of Helmholtz's equation. Therefore, with (3.31)/(3.34) we have found an appropriate approximation of the Green function related to the outer Dirichlet problem we were looking for. Its usefulness must be proven in real applications, of course.

Approximation (3.31) of the Green function G_{Γ_+} becomes especially simple if the limiting case of a spherical scatterer geometry is considered. From this limiting expression the result of the conventional Mie theory can be derived without any problems if the primary incident field is given by the plane wave (2.102), and if using the regular solutions $\psi_i(k_0 r, \theta, \phi)$ according to (2.57) as weighting functions. Due to the orthogonality relations (2.96) and (2.97) valid at the surface of a sphere with radius $r = a$ the matrices (1.39) and (1.40) defining the T-matrix become diagonal matrices of the form

$$\left[A_{\partial\Gamma}^{(\psi_0,\varphi_0)} \right]_{i,k} = \delta_{i,k} \cdot \frac{1}{a^2} \cdot j_{n(i)}^*(k_0 a) \cdot h_{n(i)}^{(1)}(k_0 a) \tag{3.37}$$

and

$$\left[B_{\partial\Gamma}^{(\psi_0,\psi_0)} \right]_{i,k} = \delta_{i,k} \cdot \frac{1}{a^2} \cdot j_{n(i)}^*(k_0 a) \cdot j_{n(i)}(k_0 a). \tag{3.38}$$

The scattering part (3.33) of the Green function reads therefore

$$G_s^{(N)}(\mathbf{x}, \mathbf{x}') = -(ik_0) \sum_{i=0}^{N} \frac{j_{n(i)}(k_0 a)}{h_{n(i)}^{(1)}(k_0 a)} \cdot \varphi_i(\mathbf{x}) \cdot \tilde{\varphi}_i(\mathbf{x}'), \tag{3.39}$$

and the corresponding approximation of the surface Green function becomes

$$G_{\partial\Gamma}^{(N)}(\mathbf{x}, \mathbf{x}') = -\frac{1}{a^2} \sum_{i=0}^{N} \left[j_{n(i)}^*(k_0 a) \cdot h_{n(i)}^{(1)}(k_0 a) \right]^{-1} \cdot \varphi_i(k_0, \mathbf{x}) \cdot \psi_i^*(k_0, \mathbf{x}'). \tag{3.40}$$

At the end of this subsection let us consider the scattering problem of a plane wave given by (2.109). Therewith we want to show that (2.286) together with (3.32) provides the representation of the scattered field we have already considered in the first chapter of this book in the context of Rayleigh's method . This ought to convince us from the equivalence of the differential and integral point of views on the level of the respective approximations.

With (1.21), (1.42), and with the radiating solutions (2.58) as expansion functions we get for the scattered field in spherical coordinates

$$u_s^{(N)}(k_0 r, \theta, \phi) = -\sum_{i,k=0}^{N} [T_{\partial\Gamma}]_{i,k} \cdot b_k \cdot \varphi_i(k_0 r, \theta, \phi). \tag{3.41}$$

b_k are the expansion coefficients of the primary incident plane wave (2.109) given by

$$b_k \;=\; E_0\, 4\pi\, \tilde{Y}_k^*(\theta_i, \phi_i) \tag{3.42}$$

according to (2.113). In conjunction with (1.8), (2.286), and (3.33) we get on the other hand

$$u_s^{(N)}(k_0 r, \theta, \phi) = -(ik_0) \sum_{i,k=0}^{N} [T_{\partial\Gamma}]_{i,k} \cdot \varphi_i(k_0 r, \theta, \phi) \cdot \int_{\Gamma_+} \tilde{\varphi}_k(k_0, \mathbf{x}') \cdot \rho(\mathbf{x}')\, dV(\mathbf{x}'). \tag{3.43}$$

Both representations become identical if the coefficients

$$\tilde{b}_k \;=\; (ik_0) \int_{\Gamma_+} \tilde{\varphi}_k(k_0, \mathbf{x}') \cdot \rho(\mathbf{x}')\, dV(\mathbf{x}') \tag{3.44}$$

calculated by use of the source distribution (2.272) are identical with the coefficients b_k of (3.42). Since we have Dirac's delta distribution in (2.272) it follows

$$\tilde{b}_k \;=\; (ik_0)\, 4\pi E_0 \cdot \tilde{\varphi}_k(k_0 r_i, \theta_i, \phi_i) \cdot r_i \cdot e^{-ik_0 r_i} \tag{3.45}$$

Please, mind the difference that we denote $|\mathbf{x}'_{\mathbf{q}}|$ with r_i in spherical coordinates. Moreover, the subindex k denotes the combined summation index and should not be confused with the parameter k characterizing the region physically. Then, it is not to difficult to show that both sets of expansion coefficients $\tilde{b}_k$ and b_k are indeed identical. For this we have to employ definition (2.85), both relations (2.62) and (2.63) (the latter is necessary because of (2.274) which provides the spherical harmonics with arguments $Y_{l,n}(\pi - \theta_i, \phi_i \pm \pi)$!), and the asymptotic behaviour resulting from (2.78) for large arguments $k_0 r_i$.

With this prove of equivalence we have established at the same time a way to arrive at expansion (2.113) for a general plane wave travelling along an arbitrary direction $\vec{k}_i$, and if starting from the integral representation (2.271) of the primary incident field. In (2.271), we must only replace G_0 by $G_0^<$ according to (2.278). In conjunction with the source distribution (2.272) and the asymptotic behaviour of the radiating expansion functions $\tilde{\varphi}_k(k_0 r_i, \theta_i, \phi_i)$ for large arguments $k_0 r_i$ we end up in a straightforward way with (2.113). This way of deriving the expansion of a plane wave foreshadows already the somehow strange nature of the physical object "plane wave". On the one hand, expansion (2.113) is assumed to hold everywhere in the entire free space Γ. On the other hand, this space must contain somewhere the source distribution (2.272). Then there exist observation points (r, θ, ϕ) nearby the source point (r_i, θ_i, ϕ_i) for which the usage of $G_0^<(\mathbf{x}, \mathbf{x}')$ is actually not allowed since the condition $|\mathbf{x}| < |\mathbf{x}'|$ is violated. From this we would infer that expansion (2.113) is not a valid representation of a plane wave everywhere in Γ. But we know also that the plane wave solves the homogeneous Helmholtz equation without any source. That is, the plane wave is smoke without fire, so to say. Isn't it a strange situation? We will come back to it in Chap. 7.

