Chapter 4
Saddle Point Problems in Hilbert Spaces

In the first chapter of this book, we introduced a large number of saddle point problems or generalisations of such problems. In most cases, the question of existence and uniqueness of solutions was left aside. In the previous chapter, we considered the solvability of *finite dimensional* problems in mixed form, together with the stability of sequences of such problems. We now introduce an abstract frame that is sufficiently general to cover all our needs, from the problems of existence and uniqueness in infinite dimension to the stability of their Finite Element discretisations.

As a first step, we shall recall some basic definitions of Functional Analysis: Hilbert spaces, continuous functionals, bilinear forms, and linear operators associated with bilinear forms.

In Sect. 4.2, we discuss conditions that ensure existence and uniqueness for mixed formulations in Hilbert spaces. Several examples of mixed formulations related to Partial Differential equations will illustrate the theoretical results. Different stability estimates will then be provided for different sets of assumptions.

The last section (Sect. 4.2.2) will be devoted to the study of perturbed problems (whose algebraic aspects were discussed in Sect. 3.6 of the previous chapter).

We shall follow essentially the analysis of [112] and [122]. We also refer the reader to other presentations, as can be found in the books [41, 106, 222, 315, 337].

4.1 Reminders on Hilbert Spaces

In this section, we recall some basic notions on Hilbert spaces. Most readers, and in particular those with a better mathematical background, will already be familiar with all the contents of the section. For them, the aim of the section will just be to fix the notation. For other people with a weaker mathematical background, it could be useful to refresh some notions. On the other hand, we do not pretend to provide a complete mastering of Hilbert spaces to people that never heard of them before.

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For these people, a superficial reading will be enough to convince them that things, in Hilbert spaces, are *very similar* to their counterparts in finite dimensional spaces.

4.1.1 Scalar Products, Norms, Completeness

We assume that the reader is familiar with the concept of *linear space over* $\mathbb{R}$. This, roughly speaking, means that you are allowed to *sum* two elements of the space, and to *multiply* each element of the space times a real number.

Let H_1 and H_2 be two linear spaces over $\mathbb{R}$. A map $a : H_1 \times H_2 \to \mathbb{R}$ is said to be **a bilinear form** on $H_1 \times H_2$ if, for every $u_1, v_1, w_1 \in H_1$, for every $u_2, v_2, w_2 \in H_2$ and for every $\lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned} a(\lambda u_1 + \mu v_1, w_2) &= \lambda a(u_1, w_2) + \mu a(v_1, w_2) \\ a(w_1, \lambda u_2 + \mu v_2) &= \lambda a(w_1, u_2) + \mu a(w_1, v_2). \end{aligned} \tag{4.1.1}$$

When both H_1 and H_2 coincide in a single linear space H, we shall often say that a is a bilinear form on H, meaning that it is a bilinear form on $H \times H$.

A bilinear form a on H is said to be **symmetric** if, for every $u, v \in H$, we have

$$a(u, v) = a(v, u). \tag{4.1.2}$$

A bilinear form s on H is said to be **a scalar product** if it is symmetric and if, moreover,

$$s(v, v) \geq 0 \quad \forall\, v \in H \qquad \text{and} \qquad s(v, v) = 0 \Rightarrow v = 0. \tag{4.1.3}$$

We assume that we have a scalar product given on $H \times H$, and from now on we shall write $(u, v)_H$ (or simply (u, v) when no confusion can occur) instead of $s(u, v)$. To a scalar product, we can always associate a **norm**

$$\|v\|_H := \Big((v, v)_H\Big)^{1/2} \qquad \forall\, v \in H. \tag{4.1.4}$$

Again, we shall simply write $\|v\|$ instead of $\|v\|_H$ when no confusion is likely to occur. It is interesting to note that the norm, as defined in (4.1.4), has the usual properties of the norms in finite dimension:

$$\begin{aligned} &\|\lambda v\| = |\lambda|\, \|v\| \qquad \forall\, v \in H,\ \forall\, \lambda \in \mathbb{R}, \\ &\|v\| \geq 0\ \forall\, v \in H \quad \text{and} \quad \|v\| = 0 \Rightarrow v = 0, \\ &\|v_1 + v_2\| \leq \|v_1\| + \|v_2\| \qquad \forall\, v_1, v_2 \in H. \end{aligned} \tag{4.1.5}$$

It is also worth noting that, even in infinite dimension, we have the Cauchy inequality

$$(u, v)_H \le \|u\|_H \|v\|_H \qquad \forall\, u, v \in H, \tag{4.1.6}$$

whose proof can be easily done mimicking the proof of Lemma 3.3.1 of the previous chapter.

It is a *strong temptation* to start defining a norm first (as a mapping from H to $\mathbb{R}$ satisfying (4.1.5)), and then getting a scalar product out of it, for instance by

$$(u, v) := \Big(\|u + v\|^2 - \|u - v\|^2\Big)/4. \tag{4.1.7}$$

Smart, isn't it? But doomed. That would work *if and only if* the norm you started with satisfies the so called *parallelogram identity*:

$$\|v + u\|^2 + \|v - u\|^2 = 2(\|u\|^2 + \|v\|^2). \tag{4.1.8}$$

A norm that satisfies (4.1.5) *and* (4.1.8) is said to be **a pre-Hilbert norm**, and induces a scalar product associated to it through (4.1.7).

A linear space H with a norm $\|\cdot\|_H$ that satisfies (4.1.5) is called **a normed space**. If, on top of that, the norm satisfies the parallelogram identity (4.1.8), then we say that H is a **pre-Hilbert space**.

As soon as we have a norm (no matter if it is a pre-Hilbert norm or not), we can talk about **convergence** and **limits**. We say that the sequence $\{v_n\}$ of elements of H converges to $v \in H$ (or that v is the limit of v_n for $n \to +\infty$) if

$$\lim_{n\to+\infty} \|v_n - v\|_H = 0. \tag{4.1.9}$$

The *limit* in (4.1.9) is obviously the one of elementary calculus (dealing with sequences of real numbers). When the type of norm to be used cannot be confused, we will also write, more simply, $v_n \to v$.

Example 4.1.1. It is immediate to see that for every integer $k \ge 1$ the space $\mathbb{R}^k$ with the usual Euclidean norm (3.1.6) used in the previous chapter is a pre-Hilbert space. Indeed, the Euclidean norm does come from a scalar product, so that

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2\mathbf{x}^T\mathbf{y}+\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\mathbf{x}^T\mathbf{y} = 2(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2).$$

On the other hand, for instance $\mathbb{R}^2$ with the norm $\|\mathbf{x}\|_1 := |x_1| + |x_2|$, already seen in (3.0.4), is *not* a pre-Hilbert space, since the norm $\|\cdot\|_1$ does not satisfy (4.1.8): try it with $u = (1, 0)$ and $v = (0, 1)$. □

Once we have a norm in H, we can measure the *distance* of two elements u and v of H by $\|u-v\|$. Given a non-empty subset $T \subseteq H$, we can measure its *diameter* by

$$diam(T) := \sup_{u,v\in T} \|u - v\|. \tag{4.1.10}$$

Loosely speaking, the diameter of T is the "maximum" distance of any two elements in T. It is obvious that for two subsets S and T, if $T \subseteq S$, then $diam(T) \leq diam(S)$ (if you increase the set of possible choices, the supremum cannot go down).

Now, for every sequence $\{v_n\}_{n\in\mathbb{N}}$ of elements of H, and for every integer $m \in \mathbb{N}$, we can consider its m-th *tail* T_m, defined as the set

$$T_m := \{v_m,\ v_{m+1},\ v_{m+2}\ldots\} \equiv \{v_n|\ n \geq m\}. \tag{4.1.11}$$

We clearly have $T_{m+1} \subseteq T_m$ for every $m \in \mathbb{N}$ (the farther you cut, the lesser is left in the tail). Hence, whatever the sequence $\{v_n\}$ from which you started, the sequence of real numbers $diam(T_m)$ that you get out of it is obviously *non increasing*: that is, $diam(T_{m+1}) \leq diam(T_m)$ for every m. Hence, the sequence $diam(T_n)$ will always have a limit, which is ≥ 0. A sequence $\{v_n\}$ of elements of a normed space H is said to be **a Cauchy sequence** in H if the sequence of real numbers $\{diam(T_m)\}$ that you get out of it verifies

$$\lim_{m\to+\infty} diam(T_m) = 0. \tag{4.1.12}$$

Note that, in order to speak about Cauchy sequences, what you need is to be able to measure the distance of two objects. This is always possible if, as in our case, you have a norm. This is also possible in more general situations, but we are not interested in them here.

A normed linear space H is said to be **complete** if for every Cauchy sequence $\{v_n\}$ in H there exists an element $v \in H$ such that $v_n \to v$ in the sense of (4.1.9). In other words, a normed linear space is complete if every Cauchy sequence has a limit. We are almost done.

Definition 4.1.1. A **Banach space** is a normed linear space that is complete.

Definition 4.1.2. A **Hilbert space** is a pre-Hilbert space that is complete.

Note that we could have defined alternatively a Hilbert space as a Banach space whose norm satisfies the parallelogram identity (4.1.8). Hence, every Hilbert space is also a Banach space, but the converse is not true: in Hilbert spaces, you have a scalar product, and in Banach spaces that are not Hilbert spaces, you do not (and can not) have one.

Example 4.1.2. It is immediate to have, from elementary Calculus, that for every integer $k \geq 1$ the space $\mathbb{R}^k$, with the usual Euclidean scalar product and norm, is a Hilbert space. In particular, $\mathbb{R}$ itself is a Hilbert space if we take the usual product of two numbers as scalar product (and hence the absolute value as norm). We also saw in the previous chapter that, for instance in $\mathbb{R}^2$, the norm $\|\mathbf{x}\|_1 := |x_1| + |x_2|$ is *equivalent* to the Euclidean norm (in the sense of (3.0.5)). On the other hand, we have already seen in Example 4.1.1 that $\mathbb{R}^2$ with the norm $\|\cdot\|_1$ is not even a pre-Hilbert space, and hence it cannot be a Hilbert space although, still by elementary Calculus, it is easily seen to be a Banach space. Actually, it is not difficult to check that the property of being complete is not lost if you exchange your norm with an equivalent norm (while the property (4.1.8) might indeed be lost). □

Example 4.1.3. Regarding the functional spaces already used in the first chapter, we can see that if Ω is a bounded open domain, then $L^1(\Omega)$ (the space of Lebesgue integrable functions over Ω), with the norm

$$\|v\|_{L^1(\Omega)} := \int_\Omega |v(x)|\, dx \tag{4.1.13}$$

is a Banach space, but not a Hilbert space. Note that, as we did already in the first chapter (and as we are going to do all over this book), we used the term *functions* in lieu of the (more precise) *classes of measurable functions*.

Instead, the space $L^2(\Omega)$ (the space of Lebesgue *square* integrable functions over Ω), with the norm

$$\|v\|^2_{L^2(\Omega)} := \int_\Omega v^2(x)\, dx \tag{4.1.14}$$

is a Hilbert space, and the corresponding scalar product is given by

$$(u, v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\, dx. \tag{4.1.15}$$

Similarly, the space $H^1_0(\Omega)$ with the scalar product

$$(u, v)_{H^1_0(\Omega)} := \int_\Omega \underline{\text{grad}}\, u(x) \cdot \underline{\text{grad}}\, v(x)\, dx \tag{4.1.16}$$

is a Hilbert space. □

In the following discussion, we shall mostly use only Hilbert spaces. Hence, from now on, we shall mainly concentrate on them, although most of the concepts and results could be extended easily to Banach spaces.

4.1.2 Closed Subspaces and Dense Subspaces

Definition 4.1.3. A subset T of a Hilbert space H is said to be **closed** if, for every Cauchy sequence $\{v_n\}_{n\in\mathbb{N}}$ of elements of T, the limit v (which surely exists in H, since H is complete) belongs to T as well.

If T is a *linear subspace* of a Hilbert space H, and if T is closed, then we will say that T is **a closed subspace** of H. Then T itself will be a Hilbert space, with the same norm as H.

Example 4.1.4. For instance, in $L^2(\Omega)$, we can consider the subspace $L^2_0(\Omega)$ made of functions that have zero mean value in Ω. It is easy to see that it is a closed subspace (since the L^2-limit of functions with zero mean value has itself zero mean value). On the other hand, $C^0(\overline{\Omega})$ is a linear subspace of $L^2(\Omega)$, but it is not closed: for instance, for $\Omega =]-1, 1[$, the sequence $f_n(x) := \arctan(nx)$ converges in $L^2(\Omega)$ to $f_\infty(x) := (\pi/2)\, sign\,(x)$, which does not belong to $C^0(\overline{\Omega})$. □

Definition 4.1.4. Let H be a Hilbert space, and let Z be a subset of H. The **closure** of Z, that we denote by $\overline{Z}$, is the set of elements $v \in H$ such that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of elements of Z that converges to v.

We abviously have that Z is closed if and only if $\overline{Z} = Z$.

Another important concept regarding subspaces is that of a *dense subspace*.

Definition 4.1.5. A subset Z of a Hilbert space H is said to be **dense** if its closure $\overline{Z}$ coincides with the whole space H. If Z is also a linear subspace of H, then we say that it is **a dense subspace**.

In other words, Z is dense in H if for every element v of H there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of elements of Z such that

$$\lim_{n\to+\infty} \|v - z_n\|_H = 0.$$

Example 4.1.5. It is not difficult to see that $Z := H^1_0(\Omega)$ is a dense subspace of $H = L^2(\Omega)$. It is also clear that Z is not a closed subspace of H: for instance, for $\Omega =]-1, 1[$, the sequence of functions defined by

$$z_n(x) := \min(1, n - n|x|) \equiv \begin{cases} 1 & \text{when } |x| \le 1 - 1/n \\ n(1-|x|) & \text{when } |x| > 1 - 1/n \end{cases}$$

verifies $z_n \in H^1_0(\Omega)$ for all n, and its limit in L^2 equals the constant 1, which is not in $H^1_0(\Omega)$ (as it does not vanish at the boundary). □

Note that a dense closed subspace of a Hilbert space H coincides necessarily with the whole space H. Hence, in general, we consider subspaces that are closed, but not dense, and subspaces that are dense, but not closed. These two categories of subspaces are both very important, and we cannot restrict our attention to just one of them. We point out however, from the very beginning, that closed subspaces are the ones that, loosely speaking, inherit most of the properties of subspaces of finite dimensional spaces. In particular, a finite dimensional subspace is *always closed* and is *never dense* (unless it coincides with the whole space).

4.1.3 Orthogonality

Some very useful instruments available in Hilbert spaces (and not in Banach spaces) are related to the concept of **orthogonality**. We say that two elements u and v of a Hilbert space H are orthogonal if $(u, v)_H = 0$. It is the same as in the finite dimensional case, with the only difference that there, the scalar product was denoted by $\mathbf{v}^T\mathbf{u}$. If Z is a linear subspace of a Hilbert space H, we can define its **orthogonal complement** $Z^\perp$

$$Z^\perp := \{w \in H \text{ such that } (w,z) = 0 \ \forall\, z \in Z\}. \tag{4.1.17}$$

It is not difficult to see that an orthogonal complement $Z^\perp$ is always closed (even when Z itself is not closed). As in (3.1.24), if Z_1 and Z_2 are both subspaces of a Hilbert space H, then

$$Z_1 \subseteq Z_2 \quad \Rightarrow \quad Z_2^\perp \subseteq Z_1^\perp. \tag{4.1.18}$$

Moreover, we have the following useful property.

Proposition 4.1.1. *Let H be a Hilbert space, let Z be a subspace of it, and let $\overline{Z}$ be its closure. Then,*

$$\overline{Z}^\perp = Z^\perp. \tag{4.1.19}$$

Proof. Since $Z \subseteq \overline{Z}$, we obviously have $\overline{Z}^\perp \subseteq Z^\perp$. On the other hand, let $w \in Z^\perp$. We want to see that $(\bar{z}, w)_H = 0$ for all $\bar{z} \in \overline{Z}$. Indeed, for every $\bar{z} \in \overline{Z}$, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of elements of Z that converges to $\bar{z}$. As $w \in Z^\perp$, we have $(z_n, w)_H = 0$ for all n. Hence,

$$(\bar{z}, w)_H = \lim_{n\to+\infty} (z_n, w) = 0. \tag{4.1.20}$$

□

Remark 4.1.1. Note that, as we had in Remark 3.1.3, the notion of orthogonal space depends heavily on the choice of the "whole space" H. Indeed, if H_1 and H_2 are Hilbert spaces, and Z is a subspace of H_1 and also a subspace of H_2, then the orthogonal of Z in H_1 will, in general, be different from the orthogonal of Z in H_2. This is rather obvious. However, the common notation (that we are using here) does not distinguish among the two (we should, for this, use something like $Z^{\perp_{H_1}}$ and $Z^{\perp_{H_2}}$, which would be tremendously ugly). As a consequence, one should be careful when confusion is possible. □

As we did in the finite dimensional case, if Z is a closed subspace, we can define the **projection operator** $\pi_Z : H \to Z$ defined for every $v \in H$ by

$$\pi_Z v \in Z \qquad \textit{and} \qquad (\pi_Z v - v) \in Z^\perp. \tag{4.1.21}$$

Compare with (3.1.31) to see that we are just extending the definitions given in the previous chapter for the finite dimensional case. As we had in the finite dimensional case, $\pi_Z v$ can be seen as the element in Z that minimises the distance from v, namely

$$\|\pi_Z v - v\|_H = \min_{z\in Z} \|z - v\|_H. \tag{4.1.22}$$

Remark 4.1.2. In the definition of π_Z, we assumed that Z was *a closed subspace* of H. Of the two properties (of being closed, and of being a subspace), the second is not very important. Indeed, it is easy to see that (4.1.22) can be used to define the projection mapping π_Z in more general cases, for instance when Z is not a subspace but simply a closed convex subset, as for instance a closed affine manifold (which, roughly speaking, is the translation of a closed subspace). On the other hand, closedness is more essential. To see what can happen when you remove it, assume that Z is a dense subspace. Then, for v in H but not in Z, the projection $\pi_Z v$ cannot be defined: indeed, we recall that, from Proposition 4.1.1, if Z is dense (and hence $\overline{Z} = H$), then $Z^{\perp} = H^{\perp} = \{0\}$. Hence, looking at the definition (4.1.21), if Z is dense the only $w \in H$ such that $(w - v) \in Z^{\perp}$ is $w = v$. However, such a w is not in Z, so that there is no element that we could choose as $\pi_Z v$ that satisfies *both* properties required in (4.1.21). Hence, $\pi_Z v$ does not exist. Note that the alternative definition (4.1.22) would not be of any help either. Actually, always for Z dense and $v \in H$ with $v \notin Z$, the *minimum* of $\|z - v\|$ for $z \in Z$ does not exist, and the *infimum* is equal to zero. □

It is easy to check that if H is a Hilbert space, and if Z is a closed linear subspace, then every element v of H can be split in a unique way into its two components in Z and in $Z^{\perp}$:

$$v = v_Z + v_{Z^{\perp}}, \tag{4.1.23}$$

just by setting $v_Z := \pi_Z v$.

Example 4.1.6. For instance, if $H := L^2(\Omega)$ and $Z := L^2_0(\Omega)$, then $Z^{\perp}$ is the (one-dimensional) space made of constant functions. The projection of $v \in L^2(\Omega)$ onto Z is given by

$$\pi_Z = v - \frac{1}{|\Omega|} \int_{\Omega} v(x)\, dx, \tag{4.1.24}$$

where $|\Omega|$ is the *Lebesgue measure* of Ω. □

We have, moreover, the following property.

Proposition 4.1.2. *Let H be a Hilbert space and Z a closed subspace of it. Then, either $Z \equiv H$ or $Z^{\perp}$ is* **not** *reduced to* $\{0\}$.

Proof. If Z does not coincide with H, then there exists a $v \in H$ such that $v \notin Z$. Hence, $\pi_Z v - v \neq 0$. As (4.1.21) also gives $\pi v - v \in Z^{\perp}$, the proof is concluded. □

We can now see the equivalent of (3.1.23) in general Hilbert spaces.

Proposition 4.1.3. *Let H be a Hilbert space, and $Z \subseteq H$ a subspace. Then,*

$$\left(Z^{\perp}\right)^{\perp} = Z \quad \textbf{iff} \quad Z \textit{ is closed.} \tag{4.1.25}$$

Proof. Indeed, if $Z \equiv (Z^{\perp})^{\perp}$, then Z, being the orthogonal of something, is closed. To see the converse, we remark first that we always have the inclusion

$Z \subseteq (Z^{\perp})^{\perp}$. If Z is closed, suppose, by contradiction, that Z does not coincide with $(Z^{\perp})^{\perp}$. From Proposition 4.1.2 (applied with $H = (Z^{\perp})^{\perp}$), we should have a $v \in (Z^{\perp})^{\perp}$, with $v \neq 0$, that is orthogonal to all z in Z. As such, v will hence be also in $Z^{\perp}$. However, $Z^{\perp} \cap (Z^{\perp})^{\perp} = \{0\}$ and this contradicts the fact that $v \neq 0$. □

Moreover, we have the following additional property.

Proposition 4.1.4. *Let H be a Hilbert space, and let Z be a subspace of H. Then,*

$$Z^{\perp} = \{0\} \quad \textbf{iff} \quad Z \textit{ is dense.} \tag{4.1.26}$$

Proof. Assume first that Z is dense. Then, $\overline{Z} \equiv H$ and hence $\overline{Z}^{\perp} = \{0\}$, and the result follows from Proposition 4.1.1. Assume conversely that $Z^{\perp} = \{0\}$. Always from Proposition 4.1.1, we have now $\overline{Z}^{\perp} = \{0\}$. However, $\overline{Z}$ is closed, and hence, by Proposition 4.1.2 $\overline{Z} = H$, and Z is therefore dense. □

4.1.4 Continuous Linear Operators, Dual spaces, Polar Spaces

We can now recall several other important definitions.

Definition 4.1.6. Let V and W be Hilbert spaces, and let M be a linear mapping from V to W. We say that M is **bounded** or that it is **continuous** if there exists a constant μ^* such that

$$\|Mv\|_W \leq \mu^* \|v\|_V \qquad \forall\, v \in V. \tag{4.1.27}$$

Note that we have two different names for that (*bounded* and *continuous*) because the two definitions do not coincide if the operator is not *linear*. Actually, for a more general operator, (4.1.27) defines a bounded operator, while continuity can be taken as in the usual Calculus books: for every $v \in V$ and for every sequence v_n converging to v, we have that $M v_n$ converges to $M v$. Here, however, we only deal with linear operators, and the two concepts coincide.

Example 4.1.7. For instance, the operator $v \to \dfrac{\partial v}{\partial x_1}$ is continuous from $H^1_0(\Omega)$ to $L^2(\Omega)$. Similarly, if ϕ is a given (fixed) bounded function, then the mapping $v \to \phi\, v$ is continuous from $L^2(\Omega)$ into itself. □

The following definition is less common but very useful.

Definition 4.1.7. Let V and W be Hilbert spaces and let M be a linear mapping from V to W. We say that M is **bounding** if there exists a constant μ_* such that

$$\|Mv\|_W \geq \mu_* \|v\|_V \qquad \forall\, v \in V. \tag{4.1.28}$$

In other words, *bounding operators* are injective operators whose inverse is continuous.

The set of all linear continuous operators from a Hilbert space V into another Hilbert space W is also a linear space (after defining, in an obvious way, the *sum* of two operators or the *multiplication* of an operator times a real number). Such a space is usually denoted by $\mathcal{L}(V, W)$. In $\mathcal{L}(V, W)$, we can also introduce a norm:

$$\|M\|_{\mathcal{L}(V,W)} := \sup_{v \in V} \frac{\|Mv\|_W}{\|v\|_V}. \tag{4.1.29}$$

When no confusion can occur, the norm in (4.1.29) is simply denoted by $\|M\|$. Hence, for instance, (4.1.29) implies

$$\|Mv\|_W \leq \|M\| \, \|v\|_V \qquad \forall\, v \in V. \tag{4.1.30}$$

One can prove that (4.1.29) actually defines a norm, and that such a norm verifies (4.1.8), so that with this norm $\mathcal{L}(V, W)$ is itself a Hilbert space.

A remarkable result concerning linear continuous and *one-to-one* operators is the following one, due to Banach.

