An Archimedean Proof of Heron's Formula for the Area of a Triangle: Heuristics Reconstructed

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Abstract I believe, as did al-Bīrūnī, that Archimedes invented and proved Heron's formula for the area of a triangle. But I also believe that Archimedes would not multiply one rectangle by another, so he must have had a another way of stating and proving the theorem. It is possible to "save" Heron's received text by inventing a geometrical counterpart to the un-Archimedean passage and inserting that before it, and to consider the troubling passage as Archimedes' own translation into terms of measurement. My invention is based on a reconstruction of the heuristics that led to the proof.

I prove a crucial lemma: If there are five magnitudes of the same kind, a, b, c, d, m, and m is the mean proportional between a and b, and $a : c = d : b$, then m is also the mean proportional between c and d.

Introduction

A triangle is the mean proportional between two rectangles, *one* of which is contained by the semiperimeter of the triangle and the semiperimeter diminished by one of the sides, whereas *the other* is contained by the semiperimeter diminished by either of the remaining sides.

The statement above is a reconstruction, seen nowhere in the received sources. In Heron's text[1] we learn that the area of the triangle is the side of a square equal to one of the said rectangles multiplied by the other one. Obviously, this kind of statement is alien to standard Greek geometry: a side, a line segment, cannot be equal to an area, and multiplication of rectangles cannot be represented by a square — so it must needs be understood arithmetically, and no wonder, since mensuration is what it was meant for. The invention of the proof is attributed to Archimedes by al-Bīrūnī, but E.J. Dijksterhuis [1956, 412] had "some doubt whether the proof in the form in which it is quoted by Heron, can really originate from Archimedes." That doubt is quite legitimate as to the form, but then it is also legitimate to guess an answer to the question: Since the theorem is proved by sound propositions from the *Elements*, how would Archimedes state the theorem geometrically?

An answer to that depends, I am sure, on a reconstruction of the heuristics that led to the proof. Below I venture such a reconstruction, and I propose a "missing" geometrical passage which will "save" the peculiar arithmetical statement. I find it quite tenable that Archimedes, after the geometrical part, himself "translated" it into arithmetic to serve its purpose of mensuration. For all we know, Archimedes did not hesitate to put (approximate) numbers to lines' lengths, e.g. in his *Mensuration of a Circle* 3.

After some typographical conventions, I present my analysis and heuristics, followed by commented translations of *Metrika* I.7 and I.8. You might want to read the translations first to form your own

A revised and extended version of my paper on this subject that appeared in *Centaurus* (which, besides being marred by a glaring erratum, contained no heuristics) [Taisbak 1980].

[1] *Metrika* I 8 [Schöne 1903, 22, ll. 15–19; 24, ll. 10–21].

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opinion about a possible analysis leading to his synthesis.

Typographical Conventions

A *triangle ABC* is denoted ABC. Its *angle B* is denoted $\angle B$ or $\angle ABC$.

A rectangle with sides $AB = l$ and $BC = w$ is denoted $(AB \cdot BC)$ or $(l \cdot w)$. If AB and BC are numbers (i.e. lengths), $AB.BC$ denotes their product. The context will guide us.

The geometrical square on (i.e. with side) $AB = q$ is denoted $AB^{\square}$ or $q^{\square}$. The arithmetical square of (the number) AB is denoted AB^2.

The *ratio* of two homogeneous magnitudes[2] A and B is written $A : B$. A *proportion* "A is to B as C to D" is written $A : B = C : D$.

