Mathematics and Her Sisters in Medieval Islam: A Selective Review of Work Done from 1985 to 1995 [1997]

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Abstract This paper surveys work done over the past decade, largely in Western Europe and North America, in the history of the mathematical sciences as practiced in medieval Islam from central Asia to Spain. Among the major topics covered, in addition to the usual branches of mathematics, are mathematical geography, astronomy, and optics. We have also given accounts of some current debates on the interpretation of important texts and, in addition, we have surveyed some of the literature dealing with the interrelation of mathematics and society in medieval Islam.

Ten years ago we surveyed recent work in the history of the mathematical sciences in medieval Is- 408
lam [10], including mathematics itself and the sciences of astronomy, astrology, cartography, optics, and music, where mathematics played an essential role. Since that time much has been published in these areas, and it seems fitting once again to survey what has been learned about medieval Islamic achievements in the mathematical sciences during the past decade.[1]

At the outset, however, two comments are called for. The first is that the present survey is largely restricted to works published in North America or Europe and, in particular, refers neither to works published in the former Soviet Union, where a long tradition of significant work exists, nor to works entirely in Arabic. Moreover, although the author has tried to reference most of the contributions of those working actively within the selected areas, it is inevitable that some important work will have been left out, for which sins of omission he can only ask forgiveness.

Second, although some cases have been noted in which a particular work has occasioned some critical comment this does not mean that all other work has been, or should be, accepted as established truth. All results in an area such as this are more or less tentative, if only because the mass of unexplored material is so large in relation to what has been studied, and the decision to report certain controversies means only that the issues raised by the controversy appear to be of particular interest.

The words "mathematical sciences" emphasize that the medieval Islamic scientists worked on such a variety of topics that to focus only on what we might call mathematics is to ignore areas in which the medievals exercised their creativity to great effect and which many of them saw as the *raison d'être* of the whole. Thus, one of the earliest (and possibly the best known) of the medieval Islamic scientists,

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[1] The author has made a serious search for all published papers from the period 1985–1995 not cited in his earlier survey [10] and has referenced such as he felt might appeal to a nonspecialist audience. As of this writing the year 1996 has not finished so a thorough survey of that year is impossible. The author has therefore referred to such papers from 1996 as have come to his attention and which add something significant to a major theme of this paper.

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Muḥammad ibn Mūsā al-Khwārizmī, contributed significantly not only to arithmetic and algebra but also to astronomy and geography. To ignore these is to impoverish the history of mathematics.[2]

Al-Khwārizmī and His Times

Of course the role of al-Khwārizmī's treatise on Hindu arithmetic in introducing that arithmetic to the Latin West in the 12th century is well known, and André Allard's study [171] of the four main relevant Latin texts seemed to take the subject just about as far as it could go. For that very reason special interest is given to Menso Folkerts's discovery [51] of a new manuscript — found in New York! —
409 of | al-Khwārizmī's introduction to Hindu arithmetic. This new manuscript not only is more detailed than the previously known Cambridge exemplar, of which an English translation [32] has recently been published, but also is more carefully written and has fewer nontrivial errors. Additionally, it has the virtue of being complete, whereas the Cambridge manuscript breaks off midway through the discussion of division of fractions and before the discussion of square roots. With Folkerts's discovery we are now closer to a knowledge of the original form of the text that introduced arithmetic to Western Europe.

As far as basic numeration itself, David King is investigating (see, for example, [94] and [95]) the history of a forgotten system of numeration. Although it has been studied by other scholars (e.g., Jacques Sesiano [142]) King has done a thorough study of its history, from its origins in Greek stenography down to the Nazi party in Germany. Never intended for use in calculation, this simple system in its final form allows one to write any numeral from 1 to 9999 as a single cipher. King has documented the use of the system in medieval astrolabes and has found in Arabic treatises[3] on cryptography characters very close to those in the European manuscripts.

Whether al-Khwārizmī, who wrote in the first half of the ninth century, was also the first Islamic writer on algebra continues to be a subject of some controversy. Further ammunition for the doubters comes from an Arabic fragment from the Yemen [89] which names the seventh-century Caliph ʿUmar as a patron of certain "algebraists" who came from the province of Fars and taught orally. It also mentions both the future Caliph, ʿAlī, as an evidently able pupil, and al-Khwārizmī as one who, much later, under the reign of al-Maʾmūn, first wrote down this knowledge. King thinks the report is dubious, and this may well be true of the part about ʿAlī, but the suggestion that algebra was initially an oral tradition, which came early to the Islamic world from Persia, fits well with arguments of Jens Høyrup [71], who has argued that medieval Islam fell heir to two ancient "sub-mathematical" traditions, and employed ancient algebraic techniques and a cut-and-paste geometry of the sort one finds systematized in *Elements*, II, to justify these techniques.[4]

Methods for solving quadratic equations were known to the Babylonians more than two millennia before al-Khwārizmī's *Algebra*. However, al-Khwārizmī's work transforms what had been with the Babylonians a systematized method for solving quadratics into the science of algebra, i.e., a method derived from explicitly stated principles which made it clear to the learner why the procedures worked. It also became a step to the solution of higher degree equations, starting with the pure equation $x^n = a$. However, since the neighboring Chinese civilization had also discovered methods for solving higher degree equations, it is inevitable that questions of dependence have arisen. In a close analysis of the
410 algorithms employed | for root extraction, Chemla [29] has used a method of Allard [4] to compare the Chinese and Arabic algorithms and to reach conclusions that upset some previously held beliefs.

Sharaf al-Dīn al-Ṭūsī's monograph on algebra [152] is by a mathematician who worked in the late

[2] King [91, 125–126], for example, stresses that "our [astronomical] material is based on mathematical procedures" and goes on to itemize the mathematical riches to be found in the astronomical texts, e.g., tabulation and graphical representations of double-argument functions, double-order interpolation, and computational devices.

[3] The author thanks Professor King for sending him relevant material from MS Leiden Or. 14, 121 and another MS in a private collection in Frankfurt, to be discussed along with other material on Arabic numerical codes in his book, *The Ciphers of the Monks: A Forgotten Number-Notation of the Middle Ages* currently in preparation.

[4] This idea of proofs is, according to Høyrup [71, 475], where the Greek element enters.

12th century and carried forward the work of Omar Khayyām's *Algebra* in two respects. First, Sharaf al-Dīn gave complete and correct discussions of the possibilities of solving each cubic equation by means of intersecting conics, whereas Omar had not been able to do so.[5] Second, he employed what is essentially the Ruffini-Horner algorithm for solving the possible cases numerically, a topic Omar does not discuss.

A debate has arisen over the ideas behind Sharaf's procedures. In discussing the possibilities of solution of a cubic equation of the form $f(x) = x^3 + ax^2 + bx = c$, Sharaf al-Dīn exhibits a quadratic polynomial equivalent to $f'(x)$ and shows that if a is the root of $f'(x) = 0$ that yields the larger value of $f(a)$ then $f(x) = c$ has no root for $c > f(a)$ and one root if $c = f(a)$.[6] Roshdi Rashed feels that this is evidence that Sharaf al-Dīn had recognized the derivative, but Jan Hogendijk [64][7] argues that this is not the case and that one can obtain Sharaf al-Dīn's conditions by arguments that do not go beyond the mathematics of *Elements*, II. The continuing efforts to understand Sharaf al-Dīn's motivation in more modern terms, found in Nicholas Farès [50] and Christian Houzel [73], show that this controversy has not yet been settled.[8]

Further hints of algebra prior to al-Khwārizmī come from Barnabas Hughes [74] in an extract from the work of Ayyūb al-Baṣrī, an estate divider, found in a treatise compiled in the 11th century by a certain Abraham or Ibrāhīm. Ayyūb gives a threefold rule (called in Latin the *regula infusa*) for solving linear equations by reducing the coefficient to one. Hughes suggests that this may be evidence of earlier methods which al-Khwārizmī replaced by a single approach.

Whatever al-Khwārizmī's priority (or lack thereof) in Arabic algebra, his theory of quadratic equations was widely influential, and a study of this influence forms part of a survey of the history of quadratic equations in medieval Islam and Europe in Yvonne Dold-Samplonius [45]. (An account with less mathematical detail may be found in Dold-Samplonius [46].)

Al-Khwārizmī is as well a seminal figure in the history of astronomy, to which he contributed a treatise on the astrolabe, one on astrology (possibly), and an | astronomical handbook, called *zīj* in 411
Arabic, consisting of tables and rules for their use. His *Zīj al-Sindhind*, one version of which is extant in a Latin translation of an edition put together ca. A.D. 1000 in Spain, is unique among surviving *zījes* in being based largely on Indian material.

