Acts of Geometrical Construction in the *Spherics* of Theodosios

Robert Thomas

Abstract Two ways of talking about mathematics, the ideal agents of Philip Kitcher and Brian Rotman, and David Wells's analogy to abstract games like chess and go, are brought together and put to the test on the Hellenistic mathematical text *Spherics* by Theodosios. The subject of agents or players, as in a game or play, is discussed.

Agents in Mathematics

In *The Wealth of Nations*, Adam Smith invented the metaphor of the invisible hand for the force that turns individually advantageous decisions into collective advantage. In economics, it is not clear who is doing what. One might think that mathematics would be free of such puzzles, but Brian Rotman's [1993] semiotic analysis of mathematical discourse identifies three characters playing roles in it as if it were a play. There is the *person* in his capacity as writer and unengaged reader of the text, totally free and in charge. If what is going on is a proof, the person takes on the task of judge, whereas if a complicated calculation is performed, then the person is reduced to a mere spectator on account of what Netz, making this point, calls "the thick texture of calculation" [Netz 2009, 40]. There is the *subject*, who obeys some of the commands in the text, for instance, "consider triangle *ABC*." While a reader can consider the triangle if she wants to enter into the spirit of the inquiry, the subject has no choice. But there is a third character in this drama, because persons such as ourselves are not able to carry out commands like letting *n* go to infinity or even joining *A* and *B*. Such obedience is the task of the *ideal agent*. Many actions in mathematical texts are not feasible for human agents, and that is why Brian Rotman [1993] and Philip Kitcher [1984] before him[1] adopted the notion of ideal agent to say who it is that does these things that we cannot do. And there is a lot of it. In geometry in particular, there is little beyond considering triangle *ABC* that the geometer *can* do. We cannot draw straight lines, and we cannot draw perfect circles. We can at best only approximate such actions; what we discuss are the lines and circles as idealized work of ideal agents. In his book *Mathematics Without Numbers* [1989], Geoffrey Hellman casts mathematics into modal terms, terms of possibility. It seems to me that talk of agents is another more concrete way of talking about possibilities, not so much logical possibilities as possibilities for someone. In this vein, I have put forward the notion that much mathematics could be regarded as basic strategic thinking for actions of ideal agents, which may be thought to be like game-playing.

I need to distance myself from two different uses of the notion of game in writing about mathematics. I am not trying to relate to the association of games and mathematics by Ludwig Wittgenstein in calling mathematics, like many other activities, a "language game." This was a rather idiosyncratic use of the

[1] And Moritz Pasch well before him (1920s); see Pollard [2010].

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term "game" on his part, and he was well aware that the concept game is not an easy one, being at best a family-resemblance concept. There are no necessary and sufficient conditions for being a game.[2]

Another writer to associate games and mathematics is Reviel Netz with his recent book title *Ludic Proof* [2009]. I want to say a couple of things about it, especially since it is about Hellenistic mathematics. He attributes a certain *playfulness* to Hellenistic mathematics in common with Hellenistic poetry — playfulness not game-playing. This observation is more about form than content, and "ludic" does not indicate in his book the level of frivolity that the word could indicate. No implication, he says, is "intended by my reference, in the title, to the 'ludic' — as if the science, or the civilization that gave birth to it, were engaged in the playful *as against* the serious. As if the ludic should be seen as a kind of holiday from the central issues of either science or poetry" [Netz 2009, 211]. One of his many examples is after all Archimedes. Mathematicians' playfulness can be a way of being serious.

Since my game analogy is not in twenty-year-old books like Kitcher's and Rotman's, I elaborate a little on it. The idea is based on and began in reaction to the idea put forward by the English writer on mathematics David G. Wells [2010] that doing mathematics is like playing abstract games like chess and go.[3] I think that it is more like the strategic thinking that distinguishes expert play from the kind of chess that I play. I don't see it as like our *playing* a game at all: no opponent, no winner, no fixed rules, no equipment, no moves even. Our aim is understanding. But what is to be understood? Possibilities, that's what. We understand things like the impossibility of finding odd composite numbers less than nine and the implications for the sides of a triangle of two angles being equal. If an ideal agent were playing a game with positive integers, "where the composite ones are" could be just what it needed to know. If it were in the business of geometrical construction, then what you can and cannot do with ruler and compasses is just what it would need to know. When I said that mathematics can be regarded as "basic strategic thinking," I meant *really* basic. Before you formulate strategy you need to know your possibilities — *that* basic. Not even the choice of ends, just the lay of the land,[4] including what means and ends are to hand. Mathematical facts are those a mathematical robot would need to know if it were carrying out rather general instructions. In practice, the *person* keeps such information (facts and strategy) in his or her head, reminds the *subject* of bits of it as required, and instructs ideal agents in great detail, for example "join *AB*."