Theorem 4.1.1 (Banach Theorem). *Let V and W be Hilbert spaces and let $M \in \mathcal{L}(V, W)$ be a one to one mapping. Then, its inverse operator M^{-1}, from W to V, is also continuous.*

Proof. The proof can be found in any book of Functional Analysis. □

As we did for finite dimensional spaces, given a subspace $Z \subseteq V$, we can consider the **extension operator** $E_{Z \to V}$, from Z to V which to every $z \in Z$ associates the same z, thought as an element of V. If there is no risk of confusion, this will, more simply, be denoted by E_Z as we did in the previous chapter. Always in agreement with the finite dimensional case, given an operator $M \in \mathcal{L}(V, W)$, we can consider the **restriction** M_Z of M to Z, that could be defined as

$$M_Z\, z = M\, E_Z\, z \qquad \forall z \in Z. \tag{4.1.31}$$

Since for every $z \in Z \subseteq V$ we have obviously $M_Z\, z = M\, z$, in several occasions, the extension operator E_Z will not be explicitly written. In other cases, however, such notation will turn out to be very useful.

If we assume that Z is a closed subspace of V, that S is a closed subspace of W and $M \in \mathcal{L}(V, W)$, we can also consider its *restriction* M_{ZS}, defined as

$$M_{ZS}\, z = \pi_S\, M\, E_Z\, z \qquad \forall z \in Z. \tag{4.1.32}$$

It is easy to check that $M_{ZS} \in \mathcal{L}(Z, S)$. Conversely, given an operator L in $\mathcal{L}(Z, S)$, we can always consider its *extension* $\tilde{L} \in \mathcal{L}(V, W)$ defined by

$$\tilde{L}\, v = E_S\, L\, \pi_Z\, v \qquad \forall v \in V. \tag{4.1.33}$$

A particular case of linear operators, *of paramount importance*, is found when the *arrival space* is $\mathbb{R}$. In this case, linear operators $V \to \mathbb{R}$ are called **linear functionals** on V. The space of all linear *continuous* functionals on a Hilbert space V is called **the dual space** of V, and is usually denoted by V'. Hence, $V' \equiv \mathcal{L}(V, \mathbb{R})$. As a particular case of the previous situation, V' is itself a Hilbert space, and its norm (often called **the dual norm** of $\|\cdot\|_V$) is given by

$$\|f\|_{V'} := \sup_{v\in V} \frac{|f(v)|}{\|v\|_V}. \tag{4.1.34}$$

We easily recognise the definition of dual norms that were given in finite dimension. The value of f at v (denoted by $f(v)$ in (4.1.34)) is often denoted in a different way: either by ${}_{V'}\langle f, v\rangle_V$ or by $\langle f, v\rangle_{V'\times V}$, or simply $\langle f, v\rangle$ when no confusion can occur. It is not too difficult to check (although we shall not do it here) that, if V is a Hilbert space, then *the dual space of* V' (often called the *bi-dual* space), actually can be identified with V itself (see the Ritz representation Theorem (4.1.37) here below).

Example 4.1.8. For instance, in one dimension, it is easy to see that the mapping $\delta_0 : v \to v(0) \in \mathbb{R}$ is continuous from $H^1_0(]-1,1[)$ to $\mathbb{R}$: indeed,

$$v(0) = \int_{-1}^{0} v'(t)\, dt \le \Big(\int_{-1}^{0} 1^2\, dt\Big)^{1/2}\Big(\int_{-1}^{0} (v'(t))^2\, dt\Big)^{1/2}$$
$$\le 1\Big(\int_{-1}^{1} \Big(v'(t))^2\, dt\Big)^{1/2} = \|v\|_{H^1_0(]-1,1[)}. \tag{4.1.35}$$

Hence, δ_0 is an element of the dual space of $H^1_0(]-1,1[)$ (usually denoted by $H^{-1}(]-1,1[)$). Note that a similar result does not hold in dimension $d > 1$. Indeed, if Ω is the disk centred at the origin O and radius $1/\sqrt{e}$, a simple explicit computation shows that the function

$$v(x,y) := \log|\log(x^2+y^2)|$$

is indeed in $H^1_0(\Omega)$. Setting

$$v_n(x,y) := \min\{n, v(x,y)\},$$

it is not difficult to see that v_n converges to v in $H^1_0(\Omega)$. However, $v_n(0,0) = n$ so that the bound

$$v_n(0,0) \le C\,\|v_n\|_{H^1_0(\Omega)}$$

cannot be true with a constant C (no matter how big) independent of n, as the left-hand side tends to $+\infty$ and the right-hand side stays finite. Similarly, the estimate (4.1.35) becomes false if we try to replace, in the right-hand side, the H^1 norm with the L^2 norm: consider, for instance, the sequence of functions defined by

$$v_n(x) = \begin{cases} 1 \text{ for } x \text{ in } [-1/n, 1/n] \\ 0 \text{ for } x \text{ outside } [-1/n, 1/n]. \end{cases}$$

We have

$$\|v_n\|^2_{L^2]-1,1[} = \int_{-1}^{1} v_n^2(x)\,dx = \int_{-1/n}^{1/n} 1\,dx = \frac{2}{n} \to 0$$

while $v_n(0) = 1$ for all n. Hence, there is no constant C (independent of n) such that

$$v_n(0) \le C\,\|v_n\|_{L^2(]-1,1[)}. \qquad \square$$

Example 4.1.9. Let us also see an example of dual norm: let n be an integer (larger than 1) and consider in $\Omega =]0,\pi[$ the function $f_n(x) := \sin(nx)$. It is immediate to check that

$$\|f_n\|_{L^2(\Omega)} = \sqrt{\pi/2} \quad \text{and that} \quad \|f_n\|_{H^1_0(\Omega)} = n\sqrt{\pi/2}.$$

To f_n we can associate an element, that we still call f_n, of $H^{-1}(\Omega)$ (that is the dual space of $H^1_0(\Omega)$) as follows

$$\langle f_n, \varphi\rangle_{H^{-1}\times H^1_0} := \int_0^{\pi} \varphi(x) f_n(x)\,dx \quad \forall \varphi \in H^1_0(\Omega).$$

Let us compute the norm of f_n in $H^{-1}(\Omega)$. For every $\varphi \in H^1_0(\Omega)$ we have (integrating by parts):

$$\begin{aligned}\langle f_n, \varphi\rangle &= \int_0^{\pi} f_n(x)\varphi(x)\,dx = \int_0^{\pi} \sin(nx)\varphi(x)\,dx \\ &= \frac{1}{n}\int_0^{\pi} \cos(nx)\varphi'(x)\,dx \le \frac{1}{n}\|\cos(nx)\|_{L^2}\,\|\varphi\|_{H^1_0} = \frac{\sqrt{\pi/2}}{n}\|\varphi\|_{H^1_0},\end{aligned}$$

giving us, always for every $\varphi \in H^1_0(\Omega)$:

$$\frac{\langle f_n, \varphi\rangle}{\|\varphi\|_{H^1_0}} \le \frac{\sqrt{\pi/2}}{n}. \qquad (4.1.36)$$

On the other hand, it is not difficult to see that, taking $\varphi \equiv \sin(nx)$, we get

$$\frac{\langle f_n, \varphi \rangle}{\|\varphi\|_{H_0^1}} = \frac{\pi/2}{n\sqrt{\pi/2}} = \frac{\sqrt{\pi/2}}{n},$$

showing that $(1/n)\sqrt{\pi/2}$ actually realises the *supremum* (over all possible φ's) of the left-hand side of (4.1.36). In conclusion, we have

$$\|f_n\|_{L^2(\Omega)} = \sqrt{\pi/2} \qquad \|f_n\|_{H_0^1(\Omega)} = n\sqrt{\pi/2} \qquad \|f_n\|_{H^{-1}(\Omega)} = \frac{\sqrt{\pi/2}}{n}.$$

This shows that the three norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{H_0^1}$, and $\|\cdot\|_{H^{-1}}$ cannot be equivalent. This also shows that a *high frequency* function can have, at the same time, an L^2-norm (and a maximum norm) of the order of 1, a huge H^1-norm ($\simeq$ energy norm), and a tiny H^{-1}-norm. This will show up in the next chapter, when the use of finer and finer grids will allow the presence of highly oscillating piecewise linear (or piecewise polynomial) functions. □

While several properties that we saw and that we will see hold in a much more general setting (for instance, in all Banach spaces), the following theorem is, in a certain sense, characteristic of Hilbert spaces.

Theorem 4.1.2 (Ritz's Theorem). *Let H be a Hilbert space, and let R_H be the operator $H \to H'$ that to each $z \in H$ associates the functional $f_z = R_H z \in H'$ defined as*

$$\langle f_z, v \rangle_{H' \times H} = (z, v)_H \quad \forall v \in H. \tag{4.1.37}$$

Then, R_H is one to one, and $\|R_H\|_{\mathcal{L}(H,H')} = \|R_H^{-1}\|_{\mathcal{L}(H',H)} = 1$. Moreover, if we identify (as it is natural) H with $(H')'$, then $R_H^{-1} = R_{H'}$.

Proof. The proof can be found in every Functional Analysis textbook. □

Another result that we are going to use later on is the following theorem, that can be seen as a particular case of a more general result, known as the *Kato Theorem*.

Theorem 4.1.3 (Kato Theorem). *Let V and W be Hilbert spaces and let T_1 and T_2 be in $\mathcal{L}(V, W)$. If T_1 is bounding, then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \le \varepsilon_0$ the perturbed operator $T_1 + \varepsilon T_2$ is also bounding, and we have moreover*

$$\|T_1^{-1} - (T_1 + \varepsilon T_2)^{-1}\|_{\mathcal{L}(W,V)} \le C\,|\varepsilon| \tag{4.1.38}$$

with C depending on ε_0 but independent of ε.

If Z is a subspace of a Hilbert space H, we can spot a special subset of H', usually called the **polar space** of Z, made of all functionals $f \in H'$ that vanish identically on Z. The polar space of Z is usually denoted by Z^0: hence, we have

$$Z^0 := \{f \in H' \text{ such that } \langle f, z \rangle_{H' \times H} = 0 \;\; \forall z \in Z\}. \tag{4.1.39}$$

It is clear that the definition of polar space of Z makes sense only when Z is considered as a subspace of another space (in this case, H). In particular, the polar space of $Z = \{0\}$ coincides with the whole H' while the polar space of $Z = H$ is reduced to the zero functional.

Remark 4.1.3. It is easy to check that a polar space is always closed. Indeed, roughly speaking, if $\langle f_n, z\rangle = 0$ for every n and for every z, and if $f_n \to f$ in H', then $\langle f, z\rangle = 0$ for all z. □

The concept of polar space is commonly used for general Banach spaces. In Hilbert spaces, however, it becomes particularly simple using the Ritz Theorem. Indeed, from (4.1.39), we immediately have

$$Z^0 \equiv R_H(Z^\perp). \tag{4.1.40}$$

From this and Proposition 4.1.3, we then have

$$\left(Z^0\right)^0 = Z \quad \textbf{iff} \quad Z \text{ is closed,} \tag{4.1.41}$$

and from (4.1.26),

$$Z^0 = \{0\} \quad \textbf{iff} \quad Z \text{ is dense.} \tag{4.1.42}$$

Remark 4.1.4. Property (4.1.42) is a particular case (or, if you want, the restriction to Hilbert spaces) of a fundamental theorem of Functional Analysis, known as the **Hahn-Banach Theorem**. □

Remark 4.1.5. As we had in Remark 4.1.1 for orthogonal spaces, if Z can be seen as a subspace of two different spaces H_1 and H_2, then the polar of Z in H_1' will be different from the polar of Z in H_2'. □

Similarly to (4.1.18), when Z_1 and Z_2 are subspaces of the same space H, then

$$Z_1 \subseteq Z_2 \quad \Rightarrow \quad Z_2^0 \subseteq Z_1^0. \tag{4.1.43}$$

4.1.5 Bilinear Forms and Associated Operators; Transposed Operators

Another important particular case is that of bilinear forms. Assume that V and Q are Hilbert spaces: we say that a bilinear form b from $V \times Q$ to $\mathbb{R}$ is **continuous** if there exists a constant μ_b such that

$$b(v, q) \le \mu_b \|v\|_V \, \|q\|_Q \qquad \forall\, v \in V,\ \forall\, q \in Q. \tag{4.1.44}$$

The **norm of the continuous bilinear form** $\|b\|_{\mathcal{L}(V\times Q,\mathbb{R})}$ is then defined as

$$\|b\|_{\mathcal{L}(V\times Q,\mathbb{R})} := \sup_{\substack{v\in V\\ q\in Q}} \frac{b(v,q)}{\|v\|_V\,\|q\|_Q}, \tag{4.1.45}$$

and it will be denoted simply by $\|b\|$ when no confusion can occur. Hence, from (4.1.44) and (4.1.45), we have

$$b(v,q) \le \|b\|\,\|v\|_V\,\|q\|_Q \qquad \forall\, v \in V,\ \forall\, q \in Q. \tag{4.1.46}$$

It is important to note that continuous bilinear forms on $V\times Q$ are strictly connected to linear continuous operators from V to Q': indeed, if b is a bilinear form on $V\times Q$, we can associate to it a linear operator B from V to Q', defined as

$$\langle Bv, q\rangle_{Q'\times Q} := b(v,q) \qquad \forall\, v \in V,\ \forall\, q \in Q. \tag{4.1.47}$$

Conversely, if B is a linear operator from V to Q', we can associate to it the bilinear form

$$b(v,q) := \langle Bv, q\rangle_{Q'\times Q} \qquad \forall\, v \in V,\ \forall\, q \in Q. \tag{4.1.48}$$

It is elementary to check that B is continuous (from V to Q') *if and only if* the associated bilinear form b is continuous from $V\times Q$ to $\mathbb{R}$. To $B : V \to Q'$ we can also associate another operator, that we call **transposed operator** $B^t : Q \to V'$, given by

$$\langle v, B^t q\rangle_{V\times V'} := \langle Bv, q\rangle_{Q'\times Q} = b(v,q). \tag{4.1.49}$$

Example 4.1.10. It is easy to see that if $V := \mathbb{R}^n$ and $Q := \mathbb{R}^m$, then the linear operators from V to $Q' \simeq Q$ are just $(m\times n)$ *matrices*. In particular, the *transposed operator* will simply be the *transposed matrix*. □

It is worth noting that the continuity of the three objects b, B, and B^t is just the same property. In particular we have

$$\|B\|_{\mathcal{L}(V,Q')} \equiv \|B^t\|_{\mathcal{L}(Q,V')} \equiv \|b\|_{\mathcal{B}(V\times Q,\mathbb{R})} \equiv \sup_{\substack{v\in V\\ q\in Q}} \frac{b(v,q)}{\|v\|_V\,\|q\|_Q}. \tag{4.1.50}$$

For a linear operator M from a Hilbert space V to another Hilbert space W, we can define the **kernel** and the **image** (or **range**) as we did in (3.1.7) for the finite dimensional case:

$$\begin{aligned} \mathrm{Ker}M &:= \{v \in V \text{ such that } Mv = 0\},\\ \mathrm{Im}M &:= \{w \in W \text{ such that } \exists\, v \in V \text{ with } Mv = w\}. \end{aligned} \tag{4.1.51}$$

Remark 4.1.6. Note that the *kernel* of a continuous operator M is always closed. Indeed, if $Mv_n = 0$ and $v_n \to v$ in V, the continuity of M will imply that $Mv = 0$. This is not true for the *image*. Referring to the case of Example 4.1.5, take $V = H^1_0(\Omega)$ and $W = L^2(\Omega)$, with $Mv = v$ for every $v \in V$. Clearly, M is continuous, but $\mathrm{Im}M = V$ is not a closed subspace of W. This fact (that the image might not be closed) *puts pains thousandfold upon the* mathematicians (whether *Achaians* or not). However, as you will see, one can survive. □

We concentrate now our attention on the case of linear operators B from V to $W = Q'$, with their associated bilinear form b and transposed operator B^t, as in (4.1.49). In this case, we can see that $\mathrm{Ker}B$ and $\mathrm{Ker}B^t$ can be written, respectively, as

$$\begin{aligned}\mathrm{Ker}B :&= \{v \in V \text{ such that } b(v,q) = 0\ \forall\, q \in Q\}\\ &= \{v \in V \text{ such that } \langle v, B^t q\rangle_{V\times V'} = 0\ \forall\, q \in Q\}\end{aligned} \tag{4.1.52}$$

and

$$\begin{aligned}\mathrm{Ker}B^t :&= \{q \in Q \text{ such that } b(v,q) = 0\ \forall\, v \in V\}\\ &= \{q \in Q \text{ such that } \langle Bv, q\rangle_{Q'\times Q} = 0\ \forall\, v \in V\}.\end{aligned} \tag{4.1.53}$$

In finite dimensional problems (see Proposition 3.1.2), we did interpret (4.1.52) and (4.1.53) as

$$\mathrm{Ker}B = (\mathrm{Im}B^T)^{\perp} \text{ and } \mathrm{Ker}B^T = (\mathrm{Im}B)^{\perp} \tag{4.1.54}$$

respectively. This, however, cannot be done in the present infinite dimensional case, because, for instance, $\mathrm{Im}B$ is not a subset of Q but a subset of Q' (the two spaces were *identified* in finite dimension without telling you anything; sorry for that!). We have, however introduced a special definition for that: the polar space (see (4.1.39)). Hence, we can interpret (4.1.52) and (4.1.53) as

$$\mathrm{Ker}B = (\mathrm{Im}B^t)^0 \quad \text{and} \quad \mathrm{Ker}B^t = (\mathrm{Im}B)^0 \tag{4.1.55}$$

respectively. In finite dimension, in Theorem 3.1.1, we also had $\mathrm{Im}B^T = (\mathrm{Ker}B)^{\perp}$ and $\mathrm{Im}B = (\mathrm{Ker}B^T)^{\perp}$. Here we might hope to have

$$(\mathrm{Ker}B)^0 = \mathrm{Im}B^t \quad \text{and} \quad (\mathrm{Ker}B^t)^0 = \mathrm{Im}B. \tag{4.1.56}$$

Actually, for instance, the equality

$$(\mathrm{Ker}B)^0 = \mathrm{Im}B^t \tag{4.1.57}$$

will follow easily from the second of (4.1.55) using (4.1.41) if we only knew that $\mathrm{Im}B$ is closed. However, unfortunately, this is not always the case. On the other hand, if $\mathrm{Im}B$ is not closed, then (4.1.57) is hopeless, as a polar space is always closed. Indeed, we can see that we have the following generalisation of Corollary 3.1.1 and Theorem 3.1.1 to the infinite dimensional case.

Theorem 4.1.4. *Let V and Q be Hilbert spaces, and B a linear continuous operator from V to Q' (that is: $B \in \mathcal{L}(V, Q')$). Then, the following three properties are equivalent:*

$$\mathrm{Im}B \text{ is closed in } Q' \tag{4.1.58}$$

$$\mathrm{Im}B = (\mathrm{Ker}B^t)^0 \tag{4.1.59}$$

$\exists\, L_B \in \mathcal{L}(\mathrm{Im}B, (\mathrm{Ker}B)^{\perp})$ *and* $\beta > 0$ *such that:*

$$B\,L_B\,g = g \quad \forall\, g \in \mathrm{Im}B \quad \textit{and} \quad \beta \|L_B\,g\|_V \le \|g\|_{Q'} \quad \forall\, g \in \mathrm{Im}B. \tag{4.1.60}$$

Proof. We already discussed the equivalence of (4.1.58) and (4.1.59). Moreover, if (4.1.58) holds, then B (or actually its restriction to $(\mathrm{Ker}B)^{\perp}$) becomes (with the same argument used in Proposition 3.1.1) a continuous one-to-one operator between the two Hilbert spaces $(\mathrm{Ker}B)^{\perp}$ and $\mathrm{Im}B$, and Theorem 4.1.1 gives us (4.1.60). Finally, (4.1.60) easily implies (4.1.58): if $g_n = B\,v_n$ is a Cauchy sequence in Q' then, using (4.1.60), we have that v_n (equal to $L\,g_n$) is a Cauchy sequence in V. Then, it converges to a $v \in V$, and the continuity of B implies that g_n converges to $B\,v$ in Q'. Hence, the limit of g_n is in $\mathrm{Im}B$. □

Exchanging B and B^t, we immediately have the equivalence of the three properties

$$\mathrm{Im}B^t \text{ is closed in } V' \tag{4.1.61}$$

$$\mathrm{Im}B^t = (\mathrm{Ker}B)^0 \tag{4.1.62}$$

$\exists\, L_{B^t} \in \mathcal{L}(\mathrm{Im}B^t, (\mathrm{Ker}B^t)^{\perp})$ and $\beta > 0$ such that:

$$B^t\,L_{B^t}\,f = f \quad \forall\, f \in \mathrm{Im}B^t \quad \text{and} \quad \beta\|L_{B^t}\,f\|_Q \le \|f\|_{V'}\ \forall\, f \in \mathrm{Im}B^t. \tag{4.1.63}$$

What is somehow remarkable is that, actually, the two triplets of properties (4.1.58)–(4.1.60) and (4.1.61)–(4.1.63) are equivalent to each other. This actually follows easily from the following proposition.

Proposition 4.1.5. *Let V and Q be Hilbert spaces, and B a linear continuous operator from V to Q' (that is: $B \in \mathcal{L}(V, Q')$). Then, ImB is closed* ***iff*** *ImB^t is closed.*

Proof. In view of the above equivalences, we only need to prove that (4.1.58)–(4.1.60) imply (4.1.61). For this, consider $q \in (\mathrm{Ker}B^t)^{\perp}$ and set $g = R_Q q$ where R_Q is the Ritz operator $Q \to Q'$. Using (4.1.40) we have $g \in (\,\mathrm{Ker}B^t)^0$. Hence, using (4.1.59), we have $g \in \mathrm{Im}B$ so that $g = Bx$ for $x = L_{Bg}$ and from (4.1.60): $\beta\|x\|_V \le \|g\|_{Q'} = \|q\|_Q$. Then, we have

$$\begin{aligned}\|q\|_Q^2 = {}_{Q'}\langle R_Q q, q\rangle_Q = {}_{Q'}\langle g, q\rangle_Q = {}_{Q'}\langle Bx, q\rangle_Q \\ = {}_V\langle x, B^t q\rangle_{V'} \le \|x\|_V\,\|B^t q\|_{V'} \le \frac{1}{\beta}\|q\|_Q\,\|B^t q\|_{V'}\end{aligned} \tag{4.1.64}$$

which easily gives

$$\beta\|q\|_Q \le \|B^t q\|_{V'} \qquad \forall q \in (\mathrm{Ker}B^t)^\perp \tag{4.1.65}$$

which, in turn, proves that $\mathrm{Im}B^t$ is closed by the same argument used in the proof of Theorem 4.1.4 □

We can summarise the above results in the following theorem, that is a particular case of a more general (and important) theorem, also due to Banach, and mostly known as the *Closed Range Theorem*.

Theorem 4.1.5 (Banach Closed Range Theorem). *Let V and Q be Hilbert spaces and let B be a linear continuous operator from V to Q'. Set*

$$K := \mathrm{Ker}B \subset V \quad \textit{and} \quad H := \mathrm{Ker}B^t \subset Q. \tag{4.1.66}$$

Then, the following statements are equivalent:

- $\mathrm{Im}B$ *is closed in* Q',
- $\mathrm{Im}B^t$ *is closed in* V',
- $K^0 = \mathrm{Im}B^t$,
- $H^0 = \mathrm{Im}B$,
- $\exists\, L_B \in \mathcal{L}(\mathrm{Im}B, K^\perp)$ *and* $\exists\,\beta > 0$ *such that* $B(L_B(g)) = g \quad \forall\, g \in \mathrm{Im}B$ *and moreover* $\beta\|L_B g\|_V \le \|g\|_{Q'} \quad \forall\, g \in \mathrm{Im}B$,
- $\exists\, L_{B^t} \in \mathcal{L}(\mathrm{Im}B^t, H^\perp)$ *and* $\exists\,\beta > 0$ *such that* $B^t(L_{B^t}(f)) = f \quad \forall\, f \in \mathrm{Im}B^t$ *and moreover* $\beta\|L_{B^t} f\|_Q \le \|f\|_{V'} \quad \forall\, f \in \mathrm{Im}B^t$. □

In the following treatment, we shall often assume that B is surjective. Let us see what the Closed Range Theorem has to say in this case.

Corollary 4.1.1. *Let V and Q be Hilbert spaces, and let B be a linear continuous operator from V to Q'. Then, the following statements are equivalent:*

- $\mathrm{Im}B = Q'$,
- $\mathrm{Im}B^t$ *is closed and* B^t *is injective,*
- B^t *is bounding:* $\exists\,\beta > 0$ *s.t.* $\|B^t q\|_{V'} \ge \beta\|q\|_Q \quad \forall\, q \in Q$,
- $\exists\, L_B \in \mathcal{L}(Q', V)$ *such that* $B(L_B(g)) = g \quad \forall\, g \in Q'$, *with* $\|L_B\| = 1/\beta$.