Al-Khwārizmī's astronomy shows that he learned more from the Hindus than simply the decimal arithmetic of whole numbers and sexagesimal fractions. Recent evidence of this comes from a reconstruction of the Sine[9] table underlying a table of an astronomical function compiled by al-Khwārizmī. Hogendijk [66] has shown that the underlying radius for the Sine table is 150, a known Indian parameter, and that the values for the Sines of multiples of 15° are also Indian. To interpolate between these values al-Khwārizmī used a modified version of linear interpolation that takes account of the fact that the values of the Sine increase most rapidly near 0° and less so as the argument approaches 90°. Adolphe Rome discovered this sensible, if crude, method of interpolation in the works of Ptolemy (ca. A.D. 150), and it has recently been found in the works of al-Battānī, Kūshyār ibn Labbān, and al-Khalīlī (Van Brummelen [154; 156]).

More sophisticated methods of interpolation were also used. Glen Van Brummelen [154] studies those of al-Khalīlī, and a general (though very concise) survey may be found in [53]. A recent addition to the list has been studied by Rashed [123], who has found in a work by a 12th-century scientist,

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[5] It is enlightening to compare Omar's and Sharaf's treatments of the equation $x^3 + c = a^2$.

[6] Readers not familiar with medieval Arabic texts should be told that symbolic expressions such as $f(x), f(a), f'(x)$ and symbolism such as = and > do not appear in these texts, although a limited use of symbolism does occur in certain late texts from northwest Africa (al-Maghrib). Given the lack of symbolism some question may arise about our reference to what are, in the texts, purely verbal expressions (such as "cubes and squares and things") as "polynomials." However, since medieval Islamic mathematicians had developed tabular methods even for extracting roots of such expressions when the number of cubes, squares, etc., were specified (i.e., the coefficients were known) we feel the description of such expressions as "polynomials" is justified.

[7] This paper also contains a useful and concise summary of the contents of Sharaf's treatise.

[8] It is regrettable that Farès makes no reference to Hogendijk [64].

[9] We capitalize the names of the trigonometric functions to remind the reader that the medieval trigonometric functions denoted the lengths of certain lines, not their ratios.

al-Samawʾal, against the astrologers an extract from a work of al-Bīrūnī that shows that the latter knew of Brahmagupta's method of interpolation as well as one other Indian method and tried out both on the Cotangent function.

A problem that challenged the ingenuity of Islamic scientists is that of calculating the aspects between two planets in an astrological prediction. The solution that Hogendijk [63] studied in two versions of material from al-Khwārizmī's *zīj* is calculated in a set of tables based on a complex geometrical projection from the ecliptic to the equator and back again.

Another problem that astrology suggested to the astronomer-mathematicians is that of dividing the ecliptic into 12 astrological "houses."[10] Two papers by Edward S. Kennedy [81; 82] give accounts of two popular methods for accomplishing this division, the one using the prime vertical and the other using the equator.[11] An interesting problem that was solved in this connection is to find two angles, given their sum or difference and the ratio of their Sines.

Hogendijk [62] has found evidence of considerable ingenuity in the methods used in astronomical investigations as early as A.D. 780 when Yaʿqūb ibn Ṭāriq, in order to calculate the brightness of the
412 moon in terms of its position relative to the sun, | needed to calculate the values $\mathrm{Sin}(d)$, where d is the great circle arc between the sun and the moon. He had no spherical trigonometry, not even Menelaus's theorem, so he approximated $\mathrm{Sin}(d)$ as $\sqrt{(\mathrm{Sin}^2(\beta) + \mathrm{Sin}^2(\eta))}$, where β is the lunar latitude and η is the elongation between the sun and the moon. The idea of approximating a spherical triangle by a plane triangle and applying the Pythagorean theorem works because β does not exceed 5° and at the time when one is interested in computing the brightness the elongation η is not very great either, so approximating arcs by their Sines is reasonable. Although the Sine table Yaʿqūb used has $R = 3438'$, another Indian parameter for the radius of the reference circle, his use of d to compute lunar brightness does not seem to be Indian.

A valuable collection of sources and studies of early Islamic astronomy is Regis Morelon's collection of the astronomical works by or attributed to Thābit ibn Qurra [75], the great scholar from ninth-century Ḥarrān, who was employed as translator by the Banū Mūsā, one of whom recommended his appointment as astrologer to the Calif al-Muʿtaḍid. A summary of the main problems attacked in these treatises and some of the mathematical problems these involve — from the analysis of apparent solar motion to the study of conic sections — has been published by Morelon [114].

Finally, in regards to al-Khwārizmī's geography, King [87] has published a study of material, whose discovery he announced in the early 1980s, which he felt was arguably due to al-Khwārizmī but, in any case, was written in the early 9th century. In this material we have the earliest exact method for computing the direction of Mecca (the *qibla*) relative to one's locality, the direction in which Muslims are to pray. The problem of computing this direction can be described in a variety of ways, most directly perhaps as the geographical problem of finding the angle between two great circles: that joining one's locality to Mecca and the local meridian.

This solution employs methods using given points and arcs on the surface of a sphere to obtain lines and triangles inside the sphere, which one in turn uses to calculate or construct other angles and arcs on the surface. These methods, known from Indian sources — where they are used particularly for time reckoning — involve three basic tools (the Pythagorean theorem, the theory of similar triangles, and the Sine and Cosine) and differ considerably from the more elegant and popular solution known as the "method of the *zīj*es." However, there are intriguing similarities between the two methods, and there seems to be some agreement that this early method was modified by the great 10th-century astronomer Ḥabash al-Ḥāsib, on the basis of his researches on sundials, to create the latter method.[12]

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[10] These are not the same as the zodiacal signs. In particular they depend on both the locality and the time for which they are calculated.

[11] In the West, these methods were ascribed to Campanus and Regiomontanus, respectively, but Kennedy quotes recent work showing that al-Bīrūnī claimed credit for the method using the prime vertical and that the equatorial method was known to Ibn Muʿādh al-Jayyānī of Jaen in 11th-century Spain.

[12] See Berggren [9] for an argument that this method originated in gnomonics. See also Lorch and Kunitzsch [107] for the publication of Ḥabash's treatise on the uses of a celestial sphere set in graduated horizontal and vertical rings. (A survey of surviving celestial spheres in medieval Islam is Savage-Smith [141].)

Two other studies on the derivation of the *qibla* are those of Juan Carandell [27] and Ahmad Dallal [36]. The former deals with a method for the *qibla* which breaks down into two analemmas by Ibn al-Raqqām, who died in Granada in A.D. 1315. (The discovery of this method in a treatise on the construction of sundials for | religious purposes shows how the historian of mathematics must be 413
willing to expand the search for sources of material beyond the obvious ones.) Ibn al-Raqqām's method is interesting in that not only is it an analemma but it is one that can be derived from the method of al-Bīrūnī, though the error in al-Bīrūnī's exposition has been corrected. Dallal's study [36] shows that the working out of the details of a solution to the *qibla* problem valid for all localities must be attributed not to al-Kāshī, as the present author once thought, but to Alhazen (Ibn al-Haytham), who lived some four centuries earlier.

The Islamic Study of Euclid's Works

One of the Islamic mathematical sciences whose theoretical traditions were just being formed when al-Khwārizmī wrote was geometry, for al-Khwārizmī's lifetime coincided with the period when Euclid's *Elements* was first being translated into Arabic. A knowledge of the various Arabic translations of this key work is essential not only to an understanding of an important genre of Arabic mathematical literature, the commentaries on the *Elements*, but perhaps to a more secure knowledge of the Greek text of that work.

Yet, the complicated, and often obscure, story of how medieval Islam came to master the contents and mathematical spirit of this seminal work has not yet been written, and only recently have some essential elements of the history become available. For example, historians have long taken at face value al-Nairīzī's claim that his commented edition of the first six books was based on the second translation of the *Elements* into Arabic by al-Ḥajjāj ibn Yūsuf, one done for the Calif Harūn al-Rashīd.[13] However, Sonja Brentjes, in her studies of the text of *Elements*, I and II [22] found in al-Nairīzī's work, has strengthened doubts that this is the case raised by Engroff in his earlier study of *Elements*, V. We now realize that although al-Nairīzī's text does represent one of al-Ḥajjāj's two versions it is one with many interventions from a later translation, one by Isḥāq ibn Ḥunain and revised by Thābit ibn Qurra, whose astronomical works we mentioned earlier. It cannot be used uncritically as a guide either to al-Ḥajjāj's methods of translation or to his vocabulary.[14]

Vocabulary is a key element in two recent studies aimed at uncovering which features of existing translations of the *Elements* are due to al-Ḥajjāj. One of these studies is that of Gregg De Young [161], who shows that earlier material is mixed into manuscripts of the *Elements* from Northwest Africa (al-Maghrib) and Spain (al-Andalus) as: (1) marginalia giving al-Ḥajjāj's version of the enunciations of the propositions for *Elements*, II, where one encounters, e.g., *ḍarb* (product) instead of the usual *saṭḥ* (area); (2) the addition of further cases to *Elements*, III, — including an elegant variation of the construction in III, 32; (3) alternate proofs of *Elements*, VIII, 20–21; and (4) three condensed and altered proofs following *Elements*, X, 67. | De Young raises, without proposing any definite answer, the question of 414
why al-Ḥajjāj was closer to the Greek in *Elements* III, IV, and VIII than in II and X.