I have summarized this game analogy as follows:

1. The agent's mathematical activity (not playing a game) is analogous to the activity of playing a game like chess where it is clear what is possible and what is impossible — the same for every player — often superhuman but bound by rules...
2. Our mathematical activity is analogous to
   a. game invention and development,
   b. the reflection on the playing of a game like chess that distinguishes expert play from novice play, or
   c. consideration of matters of play for their intrinsic interest apart from playing any particular match — merely human but not bound by rules [Thomas 2009].

This analogy brought me face to face with a question I had not seen asked except in the most general terms: what are the powers of the ideal agent in a piece of mathematics?[5]

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[2] Wittgenstein did, however, have something like the idea that mathematical conventions and deductions from them were norms for how mathematical concepts could be deployed in applications [Friederich 2011].

[3] Only *like*; no identification.

[4] Consider the example of chess. Corroboration of my amateur impression of the difference between amateur and expert perceptions in chess is given by research Chase and Simon [1973] into the different capacities of amateurs and masters in simply remembering game configurations. Such perceptions are of course fundamental to all strategic thinking. Amateurs were able to reproduce 30% and masters 70% of a configuration they had seen for five seconds, according to Sweller, Clark and Kirschner [2010]. They were equally unable to reproduce random non-game configurations.

[5] That the powers need to be superhuman has been explicitly recognized since Frege and I suppose has always been obvious in geometry if not arithmetic. What *exactly* the powers need to be has been a subject of logical/axiomatic explanation since Engeler [1967], according to Pambuccian [2008].

Introduction

It happened that with this question in the back of my mind I began work on translating the *Spherics* of Theodosios for a project on that work with Nathan Sidoli. As I read the first of the three books, I was struck by the difference between two sorts of thing that someone in it is expected to do or already to have done. It is a characteristic of the Hellenistic tradition in mathematics that many constructions are presumed to have been done. And other constructions are ordered. None of these constructions are things that humans can do, but they are not all of the same order of impossibility. It's a bit like numbers: among infinite numbers some are more infinite than others. Let me illustrate what I mean with the example of the very first proposition. On the one hand we have the situation of someone's having cut a sphere with a plane to produce a curve *ABC* in the surface of the sphere (to be proved a circle), and on the other hand we have the instruction to join the two points *A* and *B* to the foot of the perpendicular from the centre of the sphere to the plane at *D*. This latter instruction is the sort of thing to be done with a ruler in the first book of the *Elements*, whereas the slicing of a sphere with a plane to produce a circle requires a chef in Plato's heaven to wield a planar cleaver. With our compasses we can approximate a circle even on a sphere, but we have no instrument for slicing spheres. I hope that you see the distinction I am trying to draw and why, in the context of my reflection on who does what. I found this provocative enough to write about it early in 2009 when I had translated only Book One. I had just done so when I received a manuscript from Nathan in Japan called "The role of geometrical construction in Theodosius's *Spherics*," written with Ken Saito and since published [Sidoli and Saito 2009]. They had fastened on *exactly* the same difference between two sorts of thing done in the book.

Every schoolboy knows that the *Elements* of Euclid concerns lines and circles in the plane and ultimately in space — at least the better-educated schoolboys — since it ends with the construction of the Platonic solids. What many schoolboys think they know, incidentally, is that the *Elements* is a compendium of the known geometry at the time. One sees this written by adults that ought to know better. These schoolboys — and adults — are wrong. Euclid himself, if he wrote the book *Phenomena* that goes under his name, knew and used a substantial amount of geometry that is not in the *Elements*. And even if Euclid did not write that book, Autolykos used the same raw material in much the same way at about the same time. This extra mathematics considers, among other things but chiefly, plane sections of spheres. Conic sections for dummies, as it were. Yes, all the sections are circles; so what can be interesting? Well, to be honest it isn't as fascinating as Book One of the *Elements*, but it does form a gentle introduction to the topic — carefully organized.

The constructions in Book One of the *Spherics* are important, forming the apparent goal of the book. As Sidoli and Saito [2009] observe, the constructions at the end of Book One (problems 20 and 21) are used throughout the remainder of the work. They say 23 and 14 times in the remaining 23 plus 14 propositions of books two and three. The constructive goal is plainly vital to the remainder of the work. They also point out that the previous problem, 19, is called upon in the *Elements* in five of the six propositions XIII.13–18 without comment by either Euclid or Heath. The exception, 17, uses 15 and so is implicitly dependent on problem 19 — obviously not on Theodosios himself, who reworked this material after Euclid was dead. For the sake of brevity and sticking to the point, I'm going almost to ignore Book Two as the same things go on there as in Book One.