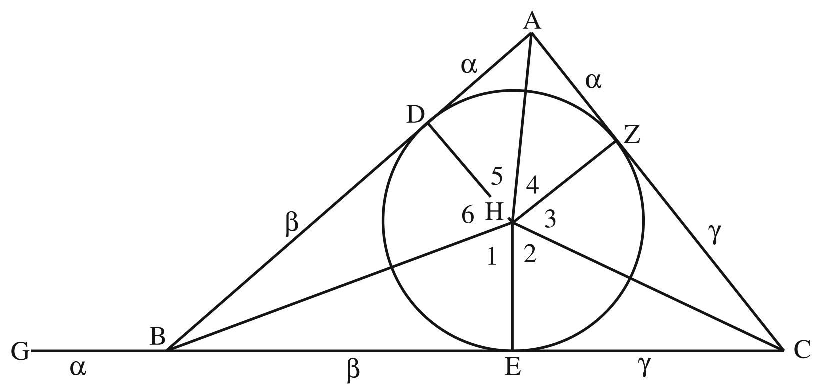

Figure 1: Relations of the semiperimeter.

The *semiperimeter* of a triangle ABC with sides a, b, c, is the sum $(a + b + c)/2 = s$. In Figure 1, $\beta + \gamma = a$, $\gamma + \alpha = b$, $\alpha + \beta = c$. Adding these equations we have $2\alpha + 2\beta + 2\gamma = a + b + c = 2s$, so that $\alpha + \beta + \gamma = s$, and

$$\alpha = s - a, \beta = s - b, \gamma = s - c.$$

To visualize s, CB is prolonged to G with $BG = AD$, so we get $BG = AD = \alpha$, $BE = \beta$, $CE = \gamma$, and therefore $CG = s$.

Historical warning: The lower-case letters a, b, c, l, s, w, α, β, γ should be understood as names of line segments, not (real) numbers. The Greeks would use BC, CA, AB, etc. In certain parts of his propositions, Heron will think of them as (approximate) lengths. I prefer lower-case letters for the sake of readability, running the risk of misinterpretation.

Analysis and Heuristics

With these conventions the opening statement — a triangle ABC is the mean proportional between two rectangles — can now be written as

$$(s \cdot \alpha) : ABC = ABC : (\beta \cdot \gamma),$$

[2] The homogeneous magnitudes involved are straight line segments, triangles, or rectangles (squares included).

and it is time to disclose that, arithmetically, this is equivalent to the formula

$$\text{Area of } ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

Lemma 1: If the radius of the triangle's incircle is r, the triangle is equal to (i.e. has the same area as) $(r \cdot s)$. This is well known, and was proved by Heron [Schöne 1903, 22, ll. 2–10].

Lemma 2: A rectangle is the mean proportional between the squares on its sides. This can be inferred from *Elements* X.53, lemma, which states that the mean proportional between two squares is the rectangle contained by their sides. *Lemma* 2 can also be proved by *Elements* VI.1 (see Figure 2):

$$l : w = l^{\square} : (l \cdot w) = (l \cdot w) : w^{\square}.$$

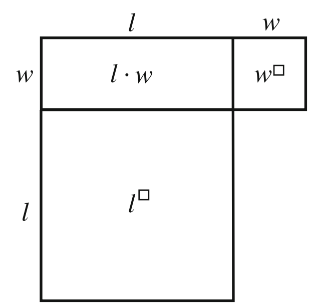

Figure 2: *Lemma 2.*

Lemma 3.1: If m is the mean proportional between a and b, there exist (infinitely many) magnitudes c and d, such that m is also the mean proportional between c and d. For line segments and rectangles this can be proved by *Elements* VI.12, which shows how to find a fourth proportional to three given line segments. Hence, we need not bother about the existence of a fourth proportional; the Greeks never did.[3]

Lemma 3.2: If m is the mean proportional between a and b, and if m is also the mean proportional between c and d, then $a : c = d : b$ (inverse proportion).

Suppose that

$$a : m = m : b, \tag{1}$$

and that

$$c : m = m : d,$$

then *enallax*[4] [*Elements* V def. 12],

$$m : c = d : m,$$

[3] While discussing "The Distinctive Assumptions of Book V" (of the *Elements*), Ian Mueller wrote [1981, 139, n. 24], "This explanation is put forward and developed by Becker in 'Warum haben die Griechen die Existenz der vierten Proportionale angenommen?'"

It seems clear that no Greek ever questioned this "assumption of the existence of a fourth proportional," perhaps because the use was not noticed, but more probably because the existence of such a proportional to three given geometrical objects was considered obvious on the basis of intuitive ideas about continuity.

