Chapter 10
Complements on Plate Problems

In this chapter, we shall present a few among many applications of mixed methods to plate problems. In the first section, we shall describe a mixed method for the linear thin plates theory and in the second, a dual hybrid method. In the last section, we shall report some recent results on the discretisation of the Mindlin-Reissner formulation for moderately thick plates.

10.1 A Mixed Fourth-Order Problem

10.1.1 The $\psi - \omega$ Biharmonic Problem

Let us now see, as a new example of application of the abstract results of Chaps. 4 and 5, some simple cases of fourth-order problems. We shall start with formulation (1.3.65) which we may now rewrite in the form (4.2.6) by setting

$$V := H^1(\Omega),\ Q := H^1_0(\Omega), \tag{10.1.1}$$

$$a(\omega,\phi) := \int_\Omega \omega\phi\,dx \qquad \forall\omega,\ \phi \in V, \tag{10.1.2}$$

$$b(\mu,\phi) := \int_\Omega \underline{\mathrm{grad}}\,\mu \cdot \underline{\mathrm{grad}}\,\phi\,dx \qquad \forall\mu \in Q,\ \phi \in V. \tag{10.1.3}$$

We shall denote by (ω,ψ) instead of (u,p) the solution of the problem in order to be consistent with the usual physical notations. It is easy to see that we are now in the situation of Sect. 3.6: the bilinear form $a(\omega,\phi)$ is not coercive on V (nor is it on Ker$B$ but only on $H := L^2(\Omega)$). A loss of accuracy is therefore to be expected. Another pitfall is that we cannot use the abstract existence results of Chap. 4 for the continuous problem and that we must deduce the existence of a

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solution through another channel. In the present case, we know that the solution of our mixed problem: *find* $\psi \in H_0^1(\Omega)$ *and* $\omega \in H^1(\Omega)$ *such that*

$$\begin{cases} \int_\Omega \omega\phi\, dx + \int_\Omega \underline{\mathrm{grad}}\,\psi \cdot \underline{\mathrm{grad}}\,\phi\, dx = 0 \quad \forall \phi \in H^1(\Omega), \\ \int_\Omega \underline{\mathrm{grad}}\,\omega \cdot \underline{\mathrm{grad}}\,\mu\, dx = \int_\Omega f\mu\, dx \quad \forall \mu \in H_0^1(\Omega), \end{cases} \tag{10.1.4}$$

should be a solution of a biharmonic problem

$$\triangle^2\psi = f,\ \psi \in H_0^2(\Omega). \tag{10.1.5}$$

From a regularity result on the biharmonic problem, we know, for instance, that if Ω is a convex polygon [234, 281, 362], for $f \in H^{-1}(\Omega)$, the solution of (10.1.5) belongs to $H^3(\Omega)$ so that $\omega = -\triangle\psi$ belongs to $H^1(\Omega)$. It is then direct to verify that we have thus obtained a solution of (10.1.4). This is an example of an "ill-posed" mixed problem. It should be remarked that the discussion of existence made above does not apply when the right-hand side of the first equation of (10.1.4) is not equal to zero.

To get a discrete problem, we take, following the notations of Chap. 2,

$$V_h := \mathcal{L}_k^1,\ Q_h := \mathcal{L}_k^1 \cap H_0^1(\Omega),\ k \geq 2. \tag{10.1.6}$$

The case $k = 1$ requires a more special analysis [197, 226, 344]. We then have that the constant $S(h)$, appearing in (5.2.40), can now be bounded by $S(h) \leq ch^{-1}$ so that a direct application of Proposition 5.2.6 gives

$$\|\omega - \omega_h\|_0 + \|\psi - \psi_h\|_1 \leq ch^{k-1}. \tag{10.1.7}$$

Indeed, the *inf-sup* condition is quite straightforward. The operator B is nothing here but the Laplace operator from $H^1(\Omega)$ to $H^{-1}(\Omega)$, which is obviously surjective. To check the discrete condition, we use the criterion of Proposition 5.4.3: *given* $\omega \in H^1(\Omega)$, *we want to build* $\omega_h \in V_h$ *such that*

$$\int_\Omega \underline{\mathrm{grad}}\,\omega_h \cdot \underline{\mathrm{grad}}\,\mu_h\, dx = \int_\Omega \underline{\mathrm{grad}}\,\omega \cdot \underline{\mathrm{grad}}\,\mu_h\, dx, \quad \forall \mu_h \in Q_h. \tag{10.1.8}$$