In a recent study Ahmed Djebbar [40] calls attention to other linguistic peculiarities of the al-Ḥajjāj material in the above-mentioned sources and elsewhere and suggests that perhaps the first of al-Ḥajjāj's translations was done under the influence of the Arabic mathematical vocabulary that developed furthest by the early ninth century, that of treatises on reckoning.

Brentjes [21] identifies material in a Paris manuscript as the first connected fragment we have either of al-Ḥajjāj's work or of something very closely connected with it. She argues, on the basis of this study, that al-Ḥajjāj's second so-called "translation" was, rather, a reworking of his first, that he did not work from a Syriac but a Greek text, and that selections from the *Elements* in secondary sources, such as the

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[13] Al-Ḥajjāj's two translations were the first translations of the *Elements* into Arabic. The first of the two was done for the Calif Harūn al-Rashīd and the second for the Calif al-Ma'mūn.

[14] On an addition to *Elements*, I, ascribed to Thābit, see Brentjes [20].

Epistles of the Brethren of Purity,[15] have more significance for the study of Islamic acquisition of the *Elements* than has heretofore been thought. Her work again rests on, among other methods, a detailed study of the vocabulary.[16]

Roger Herz-Fischler [56] investigates the Arabic tradition of a proposition on mean and extreme ratio that is implicit in the proofs of Theorems 2 and 7 of the Supplement to the *Elements* known as Book XIV, with a view to finding out whether the proposition itself was originally in the Greek text. He concludes, on the basis of evidence from Pappus, Arabic editions of the *Elements*, and the Arabo-Latin tradition that the Greek text did originally contain such a proposition. This is but one example of a larger point made by Wilbur Knorr [98] that Heiberg erred in his negative judgment of Klamroth's position that the Arabic text of the *Elements* is an essential witness to the original Greek text.

In addition to the history of the text of the *Elements* in Arabic, there is also the history of individual topics found in Euclid's work. For example, there is a rich tradition of Arabic commentaries on the theories of proportion found in Euclid, the standard study of which is the thesis by Edward Plooij [116]. On this topic, Bijan Vahabzadeh [153] draws attention to the close similarities between two attempts to use the notion of "part" to justify Euclid's rather nonintuitive definition of proportion by means of multiples — one of them by the 11th-century, Spanish scientist al-Jayyānī and the other by the British mathematician Nicolas Saunderson (1682–1739).

Another piece of Euclideana is, of course, the parallel postulate, and recently many Arabic texts dealing with the problem of the status of Postulate 5 of *Elements*, I, have been collected, translated into French, and published by Khalil Jaouiche [76], a work which will be of use not only to historians of mathematics but to those interested in the philosophy of mathematics as well.[17] Another important
415 survey | of the Arabic theory of parallels is by Boris Rosenfeld [128]. Whereas Jaouiche emphasizes how the Islamic mathematicians broke with most of the ancient commentators on Euclid, to create a true theory of parallels, Rosenfeld emphasizes the continuity of the whole endeavor and places it within the context of the history of non-Euclidean geometry.

Questions of parallel lines raise the problem of infinity in mathematics, but the problem arose in other ways as well. A discussion of some medieval Islamic thought on infinity and questions of atomism is found in Rachid Bebbouchi's work [6], and a late testimony to mathematical atomism is found in Rosenfeld [129, 97–101]. (The earlier cited work by the same author [128, 193–195] also contains a discussion of atomism in medieval Islam.) For the very important religious overtones of atomism, see Alnoor Dhanani's work [38].

Of the Euclidean works lost in the Greek, Hogendijk [60] identifies fragments of the *Porisms* in the works of two 10th-century geometers on the strength of Pappus's description of this work, and his auxiliary lemmas thereto, in his *Collection*, VII. In another study [67], he presents al-Sijzī's version of the problems in Euclid's *On Divisions*. Because he found them quite simple, al-Sijzī omitted the solutions and proofs for all but four of the problems from his source, which Hogendijk argues was a revised translation by Thābit ibn Qurra.

A witness to the active approach the medieval mathematicians took toward the ancient texts is Abū Sahl al-Kūhī's reworking of *Elements*, II. Al-Kūhī was one of the leading geometers of the late 10th century, and De Young's study [162] shows how he rewrote *Elements*, II, 1–10, to give the work greater unity and then added 17 propositions of his own.

A recent contribution by Sesiano [145] illuminates the motives of the early translators and shows how the best of them read the ancient sources actively, pen in hand. It presents a short piece by the above-mentioned Thābit solving a problem that Euclid suggested in his *On Divisions*. Thābit showed that if the side of a regular hexagon in a circle is contained in the smaller segment of that circle cut off by a side of an equilateral triangle, then the area between the two sides is one-sixth the area of the

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[15] This was a group of Muslim scholars of the late 10th century who composed an encyclopedic treatment of the sciences in support of a particular religious or political cause.

[16] A striking example is the use in the Paris manuscript of "talbīn" for "rectangle" or "square." Brentjes is able to point to the ascription of the use of the word to al-Ḥajjāj by a writer of the first half of the 12th century, Ibn al-Sarī.

[17] Remarks of both historical and didactic interest on the parallel postulate may be found in Hogendijk's review [61] of this work.

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Number Theory and Recreational Mathematics

The above "miniature" of Thābit at work on a text fits well with Hogendijk's argument [58] that Thābit not only discovered a rule for generating amicable numbers but used it to find a new pair of amicable numbers (17296 and 18416). Thābit's theorem on amicable numbers inspired the Persian mathematician Kamāl al-Dīn al-Fārisī (d. ca. 1320) to give a new proof. On the basis of al-Fārisī's work, however, the question arose in the early 1970s of whether medieval Muslim mathematicians proved the Fundamental Theorem of Arithmetic. Ahmet Ağargün and Colin Fletcher [3] argue convincingly that, despite a recent claim, al-Fārisī's treatise does not contain a proof of that theorem but something quite different, an explicit construction of the divisors of a number in terms of its prime factorization. | 416

Another topic in number theory with a long history in medieval Islam is that of magic squares, and Sesiano's [149] is an exposition of an 18th-century text by al-Kishnāwī containing a clear exposition of much earlier methods for the construction of simple magic squares of odd order, bordered magic squares,[18] partitioned magic squares of order $r \cdot n$, consisting of r^2 magic squares of order n, and squares in which all odd numbers appear in the central part and all even numbers appear near the corners. There is also a treatment of magic squares of odd order in which the center square is empty. Sesiano's most recent publication on this topic is [174], an edition of a work which he argues dates from the 11th century, since the obviously very thorough (but anonymous) author knows the 10th-century discoveries but is unaware of the full solution of constructing magic squares for numbers of the form $4k + 2$, an achievement of the 12th century.

Magic squares stand at the border of number theory and recreational mathematics, but a work part of which is solidly in this latter tradition is the *Liber mahameleth*, written in Latin in 12th-century Spain on the basis of Arabic sources. Sesiano [144] surveys a group of problems from this source, such as ladder and tree problems,[19] most of whose solutions depend on the Pythagorean theorem, and points out that the three approaches to the most common of these problems (solution by formula, proof by geometry, and solution by algebra) correspond to a traditional periodization of the history of mathematics: ancient Mesopotamia, Greece, and medieval Islam.

Other Aspects of the Hellenistic Tradition in Medieval Islam

An older contemporary of al-Kūhī (whose work on *Elements*, II, we discussed earlier) was Thābit's grandson, Ibrāhīm ibn Sinān, a talented mathematician of the first half of the 10th century. His treatise *On Analysis and Synthesis* is one of several written by prominent Islamic mathematicians on these two classical geometrical methods. Hélène Bellosta [7] shows that Ibrāhīm had two concerns in this work: to lay out an elaborate classification of geometrical problems according to the number and type of solutions they had, and to show how to do analyses and syntheses so that there is a complete correspondence between the two procedures. Ibrāhīm says he does this because of those who criticize analyses done in the usual manner, where one is always being surprised by the appearance of lines, etc., in the synthesis not found in the analysis.

The seriousness with which the classification of problems and the study of analysis and synthesis

[18] Sesiano [143] has edited the Arabic text of an earlier work, that of the 13th-century scholar al-Zanjānī on bordered-magic squares, a summary of which he had published earlier. For a recent publication in English on this topic, see Sesiano [147]. Sesiano [150] has published the Arabic text and a German paraphrase of a treatise on magic squares by Jamāl al-Zamān al-Kharaqī, who died in 1138/9.

[19] Such denomination of problems corresponds to medieval usage. For example, Sesiano [144, n. 17] cites al-Bīrūnī putting "ladder problems" into the domain of recreational mathematics.