[4] Heath [1926] refers to this operation with the expression "alternately."

and by *perturbed analogy* [*Elements* V.23]

$$a : c = d : b. \tag{2}$$

And conversely (the crucial lemma in the heuristics of *Metrika* I.8): If there are five magnitudes of the same kind, a, b, c, d, m, and m is the mean proportional between a and b, and $a : c = d : b$, then m is also the mean proportional between c and d. This is proved by taking (2) alternatively with (1) and using *Elements* V.23.[5]

Lemma 4: In Figure 1, $\angle BHC + \angle AHD = 2$ right angles. Since $\angle 1 = \angle 6$, $\angle 2 = \angle 3$, and $\angle 5 = \angle 4$, thus,

$$\angle 1 + \angle 2 + \angle 5 = \angle 6 + \angle 3 + \angle 4.$$

But since the six are equal to 4 right angles,

$$\angle 1 + \angle 2 + \angle 5 = \angle BHC + \angle AHD = 2 \text{ right angles.}$$

This lemma is proved by Heron [Schöne 1903, 22, ll. 23–28].

Lemma 5: [*Elements* VI.8, corollary] If, in a right triangle, a perpendicular is drawn from the right angle to the base (hypotenuse), the perpendicular is the mean proportional between the segments of the base. That is, the square on the perpendicular is equal to the rectangle contained by the segments of the base.

In Figure 3, let $\angle CHK$ be a right angle. Then $HE^{\square} = (KE \cdot EC)$, which we rename (for readability)

$$r^{\square} = (\varepsilon \cdot \gamma).$$

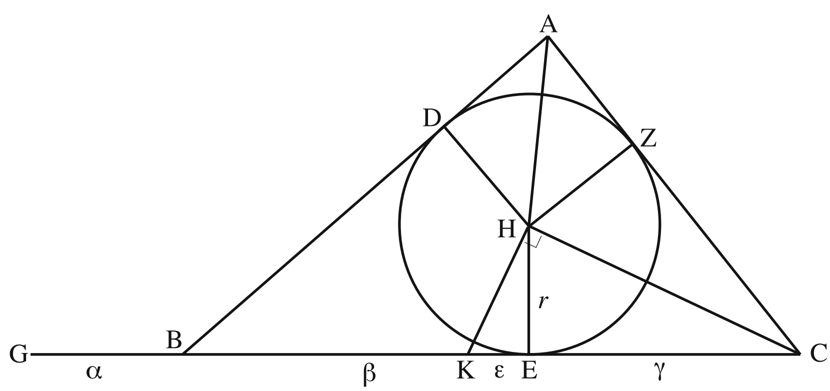

Figure 3: *Metrika* I 7 & 8, preliminary figure.

With these lemmas in mind (and let me emphasize that *Lemma* 3, as far as I know, is not known from any received text, but inspired by Book X of the *Elements*) we may turn to the heuristic proper: We learned by *Lemma* 1 that the triangle ABC is equal to $(r \cdot s)$, and therefore, by *Lemma* 2, that ABC is the mean proportional between $s^{\square}$ and $r^{\square}$. That is,

$$s^{\square} : ABC = ABC : r^{\square},$$

so that, by *Lemma* 5,

[5] When real numbers are invented, *Lemma* 3.2 becomes trivially obvious.

$$s^{\square} : ABC = ABC : (\varepsilon \cdot \gamma). \tag{1.0}$$

This first result of our heuristics shows a rectangle involving $\gamma = CE = s - c$. We might want to involve $\beta = BE = s - b$, so we use *Lemma* 3 to see what will happen to $s^{\square}$ if, in (1.0), we substitute $(\beta \cdot \gamma)$ for $(\varepsilon \cdot \gamma)$, to get the proportion

$$(s \cdot ?) : ABC = ABC : (\beta \cdot \gamma).$$

Now, since $(\beta \cdot \gamma) > (\varepsilon \cdot \gamma)$, the rectangle $(s \cdot ?) < s^{\square}$, according to *Lemma* 3.2 (inverse proportion). Let us consider a rectangle $(s \cdot z)$ with $z < s$, so that

$$(s \cdot z) : ABC = ABC : (\beta \cdot \gamma). \tag{1.1}$$

Is z given by this proportion if the sides of ABC are given?