We recall, however, that we have chosen $Q_h \subset V_h$ so that (10.1.8) will, a fortiori, hold if we take $\mu_h \in V_h$. However, (10.1.8) is then nothing but a discrete Neumann problem for which a solution exists and can be chosen (it is defined up to an additive constant) so that

$$\|\omega_h\|_1 \leq c\, \|\omega\|_1. \tag{10.1.9}$$

It must be noted that the condition $Q_h \subset V_h$ is essential to the above result. In practice, this is not a restriction as (10.1.6) is a natural and efficient choice.

Result (10.1.7) is far from optimal and may suggest at first sight that the method is not worth being used. It can however be sharpened in two ways. First it is possible to raise the estimate on $|\omega - \omega_h|_0$ by half an order [197, 345] by a quite intricate analysis using L^∞-error estimates. The second way is a more direct variant of the duality method of Sect. 5.5.5 and shows that the expected accuracy can be obtained for $\psi \in H^3(\Omega)$, that is,

$$\|\psi - \psi_h\|_1 \leq ch^k, \tag{10.1.10}$$

and under a supplementary regularity assumption

$$\|\psi - \psi_h\|_0 \leq ch^{k+1}. \tag{10.1.11}$$

We refer the reader to [107, 189, 342, 345] and [192] for this analysis.

On the other hand, the particular structure of problem (10.1.4) allows the use of sophisticated but effective techniques for the numerical solution [150, 225, 227], so that this method and its variants have a considerable practical interest. In fact, it provides a correct setting for the widely used $\psi - \omega$ approximations in numerical fluid dynamics. We refer to [222] for more informations on this subject. Still in the case of fourth-order problems, we could also consider instead formulation (1.3.70) which is more related to plate bending problems. We now set

$$V := (H^1(\Omega))_s^{2\times 2},\ Q := H_0^1(\Omega), \tag{10.1.12}$$

and we define, following (1.3.70) for $\underline{\underline{\sigma}}$ and $\underline{\underline{\tau}}$ in V,

$$a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) := \frac{12(1-\nu^2)}{Et^3} \int_\Omega [(1+\nu)\,\underline{\underline{\sigma}} : \underline{\underline{\tau}} - \nu\, tr(\underline{\underline{\sigma}})\, tr(\underline{\underline{\tau}})]\, dx. \tag{10.1.13}$$

In order to consider a weaker form of the saddle point problem (1.3.70), we introduce

$$b(v, \underline{\underline{\tau}}) := \int_\Omega (\mathrm{d}\underline{\mathrm{i}}\mathrm{v}\, \underline{\underline{\tau}}) \cdot \mathrm{gr}\underline{\mathrm{a}}\mathrm{d}\, v\, dx = \int_\Omega \sum_{i,j} \frac{\partial \tau_{ij}}{\partial x_j} \frac{\partial v}{\partial x_i} dx. \tag{10.1.14}$$

This enables us to look for $w \in H_0^1(\Omega)$ instead of $H_0^2(\Omega)$, the second boundary condition being implied by this variational formulation as a natural condition. This is again an “ill-posed” mixed problem: we must obtain existence of a solution through a regularity result on the standard problem. Two approaches have been followed in the approximation of this mixed problem. One of them consists in taking (see [300])

$$V_h := (\mathcal{L}_k^1)_s^{2\times 2},\ Q_h := \mathcal{L}_k^1 \cap H_0^1(\Omega). \tag{10.1.15}$$

With respect to (10.1.14), it is, however, possible to use a second approach and to work not in $V = (H^1(\Omega))_s^{2\times 2}$ but in the weaker space

$$\underline{\underline{H}}(\mathrm{d}\underline{\mathrm{i}}\mathrm{v};\Omega)_s := \{\underline{\underline{\tau}} \mid \tau_{ij} = \tau_{ij},\ \tau_{ij} \in L^2(\Omega),\ \mathrm{d}\underline{\mathrm{i}}\mathrm{v}\, \underline{\underline{\tau}} \in (L^2(\Omega))^2\}. \tag{10.1.16}$$

Discretisations of this space can be built through composite elements. We refer to [262] and [27] for the analysis of this case.