Book One of the *Spherics*

I paraphrase the definitions and propositions of Book One for the reader not already familiar with this work. First the definitions:

1. A *sphere* is a solid figure bounded by a single surface, all straight lines to which from a single point lying within the figure are equal to one another.
2. The point is the *centre* of the sphere.

3. An *axis* of the sphere is a straight line passing through the centre and bounded in each direction by the surface of the sphere and around which immobile straight line the sphere rotates.
4. The *poles* of the sphere are the endpoints of the axis.

Since I must interrupt here to say something about the next definition, I mention that the rotation of the sphere in definition 3 is never mentioned again.

The next definition takes a "circle in a sphere" as undefined. Its meaning was more obvious to its original audience than it is to us. A Greek circle is a circular disk,[6] so that a circular section does not lie on a sphere but lies in it. When they say a circle is in a sphere they mean two things, one that the disk is in the solid sphere (definition 1) but also that the circumference of the circle lies on the surface of the sphere. Any circle *in* a sphere is a plane section of a sphere; that any plane section of a sphere is a circle is the first proposition.

5. *Pole* of a circle in a sphere names a point on the surface of the sphere from which all straight lines meeting the circumference of the circle are equal to one another.

Now the theorems (for background, not essential to the argument):

1. If a spherical surface is cut by a plane, the curve produced in the surface of the sphere is the circumference of a circle.
   Corollary. If a circle is in a sphere, the perpendicular dropped from the centre of the sphere to it falls at its centre.
3. If a sphere Σ with centre B touches at A a plane Π not cutting Σ, then the line joining the point of contact A to the centre B is perpendicular to Π.
4. If a sphere Σ with centre B touches at A a plane Π not cutting Σ, then B will be on a perpendicular to Π erected into the sphere at A.
5. Circles through the centre of a sphere are great circles. Other circles in a sphere are equal to one another if equidistant from the centre of the sphere, and the farther away from the centre the smaller the circle.
6. If a circle C is in a sphere Σ, a straight line joining the centre of Σ to the centre of C is perpendicular to C.
7. If a perpendicular is dropped from the centre of a sphere to a circle in the sphere and produced in both directions, it will meet the sphere at the poles of the circle.
8. If a perpendicular is dropped to a circle in a sphere from one of its poles, it will fall at the centre of the circle, and produced it will meet the sphere at the other pole of the circle.
9. If a circle is in a sphere, the line joining its poles is perpendicular to the circle and will pass through the centers of the circle and of the sphere.
10. In a sphere, great circles bisect each other.
11. In a sphere, circles that bisect each other are great.
12. If a great circle in a sphere cuts a circle in the sphere at right angles, it will bisect it and pass through its poles.
13. If a great circle in a sphere bisects a small circle in the sphere, it will cut it at right angles and pass through its poles.
14. If a great circle in a sphere cuts a circle in the sphere through its poles, it will bisect it at right angles.
15. The secants to the circumference of a great circle in a sphere from its pole are equal to the side of the square inscribed in a great circle.
16. If the secants to the circumference of a circle C in a sphere from its pole are equal to the side of a square inscribed in a great circle, then C will be great.

There are as well construction problems, one at the start, and four at the end as the apparent goal of the book:

[6] The definition of a circle in Heath's *Elements* is "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure [the centre] are equal to one another." The wording of the first definition here appears to be modeled on this one.

2. To find the centre of a given sphere.
18. To set out a line equal to the diameter of a given circle in a sphere.
19. To set out a line equal to the diameter of a given sphere.
20. To draw a great circle through two given points on a spherical surface.
21. To find the pole of a given circle in a sphere.

You see that we find points, draw circles on the sphere, and set out — that is set out on a plane exterior to the sphere — lines of a specific length.

Constructions in Book One

With a pair of compasses, perhaps bowlegged, we can approximate a circle on an approximate sphere, and we know that the circle we approximate is the circumference of a plane section of the sphere. We are not bothered by the approximation here because we know that the ideal agent in the Platonic heaven has a perfectly flat plane that will cut the perfect sphere in the perfect circle that the agent can draw on its surface. These are the sort of operations that one carries out in the constructions of the *Elements*; we are used to them.[7] But in the *Spherics* there is another class of construction altogether that can be carried out also only by an ideal agent but that we are unable even to approximate. To find the centre of a given sphere, Theodosios begins by simply requiring that the sphere be cut by a plane. That it should have *been cut* by a plane, by someone unspecified somehow. In that plane we can carry out the construction of finding the centre of the circle, but getting the plane is a real difficulty. Nowadays one might obviate the difficulty by allowing some cutting plane to be given along with the sphere, as was the case in Theorem 1, but I think Theodosios would have regarded that as cheating. Theodosios seems to call upon an extra-ideal agent to perform this feat that is not just an idealization of what we can approximate but goes altogether beyond it in the manner of completed arithmetic infinities. Such a cut in a sphere is itself a completed geometric infinity of points, but so are circles drawn with compasses and lines drawn with straightedges.

