Reading the Lost Folia of the Archimedean Palimpsest: The Last Proposition of the *Method*

Ken Saito and Pier Daniele Napolitani

Abstract We examine the determination of the volume of solids by means of a virtual balance, in Archimedes' *Method*, a work preserved only in the Archimedes palimpsest.

We concentrate on the intersection of two cylinders, whose volume is correctly stated in the preface, although the folia containing the proof are lost. All the reconstructions of Archimedes' argument by previous scholars concerning the volumes of the intersection of cylinders are unsatisfactory, for the description in the preface states that the argument will contain a demonstration (not only an heuristic argument by the virtual balance), while the reconstruction of the configuration of the quires of the palimpsest shows that the space for this argument was no more than three pages — too short for a rigorous demonstration.

We take up Reinach's remark that the intersection of cylinders can be divided into eight equal hoofs whose volume has been determined in previous propositions, and we argue that Archimedes' demonstration consisted of decomposing the intersection of cylinders into hoofs.

This means that Archimedes was not aware of the fact that the hoof, the sphere and the intersection of cylinders can be treated in much the same way by virtual balance, although the process of the determination of their volumes is similar. Archimedes did not try to extract some quantitative property common to these three solids and to treat them in a similar way. Thus, Archimedes was a much more "classical" geometer than we tend to assume.

In the final part, we suggest that Archimedes' invention of the problem of determination of the intersection of cylinders may have been inspired by some building existing at his time. Indeed, excavations in Morgantina, Sicily, have revealed two barrel vaults arranged at a right angle, which can be dated to Archimedes' time. This construction may well have given him the idea to consider the intersection of two cylinders.

Appendix 1 shows the outline of propositions in the *Method* concerning the sphere and the hoof, with suggestions of how Archimedes may have found certain arguments. It also contains reconstructions by previous scholars as to the volume of intersection of cylinders.

Appendix 2 explains how one can infer, with pretty high certainty, the number of lost folia by the reconstruction of the quires of parchment.

Appendix 3 shows a schematic presentation of the three quires of the palimpsest (with the reconstruction of lost folia) which contains the *Method*.

Introduction

The *Method* is the work in which Archimedes sets out his way of finding the areas and volumes of various figures. It can be divided into three parts. The first part is the preface addressed to Eratosthenes, in

The authors thank Paolo d'Alessandro for his valuable advices on codicological issues treated in this article.

N. Sidoli and G. Van Brummelen (eds.), *From Alexandria, Through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J.L. Berggren*, DOI 10.1007/978-3-642-36736-6_10,

which Archimedes explains his motivation for writing the work. We find that he was sending demonstrations of results that he had communicated before — the volume of two novel solids, which we call "hoof" and "vault" in this article.[1]

As Archimedes thought that it was a good occasion to reveal his way of finding results that he had previously published with rigorous demonstration, he decided to include an exposition of this "way" (*tropos* in Greek, not method, as is usually assumed in modern accounts.)[2]

Thus, the first eleven propositions show how the results in his previous works (*Quadrature of the Parabola*, *Sphere and Cylinder* and *Conoids and Spheroids*) were found. We call this group of propositions the second part of the work.

The third and last part, beginning with Prop. 12, treats the two novel solids and gives a demonstration of their volumes. Unfortunately, the end of the *Method* is lost. As is well known, the *Method* is known only through the palimpsest found in 1906, and some pages had already been lost. The text of the *Method* breaks off definitively near the end of the demonstration of the volume of the hoof, the first of the two novel solids announced in the preface. We have no testimony concerning how Archimedes demonstrated the volume of the vault, the second novel solid.

In this article, we try to reconstruct this lost demonstration, based on recent studies made after the reappearance of the palimpsest in 1998.

The Archimedean "Way" of Finding Results

First, let us briefly look at the "way" that Archimedes presents in this work. In this section, we will see the simplest case of the paraboloid, and an application to the sphere.

The Simplest Example: Paraboloid (Prop. 4)

The simplest example can be found in Prop. 4, where Archimedes compares a paraboloid to the cylinder circumscribed about it[3] The paraboloid *BAG* having axis *AD*, is cut by a plane *MN*, perpendicular to its axis.[4] Archimedes shows that the segment *BAG* is half the cylinder circumscribed about it. By the property of the parabola, the following proportion holds:

$$\text{circle } CO : \text{ circle } MN = \text{sq}(CS) : \text{sq}(MS) = SA : AD.$$

Let us imagine a cylinder *BGHE*, circumscribed about the segment of the paraboloid *BAG*. Prolong axis *DA* to *Q*, so that *QA* is equal to *DA*, and imagine a balance *DAQ* whose fulcrum is the point *A*. Then, the above proportion means that if the circle *CO* (section of the segment of parabola) is moved to point *Q*, it is in equilibrium on the balance with the circle *MN* (section of the cylinder) remaining in place.

[1] We use the words "area" and "volume" only for the sake of convenience. Archimedes did not use these words, and his results were stated as comparisons between figures. For example he says, "the surface," not the area of the surface, "of any sphere is four times its greatest circle." This is a characteristic feature of Greek theoretical geometry, common at least to Euclid, Archimedes and Apollonius. The use of words like length, area and volume is based on the possibility of expressing geometrical magnitudes by real, positive numbers, taking any magnitude of the same dimension as unit. However, this was not possible for Greek geometers who did not have the concept of real number.

[2] The manuscript gives the title *ephodos*, not *methodos*, to this work which is usually called the *Method*, following Heiberg. Moreover, neither *ephodos* nor *methodos* appears in the preface or the text of this work. Archimedes always uses the word *tropos* to refer to his "method" of virtual balance which is discussed in the present paper. See Knobloch [2000, 83].

[3] The manuscript does not have proposition numbers. We use the propositions numbers in Heiberg [1910–1915].

[4] The diagrams of solid figures found in the manuscript are always planar, like Figure 1 (A), and we have often provided perspective drawings like Figure 1 (B).

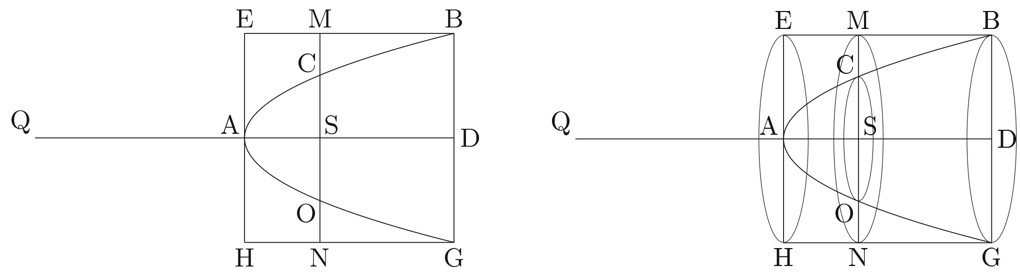


Figure 1: (A) *Method* Prop. 4. (B) Perspective diagram.

If all the sections of the paraboloid are thus moved and balanced, then the whole paraboloid, moved to point *Q*, is in equilibrium with the cylinder remaining in place.[5] As the barycenter of the cylinder is the midpoint of the axis *AD*, it follows that the (volume of the) paraboloid is half the cylinder.

This argument works because all the sections of the paraboloid are moved to one and the same point *Q*, while all the sections of the cylinder remain in place. This is possible because the circle sections of the paraboloid, such as the circle *CO*, increase in direct proportion with the distance *AS* from the vertex *A*, which is the fulcrum of the balance.[6]

Sphere: Invention of an Auxiliary Solid (Prop. 2)

Let us look at another, slightly more complicated proposition. Prop. 2 determines the volume of the sphere. In the following, we present the outline of Archimedes' argument, which is described in more in detail in Appendix 1, Prop. 2: Sphere.

Let the sphere *AG* be cut by a plane *MN*, perpendicular to the diameter *AG*. The section of the sphere is circle *CO*. This section is by no means in proportion to the distance from the point *A*.

Archimedes then adds a cone, *AEZ*, having as height the diameter of the sphere, *AG*, and as base a circle *EZ*, whose diameter is twice the diameter of the sphere. Then, the sum of the sections of the sphere and the cone, that is the circle *CO* together with the circle *PR*, is in direct proportion to the distance from *AS*, and these two circles moved to the other end of the balance, *Q*, are in equilibrium with the circle *MN*, remaining in place. The barycenter of the cylinder is the point *K*, the midpoint of its axis *AG*, and *AK* is half *AQ*. Therefore, by virtue of the law of the lever, the cylinder is twice the sphere and the cone taken together. The rest of the proposition is quite simple (for details, see Appendix 1, Prop. 2: Sphere).

By adding an auxiliary solid (cone *AEZ* in this case), Archimedes succeeds in extending the use of the virtual balance to the sphere and other solids.[7]

---

[5] Archimedes, carefully enough, does not say that the sections of the segment of the paraboloid moved to point *Q* makes up again the segment itself. He says that the segment of paraboloid is "filled" by its sections.

[6] The expression "increase in direct proportion" is modern, with an algebraic background. This is never found in Greek geometry and we use it for the sake of convenience.

[7] The volume of the hyperboloid, for which Archimedes does not describe the details of the argument in Prop. 11, can be determined by the same technique of adding an auxiliary solid. For a reconstruction of this argument, see Hayashi [1994].

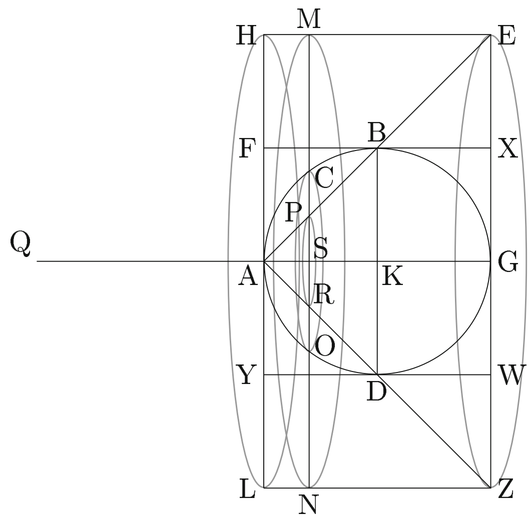


Figure 2: *Method* Prop. 2: Perspective diagram.

The First Novel Solid: Hoof

The hoof is one of the two novel solids that induced Archimedes to write the letter to Eratosthenes, now known as the *Method*. The hoof is generated by cutting a cylinder with an oblique plane passing through the diameter of the base circle.

Let there be a prism with a square base, and let a cylinder be inscribed in it. And let a diameter of the base circle of the cylinder be drawn, parallel to a side of the square, and let the cylinder be cut by an oblique plane passing through this diameter and one of the sides of the square opposite to the base of the prism. The hoof is the solid contained by the semicircle in the base of the cylinder, the semi-ellipse in the cutting plane, and the surface of the cylinder.