In the first case, the results are the same as for the $\psi-\omega$ approximation discussed above. We get, by Proposition 5.2.6, an error estimate which is $O(h^{k-1})$. Duality methods (see [192]) would enable us to lift the estimate on ψ at the right level. For the second case, we can have optimal error estimates (see the above references).

10.1.2 Eigenvalues of the Biharmonic Problem

We now briefly consider the possibility of computing eigenvalues of the biharmonic problem using the elements introduced above. If we refer to Sect. 1.2.1 of Chap. 6, we are considering a $(0, g)$ situation. This means that, fortunately for us, we do not need a coercivity condition. Our eigenvalue problem can indeed be written as: *find* $\psi \in H_0^1(\Omega)$ *and* $\omega \in H^1(\Omega)$ *such that*

$$\begin{cases} \displaystyle\int_\Omega \omega\phi\, dx + \int_\Omega \underline{\operatorname{grad}}\, \psi \cdot \underline{\operatorname{grad}}\, \phi\, dx = 0\ \forall \phi \in H^1(\Omega), \\ \displaystyle\int_\Omega \underline{\operatorname{grad}}\, \omega \cdot \underline{\operatorname{grad}}\, \mu\, dx = \lambda \int_\Omega \psi\mu\, dx\ \forall \mu \in H_0^1(\Omega). \end{cases} \tag{10.1.17}$$

In the notation of Sect. 6.5.5, we have $V = H^1(\Omega)$ and $Q = H_0^1(\Omega)$. We take $H_Q = L^2(\Omega)$ and we assume that Ω is a convex polygon. We then have

$$\begin{aligned} V_{Q'}^0 =&\{z \in H^1(\Omega) :\ \exists v \in H_0^2(\Omega) \text{ with } z = \Delta v\} \\ =&\{z \in H^1(\Omega) :\ (z, \mu) = 0\ \forall \mu \in L^2(\Omega) \text{ with } \Delta\mu = 0\} \end{aligned} \tag{10.1.18}$$

so that with obvious notation

$$V_{Q'}^0 = H^3(\Omega) \cap H_0^2(\Omega). \tag{10.1.19}$$

For any given polygon, V_H^0 and Q_H^0 will be slightly more regular, according to the maximum angle (see e.g. [233]).

For every given regular sequence $\{\mathcal{T}_h\}$ of triangulations of Ω and for every integer $k \geq 2$, we can take as in [152, 224, 298]:

$$\begin{aligned} V_h^k &:= \mathcal{L}_k^1 \\ Q_h^k &:= \mathcal{L}_k^1 \cap H_0^1(\Omega). \end{aligned} \tag{10.1.20}$$

Notice that $Q_h^k = V_h^k \cap H_0^1(\Omega)$. We can now define $\Pi_h w$ in V_h as the solution of:

$$(\Delta \Pi_h w, \Delta v_h) = (\Delta w, \Delta v_h) \quad \forall v_h \in V_h^k. \tag{10.1.21}$$

Clearly, (6.5.54)–(6.5.56) hold. Similarly, (6.5.53) holds by taking p^I (here ψ^I) as the usual interpolant. On the other hand, to check (6.5.52), we have to assume quasi-uniformity of the decomposition and then proceed, as we did for Dirichlet's problem in (7.1.43), using an inverse inequality to obtain: *for* $v_h \in KerB_h$ *and* $q \in H^3(\Omega) \cap H_0^2(\Omega)$,

$$(\Delta v_h, \Delta q) = (\Delta v_h, \Delta q - \Delta q^I) \leq C h^{-1} ||v_h||_a \, C h^2 ||q||_3.$$

This shows the utility of the requirement $k \geq 2$. However, a more sophisticated proof, following the arguments of Scholz [344], shows that (1.2.50) also holds for $k = 1$.