According to *Lemma* 3.2

$$s^{\square} : (s \cdot z) = (\beta \cdot \gamma) : (\varepsilon \cdot \gamma), \tag{1.2}$$

and by "cancelling" [*Elements* VI.1],

$$s : z = \beta : \varepsilon. \tag{1.3}$$

To a Greek experienced in handling proportions, this is very inviting because ε is part of β (KE is part of BE). Therefore, *diairesis logou* [*Elements* V def. 15], subtraction within the ratio, will render a new proportion:

$$s - z : z = \beta - \varepsilon : \varepsilon = BK : EK. \tag{1.4}$$

I imagine that here the analyser gets the (b)right idea: to make BK the side of a right triangle similar to KEH by prolonging HK to meet the perpendicular to CG from B in L (see Figure 4). Then, $BL : EH = BK : EK$, which we rename

$$\delta : r = \beta - \varepsilon : \varepsilon. \tag{1.5}$$

The geometer will see immediately that CL, if joined, subtends two right angles, $\angle LBC$ and $\angle LHC$, and thus is a diameter in a circle passing through H and B, inviting the following arguments about angles and similar triangles. Since the quadrilateral $CHBL$ is inscriptible in a circle, the opposite angles $\angle BHC$ and $\angle BLC$ are together 2 right angles [*Elements* III.22], but so are also $\angle BHC + \angle AHD$, by *Lemma* 4 [Schöne 1903, 22, 22–28].

Therefore $\angle AHD = \angle CLB$, and the triangle AHD is similar to CLB. Among other properties, this renders

$$BL : DH = BC : DA,$$

that is

$$\delta : r = \beta + \gamma : \alpha. \tag{1.6}$$

Proportions (1.4), (1.5), and (1.6) and the transitivity of ratio [*Elements* V.11] ensure that

$$s - z : z = \beta + \gamma : \alpha,$$

inviting *synthesis logou* [*Elements* V def. 14], to get a new proportion,

$$s : z = \alpha + \beta + \gamma : \alpha.$$

But $\alpha + \beta + \gamma = s$, and therefore the unknown $z = \alpha$, which was probably what the analyser hoped for, to be able to rewrite (1.1), in our terms,

$$(s \cdot \alpha) : ABC = ABC : (\beta \cdot \gamma).$$

That is, the triangle *ABC* is the mean proportional between two rectangles:

$$R_1 = (s \cdot \alpha) \text{ and } R_2 = (\beta \cdot \gamma).$$

We are now ready to read *Metrika* I.7 and I.8, and to put them into a form that both respects geometry and is useful for mensuration — as we are entitled to believe was Archimedes' (or Whoever's) intention.

Comment: *Metrika* I.7 is a parenthesis in a series of theorems about how to find the area of triangles. It is meant to explain a surprising passage in *Metrika* I.8. At the same time, it gives us an idea of how Heron thought of, and handled, numbers. It is well-known that some of our numerical terminology was born in geometry: numerical multiplication is *thought of*, but not *illustrated* geometrically; the numbers are lengths of straight line segments, their product is thought of as the rectangle "contained by" the straight lines, as defined in *Elements* II def. 1. Particularly, the square *on* a line represents the square *of* the number;[6] "square root" translates πλευρά, literally "side of the square." But the crux in this proposition, and the next, is that the operations transcend the geometrical representation. How can the product of squares be represented, since geometry has no fourth dimension? That is what this proposition is about: behind these arithmetical statements lurks our *Lemma* 2, that any rectangle is the mean proportional between two squares, the squares on its sides. It is worth mentioning that Euclid, in his number-theoretical books *Elements* VII–IX never illustrates products by rectangles, but always by line segments — in that way, using lines as we use the alphabet to denote random numbers.