Note that the culminating constructions of the book are trivial if one is prepared to use the full capabilities of such an extra-ideal agent. A great circle through two given points (20) is the plane section of the sphere through them and its centre, no challenge if one can cut the sphere to order. And the pole of a circle (21) is even easier with those capabilities, since the perpendicular erected at the centre of the circle cuts the sphere in its poles according to Theorem 8. But as humans outside the sphere we cannot approximate those operations. What the constructions of the book tell us is how to draw the great circle and find the pole using only operations that we can perform in our approximate way and that an agent merely idealizing us can carry out by doing perfectly what we can do imperfectly. A difficult question posed to me is why we subordinate or prefer the one to the other. I have no answer to that question apart from the origin of geometry in operations carried out by humans approximating the ideal actions that would, if possible, give exactly accurate determinations.[8] The actions of ideal agents are idealized from our human actions; the less far-fetched or magical the better.

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[7] We are roughly used to them, but roughly is not necessarily good enough. Presumably dissatisfied with the extant versions of the Baghdad Arabic translation of the *Spherics*, al-Ṭūsī composed a version that was not a slavish copy but was meant to be read and understood. This has been published [al-Ṭūsī, 1940] and studied by, among others, Sidoli and Kusuba. He added three postulates.

1. That we make any point that happens to be on the surface of the sphere a pole and we draw about it with any distance, less than the diameter of the sphere, a circle on that surface; and
2. that we produce any arc that there is until it completes its circle; and
3. that we cut off what is equal to a known arc from an arc greater than it, when they belong to equal circles; …[Sidoli and Kusuba 2008, 15 ff.; *cf.* al-Ṭūsī, 1940, 3]

[8] Such god-like idealization seems to be at work in the appropriation of measuring rod and rope as royal insignia in ancient Iraq [Robson 2008].

Problem 21 finds a pole of a small circle by drawing a great circle through diametrically opposite points *A* and *Z* on it and bisecting the arc *AZ* of the great circle inside the small circle.[9] So 21 depends on 20. Problem 20 is more complicated, even when the two points *A* and *B* are not the ends of a diameter so that the circle wanted is unique: The procedure is to draw two great circles with each of *A* and *B* as poles. This can be done with compasses, the proper polar distance to allow it being determined in Proposition 17 and available by Problem 19 and plane geometry. Each point of intersection *C* and *D* of the two great circles is the right distance from each of *A* and *B* because it is on those circles, and so it is a pole from which a great circle can be drawn through *A* and *B*. Either *C* or *D* will of course produce the same unique great circle through *A* and *B*. The aim of these constructions is to give approximating and ideal procedures for doing with real and ideal compasses what can in principle be easily done by magic.

Problem 2 is not at all like that. It is a construction entirely to be imagined, saying that to find the centre of a sphere you take the mid-point of a diameter of the sphere obtained by setting up a perpendicular at the centre of the circular disk obtained by cutting the sphere with some plane. These actions are those of an ideal agent of the higher order. You have to start somewhere, and you start conceptually. This problem is not used after the early theorems; it is a sort of scaffolding — needed at first but not kept.

The other two constructions, 18 and 19, are more like 20 and 21. Their object is to obtain, as line segments on a given plane, the diameter of a given circle in a sphere and the diameter of the sphere itself, that is, to extract from the surface of the sphere information that can be used for the construction in 20 of the great circle through two points. The method in 18 requires only knowing three points, *A*, *B*, and *C*, on the circle at issue. One takes the distances between the points of a triangle imagined inscribed in the embedded circle (there is no need to do anything but *imagine* this triangle inside the sphere) and uses them to construct a congruent triangle *DEF* in the plane (Figure 1). Perpendiculars at two vertices *E* and *F* of that triangle will meet at a point *H* that not only lies on the congruent circle through the vertices of the congruent triangle but is the opposite end of the diameter of that circle having the third vertex *D* as its other end. Presto, the diameter *DH* of the circle without even drawing the circle in the plane. The method in 19 also uses information obtainable on the surface of the sphere. One draws a circle with any pole and any radius. Using the method of 18 to obtain the diameter of that circle, one constructs a triangle in the plane with the diameter of the circle as one side and the polar distance as the other two sides. That triangle is congruent to a triangle inside the sphere with the pole and points at opposite ends of a diameter of the circle as its vertices. One knows by Theorem 13 that the great circle that cuts the circle at right angles passes through these three points. One then has a triangle in the plane congruent to an undrawn triangle inscribed in that undrawn great circle. The method of 18 is being used a second time and has proceeded as far as triangle *DEF*. If *H* is the intersection of perpendiculars erected at *E* and *F*, then again *DH* is the diameter of the circle, which being great has the diameter of the sphere as its diameter. The diameter of the sphere is available in a plane diagram; "presto" would be an exaggeration, but "efficiently" would not.