The proof is immediate.

A useful consequence of Corollary 4.1.1 is the well known Lax-Milgram Lemma:

Theorem 4.1.6 (Lax-Milgram Lemma). *Let V be a Hilbert space, and let $a(\cdot\,,\cdot)$ be a bilinear continuous form on V. Assume that a is coercive, that is*

$$\exists\,\alpha > 0 \textit{ such that } a(v,v) \ge \alpha\,\|v\|_V^2,\ \forall\, v \in V. \tag{4.1.67}$$

Then, for every $f \in V'$, the problem: find $u \in V$ such that

$$a(u,v) = \langle f, v\rangle_{V'\times V} \quad \forall\, v \in V \tag{4.1.68}$$

has a unique solution.

Proof. Note that (4.1.68) is equivalent to $Au = f$, where $A \in \mathcal{L}(V, V')$ is the operator associated to the bilinear form a. We have to prove that A is injective and surjective. Condition (4.1.67) immediately implies that A is *bounding*:

$$\|Av\|_{V'} = \sup_{w\in V\setminus\{0\}} \frac{a(v,w)}{\|w\|_V} \geq \frac{a(v,v)}{\|v\|_V} \geq \alpha \|v\|_V. \tag{4.1.69}$$

Hence, A is injective. With an identical proof, we see that A^t is also bounding. Hence, A^t is injective and (due to Corollary 4.1.1) A is surjective. □

Remark 4.1.7. Roughly speaking, we can summarise the result of the Closed Range Theorem by saying that operators with a closed range have essentially all the well-known properties of operators in finite dimensional spaces (whose range is always trivially closed) that we have seen in the previous chapter. In particular, Corollary 4.1.1 is *exactly what we need* to extend the properties and the results of Sect. 3.4 to the infinite dimensional case. See in particular Proposition 3.4.4. □

4.1.6 Dual Spaces of Linear Subspaces

We have seen two (very different) types of subspaces: closed subspaces and dense subspaces. We shall see now that they also behave quite differently when we consider their *dual spaces*. Let us see the difference.

Assume first that Z is a *closed* subspace of a Hilbert space H. Then, we already pointed out that, using on Z *the same norm* that we already have on H, the space Z becomes itself a Hilbert space, and, as such, it will have a dual space Z' of its own. It is easy to see that Z' could be identified with a particular subset of H', made of all functionals $f \in H'$ that vanish identically on the orthogonal complement $Z^\perp$ of Z. Note that we already have a name for that space, that is $(Z^\perp)^0$. We have therefore, in a natural way,

$$Z' \equiv (Z^\perp)^0 \equiv R_H(Z) \subset H' \quad \text{and} \quad (Z^\perp)' \equiv Z^0 \equiv R_H(Z^\perp) \subset H'. \tag{4.1.70}$$

Hence, *the dual space Z' of a closed subspace $Z \subset H$ can be identified with a closed subspace of H'*. Once this identification is made, we can also consider the extension operator $E_{Z'\to H'}$ (that we shall often denote simply as $E_{Z'}$), and the projection operator $\pi_{Z'}$ from H' to Z'. Note that, for $\phi \in Z'$, the functional $E_{Z'\to H'}\phi$ can also be described as

$$_{H'}\langle E_{Z'\to H'}\phi, v\rangle_H := {}_{Z'}\langle \phi, \pi_Z v\rangle_Z \tag{4.1.71}$$

while for $\psi \in H'$ the functional $\pi_{Z'}\psi$ can be described as

$$ {}_{Z'}\langle \pi_{Z'}\psi, z\rangle_Z := {}_{H'}\langle \psi, E_{Z\to H} z\rangle_H . \tag{4.1.72} $$

In other terms

$$ (\pi_{Z'})^t \equiv (E_Z) \tag{4.1.73} $$

and

$$ (\pi_Z)^t \equiv (E_{Z'}). \tag{4.1.74} $$

Example 4.1.11. For instance, if $H = L^2(\Omega)$ and Z is the space of constant functions, it is not difficult to see that $Z^\perp = L^2_0(\Omega)$ (the space of functions having zero mean value). Now, the dual space of Z will be the space of functionals that can be written as

$$ q \to k \int_\Omega q\,dx \qquad k \in \mathbb{R} $$

(meaning that for each $k \in \mathbb{R}$ we have a different functional). On the other hand, the dual space of $Z^\perp$ will be the space of functionals that can be written as

$$ q \to \int_\Omega k\,q\,dx \qquad\qquad k \in Z^\perp $$

(meaning that for each $k \in Z^\perp$ we have a different functional). On the other hand, $(Z^\perp)'$ could also be identified with the subset of H' made of functionals that vanish identically on constant functions (that is, with the polar set of the space of constants, which is the polar set of Z, as in (4.1.70)). Using the Ritz operator R_H of Theorem 4.1.2, we could write $Z' = R_H(Z)$ and $(Z^\perp)' = R_H(Z^\perp)$. If, as is done almost every time, we identify $L^2(\Omega)$ with its dual space, then we could write $Z' = Z$ and $(Z^\perp)' = Z^\perp$. □

Let us consider now the case of a *dense* subspace $S \subset H$ of a Hilbert space H. If we take on S the same norm as on H, we cannot (in the present setting) consider its dual space, as S will not be closed (unless $S \equiv H$, a case without any interest). Hence, we assume that on S we take a *different norm*. More precisely, we assume that on S we are given another norm, $\|\cdot\|_S$, that makes S a Hilbert space. We assume, moreover, that this other norm is (up to a multiplicative constant) *bigger* than the $\|\cdot\|_H$ norm:

$$ \exists\, C_{SH} > 0 \text{ such that } \|s\|_H \le C_{SH}\|s\|_S \quad \forall\, s \in S. \tag{4.1.75} $$

In this case, we will say that S *is continuously embedded in* H. Indeed, (4.1.75) means exactly that the identity operator is continuous from S into H. There is a special symbol for that: instead of $S \subset H$, we write $S \hookrightarrow H$.

Example 4.1.12. If we take, as in Example 4.1.5, $S = H^1_0(\Omega)$ and $H = L^2(\Omega)$, then inequality (4.1.75) is just the Poincaré inequality. □

Now S, being a Hilbert space, has a dual space S'. Let us see the relationship between S' and H'. As $S \subset H$, for each element $g \in H'$, we can consider its restriction $g_{|S} \in S'$ defined by $\langle g_{|S}, s\rangle_{S'\times S} = \langle g, s\rangle_{H'\times H}$ for all $s \in S$. Indeed, from (4.1.75) we have easily for every $s \in S$

$$\langle g_{|S}, s\rangle_{S'\times S} \equiv \langle g, s\rangle_{H'\times H} \leq \|g\|_{H'}\,\|s\|_H \leq C_{SH}\,\|g\|_{H'}\,\|s\|_S, \tag{4.1.76}$$

implying the continuity of $g_{|S} : S \to \mathbb{R}$, as well as the *continuity of the restriction operator*: namely,

$$\|g_{|S}\|_{S'} \leq C_{SH}\|g\|_{H'}. \tag{4.1.77}$$

Using the Hahn-Banach theorem (here simplified to (4.1.42)), we see that we cannot have in H' two different g's having the same restriction to S: indeed, if g^1 and g^2 have the same restriction to S (that is, if $g^1_{|S} = g^2_{|S}$), then the difference $g^1 - g^2$ is in S^0, and hence it must be zero.

We can then summarise the above discussion by saying that: every $g \in H'$ has a restriction $g_{|S}$ in S' and the mapping $g \to g_{|S}$ from H' to S' is *injective*. This allows us to *identify* H' with a subset of S':

$$H' \subseteq S'. \tag{4.1.78}$$

On the other hand, there are, in general, elements in S' that cannot be presented as the restriction of any $g \in H'$: indeed, S has a norm which is bigger than that of H, and g could be continuous from S to $\mathbb{R}$ and not from H to $\mathbb{R}$. As we have seen for instance in Example 3.1.6, for $I :=]-1,1[$, taking $H := L^2(I)$ and $S := H^1_0(I)$, the mapping $v \to v(0)$ belongs to S' but *cannot be seen* as the restriction to S of an element of H'

In other words, (4.1.77) cannot be reversed. Hence, we have $H' \subset S'$, and using (4.1.77), we see that we actually have $H' \hookrightarrow S'$, and in general the inclusion is *strict*. On the other hand, one can also prove that H' is dense in S'. Moreover, out of the previous discussion, we easily have that

$$\langle g, s\rangle_{S'\times S} = \langle g, s\rangle_{H'\times H} \qquad \text{whenever } g \in H' \textbf{ and } s \in S. \tag{4.1.79}$$

Hence, if we have two Hilbert spaces S and H with $S \subset H$ and S *dense* in H, then

$$S \hookrightarrow H \quad \Rightarrow \quad H' \hookrightarrow S'. \tag{4.1.80}$$

The difference between the two cases, (4.1.70) and (4.1.80), that might be surprising at first sight, is due to the fact that in the first case we used on S the same norm that we had on H, while in the second case we used a different, stronger norm.

Example 4.1.13. We have already seen the example of δ_0, which belongs to the dual space of $S := H^1_0(]-1,1[)$ but not to the dual space of $H := L^2(]-1,1[)$. Let us see another simple example. For a general domain Ω, taking always $S = H^1_0(\Omega)$ and $H = L^2(\Omega)$, and taking in H' the functional

$$v \to \int_{\Omega} v\, dx,$$

it is clear that its restriction to S leaves the functional (essentially) unchanged. On the other hand, for f fixed in $L^2(\Omega)$, the functional

$$v \to \int_{\Omega} f\, \frac{\partial v}{\partial x}\, dx,$$

linear and continuous on S, cannot be extended to a continuous functional on H.

□

4.1.7 *Identification of a Space with its Dual Space*

It is usually a strong temptation, when dealing with Hilbert spaces, to use the Ritz Theorem 4.1.2 to **identify a Hilbert space with its dual**. After all, this is what is done most of the times when dealing with finite dimensional spaces. However, when dealing with *functional spaces* (that is, spaces made of functions), it is **highly recommended** to limit such identification to L^2 with its dual (or of a closed subspace Z of L^2 with its dual Z'). Every other identification will be *calling for a total disaster*. Let us see why. Assume that in (4.1.80) we have $H = L^2(\Omega)$ and $S = H_0^1(\Omega)$. Identifying L^2 with its dual space, we would have $H \equiv H'$, and (4.1.80) will become

$$S \hookrightarrow H \equiv H' \hookrightarrow S'. \tag{4.1.81}$$

So far, so good. Everybody does that, and nobody suffers. Assume, however, that, in spite of all recommendations, you **also** identify S with S'. Then, in (4.1.81), you compress the four spaces $S \equiv H \equiv H' \equiv S'$ into one, identifying at the same time a function with itself **and** with its Laplacian. This is *the beginning of the end*. Now, the question that everybody asks (the first time one hears about that) is "What is so special with L^2?". It is a very good question. Actually, there is nothing special, *mathematically*, about it, apart from the fact that we are so used to identify a function $f \in L^2(\Omega)$ with the mapping (defined for $\varphi \in L^2(\Omega)$):

$$\varphi \to \int_{\Omega} f\, \varphi\, dx \tag{4.1.82}$$

that we do it all the time, without even realising it. In principle, we might as well identify a function $f \in H_0^1(\Omega)$ with the mapping (defined for φ in $H_0^1(\Omega)$):

$$\varphi \to \int_{\Omega} \underline{\text{grad}}\, f \cdot \underline{\text{grad}}\, \varphi\, dx \tag{4.1.83}$$

and don't use the identification (4.1.82). This will be mathematically correct but psychologically very, very difficult; and before the rooster crows, you will have used (4.1.82) three times. Hence, our advice is: *No matter whether the above discussion was clear or not, just avoid any identification of a functional space that is not L^2 (or a multiple copy of it, or, exceptionally, a closed subspace of it) with its dual space!* This, of course, unless you are very skilled in Functional Analysis. Although, if you are . . . why are you reading all this? □

4.1.8 Restrictions of Operators to Closed Subspaces

We shall now deal briefly with a situation that we will meet constantly in the following chapter. We have (as before) two Hilbert spaces V and Q, we have a linear continuous operator $B \in \mathcal{L}(V, Q')$, and we have two closed subspaces $Z \subset V$ and $S \subset Q$. In the applications of the next chapter, Z and S will typically be *finite dimensional spaces* (and hence automatically closed).

As we have seen, B (and its transposed operator B^t) can be associated to a bilinear form b defined on $V \times Q$. It is not difficult to see that, restricting the bilinear form to $Z \times S$, we have as associated operators

$$B_{ZS'} \equiv \pi_{S'} B E_Z \quad \text{and} \quad B^t_{SZ'} \equiv \pi_{Z'} B^t E_S \tag{4.1.84}$$

and obviously $(B_{ZS'})^t = B^t_{SZ'}$.

Remark 4.1.8. As we have already pointed out in Remark 3.1.11 of the previous chapter, in general, we cannot expect the kernel of $B_{ZS'}$ to be a subspace of the kernel of B, nor the image of $B_{ZS'}$ to be a subset of the image of B. The same is obviously true for the images and the kernels of $B^t_{SZ'}$ and B^t. □

Proposition 4.1.6. *Let V and Q be Hilbert spaces, let $B \in \mathcal{L}(V, Q')$, and let $Z \subset V$ and $S \subset Q$ be closed subspaces, with S finite dimensional. Then, the inclusion*

$$\mathrm{Ker} B^t_{SZ'} \subseteq \mathrm{Ker} B^t \tag{4.1.85}$$

holds **iff** *we have*

$$\pi_{S'}(\mathrm{Im} B) \subseteq \mathrm{Im} B_{ZS'}. \tag{4.1.86}$$

Proof. Assume first that (4.1.85) (that, to be precise, we should actually write as $E_S \mathrm{Ker} B^t_{SZ'} \subseteq \mathrm{Ker} B^t$) holds, and let $g = Bv \in \mathrm{Im} B$. As $\mathrm{Im} B_{ZS'}$ is closed (since S is finite dimensional), to show that $\pi_{S'} g \in \mathrm{Im} B_{ZS'}$, we just have to check that $\pi_{S'} g \in (\mathrm{Ker} B^t_{SZ'})^0$, that is,

$$_{Q'}\langle g, q \rangle_Q = 0 \quad \forall q \in \mathrm{Ker} B^t_{SZ'}. \tag{4.1.87}$$

If the inclusion (4.1.85) is satisfied, then every $q \in \mathrm{Ker}B^t_{SZ'}$ will also be in $\mathrm{Ker}B^t$. However, for $q \in \mathrm{Ker}B^t$ we have

$$_{Q'}\langle g, q\rangle_Q = {_{Q'}\langle Bv, q\rangle_Q} = {_V\langle v, B^t q\rangle_{V'}} = 0, \tag{4.1.88}$$

giving (4.1.87) and ending the first part of the proof.

Assume now that (4.1.86) holds, and let $q_s \in S$ be in $\mathrm{Ker}B^t_{SZ'}$, that is: $\pi_{Z'}B^t q_s = 0$. For such a q_s we have, for every $z \in Z$, that

$$\begin{aligned} &_S\langle q_s, \pi_{S'}Bz\rangle_{S'} \\ &= {_Q\langle q_s, Bz\rangle_{Q'}} = {_{V'}\langle B^t q_s, z\rangle_V} = {_{Z'}\langle \pi_{Z'}B^t q_s, z\rangle_Z} \\ &= 0, \end{aligned} \tag{4.1.89}$$

meaning that q_s is in the polar space of $\mathrm{Im}B_{ZS'}$. Inclusion (4.1.86) together with (4.1.43) implies then that q_s is in the polar space of $\pi_{S'}\mathrm{Im}B$, so that for all $v \in V$ we have $_S\langle q_s, \pi_{S'}Bv\rangle_{S'} = 0$, hence $_{V'}\langle B^t q_s, v\rangle_V = 0$ and therefore $q_s \in \mathrm{Ker}B^t$. □

Remark 4.1.9. The assumption that S is finite dimensional, in Proposition 4.1.6, is clearly stronger than necessary. Indeed, looking at the proof, we see that for the first part we only need $\mathrm{Im}B_{ZS'}$ to be closed, while the second part does not even need that. However, as we said, we are going to use the result in the case of Z and S being finite dimensional, so that we did not struggle to minimise this type of assumptions. □

Exchanging the roles of B and B^t, we have, moreover, in the same assumptions of Proposition 4.1.6 (but requiring Z to be finite dimensional instead of S), that

$$\mathrm{Ker}B_{ZS'} \subseteq \mathrm{Ker}B \tag{4.1.90}$$

is equivalent to

$$\pi_{Z'}(\mathrm{Im}B^t) \subseteq \mathrm{Im}B^t_{SZ'}. \tag{4.1.91}$$

The case in which the subspaces Z and S are related to the kernels and images of a linear operator $B \in \mathcal{L}(V, Q')$ (and of its transposed) is obviously of special interest. In particular, we can present a corollary of the Closed Range Theorem 4.1.5 that will often be useful.

Corollary 4.1.2. *In the same assumptions of Theorem 4.1.5, if one of the six equivalent properties is satisfied, then $L_B \in \mathcal{L}(K^\perp, H^0)$ is the transposed operator of $L_{B^t} \in \mathcal{L}(H^\perp, K^0)$ and in particular,*

$$\|L_B\|_{\mathcal{L}(K^\perp,H^0)} = \|L_{B^t}\|_{\mathcal{L}(H^\perp,K^0)} =: \mu. \tag{4.1.92}$$

Moreover, setting $\beta := 1/\mu$ we have

$$\beta\|v\|_V \le \|Bv\|_{V'} \qquad \forall\, v \in K^{\perp}, \tag{4.1.93}$$

and

$$\beta\|q\|_Q \le \|B^t q\|_{V'} \qquad \forall\, q \in H^{\perp}. \tag{4.1.94}$$

Proof. If, say, ImB is closed, then B will be an isomorphism from $K^{\perp}$ to ImB which, however, coincides with H^0. Similarly, B^t will be an isomorphism from $H^{\perp}$ to ImB^t that coincides with K^0. Hence, L_B coincides with $(B_{K^{\perp}H^0})^{-1}$ and L_{B^t} coincides with $(B^t_{H^{\perp}K^0})^{-1}$. We also recall from (4.1.70) that

$$(K^{\perp})^0 = K' \quad K^0 = (K^{\perp})' \qquad (H^{\perp})^0 = H' \quad H^0 = (H^{\perp})' \tag{4.1.95}$$

so that it is immediate to see that L_{B^t} is the transposed operator of L_B. Now (4.1.92) will follow immediately from (4.1.50). Finally, (4.1.93) and (4.1.94) are now immediate since, for $v \in K^{\perp}$, we have $v = L_B(Bv)$ and for $q \in H^{\perp}$, we have $q = L_{B^t}(B^t q)$. □

4.1.9 Quotient Spaces

Assume that Q is a Hilbert space and let H be a closed subspace of Q. We also assume that H is a *proper* subspace, meaning that H does not coincide with Q. We consider then *the quotient space* $Q_{/H}$ defined as *the space whose elements are the equivalence classes induced by the equivalence relation:*

$$v_1 \cong v_2 \quad \text{ if and only if } (v_1 - v_2) \in H. \tag{4.1.96}$$

In other words, two elements are equivalent if their difference belongs to H. It is immediate to see that all the elements of H will then be equivalent to 0. In view of this definition, an element of $Q_{/H}$ will then be a subset of Q made by elements that are all equivalent to each other.

Example 4.1.14. For a bounded domain $\Omega \subset \mathbb{R}^d$, we take $Q := L^2(\Omega)$ and we consider the (one-dimensional) subspace H made of *constant functions*. Then, $Q_{/H}$ will be made of *classes of functions that differ from each other by a constant function*. □

Note that *if two classes C_1 and C_2 have an element v^* in common, then they must coincide*. Indeed, for every v_1 in C_1, we have $v_1 - v^* \in H$ and, for every $v_2 \in C_2$, we have $v_2 - v^* \in H$. As a consequence, for every $v_1 \in C_1$ and every $v_2 \in C_2$, we have $v_1 - v_2 = (v_1 - v^*) - (v_2 - v^*) \in H$ (as difference of two elements of H). This implies that for every $v_1 \in C_1$ and for every $v_2 \in C_2$, we have $v_1 \cong v_2$, which is to say that the two classes C_1 and C_2 coincide. We conclude that two *different* classes have no elements in common.

It is then easy to verify that *there is a one-to-one correspondence between* $Q_{/H}$ *and the orthogonal complement* $H^{\perp}$ *of* H *in* Q. Let us see it in more detail. Let q^* be an element of $H^{\perp}$: to it we associate the class C_{q^*} defined by

$$C_{q^*} := \{v \in Q \,|\, v \cong q^*\} \equiv \{v \in Q \,|\, v - q^* \in H\}. \tag{4.1.97}$$

It is clear that the mapping $q^* \to C_{q^*}$, from $H^{\perp}$ to $Q_{/H}$, is *injective*: indeed, assume that q^* and q^{**} are two elements in $H^{\perp}$ such that the two corresponding classes C_{q^*} and $C_{q^{**}}$ coincide. This implies that, say, $q^{**} \in C_{q^*}$, that is $q^{**} - q^* \in H$. Since $q^{**} - q^*$ must also belong to $H^{\perp}$ (as difference of two elements both in $H^{\perp}$), we conclude that $q^{**} = q^*$.

Let us see that the mapping $q^* \to C_{q^*}$ is also *surjective*: let therefore the class C^* be an element of $Q_{/H}$ and let $\bar{q} \in C^*$. The class C^* could then be characterised as

$$C^* := \{v \in Q \,|\, v \cong \bar{q}\} \equiv \{v \in Q \,|\, v - \bar{q} \in H\}. \tag{4.1.98}$$

At this point, it is not difficult to see that C^* is a closed convex subset of Q and hence (see (4.1.22) and Remark 4.1.2) we can define $q^*_{C^*}$ as the projection $\pi_C 0$ of 0 on C^* (that can also be seen as the element of C^* having minimum norm). It is then elementary to check that

$$(q^*_{C^*}, v) = 0 \qquad \forall\, v \in H, \tag{4.1.99}$$

implying that $q^*_{C^*} \in H^{\perp}$ and that, actually, $C^* \equiv C_{q^*_{C^*}}$. This also allows us to define *a norm* in $Q_{/H}$: for every $C \in Q_{/H}$, we define

$$\|C\|_{Q_{/H}} := \|\pi_C 0\|_Q \equiv \|q^*_C\|_Q. \tag{4.1.100}$$

Hence, if we prefer, we could choose in each class (= element of $Q_{/H}$) the unique element, in the class, which belongs to $H^{\perp}$, and identify $Q_{/H}$ with $H^{\perp}$.

Example 4.1.15. Let us go back to the case of Example 4.1.14 where $Q := L^2(\Omega)$ and H is the subspace made of *constant functions*. We recall that $Q_{/H}$ is made of classes of functions that differ from each other by a constant function. For every such class, we could always take one function q in the class, and describe the class as the set of all functions of the form $q + c$ with c constant. In doing so, we could however decide to choose as "representative" the unique function, in the class, that has zero mean value. This is the same as picking $q^* \in H^{\perp}$, since $H^{\perp}$ is clearly the subspace of Q made of functions having zero mean value. □

4.2 Existence and Uniqueness of Solutions

4.2.1 Mixed Formulations in Hilbert Spaces

From here to the end of this chapter, we will consistently remain in the same notational framework. As this framework will also include some assumptions, we summarise all these assumptions under the name of *Assumption* $\mathcal{AB}$.

Assumption $\mathcal{AB}$: *We are given two Hilbert spaces, V and Q, and two continuous bilinear forms: $a(\cdot,\cdot)$ on $V \times V$ and $b(\cdot,\cdot)$ on $V \times Q$. We denote by A and B, respectively, the linear continuous operators associated with them. We also set*

$$K := \mathrm{Ker}B \qquad \text{and} \qquad H := \mathrm{Ker}B^t. \tag{4.2.1}$$

We recall from the previous subsection that we have

$$|a(u,v)| \leq \|a\| \, \|u\|_V \, \|v\|_V, \tag{4.2.2}$$

and that the two linear continuous operators $A : V \to V'$ and $A^t : V \to V'$ satisfy

$$\langle Au, v\rangle_{V'\times V} = \langle u, A^t v\rangle_{V\times V'} = a(u,v), \qquad \forall\, v \in V \;\forall\, u \in V. \tag{4.2.3}$$

Similarly,

$$|b(v,q)| \leq \|b\| \, \|v\|_V \, \|q\|_Q, \tag{4.2.4}$$

and the two linear operators $B : V \to Q'$, and $B^t : Q \to V'$ satisfy

$$\langle Bv, q\rangle_{Q'\times Q} = \langle v, B^t q\rangle_{V\times V'} = b(v,q) \qquad \forall\, v \in V,\; \forall\, q \in Q. \tag{4.2.5}$$

We now consider our **basic problem**. *Given $f \in V'$ and $g \in Q'$, we want to find $(u,p) \in V \times Q$ solution of*

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle_{V'\times V}, \;\; \forall\, v \in V, \\ b(u,q) = \langle g, q\rangle_{Q'\times Q}, \;\; \forall\, q \in Q. \end{cases} \tag{4.2.6}$$

Note that problem (4.2.6) can also be written as

$$\begin{cases} Au + B^t p = f \;\; \text{in } V', \\ Bu = g \;\; \text{in } Q', \end{cases} \tag{4.2.7}$$

and from now on we shall consider the formulations (4.2.6) and (4.2.7) to be the same, referring to one or the other according to the convenience of the moment. We now want to find conditions implying existence and possibly uniqueness of solutions to this problem.