Heron's *Metrika* I.7

We turn now to a commented translation of Heron's text [Schöne 1903, 16–24].

> If there be two numbers *AB* and *BC*, then the square root of the-square-of-*AB*-multiplied-by-the-square-of-*BC* will be the product $(AB \cdot BC)$.[7] For, since [**S** 18] as *AB* is to *BC*, so is the square of *AB* to the product $(AB \cdot BC)$, and also the product $(AB \cdot BC)$ to the square of *BC*, therefore also as the square of *AB* is to the product $(AB \cdot BC)$, so will the product $(AB \cdot BC)$ be to the square of *BC*.
>
> But if three numbers are in proportion (ἀνάλογον), the product of the extremes will be equal to the square of the mean. Therefore the square of *AB* multiplied by the square of *BC* will be equal to the number $(AB \cdot BC)$ multiplied by itself. Therefore the square root of the product of the-square-of-*AB*-and-the-square-of-*BC* will be the number $(AB \cdot BC)$.

Assertion: $\sqrt{AB^2 \cdot BC^2} = (AB \cdot BC)$. (Heron tells us that the square root of a square number n^2 is n.)

Proof: Since $AB : BC = AB^2 : (AB \cdot BC)$ and $AB : BC = (AB \cdot BC) : BC^2$, therefore $AB^2 : (AB \cdot BC) = (AB \cdot BC) : BC^2$ (transitivity of ratio [*Elements* V.11]).

Definition: If $p^2 : pq = pq : q^2$, then the three numbers p^2, pq, q^2 are said to be *analogon*, in (continuous) proportion. That is, pq is the mean proportional between p^2 and q^2, and $(pq)^2 = p^2 \cdot q^2$.

By *Elements* VI.17 and VII.19,

$$AB^2 \cdot BC^2 = (AB \cdot BC) \cdot (AB \cdot BC)$$

That is,

$$\sqrt{AB^2 \cdot BC^2} = (AB \cdot BC).$$

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[6] If, in Heron's text, *AB* is understood to be a number, I translate with the "square *of AB*."

[7] Literally, "the number contained by ABC" (τὸν ὑπὸ ΑΒΓ περιεχόμενον ἀριθμόν). In the next sentence, by a fairly standard practice of ellipsis, this becomes "the by ABC" (τὸν ὑπὸ ΑΒΓ).

Heron's *Metrika* I.8

> There is, however, a general method[8] to find the area of any triangle without [knowing] a height if [the] three sides are given. An example: let the [lengths of the] sides of the triangle be 7, 8, and 9 units. Add together $7 + 8 + 9$, that is 24. Take half of them: 12. Subtract 7 units, 5 left; then, subtract 8 from 12, 4 left. And also [subtract] 9, 3 left. Multiply 12 by 5, result 60; and those by 4, result 240; and those by 3, total 720. Extract the square root, which will be the area of the triangle. Since, however, 720 has no rational root, we will take its root with the least difference in the following way: Because the nearest square to 720 is 729 and has root 27, divide 720 by 27, that is 26 and two thirds; add the 27, that is 53 two thirds. Take half of that, 26 ½ ⅓. Therefore the square root of 720 is approximately 26 ½ ⅓, for 26 ½ ⅓ multiplied by itself makes 720 1/36, such that the difference [**S** 20] is 1/36. But if we want the difference expressed in a lesser part than 1/36, we may use the value just found, 720 1/36 instead of 729; and by so doing we will find that the difference becomes much less than 1/36.[9] The geometrical proof for that is the following:
>
> To find the area of a triangle, given its sides.
>
> It is of course possible to draw one height and calculate its length and find the area of the triangle, but now we must calculate the area without [knowing] the height. [**S** 22] Let the given triangle be *ABC*, and let each of [the sides] *AB*, *BC*, *CA* be given; to find the area. [See Figure 4.]
>
> Let the incircle *DEZ* with centre *H* be inscribed in the triangle, and let *AH*, *BH*, *CH*, *DH*, *EH*, *ZH* be joined.
>
> Now the rectangle ($BC \cdot EH$) is double the triangle *BHC* [*Elements* I 41], the rectangle ($CA \cdot ZH$) is double the triangle *CHA*, and the rectangle ($AB \cdot DH$) is double the triangle *AHB*. Therefore the rectangle contained by the perimeter of the triangle *ABC* and *EH*, viz. the radius of circle *DEZ*, is double the triangle *ABC*.
>
> Let *CB* be produced [to *G*] and let *BG* be made equal to *AD*; thus [the straight line] *CBG* is half the perimeter of the triangle *ABC* because $AD = AZ$, $BD = BE$, and $CZ = CE$, and so the rectangle ($CG \cdot EH$) is equal to the triangle *ABC*.