417 were taken in the 10th century is shown by the fact that not only did | Ibrāhīm in the early part of the century write on the subject, but late in the century Alhazen, whose universal solution to the problem of finding the *qibla* we mentioned above, also wrote on it, in a work which has been studied by Jaouiche [77]. Jaouiche points out that the Arabic writers saw analysis as not being confined to geometry, viewing algebra too as a kind of analysis, and Alhazen showed how analysis could be applied to all four classical branches of mathematics: geometry, arithmetic, astronomy, and music. In addition he stressed the need for creativity in analysis, that it was not a simple algorithmic procedure. (A subsequent study of Alhazen on analysis and synthesis is Rashed [181]; a critical edition of the text, with translation and commentary by A. Djebbar and Kh. Jaouiche, is to appear.)

Much of the best of the medieval Arabic work, however, was not inspired by Euclid but by Archimedes, and in our previous survey [10] we called attention to the need for a study of the Archimedean tradition in the Islamic middle ages. Happily this gap in the literature has begun to be filled, and two studies of the Archimedean tradition in medieval Islam are (1) a philological study, by Richard Lorch [109], of Arabic, Hebrew, and Latin material germane to the problem of the direct transmission of the text of *On the Sphere and Cylinder* and (2) a study, also by Lorch [108], of the indirect transmission of the same work based on the theorems and proofs of a treatise on isoperimetry by the 10th-century astronomer Abū Jaʾfar al-Khāzin, who, according to Omar Khayyām, used conics to solve the cubic equation to which al-Māhānī had reduced the problem Archimedes posed in *On the Sphere and Cylinder*, II, 4.

A curious feature of the Archimedean tradition in medieval Islam is that only two works of the present Archimedean canon were transmitted to medieval Islam, namely both parts of *On the Sphere and Cylinder* and *On the Measurement of the Circle*. If, however, one accepts the view, for which a case can be made,[20] that the medieval period in Islamic science extended into the 19th century, it is relevant to note here the recension of Archimedes' *On Spiral Lines* done by al-Yanyawī[*] ca. 1700 (Berggren [16]).

The other Archimedean works known in medieval Islam are works which, in their entirety, are spurious but may well contain parts of authentic Archimedean works. Thus it is entirely consistent with the general character of the Archimedean tradition in medieval Islam that Sesiano [146] should have found, among the theorems in an Arabic manuscript in Teheran, part of a lost work in which Archimedes established the formula for the area of a triangle in terms of its sides.

Other recent explorations of the Archimedean tradition in medieval Islam are in Rashed [124; 125]. The former, on the tradition in planimetry and stereometry, shows how Islamic mathematicians from the 9th to the 11th centuries modified Archimedes' proofs and extended his methods to solve what were for them new problems. The latter makes available the commentary of the Arab savant al-Kindī
418 on Prop. 3 (on the circumference and diameter of a circle) of Archimedes' *Measurement of the Circle*. |

An extended, close examination of the medieval (Arabic and Latin) textual tradition of Archimedes' *Measurement of the Circle*, together with Arabic treatments of the other two famous classical problems of cube duplication and angle trisection, is found in Knorr [97].

Also in the Archimedean tradition is al-Kūhī's solution of a problem inspired by Archimedes' *On the Sphere and Cylinder*, II, 5–6: to construct a spherical segment whose (curved) surface is equal to that of one spherical segment and whose volume is equal to that of another, recently published by Berggren [19]. In modern terms this problem demands the solution of two cubic equations, but al-Kūhī did not phrase the problem in algebraic terms. Rather, he viewed it as a geometrical problem of constructing line segments satisfying certain relations, which he constructed by intersecting conic sections — in this case a hyperbola and a parabola, the same two types of conics that Omar Khayyām used 70 years later to solve the type of cubic equations arising from al-Kūhī's problem.

One of al-Kūhī's contemporaries was al-Sijzī, whose writings have been an important source for Hogendijk's successful searches for material from lost works of Euclid and Apollonius [60] and [59].[21]

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[20] Certainly there was a tradition of studying and copying medieval works until that time.

[*] Editors: In the original paper the name is spelled al-Yānināwī; however, al-Yanyawī is the more common spelling.

[21] Kunitzsch and Lorch [106] give convincing codicological evidence that a Paris manuscript long supposed to have been copied by al-Sijzī was in fact copied by him.

(The latter work [59] contains traces of the *Neuses*, the *Plane Loci*, and *On Tangencies*, which must therefore have been available in Arabic.) Al-Sijzī's works have also been sources for study in their own right. Two examples are Rashed [121], devoted to an influential treatise by al-Sijzī on the asymptotes of the hyperbola, and Crozet [33], which is devoted to a treatise in which al-Sijzī attempts to calculate the volumes between, for example, three spheres, a pair of which are disjoint but contained in the third. Crozet is certainly right in saying that the point of the work is not easy to see, but his conjecture that it is a semi-intuitive exploration of ideas of "dimension" going beyond those of al-Sijzī's time into the idea of four-dimensional spheres is unwarranted.

Much of the medieval work in the Archimedean tradition, to say nothing of that on trisection, cube duplication, and related problems, required considerable knowledge of conic sections, and a major new source for the study of the Islamic tradition of conics is the publication of an Arabic translation, by the Banū Mūsā and Thābit ibn Qurra, of Apollonius's *Conics*, V–VII, the Greek text of which is lost, available in Gerald Toomer's edition, translation, and study [151].

In view of the theoretical importance ellipses assumed in astronomy as a result of the work of Kepler it is interesting to note that the 11th-century, Spanish astronomer Azarquiel (Ibn al-Zarqālluh) used them to approximate the path of the center of Mercury's epicycle in his design of an equatorium. Mercè Comes [31] has published the treatise in which this occurs, along with another by Ibn al-Samḥ.[22]
As Julio Samsò and Honorino Mielgo [140, 292] point out, in using the ellipse Azarquiel | had no 419
"theoretical pretentions," and indeed they establish that none of the writers of Spanish or northwest African *zij*es used the elliptical approximation as a basis for the difficult task of computing the positions of Mercury.

Another medieval application of conic sections, which has been published in Rashed [126],[23] is Abū Saʿd al-ʿAlā ibn Sahl's application of hyperboloids of revolution in the 10th century to design (1) a plano-convex lens that would focus an incoming bundle of parallel light rays to cause burning at a given distance and (2) a biconvex lens that would focus a pencil of rays emanating from a point source at a finite distance.

Other evidence of medieval Islamic investigations of the hyperbola is al-Kūhī's discovery of the focus-directrix property of that curve [57].[24] He used this property for the more difficult of the two cases of inscribing an equilateral pentagon in a given square, which is equivalent to constructing a solution of a fourth-degree equation.

A more recent publication of one of al-Kūhī's works that makes essential use of conic sections is Philippe Abgrall's publication [2] of a treatise in which al-Kūhī uses the method of analysis and conic sections to find on a line given in position the center of a circle tangent to two objects, each of which may be a point,[25] a line, or a circle.

A fourth Hellenistic writer (other than Euclid, Archimedes, and Apollonius) whose works are of first importance for the history of the mathematical sciences is Ptolemy, and of his works the most influential in Islamic science was the *Almagest*. During the past decade Paul Kunitzsch has followed up his history of the Arabic–Latin tradition of the *Almagest* in medieval Islam [101] with a study [117] of the fate of the catalog of 1025 stars, which forms VII, 5, and VIII, 1, of that work, during medieval times. On a more popular level Kunitzsch and Smart [102] provide reliable accounts of the derivations of the names of 254 of the brightest stars.

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[22] Both treatises concern the equatorium, a medieval instrument for analogue computation of the position of the planets. The fact that the only known application of conics prior to the time of Kepler occurs in a treatise on an astronomical instrument is just one instance of many that could be adduced of the importance of the history of instruments for the history of mathematics.

[23] This was discussed earlier in Rashed [122].

[24] This property was known to Pappus and, doubtless, to other Greek geometers, but there is no evidence that the Islamic mathematicians learned this from the Greeks.

[25] A circle is, of course, tangent to a point when it passes through the point.

Projections of the Sphere in Medieval Islam

One of the most ancient treatments of the projection of a sphere on a plane was that of Ptolemy in *The Analemma*, where he laid out the mathematical apparatus for a theoretical consideration of the design of sundials. In the horizontal sundial with a vertical gnomon the celestial sphere is projected onto the plane of the horizon, and in the medieval Islamic world the question arose among specialists in gnomonics whether the hour lines are straight or not. In a brief but informative paper, Hogendijk [68] summarizes the history of the problem and the contribution of Alhazen, who was the first to show that all hour lines except the noon line on the sundial differ from straight lines.

Another work of considerable consequence for Islamic civilization was Ptolemy's *Planisphaerium*,
420 and the notes on this treatise by an 11th-century, Spanish astrono|mer, Maslama of Madrid, together
with some related texts have been published and studied by Kunitzsch and Lorch [105]. One year later, Lorch [111] contributed an account of material found in Maslama's notes that one may interpret as being a proof of the circle-preserving property of stereographic projection, a proof which he suggests may antedate what has been considered as the oldest proof of that result, that of al-Farghānī.