Remark 4.2.1. If the bilinear form $a(\cdot,\cdot)$ is symmetric, the equations (4.2.6) are the optimality conditions of the minimisation problem

$$\inf_{Bv=g} \frac{1}{2} a(v,v) - \langle f, v\rangle_{V'\times V}. \tag{4.2.8}$$

The variable p is then the Lagrange multiplier associated with the constraint $Bu = g$, and the associated saddle point problem is

$$\inf_{v\in V}\sup_{q\in Q} \left\{\frac{1}{2} a(v,v) + b(v,q) - \langle f, v\rangle_{V'\times V} - \langle g, q\rangle_{Q'\times Q}\right\}. \tag{4.2.9}$$

This is the reason for the title of this chapter, in spite of the fact that we deal in fact with a more general case. □

Remark 4.2.2. The two equations in (4.2.6) can sometimes be written as a unique variational equation, setting

$$\mathcal{A}((u,p),(v,q)) = a(u,v) + b(v,p) - b(u,q) \qquad \forall (u,p),\ (v,q)\ \in V\times Q \tag{4.2.10}$$

and then requiring that

$$\mathcal{A}((u,p),(v,q)) = \langle f, v\rangle_{V'\times V} - \langle g, q\rangle_{Q'\times Q} \qquad \forall (v,q)\ \in V\times Q. \tag{4.2.11}$$

One can obviously go from (4.2.6) to (4.2.11), subtracting the two equations, and from (4.2.11) to (4.2.6) by considering separately the pairs $(v,0)$ and $(0,-q)$. □

It is clear from the second equation of (4.2.7) that, in order to have existence of a solution for every $g \in Q'$, we must have $\mathrm{Im}B = Q'$. Following the path of the previous chapter, we first consider a simpler case, in which we have *sufficient conditions* on a and b for having a unique solution.

Theorem 4.2.1. *Together with Assumption $\mathcal{AB}$, assume that $\mathrm{Im}B = Q'$ and that the bilinear form $a(\cdot,\cdot)$ is coercive on K, that is*

$$\exists\, \alpha_0 > 0 \text{ such that } a(v_0,v_0) \ge \alpha_0\, \|v_0\|_V^2,\ \forall\, v_0 \in K. \tag{4.2.12}$$

Then, for every $(f,g) \in V'\times Q'$, problem (4.2.6) has a unique solution.

Proof. Let us first prove the *existence* of a solution. From the surjectivity of B and Corollary 4.1.1, we have that there exists a lifting operator L_B such that $B(L_B g) = g$ for every $g \in Q'$. Setting $u_g := L_B g$, we therefore have $Bu_g = g$. We now consider the new unknown $u_0 := u - u_g$ and, in order to have $Bu = g$, we require $u_0 \in K$. For every $v_0 \in K$, we obviously have $b(v_0,q) = 0$ for every $q \in Q$, so that the first equation of (4.2.6) now implies

$$a(u_0,v_0) = \langle f, v_0\rangle_{V'\times V} - a(u_g,v_0),\ \forall v_0 \in K, \tag{4.2.13}$$

and the Lax-Milgram Lemma, using (4.2.12), ensures that we have a unique $u_0 \in \mathrm{Ker}B$ satisfying (4.2.13). Remark now that the functional

$$v \to \ell(v) := \langle f, v\rangle_{V'\times V} - a(u_g + u_0, v), \tag{4.2.14}$$

thanks to (4.2.13), vanishes identically for every $v \in K$. Hence, $\ell \in K^0$ (the polar space of K), which, due to Theorem 4.1.4, coincides with $\mathrm{Im}B^t$. Hence, ℓ is in the image of B^t, and there exists a $p \in Q$ such that $B^t p = \ell$. This means that

$$\langle B^t p, v\rangle_{V'\times V} = \langle \ell, v\rangle_{V'\times V} = \langle f, v\rangle_{V'\times V} - a(u_g + u_0, v) \tag{4.2.15}$$

for every $v \in V$, and since $u = u_g + u_0$, the first equation is satisfied. On the other hand, $Bu = Bu_g + Bu_0 = g$ and the second equation is also satisfied.

We now prove *uniqueness*. By linearity, assume that $f = 0$ and $g = 0$: then, $u \in K$. Testing the first equation on $v = u$ we get $a(u, u) = 0$ and then $u = 0$ from (4.2.12). Using $u = 0$ and $f = 0$ in the first equation of (4.2.7), we have then $B^t p = 0$, and from Corollary 4.1.1 we have $p = 0$. Then, problem (4.2.6) has a unique solution. □

Remark 4.2.3. The coercivity of $a(\cdot,\cdot)$ on K may hold while there is no coercivity on V. We have already seen examples of this situation in finite dimension and we shall see in the next chapters many other examples coming from partial differential equations. □

The result of Theorem 4.2.1 will be *the most commonly used* in our applications. However, as we had in the finite-dimensional case of the previous chapter, it is clear that it does not give a *necessary and sufficient* condition. To get it, we must weaken the coercivity condition (4.2.12). For this, we recall that K is a closed subspace of V, and hence it is itself a Hilbert space (with the same norm as V). As such, as we have seen, K has a dual space, that we denote by K'. Moreover, we note that, *restricting the bilinear form* $a(\cdot\,,\cdot)$ *to* K, we have two operators, which, according to the notation (4.1.84), we denote by $A_{KK'}$ and $A^t_{KK'}$, from K to K', given as in (4.2.3) by

$$\langle A_{KK'}u_0, v_0\rangle_{K'\times K} = \langle u_0, A^t_{KK'}v_0\rangle_{K\times K'} = a(u_0, v_0),\ \forall\, u_0, v_0 \in K. \tag{4.2.16}$$

We also recall that K' could be identified, through (4.1.70), to a subspace of Q', and precisely to $(K^\perp)^0$ (the polar space of $K^\perp$). Moreover, it is easy to check that

$$A_{KK'} = \pi_{K'} A\, E_K \tag{4.2.17}$$

coincides, in the finite dimensional case, with the operator that (identifying K and K') was denoted by A_{KK} in the previous chapter.

We are now ready to state and prove the following theorem, which is, from the theoretical point of view, the most relevant of this section. As we shall see, it generalises Theorem 4.2.1 and gives the required *necessary and sufficient conditions*.

Theorem 4.2.2. *Assume that* $\mathcal{AB}$ *holds, and let* $A_{KK'}$ *be defined as in* (4.2.16)*. Then, problem* (4.2.6) *has a unique solution for every* $(f, g) \in V' \times Q'$ ***if and only if*** *the two following conditions are satisfied:*

$$A_{KK'} \text{ is an isomorphism from } K \text{ to } K', \tag{4.2.18}$$

$$ImB = Q'. \tag{4.2.19}$$

Proof. Assume first that (4.2.18) and (4.2.19) are satisfied. The existence and uniqueness of the solution of (4.2.6) follow as in the proof of Theorem 4.2.1. The only difference is in the solution of (4.2.13), which in the present notation can be written as

$$A_{KK'}u_0 = \pi_{K'} f - \pi_{K'} A u_g. \tag{4.2.20}$$

Indeed, here we now have to use (4.2.18) (in order to get the existence of a solution u_0) instead of Lax-Milgram as we did there.

Assume conversely that the problem (4.2.6) has a unique solution for every $(f, g) \in V' \times Q'$. It is clear that, in particular for every $g \in Q'$, we can take $(0, g)$ as right-hand side in (4.2.6) and have a $u \in V$ such that $Bu = g$ (from the second equation of (4.2.7)). Hence, $\text{Im}B = Q'$ and therefore (4.2.19) holds. To show that (4.2.18) also holds, we proceed as follows.

First, *for every* $\phi \in K'$, we take in (4.2.7) $f = E_{K' \to V'}\phi$ (as defined in (4.1.71)), and $g = 0$. By assumption, we have a unique solution (u_ϕ, p_ϕ), and we observe that $u_\phi \in K$ since $g = 0$. Testing the first equation of problem (4.2.6) on $v_0 \in K$, and using (4.1.71), we have

$$a(u_\phi, v_0) = \langle f_\phi, v_0\rangle_{K' \times K} \equiv \langle \phi, v_0\rangle_{K' \times K} \qquad \forall\, v_0 \in K. \tag{4.2.21}$$

This implies that $A_{KK'}u_\phi = \phi$, and hence that $A_{KK'}$ is *surjective*. Hence, we are left to show that $A_{KK'}$ is also *injective*. Assume, by contradiction, that we had $A_{KK'}w = 0$ for some $w \in K$ *different from zero*. Then, we would have $a(w, v_0) = 0$ for all $v_0 \in K$, implying that $Aw \in K^0$. Due to Corollary 4.1.1 (as we already saw that (4.2.19) holds), this would imply $Aw \in \text{Im}B^t$ and we would have the existence of a $p_w \in Q$ such that $B^t p_w = Aw$. Then, the pair $(w, -p_w)$ (different from zero) would satisfy the homogeneous version of problem (4.2.6), and uniqueness would be lost. Hence, such a $w \neq 0$ cannot exist. This shows that $A_{KK'}$ must also be injective, and hence (4.2.18) holds. □

4.2.2 Stability Constants and inf-sup *Conditions*

In this subsection, we would like to express condition (4.2.19) and condition (4.2.18) in a different way, to emphasise the role of the stability constants.

Let us start from condition (4.2.19). According to Corollary 4.1.1 of the Closed Range Theorem, we already know that (4.2.19) holds if and only if the operator B^t is bounding, that is, if and only if there exists a constant $\beta > 0$ such that

$$\|B^t q\|_{V'} \geq \beta \|q\|_Q \qquad \forall\, q \in Q. \tag{4.2.22}$$

Always from the same corollary, we have that this is also equivalent to the existence of a lifting $L_B : Q' \to V$ of the operator B

$$B(L_B(g)) = g \qquad \forall\, g \in Q', \tag{4.2.23}$$

with its norm being bounded by:

$$\|L_B\|_{\mathcal{L}(Q',V)} \le \frac{1}{\beta}, \tag{4.2.24}$$

where β is the same constant as in (4.2.22) and $\mathrm{Im} L_B = K^{\perp}$.

We now want to define, somehow, the *best possible constant that would fit in* (4.2.22). For this, we note that (4.2.22) is equivalent to

$$\inf_{q\in Q} \frac{\|B^t q\|_{V'}}{\|q\|_Q} \ge \beta, \tag{4.2.25}$$

which, recalling the definition of norm in a dual space (4.1.34) and (4.1.49), becomes

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q} \ge \beta, \tag{4.2.26}$$

which is possibly the most commonly used among the many equivalent formulations of assumption (4.2.19).

With similar arguments, we see that condition (4.2.18) is equivalent to saying that there exists an $\alpha_1 > 0$ such that

$$\inf_{v_0\in K} \sup_{w_0\in K} \frac{a(v_0,w_0)}{\|v_0\|_V \, \|w_0\|_V} \ge \alpha_1$$
$$\inf_{w_0\in K} \sup_{v_0\in K} \frac{a(v_0,w_0)}{\|v_0\|_V \, \|w_0\|_V} \ge \alpha_1. \tag{4.2.27}$$

Remark 4.2.4. Note that in (4.2.25), in (4.2.26), and in (4.2.27), as we did in the previous chapter and we shall do in the rest of the book, we assumed *implicitly* that for fractions of the type

$$\frac{\ell(v)}{\|v\|_V} \quad \text{or} \quad \frac{|\ell(v)|}{\|v\|_V}$$

where $\ell(\cdot)$ is a linear functional on a Banach space V, the supremum and the infimum are taken for $v \neq \{0\}$, and therefore we wrote the supremum (or infimum) for $v \in V$ rather than for $v \in V \setminus \{\mathbf{0}\}$ (as it would have been more correct, since these fractions do not make sense for $v = \{\mathbf{0}\}$). □

We now want to point out, for future use, the following extension of Lemma 3.3.1 of the previous chapter, that is an important ingredient in the proof of the present Theorem 4.2.3.

Lemma 4.2.1. *Let V be a Hilbert space and let $a(\cdot\,,\cdot)$ be a symmetric bilinear continuous form on V. Assume that*

$$a(v,v) \geq 0 \qquad \forall\, v \in V. \tag{4.2.28}$$

Then, we have

$$(a(v,w))^2 \leq a(v,v)\, a(w,w) \qquad \forall\, v,w \in V \tag{4.2.29}$$

and, for the associated operator A,

$$\|Av\|^2_{V'} \leq \|a\|\, a(v,v) \equiv \|A\| \langle Av, v\rangle \qquad \forall\, v \in V. \tag{4.2.30}$$

Apart from the different notation, the proof is identical to that of Lemma 3.3.1.

Moreover, under the assumptions of the previous lemma, we also have the following result.

Lemma 4.2.2. *Let V be a Hilbert space, and let $a(\cdot\,,\cdot)$ be a symmetric bilinear continuous form on V. Assume that*

$$a(v,v) \geq 0 \qquad \forall\, v \in V. \tag{4.2.31}$$

Then, (4.2.27) *implies ellipticity on the kernel* (4.2.12).

Proof. Indeed, from (4.2.27), we have for $v_0 \in K$, using (4.2.29),

$$\alpha_1^2 \|v\|^2_V \leq \sup_{w\in K} \frac{a(v,w)^2}{\|w\|^2_V} \leq \sup_{w\in K} \frac{a(v,v)a(w,w)}{\|w\|^2_V} \leq \|a\| a(v,v), \tag{4.2.32}$$

hence the result with $\alpha_0 = \alpha_1^2/\|a\|$. □

4.2.3 The Main Result

As we had in the previous chapter (in Theorems 3.4.1 and 3.4.2), we have here the following final result, that could be considered as *the main result of this chapter*.

Theorem 4.2.3. *Together with $\mathcal{AB}$, assume that there exist two positive constants α and β such that the inf-sup condition* (4.2.26) *on $b(\cdot,\cdot)$, and the double-inf-sup condition* (4.2.27) *on the restriction of $a(\cdot,\cdot)$ to K are satisfied. Then, for every $f \in V'$ and for every $g \in Q'$, problem* (4.2.6) *has a unique solution that is bounded by*

$$\|u\|_V \leq \frac{1}{\alpha_1}\|f\|_{V'} + \frac{2\|a\|}{\alpha_1\beta}\|g\|_{Q'}, \tag{4.2.33}$$

$$\|p\|_Q \leq \frac{2\|a\|}{\alpha_1\beta}\|f\|_{V'}F + \frac{2\|a\|^2}{\alpha_1\beta^2}\|g\|_{Q'}. \tag{4.2.34}$$

If, moreover, $a(\cdot,\cdot)$ is symmetric and satisfies

$$a(v,v) \geq 0 \qquad \forall\, v \in V, \tag{4.2.35}$$

then we have the improved estimates

$$\|u\|_V \leq \frac{1}{\alpha_0}\|f\|_{V'} + \frac{2\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|g\|_{Q'}, \tag{4.2.36}$$

$$\|p\|_Q \leq \frac{2\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|f\|_{V'} + \frac{\|a\|}{\beta^2}\|g\|_{Q'}, \tag{4.2.37}$$

where α_0 is the constant appearing in (4.2.12).

The proof is identical to that given in the previous chapter for Theorems 3.4.1 and 3.4.2. This is indeed the gift of the Closed Range Theorem, which allows us to extend all the instruments that were used in finite dimension to the more general case of Hilbert spaces.

Remark 4.2.5. As we did in Theorem 3.5.2, we could restate Theorem 4.2.2 in terms of necessary and sufficient conditions. In the present context, this means that if the bounds (4.2.33) and (4.2.34) hold for all right-hand sides f and g, then (4.2.27) and (4.2.26) hold. Indeed, for an arbitrary $u_0 \in K$, let us define $f_0 \in K'$ by

$$\langle f_0, v_0\rangle = a(u_0, v_0) \quad \forall\, v_0 \in K. \tag{4.2.38}$$

We then use the prolongation $E_{K'} f_0$ of f_0 to V', as in (4.1.71), and we take $g = 0$. We now have that $(u_0, 0)$ is solution of (4.2.6) with $f = f_0$, and by (4.2.33) we have

$$\|u\|_V \leq \frac{1}{\alpha_1}\|f_0\|_{V'} = \frac{1}{\alpha_1}\sup_{w_0\in K}\frac{a(u_0,w_0)}{\|w_0\|}. \tag{4.2.39}$$

Similarly, taking $\langle f_p, v\rangle = b(v,p)$ and again $g = 0$, we have that $(0,p)$ is solution of (4.2.6) with $f = f_p$, and (4.2.34) implies (4.2.26). All this can be seen as the natural extension of Lemma 3.5.2 to the infinite dimensional case. □

If the bilinear form $a(\cdot,\cdot)$ is coercive on the whole space, we have immediately the following corollary (particularly useful for Stokes addicts that do not even want to know what a kernel is).

Corollary 4.2.1. *Let the assumptions $\mathcal{AB}$ hold. Suppose that there exist two positive constants α and β such that the inf-sup condition* (4.2.26) *on $b(\cdot,\cdot)$, and the global coercivity condition* (4.1.67) *on $a(\cdot,\cdot)$ are satisfied. Then, for every $f \in V'$ and for every $g \in Q'$, problem* (4.2.6) *has a unique solution that is bounded by*

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \frac{2\|a\|}{\alpha\beta}\|g\|_{Q'}, \tag{4.2.40}$$

$$\|p\|_Q \leq \frac{2\|a\|}{\alpha\beta}\|f\|_{V'} + \frac{2\|a\|^2}{\alpha\beta^2}\|g\|_{Q'}. \tag{4.2.41}$$

If, moreover, $a(\cdot,\cdot)$ *is symmetric, then we have the improved estimates*

$$\|u\|_V \le \frac{1}{\alpha}\|f\|_{V'} + \frac{2\|a\|^{1/2}}{\alpha^{1/2}\beta}\|g\|_{Q'}, \tag{4.2.42}$$

$$\|p\|_Q \le \frac{2\|a\|^{1/2}}{\alpha^{1/2}\beta}\|f\|_{V'} + \frac{\|a\|}{\beta^2}\|g\|_{Q'}. \tag{4.2.43}$$

4.2.4 The Case of Im$B \neq Q'$

We now want to discuss briefly the case in which the *inf-sup* condition on the bilinear form b does not hold.

Essentially, if ImB does not coincide with Q', we can distinguish two cases. Either ImB is closed in Q', or it is not (everybody surely agrees with that).

If ImB is not closed, then we are in *deep trouble*. Generally speaking, *we should better look for a different formulation*.

If instead the image of B is closed, we survive rather easily. Let us analyse the situation. We first observe that in this case $H = \text{Ker}B^t$ will be a closed subspace of Q that is *not* reduced to $\{0\}$. In this case, it is clear that problem (4.2.7) *cannot* have a unique solution for every $f \in V'$ and for every $g \in Q'$. To start with, if Im$B \neq Q'$, and if $g \in Q'$ does not belong to ImB, we cannot have a solution. Hence, the *existence* of the solution will *not always* hold. Moreover, if by chance we have $g \in \text{Im}B$ and we have a solution (u, p), then for every $p^* \in H$ with $p^* \neq 0$, we easily have that $(u, p + p^*)$ is another, different solution. Hence, the *uniqueness* of the solution will *never* hold. Apparently, we are not so well off.

However, if we have $g \in \text{Im}B$, then there is an easy way out. Indeed, we observe first that if A_{KK} is non-singular, we could proceed as we did in the finite dimensional case (see Proposition 3.2.1) and deduce that we still have at least one solution, whose first component is unique and whose second component is unique only up to an element of H. Moreover, we could note that $b(v, q) = 0$ for every $q \in H$. Hence, following what has been done in Remark 3.2.4 (for the finite-dimensional case), we can consider the restriction $\tilde{b}$ of b to $V \times H^\perp$ without loosing any information. However, this time, $\tilde{B}$ will be surjective from V to $(H^\perp)'$. Indeed, using (4.1.70) we have that $(H^\perp)' = H^0$ $(= (\text{Ker}B^t)^0)$. On the other hand, from the Closed Range Theorem 4.1.5, we have $(\text{Ker}B^t)^0 = \text{Im}B$, and joining the two we get $(H^\perp)' = \text{Im}B$ and everything works.

Hence, *the theory developed so far in the case of B surjective applies to the case where* ImB *is closed and* $g \in \text{Im}B$, *by just replacing* Q *with* $H^\perp$.

An alternative path (whose difference from the one above is mainly psychological) consists in replacing Q with the *quotient space*

$$\tilde{Q} := Q_{/H}. \tag{4.2.44}$$

We recall that the elements of $\tilde{Q}$ are subsets of elements of Q that differ from each other by an element of H. As we have seen in Sect. 4.1.9, $\tilde{Q}$ can be identified to $H^{\perp}$. Hence, as we said, the difference between using $V \times H^{\perp}$ and using $V \times Q_{/H}$ is mainly psychological. Nevertheless, some people seem to be in love with this second option and dislike the first. Just for them, we re-state one of our previous results in terms of the original space Q and the original bilinear form b in the following theorem, which is just Theorem 4.2.3 applied to $V \times Q_{/H}$, and stated in terms of V and Q.

Theorem 4.2.4. *Together with Assumption $\mathcal{AB}$, assume that ImB is closed and that the double-inf-sup condition (4.2.27) is satisfied. Then, for every $f \in V'$ and for every $g \in ImB$, problem (4.2.6) has a solution (u, p) where u is uniquely determined, and p is determined up to an element of H. Moreover, setting*

$$\tilde{\beta} := \inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \, \|q\|_{Q/H}} \tag{4.2.45}$$

we have

$$\|u\|_V \leq \frac{1}{\alpha_1} \|f\|_{V'} + \frac{2\|a\|}{\alpha_1 \tilde{\beta}} \|g\|_{Q'}, \tag{4.2.46}$$

$$\|p\|_{Q/H} \leq \frac{2\|a\|}{\alpha_1 \tilde{\beta}} \|f\|_{V'} + \frac{2\|a\|^2}{\alpha_1 \tilde{\beta}^2} \|g\|_{Q'}. \tag{4.2.47}$$

If, moreover, $a(\cdot,\cdot)$ is symmetric and satisfies

$$a(v, v) \geq 0 \qquad \forall\, v \in V, \tag{4.2.48}$$

then we have the improved estimates

$$\|u\|_V \leq \frac{1}{\alpha_0} \|f\|_{V'} + \frac{2\|a\|^{1/2}}{\alpha_0^{1/2} \tilde{\beta}} \|g\|_{Q'}, \tag{4.2.49}$$

$$\|p\|_{Q/H} \leq \frac{2\|a\|^{1/2}}{\alpha_0^{1/2} \tilde{\beta}} \|f\|_{V'} + \frac{\|a\|}{\tilde{\beta}^2} \|g\|_{Q'}, \tag{4.2.50}$$

where again α_0 is the constant appearing in (4.2.12). □

Remark 4.2.6. We point out that the estimates (4.2.46) and (4.2.47), valid for every $f \in V'$ and for every $g \in \mathrm{Im}B$, imply in particular that, under the assumptions of Theorem 4.2.4, the image of the operator $\mathbb{M} : (u, p) \to (Au + B^t p, Bu)$ from $V \times Q$ to $V' \times Q'$ *is also closed.* □

Remark 4.2.7. Another type of generalisation was considered in [312] and [68]. They consider a problem of type (4.3.1) but employing two bilinear forms $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ on $V \times Q$, that is,

$$\begin{cases} a(u,v) + b_1(v,p) = \langle f,v\rangle_{V'\times V}, \ \forall\, v \in V \\ b_2(u,q) = \langle g,q\rangle_{Q'\times Q}, \ \forall\, q \in Q. \end{cases} \tag{4.2.51}$$

Conditions for existence of a solution are now that both $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ should satisfy an inf-sup condition of the type (4.2.45), and $a(u,v)$ should satisfy an invertibility condition from $\mathrm{Ker}B_2$ on $(\mathrm{Ker}B_1)'$, that is,

$$\inf_{u_0\in \mathrm{Ker}B_1} \sup_{v_0\in \mathrm{Ker}B_2} \frac{a(u_0,v_0)}{\|u_0\|\ \|v_0\|} \geq \alpha_1, \tag{4.2.52}$$

$$\inf_{v_0\in \mathrm{Ker}B_1} \sup_{u_0\in \mathrm{Ker}B_2} \frac{a(u_0,v_0)}{\|u_0\|\ \|v_0\|} \geq \alpha_1. \tag{4.2.53}$$

This condition is in general rather hard to check, and the ellipticity on the whole space V, when applicable, can bring a considerable relief.