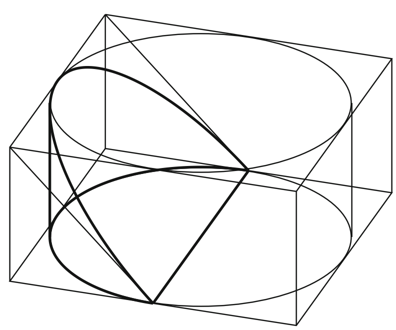


Figure 3: A hoof.

Archimedes found that the hoof is one sixth of the prism. This is the first solid found to be equal to some other solid contained by planes only. Although he had obtained several results concerning solids contained by various curved surfaces of different solids, as sphere, paraboloid, etc., they were always compared with other solids contained by at least one curved surface, as cone and cylinder (see note 1, above). The fact that Archimedes was very proud of this novel result can be seen from what he says about it in the preface of the *Method*.

He gives no less than three arguments for the volume of this solid:

(1) First, in Prop. 12+13, by using the virtual balance, which was already familiar to him (for details, see Appendix 1, Volume of the Hoof: Part 2).[8] What is strange to us in this argument is that Archimedes does not see that the virtual balance could be used in much the same way as in Prop. 2, where he determined the volume of the sphere; and the whole argument would have been much simpler. We have reconstructed this argument in Appendix 1, Volume of the Hoof: Part 1.

(2) Then, Archimedes gives another argument using plane sections without breadth (like Cavalieri's indivisibles) in Prop. 14.

(3) In the following Prop. 15, this argument is transformed into a rigorous demonstration using *reductio ad absurdum* (Appendix 1, Volume of the Hoof: Part 3) twice. We shall call this kind of argument, often called the "method of exhaustion," simply double *reductio ad absurdum*.[9]

A large portion of Prop. 15, the last extant proposition of the *Method*, is lost, and there is no further folium which contains text from this work. So we have no direct textual witness for the reconstruction of Archimedes' arguments for the other solid, the vault.

Possible Propositions for the Vault

We now proceed to the second novel solid in the *Method* which we call "vault." It is the solid bounded by the surfaces of two cylinders having equal bases whose axes meet at a right angle each other. In short, it is the intersection of two equal cylinders.

Let AA' and BB' be axes of cylinders, meeting at point K. Their base circles are $EZHZ'$ and $EYHY'$ respectively. (Archimedes describes the solid within the cube to which the intersection is inscribed, but we have prolonged both cylinders in our figure to make the intersection clearly visible. Note that we have only Archimedes' verbal presentation and no diagram for this solid is extant in the manuscript.) In the figure, only the half of the intersection is shown; the other half of the solid, behind the plane of $YZY'Z'$, is symmetrical to the part shown in the figure.

Archimedes states, in the preface, that this solid is two-thirds of the cube circumscribed about the intersection.

For us, the most important property of the vault is a square section formed by passing a plane parallel to the axes of the two cylinders (hatched in the figure). Indeed, all of the reconstructions hitherto proposed for Archimedes' lost arguments of the vault make use of this square section.

Our conclusion in the present paper, however, is that Archimedes cannot have argued in this way. Let us first look at the mathematically plausible reconstructions hitherto proposed.

Scholars have unanimously claimed that there were at least two different arguments: first a mechanical and heuristic one, then a geometric and rigorous one. This assumption seems to be natural, for there were three arguments for the hoof: besides the mechanical arguments in Prop. 12+13, there were two geometrical arguments, one using "indivisibles" (Prop. 14) and another by a rigorous *reductio ad absurdum* (Prop. 15).

Let us now consider the reconstructed arguments or demonstrations, bearing in mind that they are all mathematical reconstructions, with no direct textual evidence, as Ver Eecke rightly observed.[10]

---

[8] Heiberg divided this argument into two propositions, probably because there are diagrams at the end of what he named Prop. 12. We follow his numbering, and write Prop. 12+13 when we refer to the whole argument.

[9] On the so-called method of exhaustion, and its appearance as a real method in Western mathematics, see Napolitani and Saito [2004].

[10] Ver Eecke, referring to the reconstructions in Heiberg and Zeuthen [1907], Reinach [1907] and Heath [1912], expresses his doubts about their significance as historical research:

> Ces reconstitutions, qui pourraient du reste être étendues à un grand nombre d'autres propositions, n'intéressent que comme applications de la méthode mécanique d'Archimède, ou comme exercices d'archéologie mathématique [Ver Eecke 1921, vol. 2, 519].

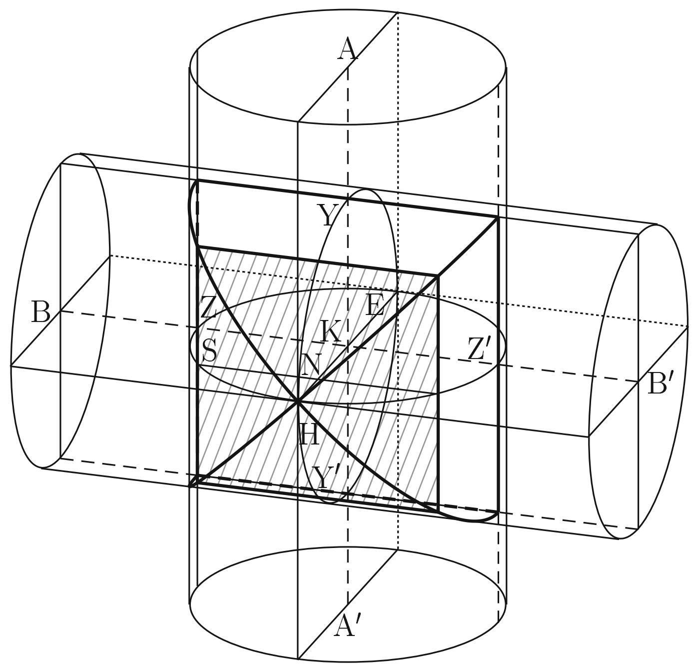


Figure 4: A "vault" (intersection of cylinders).

Reconstruction by Virtual Balance

As for the use of virtual balance, the argument for the sphere (Prop. 2) is valid also for the vault. This mathematical fact was pointed out as early as 1907, only one year after the discovery of the *Method* [Heiberg and Zeuthen 1907; Reinach 1907]. Here we give the basic idea of the argument (for more details see Appendix 1, Volume of the Vault: Part 1).

In the preceding Figure 4, imagine a sphere of which $EZHZ'$ and $EYHY'$ are great circles. Then, its section by the plane which cuts the hatched square from the vault, is the circle inscribed in the square (or the square section of the intersection is circumscribed about the section of the sphere). So, if one substitutes the sphere, the cylinder and the cone appearing in the argument of the volume of the sphere (Appendix 1, Prop. 2) by the vault, the prism and the pyramid respectively, then the rest of the argument is practically the same and the vault turns out to be two-thirds of the cube.

Rufini [1926] refers to this argument as proposition 16. So we call this hypothetical proposition "Rufini 16."

Note that the arguments in Archimedes' Prop. 12+13 for the hoof are completely different from this reconstruction for the vault. We will return to this point later.

Reconstruction by Indivisibles, and by Double Reductio ad Absurdum

For the vault, an argument by indivisibles, like Prop. 14 for the hoof is, of course, also possible. This argument is based on the fact that the square section of the vault is eight times the triangular section of a particular hoof (see Appendix 1, Volume of the Vault: Part 2 and Figure 13 for details).

Thus, if one compares the vault with the circumscribed cube, just as Archimedes compared the hoof with the triangular prism circumscribed about it in Prop. 14, the rest of the proposition is so similar to

Prop. 14 that Sato [1986–1987] even tried to reconstruct the Greek text of this hypothetical argument, which he named proposition 17, depending heavily on the extant text of Prop. 14.[11] We call this Sato 17.

Once a proposition by indivisibles — similar to the extant Prop. 14 — has been reconstructed, it is no more than routine (though tedious) work to convert this argument by indivisibles, to a demonstration by double *reductio ad absurdum*. Rufini numbered this hypothetical proposition 17, (different from Sato 17), and described its outline [Rufini 1926, 174–178].

Thus, one can reconstruct three arguments for the lost pages at the end of the *Method*: (1) an heuristic argument for finding the volume of the vault by way of the virtual balance, modeled after Prop. 2 (Rufini 16), (2) then an argument by indivisibles like Prop. 14 (Sato 17), (3) and a rigorous demonstration like Prop. 15 (Rufini 17). In the preface, Archimedes promises only the last one, the rigorous demonstration. However, at least one of the former two arguments can also be expected, as in the case of the hoof. This has been the consensus of the scholars up to now.

	balance	indivisible	*reductio ad absurdum*
Hoof	Prop. 12+13 (Appen. 1, Hoof 2)	Prop. 14 (Appen. 1, Hoof 3)	Prop. 15
Vault	Rufini 16 (Appen. 1, Vault 1)	Sato 17 (Appen. 1, Vault 2)	Rufini 17

Table 1: Extant propositions for the hoof, and reconstructed propositions for the vault.

The Problem with the Current Reconstruction

In Table 1, above, three approaches are shown (balance, indivisibles and *reductio ad absurdum*) for each of the two novel solids, namely, the hoof and the vault. The arguments for the hoof are extant in the palimpsest either partially or fully, while those for the vault are completely lost, and are reconstructions. Among these, the indivisible argument (Sato 17) and the demonstration by *reductio ad absurdum* (Rufini 17) are simple adaptations of the extant propositions for the hoof (Prop. 14 and 15, respectively). This was made possible by the fact that the square section of the vault is always eight times the triangular section of the hoof.[12]

However, the arguments by virtual balance for the two solids are completely different. Archimedes applies the virtual balance to the hoof in Prop. 12+13, and his argument depends on a particular property of the hoof, that its height is in direct proportion to the distance from the diameter of the base.

So what would happen if we accepted the reconstructions for the vault? If at least one of the two reconstructions that do not use the virtual balance (i.e., Sato 17 or Rufini 17 in Table 1) corresponded to what Archimedes really wrote, then the parallelism between the arguments between the hoof and vault would have been obvious to any careful reader, to say nothing of Archimedes himself, for the square section of the latter is eight times the triangular section of the former, and the structure of the arguments is the same.

And if, in addition, the manuscript had also contained the argument for the vault by means of a virtual balance (like Rufini 16 which uses the same square section as in Sato 17 or Rufini 17), then

[11] He assumed another proposition, 16 (equivalent to Rufini 16), before it, which would have had recourse to the virtual balance.

[12] This relation between the sections of the two solids is visually represented in Figure 13 in Appendix 1. Compare this figure with Figure 4.

it would have been rather difficult not to wonder if an argument by virtual balance, similar to Rufini 16, would not be possible for the hoof, too. This is mathematically possible, indeed, as is shown in Appendix 1, Volume of the Hoof: Part 1.