The instrument derived from Ptolemy's *Planisphaerium* is the astrolabe,[26] and al-Kūhī's work *On the Construction of the Astrolabe with Proof* has been published both by Berggren [17] and by Rashed [126]. Al-Kūhī begins with a comprehensive, theoretical discussion of the possible projections of the sphere onto the plane and the kinds of curves one obtains by such projections. After giving a general proof that stereographic projection maps circles onto circles or straight lines, he devotes the bulk of the work to solving problems in which certain parts of an astrolabe have been, as it were, effaced and one is required to reconstruct the whole from what remains.[27]

This treatise shows the importance of studying a person's work in its entirety before coming to any firm conclusions on it. For example, in his work *On the Division of a Line According to the Ratio of Areas* al-Kūhī poses the following problem: Given four points A, G, D, and B on a given line segment AB, divide GD at E so that $AE \cdot ED / GE \cdot EB$ is equal to a known ratio. Had we not known of his treatise on the astrolabe we would hardly have suspected that a problem forming the subject of Apollonius's *On the Determinate Section* would play a role in the problem of constructing an astrolabe given a circle parallel to a horizon of known latitude and the image of a point of known declination.

One reason that astrolabes are important in the history of mathematics is that they provided a ready challenge to geometers to find new methods of representing spherical arcs and circles on the plane. One of these circles, of course, is the ecliptic which, as the annual course of the sun, is fundamental for the astrolabe, and Christopher Anagnostakis [5] has studied some methods of representing it. Among the most difficult of the curves to draw on the astrolabe, however, are ones that medieval Islam contributed to that ancient instrument, the azimuth circles. Berggren [13] has given a survey and mathematical analysis of 10 medieval Islamic geometric methods for drawing these curves on the astrolabe. The study shows, among other results, that al-Kūhī's method for drawing azimuth circles may well have been discovered by him. In a subsequent study, Berggren [14] argues that Ḥabash al-Ḥāsib's method for drawing the azimuth circles was originally based on a geometric method, popular in both Greek and
421 Islamic mathematics, known as an analemma. |

The Muslim geometers realized that one could exploit the symmetry resulting from stereographic projection to replace redundant parts from projection from the north pole with necessary parts for projection from the south pole. This gave rise to a variety of "mixed" astrolabes, which the geometers were as ingenious in naming as they were in designing. One wanting a guide through the maze of terms ("the myrtle-form," "the crab-form," etc.) can consult the nicely illustrated, short piece by Lorch [110].

Another mapping of the sphere that stimulated considerable discussion in the 10th century is one that al-Bīrūnī in his *Treatise on Projection of the Constellations* [8, 52] referred to as producing a "melon-

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[26] Several papers not bearing directly on mathematical issues, but highly informative about the history of the astrolabe both in medieval Islam and the West, may be found in a collection of some of Kunitzsch's papers [103].

[27] The tradition of such problems stems from Ptolemy's *Planisphaerium*, where he solves the problem of finding the radius of the equator given the greatest circle in the plane (typically that representing the Tropic of Capricorn) and the distance from the south pole of the circle corresponding to it on the sphere.

shaped" astrolabe, and of which he said that eminent astronomers were counted both among the supporters and the detractors. The name "melon-shaped" seemed so strange that some doubted the validity of the reading of the Arabic, but Kennedy and Lorch [83] describe the mathematics behind this instrument as it is found in a treatise by Ḥabash, from which it emerges that the melon-shaped astrolabe is based on the polar azimuthal projection, which preserves distances between circles parallel to the equator and distances along the meridians. The great difficulty with the projection, which Ḥabash solved, is drawing the horizon and its almucantars and azimuths. Evidently, the astrolabe was called "melon-shaped" because the image of the horizon resembles a melon.

A third publication of an Arabic treatise on projections is Dallal's work [35] devoted to al-Bīrūnī's treatise, *Book of Pearls on the Projection of Spheres*. This work is devoted to the construction and use of the planispheric astrolabe for astrology and concludes with a discussion of such unusual types of astrolabes as "the boat astrolabe" and "the spiral astrolabe."

A more direct method of modeling the heavens — for purposes of visualization, instruction, and solving problems — is by means of a sphere. Such a sphere is marked with the principal celestial circles and (usually) some of the principal stars and/or constellations, and the surviving examples of such instruments have been studied carefully by Emilie Savage-Smith [141]. Such a sphere could rotate about its poles within a frame of two perpendicular circles, which represent the horizon and meridian of an observer. Lorch and Kunitzsch [107] have published both the text and translation of Ḥabash al-Ḥāsib's treatise on this instrument.

Important work on the Spanish and Northwest African tradition of astrolabes and other instruments has been done by the Barcelona school, under the direction of Samsò, whose students have published a series of works on the subject. Although much of their work has been published in Catalan, for which reason we do not cite it here, two recent English pieces are by Emilia Calvo [25; 26], on the construction and use of the universal plate of Ibn Bāṣo, who died in Granada in 1316. (The epithet "universal" refers to the fact that, unlike an ordinary astrolabe, Ibn Bāṣo's instrument can be used at all latitudes.) This ingenious device combines stereographic projections from two different points and gives the same set of curves different interpretations according to the problem to be solved. (Calvo has pointed out that the plate was rediscovered by Latin astronomers.) One also notes two | papers by Roser Puig [118; 119], 422
in the first of which she argues that the plate of the shakkāziyya astrolabe, one of the two universal astrolabes known from 11th-century Spain, was conceived as a simplification of the universal astrolabe invented by the Spanish astronomer Azarquiel, whose equatorium for Mercury we mentioned earlier. In the other, she studies the mathematics of a device found on the back of Azarquiel's astrolabe, where one applies an Indian technique to determine lunar parallax according to Ptolemy's theory.

Cartography and Geometry

In addition to their astronomical uses, projections were of cartographic interest, and one of the joys of the history of cartographic projections in Islam is that new material keeps turning up. A recent example is a *qibla*-finder which was sold at Sotheby's in London in 1989 and which consists of a projection of the medieval Islamic world onto a disk centered at Mecca. It is equipped with a rotatable ruler, and when the ruler is positioned on the square corresponding to a given city the *qibla* for that city may be read from the rim of the instrument and the distance to Mecca from the scale on the ruler.

The instrument, which was made in Isfahan, probably in the early 18th century, incorporates a projection which was thought to have been invented by the German historian of Islamic science, Carl Schoy, in the 1920s. But the instrument dates from 1710, and specialists believe that it is a descendant of a medieval prototype. Indeed, King [96, 8] argues that "al-Bīrūnī himself produced a Mecca-centered map something like the Isfahan world-map." Very much in the medieval tradition, the instrument maker represented the slightly curved meridians on either side of Mecca by straight lines. Had such an instrument not been found, no one would have suspected the existence of this ingenious cartographic solution to the problem of finding the *qibla*.

Most of the coordinates of localities on the *qibla*-finder come ultimately from a *Book of Longitudes and Latitudes of the Persians* (via the *zīj*es of Naṣīr al-Dīn al-Ṭūsī and Ulugh Beg). A valuable source for these and other coordinates is E. S. and M. H. Kennedy's publication [79] of thousands of coordinates of localities as recorded in medieval sources. For years it existed only as computer printouts which one could inspect in Beirut, Providence, or New Haven, so its publication in any form would have been welcome. However, E. S. Kennedy's long experience with using the catalog prompted four different sortings of the data (by locality, by source, by increasing longitude, and by increasing latitude), so the wait has resulted in a volume of much-increased usefulness.

An interesting mathematical problem arising from the ancient geographical doctrine of climata is the subject of Dallal [34]. The climata of classical geography were belts of the Earth's surface parallel to the equator determined by the maximum length of daylight, and in V, 9 of his *zīj*, the *Masʿūdīc Canon*, al-Bīrūnī calculates their areas. In his study of this treatise, Dallal argues that al-Bīrūnī used Archimedes' exact result for the calculation of areas of spherical segments — which leads to the expression $\pi \cdot$
423 $[(180 \cdot 56\ 2/3)/\pi]^2(\sin\varphi' - \sin\varphi)$ for the area of the belt bounded | by latitudes φ and φ' — and the
approximation of 22/7 for π. This same problem appears in a different context, with a different solution, as we shall see below.

Optics in Medieval Islam

In addition to cartography, another science of considerable interest to medieval Muslim scientists was optics, and several recent publications have considerably advanced our knowledge of medieval Islamic achievements in this area. Foremost among these is Abdulhamid Sabra's [133], an English translation and study of Alhazen's *Optics*, I–III, which treats the subject of direct vision.[28] (Sabra's translation is based on his earlier publication [55] of the Arabic text.) As an attempt "to examine afresh, and in a systematic manner, the entire science of vision and to place it on new foundations" Alhazen's work synthesizes physics and mathematics in a way that was radically new.[29]

An earlier study of Alhazen's optical writings is Sabra's discussion [132], which deals with his treatise, *On Seeing the Stars*, in the context of other writings by that author and Ptolemy. In this treatise Alhazen attempts to reconcile Ptolemy's treatment of what has come to be called the moon illusion[30] in his *Almagest* with that presented in his *Optics*. Alhazen shows mathematically that, on the basis of the principle of refraction and the assumption that there is a layer of air above the denser, vaporous atmosphere surrounding us but below the rarer aether, it is possible that the celestial luminaries will appear larger to our eyes than they ought.[31] Sabra has recently carried further his study of Alhazen's optical writings concerned with astronomical phenomena with [173], an edition, translation, and study of some of the questions raised in three sections (as Sabra calls them) of a collection of Alhazen's writings called *Solution of Doubts in the Book of the Almagest, Which a Certain Scholar Has Raised*.