3.3.2 The Outer Transmission Problem

With the following two steps we arrive at the approximation of the Green function $G_{\Gamma_+}^{(d)}$ of the outer transmission problem:

First step:

We go back to the transmission conditions (1.46) and (1.47) but by cancelling the additional τ-summation therein due to the restriction to the scalar case. For the expansion functions (1.48)–(1.53) appearing in these conditions we use the regular as well as the radiating eigensolutions of Helmholtz's equation defined in Sect. 2.3.1. Employing the shorter matrix notation introduced in (1.44) we thus have the two equations

$$\vec{\varphi}_0(\mathbf{x}) \cdot \vec{a}^{\,(N)^{tp}} - \vec{\psi}(\mathbf{x}) \cdot \vec{c}^{\,(N)^{tp}} = -\vec{\psi}_0(\mathbf{x}) \cdot \vec{b}^{\,tp} \tag{3.46}$$

$$\partial_{\hat{n}}\vec{\varphi}_0(\mathbf{x}) \cdot \vec{a}^{\,(N)^{tp}} - \partial_{\hat{n}}\vec{\psi}(\mathbf{x}) \cdot \vec{c}^{\,(N)^{tp}} = -\,\partial_{\hat{n}}\vec{\psi}_0(\mathbf{x}) \cdot \vec{b}^{\,tp}. \tag{3.47}$$

Please, have in mind that the regular functions $\psi_i(\mathbf{x})$ contain the parameter k in their arguments to characterize the physical property of the scatterer. Contrariwise, the regular functions $\psi_{0_i}(\mathbf{x})$ as well as the radiating solutions $\varphi_{0_i}(\mathbf{x})$ contain the parameter k_0 related to the free space which is assumed to be vacuum. According to the procedure described in Sect. 1.3.2. we could apply a scalar multiplication with the weighting functions $g_j(\mathbf{x})$ and $h_j(\mathbf{x})$ ($j = 0, \ldots, N$) to these two equations to generate the two equation systems (1.62) and (1.63). Eliminating the expansion coefficients $\vec{c}^{\,(N)}$ belonging to the approximation of the internal field would produce the T-matrix to interrelate the expansion coefficients $\vec{a}^{\,(N)}$ of the scattered field we sought-after to the known expansion coefficients $\vec{b}$ of the primary incident field. But here we will take the other way which was already introduced when discussing the transformation character of the T-matrix in Sect. 2.2.3. (see especially the discussion concerning relations (2.24)–(2.32) in this section). As a result we get **one** equation from both transmission conditions (3.46) and (3.47) which can be treated as a modification of the Dirichlet condition related to the outer Dirichlet problem. However, this modified condition contains the real Dirichlet condition of the outer Dirichlet problem as a limiting case. For this we must first eliminate the unknown expansion coefficients $\vec{a}^{\,(N)}$ of the scattered field from the equation systems (1.62) and (1.63). Thus we get the relation

$$\vec{c}^{\,(N)^{tp}} = \mathbf{T}_{\psi} \cdot \vec{b}^{\,tp} \tag{3.48}$$

between the expansion coefficients of the internal and primary incident field. The quantity $\mathbf{T}_\psi$ therein is given by expression (2.29). Inserting (3.48) into (3.46) provides

$$\vec{\varphi}_0(\mathbf{x}) \cdot \vec{a}^{\,(N)^{tp}} \;=\; \vec{\psi}(\mathbf{x}) \cdot \mathbf{T}_\psi \cdot \vec{b}^{\,tp} \;-\; \vec{\psi}_0(\mathbf{x}) \cdot \vec{b}^{\,tp} . \tag{3.49}$$

Next, we approximate the functions $\psi_i(\mathbf{x})$ at the scatterer surface by linear combinations of the functions $\psi_{0_i}(\mathbf{x})$ according to relation (2.31). The latter functions are also considered at the scatterer surface only. Thus, we obtain

$$\vec{\varphi}_0(\mathbf{x}) \cdot \vec{a}^{\,(N)^{tp}} \;=\; -\,\vec{\psi}_0(\mathbf{x}) \cdot \left[\mathbf{E} \;-\; \mathbf{T}_{\psi_0/\psi} \cdot \mathbf{T}_\psi\right] \cdot \vec{b}^{\,tp} \tag{3.50}$$

or alternatively

$$\vec{\varphi}_0(\mathbf{x}) \cdot \vec{a}^{\,(N)^{tp}} \;=\; -\,\vec{\psi}_0(\mathbf{x}) \cdot \vec{\tilde{b}}^{\,(N)^{tp}} \tag{3.51}$$

with the new coefficients

$$\vec{\tilde{b}}^{\,(N)^{tp}} \;=\; \left[\mathbf{E} \;-\; \mathbf{T}_{\psi_0/\psi} \cdot \mathbf{T}_\psi\right] \cdot \vec{b}^{\,tp} . \tag{3.52}$$

These new coefficients are dependent on the upper summation index N (they are not final any more!). They can be considered as expansion coefficients of a modified primary incident field at the scatterer surface. But this results in the modification

$$u_s(\mathbf{x}) \;=\; -\,\tilde{u}_{inc}(\mathbf{x}) \; ; \quad \mathbf{x} \in \partial\Gamma \tag{3.53}$$

of condition (1.10). That is, coefficients $\vec{\tilde{b}}^{\,(N)}$ are the expansion coefficients of the approximation of the modified field $\tilde{u}_{inc}$ at the scatterer surface. Equations (3.51) and (3.53) are thus a representation of the modified outer Dirichlet problem.