However, the extant text of Prop. 12+13 for the hoof, which is much more complicated than this reconstruction, does not show awareness of this fact on Archimedes' part. So if one accepts the current reconstructions treating the vault, one has difficulty in explaining the structure of the argument of Prop. 12+13.

Anticipating the conclusion of the present article, we reply that none of the reconstructed arguments for the vault existed in the palimpsest, and that Archimedes' approach to this solid was completely different.

The Space for the Lost Propositions: Mathematical Estimates vs. Codicological Arguments

We have pointed out a problem in accepting the reconstructions concerning the vault, which are mathematically fully acceptable (and have been accepted), consisting only of techniques used by Archimedes himself.

Now let us look at the problem from another point of view: how many pages of the manuscript were occupied by the lost proposition(s) for the vault?

Before entering into codicological arguments, let us estimate the length of the three hypothetical propositions in Table 1. The argument by virtual balance (Rufini 16) would have been approximately of the same length as Prop. 2, of which it is an adaptation. Prop. 2 occupies a little more than 2 pages.[13] The 'indivisible' argument for the hoof (Prop. 14) has 2 pages and some lines, while the rigorous proof by double *reductio* (Prop. 15) occupies about 6 pages.[14] The corresponding propositions to each of these (Sato 17 and Rufini 17, respectively) would have been more or less of the same length. So we would expect about ten pages in total for three propositions concerning this solid, and at the least six pages, because this would correspond to the rigorous demonstration that Archimedes promised in the preface.

Codicological arguments, however, show that there cannot have been even six pages at the end of the *Method* for these proposed propositions on the volume of the vault. The space is only about three pages, against any mathematical expectations — ten pages for all the three propositions, and the demonstration alone would require six pages!

Indeed, the folium which contains the last extant word of the text of the *Method* is followed (not immediately) by the folium containing the initial part of the *Spiral Lines*, and four folia or eight pages are lost between them — this is what the codicological argument shows.

Of the lost eight pages, the first half page, that is, one column, should be occupied by the concluding arguments of Prop. 15, which is not complete in the extant folium, and the last four pages and a half are necessary to accommodate the beginning of the *Spiral Lines* (the part preceding the text in the extant folium 168), so that only three pages are left for the last proposition of the *Method*, which is completely lost.[15]

---

[13] The 'page' is that of the codex, and consists of 2 columns, 34–36 lines, each line containing about 25 characters. One page of the codex corresponds to about three pages of the Greek text in Heiberg's edition.

[14] Only a part of this proposition is extant, and this estimate depends on the reconstruction of the quires of the codex, which we will discuss later.

[15] See the reconstruction of the quires proposed by Abigail Quandt in Netz, Noel, Tchernetska and Wilson [2011, 41–49]. The quire at issue is quire 5 (p. 41). Since this reconstruction is vital to our argument, we explain it in detail in Appendix 2.

What Demonstration Would Fit in Only Three Pages?

This space is surprisingly short. As we have argued, the lost text must contain a rigorous demonstration for the volume of the vault, and such a space is too small for the usual lengthy Archimedean arguments by double *reductio ad absurdum*, as the demonstration of the volume of a solid contained by curved surfaces (in this case, like Rufini 17). Judging from the extant Prop. 15, this proposition would be as long as six pages.

The lost pages at the end of the *Method* could not contain such a demonstration, let alone a set of three propositions as those for the hoof.

Then, in this short space, what kind of argument can we imagine for the vault that would be consistent with Archimedes' words in the preface where he promised to give its *demonstration*?

We seem to be at an impasse, but there is a very simple solution.[16] The vault can be divided into eight hoofs, all equal to each other. Figure 5 shows one of the eight such hoofs cut from the vault. One only has to divide the vault by two planes passing through the border lines of the surfaces of the two cylinders (shown with dotted lines in the figure), then by two planes, each through one axis of a cylinder and perpendicular to the other axis. We have argued above that the square section of the vault is eight times the triangular section of the hoof, but similar relations also hold between the entire solids. This fact was already pointed out by Heiberg and Zeuthen [1907, 357] and Reinach [1907, 960–61], but only *en passant*, after showing a reconstruction of a proposition by virtual balance (Rufini 16).

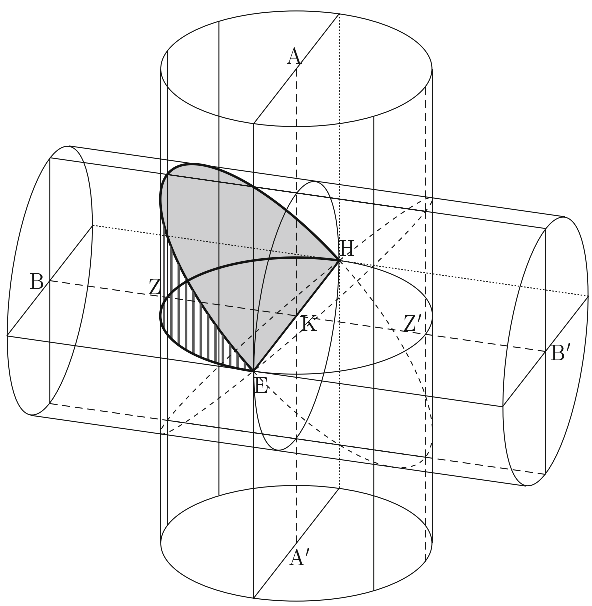


Figure 5: Decomposition of the vault into eight hoofs.

There may be some doubt about whether a proof by decomposition of the vault into hoofs would be too simple and straightforward to fill three manuscript pages.

However, the vault is not a simple solid like the sphere, and mere description of the solid requires some space. In the preface of the *Method*, Archimedes states the volume of the vault as follows:

16 Our conclusion is also suggested in Netz, Noel, Tchernetska and Wilson [2011, vol. 1, 230]. Most of our arguments come from Hayashi and Saito [2009], from which we have borrowed the figures (except Figures 6, 15 and the plate of appendix 3).

> If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube. [Heath 1912, suppl. p. 12]

This enunciation occupies sixteen lines in the manuscript, almost one fourth of a page (a page consists of two columns, which has usually 36 lines). The lost proposition must have begun with a description like this, then there must have been the exposition (*ekthesis*) referring to the diagram by the names of the points. To describe the solid, it is necessary to identify which of the cylinder surfaces appear as the surface of the intersection. Since Archimedes does not use perspective drawing in the *Method*, probably he drew a plane diagram like Figure 6, and developed some argument purporting to establish that the lines *EG* and *FH* are the borders of the two cylindrical surfaces constituting the surface of the solid of intersection, and that the areas *EKF* and *GKH* are the surface of the cylinder having the axis *AB*, while the areas *FKG* and *HKE* represent the surface of the cylinder having the axis *CD*, and so on. Such an affirmation must have been accompanied by some justificative statements. Only after such descriptions and arguments, is it possible to assert that the vault is decomposable into eight hoofs which are equal to one another. He may well have used another diagram to show the hoof obtained by decomposition.

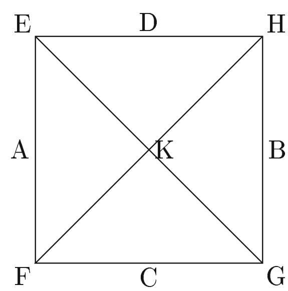


Figure 6: A possible planar diagram for the vault.

All these arguments and diagrams seem sufficient to fill most of the space of three pages. Moreover, the concluding part of Prop. 15, which we assumed to occupy just one column, may have been longer, and some concluding remarks pertaining to the whole work may have existed after the demonstration of the volume of the vault. Thus three pages seem to be just enough to contain the proof we propose.

Archimedes as Ancient Geometer: A Revised Portrait

We have argued that the lost demonstration of the volume of the vault was probably its decomposition into eight hoofs, whose volume had already been determined in Prop. 12–15.

In this section, we argue that this interpretation suggests a considerable change of the image of Archimedes as mathematician, which has been excessively modernized.

Difficulty Resolved: An Interpretation of the Method

We have pointed out the difficulty with the current reconstruction of the determination of the volume of the vault, above. If we assume that Archimedes cut the solid in such a way to obtain square sections and used the virtual balance in much the same way as in Prop. 2 for the sphere (Rufini 16; see Appendix

1, Volume of the Vault: Part 1), it is difficult to explain why he did not adopt a similar argument for the hoof, for which he developed a series of very complicated arguments in Prop. 12+13.[17]

In our interpretation, Archimedes did not cut the vault by such planes. He first observed that its surface consists of two parts — the surface of one of the intersecting cylinders, or that of the other — and cut the solid of intersection according to the border line of the two parts. Thus he gets four segments, each of which is one fourth of the whole solid. If one looks at this segment along the line KZ (see Figure 5 above), it is easy to see that its "height" is proportional to the distance from K. This is the convenient condition for an approach by the virtual balance. At this point, or earlier, he may have realized that he could divide the segment into two symmetrical parts cutting it by the circle $EZHZ'$, obtaining the hoof.

Thus the question of the volume of the vault is reduced to that of the hoof, and the natural approach is to introduce the virtual balance whose arm is KZ with fulcrum K (see Appendix 1, Volume of the Hoof: Part 2). This is Prop. 12 of the *Method*, and although this approach led to the apparently no less difficult problem of determining the barycenter of a semicircle, Archimedes somehow circumvented it in Prop. 13, and obtained the result. In the latter proposition, he cut the solids by planes parallel to the arm of the balance, and this new way of cutting the solid, through which he obtained the result he was looking for, probably suggested the "indivisible" solution (Prop. 14), which could easily be transformed into a rigorous demonstration by double *reductio ad absurdum* (Prop. 15).[18]

With this interpretation, the difficulty with Prop. 12+13 disappears. Archimedes did not cut the vault in the manner of generating square sections, for he first divided it into hoofs. We should add that his approach was rather natural. For us moderns, equipped with the diabolic effectiveness of integral calculus, the volume of a solid has little to do with its shape or appearance. One only has to find a set of parallel planes which generates "simple" sections (or more precisely, the sections whose areas can be expressed by integrable functions). And since Archimedes cuts the conoids and spheroids (paraboloids, hyperboloids and ellipsoids in our terms) always by planes perpendicular to their axis, we tacitly assume that Archimedes shared our idea, that is, to find the volume of a solid is to find appropriate parallel sections.

In short, we have thus overestimated the "modern" ingredients in Archimedes' works. If he treated the paraboloid, spheroid and the hyperboloid in the same manner in his preceding work *Conoids and Spheroids*, this was because they were all generated by rotation, and the same approach was valid for all of them. However, the vault is not a solid of rotation, and he observed its shape and appearance to find an appropriate approach. According to our interpretation, he first cut this solid by the planes through the border lines of the surfaces of two cylinders, so that the segments are part of one cylinder, not some entangled mixture of two cylinders. For him, this simplifies the situation. Should one ask why he did not cut the solid by planes that would generate square sections, the answer is now clear. First, he did not share our concept that determining volume implies finding appropriate parallel sections. There was no reason to cut the intersection of two cylinders in such a way as to mix up the two cylinders, while it can obviously be divided into segments, each of which consists of "one" cylinder, not of "two."[19]

---

[17] R. Netz suggests that Archimedes was "playful" and "sly" (Netz and Noel [2007, 37], though not in this context). Such an interpretation would resolve this difficulty, for Archimedes might well have written confusing and unnecessarily complicated arguments anywhere on purpose. Our arguments, however, try to defend an honest Archimedes.