We thus have checked all the hypotheses of Theorem 6.5.3 and our eigenvalue problem is properly posed.

10.2 Dual Hybrid Methods for Plate Bending Problems

We now consider as a final example an application of our general theory to hybrid methods. We go back again to Example 1.3.8 and set, for the sake of simplicity, $\nu = 0$ and $Et^3/12 = 1$. The consideration of the true values would not change the mathematical structure of the problem, but would result in more lengthy formulae. The condition $D_2^*(\underline{\underline{\tau}}) = f$ in (1.3.74) is, in general, difficult to enforce directly. Hence, following [321], we may think of working with stresses satisfying $D_2^*(\underline{\underline{\tau}}) = f$ inside each element of a given decomposition. This will imply that we have to enforce some continuity of the stresses by means of a Lagrangian multiplier; moreover, it will be convenient to assume $f \in L^2(\Omega)$. In order to make the exposition clearer, we need some Green's formulae. We have indeed, on any triangle K of a triangulation $\mathcal{T}_h$ of Ω,

$$\int_K \underline{\underline{\tau}} : \underline{\underline{D}}_2(v)\, dx = \int_K D_2^*(\underline{\underline{\tau}}) v \, dx + \int_{\partial K} [M_{nn}(\underline{\underline{\tau}}) \frac{\partial v}{\partial n} - K_n(\underline{\underline{\tau}}) v] \, ds \qquad (10.2.1)$$

for all $\underline{\underline{\tau}} \in (H^2(T))_s^{2\times 2}$ and $v \in H^2(T)$, where

$$M_{nn}(\underline{\underline{\tau}}) := (\underline{\underline{\tau}} \cdot \underline{n}) \cdot \underline{n}, \qquad (10.2.2)$$

$$K_n(\underline{\underline{\tau}}) := \frac{\partial}{\partial n} \operatorname{tr}(\underline{\underline{\tau}}) - \frac{\partial}{\partial t}[(\underline{\underline{\tau}} \cdot \underline{n}) \cdot \underline{t}], \; \underline{t} = \text{tangent unit vector.} \qquad (10.2.3)$$

It is essential, in the definition of K_n, to consider the derivative $\partial/\partial t$ *in the distributional sense*, that is, to take into account the *jumps* of $(\underline{\underline{\tau}} \cdot \underline{n}) \cdot \underline{t}$ at the corners of K (the so-called *corner forces*).

It is easy to check that the condition $D_2^*(\underline{\underline{\tau}}) = f$ in Ω is equivalent to

$$\begin{cases} D_2^*(\underline{\underline{\tau}}) = f \text{ in each } T, \\ \displaystyle\sum_K \int_{\partial K} [M_{nn}(\underline{\underline{\tau}}) \frac{\partial v}{\partial n} - K_n(\underline{\underline{\tau}}) v]\, ds = 0, \qquad \forall v \in H_0^2(\Omega). \end{cases} \tag{10.2.4}$$

Setting

$$\begin{aligned} b(\underline{\underline{\tau}}, v) &:= \sum_K \int_{\partial K} [M_{nn}(\underline{\underline{\tau}}) \frac{\partial v}{\partial n} - K_n(\underline{\underline{\tau}}) v]\, ds \\ &\equiv \int_\Omega \underline{\underline{\tau}} : \underline{\underline{D}}_2(v)\, dx - \sum_T \int_T D_2^*(\underline{\underline{\tau}}) v\, dx, \end{aligned} \tag{10.2.5}$$

$$V_f(\mathcal{T}_h) := \{\underline{\underline{\tau}} \,|\, \underline{\underline{\tau}} \in (L^2(\Omega))_s^{2\times 2},\ D_2^*(\underline{\underline{\tau}}) = f \text{ in each } K\}, \tag{10.2.6}$$

the problem can now be written as

$$\inf_{\underline{\underline{\tau}} \in V_f(\mathcal{T}_h)} \sup_{v \in H_0^2} \frac{1}{2} \|\underline{\underline{\tau}}\|_0^2 - b(\underline{\underline{\tau}}, v). \tag{10.2.7}$$