Sabra's views that (1) Ptolemy's *Optics* was of limited influence (because it was largely unavailable) in medieval Islam and (2) Alhazen's "science of optics" was a science of vision and not a science of instruments designed to focus light, such as lenses or burning mirrors, have both been contested by, among others, Rashed, whose publication [126] makes available treatises dealing with, among other topics, mirrors and lenses that cause burning and optical magnification. One of the most interesting of these is a treatise by Abū Saʿd al-ʿAlā ibn Sahl,[32] which, according to Rashed, forces a radical revision of our view of medieval optics because it contains the first statement of the Law of Refraction (Snell's
424 Law). Sabra [134], among others, has dissented from this conclusion, but there is no doubt that it is |

[28] See the reviews by Berggren [18] and Kheirandish [84] (who raises some interesting historiographic issues).

[29] Sabra [133, II: liv].

[30] This refers to the apparently greater size of the full moon near the horizon than when it is high in the heavens.

[31] A critical edition of the text and an English translation of Ibn al-Haytham's *On Seeing the Stars* appears in Sabra and Heinen [131].

[32] See Rashed's earlier study of this treatise in [122].

an important text for a variety of reasons, including its mathematical treatment of the focal properties of lenses.[33]

Another point at issue between Sabra and Rashed is the strength of the Ptolemaic optical tradition in medieval Islam. But, whatever medieval Islam's acquaintance with Ptolemy's *Optics*, there is no doubt that Euclid's work of the same name had a strong Islamic tradition. In her study [85] of the problems of transmission of Euclidean optics, Elaheh Kheirandish has suggested that linguistic transformations, rather than conceptual innovations, either in the late antique versions of the *Optics* or in the Arabic translation, led to variants of the Euclidean optical theory such as that found in al-Kindī's *De aspectibus*, one which we find developed in Alhazen's *Optics*. She argues that the several possible meanings for many of the Arabic translations of terms that had come to have precise technical meanings in Greek may have provided parts of what one had come to think of as conceptual innovations in later medieval optics (in Arabic and Latin).

Kheirandish's forthcoming monograph[34] on the Arabic version of Euclid's *Optics* will, with Lorch's forthcoming study of the Arabic tradition of Theodosius's *Spherics*, contribute valuable material to the study of the history of what the Arabic writers called "the middle books" in medieval Islam, i.e., those books that were to be studied after one had mastered the *Elements* and before one began a study of the *Almagest*.[35]

Islamic Spain and Northwest Africa

Although Alhazen's *Optics* appears to have attracted little attention in the eastern part of the Islamic world prior to the work of al-Fārisī (whose work on number theory we discussed above), it appears that it was read in Islamic Spain (al-Andalus) soon after it was written, a fact which challenges the oft-repeated view that the western part of the Islamic world lagged scientifically behind the eastern part.[36] Indeed, one of the most interesting documents to come to light in the past decade has been that discovered by Djebbar and Hogendijk, a substantial part of the mathematical compendium, *Book of Perfection* (*Istikmāl*), written in the 11th century by Yūsuf al-Muʾtaman ibn Hūd, a savant who ruled the Spanish kingdom of Saragossa from 1081 until his assassination in 1085. A striking feature of the structure of the treatise is its philosophically based classification of theorems according to the Aristotelian concepts of genera and species. From Hogendijk's study [65] of the geometrical parts, we learn that Islamic mathematicians were aware of Ceva's | theorem as well as what we would now call 425
the projective invariance of the cross ratio. Djebbar's study of the arithmetical parts will appear soon.[37]

In his earlier study of algebra in the western part of the Islamic world, Djebbar [42] makes clear the considerable influence of the school of the algebraist of the eastern part, al-Karajī, who wrote in the latter part of the 10th century. He also makes some interesting conjectures about the sociopolitical reasons for the almost total lack of evidence of research in algebra in the western part of Islam prior to the 12th century.

The first part of another large mathematical treatise from the Islamic west has recently come to light. Aballagh and Djebbar [1] announced the discovery of 117 folios of the book by the (probably) 12th-century mathematician al-Ḥaṣṣār titled *The Complete* [*Book*] *on the Art of Number*.

Al-Ḥaṣṣār, Ibn al-Yāsamīn, Ibn Munʿim, and Ibn al-Bannāʾ are the mathematicians whose work is

[33] A clear account of what Ibn Sahl accomplished, along with some interesting speculations on how he might have discovered his result (and other matters of interest), may be found in Hogendijk's forthcoming review in *Physis*.

[34] This work is based on her Harvard thesis, *The Medieval Arabic Tradition of Euclid's Optika*, and will be published in [86].

[35] An important work in this corpus, Euclid's *Phaenomena*, has recently appeared. See [166].

[36] For this last point and for knowledge of the *Optics* in the eastern part of the Islamic world, see Sabra [133 II: lxiv]. For knowledge of at least some mathematical parts of the *Optics* in Islamic Spain, see Hogendijk [65, 220–222].

[37] See Djebbar [44]. A complete text of al-Muʾtaman's work has recently been found in a commentary by the mathematician Ibn Sartāq, his *K. al-Ikmāl*. Hogendijk and Djebbar plan a joint study of it.

highlighted in Djebbar's survey [43] of mathematical activity in northwest Africa from the 8th to the 16th centuries. The survey touches on arithmetic, combinatorics, and algebra as well as mathematics in the life of the medieval Muslim city; it emphasizes the impact of the teaching activity in mathematics, and it speculates on sociopolitical factors that may have influenced the growth and decline both of mathematical activities and of the reputations of certain mathematicians. The survey also provides a bibliographic entry to the work on the history of mathematics in northwest Africa by Djebbar's students over the past decade. The astronomical literature of that region is surveyed by King [91].

Djebbar's survey brings out the important role Islamic Spain played for the above region as a source of both texts and scholars. Further evidence of the important role Spain played in the history of mathematics comes from Lorch's publication [113] of the contents of those sections of the Sevillian astronomer Jābir ibn Aflaḥ's critique of the *Almagest* dealing with plane and spherical trigonometry. Particularly interesting are Jābir's different approaches to the exposition of plane and spherical trigonometry, the mystery of the sources of his work, and its considerable influence on the Latin West.

The Problem of Decline in The Islamic Mathematical Sciences

With al-Ḥaṣṣār and the other writers mentioned above we have entered what was at one time supposed to be the period of decline in Islamic science. However, recent research in the history of astronomy has shown the extent to which such a periodization of Islamic history does violence to historical reality. For over 40 years scholars have known that highly sophisticated alternatives to the Ptolemaic models for planetary motion were developed by the Damascus astronomer Ibn al-Shāṭir. More recent research has shown, however, that his work is simply one development in a tradition of reform of Ptolemaic
426 astronomy that began with Alhazen in the | 11th century, one whose ultimate goal was to produce planetary models that yielded predictions at least as good as Ptolemy's but, unlike his, did not require physical impossibilities, such as uniform motion of a sphere around a point other than its center. This reform movement developed principally at Marāgha in Iran under the patronage of Hūlāgū Khān, and the first serious models, which were proposed by al-ʿUrḍī in the mid-13th century, were developed later in that century by Naṣīr al-Dīn al-Ṭūsī and his student Quṭb al-Dīn al-Shirāzī. These non-Ptolemaic models continued to be explored through the 17th century. A primary source for these developments is al-Ṭūsī's exposition of cosmology, called the *Tadhkira*, in which work, recently translated and published by Ragep [120], al-Ṭūsī gives a detailed explanation of his modifications of Ptolemy's models.

The basic device that allowed this successful reform of astronomy was one that Kennedy has called the Ṭūsī couple: a linkage of two segments of equal and constant length, initially pointing in the same direction and rotating with constant angular velocities in the same plane. The angular velocity of the second segment relative to the first is twice that of the first around its center, and the effect is that the end of the second segment produces rectilinear motion. George Saliba and Kennedy [80] give an account of an extension of the device, also by al-Ṭūsī, which allowed it to operate not in a plane but on the surface of a sphere in order to model the planet's change in latitude during its course around the ecliptic.

Al-Ṭūsī confidently predicted that the spherical device worked just as the planar one did in changing rotating motion to rectilinear motion, but al-Shirāzī recognized that this was only approximately the case, and another of Naṣīr al-Dīn's students, al-Nīsābūrī, proved, using Menelaus's *Sphærica*, I, 11, that in fact one has what is now called hippopede motion (evidently first used in astronomy by Eudoxus of Knidus (400 B.C.) and recently studied by John North [115]). However, the width of the hippopede is very slight and all who wrote on it after al-Ṭūsī, right up through the early 16th century, assert that it is a good approximation to rectilinear motion.