[18] It seems that Prop. 14 offers a new, powerful approach for the determination areas and volumes, though we know nothing about its application to other figures. Probably, Archimedes did not have time to develop its potentiality after he wrote *Method*, which was very probably written after all his other major works had been sent to Alexandria, from *Quadrature of Parabola* to *Conoids and Spheroids*.

[19] It should be remembered, that eighteen centuries later, Piero della Francesca treated the vault, and he cut this solid and the circumscribed cube by a plane through the straight line passing through the intersection of the axes of cylinders, and perpendicular to both axes. (Then the cutting plane can be rotated around this straight line.) The section of the solid is always that of one cylinder, and is an ellipse. Then he compared this section with the circumscribed rectangle, which is the section of the cube produced by the same cutting plane. By ingenious, but not very rigorous inferences, he concluded correctly that the vault is two-thirds of the circumscribed cube. For details, see Gamba, Montbelli and Piccinetti [2006].

This is historical evidence that cutting the vault by planes which generate square sections is not a universally obvious approach.

Moreover, cutting the solid through the curve of the borders of the cylindrical surfaces, Archimedes obtains a segment whose height is proportional to the distance from the center of original solid as we have seen above. Then there is no reason to make other trials other than to introduce the usual tool of virtual balance, unless this approach happens to prove impracticable. This approach led him to a very difficult problem as we have seen, but fortunately, his genius found a solution to it in Prop. 13.

Our interpretation, then, suggests the figure of a mathematician much less modern than we are used to imagine. He did not recognize the general approach of cutting the solid by appropriate planes to determine its volume. His approach was much less general, and the appearance of a solid was a non-negligible factor in his investigation. Losing much of Archimedes' "modernity," we have instead recovered his honesty and sincerity at least in the *Method*, for we no longer have to ascribe to him a playful or sly character when he develops the complicated arguments in Prop. 12+13 of the *Method*. If this conclusion seems strange, it is because of what has been said about the *Method* since its discovery. Modern scholars have been misled by the title of the work "*Method*", invented by Heiberg (see note 2, above), and by his remarks that Archimedes' method in this work was equivalent to the integral calculus.[20]

Archimedes as Ancient Geometer

We should rather look at Archimedes in the context of Greek geometry, of which at least a basic part is still taught at schools. Indeed, our school geometry — with its theorems on congruence of triangles, similar figures, and so on — is an adaptation of Euclid's *Elements* which were directly used in the classrooms until the 19th century. The objects are figures which are described by words and shown in the diagrams. The demonstration is directly referred to the objects shown in the diagrams or at least connected to them by labels (e.g., the expression "square on *AB*," where the square is not always really drawn in the diagram). Geometry and arithmetic were clearly separated; there is no symbolic language similar to our symbolic algebra.

The objects (figures) are formalizations of either concrete objects, or of effective solution procedures. Let us illustrate this last point. For us, an ellipse is the locus of zeros of a polynomial of second degree with two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0; \text{ where } B^2 - 4AC < 0.$$

In other words, a curve is defined by an abstract property of an algebraic nature, which precedes the object itself. For the Greeks, however, the ellipse is the curve obtained by cutting a cone with a plane that intersects all of its generatrices. A curve is defined by a specific procedure, then its properties are derived thereof. Greek mathematics is thus a mathematics of individual objects, each generated from a suitable constructive process.[21]

So every argument is referred to some figure shown in the diagram, and this means that there was no way of describing a general method of solution. If we have discussed the double *reductio*, not the *method* of exhaustion in the present article, this is because neither Euclid, nor Archimedes, nor any Greek mathematician has ever spoken of this demonstration technique in general terms. We have several propositions in which we find similar sequences of particular arguments, and it is we that give a name to this pattern of arguments. The same is true for what we have called the virtual balance of Archimedes in the present article.

If we adopt this point of view, some consequences immediately follow:

- Greek mathematics is not a general mathematics, unlike post-Cartesian mathematics.
- There are no general objects, still less general methods.

[20] "Die neue Methode des Archimedes ist tatsächlich mit der Integralrechnung identisch" [Heiberg 1907, 302].

[21] This is the view on the objects of Greek mathematics given by Giusti [1999] (esp. capitolo 5).

- The procedure of measuring an object is a formalization of some concrete process; indirect confrontations are applied only if the direct one is proved impossible.

Archimedes' works fits these general characteristics well. His extant works are divided into two groups according to two major themes: geometry of measure and mechanics. In the works of geometry (*Measurement of a Circle*, and the four works sent to Alexandria: *Quadrature of the Parabola*, *On the Sphere and the Cylinder*, *Spiral Lines*, *Conoids and Spheroids*), Archimedes deals with the problem of measuring, that is, determining the size of geometrical objects, through direct comparison between an "unknown" figure (e.g., a sphere or a paraboloid) and a better known one (e.g., a cylinder or a cone), and shows, for example, that the sphere is two-thirds of the circumscribed cylinder, or that the paraboloid is one and a half times the cone inscribed in it, and so on. This is why Archimedes was so proud to tell Eratosthenes, in the preface of the *Method*, that he had succeeded in demonstrating for the first time the equivalence between a solid curved figure and a "straight" one (a parallelepiped). Quadrature (or cubature in this case) of a figure was not the result of finding a formula, like $V = \frac{4}{3}\pi r^3$, but of finding the simplest known figure equal to it.

Our investigation in the present article confirms that Archimedes was working within the framework of Greek geometry, despite of his numerous and marvelous results.

Concluding Remarks: How Did Archimedes Come to Consider the Novel Solids?

Before concluding this article, we should mention a problem which is brought about by our interpretation of the last proposition of the *Method*.

According to the hitherto prevailing interpretation, Archimedes used a parallel argument for the hoof and the vault. Then, it was not so important to decide how he came to consider these particular solids. He might even have started from the method of determining the volume. A possibility was that he was perhaps looking for some solid for which the same argument for the volume of the sphere was valid, and found the vault which can be obtained by replacing the circles (section of the sphere by parallel planes) with squares. Then, replacing these square sections by similar triangles, a hoof can be obtained. Though one could only speak of a possibility, Archimedes' novel solids may have been invented "inversely" from the way of determining their volume.[22]

Our interpretation in the present article, however, has confirmed a "classical" interpretation of Archimedes, denying his recognition of the common method between the sphere and the two novel solids. But if Archimedes was not induced to consider the novel solids because of the common method used to determine their volume, how did he come to consider these novel solids? As we have proposed that Archimedes found the hoof during his investigation of the vault, the question is reduced to that of considering the vault.

Concerning this question, we have a very interesting piece of archaeological evidence. Recent excavations of a bath at Morgantina, in Sicily, have revealed the existence of two barrel vaults arranged at a right angle, although without intersecting.[23] This reminds us of the discovery of remains of a "hydraulic establishment" in Syracuse by the Italian archaeologist G. Cultrera in the 1930s, where the same technique was used [Cultrera 1938]. The existence of this type of construction at Morgantina — at that time part of the Syracusan kingdom of Hieron II — and the existence of at least one similar building in Syracuse itself, suggest the possibility that Archimedes was inspired by something to consider the volume of the vault. If we are allowed to put it dramatically, Archimedes, lying in the bath or having a massage, asked himself the question: what if those two vaults were to intersect? What kind of shape would result? It should be noted that the construction techniques used at Morgantina (and most likely at Syracuse) were not such that would easily have allowed the construction of a cross vault. However, it is not really a question of whether Archimedes knew this public bath directly or indirectly. What is

[22] One of the authors was once inclined to this position. See Saito [2006].

[23] For a more detailed description of the excavation, see Lucore [2009].

important is that there is a possibility that Archimedes may have found the inspiration of considering the vault from some real and existing objects like vaults, so that we do not have to assume that he started from some established method of determining the volume of a solid, and worked backwards to other solids for which the same method was valid.

In short, Archimedes was not so modern as we have been prone to imagine. He was an ancient. His arguments about the volume of solids always began with some concrete solid; he did not invent a solid from a method; the recognition, evident for us, that the volume of a solid is determined by its sections, was not necessarily evident to him.

Thus the inquiry into the number of lost pages at the end of the *Method* has revealed an Archimedes less modern but at the same time less sly and more honest and serious.

Appendix 1: Archimedes' Propositions and Reconstructions

Prop. 2: Sphere

The use of the virtual balance is based on the equilibrium between sections of figures whose volume (or area) is unknown, and corresponding sections of a known figure. To determine the volume of a solid (or the area of a plane figure in Prop. 1), it is necessary to carry its sections to the other end of the virtual balance, and find the section of another solid, which, in its place, is in equilibrium with it. Both the volume and the barycenter of the second solid must be known. In many cases, this second solid is a cylinder.

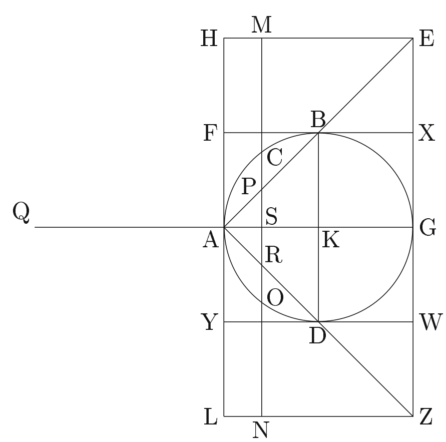


Figure 7: *Method* Prop. 2: Volume of the sphere.

In Figure 7, *ABGD* is a great circle of the sphere, *AG* and *BD* are two of its diameters, perpendicular to each other. Archimedes conceives another great circle in the sphere with diameter *BD* and perpendicular to the plane of *ABGD*, and constructs a cone having this circle as the base and the point *A* as vertex. This cone is extended to the plane through *G* and parallel to the base of the cone. This cone makes a circle whose diameter is *EZ*. Then a cylinder is constructed with the circle *EZ* as base, and the straight line *AG* as height. Finally, a balance *QG* is conceived, *QA* being equal to *AG*.

If one cuts the sphere, cone and cylinder by a plane *MN*, perpendicular to *AG*, then the sections of the sphere, the cone and the cylinder are all circles whose diameter are *CO*, *PR*, *MN*, respectively.