Second step:

For the Green function related to the outer Dirichlet problem we could derive approximation (3.32)/(3.33) in the foregoing section. Now, if replacing matrix $\mathbf{T}_{\partial\Gamma}$ in approximation (3.33) by matrix

$$\mathbf{T}_{\partial\Gamma}^{(d)} \;=\; \mathbf{T}_{\partial\Gamma} \cdot \left[\mathbf{E} \;-\; \mathbf{T}_{\psi_0/\psi} \cdot \mathbf{T}_\psi\right] \tag{3.54}$$

we obtain

$$G_{\Gamma_+}^{(d,N)}(\mathbf{x}, \mathbf{x}') \;=\; G_0(\mathbf{x}, \mathbf{x}') \;+\; G_s^{(d,N)}(\mathbf{x}, \mathbf{x}') \tag{3.55}$$

as an approximation of the Green function related to the outer transmission problem. The scattering contribution of this Green function is then given by

$$G_s^{(d,N)}(\mathbf{x}, \mathbf{x}') = \;-(ik_0) \sum_{i,k=0}^{N} \left[T_{\partial\Gamma}^{(d)}\right]_{i,k} \cdot \varphi_i(k_0, \mathbf{x}) \cdot \tilde{\varphi}_k(k_0, \mathbf{x}') . \tag{3.56}$$

The transformation matrix (3.54) was already derived in conjunction with equations (2.28) and (2.32). Obviously, if we choose $\mathbf{T}_\psi \equiv 0$, approximation (3.55) of the Green function related to the outer transmission problem becomes identical with the approximation of the Green function G_{Γ_+} related to the outer Dirichlet problem. Equation (3.55) in conjunction with (3.56) is moreover a solution of the inhomogeneous Helmholtz equation (2.279) subject to the radiation condition with respect to the variable $\mathbf{x}$. The prove of the fulfilment of boundary conditions (2.281) and (2.282) in the sense of this approximation will be shifted to Chap. 4.

Let us now consider the corresponding approximations of the dyadic Green functions.

3.4 The Dyadic Delta Distribution at the Scatterer Surface

The dyadic delta distribution $\mathbf{D}(\mathbf{x}-\mathbf{x}') = \mathbf{I}\delta(\mathbf{x}-\mathbf{x}')$ as the relevant inhomogeneity of the dyadic free-space Green function was already introduced in Sect. 2.6.3. In close analogy to (3.6) we are also able to define a corresponding dyadic delta distribution at the scatterer surface by the integral relation

$$\oint_{\partial\Gamma} \mathbf{D}_{\partial\Gamma}(\mathbf{x}'-\mathbf{x}) \cdot \vec{f}(\mathbf{x}')\, dS(\mathbf{x}') \;:=\; \vec{f}(\mathbf{x}); \quad \mathbf{x}, \mathbf{x}' \in \partial\Gamma. \tag{3.57}$$

But due to the boundary conditions (1.12) and (1.18) we are rather interested in a dyadic delta distribution for the special case of the tangential projections $\vec{f}^{\,\hat{n}}$ of the vector functions $\vec{f}$ at the surface $\partial\Gamma$. For our purposes it is therefore more convenient to define a dyadic delta distribution $\mathbf{D}_{\partial\boldsymbol{\Gamma}}^{(\hat{\mathbf{n}})}$ at the scatterer surface according to

$$\oint_{\partial\Gamma} \mathbf{D}_{\partial\boldsymbol{\Gamma}}^{(\hat{\mathbf{n}})}(\mathbf{x}'-\mathbf{x}) \cdot \vec{f}^{\,\hat{n}'}(\mathbf{x}')\, dS(\mathbf{x}') \;:=\; \vec{f}^{\,\hat{n}}(\mathbf{x}); \quad \mathbf{x}', \mathbf{x} \in \partial\boldsymbol{\Gamma}. \tag{3.58}$$

As already demonstrated in the scalar case we can approximate this specific dyadic delta distribution by a finite series expansion so that

$$\oint_{\partial\Gamma} \mathbf{D}_{\partial\boldsymbol{\Gamma}}^{(\hat{\mathbf{n}},\mathbf{N})}(\mathbf{x}'-\mathbf{x}) \cdot \vec{f}^{\,\hat{n}'}(\mathbf{x}')\, dS(\mathbf{x}') \;=\; \vec{f}^{\,(\hat{n},N)}(\mathbf{x}); \quad \mathbf{x}', \mathbf{x} \in \partial\Gamma \tag{3.59}$$

holds. To derive its approximation we go back to the results of Sect. 2.2. First we expand the tangential projections $\vec{f}^{\,\hat{n}}(\mathbf{x})$ of the vector functions $\vec{f}(\mathbf{x})$ at the scatterer surface into a finite series in terms of the vector functions $\vec{\varphi}_{i,\tau}^{\,\hat{n}}(\mathbf{x})$ according to (2.1). The expansion functions are not necessarily the radiating solutions of the vector-wave equation but they are assumed to be linearly independent at the scatterer surface. This provides

$$\vec{f}^{\,(\hat{n},N)}(\mathbf{x}) \;=\; \sum_{\tau=1}^{2} \sum_{i=0}^{N} b_{i,\tau}^{(N)} \cdot \vec{\varphi}_{i,\tau}^{\,\hat{n}}(\mathbf{x}); \quad \mathbf{x} \in \partial\Gamma \tag{3.60}$$

Please, note that the τ-summation cannot be neglected in the vector case. The expansion coefficients therein are again calculated from relation (2.14). With the definition (1.35) of the relevant scalar product we obtain the explicit expression

$$b_{i,\tau}^{(N)} = \sum_{\tau'=1}^{2} \sum_{j=0}^{N} [A_{\partial\Gamma}^{(g,\varphi)^{-1}}]_{i,j}^{\tau,\tau'} \cdot \oint_{\partial\Gamma} \vec{g}_{j,\tau'}^{\,*}(\mathbf{x}') \cdot \vec{f}^{\,\hat{n}'}(\mathbf{x}')\, dS(\mathbf{x}'). \qquad (3.61)$$

Inserting these coefficients into (3.60), interchanging summation and integration, and comparing the result with (3.59) provides finally

$$\mathbf{D}_{\partial\mathbf{\Gamma}}^{(\hat{\mathbf{n}},\mathbf{N})}(\mathbf{x}' - \mathbf{x}) = \sum_{\tau,\tau'=1}^{2} \sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi)^{-1}}\right]_{i,j}^{\tau,\tau'} \cdot \left\{\vec{\varphi}_{i,\tau}^{\,\hat{n}}(\mathbf{x}) \odot \vec{g}_{j,\tau'}^{\,*}(\mathbf{x}')\right\} \qquad (3.62)$$

as the appropriate approximation of the dyadic delta distribution (3.58).