For more details, we refer to [68]. □

Remark 4.2.8 (Special cases $(\mathbf{f},0)$ *and* $(0,\mathbf{g})$*).* We have considered these special cases in the Sect. 3.5.3 in the finite dimensional framework. In these cases, it is possible to obtain existence and stability results under weakened assumptions. We shall not make them explicit here. However, we refer to the proofs of Theorem 3.4.1 and the following ones in Sect. 3.4 where detailed proofs of related situations are presented. We just want to point out here that in the case $(\mathbf{f},0)$, the a priori estimates (e.g. (4.2.46)) on u do not depend on the inf-sup constant of B. Conversely, in the case $(0,\mathbf{g})$, for $a(\cdot,\cdot)$ symmetric and positive semi-definite, the estimates on p (e.g. (4.2.50)) do not depend on the constant α_0. □

4.2.5 Examples

To fix ideas, we shall apply the results just obtained to some of the examples introduced in Chap. 1.

Example 4.2.1 (Mixed formulation of the Poisson problem). We consider here the case of Example 1.3.5. Given f in $L^2(\Omega)$, we look for $\underline{u} \in H(\mathrm{div};\Omega) =:$ V *and* $p \in L^2(\Omega) =: Q$ such that:

$$\begin{cases} \displaystyle\int_\Omega \underline{u}\cdot\underline{v}\,dx + \int_\Omega p\ \mathrm{div}\ \underline{v}\,dx = 0,\ \forall\,\underline{v} \in H(\mathrm{div};\Omega), \\ \displaystyle\int_\Omega (\mathrm{div}\ \underline{u} + f)q\,dx = 0,\ \forall\, q \in L^2(\Omega). \end{cases} \tag{4.2.54}$$

Here we have

$$b(\underline{v}, q) = \int_{\Omega} \operatorname{div} \underline{v}\, q\, dx, \tag{4.2.55}$$

and B is the divergence operator from $H(\operatorname{div}; \Omega)$ into $L^2(\Omega)$. It is not difficult to check that it is surjective: for instance, for every $g \in L^2(\Omega)$, consider the auxiliary problem: *find* $\psi \in H_0^1(\Omega)$ *such that* $\Delta\psi = g$. Its (traditional) variational formulation is

$$\int_{\Omega} \underline{\operatorname{grad}}\, \psi \cdot \underline{\operatorname{grad}}\, \phi\, dx = - \int_{\Omega} g\, \phi\, dx \qquad \forall\, \phi \in H_0^1(\Omega), \tag{4.2.56}$$

and it has a unique solution thanks to the Lax-Milgram lemma. Then, take $\underline{v}_g := \underline{\operatorname{grad}}\, \psi$ and you immediately have $\underline{v}_g \in H(\operatorname{div}; \Omega)$ and $\operatorname{div} \underline{v}_g = g$ as wanted. The kernel of B is made of the vectors $\underline{v}_0 \in H(\operatorname{div}; \Omega)$ such that $\operatorname{div} \underline{v}_0 = 0$. The bilinear form a is given by

$$a(\underline{u}, \underline{v}) = \int_{\Omega} \underline{u} \cdot \underline{v}\, dx, \tag{4.2.57}$$

while we remember that in (1.3.44) the norm in $H(\operatorname{div}; \Omega)$ was defined as

$$\|\underline{v}\|^2_{H(\operatorname{div};\, \Omega)} := \|\underline{v}\|^2_{(L^2(\Omega))^2} + \|\operatorname{div} \underline{v}\|^2_{L^2(\Omega)}. \tag{4.2.58}$$

Hence, a is coercive on Ker$B$ (although it is *not coercive* on $H(\operatorname{div}; \Omega)$). Our abstract theory (in particular Theorem 4.2.1) applies immediately, and we have existence and uniqueness of the solution. □

Example 4.2.2 (The Stokes problem). Let us go back to Example 1.3.1. We take $V := (H_0^1(\Omega))^2$, $Q := L^2(\Omega)$, and, given $\underline{f} \in V'$, we look for $(\underline{u}, p) \in V \times Q$, solution of

$$\begin{cases} 2\mu \displaystyle\int_{\Omega} \underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{v})\, dx - \int_{\Omega} p\ \operatorname{div} \underline{v}\, dx = \int_{\Omega} \underline{v} \cdot \underline{f}\, dx,\ \forall\, v \in V, \\ \displaystyle\int_{\Omega} q\ \operatorname{div} \underline{u}\, dx = 0,\ \forall\, q \in Q. \end{cases} \tag{4.2.59}$$

Here, we have $g = 0$. Moreover, the bilinear form $a(u, v) = 2\mu \int_{\Omega} \underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{v})\, dx$ is coercive on V, due to the *Korn inequality* [183, 362]

$$\exists\, \kappa = \kappa(\Omega) > 0 \text{ s.t. } \|\underline{\underline{\varepsilon}}(\underline{v})\|^2_{(L^2(\Omega))^4} \geq \kappa |\underline{v}|^2_{1,\Omega} \qquad \forall \underline{v} \in (H_0^1(\Omega))^2. \tag{4.2.60}$$

On the other hand, we have

$$b(\underline{v}, q) = - \int_{\Omega} q\ \operatorname{div} \underline{v}\, dx \tag{4.2.61}$$

and B is the divergence operator from $(H_0^1(\Omega))^2$ into $L^2(\Omega)$. This time, the study of its image is much harder than in the previous example. Due to a non-trivial result by O. Ladyzhenskaya, we have [272, 362] that

$$\mathrm{Im}B = L_0^2(\Omega) = \{q \mid q \in L^2(\Omega), \int_\Omega q\,dx = 0\} \tag{4.2.62}$$

and that this subspace of $L^2(\Omega)$ is closed and has co-dimension one. In agreement with the Closed Range Theorem, $\mathrm{Ker}B^t$ has also dimension 1:

$$\mathrm{Ker}B^t = \mathrm{Ker}(-\underline{\mathrm{grad}}) = \{q \mid q \text{ is constant on } \Omega\}. \tag{4.2.63}$$

We are therefore in the case where B^t is not injective. As we did in the last subsection (see Sect. 4.2.4), we can easily survive by considering $\tilde{Q}$, defined as in (4.2.44), instead of Q. However, in this case, the space $\tilde{Q}$ (that is the space of *classes* of functions in $L^2(\Omega)$ that differ from each other by an additive constant) is often identified with the space $H^\perp$ of functions in $L^2(\Omega)$ having zero mean value, as discussed in Example 4.1.14. Actually, in practice, we simply take

$$Q := \left\{q \middle| q \in L^2(\Omega), \int_\Omega q\,dx = 0\right\} \equiv L_0^2(\Omega) \tag{4.2.64}$$

and we can apply directly Theorem 4.2.1, which will give the existence and uniqueness of the velocity $\underline{u}$, together with the existence of a pressure p that is unique up to an additive constant.

The example of Stokes' problem is paradigmatic of the typical escape that is usually performed when $\mathrm{Im}B$ is closed but different from Q'. □

Example 4.2.3 (Domain decomposition for the Poisson problem). Referring to Example 1.4.2, we have to solve the following problem: find $(p, \underline{u})$ with $p \in X(\Omega) =: V$, $\underline{u} \in H(\mathrm{div};\Omega) =: Q$, solution of

$$\begin{cases} \displaystyle\int_{K_i} \underline{\mathrm{grad}}\, p_i \cdot \underline{\mathrm{grad}}\, q_i \,dx - \int_{\partial K_i} \underline{u}\cdot\underline{n}_i\, q_i\,d\sigma = \int_{K_i} f\, q_i\,dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, q_i \in H^1(K_i),\ \forall\, K_i, \\ \displaystyle\sum_i \int_{\partial K_i} \underline{v}\cdot\underline{n}_i\, p_i\,d\sigma = 0, \forall \underline{v} \in H(\mathrm{div};\Omega). \end{cases} \tag{4.2.65}$$

We thus have $b(q,\underline{v}) = -\sum_i \int_{\partial K_i} \underline{v}\cdot\underline{n}_i\, q_i\,d\sigma$, and the operator B, roughly speaking, associates to $q \in X(\Omega)$ its "DG jumps" $q_i\underline{n}_i + q_j\underline{n}_j$ on the interfaces $e_{ij} = \partial K_i \cap \partial K_j$. The kernel of B is nothing but $H_0^1(\Omega)$ and the problem corresponding to (4.2.65) is the standard Poisson problem. To prove the existence of u, we shall have to prove that $\mathrm{Im}B$ is closed in $(H(\mathrm{div};\Omega))'$ and we shall have to characterise $\mathrm{Ker}B^t$. This will be done in Chap. 7. □

We shall of course come back to these problems when studying more precisely mixed and hybrid methods. Checking the closedness of ImB, even if existence proofs can be obtained through other considerations, is a crucial step ensuring that we have a well-posed problem and that we are working with the right functional spaces. This last fact is essential to obtain "natural" error estimates.

We end this subsection with a few rather academic examples, just in order to see formulations that *do not* work (or present some sort of difficulty) and understand why.

We shall consider the problem (*very loosely* related to plate bending problems, as in Example 1.2.4, or to the Stokes problem in the so-called *streamline-vorticity* formulation):

$$\Delta^2\psi = f \qquad \text{in } \Omega \tag{4.2.66}$$

on a reasonably smooth domain Ω (for instance, a convex polygon). We introduce $\omega := -\Delta\psi$, and we are going to consider *various boundary conditions, and different possible mixed formulations*.

- We start with the easiest choice of boundary conditions, that is

$$\psi = \omega = 0 \quad \text{on } \Gamma. \tag{4.2.67}$$

In this case, we can set $V \equiv Q := H^1_0(\Omega)$, and consider the formulation

$$\begin{cases} \displaystyle\int_\Omega \omega\,\mu\,dx - \int_\Omega \underline{\text{grad}}\,\mu\ \underline{\text{grad}}\,\psi\,dx = 0, & \forall\,\mu\in V,\\ \displaystyle -\int_\Omega \underline{\text{grad}}\,\omega\ \underline{\text{grad}}\,\varphi\,dx = -\langle f,\varphi\rangle, & \forall\,\varphi\in Q. \end{cases} \tag{4.2.68}$$

In this case, both the operators B and B^t coincide with the Laplace operator $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$, which is an isomorphism. In particular, Im$B = Q'$ and Ker$B = \{0\}$, so that the ellipticity in the kernel (4.2.12) is also trivially satisfied. All is well and good. However, one could object that, with the boundary conditions (4.2.67), we are *almost cheating*. Indeed, the problem is equivalent to the cascade of sub-problems: $-\Delta\omega = f$ and $-\Delta\psi = \omega$ which are both well posed if we look for $\omega \in H^1_0$ and $\psi \in H^1_0$.
- We now consider the "clamped plate" boundary conditions

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma. \tag{4.2.69}$$

Setting $V := L^2(\Omega)$ and $Q := H^2_0(\Omega)$, it is immediate to see that (ω,ψ) satisfies the equations

$$\begin{cases} \displaystyle\int_\Omega \omega\,\mu\,dx + \int_\Omega \mu\,\Delta\psi\,dx = 0, & \forall\,\mu\in V,\\ \displaystyle\int_\Omega \omega\,\Delta\varphi\,dx = -\langle f,\varphi\rangle, & \forall\,\varphi\in Q. \end{cases} \tag{4.2.70}$$

Here, the operator B^t is just the Laplace operator from $H^2_0(\Omega)$ to $L^2(\Omega)$, and it is clearly injective and bounding, since $\|\varphi\|_{2,\Omega} \le C\,\|\Delta\varphi\|_{0,\Omega}$ for some constant C. Hence, the image of B coincides with Q'. The kernel of B is a little more sophisticated. With some work, we discover that it can be characterised as

$$\mathrm{Ker}B = (L^2_{harm})^{\perp}, \tag{4.2.71}$$

where L^2_{harm} is the (closed) subspace of $L^2(\Omega)$ made of harmonic functions (that is, functions w such that $\Delta w = 0$ in the distributional sense). Indeed, for such functions, it is not difficult to see that $(w, \Delta\varphi) = 0$ for all $\varphi \in H^2_0(\Omega)$. In any case, we don't care too much about $\mathrm{Ker}B$, since the bilinear form a is coercive on the whole V. Our theory applies, and we are happy.

- Still with the "clamped plate" boundary conditions (4.2.69), if we are not willing to use spaces involving two derivatives (as $H^2_0(\Omega)$), we could take $V := H^1(\Omega)$ and $Q := H^1_0(\Omega)$. It is not difficult to see that (ω, ψ) solves

$$\begin{cases} \displaystyle\int_\Omega \omega\,\mu\,dx - \int_\Omega \underline{\mathrm{grad}}\,\mu\ \underline{\mathrm{grad}}\,\psi\,dx = 0, & \forall\,\mu \in V,\\ \displaystyle -\int_\Omega \underline{\mathrm{grad}}\,\omega\ \underline{\mathrm{grad}}\,\varphi\,dx = -\langle f, \varphi\rangle, & \forall\,\varphi \in Q. \end{cases} \tag{4.2.72}$$

This time, B is the Laplace operator from $H^1(\Omega)$ to $H^{-1}(\Omega)$ (dual space of H^1_0), which is clearly surjective. However, its kernel is made of the harmonic functions in $H^1(\Omega)$ and the bilinear form a (which in this case is just the L^2-inner product) cannot be coercive (in $H^1(\Omega)$), not even if you restrict it to the harmonic functions. If you are not convinced of that, consider in $\Omega :=]0,\pi[\times]0,1[$ the sequence of functions

$$\phi_k := \sin(kx)\,e^{ky}.$$

Clearly, $\Delta\phi_k = 0$ for all k. However, a simple computation shows that

$$\|\,\underline{\mathrm{grad}}\,\phi_k\|^2_{0,\Omega} = 2k^2\|\phi_k\|^2_{0,\Omega}$$

and you cannot bound $\|\phi_k\|^2_V$ (that is $\|\,\underline{\mathrm{grad}}\,\phi_k\|^2_{0,\Omega}$) with $a(\phi_k, \phi_k)$ (that is $\|\phi_k\|^2_{0,\Omega}$) uniformly in k. Hence, formulation (4.2.72) is not really healthy, as the ellipticity in the kernel fails. Indeed, if for instance the domain Ω is not convex, you are likely to have a problem without existence, as ψ usually will not be in $H^3(\Omega)$ and therefore ω might not be in $H^1(\Omega)$. We shall see in the following chapters that methods based on this formulation might exhibit a suboptimal rate of convergence.

- We now consider the boundary conditions

$$\frac{\partial\psi}{\partial n} = \frac{\partial\omega}{\partial n} = 0 \quad \text{on } \Gamma. \tag{4.2.73}$$

Here, we can take $V := L^2(\Omega)$ and $Q := {}_0H^2(\Omega)$ defined as

$${}_0H^2(\Omega) := \{\varphi \,|\, \varphi \in H^2(\Omega),\ \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma\}$$

and use the formulation (4.2.70). We see that B^t is again the Laplace operator, but this time ${}_0H^2(\Omega) \to (L^2(\Omega))' \equiv L^2(\Omega)$. If, for instance, the domain Ω is convex, then $H := \mathrm{Ker}B^t$ is the space of constants, and $\mathrm{Im}B^t$ will be the subset of $V' \equiv V = L^2(\Omega)$ made of functions with zero mean value. The kernel of B will also be the space of constants, and the image of B will be the polar set of $\mathrm{Ker}B^t$, made of those functionals that vanish on constants. We are in a situation similar to that faced for the Stokes problem in Example 4.2.2. Here, we can adjust everything by redefining Q as the subset of ${}_0H^2(\Omega)$ made of functions with zero mean value (that is, $Q = H^{\perp}$). The compatibility condition $\langle f, c\rangle = 0$ for every constant c will still have to be required, in order to have $f \in Q'$ (now $= (H^{\perp})'$). Doing that, we have that a is elliptic on V and $\mathrm{Im}B = Q'$ (the new Q', of course), and everything will work.

- It is time to see a *really weird* case. Consider, to fix the ideas, the case of $\Omega :=]0, \pi[\times]0, 1[$, and split its boundary into the bottom part $\Gamma_b :=]0, \pi[\times\{0\}$, the top part $\Gamma_t :=]0, \pi[\times\{1\}$, and the lateral part $\Gamma_\ell := \partial\Omega \setminus (\Gamma_b \cup \Gamma_t)$. Consider now the boundary conditions

$$\psi = \frac{\partial \psi}{\partial n} = 0 \ \text{ on } \Gamma_b; \quad \omega = \frac{\partial \omega}{\partial n} = 0 \ \text{ on } \Gamma_t; \quad \frac{\partial \psi}{\partial n} = \frac{\partial \omega}{\partial n} = 0 \ \text{ on } \Gamma_\ell \tag{4.2.74}$$

and the spaces $V := L^2$ and $Q := \tilde{H}^2$ defined as

$$\tilde{H}^2 := \{\varphi \,|\, \varphi \in H^2(\Omega),\ \varphi = \frac{\partial \varphi}{\partial n} = 0 \ \text{ on } \Gamma_b \text{ and } \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_\ell\}.$$

It is clear that, if you have a solution (ω, ψ) of the problem, then it will satisfy

$$\begin{cases} \displaystyle\int_\Omega \omega\, \mu\, dx + \int_\Omega \mu\, \Delta\psi\, dx = 0, & \forall\, \mu \in V, \\ \displaystyle\int_\Omega \omega\, \Delta\varphi\, dx = -\langle f, \varphi\rangle, & \forall\, \varphi \in Q. \end{cases} \tag{4.2.75}$$

This time, B^t will be the Laplace operator from $\tilde{H}^2$ to $L^2(\Omega)$. A non-trivial result of complex analysis (Cauchy-Kovalewskaya Theorem) ensures that $\mathrm{Ker}B^t = \{0\}$ (the boundary conditions at the bottom are enough to give you that). However, we can check that B^t *is not bounding*. To see that, consider the sequence

$$\phi_k := \frac{1}{k^2}\cos(kx)(1 - \cosh(ky)).$$

It is clear that $\phi_k \in \tilde{H}^2$ for all k. A simple computation shows that $-\Delta\phi_k = \cos(kx)$, so that

$$\|\Delta\phi_k\|^2_{L^2(\Omega)} = \frac{\pi}{2}.$$

On the other hand, it is also simple to check that

$$\|\phi_k\|^2_{L^2(\Omega)} \simeq \frac{e^{2k}}{k^4}$$

goes to $+\infty$ for $k \to +\infty$, so that a uniform bound (in k) of the form

$$\|\Delta\varphi\|_{L^2(\Omega)} \geq \beta\|\varphi\|_{\tilde{H}^2} \qquad \forall\varphi \in \tilde{H}^2$$

is hopeless. Hence, $\mathrm{Im}B^t$ *is not closed* and therefore $\mathrm{Im}B$ is not closed either. Indeed, the problem is severely ill posed, and you cannot solve it in practice unless you add some sort of regularisation.

4.3 Existence and Uniqueness for Perturbed Problems

Some applications, in particular nearly incompressible materials (Sect. 8.13), will require a more general formulation than Problem (4.2.6). Although the first generalisation introduced will appear to be simple, we shall see that its analysis is rather more intricate.

4.3.1 Regular Perturbations

We assume that we are also given a continuous bilinear form $c(\cdot\,,\cdot)$ on $Q \times Q$, and we denote by C its associated operator $Q \to Q'$.

We now consider the following extension of problem (4.2.6): *given* $f \in V'$ *and* $g \in Q'$, *find* $u \in V$ *and* $p \in Q$ *such that*

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle_{V'\times V}, & \forall\, v \in V, \\ b(u,q) - c(p,q) = \langle g, q\rangle_{Q'\times Q}. & \forall\, q \in Q. \end{cases} \tag{4.3.1}$$

Remark 4.3.1. Whenever $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric, this problem corresponds to the saddle point problem

$$\inf_{v\in V}\sup_{q\in Q} \frac{1}{2}a(v,v) + b(v,q) - \frac{1}{2}c(q,q) - \langle f, v\rangle + \langle g, q\rangle$$

and it is no longer equivalent to a minimisation problem on u. □

Remark 4.3.2. As in Remark 4.2.2, the two equations in (4.3.1) can sometimes be written as a unique variational equation, setting

$$\mathcal{A}((u,p),(v,q)) = a(u,v) + b(v,p) - b(u,q) + c(p,q) \qquad \forall (u,p),\ (v,q) \in V \times Q \tag{4.3.2}$$

and then requiring again that

$$\mathcal{A}((u,p),(v,q)) = \langle f, v\rangle_{V'\times V} - \langle g, q\rangle_{Q'\times Q} \qquad \forall (v,q) \in V \times Q. \tag{4.3.3}$$

□

We now want to look for conditions on a, b and c ensuring the existence and uniqueness of a solution to (4.3.1), together with the proper stability bounds.

Let us first consider a special case. We assume that $c(\cdot,\cdot)$ is coercive on Q, that is

$$\exists\, \gamma > 0 \text{ such that } c(q,q) \geq \gamma\ \|q\|_Q^2,\ \forall\, q \in Q \tag{4.3.4}$$

and that $a(\cdot,\cdot)$ is also coercive on V:

$$\exists\, \alpha > 0 \text{ such that } a(v,v) \geq \alpha\ \|v\|_V^2,\ \forall\, v \in V. \tag{4.3.5}$$

Then, we have the following proposition.

Proposition 4.3.1. *Together with Assumption $\mathcal{AB}$, assume that* (4.3.4) *and* (4.3.5) *hold. Then, for every $f \in V'$ and $g \in Q'$, problem* (4.3.1) *has a unique solution* (u,p). *Moreover, we have:*

$$\frac{\alpha}{2}\|u\|_V^2 + \frac{\gamma}{2}\|p\|^2 \leq \frac{1}{2\alpha}\|f\|_{V'}^2 + \frac{1}{2\gamma}\|g\|_{Q'}^2. \tag{4.3.6}$$

Proof. The proof is elementary (using, for instance, Lax-Milgram Lemma (4.1.6) on the bilinear form (4.3.2)). □

The estimate (4.3.6) is unsatisfactory. Actually, in many applications, we will deal with a bilinear form $c(\cdot,\cdot)$ defined by

$$c(p,q) = \lambda(p,q)_Q,\ \lambda \geq 0, \tag{4.3.7}$$

and we would like to get estimates that provide uniform bounds on the solution for λ small (say $0 \leq \lambda \leq 1$). Clearly, if $c(\cdot,\cdot)$ has the form (4.3.7), one has $\gamma = \lambda$ in (4.3.4) and the bound (4.3.6) explodes for vanishing λ. This fact has practical implications, as we shall see, on the numerical approximations of some problems, for instance when dealing with nearly incompressible materials. On the other hand, Proposition 4.3.1 makes no assumptions on $b(\cdot,\cdot)$ (except the usual (4.2.4)) and it is then quite natural for the choice $c \equiv 0$ to be forbidden.

It is then natural to start assuming, as we did in Sect. 3.6 of the previous chapter, that the corresponding *unperturbed* problem (corresponding to the case $c \equiv 0$) is well posed, and try to find sufficient conditions on c that ensure the well-posedness of the perturbed problem.

We shall start with the simplest case, generalising Proposition 3.3.1 in an obvious manner.

Proposition 4.3.2. *Together with Assumption $\mathcal{AB}$, assume that $A_{KK'}$ is an isomorphism from K to K' and that $\mathrm{Im}B = Q'$. Then, there exists an $\varepsilon_0 > 0$ such that, for every ε with $|\varepsilon| \le \varepsilon_0$, condition $\|c\| \le \varepsilon$ implies that problem (4.3.1) has a unique solution for every $f \in V'$ and for every $g \in Q'$.* □

As for Proposition 4.3.1, the proof is immediate, this time using the Kato Theorem 4.1.3.