Saliba has published a collection of his papers [137] treating many aspects of the whole reform movement, both the technical details of the theories and the wider issues of periodization in the history of science, the relation between theory and observation and the motivation for the reform. Of particular interest is the essential role that two pieces of mathematics, the Ṭūsī couple and ʿUrḍī's Lemma, played in the whole development, as well as the discussion of the mathematical equivalence of models developed by this school to those of Copernicus.

Further evidence of important later work in medieval Islamic science comes from one of the most brilliant periods of the whole enterprise in the late 14th and early 15th centuries. During this time the Timurid prince, Ulugh Beg in Samarqand, patronized a group of scholars, chief among them being the human microchip, Jamshīd al-Kāshī.

One of al-Kāshī's great achievements is the *Khāqānī Zīj*, based on an earlier *zīj*, the *Ilkhānī*, of Naṣīr al-Dīn al-Ṭūsī, and it is regrettable that this has not yet been published. Yet there have been a number of studies of various parts of it, among them Kennedy's study of its spherical trigonometry [78]. By al-Kāshī's time, over | four centuries had passed since the contemporaries of al-Bīrūnī had worked out 427
the basic theorems of spherical trigonometry, such as the Sine Law and the spherical analog of the Pythagorean theorem: $\cos a \cdot \cos b = \cos c$. Despite this, however, al-Kāshī uses not these theorems but the theorem of the complete quadrilateral, known to Menelaus ca. A.D. 100. Nor was he alone in this. Al-Bīrūnī's older contemporary, al-Kūhī, wrote a separate treatise on the calculation of rising times by Menelaus's theorem, which he recognized as being "old-fashioned" but which he defended on other grounds, and Dallal [36, 151] calls attention to this feature of Alhazen's treatment of the problem of determining the *qibla* as well.

Perhaps one has here an example of different conceptions of what is elegant: ours might be to use the more recent theory whereas al-Kāshī, al-Kūhī, and Alhazen evidently preferred a unified approach involving repeated applications of a single theorem.

Dold-Samplonius [48] studies al-Kāshī's work on the measurement of *qubba*'s (domes) in IV, Ch. 9, of his *Calculator's Key*. According to Dold-Samplonius [49, 94], earlier writers on the subject, such as al-Būzjānī, had also treated the topic of *qubba*'s, but architecture had advanced sufficiently by al-Kāshī's time that the earlier treatments no longer sufficed, for the vertical cross sections were no longer simple segments of spheres or cones. They were, however, still constructed by rotating an arc of a circle about an axis, which allowed al-Kāshī to approximate their volumes and surface areas by cones and their frusta. For the surface areas he uses results from Archimedes' *Sphere and Cylinder*, I, to calculate a factor (1.775) by which one should multiply the square of the diameter of the base to obtain the surface area. In fact, Dold-Samplonius points out that modern methods produce a value of 1.784, so al-Kāshī's value is only about 0.5% too small.

In the case of the volumes, al-Kāshī recommends multiplying the cube of the diameter by 0.306, whereas modern methods yield the factor 0.313. Once again, and for the same reasons, al-Kāshī has an underestimate, but this time the error is on the order of 2.3%, still a very moderate error, as Dold-Samplonius points out, compared with earlier manuals on the subject which, for the much easier case of the hemispheric dome, erred by almost 18% in the volume.

Dold-Samplonius's [47] is a study of al-Kāshī's directions for measuring the surface of the *muqarnas*, a honeycomb of squinches that gave an aesthetic solution to the problem of putting a round dome on a square base that so strikes the visitor to Islamic shrines and other buildings.[38] Al-Kāshī approaches this seemingly complex structure by analyzing it into simple polygonal shapes and providing factors that may be used to calculate the area of those shapes from dimensions that one can measure.

Another study showing al-Kāshī's virtuosity as a calculator is Kennedy's study [178] of two methods for calculating the equation of time,[39] those of Kūshyār ibn | Labbān and al-Kāshī. Although the part 428
dealing with Kūshyār must be modified according to the study of van Dalen [158], it is the study of al-Kāshī's method (in the Khāqānī *zīj*) that concerns us here. Al-Kāshī calculated for each integral value of the solar longitude λ the value of $\bar{\lambda}$, where $\bar{\lambda}$ satisfies the equation $\lambda = \bar{\lambda} + eq(\bar{\lambda} - \lambda_a)$, where λ_a is a constant and *eq* is a term correcting for the fact that the earth is not the center of the sun's orbit. He solves the above equation 360 times by an iterative method, and then calculates the value of the equation of time in each case. In every case but one, al-Kāshī's value is within one second of that calculated by a

[38] For a study of the construction of this feature of Islamic architecture by Mohammad al-Asad as well as a geometric analysis of many features of Islamic architectural decoration, see the recent work by Gülru Necipoğlu [172].

[39] This is the number of minutes that, depending on the time of the year, one must add or subtract to the time shown on an accurate sundial in order to obtain mean local time.

computer, and Kennedy supposes that the one exception is a scribal error rather than a computational slip.

Science and Society in Medieval Islam

A topic that cuts across the various branches of the mathematical sciences in medieval Islam is that of the interaction of these sciences with medieval Islamic society, an interaction that began with the formation of Islamic science. As we indicated earlier, an important part of this formation was what has been traditionally called the "reception" of foreign science — in this case principally Greek, but also Indian and Sassanid Persian — which several recent papers treat in different ways. Sabra [130] argues that the process must be seen not as a passive one of simple "reception" of ancient knowledge but, rather, as an active process of appropriation, in which Islamic scholars played a crucial role. Following on the appropriation, Sabra claims, came assimilation and naturalization of the formerly "foreign" sciences, and it is in this process, he argues, that we must search for answers to questions about the decline of science in medieval Islam. Berggren [15] adopts Sabra's concept of active recipients and explores how the Islamic features of the society affected both the acquisition and the subsequent development of the foreign sciences, including even such a basic science as arithmetic. Høyrup [72] emphasizes the unique character of the end result of this process and writes of the Islamic "miracle" of integrating "subscientific" mathematical traditions with the scientific traditions of Greek and Indian mathematics, an integration whose roots, he says, lay in the close connection of the religious and the secular in early Islamic fundamentalism.

The attitude of Muslim society, especially that of its religiously orthodox intellectuals, toward this foreign learning has been variously assessed since Goldziher's fundamental study of 1915 (translation in [167]), and Sabra discusses it at some length in his paper cited above. It also figures importantly in Sabra's study [135] of the development and role of an influential school of theology in the intellectual life of medieval Islam as reflected in the writings of, among others, Ibn Khaldūn and the 14th-century theologian al-Ījī. Particularly germane to the present essay are this school's views of scientific (in particular, astronomical) knowledge, for example, the question of the reality of the entities considered in mathematical astronomy. Consistently with his theological position, al-Ījī takes a fictionalist view of these entities and assures his readers [135, 37] that "... once you see these things [ecliptic, solsticial points, etc.] as mere imaginings which are more tenuous than a spider's web you will cease to be fright-
429 ened by the clanking noise of these words." | Brentjes [23] urges a re-examination of the question of the status of the foreign sciences in medieval Islam and offers, as a beginning, a study of al-Nuʿaimī's history of institutions of higher learning (*madrasas*) in Damascus between the 12th and 15th centuries. On the basis of this she argues that both the traditional picture of orthodox Islam's hostility to the ancient sciences and a belief in the exclusion of these sciences from the regular curriculum at the *madrasas* are mistaken.[40]

Another aspect of science in Islamic society is the existence of parallel traditions, the one "Islamic" and the other "foreign," of methods for attacking a range of problems with scientific overtones. Over the past 25 years King has explored the coexistence in medieval Islam of a strong tradition of a folk astronomy which employed no advanced mathematics and a mathematical astronomy stemming ultimately from foreign sources. As King points out in his introduction to a collection of his papers [180], the two traditions were virtually independent, and both offered methods to support three of the "five pillars of Islam" (prayer, fasting during the month of Ramadan, and the *ḥajj*). A broadly based survey of an important aspect of the folk tradition is in King [90], devoted to the use of shadows in time-keeping. Still to appear is King's *Survey of Astronomical Timekeeping in Medieval Islam*, I & II.[41]

In reckoning, too, a tradition of mental computation indigenous to many of the Islamic lands[42]

[40] On these points, see also Berggren [182, 314–318].

[41] To be published by Springer-Verlag. (Editors: The books were eventually published by Brill.)

[42] Because intermediate results in the mental calculations are stored by holding the fingers in certain positions, the system

coexisted with foreign systems stemming ultimately from ancient Mesopotamia and India. The book by Ulrich Rebstock [127] is a study of arithmetic in the life of Islamic society — as it was used by the merchants, functionaries, builders, and legal scholars — and it surveys the various ramifications of arithmetic in daily practice of a Muslim society on the basis of an extraordinary range of texts. Two other studies of the various arithmetic traditions in the Islamic world are Djebbar's study [41] of fractions in the Western part of the Islamic world, and Chemla's and Djebbar's comparisons [30] of Arabic, Chinese, and Indian treatments of fractions.