The next passage[10] is the one that troubled Dijksterhuis [1956, 412], among others, although we should be warned by *Metrika* I.7, above. Dijksterhuis rightly commented that the "squares on *CG* and *EH*" have lost their direct geometrical meaning and are looked upon as dimensionless magnitudes (or numbers) which can be squared in their turn. I think that the passage can be understood by inventing its geometrical counterpart (marked < ... >), inserting that before it, and considering the troubling passage as Archimedes' own translation into terms of measurement, in accordance with what we learned in *Metrika* I.7.

> But < since any rectangle is the mean proportional between the squares on its sides, the rectangle ($CG \cdot EH$) is the mean proportional to the square on *CG* and the square on *EH*. Thus the triangle *ABC* is the mean proportional to the square on *CG* and the square on *EH*. Therefore > the rectangle ($CG \cdot EH$) is the side [i.e. square root] of the square of *CG* multiplied by the square on *EH*; thus the area of the triangle *ABC* multiplied by itself is equal to the square of *CG* multiplied by the square on *EH*.

As a matter of fact, it is safe to invent a geometrical counterpart because the following reasonings are perfectly geometric and in accordance with the theory of magnitudes and proportion in the *Elements*. In most texts in Greek geometry, analysis and heuristics are suppressed and only a synthesis is presented; such is also the case here: Heron now starts a construction at random, it seems, conjuring up a very informative diagram (Figure 4) that sequentially proves the whole thing. It is, however, instructive (and often very entertaining) to try to reconstruct the heuristics by turning the synthesis upside down. I hope to have done so above in the introduction.

> Let *HL* have been drawn at right angles with *CH*, and *BL* with *CB*, and let *CL* be joined. Since either of the angles *CHL*, *CBL* is right, [*CL* is a diameter and] the quadrilateral *CHBL* is [inscriptible] in a circle, and so the angles *CHB* and *CLB* are [together] equal to two right angles [*Elements* III.22].

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[8] That is, besides the various methods shown in the previous theorems.

[9] Heron's method can be understood as follows: If q^2 is the square nearest to n, we have $n = q^2 \pm r$, and $\sqrt{n} = q \pm f$, ($f < 1$). So $n = q^2 \pm 2qf + f^2 = q^2 \pm r$. If we ignore f^2, we have $r \approx 2qf$, and so $f \approx r/2q$. This method was known from Babylonian mathematics and was probably used at all times.

Heron uses the Egyptian concept of unit fractions instead of 26 5/6, a normal practice in Hellenistic arithmetic.

[10] Schöne [1903, 22, ll. 15–19].