The result of Proposition 4.3.2 is also unsatisfactory. For once, it does give us a result only for ε small enough. Besides, ε_0 will be very difficult to compute in practice. Without it, we basically never know, in every particular case, whether we are solving a well posed problem or not, which is clearly a quite unhappy situation.

We therefore have to look for better results. We could start, as in the previous subsection, by assuming that $\mathrm{Im}B = Q'$, and then try to adapt the results to the case in which $\mathrm{Im}B$ is closed but not equal to Q'. We remark, however, that, this time, the passage from the case when $\mathrm{Im}B = Q'$ (when $H = \{0\}$) and the case when $\mathrm{Im}B$ is simply closed is no longer so simple, as the bilinear form c could mix together the components of p in H and in $H^{\perp}$. Therefore, it is better to look directly at the case where we simply have $\mathrm{Im}B$ closed. On the other hand, we have already seen in the previous chapter that assuming symmetry of *both* a and c gives much better stability bounds. Hence, we decide to concentrate on that case. This is particularly reasonable since, in most applications, the symmetry assumptions are satisfied.

Therefore, to start with, we enlarge our Assumption $\mathcal{AB}$ to include the additional bilinear form c and the additional properties that we are going to use throughout this subsection.

Assumption $\mathcal{ABC}$: *Together with Assumption $\mathcal{AB}$, we assume that we are given a continuous bilinear form $c(\cdot,\cdot)$ on $Q \times Q$, and we denote by C its associated operator $Q \to Q'$. We assume, moreover, that $\mathrm{Im}B$ is closed, and that both $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric and positive semi-definite:*

$$a(v,v) \ge 0, \ \forall\, v \in V \qquad c(q,q) \ge 0, \ \forall\, q \in Q. \tag{4.3.8}$$

We now introduce some additional notation, and a few related properties that hold when a and c are symmetric and positive semi-definite, and $\mathrm{Im}B$ is closed.

We define the semi-norms

$$|v|_a^2 := a(v,v) \qquad |q|_c^2 := c(q,q), \tag{4.3.9}$$

and we note that, thanks to the continuity of a and c,

$$|v|_a^2 \leq \|a\|\,\|v\|_V^2 \;\; \forall\, v \in V \quad \text{and} \quad |q|_c^2 \leq \|c\|\,\|q\|_Q^2 \;\; \forall\, q \in Q. \tag{4.3.10}$$

We also note that, from (4.2.29), we have

$$a(u,v) \leq |u|_a\,|v|_a \quad \text{and} \quad c(p,q) \leq |p|_c\,|q|_c, \tag{4.3.11}$$

and from (4.2.30),

$$\|Au\|^2 \leq \|a\|\,|u|_a^2 \quad \text{and} \quad \|Cp\|^2 \leq \|c\|\,|p|_c^2. \tag{4.3.12}$$

Setting again $K = \mathrm{Ker}B$ and $H = \mathrm{Ker}B^t$ as in (4.1.66), we can split each $v \in V$ and each $q \in Q$ as

$$v = v_0 + \overline{v} \qquad q = q_0 + \overline{q}, \tag{4.3.13}$$

with $v_0 \in K$, $\overline{v} \in K^{\perp}$, $q_0 \in H$, and $\overline{q} \in H^{\perp}$, and we note that

$$b(v,q) = b(\overline{v},q) = b(\overline{v},\overline{q}) = b(v,\overline{q}). \tag{4.3.14}$$

In a similar way, we can split each $f \in V'$ and each $g \in Q'$ as

$$f = f_0 + \overline{f} \qquad g = g_0 + \overline{g} \tag{4.3.15}$$

with $f_0 \in K'$, $\overline{f} \in (K^{\perp})' \equiv K^0$, $g_0 \in H'$ and $\overline{g} \in (H^{\perp})' \equiv H^0$, and we note that

$$\langle f, v\rangle = \langle f_0, v_0\rangle + \langle \overline{f}, \overline{v}\rangle \quad \langle g, q\rangle = \langle g_0, q_0\rangle + \langle \overline{g}, \overline{q}\rangle \tag{4.3.16}$$

with obvious meaning of the duality pairings.

We therefore have the following result, in which the roles of a and c are perfectly interchangeable.

Theorem 4.3.1. *Together with Assumption $\mathcal{ABC}$, assume that $a(\cdot\,,\cdot)$ is coercive on K and $c(\cdot\,,\cdot)$ is coercive on H. Let therefore α_0, β, and γ_0 be positive constants such that*

$$\alpha_0\|v_0\|_V^2 \leq a(v_0,v_0) \qquad \forall\, v_0 \in K, \tag{4.3.17}$$

$$\inf_{q\in H^{\perp}} \sup_{v\in V} \frac{b(v,q)}{\|q\|_Q\,\|v\|_V} = \inf_{v\in K^{\perp}} \sup_{q\in Q} \frac{b(v,q)}{\|q\|_Q\,\|v\|_V} = \beta > 0, \tag{4.3.18}$$

$$\gamma_0\|q_0\|_Q^2 \leq c(q_0,q_0) \qquad \forall\, q_0 \in H. \tag{4.3.19}$$

Then, for every $f \in V'$ and $g \in Q'$, we have that the problem

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle_{V'\times V}, & \forall\, v \in V, \\ b(u,q) - c(p,q) = \langle g, q\rangle_{Q'\times Q}, & \forall\, q \in Q \end{cases} \tag{4.3.20}$$

has a unique solution, that moreover satisfies

$$\|u\|_V + \|p\|_Q \le C\Big(\|f\|_{V'} + \|g\|_{Q'}\Big) \qquad (4.3.21)$$

with C constant depending only on the stability constants α_0, β, γ_0 and on the continuity constants $\|a\|$ and $\|c\|$. More precisely, we have:

$$\|\overline{u}\|_V \le \frac{\|c\|\|\overline{f}\|}{\beta^2} + \frac{\sqrt{2\beta^2+\mu^2}\|c\|^{1/2}\,\|f_0\|}{\alpha_0^{1/2}\beta^2} + \frac{(\beta+\mu)\|\overline{g}\|}{\beta^2} + \frac{3\sqrt{\beta^2+\mu^2}\|c\|^{1/2}\|g_0\|}{\gamma_0^{1/2}\beta^2}, \qquad (4.3.22)$$

$$\|u_0\|_V \le \frac{\|c\|\,\|a\|^{1/2}\,\|\overline{f}\|}{\alpha_0^{1/2}\beta^2} + \frac{2(\beta^2+\mu^2)\|f_0\|}{\alpha_0\beta^2} + \frac{(\beta+\mu)\|a\|^{1/2}\|\overline{g}\|}{\alpha_0^{1/2}\beta^2} + \frac{3\mu\sqrt{\beta^2+\mu^2}\|g_0\|}{\gamma_0^{1/2}\alpha_0^{1/2}\beta^2}, \qquad (4.3.23)$$

$$\|\overline{p}\|_Q \le \frac{(\beta+\mu)\|\overline{f}\|}{\beta^2} + \frac{3\sqrt{\beta^2+\mu^2}\|a\|^{1/2}\,\|f_0\|}{\alpha_0^{1/2}\beta^2} + \frac{\|a\|\|\overline{g}\|}{\beta^2} + \frac{\sqrt{2\beta^2+\mu^2}\|a\|^{1/2}\,\|g_0\|}{\gamma_0^{1/2}\beta^2}, \qquad (4.3.24)$$

$$\|p_0\|_Q \le +\frac{(\beta+\mu)\|c\|^{1/2}\|\overline{f}\|}{\gamma_0^{1/2}\beta^2} + \frac{3\mu\sqrt{\beta^2+\mu^2}\|f_0\|}{\alpha_0^{1/2}\gamma_0^{1/2}\beta^2} + \frac{\|a\|\,\|c\|^{1/2}\|\overline{g}\|}{\gamma_0^{1/2}\beta^2} + \frac{2(\beta^2+\mu^2)\|g_0\|}{\gamma_0\beta^2}, \qquad (4.3.25)$$

where μ is defined by

$$\mu^2 := \|a\|\,\|c\|. \qquad (4.3.26)$$

Proof. As the problem is symmetric, we just have to prove that the mapping $M : (u,p) \to (f,g)$ is bounding (that is, we have to prove that the bounds (4.3.22)–(4.3.25) hold true). Then, M will be injective and $M^t \equiv M$ will be surjective, and the theorem will be proved.

Then, we note that there is another (fundamental) symmetry in our assumptions when we exchange a with c, u with p and B with B^t. Hence, we can start proving our bounds for the case, say, $f = 0$. These bounds, due to the above symmetry, will imply similar ones for the case $g = 0$ (exchanging u with p, α_0 with γ_0, and so on). Then, by linearity, we will sum the estimates for $f = 0$ and those for $g = 0$, and obtain the final estimates for the general case.

Hence, we proceed by assuming $f = 0$. We first observe that, for $f = 0$, we have from the first equation

$$a(u, u_0) = -b(u_0, p) = 0 \tag{4.3.27}$$

since $u_0 \in \text{Ker}B$. Hence, using (4.3.9), $u_0 = u - \overline{u}$, and (4.3.11),

$$|u_0|_a^2 = a(u_0, u_0) = -a(\overline{u}, u_0) \le |\overline{u}|_a\, |u_0|_a, \tag{4.3.28}$$

which, combined with the ellipticity condition (4.3.17) and then with (4.3.10), gives

$$\|u_0\|_V \le \frac{1}{\alpha_0^{1/2}}|u_0|_a \le \frac{1}{\alpha_0^{1/2}}|\overline{u}|_a. \le \frac{\|a\|^{1/2}}{\alpha_0^{1/2}}\|\overline{u}\|_V. \tag{4.3.29}$$

We also note that, in operator form, Eqs. (4.3.20), for $f = 0$, give

$$Au = -B^t p \tag{4.3.30}$$

and

$$Bu = Cp + g. \tag{4.3.31}$$

Moreover, taking in (4.3.20) $v = u$ in the first equation, $q = p$ in the second equation, and subtracting, we have

$$a(u, u) + c(p, p) = -\langle g, p\rangle, \tag{4.3.32}$$

implying through (4.3.12) that

$$\frac{\|Au\|_{V'}^2}{\|a\|} + \frac{\|Cp\|_{Q'}^2}{\|c\|} \le -\langle g, p\rangle. \tag{4.3.33}$$

At this point, it will be convenient to further distinguish the cases $g_0 = 0$ and $\overline{g} = 0$, to make the estimates separately, and then sum them. We start with the easier case $g_0 = 0$. Then, (4.3.33) becomes

$$\frac{\|Au\|_{V'}^2}{\|a\|} + \frac{\|Cp\|_{Q'}^2}{\|c\|} \le -\langle \overline{g}, \overline{p}\rangle. \tag{4.3.34}$$

On the other hand, $p - \overline{p} = p_0 \in H$ so that $B^t\overline{p} = B^t p$. Hence, using (4.3.30), then (4.3.34), and then (4.1.94), we have

$$\begin{aligned}\|B^t\overline{p}\|^2_{V'} &= \|B^t p\|^2_{V'} = \|Au\|^2_{V'} \\ &\le \|a\|\,\|\overline{g}\|_{Q'}\|\overline{p}\|_Q \le \|a\|\,\|\overline{g}\|_{Q'}\frac{1}{\beta}\|B^t\overline{p}\|_{V'}\end{aligned} \tag{4.3.35}$$

which, using again (4.1.94), gives

$$\|\overline{p}\|_Q \le \frac{1}{\beta}\|B^t\overline{p}\|_{V'} \le \frac{\|a\|}{\beta^2}\|\overline{g}\|_{Q'}. \tag{4.3.36}$$

At this point, we remark that, from the second equation of (4.3.20) tested on $q = p_0$, we have

$$c(p, p_0) = b(u, p_0) - \langle\overline{g}, p_0\rangle = 0 - 0 = 0, \tag{4.3.37}$$

since $B^t p_0 = 0$ and $\langle\overline{g}, p_0\rangle = 0$ as in (4.3.16). Proceeding exactly as in (4.3.27)–(4.3.29), we then have

$$|p_0|_c \le |\overline{p}|_c. \tag{4.3.38}$$

Using the ellipticity condition (4.3.19), then (4.3.38), (4.3.10) and the previous estimate (4.3.36) on $\overline{p}$, we have

$$\|p_0\|_Q \le \frac{1}{\gamma_0^{1/2}}|p_0|_c \le \frac{1}{\gamma_0^{1/2}}|\overline{p}|_c \le \frac{\|a\|\,\|c\|^{1/2}}{\gamma_0^{1/2}\beta^2}\|\overline{g}\|_{Q'}. \tag{4.3.39}$$

The estimates on $\overline{u}$ and u_0 can be obtained in a similar way: indeed we can use (4.3.31), then (4.3.34), and then again (4.3.36) to obtain

$$\|Bu - \overline{g}\|^2_{Q'} = \|Cp\|^2_{Q'} \le \|c\|\,\|\overline{g}\|_{Q'}\|\overline{p}\|_Q \le \frac{\|c\|\,\|a\|}{\beta^2}\,\|\overline{g}\|^2_{Q'},$$

giving

$$\|Bu\|_{Q'} \le \frac{(\|c\|\,\|a\|)^{1/2}}{\beta}\,\|\overline{g}\|_{Q'} + \|\overline{g}\|_{Q'} \le \frac{\mu + \beta}{\beta}\|\overline{g}\|_{Q'}, \tag{4.3.40}$$

where in the last step we used the definition of μ given in (4.3.26). Hence, using (4.1.93), $Bu = B\overline{u}$, and (4.3.40), we have

$$\|\overline{u}\|_V \le \frac{1}{\beta}\|B\overline{u}\|_{Q'} = \frac{1}{\beta}\|Bu\|_{Q'} \le \frac{\mu + \beta}{\beta^2}\|\overline{g}\|_{Q'}. \tag{4.3.41}$$

Finally, we can use (4.3.29) to obtain

$$\|u_0\|_V \leq \frac{\|a\|^{1/2}}{\alpha^{1/2}}\|\overline{u}\|_V \leq \frac{(\mu+\beta)\|a\|^{1/2}}{\alpha^{1/2}\beta^2}\|\overline{g}\|_{Q'}. \tag{4.3.42}$$

The estimates in the case $g_0 = 0$ are therefore completed.

We now consider the case $\overline{g} = 0$. Using the definition (4.3.9), then (4.3.13), and then the second equation of (4.3.20) with $q = p_0$, we have

$$\begin{aligned} |p_0|_c^2 &= c(p_0, p_0) = c(p, p_0) - c(\overline{p}, p_0) \\ &= b(u, p_0) - \langle g_0, p_0\rangle - c(\overline{p}, p_0) = -\langle g_0, p_0\rangle - c(\overline{p}, p_0) \\ &\leq \|g_0\|_{Q'}\|p_0\|_Q + |\overline{p}|_c\,|p_0|_c \end{aligned} \tag{4.3.43}$$

where the last equality holds since $p_0 \in KerB^t$. We also note that, due to (4.3.19),

$$\|p_0\|_Q \leq \frac{1}{\gamma_0^{1/2}}|p_0|_c. \tag{4.3.44}$$

Joining (4.3.43) and (4.3.44), we then have

$$|p_0|_c^2 \leq \frac{\|g_0\|_{Q'}}{\gamma_0^{1/2}}|p_0|_c + |\overline{p}|_c|p_0|_c, \tag{4.3.45}$$

implying

$$|p_0|_c \leq \frac{\|g_0\|_{Q'}}{\gamma_0^{1/2}} + |\overline{p}|_c, \tag{4.3.46}$$

and using once more (4.3.44), and then (4.3.10),

$$\|p_0\|_Q \leq \frac{1}{\gamma_0^{1/2}}\Big(\frac{\|g_0\|_{Q'}}{\gamma_0^{1/2}} + |\overline{p}|_c\Big) \leq \frac{\|g_0\|_{Q'}}{\gamma_0} + \frac{\|c\|^{1/2}}{\gamma_0^{1/2}}\|\overline{p}\|_Q. \tag{4.3.47}$$

Proceeding as in (4.3.35), and then using (4.3.47), and finally (4.1.94), we now have

$$\begin{aligned} \|B^t\overline{p}\|_{V'}^2 &= \|B^t p\|_{V'}^2 = \|Au\|_{V'}^2 \leq \|a\|\,\|g_0\|_{Q'}\|p_0\|_Q \\ &\leq \frac{\|a\|\,\|g_0\|_{Q'}^2}{\gamma_0} + \frac{\|a\|\,\|g_0\|_{Q'}}{\gamma_0^{1/2}}\|c\|^{1/2}\|\overline{p}\|_Q \\ &\leq \frac{\|a\|\,\|g_0\|_{Q'}^2}{\gamma_0} + \frac{\|a\|\,\|g_0\|_{Q'}\|c\|^{1/2}}{\beta\gamma_0^{1/2}}\|B^t\overline{p}\|_{V'}. \end{aligned} \tag{4.3.48}$$

Using the classical inequality $xy \leq (x^2+y^2)/2$ on the last term of (4.3.48), we obtain

$$\frac{1}{2}\|B^t\overline{p}\|^2_{V'} \leq \frac{\|a\|\,\|g_0\|^2_{Q'}}{\gamma_0} + \frac{\|a\|^2\,\|g_0\|^2_{Q'}\|c\|}{2\beta^2\gamma_0} \leq \frac{(2\beta^2+\mu^2)\|a\|\,\|g_0\|^2_{Q'}}{2\gamma_0\beta^2}, \tag{4.3.49}$$

implying easily

$$\|B^t\overline{p}\|_{V'} \leq \frac{(2\beta^2+\mu^2)^{1/2}\|a\|^{1/2}\,\|g_0\|_{Q'}}{\gamma_0^{1/2}\beta}. \tag{4.3.50}$$

From (4.3.50), using once more (4.1.94), we obtain the estimate on $\overline{p}$

$$\|\overline{p}\|_Q \leq \frac{(2\beta^2+\mu^2)^{1/2}\|a\|^{1/2}\,\|g_0\|_{Q'}}{\gamma_0^{1/2}\beta^2}, \tag{4.3.51}$$

which, using (4.3.47), also gives the bound on p_0:

$$\begin{aligned}\|p_0\|_Q &\leq \frac{\|g_0\|_{Q'}}{\gamma_0} + \frac{\|c\|^{1/2}}{\gamma_0^{1/2}}\frac{(2\beta^2+\mu^2)^{1/2}\|a\|^{1/2}\,\|g_0\|_{Q'}}{\gamma_0^{1/2}\beta^2}\\ &= \frac{\beta^2+\mu(2\beta^2+\mu^2)^{1/2}}{\gamma_0\beta^2}\|g_0\|_{Q'} \leq \frac{2(\beta^2+\mu^2)}{\gamma_0\beta^2}\|g_0\|_{Q'}.\end{aligned} \tag{4.3.52}$$

This, in turn, gives us a bound on $\|Cp\|_{Q'}$. Indeed, using (4.3.33) and remembering that in this case

$$\langle g, p\rangle = \langle g_0, p\rangle = \langle g_0, p_0\rangle \tag{4.3.53}$$

and then using (4.3.52), we easily have

$$\|Cp\|^2_{Q'} \leq -\|c\|\,\langle g_0, p_0\rangle \leq \frac{\|c\|\,2(\beta^2+\mu^2)}{\gamma_0\beta^2}\,\|g_0\|^2_{Q'}. \tag{4.3.54}$$

On the other hand, the second equation of (4.3.20) gives $Bu = Cp + g_0$, so that using (4.3.54),

$$\|Bu\|_{Q'} \leq \Big(\frac{\|c\|^{1/2}\,\sqrt{2(\beta^2+\mu^2)}}{\gamma_0^{1/2}\beta} + 1\Big)\|g_0\|_{Q'} \leq \frac{3\sqrt{\beta^2+\mu^2}\|c\|^{1/2}}{\gamma_0^{1/2}\beta}\|g_0\|_{Q'}, \tag{4.3.55}$$

where we used the fact that $\gamma_0 \leq \|c\|$. We now note that $Bu = B\overline{u}$, so that, using (4.1.93) and (4.3.55), we have the estimate on $\overline{u}$

$$\|\overline{u}\|_V \leq \frac{1}{\beta}\|Bu\|_{Q'} \leq \frac{3\sqrt{\beta^2+\mu^2}\|c\|^{1/2}}{\gamma_0^{1/2}\beta^2}\|g_0\|_{Q'}. \tag{4.3.56}$$

The estimate on u_0 then follows from (4.3.29), that is,

$$\|u_0\|_V \leq \frac{\|a\|^{1/2}}{\alpha_0^{1/2}} \|\overline{u}\|_V \leq \frac{3\mu\sqrt{\beta^2+\mu^2}}{\gamma_0^{1/2}\alpha_0^{1/2}\beta^2} \|g_0\|_{Q'}. \tag{4.3.57}$$

As already discussed, the estimates for the cases $g = 0$ and $f = \overline{f}$ or $f = f_0$ are "symmetrical", and the proof is completed. □

Remark 4.3.3. Following the path of Theorem 3.6.1, we could have proved stability also for the case in which a or c are not symmetric (at least in the case $\mathrm{Im}B = Q'$). However, the dependence of the stability constants upon α_0 and β would have been *much worse*. □

A very particular (but important) case is met when c has the form, as in (4.3.7),

$$c(p,q) = \lambda(p,q)_Q, \ \lambda \geq 0 \tag{4.3.58}$$

where $(\cdot\,,\,\cdot)_Q$ is the scalar product in Q. We decided therefore to dedicate a theorem especially to it.

Theorem 4.3.2. *In the framework of Assumption $\mathcal{ABC}$, assume further that the inf-sup condition* (4.2.26) *and the ellipticity requirement (4.2.12) are satisfied, and that c is given by* (4.3.58) *with $\lambda > 0$. Then, for every $f \in V'$ and for every $g \in Q'$, problem* (4.3.20) *has a unique solution, and we have the estimate*

$$\|u\|_V \leq \frac{\beta^2 + 4\lambda\,\|a\|}{\alpha_0\beta^2}\|f\|_{V'} + \frac{2\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|g\|_{Q'} \tag{4.3.59}$$

and

$$\|p\|_Q \leq \frac{2\,\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|f\|_{V'} + \frac{4\,\|a\|}{\lambda\|a\| + 2\,\beta^2}\|g\|_{Q'}. \tag{4.3.60}$$

Proof. As we are already used to, we shall split the two cases $f = 0$ and $g = 0$, and then combine the estimates by linearity. Let us first consider the case $f = 0$, and assume that u, p and g satisfy

$$\begin{cases} a(u,v) + b(v,p) = 0, & \forall\, v \in V, \\ b(u,q) - \lambda(p,q)_Q = \langle g,q\rangle_{Q'\times Q}, & \forall\, q \in Q. \end{cases} \tag{4.3.61}$$

In operator form, (4.3.61) can be written as

$$\begin{cases} Au + B^t p = 0, \\ Bu - \lambda R_Q p = g, \end{cases} \tag{4.3.62}$$

where R_Q is the Ritz operator $Q \to Q'$ (see (4.1.37)).