The calculation of the elements of matrix $\mathbf{A}_{\partial\Gamma}^{(g,\varphi)}$ by use of (1.39) requires the calculation of the scalar product (1.35) of the weighting functions $\vec{g}_{j,\tau'}$ with the tangential projections $\vec{\varphi}_{i,\tau}^{\,\hat{n}}$ of the vector functions $\vec{\varphi}_{i,\tau}$. To distinguish the vector case from the scalar case in what follows and to avoid misunderstandings we will introduce an additional mark "$\hat{n}$" in the upper indices attached to the matrices if the tangential projections of vector functions are used. Mark "$\hat{n}$" is replaced by "$\hat{n}_-$" if the scatterer surface is considered to be the inner boundary surface of the outer region Γ_+. Instead of (3.62) we write therefore

$$\mathbf{D}_{\partial\mathbf{\Gamma}}^{(\hat{\mathbf{n}},\mathbf{N})}(\mathbf{x}' - \mathbf{x}) = \sum_{\tau,\tau'=1}^{2} \sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi^{\hat{n}})^{-1}}\right]_{i,j}^{\tau,\tau'} \cdot \left\{\vec{\varphi}_{i,\tau}^{\,\hat{n}}(\mathbf{x}) \odot \vec{g}_{j,\tau'}^{\,*}(\mathbf{x}')\right\}. \qquad (3.63)$$

The dyadic product in (3.62) and (3.63) is a consequence of the definition (2.297) of a scalar product of a vector with a dyadic. In the above discussion this vector is given by the approximation $\vec{f}^{\,(\hat{n}',N)}(\mathbf{x}')$ according to (3.60).

3.5 The Dyadic Green Functions Related to the Vector-Wave Equation

3.5.1 The Outer Dirichlet Problem

Here too we want to show at the beginning that the expansion coefficients $a_{i,\tau}^{(N)}$ of approximation (1.21) for the scattered field $\vec{u}_s^{\,(N)}(\mathbf{x})$ which holds everywhere in the outer region Γ_+ are identical with the expansion coefficients $\alpha_{i,\tau}^{(N)}$ of the corresponding approximation

$$\vec{u}_s^{\,(\hat{n}_-,N)}(\mathbf{x}) \;=\; \sum_{\tau=1}^{2}\sum_{i=0}^{N} \alpha_{i,\tau}^{(N)} \cdot \vec{\varphi}_{i,\tau}^{\,\hat{n}_-}(k_0,\mathbf{x}) \;; \quad \mathbf{x} \in \partial\Gamma \tag{3.64}$$

at the scatterer surface calculated from the application of the continuity condition (1.29). This proof requires moreover that the radiating vector solutions (2.122)/(2.123) of the vector-wave equation and their tangential projections, respectively, are used as expansion functions in both of these approximations. First we use the vectorial form (2.316) of Green's theorem with the two vector functions $\vec{\Psi}(\mathbf{x}) = \vec{u}_s(\mathbf{x})$ and $\vec{\Phi}(\mathbf{x}) = \vec{\varphi}_{i,\tau}(k_0,\mathbf{x})$. Since $\vec{u}_s$ as well as $\vec{\varphi}_{i,\tau}$ are solutions of the homogeneous vector-wave equation, and since both vector functions are in correspondence with the radiation condition we get with identity

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) \;=\; \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} \tag{3.65}$$

the equation

$$\oint_{\partial\Gamma} \Big\{ \vec{\varphi}_{i,\tau}^{\,\hat{n}_-}(k_0,\mathbf{x}) \cdot \left[\nabla \times \vec{u}_s(\mathbf{x})\right] - \vec{u}_s^{\,\hat{n}_-}(\mathbf{x}) \cdot \left[\nabla \times \vec{\varphi}_{i,\tau}(k_0,\mathbf{x})\right] \Big\}\, dS(\mathbf{x}) \;=\; 0. \tag{3.66}$$

The tangential projections $\vec{\varphi}_{i,\tau}^{\,\hat{n}_-}$ are defined in (2.159). Next we replace $\vec{u}_s$ in the first term of the boundary integral on the left hand side by its approximation (1.21). As already mentioned in the scalar case, this is justified by the fact that the operation $\nabla \times \vec{u}_s$ must be performed first in Γ_+ before moving the argument $\mathbf{x}$ to the scatterer surface $\partial\Gamma$. But for the quantity $\vec{u}_s^{\,\hat{n}_-}(\mathbf{x})$ in the second term we can apply approximation (3.64). It follows

$$\sum_{\tau'=1}^{2}\sum_{j=0}^{N} \oint_{\partial\Gamma} \Big\{ a_{j,\tau'}^{(N)} \cdot \vec{\varphi}_{i,\tau}^{\,\hat{n}_-}(k_0,\mathbf{x}) \cdot \left[\nabla \times \vec{\varphi}_{j,\tau'}(k_0,\mathbf{x})\right] - \alpha_{j,\tau'}^{(N)} \cdot \vec{\varphi}_{j,\tau'}^{\,\hat{n}_-}(k_0,\mathbf{x}) \cdot \left[\nabla \times \vec{\varphi}_{i,\tau}(k_0,\mathbf{x})\right] \Big\}\, dS(\mathbf{x}) \;=\; 0. \tag{3.67}$$