Since

$$\mathrm{sq}(CS) + \mathrm{sq}(PS) = \mathrm{sq}(CS) + \mathrm{sq}(AS) = \mathrm{sq}(AC),$$

and

$$\mathrm{sq}(GA) : \mathrm{sq}(AC) = GA : AS = QA : AS,$$

therefore

$$\mathrm{sq}(GA) : \mathrm{sq}(CS) + \mathrm{sq}(AS) = QA : AS.$$

The squares can be replaced by the circles having the sides of the square as radius. Therefore,

$$\text{circle } MN : \text{circle } CO + \text{circle } PR = QA : AS. \quad (1)$$

This means that the circle CO (section of the sphere) and the circle PR (section of the cone), taken together and carried to the point Q, are in equilibrium with the circle MN (section of the cylinder) remaining in place. Doing the same for other parallel planes, like MN, we have an equilibrium between the solids: the sphere and cone moved to point Q are in equilibrium with the cylinder remaining in place. From this equilibrium, the volume of the sphere is easily determined.

Volume of the Hoof

Part 1: A Possible Use of the Virtual Balance

The volume of the hoof can easily be determined in substantially the same way as that of the sphere (Prop. 2), though Archimedes did not take this approach.

Imagine a hoof, cut from a cylinder whose base is the circle EH, by the oblique plane through EH and FV.

Extend HZ and HW until they meet ED and EF, extended, at points D' and F', respectively. Imagine a pyramid having base $ED'F'$ and vertex H, and a triangular prism $D'EF' - G'HV'$. Their role corresponds to that of the cone AEZ (Figure 7) and cylinder $EZLH$ in Prop. 2, which treats the sphere.

Take the barycenter of the triangle $ED'F'$ and $HG'V'$, E_0 and H_0 respectively, and extend E_0H_0 to Q_0 so that $E_0H_0 = H_0Q_0$, and imagine the balance $Q_0H_0E_0$ with fulcrum H_0. The section of the hoof by any plane, $NM'Y'$, perpendicular to the arm of the balance is triangle NSX (shown by the shadowed triangle in the figure). In the case of the sphere, the section was the circle having center N and radius NS. Instead of the sections of the sphere, the cone and the cylinder in the case of the sphere, consider the sections of the hoof, the pyramid and the prism cut by the plane $NM'Y'$. The sections are triangles NSX, $NM'Y'$ and $NM''Y''$, respectively, and they are similar to each other (the section of the hoof is shadowed, and those of the pyramid and the cylinder are shown by dashed lines in the figure).

The rest of the argument is similar to that in Prop. 2. For the circle sections of the sphere, the cone and the cylinder, the proportion (1) was deduced; now between the similar triangles, which are sections of the hoof, the pyramid and the cylinder, one can deduce:

$$\text{triangle } NM'Y' : \text{triangle } NSX + \text{triangle } NM''Y'' = QH_0 : H_0N_0.$$

This proportion means an equilibrium on the virtual balance: the triangles NSX and $NM''Y''$, that is, the sections of the hoof and of the pyramid, taken together and carried to Q_0 are in equilibrium with the triangle $NM'Y'$, the section of the prism, remaining in place.[24] From this equilibrium of the sections follows the equilibrium between the solids — the hoof and the pyramid carried to Q_0 is in equilibrium with the prism remaining in place. This equilibrium means that the hoof and the pyramid taken together are half the prism, and it can be easily deduced that the hoof is one-sixth the prism $D'EF' - G'HV'$, or two-thirds the prism $DEF - GHV$.

[24] We have imagined a balance E_0Q_0 which passes the barycenter of each of the section of the prism (e.g., N_0 is the barycenter of the triangle $NM'Y'$).

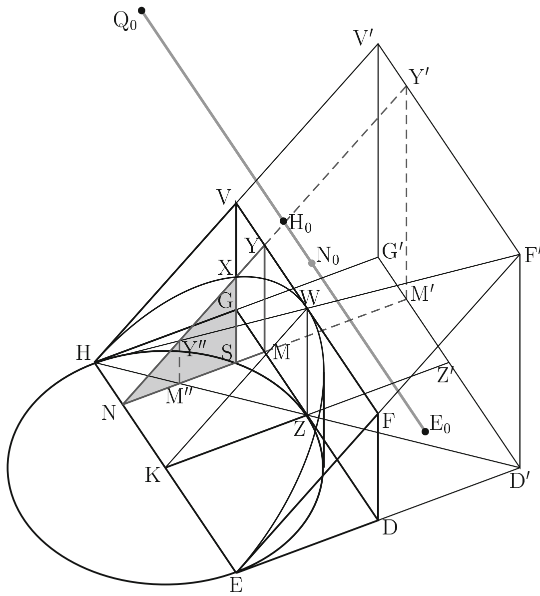


Figure 8: Volume of the hoof: A possible use of virtual balance.

Part 2: Archimedes' Use of the Virtual Balance, Prop. 12+13

Archimedes' approach to the hoof, however, was completely different from the above reconstruction. He imagines a balance *HJ* (Figure 9), perpendicular to the section of the half cylinder which contains the hoof, and passing through the center *O* of the section. Probably, Archimedes first saw that the "height" of the hoof is proportional to the distance from the diameter of the base *AC*. Indeed, if one cuts the hoof by a plane *LM*, perpendicular to the base and parallel to the diameter *AC* of the semicircle, the section is the parallelogram *MF*, whose height *PR* is proportional to *KR*. It is obvious that the section of the hoof (parallelogram *MF*), carried to point *H*, is in equilibrium with the section of the semicylinder (*ML*) remaining in its place. This is, indeed, the repeated pattern of the argument by virtual balance.

Considering all the sections by parallel planes, the hoof moved to point *H* is in equilibrium with the semicylinder having base *ABC* and height *BD*, left in its place. If one knew the barycenter of the semicylinder (this is, of course, equivalent to the barycenter of the semicircle) the volume of the hoof would be determined at once.

However, this was not the case, of course. Archimedes then finds another solid, whose volume and the barycenter is known, and in equilibrium with the semicylinder. The solid is a triangular prism (Figure 10A). The semicylinder and the prism are in equilibrium on the balance *CP* whose fulcrum is the point *Q*.

No attempt has been made, as far as the authors know, to explain how Archimedes discovered the triangular prism, but it is fairly easy to find a reasonable hypothesis. Obviously, the problem is reduced to finding a plane figure in equilibrium with a semicircle. In Figure 10B, *CP* is the arm of a virtual balance having the fulcrum at the point *Q*. It is required to find some figure on the left side of *RO*, which would be in equilibrium with the semicircle *OPR*.

Archimedes always cuts the figure by lines or planes perpendicular to the arm of the balance, but this approach was useless in this situation, for it would have taken him back to the hoof from which he started. Confronted with this difficulty, his genius invented another way of cutting the figure. Imagine that the semicircle is cut by a line *SK*, parallel to the arm *CP* of the balance, and look for the section *LX* which would be in equilibrium with *SK*, around the point *S*. If such a line *LX* is found for each section

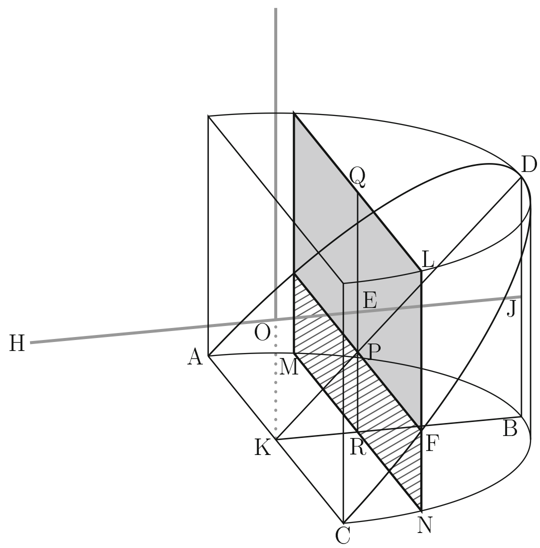


Figure 9: Prop. 12: Hoof and semicylinder.

SK, then the figure filled by all the lines *LX*, would be a figure in equilibrium with the semicircle.

Let the length of *LX* be *m* and the distance of the barycenter of *LX* (midpoint of *LX*) from *S* be *l*, Since this is in equilibrium with *SK*,

$$SK : m = l : \frac{SK}{2}.$$

That is,

$$\mathrm{rec}(l, m) = \frac{\mathrm{sq}(SK)}{2}.$$

Now, every Greek mathematician knew that $\mathrm{sq}(SK) = \mathrm{rec}(RS, SO)$ in a circle, so that

$$\mathrm{rec}(l, m) = \frac{\mathrm{sq}(SK)}{2} = \frac{\mathrm{rec}(RS, SO)}{2}. \tag{2}$$

Then, one might as well try assigning *RS*, *SO* and ½ to *l* and *m*, so that the equality (2) holds. There are not so many possibilities for such an assignment, and the assignment $l = SO/2$, $m = RS$ for sections between *R* and *Q* would create triangle *CHQ*. For the sections between *Q* and *O*, an assignment symmetrical to those between *R* and *Q* would create triangle *CMQ*. As a whole, triangle *HMQ* is found to be in equilibrium with the semicircle *OPR*. Then, considering the prism and the semicylinder having these plane figures as base, the semicylinder is in equilibrium with the triangular prism, so that the triangular prism is in equilibrium with the hoof, carried to the endpoint of the balance.

The whole argument of Prop. 12+13 is very long and complicated, but the first step of introducing the semicylinder in Prop. 12 is quite natural for someone who has become accustomed to the use of the virtual balance as seen in the other figures. The only impressive leap is found in Prop. 13, where Archimedes cuts the figure by planes parallel to the arm of the balance, while in the previous propositions he cut the figure by planes perpendicular to the arm of the balance.[25] Once this unusual way of cutting is found, then it must have been easy for any Greek geometer to find a section of some

[25] In our exposition above, we considered the semicircle which is the base of the semicylinder, so we cut the semicircle by lines parallel to the arm of the balance.

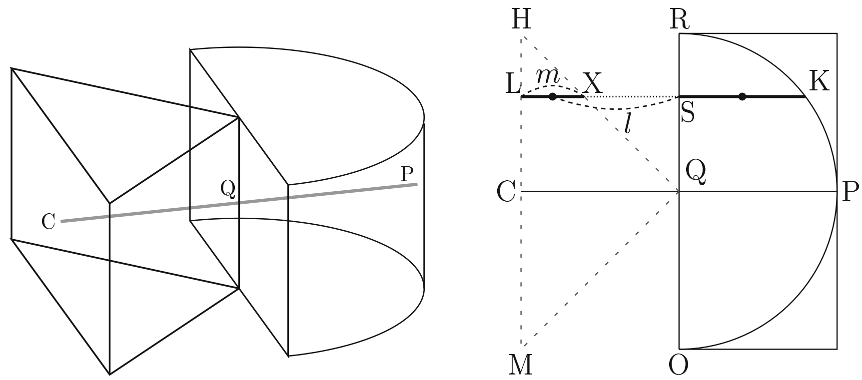


Figure 10: (A) Prop. 13: Semicylinder and triangular prism in equilibrium; (B) Look for a shape in equilibrium with a semicircle.