If now $\underline{\underline{\sigma}}^f$ is a given element of $V_f(\mathcal{T}_h)$, that is, a *particular solution* of $D_2^*(\underline{\underline{\sigma}}) = f$ in each K, we have

$$\begin{cases} (\underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}^f, \underline{\underline{\tau}}) - b(\underline{\underline{\tau}}, w) = 0 \qquad \forall \underline{\underline{\tau}} \in V_0(\mathcal{T}_h), \\ \qquad b(\underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}^f, v) = 0 \qquad \forall v \in H_0^2(\Omega), \end{cases} \tag{10.2.8}$$

where obviously $\underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}^f := \underline{\underline{\sigma}}$. Problem (10.2.8) has now the form (4.2.6), where $V = V_0(\mathcal{T}_h)$, $Q = H_0^2$, $a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) = (\underline{\underline{\sigma}}, \underline{\underline{\tau}})$, and $b(\underline{\underline{\tau}}, v)$ is given by (10.2.5). The right-hand side is obviously $-(\underline{\underline{\sigma}}^f, \underline{\underline{\tau}})$ for the first equation and $-b(\underline{\underline{\sigma}}^f, v)$ for the second equation. It is natural to use in V the L^2-norm, and in Q the norm $\|v\|_Q = \|\underline{\underline{D}}_2 v\|_V = \|\underline{\underline{D}}_2 v\|_0$. It is clear that condition (4.2.12), that is, the ellipticity of $a(\cdot,\cdot)$, is trivially satisfied in the whole V (and not only in Ker$B$) with $\alpha = 1$. A different value for E, t, ν would obviously yield a different value for α, but the V-ellipticity will still be true. It is clear that KerB^t cannot be empty; indeed, any v with support in a single K will satisfy $b(\underline{\underline{\tau}}, v) = 0$ for all $\underline{\underline{\tau}}$, and hence is a zero energy mode. However, it is not difficult to see that ImB is closed.

Proposition 10.2.1. *The image of B is a closed subset of* $Q' := H^{-2}(\Omega)$.

Proof. We have to show that if a sequence $\chi_n := B\underline{\underline{\tau}}_n$ converges to χ in H^{-2}, then $\chi = B\underline{\underline{\tau}}$ for some $\underline{\underline{\tau}} \in V_0(\mathcal{T}_h) =: V$. We first note that

$$\text{if } \underline{\underline{\tau}} \in V_0(\mathcal{T}_h) \text{ and } \phi \in H_0^2(\Omega), \text{ then } b(\underline{\underline{\tau}}, \phi) \equiv (\underline{\underline{\tau}}, \underline{\underline{D}}_2 \phi), \tag{10.2.9}$$

which is quite obvious from (10.2.5) and (10.2.6). Now let $\phi \in H_0^2(\Omega)$ be such that $\triangle^2\phi = \chi$ and let $\underline{\underline{\tau}} := \underline{\underline{D}}_2\phi$ (so that $D_2^*\underline{\underline{\tau}} = \chi$). For every $\phi \in H_0^2$, we have

$$\langle \chi, \phi\rangle_{H^{-2}\times H_0^2} = \langle D_2^*\underline{\underline{\tau}}, \phi\rangle_{H^{-2}\times H_0^2} = (\underline{\underline{\tau}}, \underline{\underline{D}}_2\phi). \tag{10.2.10}$$

Now, since $\chi_n = B\underline{\underline{\tau}}_n \to \chi$ in H^{-2}, we have

$$(\underline{\underline{\tau}}_n, \underline{\underline{D}}_2\phi) = b(\underline{\underline{\tau}}_n, \phi) = \langle B\underline{\underline{\tau}}_n, \phi\rangle = \langle \chi_n, \phi\rangle \to \langle \chi, \phi\rangle = (\underline{\underline{\tau}}, \underline{\underline{D}}_2\phi), \tag{10.2.11}$$

that is, $(\underline{\underline{\tau}}_n - \underline{\underline{\tau}}, \underline{\underline{D}}_2\phi) \to 0$ for all $\phi \in H_0^2(\Omega)$. This easily implies $D_2^*\underline{\underline{\tau}} = 0$ in each T, so that $\underline{\underline{\tau}} \in V_0(\mathcal{T}_h)$. Hence, $\langle \chi, \phi\rangle = (\underline{\underline{\tau}}, \underline{\underline{D}}_2\phi) = b(\underline{\underline{\tau}}, \phi) = \langle B\underline{\underline{\tau}}, \phi\rangle$, that is, $\chi \in \mathrm{Im}B$. □

Proposition 10.2.2. *We have* $\mathrm{Ker}B^t = \prod_K H_0^2(K)$.