Using (4.1.94) together with the first equation of (4.3.62), we obtain

$$\beta\|p\|_Q \le \|B^t p\|_{Q'} = \|Au\|_{V'}. \tag{4.3.63}$$

On the other hand, we already noted (see (4.3.32)) that

$$a(u,u) + \lambda\|p\|_Q^2 = -\langle g, p\rangle_{Q'\times Q}. \tag{4.3.64}$$

Using (4.3.12), Eq. (4.3.64) and finally (3.4.17), we have

$$\|Au\|_{V'}^2 \le \|a\|a(u,u) \le \|a\|\|p\|_Q\,\|g\|_{Q'}, \tag{4.3.65}$$

which, combined with (4.3.63), yields

$$\|Au\|_{V'} \le \frac{\|a\|}{\beta}\|g\|_{Q'}, \tag{4.3.66}$$

and using again (4.3.63),

$$\|p\|_Q \le \frac{\|a\|}{\beta^2}\|g\|_{Q'}. \tag{4.3.67}$$

Using the lifting operator L_B defined in Theorem 4.1.5, we set

$$\tilde{u} := L_B(g + \lambda R_Q^{-1} p) \tag{4.3.68}$$

and we have from (3.4.43)

$$B\tilde{u} = g + \lambda R_Q^{-1} p. \tag{4.3.69}$$

Setting now

$$u_0 := u - \tilde{u}, \tag{4.3.70}$$

we have from (4.3.69) and the second equation of (4.3.62) that $u_0 \in K$. We then note that, testing the first equation of (4.3.61) with $v = u_0$, we have, as in (4.3.27):

$$a(u,u_0) = -b(u_0,p) = 0. \tag{4.3.71}$$

Moreover, using (4.3.70), (4.3.71) and (4.3.11), we have as in (4.3.28)

$$a(u_0,u_0) = -a(u_0,\tilde{u}) \le |u_0|_a|\tilde{u}|_a, \tag{4.3.72}$$

which easily gives

$$|u_0|_a \le |\tilde{u}|_a. \tag{4.3.73}$$

Hence, we can use (4.2.12) and (4.3.73) to obtain

$$\alpha_0\|u_0\|_V^2 \le |u_0|_a^2 \le |\tilde{u}|_a^2, \tag{4.3.74}$$

and finally from (4.3.74) and (4.3.10),

$$\|u_0\|_V \le \left(\frac{\|a\|}{\alpha_0}\right)^{1/2} \|\tilde{u}\|_V. \tag{4.3.75}$$

Finally, we can collect (4.3.70) and (4.3.75) and have an estimate for u:

$$\|u\|_V \le \|u_0\|_V + \|\tilde{u}\|_V \le \left(1 + \left(\frac{\|a\|}{\alpha_0}\right)^{1/2}\right) \|\tilde{u}\|_V. \tag{4.3.76}$$

We now consider the first equation of (4.3.61) with $v = u$, getting

$$a(u, u) + b(u, p) = 0. \tag{4.3.77}$$

Recalling that a is positive semi-definite (see (4.3.8)), we obtain

$$b(u, p) \le 0,$$

and substituting $p = \lambda^{-1} R_Q^{-1}(Bu - g)$:

$$\begin{aligned} 0 &\ge \langle Bu, \lambda^{-1} R_Q^{-1}(Bu - g)\rangle_{Q' \times Q} \\ &= \lambda^{-1}\Big(\|Bu\|^2_{Q'} - \langle Bu, R_Q^{-1} g\rangle_{Q' \times Q}\Big), \end{aligned} \tag{4.3.78}$$

which easily implies

$$\|Bu\|^2_{Q'} \le \langle Bu, R_Q^{-1} g\rangle_{Q' \times Q} \le \|Bu\|_{Q'}\, \|g\|_{Q'}, \tag{4.3.79}$$

giving

$$\|Bu\|_{Q'} \le \|g\|_{Q'}. \tag{4.3.80}$$

Using once more the *inf-sup* condition (4.1.93),

$$\|\tilde{u}\|_V \le \frac{1}{\beta}\|B\tilde{u}\|_{Q'} \le \frac{1}{\beta}\|Bu\|_{Q'} \le \frac{1}{\beta}\|g\|_{Q'}, \tag{4.3.81}$$

and inserting (4.3.81) in (4.3.76), then using $\alpha_0 \le \|a\|$, gives

$$\|u\|_V \le \left(1 + \left(\frac{\|a\|}{\alpha_0}\right)^{1/2}\right) \|\tilde{u}\|_V \le \frac{2\|a\|^{1/2}}{\beta \alpha_0^{1/2}} \|g\|_{Q'}. \tag{4.3.82}$$

We note at this point that we have another way to obtain an estimate for p, apart from (4.3.67); actually, from the second equation of (4.3.62), and (4.3.80):

$$\|p\|_Q \leq \frac{1}{\lambda}\|Bu - g\|_{Q'} \leq \frac{2}{\lambda}\|g\|_{Q'}. \tag{4.3.83}$$

With some manipulations, we see that (4.3.67) and (4.3.83) can be combined into

$$\|p\|_Q \leq \frac{4\,\|a\|}{\|a\|\lambda + 2\,\beta^2}\|g\|_{Q'}. \tag{4.3.84}$$

We now consider the case in which $g = 0$ and assume that u, p and f satisfy

$$\begin{cases} a(u, v) + b(v, p) = \langle f, v\rangle_{V'\times V}, & \forall\, v \in V, \\ b(u, q) - \lambda(p, q)_Q = 0, & \forall\, q \in Q, \end{cases} \tag{4.3.85}$$

which in operator form reads:

$$\begin{cases} Au + B^t p = f \\ Bu - \lambda R_Q\, p = 0, \end{cases} \tag{4.3.86}$$

where again R_Q is the Ritz operator $Q \to Q'$ (see (4.1.37)). We use again the lifting operator L_B of Theorem (4.1.5), this time setting $\tilde{u} := L_B \lambda R_Q\, p$ so that

$$B\tilde{u} = Bu = \lambda R_Q\, p, \tag{4.3.87}$$

and, defining again u_0 as in (4.3.70), we still have $u_0 \in K$. Taking $v = \tilde{u}$ as test function in the first equation of (4.3.86), and substitute $p = R_Q^{-1}\lambda^{-1}Bu$:

$$a(u, \tilde{u}) + b(\tilde{u}, R_Q^{-1}\lambda^{-1}Bu) = \langle f, \tilde{u}\rangle. \tag{4.3.88}$$

As $B\tilde{u} = Bu$, we can rewrite (4.3.88) as follows

$$\lambda^{-1}\|Bu\|^2_{Q'} = \langle f, \tilde{u}\rangle - a(u, \tilde{u}) \leq \|f\|_{V'}\,\|\tilde{u}\|_V - a(u, \tilde{u}). \tag{4.3.89}$$

We leave (4.3.89) for a while, and we estimate $-a(u, \tilde{u})$. Using the fact that $u = \tilde{u} + u_0$ and (4.3.11), we obtain

$$-\,a(u, \tilde{u}) = -a(\tilde{u} + u_0, \tilde{u}) \leq -|\tilde{u}|_a^2 + |\tilde{u}|_a\,|u_0|_a. \tag{4.3.90}$$

On the other hand, testing the first equation with $v = u_0$, we get

$$a(u, u_0) = \langle f, u_0\rangle, \tag{4.3.91}$$

yielding

$$|u_0|_a^2 = a(u_0, u_0) = a(u, u_0) - a(\tilde{u}, u_0) \leq \|f\| \, \|u_0\|_V + |\tilde{u}|_a \, |u_0|_a. \tag{4.3.92}$$

On the other hand, (4.3.17) gives

$$\alpha_0 \|u_0\|_V^2 \leq |u_0|_a^2 \tag{4.3.93}$$

which, together with (4.3.92), yields

$$|u_0|_a \leq \frac{\|f\|_{V'}}{\alpha_0^{1/2}} + |\tilde{u}|_a. \tag{4.3.94}$$

Inserting this into (4.3.90), we have

$$-a(u, \tilde{u}) \leq -|\tilde{u}|_a^2 + |\tilde{u}|_a \left(\frac{1}{\alpha_0^{1/2}} \|f\|_{V'} + |\tilde{u}|_a \right) = |\tilde{u}|_a \frac{1}{\alpha_0^{1/2}} \|f\|_{V'}. \tag{4.3.95}$$

Inserting this into (4.3.89), then using (4.3.10), and finally using (4.1.93) gives

$$\begin{aligned} \lambda^{-1} \|Bu\|_{Q'}^2 &\leq \|f\|_{V'} \, \|\tilde{u}\|_V + |\tilde{u}|_a \frac{1}{\alpha_0^{1/2}} \|f\|_{V'} \\ &\leq \left(1 + \frac{\|a\|^{1/2}}{\alpha_0^{1/2}}\right) \|f\|_{V'} \|\tilde{u}\|_V \leq \left(1 + \frac{\|a\|^{1/2}}{\alpha_0^{1/2}}\right) \|f\|_{V'} \frac{1}{\beta} \|Bu\|_{Q'} \\ &= \frac{\alpha_0^{1/2} + \|a\|^{1/2}}{\beta \alpha_0^{1/2}} \|f\|_{V'} \|Bu\|_{Q'}. \end{aligned} \tag{4.3.96}$$

Using again (4.1.93) and then (4.3.96), we have therefore

$$\|\tilde{u}\|_V \leq \frac{1}{\beta} \|Bu\|_{Q'} \leq \frac{\lambda(\alpha_0^{1/2} + \|a\|^{1/2})}{\beta^2 \alpha_0^{1/2}} \|f\|_{V'}. \tag{4.3.97}$$

Using (4.3.93), (4.3.94), and (4.3.10) and then (4.3.97), we have then

$$\begin{aligned} \|u_0\|_V &\leq \frac{1}{\alpha_0^{1/2}} |u_0|_a \leq \frac{\|f\|_{V'}}{\alpha_0} + \frac{|\tilde{u}|_a}{\alpha_0^{1/2}} \leq \frac{\|f\|_{V'}}{\alpha_0} + \left(\frac{\|a\|}{\alpha_0}\right)^{1/2} \|\tilde{u}\| \\ &\leq \left(\frac{1}{\alpha_0} + \frac{\lambda(\alpha_0^{1/2} + \|a\|^{1/2}) \|a\|^{1/2}}{\alpha_0 \beta^2} \right) \|f\|_{V'}. \end{aligned} \tag{4.3.98}$$

From the second equation of (4.3.86) and (4.3.97), we also derive the estimate for p

$$\|p\|_Q = \|\lambda^{-1}Bu\|_{Q'} \le \frac{\alpha_0^{1/2} + \|a\|^{1/2}}{\beta\alpha_0^{1/2}} \|f\|_{V'}. \tag{4.3.99}$$

We collect the results for $g = 0$, using the fact that $\alpha_0 \le \|a\|$. From (4.3.97) and (4.3.98), we have the estimate on u

$$\begin{aligned}\|u\|_V &\le \|\tilde{u}\|_V + \|u_0\|_V \\ &\le \Big(\frac{\lambda\,(\|a\|^{1/2} + \alpha_0^{1/2})}{\alpha_0^{1/2}\beta^2} + \frac{1}{\alpha_0} + \frac{\lambda(\|a\| + (\|a\|\alpha_0)^{1/2})}{\alpha_0\beta^2}\Big)\|f\|_{V'} \\ &\le \frac{\beta^2 + 4\lambda\|a\|}{\alpha_0\beta^2}\|f\|_{V'},\end{aligned} \tag{4.3.100}$$

while from (4.3.99) we have the estimate on p

$$\|p\|_Q \le \frac{2\,\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|f\|_{V'}. \tag{4.3.101}$$

The final results can then be obtained collecting (4.3.82), (4.3.84), (4.3.100) and (4.3.101). □

Corollary 4.3.1. *In the framework of Assumption $\mathcal{ABC}$, assume that ImB is closed, that the ellipticity requirement (4.2.12) is satisfied and that c is given by* (4.3.58) *with $\lambda \ge 0$. Set $g = \overline{g} + g_0$ with $\overline{g} \in H^0$ and $g_0 \in H'$ (with $H := kerB^t$, as usual), and set $p = \overline{p} + p_0$ with $\overline{p} \in H^\perp$ and $p_0 \in H$. Then, for every $f \in V'$ and for every $g \in Q'$, problem* (4.3.20) *has a unique solution, and we have the estimates*

$$\|u\|_V \le \frac{\beta^2 + 4\lambda\,\|a\|}{\alpha_0\beta^2}\|f\|_{V'} + \frac{2\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|\overline{g}\|_{Q'}, \tag{4.3.102}$$

$$\|\overline{p}\|_Q \le \frac{2\,\|a\|^{1/2}}{\alpha_0^{1/2}\beta}\|f\|_{V'} + \frac{4\,\|a\|}{\lambda\|a\| + 2\,\beta^2}\|\overline{g}\|_{Q'}, \tag{4.3.103}$$

$$\|p_0\|_Q \le \frac{1}{\lambda}\|g_0\|_{Q'}. \tag{4.3.104}$$

Proof. It is immediate to check that, actually, the problem splits into two subproblems: *find* $(u, \overline{p}) \in V \times H^\perp$ *such that*

$$\begin{cases} a(u, v) + b(v, p) = \langle f, v\rangle_{V'\times V}, & \forall\, v \in V, \\ b(u, q) - \lambda(\overline{p}, \overline{q})_Q = \langle g, \overline{q}\rangle_{(H^\perp)'\times H^\perp}, & \forall\, \overline{q} \in H^\perp, \end{cases} \tag{4.3.105}$$

and

$$\lambda p_0 = g_0. \tag{4.3.106}$$

For problem (4.3.105), we can apply the results of Theorem 4.3.2 using $H^\perp$ instead of Q (with the same norm). Problem (4.3.106) is trivial. □

In the case where c has the form (4.3.58), as in Theorem 4.3.2, it is also interesting to estimate the distance between the solution of the perturbed problem (4.3.20) and the solution of the limit problem, for $\lambda \to 0$.

We have in particular the following proposition.

Proposition 4.3.3. *Together with Assumption $\mathcal{AB}$, assume that $a(\cdot,\cdot)$ is symmetric, positive semi-definite and elliptic on K, and that* ImB *is closed. Let $f \in V'$, let $g \in$* ImB*, and let (u^*, p^*) be the solution in $V \times H^\perp$ of the problem*

$$\begin{cases} a(u^*, v) + b(v, p^*) = \langle f, v\rangle_{V'\times V}, & \forall\, v \in V, \\ b(u^*, q) = \langle g, q\rangle_{Q'\times Q}, & \forall\, q \in Q. \end{cases} \tag{4.3.107}$$

Let moreover, for $\lambda > 0$, (u_λ, p_λ) be the solution in $V \times Q$ of

$$\begin{cases} a(u_\lambda, v) + b(v, p_\lambda) = \langle f, v\rangle_{V'\times V}, & \forall\, v \in V, \\ b(u_\lambda, q) - \lambda(p_\lambda, q)_Q = \langle g, q\rangle_{Q'\times Q}, & \forall\, q \in Q. \end{cases} \tag{4.3.108}$$

Then, we have

$$\|u^* - u_\lambda\|_V + \|p^* - p_\lambda\|_Q \le C\,\lambda, \tag{4.3.109}$$

where C is a constant depending only on α_0, $\|a\|$ and β.

Proof. Setting $\delta_u := u_\lambda - u^*$ and $\delta_p := p_\lambda - p^*$ and taking the difference of (4.3.108)–(4.3.107), we easily have

$$\begin{cases} a(\delta_u, v) + b(v, \delta_p) = 0, & \forall\, v \in V, \\ b(\delta_u, q) - \lambda(\delta_p, q) = \lambda\,(p^*, q)_Q, & \forall\, q \in Q. \end{cases} \tag{4.3.110}$$

Hence, we can apply estimates (4.3.59) and (4.3.60) with $g = \lambda R_Q\, p^*$. □

Remark 4.3.4. We point out that the validity of (4.3.109) *for* $\lambda \to 0$ could have been obtained directly from Theorem 4.3.2 and the Kato Theorem (4.1.3). □

We also point out the following result, that is particularly useful if one is not too keen on spotting the best dependence of the stability constants.

Proposition 4.3.4. *Together with Assumption $\mathcal{AB}$, assume that $a(\cdot,\cdot)$ is symmetric, positive semi-definite, and elliptic on K, and that* ImB *is closed. Then, for every $\chi > 0$, there exist a constant $\tilde{\alpha}$, depending on χ, $\|a\|$, α_0 and β (defined in (4.3.18)), such that*

$$\tilde{\alpha}\|v\|_V^2 \le a(v,v) + \chi\|Bv\|_{Q'}^2 \quad \forall\, v \in V. \tag{4.3.111}$$

Proof. It is easy to check that, for every $\varepsilon \in]0,1[$,

$$
\begin{aligned}
a(v,v) + \chi\|Bv\|^2_{Q'} &= |v_0|^2_a + |\overline{v}|^2_a + 2a(v_0,\overline{v}) + \chi\|Bv\|^2_{Q'} \\
&\geq |v_0|^2_a + |\overline{v}|^2_a + \chi\beta\|\overline{v}\|^2_V - 2|v_0|_a|\overline{v}|_a \\
&\geq |v_0|^2_a + |\overline{v}|^2_a + \chi\beta^2\|\overline{v}\|^2_V - \varepsilon|v_0|^2_a - \frac{1}{\varepsilon}|\overline{v}|^2_a \\
&= (1-\varepsilon)|v_0|^2_a + (1-\frac{1}{\varepsilon})|\overline{v}|^2_a + \chi\beta^2\|\overline{v}\|^2_V \\
&\geq (1-\varepsilon)|v_0|^2_a + (\|a\| - \frac{\|a\|}{\varepsilon})|\overline{v}|^2_V + \chi\beta^2\|\overline{v}\|^2_V \\
&= (1-\varepsilon)|v_0|^2_a + \frac{\chi\beta^2\varepsilon + \|a\|\varepsilon - \|a\|}{\varepsilon}\|\overline{v}\|^2_V \\
&\geq \alpha_0(1-\varepsilon)\|v_0\|^2_V + \frac{\chi\beta^2\varepsilon + \|a\|\varepsilon - \|a\|}{\varepsilon}\|\overline{v}\|^2_V,
\end{aligned}
$$

and the result follows by taking $\varepsilon = \dfrac{\chi\beta^2 + 2\|a\|}{2\chi\beta^2 + 2\|a\|}$. □

Remark 4.3.5. It is clear that, conversely, the property (4.3.111) implies the ellipticity of a on the kernel K of B. □

Remark 4.3.6. Looking at the proof of Proposition 4.3.4, we can analyse the dependence of the constant $\tilde{\alpha}$ on $\chi\beta^2$, on $\|a\|$, and on α_0. Indeed, setting $k := \chi\beta^2$ and $m := \|a\|$, for $\varepsilon = \dfrac{k+2m}{2k+2m}$ we have

$$
\frac{k\varepsilon + m\varepsilon - m}{\varepsilon} = \frac{(k/2) + m - m}{\varepsilon} = \frac{k/2}{\varepsilon} = \frac{k(k+m)}{k+2m} \tag{4.3.112}
$$

while

$$
\alpha_0(1-\varepsilon) = \frac{\alpha_0(2k+2m-k-2m)}{2k+2m} = \frac{\alpha_0 k}{2k+2m}. \tag{4.3.113}
$$

On the other hand, since $\alpha_0 \leq m = \|a\|$, we have

$$
\frac{k(k+m)}{k+2m} \geq \frac{k\alpha_0}{2k+2m} \tag{4.3.114}
$$

which finally gives (looking at the last line of the proof of Proposition 4.3.4)

$$
\tilde{\alpha} \geq \frac{\chi\beta^2\alpha_0}{2\chi\beta^2 + 2\|a\|}. \tag{4.3.115}
$$

It is easy to see (taking the derivative) that the right-hand side of (4.3.115), as a function of χ, is monotonically increasing. Hence, we can say that, for every fixed $\chi_* > 0$, we have that for every $\chi \geq \chi_*$,

$$\tilde{\alpha} \geq \alpha_* := \frac{\chi_* \beta^2 \alpha_0}{2\chi_* \beta^2 + 2\|a\|}. \tag{4.3.116}$$

□

Remark 4.3.7. In the above theorem, there is no mention of any bilinear form c, and one may wonder why the theorem has been put in this subsection. However, the bilinear form $a(u,v) + \chi(Bu, Bv)_{Q'}$ is exactly what we get from problem (4.3.20) for $c(p,q) = \lambda(p,q)_Q$ (that is, in the case of the problem (4.3.108)). Indeed, in this case, the second equation of (4.3.108) can be written as: $Bu = \lambda R_Q p + g$ where R_Q is the Ritz operator in Q, as defined in Theorem 4.1.2. Solving for p and substituting in the first equation gives

$$a(u,v) + \frac{1}{\lambda}\langle R_Q^{-1} Bu, Bv\rangle_{Q\times Q'} = \langle f, v\rangle_{V'\times V} + \frac{1}{\lambda}\langle R_Q^{-1} g, Bv\rangle_{Q\times Q'}.$$

Then, we use the fact that $R_Q^{-1} \equiv R_{Q'}$, we set $\chi = 1/\lambda$, and we obtain that the problem (4.3.108) is **equivalent** to

$$\begin{cases} a(u,v) + \chi(Bu, Bv)_{Q'} = \langle f, v\rangle_{V'\times V} + \chi(g, Bv)_{Q'} & \forall\, v \in V, \\ p = \chi\, R_{Q'}(g - Bu), \end{cases} \tag{4.3.117}$$

where clearly the first equation can be solved by itself, and its solution u used to express the solution p of the second equation. □

We conclude the subsection on regular perturbations with the following general theorem, which is often useful in these kinds of problems.

Theorem 4.3.3 (The shadow solution). *Assume that $\mathcal{H}$ is a Hilbert space, and that $\mathbb{M}$ and $\mathbb{D}$ are linear continuous operators from $\mathcal{H}$ into its dual space. Assume that* $\mathrm{Im}\mathbb{M}$ *is closed and that there exists a $\lambda^* > 0$ such that, for every λ positive with $\lambda \leq \lambda^*$, we have*

$$\lambda \|\mathbf{x}\|^2_{\mathcal{H}} \leq C\, \langle \mathbb{M}\mathbf{x} + \lambda\mathbb{D}\mathbf{x}, \mathbf{x}\rangle_{\mathcal{H}'\times\mathcal{H}} \qquad \forall\, \mathbf{x} \in \mathcal{H}, \tag{4.3.118}$$

for some C independent of λ and $\mathbf{x}$. *Let $\mathcal{F} \in \mathrm{Im}\mathbb{M}$ and consider, for every λ positive with $\lambda \leq \lambda^*$, the solution $\mathbf{x}_\lambda$ of the perturbed equation*

$$\mathbb{M}\mathbf{x}_\lambda + \lambda\mathbb{D}\mathbf{x}_\lambda = \mathcal{F}. \tag{4.3.119}$$

Then, $\mathbf{x}_\lambda$ *has a unique limit* $\mathbf{x}_*$ *for* $\lambda \to 0+$ *and*

$$\|\mathbf{x}_\lambda - \mathbf{x}_*\|_{\mathcal{H}} \leq C\lambda \tag{4.3.120}$$

where C *is independent of* λ*.*

Proof. We give a hint of the proof. As $\mathcal{F} \in \mathrm{Im}\mathbb{M}$, we have $\mathcal{F} = \mathbb{M}\overline{\mathbf{x}}$ for some $\overline{\mathbf{x}} \in (\mathrm{Ker}\mathbb{M})^\perp$ with $\|\overline{\mathbf{x}}\|_{\mathcal{H}}$ bounded by $\|\mathcal{F}\|_{\mathcal{H}'}$. Then,

$$\langle \mathbb{M}(\overline{\mathbf{x}} - \mathbf{x}_\lambda), (\overline{\mathbf{x}} - \mathbf{x}_\lambda)\rangle + \lambda\langle \mathbb{D}(\overline{\mathbf{x}} - \mathbf{x}_\lambda), (\overline{\mathbf{x}} - \mathbf{x}_\lambda)\rangle = \lambda\,\langle \mathbb{D}\overline{\mathbf{x}}, (\overline{\mathbf{x}} - \mathbf{x}_\lambda)\rangle,$$

showing that

$$\|\mathbf{x}_\lambda - \overline{\mathbf{x}}\|^2_{\mathcal{H}} \leq\; C_0\,\langle \mathbb{D}\overline{\mathbf{x}}, (\overline{\mathbf{x}} - \mathbf{x}_\lambda)\rangle\; \leq C_1\|\mathcal{F}\|_{\mathcal{H}'}\|\mathbf{x}_\lambda - \overline{\mathbf{x}}\|_{\mathcal{H}}, \tag{4.3.121}$$

with C_0 and C_1 independent of λ. Hence, $\mathbf{x}_\lambda - \overline{\mathbf{x}}$ is bounded, and (up to the extraction of a subsequence) converges weakly in $\mathcal{H}$. We define then $\mathbf{x}_*$ as the weak limit (for $\lambda \to 0+$) of $\mathbf{x}_\lambda$. Then, we can go back to the first inequality in (4.3.121), and see that the convergence is strong. Now we remark that, for every λ, equation (4.3.119) gives that $\mathbb{D}\mathbf{x}_\lambda$ belongs to the image of $\mathbb{M}$. As the image is closed, its limit $\mathbb{D}\mathbf{x}_*$ is also in the image. Let $\mathbf{y}_* \in (\mathrm{Ker}\mathbb{M})^\perp$ be such that $\mathbb{M}\mathbf{y}_* = \mathbb{D}\mathbf{x}_*$. Set now $\mathbf{y}_\lambda := \mathbf{x}_* - \mathbf{x}_\lambda$, $\overline{\mathbf{y}} := \lambda\mathbf{y}_*$ and $\mathcal{G} := \mathbb{M}\overline{\mathbf{y}}$. We easily have that

$$\mathbb{M}\mathbf{y}_\lambda + \lambda\mathbb{D}\mathbf{y}_\lambda = \lambda\mathbb{D}\mathbf{x}_* = \mathbb{M}(\lambda\mathbf{y}_*) = \mathcal{G}. \tag{4.3.122}$$