"manageable figure" in equilibrium with the section *SK* of the semicircle, and to find consequently the triangle *HMQ* (or some other appropriate figure), as we have shown.

This new way of cutting the figure by planes parallel to the arm of the balance probably opened the way for the argument without the balance in Prop. 14. Archimedes only had to try cutting the original hoof by the same plane with which he cut the semicylinder. The resulting sections of the hoof are triangles similar to each other and constructed on half chords of a circle (we will soon see it in Figure 11). At this point he could have realized that the determination of the volume of the hoof is identical with that of spheroids, but for some unknown reason he did not, and instead found another wonderful way of reducing the determination of the volume of the hoof to the quadrature of a parabola, one of his early findings. Rewriting the whole argument of Prop. 14 into a rigorous demonstration by double *reductio ad absurdum* must have been no more than routine work for Archimedes, who had already written the *Conoids and Spheroids*.

We moderns may find at once that cutting the figure by planes perpendicular to the diameter of the base gives the easiest solution, because the cubature of a solid involves finding some set of parallel planes which yield sections whose area are easily integrable. So our approach begins with cutting the solid by various parallel planes, and it is rather difficult not to find the "right" way of cutting the hoof. However, it was not possible for Archimedes to find this section before the usual and evident application of the virtual balance (Prop. 12), and the effort to resolve the difficulty he encountered (Prop. 13).

Part 3: By Indivisibles without the Balance, and Its "Exhaustion" Version (Prop. 14 and 15)

In this proposition, Archimedes cuts the hoof by a plane passing through N and perpendicular to the diameter of the semicircle EZH (Figure 11). This plane cuts, from the hoof, triangle NSX. If one considers the triangular prism $DEF-GHV$ circumscribed about the hoof, the same cutting plane cuts the triangle MNY from the prism. Archimedes compares the two sections MNY and NSX, which are two similar triangles, and shows that their ratio $MNY : NSX$ is reduced to a ratio of two line segments, $MN : NL$, where L is the point where the cutting plane meets the parabola with vertex Z, diameter ZK, and passing through E and H. That is,

$$MNY : NSX = MN : NL.$$

This proportion holds for any point N on the diameter EH, and gathering all the sections together, Archimedes concludes that[26]

[26] There are some twenty lines of text justifying this transition from the proportion of the sections to that of "all the

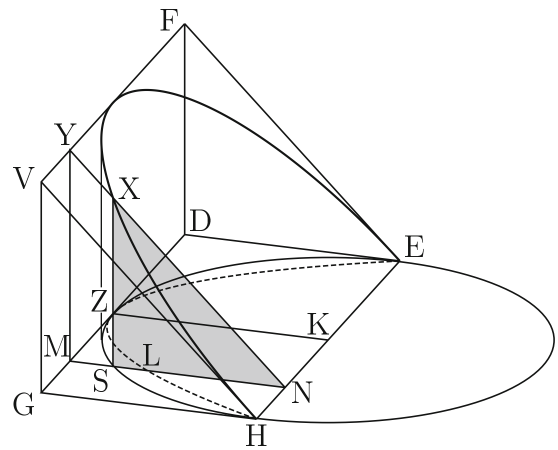


Figure 11: Prop. 14: "Indivisible" approach to the volume of the hoof.

$$\text{prism } DEF - GHV : \text{hoof} = \text{parallelogram } DH : \text{parabolic segment } EZH.$$

As the parabolic segment is two-thirds of the circumscribed parallelogram, the hoof is also two-thirds of the prism.

This argument, though not containing mechanical elements, was by no means at the level of rigor required in Greek geometry. However, it is only routine work to transform it into such a demonstration. One only has to divide the diameter *EH* into equal parts, and consider the triangular prism having as base the triangle *NSX*. Thus one can construct solids inscribed in, and circumscribed about, the hoof consisting of triangular prisms, which differ by a magnitude smaller than any assigned magnitude. Then the rest is the usual argument by *reductio ad absurdum*. This is exactly what Archimedes did in Prop. 15.

The Volume of the Vault

Part 1: By Virtual Balance (Rufini 16)

We present here an outline of the reconstruction of the argument by the virtual balance for the volume of the vault, which can be found in Heiberg and Zeuthen [1907, 357], Reinach [1907, 959–960], Heath [1912, suppl. p. 48–51], and Rufini [1926, 170–173].[27]

The outline of the argument by virtual balance is as follows. Using the diagram from Prop. 2, one only has to imagine that the circle *ABGD* is the section of the vault by the plane of this diagram (one of the axes of the intersecting cylinders is *BD*, the other axis is through *K* and perpendicular to the plane of the diagram), and that parallelogram *EL* and triangle *AEZ* are sections of a prism (or parallelopiped) and a pyramid respectively (both having square base). Then, the plane through *MN* cuts, from the

sections" (figures), which was illegible for Heiberg. Recent studies of the palimpsest has restored the text and shown that Archimedes was not developing a naïve argument by intuition, but was trying to provide a justification to this argument of "summing up" infinite sections applying a theorem valid for the proportion of the sum of a finite number of terms. See Netz, Saito and Tchernetska [2001–2002].

[27] These reconstructions are generally called "Prop. 15" except by Rufini [1926] who assigns number 16, because the current Prop. 8 did not appear in Heiberg's first report of the discovery of the palimpsest [Heiberg 1907], and the proposition numbers assigned to the subsequent propositions were less by one (see also Appendix 2). The proposition numbers we use are those in Heiberg [1910–1915].

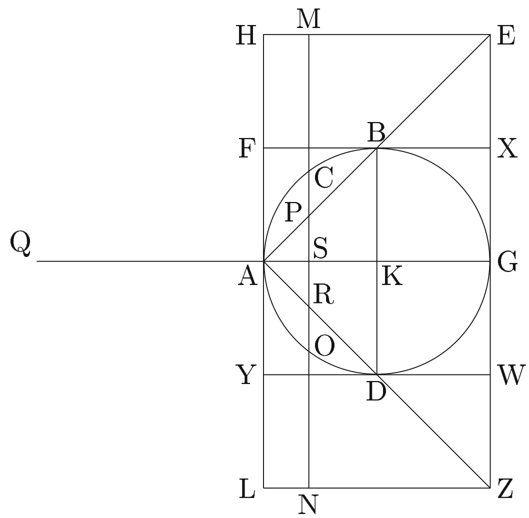


Figure 12: The diagram for the sphere, reused for the vault.

vault, the square on CO — more precisely, the line CO joins the midpoints of the opposite sides of the square — from the prism the square on MN, and from the pyramid the square on the PR. By the same argument for the sphere in Prop. 2, the square on CO and on PR, carried to the point Q are in equilibrium with the square on MN, remaining in its place. The same thing being done for other sections, it turns out that the vault and the pyramid together, carried to the other end of the balance so that their center of gravity is the point Q, are in equilibrium with the prism EH remaining in place. So (the vault) + (pyramid AEZ) is half the prism EH; and since the pyramid AEZ is one third the prism EH, the vault is one-sixth of the prism EH. And the cube FW is one-fourth the prism, so that the vault is two-thirds of the cube FW in which it is inscribed.

Part 2: By Indivisibles (Sato 17)

The volume of the vault can be determined in much the same way as Prop. 14, Archimedes' treatment of the hoof.

The reconstructed argument can best be understood by adding some lines to the hoof (Figures 11 and 13). Consider the hoof that is cut from a cylinder by a plane which makes half a right angle to the plane of the base circle, so that $ED = DF$ (Figure 11). Then construct a cube in which the vault is inscribed (Figure 13); one of the two cylinders is the cylinder of the hoof, the other (not drawn in the figure) has the axis ZKZ'. The section of the intersection by the same plane through N, which cuts the triangle NSX from the hoof, cuts a square from the vault. This square is hatched in Figure 13, and is obviously eight times the triangle NSX. The same cutting plane cuts from the circumscribed cube a square equal to the square of the surface of the cube. Then, just as in the case of the hoof, the following proportion holds (the point L is shown in Figure 11 only, not in Figure 13):

$$\text{(section of the cube)} : \text{(section of the intersection)} = MN : NL,$$

and it can be shown that the vault is two-thirds of the circumscribed cube.

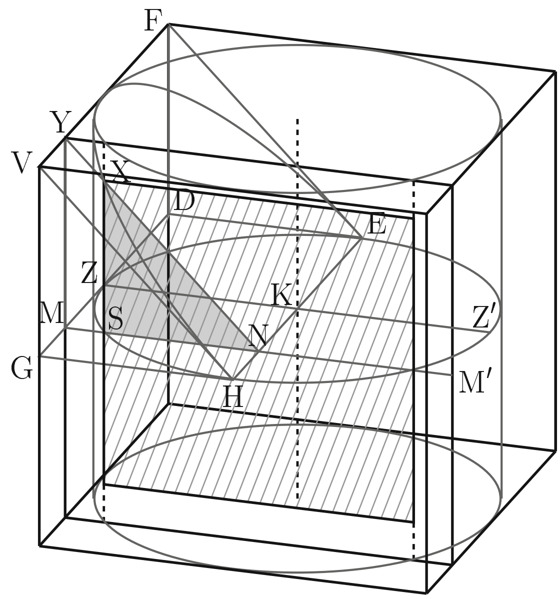


Figure 13: The volume of the vault by indivisibles.

Appendix 2: Reconstructing the Codex

The extant text of the *Method* breaks off before Prop. 15 finishes, and the remainder of this proposition and the proposition which followed it are found nowhere in the palimpsest. Nevertheless, we know for certain the number of the lost folia at the end of the *Method*, thanks to a successful reconstruction of the quires of the original Archimedean codex. In this appendix, we show how the reconstruction is made, and argue that its results are certain.

Reconstruction of the Quires of the Archimedean Codex

First, let us consider how the Archimedean codex was unbound, and its folia reused in the palimpsest.

Medieval manuscripts like the palimpsest, as well as the original Archimedean codex whose parchment was reused for the palimpsest, are materially compiled from quires. A quire consists of four parchment sheets (less often, three or five, or more), folded in half, placed one inside another other and sewn at the fold.

However, the Archimedean codex no longer exists in bound form, so the first challenge is to reconstruct its quires. When the palimpsest was made, the original codex was unbound and the folia were cut into two halves (that is, into single pages), then themselves folded in half and reused for the prayer book, which is thus half the size of the original Archimedean manuscript (Figure 14).[28] Of course, the page order of the original Archimedean manuscript is not preserved in the extant prayer book, and not all the folia that the original contained are present in it.

However, where the Archimedean text is readable, the text itself permits us to determine the order of the pages in the original codex, and now the folia of the Archimedean palimpsest are given double

[28] More precisely, the folia were trimmed in the process of making the palimpsest, so that the page of the palimpsest is smaller than the half of that of the original Archimedean codex. See Netz, Noel, Tchernetska and Wilson [2011, vol. 1, 144].