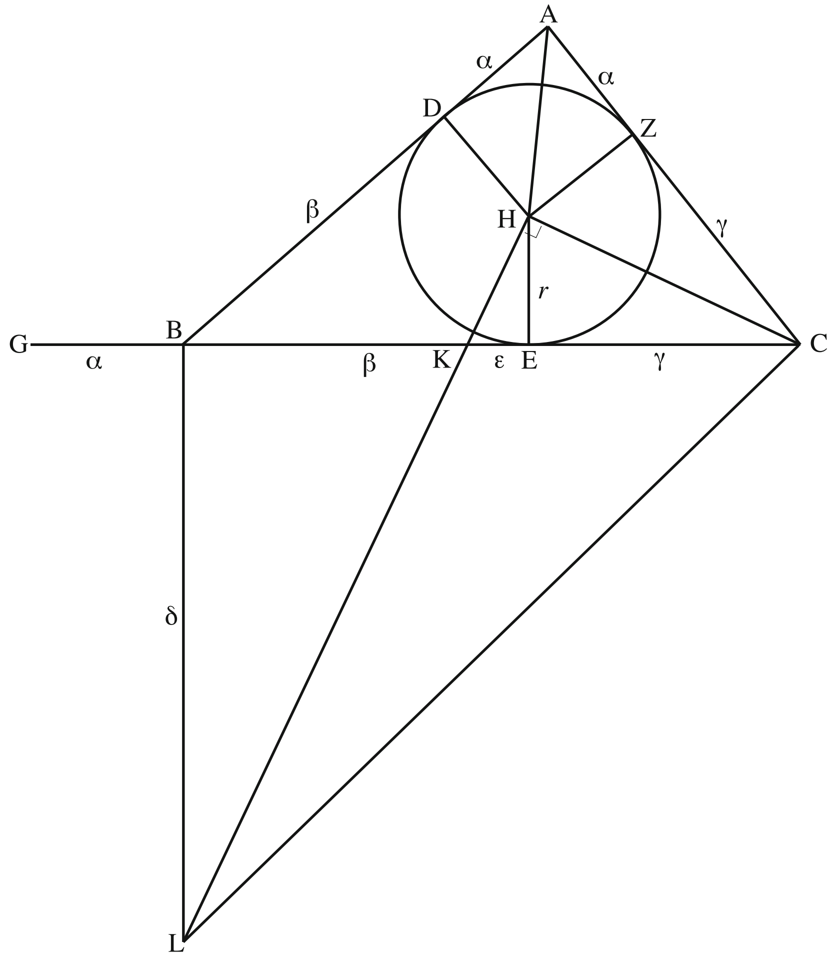

Figure 4: *Metrika* I.7 and I.8.

Why does he construct two right angles, $\angle CHL$ and $\angle CBL$? Oh, yes, to ensure an inscriptible quadrilateral, this is a good idea because the similar triangles below, *AHD* and *CLB*, seem to drop out — if not out of the blue, then out of the quadrilateral at any rate. If we only knew what they are good for. This reticence about what we are heading for is one of the most charming and irritating features in Hellenistic mathematics.[11]

> But also the angles *CHB* and *AHD* are [together] equal to two right angles; for the angles at the centre *H* are halved by *AH*, *BH*, *CH*, and the angles *CHB* and *AHD* are [together] equal to the angles *AHC* and *DHB* [together], and the sum of all of them equals four right angles. Therefore the angle *AHD* is equal to *CLB*. And the right angle *ADH* is equal to the right angle *CBL*; [S 24] thus the triangle *AHD* is similar to the triangle *CBL*. Therefore, as *CB* is to *BL*, so is *AD* to *DH*, that is as *BG* to *EH*, and *enallax* as CB is to BG, so is BL to EH [*Elements* V def. 12], that is *BK* to *KE*, because *BL* is parallel to *EH* [*Elements* VI.4]. And *synthenti*, as *CG* is to *BG*, so is *BE* to *EK* [*Elements* V.18].

That is,

$$\beta + \gamma : \delta = \alpha : r$$

and *enallax*,

[11] The reticence of the ancient mathematicians has been much discussed by early modern mathematicians and modern scholars, but Netz [2009] has recently discussed it in some detail.

$$\beta + \gamma : \alpha = \delta : r = \beta - \varepsilon : \varepsilon,$$

and *synthenti*,

$$\alpha + \beta + \gamma : \alpha = \beta : \varepsilon.$$

But, since $\alpha + \beta + \gamma = s$,

$$s : \alpha = \beta : \varepsilon.$$

Therefore also, as the square on *CG* is to the rectangle (*CG* · *GB*), so is the rectangle (*BE* · *EC*) to [the rectangle] (*CE* · *EK*) [*Elements* VI.1], that is to the square on *EH*, for *EH* is drawn in a right triangle perpendicular from the right angle to the base [i.e. hypotenuse, *Elements* VI.8, corollary].