Proceeding as in the previous part of the proof, we have then

$$\begin{aligned}\langle \mathbb{M}(\overline{\mathbf{y}} - \mathbf{y}_\lambda), (\overline{\mathbf{y}} - \mathbf{y}_\lambda)\rangle + \lambda\langle \mathbb{D}(\overline{\mathbf{y}} - \mathbf{y}_\lambda), (\overline{\mathbf{y}} - \mathbf{y}_\lambda)\rangle \\ = \lambda\,\langle \mathbb{D}\overline{\mathbf{y}}, \overline{\mathbf{x}} - \mathbf{x}_\lambda\rangle \;= \lambda^2\,\langle \mathbb{D}\mathbf{y}_*, \overline{\mathbf{y}} - \mathbf{y}_\lambda\rangle,\end{aligned}$$

showing that

$$\|\mathbf{y}_\lambda - \overline{\mathbf{y}}\|^2_{\mathcal{H}} \leq\; C_0\,\lambda\,\langle \mathbb{D}(\mathbf{y}_*), (\overline{\mathbf{y}} - \mathbf{y}_\lambda)\rangle\; \leq \lambda C_2\|\overline{\mathbf{y}} - \mathbf{y}_\lambda\|_{\mathcal{H}}, \tag{4.3.123}$$

with C_2 independent of λ. Hence, $\|\mathbf{y}_\lambda - \overline{\mathbf{y}}\|_{\mathcal{H}} = O(\lambda)$. Recalling the definition of $\mathbf{y}_\lambda$ and $\overline{\mathbf{y}}$, we have then $\|\mathbf{x}_* - \mathbf{x}_\lambda - \lambda\mathbf{y}_*\|_{\mathcal{H}} = O(\lambda)$ and finally (4.3.120). □

Remark 4.3.8. We note that $\overline{\mathbf{x}}$ and $\mathbf{x}_*$ will both solve the limit equation $\mathbb{M}\mathbf{x} = \mathcal{F}$, and they have the same component in $(\mathrm{Ker}\mathbb{M})^\perp$. However, the perturbation $\lambda\mathbb{D}$, although vanishing in the limit, leaves a unique choice of the part of the solution that belongs to $(\mathrm{Ker}\mathbb{M})$: it is *the shadow* of the perturbation. □

Remark 4.3.9. The above theorem applies for instance to perturbed mixed formulations as (4.3.20) when a and c are positive definite, with $\mathcal{H} = V \times Q$. In this case, we can set $\mathbb{M}(u, p) = (Au + B^t p, -Bu)$ and $\mathbb{D}(u, p) = (0, Cp)$ and the theorem applies. Note that $\mathrm{Im}\mathbb{M}$ will be closed due to Remark (4.2.6). □

4.3.2 Singular Perturbations

An important variant of problem (4.3.20) will occur in applications (cf. Sect. 10.4). Assume that we are given a Hilbert space W continuously embedded in Q (that is $W \hookrightarrow Q$) and dense in Q. We recall that, as in (4.1.75), the continuous embedding means that $W \subseteq Q$ and, moreover,

$$\|w\|_Q \le C_{WQ}\|w\|_W \qquad \forall\, w \in W \tag{4.3.124}$$

(and without loss of generality we can assume here that $C_{WQ} = 1$). As discussed in Sect. 4.1.6, the density implies that $Q' \hookrightarrow W'$, that Q' is dense in W', the inequality

$$\|w\|_{W'} \le \|w\|_{Q'} \qquad \forall\, w \in Q', \tag{4.3.125}$$

and finally that

$$\langle g, q\rangle_{W'\times W} = \langle g, q\rangle_{Q'\times Q} \quad \text{whenever } g \in Q' \textbf{ and } q \in W. \tag{4.3.126}$$

Remark 4.3.10. Having assumed already that $C_{WQ} = 1$, and also in order to keep the formulae reasonably simple, throughout this subsection, we implicitly assume that the problem has been **adimensionalised**, so that all the quantities we deal with are pure numbers. □

We now consider for every $\lambda > 0$ a perturbation of the type $c(p,q) = \lambda\,(p,q)_W$, that is, we consider problems of the form: *find* (u_λ, p_λ) in $V \times W$ *such that*:

$$a(u_\lambda, v) + b(v, p_\lambda) = \langle f, v\rangle_{V'\times V},\ \forall\, v \in V, \tag{4.3.127}$$

$$b(u_\lambda, q) - \lambda\,(p_\lambda, q)_W = \langle g_1, q\rangle_{Q'\times Q} + \langle g_2, q\rangle_{W'\times W},\ \forall\, q \in W. \tag{4.3.128}$$

Depending on which space is identified to its dual space, we shall meet cases where $W \hookrightarrow Q \equiv Q' \hookrightarrow W'$ or where $Q' \hookrightarrow W' \equiv W \hookrightarrow Q$. In all cases, roughly speaking, *the solution of a problem in* $V \times Q$ *is approximated by the (smoother) solution of a problem in* $V \times W$. To put the problem in the right frame, we suppose first, for simplicity, that $a(\cdot,\cdot)$ is coercive on V and $b(\cdot,\cdot)$ continuous on $V \times Q$ (hence on $V \times W$) with $\mathrm{Im}B$ closed in Q'. We suppose in (4.3.128) that $g_1 \in \mathrm{Im}B$. Taking as usual (4.3.127) with $v = u_\lambda$ and subtracting (4.3.128) with $q = p_\lambda$, and then using the coercivity of a, we have immediately

$$\alpha\|u_\lambda\|^2 + \lambda\|p_\lambda\|_W^2 \le \|f\|_{V'}\|u_\lambda\|_V + \|g_1\|_{Q'}\|\overline{p}_\lambda\|_Q + \|g_2\|_{W'}\|p_\lambda\|_W, \tag{4.3.129}$$

where, as usual, $\overline{p}$ is the component of p in $H^\perp$, with $H = \mathrm{Ker}B^t$. On the other hand, with the usual arguments, one still has from (4.3.127) that $\beta\|\overline{p}_\lambda\|_Q \le \|a\|\,\|u_\lambda\| + \|f\|_{V'}$. By classical arguments, one then gets the estimate

$$\|u_\lambda\|_V^2 + \|\overline{p}_\lambda\|_Q^2 + \lambda\|p_\lambda\|_W^2 \le C\,(\|f\|_{V'}^2 + \|g_1\|_{Q'}^2 + \frac{1}{2\lambda}\|g_2\|_{W'}^2), \tag{4.3.130}$$

where (here and in the sequel of this subsection) we denote by C any constant that depends only on the bilinear forms a and b. If we have $g_2 = g_2(\lambda)$ with $\|g_2(\lambda)\|^2_{W'}/\lambda$ bounded independently of λ, then the solution will become unbounded in W for $\lambda \to 0$ but will remain bounded in $Q/\mathrm{Ker}B^t$, and we expect it to converge to the solution of problem (4.2.6).

Before discussing further this matter, we would like to relax the ellipticity condition on a, assuming ellipticity only in the kernel of B. This, however, will produce unnecessary technical difficulties, so that we will compromise on a slightly stronger condition: We have seen in Proposition 4.3.4 and in Remark 4.3.5 that, when $\mathrm{Im}B$ is closed and a is symmetric, the ellipticity in the kernel of a is equivalent to the property (4.3.111). Here, taking into account Remarks 4.3.7 and 4.3.6 as well, we are going to assume that *for every* $\chi_* > 0$ *there exists an* $\alpha_* > 0$ *such that:*

$$\forall \chi > \chi_* \; \exists \tilde{\alpha} > \alpha_* \text{ s. t. } \tilde{\alpha}\|u\|^2_V \le a(v,v) + \chi\|Bv\|^2_{W'} \; \forall\, v \in V. \tag{4.3.131}$$

Note that, as W' is bigger than Q' (and has a smaller norm), condition (4.3.131) is *stronger* than the corresponding condition (4.3.111).

Finally, as we are interested in the case of λ small, we will not care about the possible behaviour for $\lambda \to +\infty$, and we can limit ourselves to the case $\lambda \le \lambda_0$ (implying that χ is bigger that some fixed χ_*). In the next theorem, it will be convenient to take $\lambda_0 = 1/2$, just to have slightly nicer formulae.

Theorem 4.3.4. *Together with Assumption $\mathcal{AB}$, assume that* $\mathrm{Im}B$ *is closed in* Q *and that* $a(\cdot,\cdot)$ *is positive semi-definite and verifies* (4.3.131)*. Assume moreover that* W *is a Hilbert space, continuously embedded in* Q *and dense in* Q*. Then, for every* λ *with* $0 < \lambda \le 1/2$*, for every* $f \in V'$*, for every* $g_1 \in \mathrm{Im}B$*, and for every* $g_2 \in W'$*, the problem* (4.3.127) *and* (4.3.128) *has a unique solution which, moreover, satisfies*

$$\|u_\lambda\|_V + \|\overline{p}_\lambda\|_Q + \lambda^{1/2}\|p_\lambda\|_W \le C\,(\|f\|_{V'} + \|g_1\|_{Q'} + \frac{1}{\lambda^{1/2}}\|g_2\|_{W'}), \tag{4.3.132}$$

where $\overline{p}_\lambda$ *is the component of* p_λ *in* $H^\perp$*.*

Proof. Since we do not yet have the existence of the solution, we apply a *regularisation argument*. We first substitute a with a_ε given by

$$a_\varepsilon(u,v) = a(u,v) + \varepsilon(u,v)_V,$$

with $\varepsilon > 0$. Then, we prove a-priori bounds independent of ε and we have the solution in the limit for $\varepsilon \to 0+$. For brevity, we do not re-write problem (4.3.127) and (4.3.128) with a_ε in place of a, and we do not indicate the dependence of the solution of the regularised problem on ε. Taking the first equation (4.3.127) with $v = u_\lambda$, and subtracting the second equation (4.3.128) for $q = p_\lambda$, we get

$$\begin{aligned}\varepsilon\|u_\lambda\|^2_V + &a(u_\lambda,u_\lambda) + \lambda(p_\lambda,p_\lambda)_W \\ &= \langle f,u_\lambda\rangle + \langle g_1,p_\lambda\rangle_{Q'\times Q} + \langle g_2,p_\lambda\rangle_{W'\times W}.\end{aligned} \tag{4.3.133}$$

We note that we still have from the first equation that

$$\beta\|\overline{p}_\lambda\|_Q \le C(\|u_\lambda\|_V + \|f\|_{V'}), \tag{4.3.134}$$

and since we assumed $g_1 \in \mathrm{Im}B$, we have

$$\langle g_1, p_\lambda\rangle = \langle g_1, \overline{p}_\lambda\rangle \le C\|g_1\|_{Q'}(\|u_\lambda\|_V + \|f\|_{V'}). \tag{4.3.135}$$

On the other hand, we also have

$$\langle f, u_\lambda\rangle \le \|f\|_{V'}\,\|u_\lambda\|_V \tag{4.3.136}$$

and

$$\langle g_2, p_\lambda\rangle \le \frac{1}{\lambda^{1/2}}\|g_2\|_{W'}\lambda^{1/2}\|p_\lambda\|_W \le \frac{1}{2\lambda}\|g_2\|^2_{W'} + \frac{\lambda}{2}\|p_\lambda\|^2_W. \tag{4.3.137}$$

Inserting (4.3.135), (4.3.136), and (4.3.137) in (4.3.133) and dropping the term with the ε (which is positive), we then easily have

$$\begin{aligned} &a(u_\lambda, u_\lambda) + \lambda\|p_\lambda\|^2_W \\ &\le C\left(\|g_1\|_{Q'}(\|u_\lambda\|_V + \|f\|_{V'}) + \|f\|_{V'}\,\|u_\lambda\|_V + \frac{1}{\lambda}\|g_2\|^2_{W'}\right) \\ &\le C\Big(\|u_\lambda\|_V(\|f\|_{V'} + \|g_1\|_{Q'}) + \|f\|^2_{V'} + \|g_1\|^2_{Q'} + \frac{1}{\lambda}\|g_2\|^2_{W'}\Big). \end{aligned} \tag{4.3.138}$$

On the other hand, from the second equation we have that $\lambda R_W p_\lambda$ (where R_W is the Ritz operator in W, as in Theorem 4.1.2) is equal to $Bu_\lambda - g_1 - g_2$. Hence,

$$\lambda\|p_\lambda\|^2_W = \lambda\|R_W p_\lambda\|^2_{W'} = \frac{1}{\lambda}\|Bu_\lambda - g_1 - g_2\|^2_{W'}. \tag{4.3.139}$$

Hence, using $(a+b)^2 \le 2a^2 + 2b^2$, the assumption $\lambda \le 1/2$, (4.3.139) and (4.3.125), we have:

$$\begin{aligned} \|Bu_\lambda\|^2_{W'} &\le 2\|Bu_\lambda - g_1 - g_2\|^2_{W'} + 2\|g_1 + g_2\|^2_{W'} \\ &\le \frac{1}{\lambda}\|Bu_\lambda - g_1 - g_2\|^2_{W'} + 4\|g_1\|_{W'} + 4\|g_2\|_{W'} \\ &\le \lambda\|p_\lambda\|^2_W + 4\|g_1\|_{Q'} + 4\|g_2\|_{W'}, \end{aligned} \tag{4.3.140}$$

which, joined with (4.3.138), gives immediately

$$\begin{aligned} & a(u_\lambda, u_\lambda) + \|Bu_\lambda\|_{W'} + \lambda \|p_\lambda\|_W^2 \\ & \leq C\Big(\|u_\lambda\|_V(\|f\|_{V'} + \|g_1\|_{Q'}) + \|f\|_{V'}^2 + \|g_1\|_{Q'}^2 + \frac{1}{\lambda}\|g_2\|_{W'}^2\Big). \end{aligned} \tag{4.3.141}$$

Finally, using (4.3.134) together with (4.3.131) and (4.3.141) gives

$$\begin{aligned} & \|u_\lambda\|_V^2 + \|\overline{p}_\lambda\|_Q^2 + \lambda \|p_\lambda\|_W^2 \\ & \leq C_1\Big(\|u_\lambda\|_V^2 + \|f\|_{V'}^2 + \lambda \|p_\lambda\|_W^2\Big) \\ & \leq C_2\Big(a(u_\lambda, u_\lambda) + \|Bu_\lambda\|_{W'} + \|f\|_{V'}^2 + \lambda \|p_\lambda\|_W^2\Big) \\ & \leq C_3\Big(\|u_\lambda\|_V(\|f\|_{V'} + \|g_1\|_{Q'}) + \|f\|_{V'}^2 + \|g_1\|_{Q'}^2 + \frac{1}{\lambda}\|g_2\|_{W'}^2\Big), \end{aligned} \tag{4.3.142}$$

which easily yields the result (4.3.132) □

As we shall see, a particularly interesting case is met when both g_1 and g_2 are zero. In fact, it is remarkable that in this case we *do not* need the *inf-sup* condition (meaning that we do not need $\mathrm{Im}B$ to be closed). We have indeed the following proposition.

Theorem 4.3.5. *Together with Assumption $\mathcal{AB}$, assume that $a(\cdot,\cdot)$ is positive semi-definite and verifies* (4.3.131)*. Assume, moreover, that W is a Hilbert space, continuously embedded in Q and dense in Q. Then, for every λ with $0 < \lambda \leq 1/2$, and for every $f \in V'$, the problem: find (u_λ, p_λ) in $V \times W$ such that*

$$a(u_\lambda, v) + b(v, p_\lambda) = \langle f, v\rangle_{V'\times V},\ \ \forall\, v \in V, \tag{4.3.143}$$

$$b(u_\lambda, q) - \lambda\,(p_\lambda, q)_W = 0,\ \ \forall\, q \in W, \tag{4.3.144}$$

has a unique solution, that moreover satisfies

$$\tilde{\alpha}\|u_\lambda\|_V^2 + \lambda\|p_\lambda\|_W^2 \leq \frac{4\|f\|_{V'}^2}{\tilde{\alpha}}, \tag{4.3.145}$$

where $\tilde{\alpha}$ is given in (4.3.131)*.*

Proof. Mimicking the proof of Theorem 4.3.4, we now have, using (4.3.131), then (4.3.139), and finally (4.3.133):

$$\begin{aligned} & \tilde{\alpha}\|u_\lambda\|_V^2 + \lambda\|p_\lambda\|_W^2 \\ & \qquad \leq a(u_\lambda, u_\lambda) + \frac{1}{\lambda}\|Bu_\lambda\|_{W'}^2 + \lambda\|p_\lambda\|_W^2 = a(u_\lambda, u_\lambda) + \frac{2}{\lambda}\|Bu_\lambda\|_{W'}^2 \\ & \qquad\qquad \leq 2\Big(a(u_\lambda, u_\lambda) + \frac{1}{\lambda}\|Bu_\lambda\|_{W'}^2\Big) \leq 2\langle f, u_\lambda\rangle_{V'\times V}, \end{aligned} \tag{4.3.146}$$

and the result follows immediately since

$$2\langle f, u_\lambda \rangle_{V'\times V} \le \frac{2\|f\|_{V'}^2}{\tilde{\alpha}} + \frac{\tilde{\alpha}\|u_\lambda\|_V^2}{2}. \qquad \square$$

As we consider the augmented problem as a perturbation, we shall now try to get an estimate on $\|u - u_\lambda\|_V$ and $\|p - p_\lambda\|_Q$ as $\lambda \to 0+$.

Proposition 4.3.5. *With the same assumptions of Theorem 4.3.4, and assuming moreover that* $g_2 = 0$*, let* $(u_\lambda, p_\lambda) \in V \times W$ *be the solution of problem* (4.3.127)–(4.3.128) *and* $(u, p) \in V \times Q$ *be the solution of problem* (4.2.6)*. We then have*

$$\|u - u_\lambda\|_V + \|\overline{p} - \overline{p}_\lambda\|_Q \le C \inf_{p_w \in W} \Big[\|p - p_w\|_Q + \sqrt{\lambda}\, \|p_w\|_W \Big]. \qquad (4.3.147)$$

Proof. Subtracting (4.3.127)–(4.3.128) from (4.2.6) with $q \in W$, one easily has

$$\begin{cases} a(u - u_\lambda, v) + b(v, p - p_\lambda) = 0, \ \forall v \in V, \\ b(u - u_\lambda, q) = \lambda\, (p_\lambda, q)_W, \ \forall q \in W. \end{cases} \qquad (4.3.148)$$

The argument of Proposition 4.3.3 cannot be applied, for it would require (in the second equation of (4.3.148)) $q \in Q$. However, let p_w be any element of W. We rewrite (4.3.148) as

$$\begin{cases} a(u - u_\lambda, v) + b(v, p_w - p_\lambda) = b(v, p_w - p), \ \forall\, v \in V, \\ b(u - u_\lambda, q) + \lambda\, (p_w - p_\lambda, q)_W = -\lambda\, (p_w, q)_W, \ \forall\, q \in W. \end{cases} \qquad (4.3.149)$$

We can now apply Theorem 4.3.4 with $\langle g_2, q \rangle = \lambda (p_w, q)_W$, and use estimate (4.3.132) to get

$$\|u - u_\lambda\|_V^2 + \|\overline{p}_w - \overline{p}_\lambda\|_Q^2 \le C\ (\|p_w - p\|_Q^2 + \lambda\ \|p_w\|_W^2). \qquad (4.3.150)$$

From the triangle inequality and the arbitrariness of p_w, one deduces (4.3.147). $\square$

Remark 4.3.11. The right-hand side of (4.3.147) will, in general, tend to zero with λ whenever p is more regular than just $p \in Q$. For instance, if $p \in W$, we can take $p_W = p$ and (4.3.147) will give

$$\|u - u_\lambda\|_V + \|\overline{p} - \overline{p}_\lambda\|_Q \le C \sqrt{\lambda}. \qquad (4.3.151)$$

See also Remark 4.3.14 here below. $\square$

Remark 4.3.12. The above result is not optimal. For instance, it does not reduce to the estimate (4.3.109) of Proposition 4.3.3 when $W = Q$. Let us suppose however, for simplicity, that $\mathrm{Im}B = Q'$, and consider the space W^+ defined as

$$W^+ := R_W^{-1}(Q'), \tag{4.3.152}$$

where R_W is as usual the Ritz operator in W as defined in Theorem 4.1.2. Since Q' is a dense subspace of W', we easily have that W^+ is a dense subspace of W, and moreover,

$$W^+ \hookrightarrow W \hookrightarrow Q. \tag{4.3.153}$$

Furthermore, for every $p^+ \in W^+$, there exists, from the definition (4.3.152), a $g \in Q'$ such that

$$(p^+, q)_W = (R_W^{-1} g, q)_W = \langle g, q\rangle_{W' \times W} \tag{4.3.154}$$

and, using (4.1.76), we have, for every $q \in W$,

$$(p^+, q)_W = \langle g, q\rangle_{W' \times W} = \langle g, q\rangle_{Q' \times Q} \le \|g\|_{Q'} \|q\|_Q \quad \forall\, q \in W, \tag{4.3.155}$$

where we also used (4.3.126). We can think of W^+ as a subspace of W made of more regular functions. Taking now $p_w = p_{w_+} \in W^+$, we can now go back to (4.3.149), considering this time that the right-hand side of the second equation (that is $\lambda\,(p_w, q)_W$) corresponds to the choice $g_2 = 0$ and $\langle g_1, q\rangle = \lambda\,(p_{w_+}, q)_W$ when using Theorem 4.3.4. From (4.3.132), we now have

$$\|u - u_\lambda\|_V + \|p - p_\lambda\|_Q \le C \left(\inf_{p_{w_+} \in W^+} \|p - p_{w_+}\|_Q + \lambda\, \|p_{w_+}\|_{W^+} \right) \tag{4.3.156}$$

where we also took into account that we assumed $\mathrm{Im}B = Q'$ and hence $\overline{p}_\lambda = p_\lambda$. Now, (4.3.156) is optimal for $W^+ = Q$. □

Remark 4.3.13. The argument of the above remark can easily be extended to the case in which $\mathrm{Im}B$ is closed but does not coincide with Q'. We simply have to take $W^+ := R_W^{-1} H^0$ (where H^0 is the polar space of $H \equiv \mathrm{Ker}B^t$), so that $(p_{w+}, q)_W \le C\,\|\overline{q}\|_Q$. In general, such a W^+ will not be dense in W, but in many applications p (belonging to $H^\perp$) will still belong to its closure (that is, you can still approximate p with a sequence of elements in W^+). □

Remark 4.3.14. In the spirit of Remark 4.3.11, we observe that, here again, the right-hand side of (4.3.156) can be bounded in terms of λ whenever p is more regular. In particular for $p \in W^+$, we would have

$$\|u - u_\lambda\|_V + \|p - p_\lambda\|_Q \le C\,\lambda. \tag{4.3.157}$$

□

Remark 4.3.15. In both (4.3.147) and (4.3.156), an **intermediate regularity** between Q and W^+ can provide an **intermediate speed of convergence** for $\lambda \to 0$. More precisely, let us suppose that p belongs to $[W^+, Q]_{\theta,\infty}$ for $0 < \theta < 1$. The space $[W^+, Q]_{\theta,\infty}$ is an interpolation space between W^+ and Q. We refer the reader to [62] for more details on these spaces. Here, we just recall that

$$\|p\|_{[W^+,Q]_{\theta,\infty}} = \sup_{\lambda>0} \inf_{p_{w_+}\in W^+} (\lambda^{-\theta} \|p - p_{w_+}\|_Q + \lambda^{1-\theta} \|p_{w_+}\|). \tag{4.3.158}$$

As a consequence, if $p \in [W^+, Q]_{\theta,\infty}$, then we have

$$\begin{aligned} &\inf_{p_{w_+}\in W^+} (\|p - p_{w_+}\| + \lambda \|p_{w_+}\|_{W^+}) \\ &\qquad = \lambda^{\theta} \inf_{p_{w_+}\in W^+} (\lambda^{-\theta} \|p - p_{w_+}\|_Q + \lambda^{1-\theta} \|p_{w_+}\|) \\ &\qquad\qquad \le \lambda^{\theta} \|p\|_{[W^+,Q]_{\theta,\infty}} \end{aligned}$$

(as in [62], Theorem 3.12). Hence, (4.3.156) can be written as

$$\|u - u_\lambda\|_V + \|p - p_\lambda\|_Q \le C\, \lambda^{\theta} \|p\|_{[W^+,Q]_{\theta,\infty}}. \tag{4.3.159}$$

Note that, in particular if $W^+ := H^1(\Omega)$ and $Q := L^2(\Omega)$, we have that

$$H^{\theta}(\Omega) \hookrightarrow [W^+, Q]_{\theta,\infty}.$$

Hence, if $p \in H^{\theta}(\Omega)$, we will also have $p \in [W^+, Q]_{\theta,\infty}$, and estimate (4.3.159) will hold true. Clearly, a similar argument could be applied to the estimate (4.3.147) for p having an intermediate regularity between Q and W. □