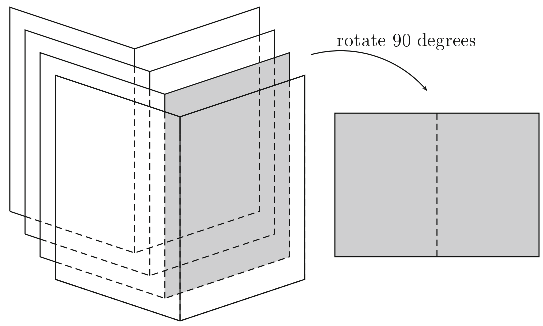


Figure 14: Recycling the parchment.

numberings. One is the folium number in the prayer book, the other is the folium number in the order of the Archimedean text. The former, like 46r-43v, is shown in the margin of Heiberg's edition of the *Method*, while the latter, like A15, can be found in the names of the digitized images of the pages of the palimpsest.[29]

Thus the order of the folia originating in the Archimedean codex is known, but the order of the text does not show where one quire began and how many folia were bound into one quire. However, a careful comparison of the folium number of the prayer book and that of the Archimedean codex can reveal, almost certainly, the composition of the quires in the original Archimedean codex. For example, we are sure that the eight folia from A14 to A21 constituted one quire, as is shown in Figure 15A.[30]

To illustrate how the construction of the quires is determined, let us take, for example, the two folia A15 and A20, which are separated by four intermediate folia, from A16 to A19. In the prayer book, they are bi-folia 46-43 and 45-44 respectively. This means that they are two consecutive folia in the prayer book. If this has not happened by chance, the only reasonable explanation is that A15 and A20 are two halves of one folium in the original Archimedean codex (see Figure 15A), and they were put one over the other to be reused in the prayer book.

This reasoning is supported by a similar examination of the arrangement of the eight folia from A14 to A21 in the prayer book (Figure 15A), and we are sure that this cannot have happened by chance. The table shows that the same thing has happened for other three folia of the same quire of the Archimedean codex, A14+A21, A16+A19 and A17+A18; although they ended up in different places in the prayer book, the two half folia obtained from one original folium are always consecutive in the prayer book. This approach turns out quite successful for all the folia of Archimedean manuscript, and the reconstruction is sometimes physically confirmed by the traces of binding [Netz, Noel, Tchernetska and Wilson 2011, vol. 1, 48].

A similar argument for the folia containing the following propositions of the *Method* (from A22) enables the reconstruction of the following quire, where one folium between A26 and A27 is missing, as is shown in Figure 15B.[31]

[29] The folia from 41 to 48 of the prayer book constitutes one quire, whose decomposition yields four sheets of parchments 41-48, 42-47, 43-46 and 44-45. The folium (or bi-folium) 43-46 is the 15th folium among the extant folia of Archimedean manuscript, and its *recto* side A15r is double pages 46r-43v of the prayer book. So A15=46r-43v. (See below.)

The page 46r is written before 43v, because, when this page is placed so that the Archimedean text is readable, the page 46r of the prayer book is the upper half of the page, and the 43v is the lower half. The images of this page on the web have the name beginning with “46r-43v Archi15r,” and in the printed edition “Arch 15r 46r+43v” [Netz, Noel, Tchernetska and Wilson 2011, vol. 1, 41]. The images of all the pages of the palimpsest are available at http://www.archimedespalimpsest.org.

[30] This is based on the figure in Netz, Noel, Tchernetska and Wilson [2011, vol. 1, 41]. We have changed the folio number “Arch14r” to “A14r” etc.

[31] We have corrected typos in Netz, Noel, Tchernetska and Wilson [2011, vol. 1, 41], where 166r+167v and 166v+167r for A24r and A24v appear as “166r+166v” and “166v+166r,” and we have described lost folia explicitly as “(lost).”

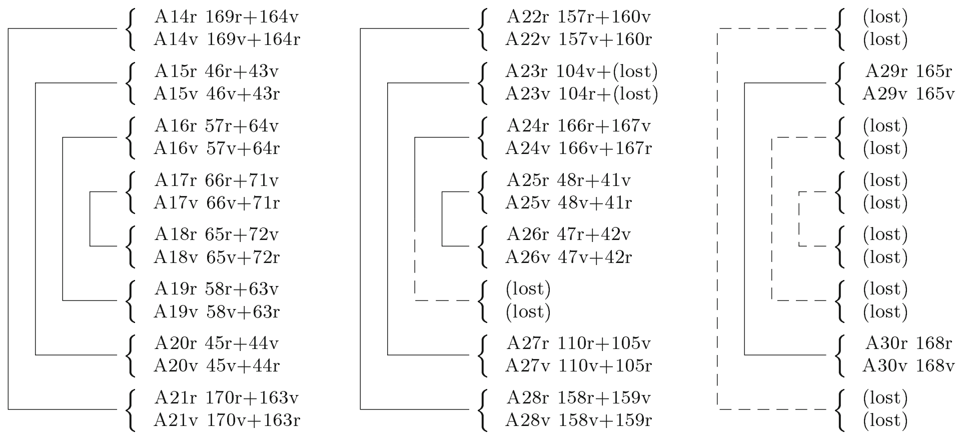


Figure 15: (A) Reconstruction of quire 3 of the Archimedes codex (the first containing the *Method*); (B) Reconstruction of quire 4 (the second containing the *Method*); (C) Reconstruction of quire 5 (the third containing the *Method*).

Much of the content of the lost folium between A26 and A27 can also be reconstructed. Prop. 13 begins in A26v, and the extant text (six lines and one whole column; see Heiberg [1910–1915, vol. 2, 492]) is just enough to infer the argument intended by Archimedes, which is fairly complicated and would have occupied most of the lost folium, and the following Prop. 14 begins exactly at the beginning of the following folium A27. So the possibility is excluded that there might have been another, unknown, proposition, between Prop. 13 and 14, except that there may have been some remarks by Archimedes like the ones between Prop. 1 and 2, or at the end of Prop. 2.

Prop. 14 continues to the next folium A28, which is the last in this quire, and ends in the middle of the first column of its recto page. Then comes Prop. 15, the last extant proposition of the *Method*, and continues into another quire of which only one folium is extant. We have thus reconstructed the first two quires containing the *Method*, and we can be sure no proposition has been completely lost up to this point.

Now let us examine the only extant folium of the following (third) quire of the *Method*, which has the folium number 165-168 in the prayer book. Figure 15C is the reconstruction of this quire. The readers may have two reasonable questions: first, why does the one folium 165-168 of the palimpsest appear in two Archimedean pages (this cannot occur if a folium of the palimpsest is one page of the original Archimedean codex), and then, how has it been possible to determine that this is the second folium from outside the quire while all other folia are lost? Now we respond to these questions.

Folium 165-168, the only one extant in this quire, is exceptional, for it is not one page of the Archimedes codex, as all other folia in the palimpsest, but spans two pages. It is, in fact, the central part of one original parchment sheet [Netz, Noel, Tchernetska and Wilson 2011, vol. 1, 45]. It was placed upside down when the text of the prayer book was written on it, apparently to minimize interference from the Archimedean text, which remains visible (Figure 16). On the left page (165v of the prayer book, then 165r), we read part of Prop. 15. Each column contains only 27 lines of the usual 36 lines in the Archimedean codex, so that about nine lines are completely lost.[32] On both pages, each line in the outer column is partly lost, either at the beginning or at the end, as is shown in Figure 16.

The opposite page (168v and 168r), contains text from the *Spiral Lines*. This means that a part of Prop. 15 of the *Method* and the beginning part of the *Spiral Lines* was in the same quire. How long was the length of the lost text between these two pages?

[32] According to Netz, Noel, Tchernetska and Wilson [2011, vol. 1, 34], six lines from the top and three lines from the bottom are lost.

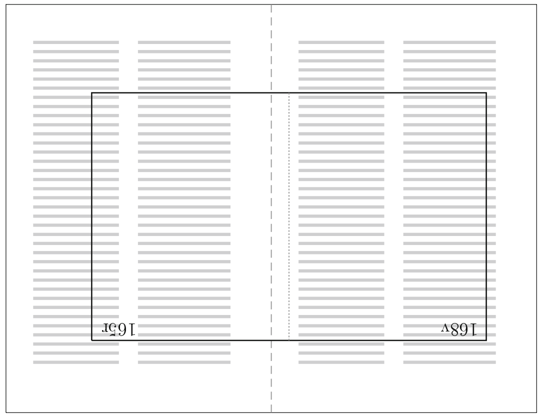


Figure 16: The last extant folium containing the *Method*.

Since most of the reconstructed quires of the Archimedean codex are quaternions, that is, quires of four folia (there are also a few ternions, quires of three folia), we may assume that this quire, of which we possess only (the central part of) one folium, was also a quaternion. We will see (note 35) that the possibility of a ternion is excluded.

Then, the number of intermediate pages depends on the position of the extant folium, 165-168, in the original quire. Fortunately, we know that this is the second folium from the outermost one. The subsequent text of the *Spiral Lines* is found in the palimpsest, after a lacuna corresponding to two pages, or one folium.[33] The part following just after this gap constitutes a ternion, then follows immediately a quaternion. All the (six plus eight) folia of these two quires are extant, and there is no gap in the text.

This means that the extant folium 165–168 is the second folium from the outside of the quire. Consequently, the lost folia of this quire are: (1) the first folium containing two pages of Prop. 15 of the *Method*, (2) four folia or eight pages between the extant text of *Method* Prop. 15 in folium 165 and the text of the *Spiral Lines* in folium 168, and (3) last folium (two pages) corresponding the lacuna in the text of the *Spiral Lines*.

We also know the length of the beginning part of the *Spiral Lines* before folium 168. In Heiberg's edition, there are about 8500 characters, which correspond to four pages and a column in the manuscript.[34] If we assume that the *Spiral Lines* begins at the top of a column as does the *Method*, the *Spiral Lines* very likely begins at the second column of the verso of the fourth folium of the quire (see Appendix 3).[35]

Therefore, there are only three pages and one column for the final part of the *Method*, of which at least one column was occupied by the concluding part of the Prop. 15. This leaves only three pages, perhaps less, for the whole set of the lost propositions on the vault.[36] In Appendix 3, we have shown in a somewhat schematic way, the content of each page of the three quires containing the text of the *Method*.

---

33 We can precisely estimate the length of this lacuna, for the complete text of *Spiral Lines* is preserved in other manuscripts.

34 In the following part of the *Spiral Lines*, where the text of the palimpsest is available, there is no discrepancy between the reading of the palimpsest and the other manuscripts that would affect the estimate of the length of the text. We assume that this is also the case in the beginning part, where the palimpsest is lost.

35 The possibility of a ternion is excluded at this point, for if it had been a ternion there would not even have been enough space for the beginning part of the *Spiral Lines*.