An interesting case of a science whose roots lay within Islamic society (e.g., in Arabic lexicography and poetry) is that of what we now call combinatorics but which, in medieval Islam, was considered simply one more branch of "reckoning" (*ḥisāb*). Djebbar [39] provides details on the many counting arguments, the progress from tabular enumeration to manipulation of rules, and the widened field of application of combinatorial methods one encounters in works such as those of the 13th-century writer Ibn Munʿim and of Ibn al-Bannāʾ in the 14th century. Despite this, however, he points out that combinatorics never became recognized as a branch of mathematics in its own right and asks, for example, whether the complete lack of mention of the work of Ibn Munʿim was in any way the result of the displeasure of an orthodox ideology which had come to power at a mathematician whose nonmathematical writings showed him to be a nonconformist. | 430

Another topic that Djebbar mentions as a field for combinatorial calculation was the theorem on the well-known figure from *Almagest*, I, 11, called "the secant figure," which asserted that for certain lines a, b, c, d, e, and f the ratio $a : b$ was equal to $c : d$ composed with $e : f$. The problem was to state and prove all 18 possible cases of this, and Sabine Koelblen [99; 100] discusses a variety of treatments of the problem in Arabic (and Latin) literature, from the separate geometric proofs of each case by Aḥmad ibn Yūsuf through the proof based on the idea of permutations by Thābit ibn Qurra to the elegant proof by Naṣīr al-Dīn al-Ṭūsī in his *Treatise on the Complete Quadrilateral*. In [99] Koelblen deals principally with the history of the problem from the point of view of combinatorics and finds in the texts she studies two different approaches to counting the number of cases. In [100] she studies the different modes of demonstration in the texts discussed in [99] and concludes with some well-taken warnings against seeing medieval work on composition of ratios as a step on the road to the conception of ratios as real numbers.

Mathematical Methods in the Study of Ancient and Medieval Sciences

In recent years a variety of techniques, combining mathematical, historical, and textual methods, have been developed for analyzing tables found in the *zīj*es. A recent example of the fruits these methods, long known, can produce is the study by José Chabás and Bernard Goldstein [28] of tables for the solar equation and planetary latitudes in a Spanish *zīj*. An ingenious suggestion the authors make is that the reference in the *zīj* to "Ebi Iusufi byn Tarach" is to Yaʿqūb ibn Ṭāriq, on the basis (among other matters) that the Biblical Yaʿqūb (Jacob) was indeed the father of Joseph (Ebi Iusufi)!

Within the recent past, two studies of ancient astronomical tables using newly developed statistical methods have been published. The first to appear was that of Benno van Dalen who, in his thesis at Utrecht [157], used four statistical estimators and ingenious *ad hoc* methods to shed new light on the parameters underlying the many tables found in ancient astronomical handbooks. Such handbooks are notoriously difficult to understand fully, not least because often the values of the parameters are not given in the text or, when they are, they may not be the ones used in the computation of the tables.

Van Dalen [158] shows the power of such methods when he determines not only the tabulated function and the underlying parameter values but even the author of a table about which no textual information is available. He shows that one of the tables appended to a version of Kūshyār's *Jāmiʿ Zīj* found in Berlin was most likely computed according to a mathematical formula known as "the method

is also called "finger reckoning" in Islamic sources.

of declinations" by Yaḥyā ibn Abī Manṣūr, a contemporary of al-Khwārizmī and the author of the solar
431 longitude table found in the Ashrafī *zīj*.[43] |

Finally, Van Brummelen [155] addresses the problem of how a medieval author computed large (on the order of 5000 entries) tables of auxiliary functions — specifically, what rounding procedures were used, on what table a given table is based, and, when interpolation is used (as it often is), how does one identify both the grid of values that are computed according to the rule given and the interpolation method used for the others? Van Brummelen was able to discover a grid as well as a possible interpolation scheme for one of the tables and to show that a second table depends on it.

General Surveys

In addition to the specialized literature referred to above there are a number of publications, whose intended audience ranges from the interested layperson to the historian of science with a specialty far from medieval Islamic mathematics but who is interested in a general picture of the field.

One such book is Berggren [11], whose focus is the history of arithmetic, algebra, geometry, and trigonometry in that part of the medieval Islamic world between Cairo and Samarqand. It also takes into account Islamic dimensions of these developments–including inheritance problems, the payment of tax, astronomical timekeeping, and the direction of prayer.

Samsò's book [139] provides coverage of the western part of the Islamic world with a history of the "ancient sciences"[44] in Islamic Spain.[45] A work in the same general area is by Samsò and Juan Vernet [138], valuable both for its text and for its beautifully reproduced high-quality color illustrations. In this work the reader will also find pieces by Mercè Viladrich (for astrolabes), Roser Puig (for the so-called universal instruments),[46] Mercè Comes (for equatoria), and David King (for sundials).[47] This work provides, in fact, a general overview of science in Islamic Spain and, to mention only the parts most relevant to this paper, contains essays on Spanish mathematics (by Ahmed Djebbar), astronomy (by Julio Samsò), navigation (by Juan Vernet), and the *anwā'* literature (by Miquel Forcada).

Berggren [165] provides a brief overview of the Islamic acquisition and development of the sciences discussed in this paper, as well as their transmission to the Latin West. The author stresses the technical depth of Islamic achievements because "it is too often emphasized that the primary interest of medieval
432 Islamic society in mathematics was due to the subject's utility."[48] |

An extensive survey of Islamic mathematics during its first three centuries is found in Sesiano's account [148], noteworthy for the number of specimens of actual mathematics included and for its discussion of recreational mathematics and magic squares. (In the same volume is Kunitzsch's survey of astronomy from the 8th to the 10th centuries [104], aimed at a general scholarly readership.) Another brief survey of pure mathematics in medieval Islam is that of Hogendijk [69].

More specialized is the volume of studies [163] and the papers therein on astronomy (King [92]), astrology (Pingree), geography (Hopkins), and al-Bīrūnī (Saliba). Mathematics proper is treated rather too briefly, with a bare five pages in a paper (Hill) whose focus is mechanical technology.

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[43] In his study [70] of the *qibla* table of this *zīj*, Hogendijk shows that, despite what had been previously thought, the table was not carelessly computed from an approximate formula but that parts of it were computed according to an exact formula, and there is evidence that the table was copied from an earlier source.

[44] By this term the Arabic writers denoted the sciences they had first learned from earlier civilizations, principally the Greek.

[45] Those who have no Spanish will find the leading ideas in this work in the paper, "Andalusian Astronomy: Its Main Characteristics and Influence in the Latin West," in Samsò [179, 1–23].

[46] The term "universal instrument" refers to an instrument that solves a certain set of problems for all latitudes.

[47] King is currently directing a survey of all known Islamic astrolabes and sundials. A recent report on this project is his [93].

[48] Berggren [165, 141]. Indeed, elsewhere [164] he gives examples to show that many medieval Islamic mathematicians were concerned not only with highly technical mathematics but with the subtle questions of what constituted a proof and how the demands for rigor squared with those for clarity.

An expert history of premodern Islamic cartography occupies about half of John Harley and David Woodward's volume [54] which contains informative and authoritative treatments of, among much else of interest, Emilie Savage-Smith on celestial mapping [54, 12–70], Gerald Tibbetts on geographical mapping [54, 90–155], S. Maqbul Ahmad on Idrīsī [54, 156–174], Raymond Mercier on geodesy [54, 175–188], and David King and Richard Lorch on *qibla* charts, maps, and related instruments [54, 189–205].

Some Conclusions

The history of medieval Islamic mathematics and her sister sciences continues to be an active area of contemporary scholarship, one that regularly produces both significant new discoveries — be they new map projections, numeration systems, or planetary models — and lively debate over their interpretation. Moreover, these discoveries are as likely to come from the study of scientific instruments and astronomical or geographical tables as from treatises on number theory or geometry. Finally, one trusts that "the best is yet to be." Al-Kūhī was regarded by his contemporaries as the best geometer of his age, but less than half of his treatises have been translated or properly studied. His successor, al-Bīrūnī, was one of the great scientists of any age, but neither his *Al-Qānūn al-Masʿūdī* nor his great treatise on the astrolabe has been translated into a European language other than Russian. And, to give only one other author among those we have mentioned, the same may be said of al-Kāshī's most famous works — his *Calculator's Key* and his Khāqānī *Zīj* (of which no translation has been published). Clearly the field still offers an abundance of material for further investigation, and there is much work yet to be done.

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Supplementary Bibliography

Since submitting the paper we have become aware of other papers relevant to this article which ought to be listed in the bibliography. The size of the bibliography made renumbering impractical so we have listed them here and numbered them consecutively with the last entry of the above bibliography.

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