On the other hand, if using Green's theorem (2.316) with the two vector functions $\vec{\Psi}(\mathbf{x}) = \vec{\varphi}_{i,\tau}$ and $\vec{\Phi}(\mathbf{x}) = \vec{\varphi}_{j,\tau'}$ we obtain the identity

$$\oint_{\partial\Gamma} \vec{\varphi}_{j,\tau'}^{\,\hat{n}_-}(k_0,\mathbf{x}) \cdot \left[\nabla \times \vec{\varphi}_{i,\tau}(k_0,\mathbf{x})\right] = \oint_{\partial\Gamma} \vec{\varphi}_{i,\tau}^{\,\hat{n}_-}(k_0,\mathbf{x}) \cdot \left[\nabla \times \vec{\varphi}_{j,\tau'}(k_0,\mathbf{x})\right] dS(\mathbf{x}) \;=\; 0 \tag{3.68}$$

so that (3.67) can be rewritten into

$$\sum_{\tau'=1}^{2}\sum_{j=0}^{N}\left[a_{j,\tau'}^{(N)} - \alpha_{j,\tau'}^{(N)}\right]\cdot\oint_{\partial\Gamma}\vec{\varphi}_{i,\tau}^{\hat{n}-}(k_0,\mathbf{x}) \cdot\left[\nabla\times\vec{\varphi}_{j,\tau'}(k_0,\mathbf{x})\right] dS(\mathbf{x}) \;=\; 0\, i=0,\ldots,N,\;\; \tau=1,2. \qquad (3.69)$$

As in the scalar case we thus obtain

$$a_{j,\tau'}^{(N)} \;=\; \alpha_{j,\tau'}^{(N)} \qquad (3.70)$$

if the matrix $\mathbf{M}$ is invertible. Its elements result from the boundary integral on the left-hand side of (3.69). For a certain geometry of the scatterer this can be proven numerically, as the case may be.

The cooking recipe for deriving the dyadic Green function related to the outer Dirichlet problem is as follows:

First step:

We expand the tangential projection of the primary incident field $\vec{u}_{inc}^{\hat{n}-}$ at the scatterer surface according to (2.1) into a series in terms of the tangential projections $\vec{\psi}_{i,\tau}^{\hat{n}-}(k_0,\mathbf{x})$ of the vector functions $\vec{\psi}_{i,\tau}(k_0,\mathbf{x})$. These could be the regular vector solutions of the vector-wave equation, for example, but not necessarily. The corresponding expansion coefficients $b_{i,\tau}^{(N)}$ are then calculated according to (2.14) and (2.15).

Second step:

Utilizing the transformation character (2.18) of the T-matrix (2.19) we accomplish the transition from the expansion functions $\vec{\psi}_{i,\tau}^{\hat{n}-}(k_0,\mathbf{x})$ to the radiating vector solutions $\vec{\varphi}_{i,\tau}^{\hat{n}-}(k_0,\mathbf{x})$ in the approximation of the tangential projection of the primary incident field at the scatterer surface. The new expansion coefficients $a_{i,\tau}^{(N)}$ are calculated by use of (2.23) from the old coefficients $b_{i,\tau}^{(N)}$. Due to the identical definitions (2.15) and (2.22) of both matrices $\mathbf{A}_{\partial\Gamma}^{(g,\psi_0^{\hat{n}-})}$ and $\mathbf{B}_{\partial\Gamma}^{(g,\psi_0^{\hat{n}-})}$ which appear in (2.14) and (2.19), from the continuity condition (1.29), from the above derived relation (3.70), and after interchanging integration and summation we get

$$\vec{u}_s^{(N)}(\mathbf{x}) = -\oint_{\partial\Gamma}\sum_{\tau,\tau'=1}^{2}\sum_{i,j=0}^{N}\left[A_{\partial\Gamma}^{(g,\varphi_0^{\hat{n}-})^{-1}}\right]_{i,j}^{\tau,\tau'} \cdot\vec{g}_{j,\tau'}^{\,*}(\mathbf{x}')\cdot\vec{u}_{inc}^{\hat{n}-}(\mathbf{x}')\, dS(\mathbf{x}')\cdot\vec{\varphi}_{i,\tau}(k_0,\mathbf{x}); \qquad \mathbf{x}\in\Gamma_+,\;\; \mathbf{x}'\in\partial\Gamma \qquad (3.71)$$

as an approximation of the scattered field u_s in the outer region Γ_+. This corresponds to (3.20) in Sect.3.3.1. As in the scalar case it holds also here that, once we have specified the primary incident field as well as the vectorial weighting functions

$\vec{g}_{j,\tau'}$, with (3.71) we have found an appropriate approximation of the outer Dirichlet problem related to the vector-wave equation.

Third step:

We use the vector-dyadic form (2.318) of Green's theorem with the two quantities $\Psi(\mathbf{x}) = \vec{u}_s(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x}, \mathbf{x}') = \mathbf{G}_{\mathbf{\Gamma}_+}(\mathbf{x}, \mathbf{x}')$. $\vec{u}_s$ is a solution of the homogeneous vector-wave equation whereas G_{Γ_+} is a solution of the inhomogeneous equation (2.341). Both quantities obey additionally the radiation condition at S_∞. We get therefore

$$\vec{u}_s(\mathbf{x}') = -\oint_{\partial\Gamma} \hat{n}_- \cdot \left\{\vec{u}_s(\mathbf{x}) \times \left[\nabla_x \times \mathbf{G}_{\mathbf{\Gamma}_+}(\mathbf{x}, \mathbf{x}')\right]\right\} dS(\mathbf{x}). \tag{3.72}$$

From this it follows

$$\vec{u}_s(\mathbf{x}') = \oint_{\partial\Gamma} \left[\nabla_x \times \mathbf{G}_{\mathbf{\Gamma}_+}(\mathbf{x}, \mathbf{x}')\right]^{tp} \cdot \vec{u}_{inc}^{\,\hat{n}_-}(\mathbf{x})\, dS(\mathbf{x}) \tag{3.73}$$

with identities (2.309) and (2.310), and with the boundary condition (1.18). Let us interchange $\mathbf{x}$ and $\mathbf{x}'$ in this expression to denote the observation point with the unprimed variable, i.e., we write

$$\vec{u}_s(\mathbf{x}) = \oint_{\partial\Gamma} \left[\nabla_{x'} \times \mathbf{G}_{\mathbf{\Gamma}_+}(\mathbf{x}', \mathbf{x})\right]^{tp} \cdot \vec{u}_{inc}^{\,\hat{n}'_-}(\mathbf{x}')\, dS(\mathbf{x}'). \tag{3.74}$$