You see that Problems 18–21 are practical constructions for use with compasses and a solid globe. What needs to be done is idealized merely in exactness. This is not the case in the proofs of the theorems, where inside the sphere actions are performed to which we could not approximate even if we were inside an empty sphere. As with all actions of the extra-ideal agent, however magical, it suffices to imagine them — as in Problem 2. I see no existential commitment in imagining things, and for the most part even imagining things is optional, although the solution of Problem 19 begins (uniquely) with the instruction to imagine a sphere.

It is the process of Problem 19, setting out on the plane the diameter of a given sphere, that Euclid calls upon in the final six propositions of the *Elements*, where, in order to construct solids that could be inscribed in a given sphere, he says simply, "Let the diameter of the given sphere be set out."[10] Heath

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[9] *Z* is found by bisecting an arc *DE* of the small circle, where the other arc *DE* is bisected by *A*. These bisections have attracted comment [Berggren 1991; Sidoli and Saito 2009] since Theodosios does not specify how they are to be done, and there are several possibilities.

[10] Netz misstates this as beginning with a given diameter [Netz 2009, 92]. He also says "It is as if Euclid has invented a

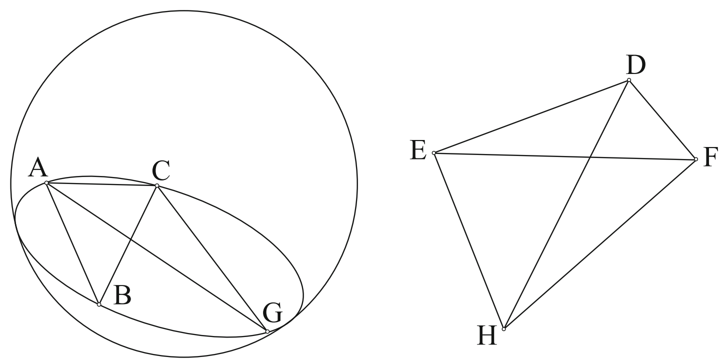

Figure 1: Congruent triangles *ABC* inscribed in a circle on the sphere and *DEF* in the plane with perpendiculars *EH* and *FH* and *DH*, diameter of the circle. Diagram due to N. Sidoli.

points out that the solution of Pappus to the problem of Proposition 13, to inscribe a tetrahedron in a given sphere, requires "a knowledge of some properties of a sphere which are of course not found in the *Elements* but belonged to treatises such as the *Sphaerica* of Theodosius." But he ignores that the method of setting out on the plane the diameter of a given sphere — required in the Euclidean text itself — is also included in such knowledge. This appears to be an instance of anti-spherics prejudice blinding an observer to the obvious.

Constructions in Book Two

Sidoli and Saito [2009] add to the above the two constructions from Book Two.

14. In a sphere, given a small circle and a certain point on its circumference, to draw a great circle through the given point touching the given circle.
15. Given a small circle in a sphere and a certain point on the surface of the sphere between it and the circle equal and parallel to it, to draw a great circle through the given point touching the given circle.

As they remark, the construction and proof of 15 are more complicated than those of 14, but in each case the aim is to enable the one sort of agent — our sort of agent — to draw the great circle with compasses as in I 20. While the constructions cannot require any constructions from Book Two, there being no other constructions in Book Two, the proofs use theorems of Book Two.

To return to the metaphor of actors in a play, the *dramatis personae* of the *Spherics* includes two characters with different responsibilities whom we may call the *construction agent*, that is, the lesser ideal agent that does exactly things we can't do exactly but can approximate, and the *imagination agent* that is the extra-ideal agent, whose doings beyond the construction agent's abilities we can imagine for the sake of learning what the construction agent can do. According to Proclus reporting Geminus, a corresponding distinction is that between postulates about what one can construct and axioms about what one can know. This is explained by Pambuccian thus:

> *postulates* ask for the production, the ποίησις of something not yet given, of a τι, whereas *axioms* refer to the γνῶσις of a given, to insight into the validity of certain relationships that hold between given notions... [Pambuccian 2008, 25]

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parlor game 'fit solids inside spheres.'" Others try their hand at other solids — and polygons inside circles [Netz 2009, 92–94].