That is,

$$s^{\square} : (s \cdot \alpha) = (\beta \cdot \gamma) : (\varepsilon \cdot \gamma) = (\beta \cdot \gamma) : r^{\square}.$$

Therefore the square of *CG* multiplied by the square of *EH*, the square root of which [product] is the area of the triangle *ABC* [because *ABC* is the mean proportional between those squares], is equal to the rectangle (*CG* · *GB*) multiplied by the rectangle (*CE* · *EB*).

That is,

$$(ABC \cdot ABC) = s^2 \cdot r^2 = (s \cdot \alpha) \cdot (\beta \cdot \gamma).$$

And each of [the segments] *CG*, *GB*, *BE*, *CE* is given, for *CG* is half the perimeter of the triangle *ABC*, *BG* is the difference between half the perimeter and *CB*, *BE* is the difference between half the perimeter and *AC*, and *EC* is the difference between half the perimeter and *AB*, because $EC = CZ$, $BG = AZ = AD$. Thus the area of the triangle *ABC* is given.

As often, Heron ends with a *synthesis*, in geometry meant as a constructive demonstration of the validity of the proposition. In this case, however, the *synthesis* is simply a numerical example, which does not prove any validity unless one calculates the area of the said triangle by another method. He may have thought of that, however, when choosing the lengths of the sides: I suspect that he knew how to find triangles with sides of integer length, by first finding two right triangles with one side of equal length; *in casu* 5, 12, 13 and 9, 12, 15. A method to find such triangles (of which there are infinitely many, even with prime lengths) was well known in Hellenistic mathematics. The length 12 is the height of the triangle on the base 14.

It is calculated in the following way: Let *AB* be 13 units, *BC* 14 units, and *AC* 15 units. Add 13, 14 and 15, and 42 results; of which half becomes 21. Subtract 13, 8 remain; the same with 14, 7 remain; and lastly 15, 6 remain. 21 by 8, and the product by 7, and yet again the product by 6, 7056 results. The square root thereof is 84; so big will the area of the triangle be.

Epilogue

I have no doubt that this theorem was meant, stated and proved as a genuine geometric proposition, and then — when applied in mensuration, which of course was its *raison d'être* — summed up in an arithmetical style. Obvious relatives are the propositions in Archimedes' *Mensuration of a Circle*, and like them it is more than probable that the text underwent several "emendations" on its way to classrooms. However, Heron seems very painstaking, in *Metrika* I.7, in preparing our minds for the obnoxius concept of multiplying a square by a square and finding the "side" of such a monster-square. It remains (to me, at least) a wonder when looking into Hellenistic mathematics why millenia had to pass before arithmetic got a footing as solid as, or more than, Euclid's geometry. Why didn't it trouble them? But then, what do I know about the troubles they've seen?

References

Becker, O., 1933. Eudoxos-Studien II. Warum haben die Griechen die Existenz der vierten Proportionale angenommen? Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik, Abteilung B, 2, 369–387.

Dijksterhuis, E.J., 1956. Archimedes. Ejnar Munksgaard, Copenhagen.

Heath, T.L., 1926. The Thirteen Books of Euclid's Elements. Cambridge University Press, Cambridge.

Mueller, I., 1981. Philosophy of Mathematics and Deductive Structure in Euclid's Elements. MIT Press, Cambridge.

Netz, R., 2009. Ludic Proof. Cambridge University Press, Cambridge.

Schöne, H., 1903. Heron von Alexandria: Vermessungslehre und Dioptra, Heronis Alexandrini opera quae supersunt omnia, vol. 3. Teubner, Leipzig.

Taisbak, C.M., 1980. An Archimedean proof of Heron's Formula for the area of a triangle; Reconstructed. Centaurus 24, 110–116.