Now, with definition

$$\mathbf{G}_{\partial\mathbf{\Gamma}}(\mathbf{x}, \mathbf{x}') := \left[\nabla_{\mathbf{x}'} \times \mathbf{G}_{\mathbf{\Gamma}_+}(\mathbf{x}', \mathbf{x})\right]^{tp} \tag{3.75}$$

we introduce the dyadic surface Green function $\mathbf{G}_{\partial\mathbf{\Gamma}}$ related to $\mathbf{G}_{\mathbf{\Gamma}_+}$. Then, we write instead of (3.74)

$$\vec{u}_s(\mathbf{x}) = \oint_{\partial\mathbf{\Gamma}} \mathbf{G}_{\partial\Gamma}(\mathbf{x}, \mathbf{x}') \cdot \vec{u}_{inc}^{\,\hat{n}'_-}(\mathbf{x}')\, dS(\mathbf{x}'). \tag{3.76}$$

Comparing this expression with (3.71) provides

$$\mathbf{G}_{\partial\mathbf{\Gamma}}^{(\mathbf{N})}(\mathbf{x}, \mathbf{x}') = -\sum_{\tau,\tau'=1}^{2} \sum_{i,j=0}^{N} \left[A_{\partial\Gamma}^{(g,\varphi_0^{\hat{n}_-})^{-1}}\right]_{i,j}^{\tau,\tau'} \cdot \left\{\vec{\varphi}_{i,\tau}(k_0, \mathbf{x}) \odot \vec{g}^{\,*}_{j,\tau'}(\mathbf{x}')\right\}; \quad \mathbf{x} \in \Gamma_+,\ \mathbf{x}' \in \partial\mathbf{\Gamma} \tag{3.77}$$

as an approximation of the dyadic surface Green function.

Fourth step:

At first we want to derive Huygens' principle expressed solely in terms of Green functions. This can be achieved by employing the relevant dyadic-dyadic Green theorem in the outer region. It interrelates the dyadic Green functions $\mathbf{G}_{\boldsymbol{\Gamma}_+}$ and $\mathbf{G}_{\partial\boldsymbol{\Gamma}}$. From this principle we are then able to derive the approximation of $\mathbf{G}_{\boldsymbol{\Gamma}_+}$ which is in correspondence with approximation (3.77). We use the two quantities

$$\mathbf{Q}(\bar{\mathbf{x}}, \mathbf{x}) = \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x}) \tag{3.78}$$

$$\mathbf{P}(\bar{\mathbf{x}}, \mathbf{x}') = \mathbf{G}_{\boldsymbol{\Gamma}_+}(\bar{\mathbf{x}}, \mathbf{x}') \tag{3.79}$$

in Green's theorem (2.319). Taking symmetry relation (2.329) into account we get

$$\mathbf{G}_{\boldsymbol{\Gamma}_+}(\mathbf{x}, \mathbf{x}') = \mathbf{G_0}(\mathbf{x}, \mathbf{x}') + \oint_{\partial\Gamma} \left[\hat{\bar{n}}_- \times \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x})\right]^{tp} \cdot \left[\nabla_{\bar{\mathbf{x}}} \times \mathbf{G}_{\boldsymbol{\Gamma}_+}(\bar{\mathbf{x}}, \mathbf{x}')\right] dS(\bar{\mathbf{x}}). \tag{3.80}$$

This can be reformulated into

$$\mathbf{G}_{\boldsymbol{\Gamma}_+}(\mathbf{x}, \mathbf{x}') = \mathbf{G_0}(\mathbf{x}, \mathbf{x}') + \oint_{\partial\Gamma} \left[\hat{\bar{n}}_- \times \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x})\right]^{tp} \cdot \mathbf{G}^{\mathbf{tp}}_{\partial\boldsymbol{\Gamma}}(\mathbf{x}', \bar{\mathbf{x}})\, dS(\bar{\mathbf{x}}) \tag{3.81}$$

by use of definition (3.75) of the dyadic surface Green function. It can be shown that the following symmetry relation holds for the boundary integral on the right-hand side of (3.81):

$$\oint_{\partial\Gamma} \left\{\mathbf{G}_{\partial\boldsymbol{\Gamma}}(\mathbf{x}', \bar{\mathbf{x}}) \cdot \left[\hat{\bar{n}}_- \times \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x})\right]\right\}^{tp} dS(\bar{\mathbf{x}}) = \oint_{\partial\Gamma} \mathbf{G}_{\partial\boldsymbol{\Gamma}}(\mathbf{x}, \bar{\mathbf{x}}) \cdot \left[\hat{\bar{n}}_- \times \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x}')\right] dS(\bar{\mathbf{x}}). \tag{3.82}$$

This can be proven by use of identity (2.315) in conjunction with the symmetry relations (2.329) and (2.346). Then, Huygens' principle reads finally

$$\mathbf{G}_{\boldsymbol{\Gamma}_+}(\mathbf{x}, \mathbf{x}') = \mathbf{G_0}(\mathbf{x}, \mathbf{x}') + \oint_{\partial\Gamma} \mathbf{G}_{\partial\boldsymbol{\Gamma}}(\mathbf{x}, \bar{\mathbf{x}}) \cdot \left[\hat{\bar{n}}_- \times \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x}')\right] dS(\bar{\mathbf{x}}) \tag{3.83}$$

if expressed solely in terms of dyadic Green functions.

Fifth step:

Utilizing approximation (3.77) in (3.83) and employing definition

$$\vec{\tilde{g}}^{\,*}_{j,\tau'}(\mathbf{x}') := \oint_{\partial\Gamma} \vec{g}^{\,*}_{j,\tau'}(\bar{\mathbf{x}}) \cdot \left[\hat{\bar{n}}_- \times \mathbf{G_0}(\bar{\mathbf{x}}, \mathbf{x}')\right] dS(\bar{\mathbf{x}}) \tag{3.84}$$

results in the following expression:

$$\mathbf{G}^{(\mathbf{N})}_{\mathbf{\Gamma}_+}(\mathbf{x},\mathbf{x}') = \mathbf{G_0}(\mathbf{x},\mathbf{x}') \;-\; \sum_{\tau,\tau'=1}^{2}\sum_{i,j=0}^{N}\left[A_{\partial\Gamma}^{(g,\varphi_0^{\hat{n}_-})^{-1}}\right]_{i,j}^{\tau,\tau'} \cdot \left\{\vec{\varphi}_{i,\tau}(k_0,\mathbf{x}) \odot \vec{\tilde{g}}^{\,*}_{j,\tau'}(\mathbf{x}')\right\}. \tag{3.85}$$