The construction agent is bound by the postulates,[11] whereas the imagination agent is bound only by our imaginations. It might be that someone with metaphysical inclinations would be tempted to argue that what the imagination agent does and creates need only be imagined and is therefore unreal or because created is therefore real. The agents, however, are unreal, imaginary, mere *façons de parler*.

Euclid

I wondered when I first thought of this topic whether the imagination agent is needed in the *Elements*. One hears about their use of ruler and compasses, and most of the books do in fact require no more. However, the dropping of a perpendicular from the centre of a sphere, as from any point not on a plane to the plane is not something that can be carried out with straightedge and compasses. This difficulty infects the whole of Books XI and XII of the *Elements* from Proposition 4 of Book XI where points on and off a plane must be "joined," as they are from the beginning in the *Spherics*. Construction agents at all like ourselves are not good enough for the late books of the *Elements*.[12] In Ian Mueller's massive examination of the logical structure of the *Elements*, he considers in great detail the constructions in Book XIII of Platonic solids to fit in a given sphere. In some ways this appears to be the culmination of the work as a whole. This is not altogether so, as there is more in the arithmetical Books VII–IX than is needed for X and XIII [Mueller 1981, 58] and more in the geometrical Books XI–XII than is needed for XIII, with the material in XII being ascribed to Eudoxos and in XIII to Theaetetus [Heath 1981, vol. 1, 212; Mueller, 1981, 207]. But one might say that geometrically that is where the work is headed. It is accordingly worth something to get as good an understanding of Book XIII as possible. I suggest that the present consideration can help with that understanding.

As Mueller notes [1981, 247, n. 1], Theodosios "uses a different definition of the sphere." Euclid's definition reads:

> A sphere is the figure comprehended when, the diameter of the semicircle remaining fixed, the semicircle is carried around and returned again to the same (position) from which it began to be moved. [XI.14. Mueller's translation, 1981, 361]

It is simply assumed that the solid so defined is the same as that of Theodosios. One can't prove it because the whole notion of motion — essential to Euclid's definition — is completely foreign to Theodosios's work, which is resolutely static in spite of being intended "for astronomical rather than geometrical purposes" [Mueller 1981, 247, n. 1]. The motion in the *Spherics* is just drawing circles with compasses and lines, as elsewhere in the *Elements*. (The astronomical purpose for which the *Spherics* is intended is the consideration of motion of a sphere as in Euclid's *Phenomena*. Sphere motion is astronomy; geometry is static.) Recall Definition 3 of "axis." The rotation mentioned there is not mentioned again.

In Propositions 13–17 of Book XIII, Euclid constructs the tetrahedron, octahedron, cube, isosahedron, and dodecahedron that a given sphere would circumscribe. He does three things given the sphere: he constructs the solid, he shows that it fits exactly into a sphere of the given size,[13] and he discusses the characteristic edge length of the solid. The earlier propositions in the book Heath calls "introductory" [1981, vol. 1, 212] and Mueller calls "lemmas" [1981, 251], criticizing their proofs rather severely. We are concerned with the constructive part, which Mueller describes thus:

> This procedure involves constructing the figure outside the [given] sphere, relative to a straight line equal to the diameter of the given sphere, and then arguing that the semicircle with the straight line as diameter will pass through all the vertices of the figure as it revolves around the diameter. There is no real mathematical difference between this procedure and inscribing the figure in the sphere. Apparently Euclid adopts the procedure and, hence, the generative definition of the sphere (XI, def. 14) as a means of avoiding a treatment of the sphere

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[11] Note the postulates of the careful al-Ṭūsī; he cannot have overlooked the actions I attribute to the imagination agent.

[12] Straightedges at all like mine are not good enough for XI.4. Mine can help me to draw lines only on a plane already there, not between arbitrary points in space.

[13] This is also called "comprehending" the solid in the sphere and is explained by Heath as "the construction of the circumscribing sphere," which must be shown actually to circumscribe the solid.