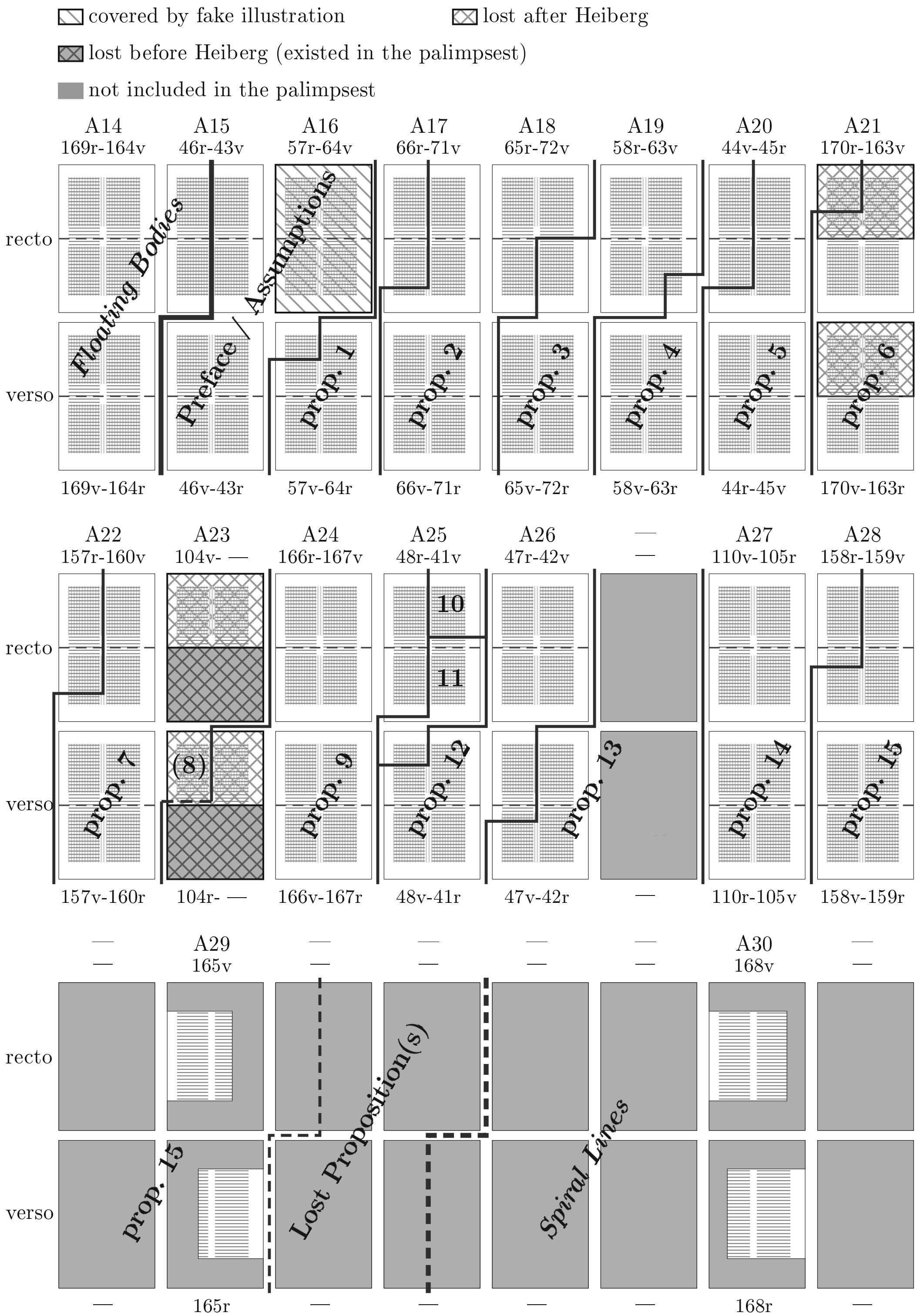
36 We have to admit that a slight possibility exists that the quire at issue was a quinternion, a quire of five folia, so that there were seven pages, instead of three, for the lost proposition(s) on the vault. However, there is no quinternion among the reconstructed quires of the Archimedean codex, and it seems arbitrary to assume here a quinternion which is not attested elsewhere in the codex.

Appendix 3: Page by Page Reconstruction of the Quires Containing the *Method*

The diagram of this appendix shows all the folia containing the *Method*, with the reconstruction of the original quires. The following are legenda and some comments:

1. The horizontal lines in each page show Archimedean text, while the vertical lines are text of the prayer book. However, these lines do not exactly correspond to the text of each page. The space occupied by the text and the number of lines in one page are different from one page to another. However, the same image (36 lines for Archimedes' text) is mechanically reproduced for all the pages in this diagram.
2. The lower half of A23 had already been lost when Heiberg consulted the palimpsest. One folium is lost between A26 and A27 in the second quire, and all the folia of the third quire are lost, except 165-168 (the central part of A29-A30). Very probably, they were simply not used in the palimpsest. In this diagram, they appear without the horizontal lines showing the text.
3. After Heiberg consulted the palimpsest, the recto page of A16 was covered by a forged illustration, and the upper half of A21 and A23 (recto and verso) were lost. These pages are indicated by different hatched lines. We have Heiberg's readings of these pages, which can no longer be examined.
4. The spaces occupied by each proposition are divided by straight lines, which are dashed when the border between propositions is not certain.
5. Prop. 8 consists only of a short enunciation without further argument, and belongs to a folium read by Heiberg but now lost. Judging from Heiberg's text, which shows the beginning of every line in the manuscript, Prop. 8 consists of eight full lines followed by seven very short lines. Very probably, the diagram of the Prop. 7 appeared beside the shortened lines of the text of Prop. 8.[37] This means that Prop. 8 was not meant as an independent proposition but was a mere corollary. Thus we have put the number of the Prop. 8 in parentheses.

[37] A similar arrangement of diagrams can be found at the ends of Prop. 3 and Prop. 4, for which the border lines between the following proposition goes through the middle of a column.


covered by fake illustration
lost after Heiberg
lost before Heiberg (existed in the palimpsest)
not included in the palimpsest
A14 169r-164v
A15 46r-43v
A16 57r-64v
A17 66r-71v
A18 65r-72v
A19 58r-63v
A20 44v-45r
A21 170r-163v
recto
verso
Floating Bodies
Preface / Assumptions
prop. 1
prop. 2
prop. 3
prop. 4
prop. 5
prop. 6
169v-164r
46v-43r
57v-64r
66v-71r
65v-72r
58v-63r
44r-45v
170v-163r
A22 157r-160v
A23 104v- —
A24 166r-167v
A25 48r-41v
A26 47r-42v
A27 110v-105r
A28 158r-159v
10
11
prop. 7
(8)
prop. 9
prop. 12
prop. 13
prop. 14
prop. 15
157v-160r
104r- —
166v-167r
48v-41r
47v-42r
110r-105v
158v-159r
A29 165v
A30 168v
prop. 15
Lost Proposition(s)
Spiral Lines
165r
168r

References

Cultrera, G., 1938. Siracusa: Rovine di un antico stabilimento idraulico in contrada Zappalà. Atti della Reale Accademia Nazionale dei Lincei. Notizie degli scavi di antichità. 261–301.

Gamba, E., Montbelli, V., Piccinetti, P., 2006. La matematica di Piero della Francesca. Lettera matematica pristem 59, 49–59.

Giusti, E., 1999. Ipotesi sulla natura degli oggetti matematici. Bollati Boringhieri, Turin. (trans. La naissance des objets mathématiques. Ellipses, Paris.)

Hayashi, E., 1994. A reconstruction of the proof of Proposition 11 in Archimedes's Method: Proofs about the volume and center of gravity of any segment of an obtuse-angled conoid. Historia Scientiarum 3, 215–230.

Hayashi E., Saito K., 2009. Tenbin no Majutsushi: Arukimedesu no Sūgaku (Sorcerer of the Scales: Archimedes' Mathematics). Kyōritsu Shuppan, Tokyo. (Reviewed by Sidoli, N., 2012, Historia Mathematica 39, 222–224.)

Heath, T.L., 1912. The Works of Archimedes Edited in Modern Notation with Introductory Chapters by T.L. Heath, with a Supplement: The Method of Archimedes Recently Discovered by Heiberg. Cambridge University Press, Cambridge.

Heiberg, J.L., 1907. Eine neue Archimedeshandschrift. Hermes 42, 236–303.

——— 1910–1915. Archimedis opera omnia cum commentariis Eutocii, 3 vols. Teubner, Leipzig. Reprinted: Stuttgart, 1972.

Heiberg, J.L., Zeuthen, H.G., 1907. Eine neue Schrift des Archimedes. Bibliotheca Mathematica III[e] Folge, 321–363.

Knobloch, E., 2000. Archimedes, Kepler, and Guldin: The role of proof and analogy. In: Thiele, R. (ed.), Mathesis: Festschrift zum siebzigsten Geburtstag von Matthias Schram. Berlin, Diephloz, pp. 82–100.

Lucore, S., 2009. Archimedes, the North Baths at Morgantina and early developments in vaulted construction. In: Kosso, C., Scott A. (eds.), The Nature and Function of Water, Baths, Bathing and Hygiene from Antiquity through the Renaissance. Brill, Leiden, pp. 43–59.

Napolitani, P.D., Saito, K., 2004. Royal road or labyrinth? Luca Valerio's De centro gravitatis solidorum and the beginnings of modern mathematics. Bollettino di Storia delle Scienze Matematiche 24, 67–124.

Netz, R., Saito, K., Tchernetska, N., 2001–2002. A new reading of "Method" Proposition 14: Preliminary evidence from the Archimedes Palimpsest (Part 1 and Part 2). SCIAMVS 2, 9–29; 3, 109–129.

Netz, R., Noel, W., Tchernetska, N., Wilson, N. (eds.), 2011. The Archimedes Palimpsest. 2 vols. Cambridge University Press, Cambridge.

Netz, R., Noel, W., 2007. The Archimedean Codex: Revealing the Secrets of the World's Greatest Palimpsest. Orion Books, London.

Reinach, T., 1907. Un traité de géométrie inédit d'Archimède. Restitution d'après un manuscrit récemment découvert. Revue générale des sciences pures et appliquées 18, 911–928, 954–961.

Rufini, E. 1926. Il "Metodo" di Archimede e le origini del calcolo infinitesimale nell'Antichità. Stock, Bologna. Reprinted: Feltrinelli, Milano, 1961.

Saito, K., 2006. Between magnitude and quantity: Another look at Archimedes' quadrature. Sugaku Expositions 19, 35–52.

Sato, T., 1986–1987. A reconstruction of "The Method 17, and the development of Archimedes' thought on quadrature — Why did Archimedes not notice the internal connection in the problems dealt with in many of his works? Historia Scientiarum 31, 61–86; 32, 75–142.

Ver Eecke, 1921. Les œuvres complètes d'Archimède, 2 vols. Desclée, De Brouwer & cie, Bruges. Reprinted: Vaillant-Carmanne, Liège. 1960.