The variable $\mathbf{x}'$ of the dyadic free-space Green function in (3.84) denotes the location of the source distribution of the primary incident field. As already done in the scalar case we assume that this source distribution is confined to an area which is located somewhere outside the smallest spherical surface circumscribing the scatterer. Moreover, since restricting our considerations to solenoidal fields only, we can replace $\mathbf{G_0}(\bar{\mathbf{x}},\mathbf{x}')$ in (3.84) by $\mathbf{G_t}^{<}(\bar{\mathbf{x}},\mathbf{x}')$ according to (2.340). From this procedure we get

$$\vec{\tilde{g}}^{\,*}_{j,\tau'}(\mathbf{x}') \;=\; (ik_0)\cdot\sum_{\bar{\tau}=1}^{2}\sum_{k=0}^{N}\left[B_{\partial\Gamma}^{(g,\psi_0^{\hat{n}_-})}\right]_{j,k}^{\tau',\bar{\tau}} \cdot \vec{\tilde{\varphi}}_{k,\bar{\tau}}(k_0,\mathbf{x}') \tag{3.86}$$

with matrix elements $\left[B_{\partial\Gamma}^{(g,\psi_0^{\hat{n}_-})}\right]_{j,k}^{\tau',\bar{\tau}}$ given by the scalar product (2.22). The approximation of the dyadic Green function related to the outer Dirichlet problem reads therefore

$$\mathbf{G}^{(\mathbf{N})}_{\mathbf{\Gamma}_+}(\mathbf{x},\mathbf{x}') \;=\; \mathbf{G_0}(\mathbf{x},\mathbf{x}') \;+\; \mathbf{G}^{(\mathbf{N})}_{\mathbf{s}}(\mathbf{x},\mathbf{x}') \tag{3.87}$$

with

$$\mathbf{G}^{(\mathbf{N})}_{\mathbf{s}}(\mathbf{x},\mathbf{x}') = -\,(ik_0)\cdot\sum_{\tau,\bar{\tau}=1}^{2}\sum_{i,k=0}^{N}\left[T_{\partial\Gamma}^{\hat{n}_-}\right]_{i,k}^{\tau,\bar{\tau}} \cdot\left\{\vec{\varphi}_{i,\tau}(k_0,\mathbf{x}) \odot \vec{\tilde{\varphi}}_{k,\bar{\tau}}(k_0,\mathbf{x}')\right\}. \tag{3.88}$$

$$\left[T_{\partial\Gamma}^{\hat{n}_-}\right]_{i,k}^{\tau,\bar{\tau}} \;=\; \sum_{\tau'=1}^{2}\sum_{j=0}^{N}\left[A_{\partial\Gamma}^{(g,\varphi_0^{\hat{n}_-})^{-1}}\right]_{i,j}^{\tau,\tau'} \cdot \left[B_{\partial\Gamma}^{(g,\psi_0^{\hat{n}_-})}\right]_{j,k}^{\tau',\bar{\tau}} \tag{3.89}$$

are again the elements of the transformation matrix (2.19). These elements differ only in the additional τ-summation and in the considered vector functions appearing in the relevant scalar product definitions. Approximation (3.88) is in agreement with the inhomogeneous equation (2.341) and the radiation condition with respect to $\mathbf{x}$. The question if it suffices the boundary condition (2.342) can be also answered in close analogy to the scalar case. From Huygens' principle (3.83) it becomes obvious that this boundary condition is fulfilled if relation

$$\hat{n}_- \times \mathbf{G}_{\partial\mathbf{\Gamma}}(\mathbf{x},\bar{\mathbf{x}}) \;=\; -\,\mathbf{D}_{\partial\mathbf{\Gamma}}^{\hat{\mathbf{n}}_-}(\bar{\mathbf{x}}-\mathbf{x}) \tag{3.90}$$

holds for $\mathbf{x} \in \partial\Gamma$, according to definition (3.58). Comparing (3.63) with (3.77) shows that this relation holds indeed for the approximations of $\mathbf{G}_{\partial\Gamma}$ and $\mathbf{D}_{\partial\boldsymbol{\Gamma}}^{\hat{\mathbf{n}}-}$. Thus, we can state that boundary condition (3.342) is fulfilled in this approximate sense.

The derived approximations become again especially simple if a spherical scatterer is considered. If choosing the tangential projections of the regular vector solutions as weighting functions, and if taking the orthogonality relations (2.179) and (2.180) at the surface of a sphere with the radius $r = a$ into account we get the following expressions for the relevant matrix elements:

$$\left[A_{\partial\Gamma}^{(\psi_0^{\hat{n}-},\varphi_0^{\hat{n}-})^{-1}}\right]_{i,k}^{\tau,\tau'} = \delta_{\tau,\tau'}\delta_{i,k} \cdot \frac{1}{a^2} \cdot \frac{1}{d_{i,\tau}^{(\psi_0,\varphi_0)}} \tag{3.91}$$

and

$$\left[B_{\partial\Gamma}^{(\psi_0^{\hat{n}-},\psi_0^{\hat{n}-})}\right]_{i,k}^{\tau,\tau'} = \delta_{\tau,\tau'}\delta_{i,k} \cdot a^2 \cdot d_{i,\tau}^{(\psi_0,\psi_0)}. \tag{3.92}$$