> analogous to the treatment of the circle in Book III. From a foundational point of view the advantage gained is only apparent, since the procedure depends upon tacit assumptions about the properties of a semicircle revolving about its diameter. [Mueller 1981, 254]

More importantly from a foundational point of view, as we have seen, the whole procedure depends on the *Spherics*, which is precisely the required treatment of the sphere analogous to the treatment of the circle in Euclid's Book III. As Heath notes:

> It will be observed that, although in these cases Euclid's construction is equivalent to inscribing the particular regular solid in a given sphere, he does not actually construct the solid *in* the sphere but constructs a solid which a sphere *equal* to the given sphere will circumscribe. [Heath 1956, vol. 3, 472]

By the time of Pappus this subtlety was water under the bridge, and he constructs the circumscribed solids inside the given sphere [*ibid.*]. It is plainly true that the same solid results whether you construct it out of whole cloth on your working plane as you would do any other construction or you try to build it inside the given sphere. The difference, as it seems to me, is partly in what it means to be *given* a sphere. If you are given a sphere, then you have to determine its diameter from outside it, and that is the first thing Euclid does, "Let the diameter of the given sphere be set out" on the working plane. As there is nothing in the *Elements* about such an operation, it must be presumed to be done à la Theodosios I 19 by operations the construction agent can perform on the surface of the given sphere and on the plane. Operations then follow à la Euclid, including the erection of a perpendicular to the working plane to get the construction into space and joining points in space that do not lie on any given plane. So in Euclid the construction agent is more powerful than the construction agent in Theodosios. Whether it has the full power of Theodosios's imagination agent — or more — I do not know. It is able, however, to create a curved surface by the rotation about the diameter of a semicircle, something neither agent in Theodosios needs to do.

I see some gain for understanding in seeing that an agent of ideal-human powers can begin with the sphere and proceed to the construction. If one began by constructing a Euclidean sphere, one would already have the diameter, as that is the beginning of the construction of the sphere. You begin, in effect, with the diameter rather than with the sphere, as Netz [2009, 92] says that Euclid does.

The conjunction of Euclid and constructions raises — in some minds — the odd question, "What are such constructions for?" As I have been suggesting that they tell how things can be constructed in accordance with agreed-upon limitations, I am unsympathetic to what Orna Harari calls "the widely accepted contention that geometrical constructions serve in Greek mathematics as proofs of the existence of the constructed figures" [Harari 2009, 1]. Since I have encountered the contention by Zeuthen [1896] only where it has been opposed [Mueller 1981; Knorr 1986; Harari 2009] and Harari supplies no supporting citation, I think it is time — despite attempts to take it seriously and successfully to show that it is wrong — to dismiss it as an anachronism that arose in the nineteenth century because of that century's concerns and not Euclid's. I cannot doubt that the ancients assumed, for example, that there was a cube twice the size of any given cube; the famous problem was to construct it.[14] Moreover, the theme of this paper undercuts the notion that problems had any single purpose, the unlikelihood of which was brought to my attention by João Caramalho Domingues.[15]

Conclusion

The view of mathematics that I put forward at the beginning was one that expressed what we learn about mathematics in terms of what someone can do — the constraints on mathematical behavior. I first encountered this idea in a posthumously published paper of Leslie Tharp in which he wrote:

> We have claimed that the modal propositions of arithmetic are primarily about concepts, and are about ordinary objects and activities in the indirect sense that the concepts may be applied to ordinary objects arising from

[14] Where existence was a concern, the case of a square root of two, they were not shy about mentioning it.

[15] Personal communication, 2011 10 15.

> ordinary activities, such as an actually constructed inscription. In particular, existential assertions such as 'there is a number...' may go far beyond anything humanly feasible. The discomfort with modal treatments of mathematics is reminiscent of the everyday interchange of 'can' and 'may.' One sometimes says 'Herr Schmidt can drive 150 kph on the Autobahn' when he actually cannot (because, say, his Volkswagen won't go that fast). Obviously, what one means is that he *may,* that is, the relevant rules permit such speed. We interpret the mathematical modalities in such a 'may' sense: one may construct an inscription with 99^{99} strokes — the concepts undeniably permit it. [Tharp 1989, 187]

Some of the mathematical behavior that is permitted by our concepts is ours directly. If we are going to look for the subgroups of a finite group, we shall look among the divisors of the order of the group for their sizes. If the order is prime, the Autobahn is closed. When there are divisors, we can look for, find, and write down existent subgroups ourselves. But much of what we seek in mathematics can only be found by us nominally. The function $y = f(x) = x^2$ is one of which we can write a formula and approximately a graph over a finite interval, but the whole graph — even approximately — and an exact graph — even over a finite interval — are things that we cannot draw. The behavior that our knowledge of such a function governs is that of an ideal construction agent capable of actions we can perform only in principle. Only an ideal agent is licensed to drive on this particular Autobahn. When in contrast we come to some exercise of the axiom of choice or some more esoteric action involving the iterative hierarchy of sets in aid of proving some theorem, then we need the imagination agent whose powers vastly exceed those of our surrogate the construction agent. Whether there is a useful distinction to be drawn here between different ranks of agent, it is clear enough that such agents are required if we are to think of mathematical actions actually being performed. And since mathematical discourse is full of demands for such action, such agents are at least implicitly required. If one is going to make sense of mathematical language using modality, then one has implicit agents that one is making use of. Just what agents are presupposed in different times and parts of mathematics — even in different books of the *Elements* — varies and so is a subject for historical research.

References

Berggren, J.L., 1991. The relation of Greek spherics to early Greek astronomy. In: Bowen, A. (ed.), Science and Philosophy in Classical Greece. Garland, New York, pp. 227–248.

Berggren, J.L., Thomas R.S.D., 2006. Euclid's Phænomena: A Translation and Study of a Hellenistic Treatise in Spherical Astronomy. Garland, New York. Reprinted: American Mathematical Society and London Mathematical Society, Providence, 2006.

Chase, W.G., Simon, H.A., 1973. Perception in chess. Cognitive Psychology 4, 55–81.

Czinczenheim, C., 2000. Édition, traduction et commentaire des Sphériques de Théodose. Lille. (Thèse de docteur de l'Université Paris IV.)

Engeler, E., 1967. Algorithmic properties of structures. Mathematical Systems Theory 1, 183–195.

Friederich, S., 2011. Motivating Wittgenstein's perspective on mathematical sentences as norms. Philosophia Mathematica 19, 1–19.

Harari, O., 2003. The concept of existence and the role of constructions in Euclid's Elements. Archive for History of Exact Sciences 57, 1–23.

Hellman, G., 1989. Mathematics Without Numbers: Towards a Modal-Structural Interpretation. Oxford University Press, Oxford.

Heath, T.L., 1921. A History of Greek Mathematics, 2 vols. Clarendon Press, Oxford. Reprinted: Dover, New York, 1951.

——— 1926. The Thirteen Books of Euclid's Elements, 3 vols., 2nd edition. Cambridge University Press, Cambridge. Reprinted: Dover, New York, 1956.

Heiberg, J.L., 1927. Theodosius Tripolites [word deleted in *corrigenda*] Sphaerica. Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Philologisch-historische Klasse (N.S.) 19, no. 3, i–xvi, 1–199.

Kitcher, P., 1984. The Nature of Mathematical Knowledge. Oxford University Press, Oxford.

Morrow, G.R., 1970. Proclus, A Commentary on the First Book of Euclid's Elements. Princeton University Press, Princeton.
Mueller, I., 1981. Philosophy of Mathematics and Deductive Structure in Euclid's Elements. MIT Press, Cambridge, MA.
Netz, R., 2009. Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic. Cambridge University Press, Cambridge.
Pambuccian, V., 2008. Axiomatizing geometric constructions. Journal of Applied Logic 6, 24–46.
Pollard, S., 2010. "As if" reasoning in Vaihinger and Pasch. Erkenntnis 73, 83–95.
Robson, E., 2008. Mathematics in Ancient Iraq: A Social History. Oxford University Press, Oxford.
Rotman, B., 1993. Ad Infinitum: The Ghost in Turing's Machine. Stanford University Press, Palo Alto.
Sidoli, N., Kusuba, T., 2008. Naṣīr al-Dīn al-Ṭūsī's revision of Theodosius's Spherics. Suhayl 8, 9–46.
Sidoli, N., Saito, K., 2009. The role of geometrical construction in Theodosius's Spherics. Archive for History of Exact Sciences 63, 581–609.
Sweller, J., Clark, R., Kirschner, P., 2010. Teaching general problem-solving skills is not a substitute for, or a viable addition to, teaching mathematics. Notices of the American Mathematical Society 57, 1303–1304.
Tharp, L., 1989. Myth and mathematics: A conceptualistic philosophy of mathematics I. Synthese 81, 167–201.
Thomas, R., 2009. Mathematics is not a game but... The Mathematical Intelligencer 31, no. 1, 4–8.
al-Ṭūsī, Naṣīr al-Dīn, 1940. Kitāb al-Ukar li-Thā'ūdhūsiyūs, Osmania Oriental Publications Bureau, Hyderabad. Reprinted: Sezgin, F. (ed.), 1998. A Collection of Mathematical and Astronomical Treatises as Revised by Naṣīraddīn aṭ-Ṭūsī, 2 vols. Institute for the History of Arabic-Islamic Science at the Johann Wolfgang Goethe University, Frankfurt am Main.
Wells, D.G.., 2010. Philosophy and Abstract Games. Rain Press, London.
Zeuthen, H.G., 1896. Die geometrischen Construction als "Existenzbeweis" in der antiken Geometrie. Mathematische Annalen 47, 222–228.