The normalization constants therein are calculated from (2.183), (2.184), (2.187), and (2.188) with κ and κ' replaced by the parameter k_0. As a result, we obtain

$$\begin{aligned}\mathbf{G}_{\boldsymbol{\Gamma}_+}^{(\mathbf{N})}(\mathbf{x},\mathbf{x}') = \mathbf{G_0}(\mathbf{x},\mathbf{x}') - ik_0 \cdot \sum_{i=0}^{N} a_{i,1} \cdot \left\{\vec{\varphi}_{i,1}(k_0,\mathbf{x}) \odot \vec{\tilde{\varphi}}_{i,1}(k_0,\mathbf{x}')\right\} \\ + a_{i,2} \cdot \left\{\vec{\varphi}_{i,2}(k_0,\mathbf{x}) \odot \vec{\tilde{\varphi}}_{i,2}(k_0,\mathbf{x}')\right\}\end{aligned} \tag{3.93}$$

with coefficients

$$a_{i,1} = \frac{j_{n(i)}(k_0 a)}{h_{n(i)}^{(1)}(k_0 a)} \tag{3.94}$$

and

$$a_{i,2} = \frac{\frac{\partial}{\partial r}\left[r \cdot j_{n(i)}(k_0 r)\right]_{r=a}}{\frac{\partial}{\partial r}\left[r \cdot h_{n(i)}^{(1)}(k_0 r)\right]_{r=a}} \tag{3.95}$$

as the approximation of the dyadic Green function of the outer Dirichlet problem. The corresponding approximation of the dyadic surface Green function becomes

$$\begin{aligned}\mathbf{G}_{\partial\boldsymbol{\Gamma}}^{(\mathbf{N})}(\mathbf{x},\mathbf{x}') = -\sum_{i=0}^{N} \frac{1}{a^2\, d_{i,1}^{(\psi_0,\varphi_0)}} \cdot \left\{\vec{\varphi}_{i,1}(k_0,\mathbf{x}) \odot \left[\vec{\psi}_{i,1}^{\,\hat{n}-}(k_0,\mathbf{x}')\right]\right\}^* \\ + \frac{1}{a^2\, d_{i,2}^{(\psi_0,\varphi_0)}} \cdot \left\{\vec{\varphi}_{i,2}(k_0,\mathbf{x}') \odot \left[\vec{\psi}_{i,2}^{\,\hat{n}-}(k_0,\mathbf{x})\right]^*\right\}.\end{aligned} \tag{3.96}$$

As already demonstrated in the scalar case we can express the scattered field by the finite series expansion

$$\vec{u}_s^{(N)}(k_0 r, \theta, \phi) \;=\; -\sum_{\tau,\bar{\tau}=1}^{2}\sum_{i,k=0}^{N}\left[T_{\partial\Gamma}^{\hat{n}_-}\right]_{i,k}^{\tau,\bar{\tau}} \cdot b_{k,\bar{\tau}} \cdot \vec{\varphi}_{i,\tau}(k_0 r, \theta, \phi) \tag{3.97}$$

which results from the integral representation (2.348) and the scattering part (3.88) of $\mathbf{G}_{\mathbf{\Gamma}_+}^{(\mathbf{N})}$. In combination with the vector source (2.332) the coefficients $b_{k,1}$ and $b_{k,2}$ therein become identical with the coefficients specified in (2.236) and (2.237). These latter coefficients belong to the series expansion of a linearly polarized plane wave travelling along an arbitrary direction $\vec{k}_i$. For the proof of equality of these both sets of expansion coefficients we need the asymptotic behaviour (2.152) and (2.153) of the radiating vector solutions $\vec{\varphi}_{i,\tau}$ for large arguments as well as relations (2.136) and (2.137). The latter relations are a consequence of the unit vector $\vec{k}_i$ pointing from the source into the direction of the plane wave propagation. Because of (2.334) it causes the vector spherical harmonics with arguments $\vec{C}_{l,n}(\pi - \theta_i, \phi_i \pm \pi)$ and $\vec{B}_{l,n}(\pi - \theta_i, \phi_i \pm \pi)$. This proof shows us, moreover, that one can derive expansion (2.235) of the general case of a linearly polarized plane wave in a straightforward way by employing the expansion (2.340) of $\mathbf{G}_\mathbf{t}^<$ and the vector source (2.332) in the integral representation (2.331).

3.5.2 The Outer Transmission Problem

In Sect. 2.2.3 we have discussed the transformation character of the T-matrix by use of an abstract notation which is independent of whether the scalar or vectorial boundary value problems are considered. This allows us to adopt the approximation of the scalar Green function belonging to the outer transmission problem derived in Sect. 3.3.2 with only slight changes for the corresponding dyadic Green function. In place of the scalar expansion and weighting functions and their normal derivatives at the scatterer surface we apply simply the tangential projections of the corresponding vector functions as defined in (1.38), (1.61), and (1.54)–(1.59). The τ-summation must additionally be taken into account. Thus, we have for the approximation of the dyadic Green function related to the outer transmission problem

$$\mathbf{G}_{\mathbf{\Gamma}_+}^{(\mathbf{d},\mathbf{N})}(\mathbf{x}, \mathbf{x}') \;=\; \mathbf{G_0}(\mathbf{x}, \mathbf{x}') \;+\; \mathbf{G}_\mathbf{s}^{(\mathbf{d},\mathbf{N})}(\mathbf{x}, \mathbf{x}') \tag{3.98}$$

with its scattering part $\mathbf{G_s}$ given by

$$\mathbf{G}_\mathbf{s}^{(\mathbf{d},\mathbf{N})}(\mathbf{x}, \mathbf{x}') = -(ik_0) \cdot \sum_{\tau,\bar{\tau}=1}^{2}\sum_{i,k=0}^{N}\left[T_{\partial\Gamma}^{(\hat{n}_-,d)}\right]_{i,k}^{\tau,\bar{\tau}} \cdot \left\{\vec{\varphi}_{i,\tau}(k_0, \mathbf{x}) \odot \vec{\tilde{\varphi}}_{k,\bar{\tau}}(k_0, \mathbf{x}')\right\}. \tag{3.99}$$

For the T-matrix itself we obtain the expression

$$\mathbf{T}_{\partial\Gamma}^{(\hat{n}_-,d)} = \mathbf{T}_{\partial\Gamma}^{\hat{n}_-} \cdot \left[\mathbf{E} - \mathbf{T}^{\hat{n}_-}_{\psi_0^{\hat{n}_-}/\psi^{\hat{n}_-}} \cdot \mathbf{T}^{\hat{n}_-}_{\psi^{\hat{n}_-}} \right]. \tag{3.100}$$

This corresponds to (3.54) in the scalar case with the difference that all matrices are now (2 $\times$ 2)-block matrices, due to the additional τ-summation.