Proof. It is obvious from (10.2.5) that if $\phi|_K \in H_0^2(K)$ for all K, then $b(\underline{\underline{\tau}}, \phi) = 0$ $\forall\underline{\underline{\tau}}$ and hence $\phi \in \mathrm{Ker}B^t$. Therefore, we only need to prove that $\mathrm{Ker}B^t \subset \prod_K H_0^2(K)$. For this, let $\phi \in \mathrm{Ker}B^t$, that is,

$$b(\underline{\underline{\tau}}, \phi) \equiv (\underline{\underline{\tau}}, \underline{\underline{D}}_2\phi) = 0 \qquad \forall\underline{\underline{\tau}} \in V_0(\mathcal{T}_h). \tag{10.2.12}$$

We want to show that $\phi \in \prod_K(H_0^2(K))$, that is,

$$\phi|_K \in H_0^2(K) \text{ for all } K. \tag{10.2.13}$$

Let ψ be defined in each K by

$$\psi \in H_0^2(K) \text{ and } \triangle^2\psi = \triangle^2\phi; \tag{10.2.14}$$

clearly, $(\underline{\underline{\tau}}, \underline{\underline{D}}_2\psi) = 0$ for all $\underline{\underline{\tau}}$ in $V_0(\mathcal{T}_0)$ so that from (10.2.12),

$$b(\underline{\underline{\tau}}, \psi - \phi) = (\underline{\underline{\tau}}, \underline{\underline{D}}_2(\psi - \phi)) = 0 \qquad \forall\underline{\underline{\tau}} \in V_0(\mathcal{T}_h). \tag{10.2.15}$$

However, $D_2^*\underline{\underline{D}}_2(\psi - \phi) = \triangle^2(\psi - \phi) = 0$ in each K, so that we can take $\underline{\underline{\tau}} = \underline{\underline{D}}_2(\psi - \phi)$ in (10.2.15) and obtain $\underline{\underline{D}}_2(\psi - \phi) \equiv 0$. Since both ψ and ϕ are in $H_0^2(\Omega)$, this implies $\psi = \phi$ so that from (10.2.14), we get (10.2.13). □

Proposition 10.2.3. *We have*

$$\|\phi\|_{Q/\mathrm{Ker}B^t} = \|\underline{\underline{D}}_2\bar{\phi}\|_0, \tag{10.2.16}$$

where $\bar{\phi}$ is the function in $H_0^2(\Omega)$ such that

$$\phi - \bar{\phi} \in H_0^2(K) \textit{ for each } K, \tag{10.2.17}$$

$$\triangle^2\bar{\phi} = 0 \qquad \textit{in each } K. \tag{10.2.18}$$

Proof. By definition, we have

$$\|\phi\|_{Q/\mathrm{Ker}B^t} = \inf_{\psi\in\mathrm{Ker}B^t} \|\phi-\psi\|_Q. \tag{10.2.19}$$

Now from Proposition 10.2.2 and the definition of $\|\chi\|_Q := \|\underline{\underline{D}}_2\chi\|_0$, we have

$$\|\phi\|_{Q/\mathrm{Ker}B^t} = \inf_{\psi\in\prod_K H_0^2(K)} \|\underline{\underline{D}}^2(\phi-\psi)\|_{0,K}. \tag{10.2.20}$$

It is now an easy matter to check that, for each K,

$$\inf_{\psi\in H_2^2(K)} \|\underline{\underline{D}}_2(\phi-\psi)\|_{0,K}^2 = \inf_{(\psi-\phi)\in H_0^2(K)} \|\underline{\underline{D}}_2\psi\|_{0,K}^2 = \|\underline{\underline{D}}_2\bar{\phi}\|_{0,K}^2, \tag{10.2.21}$$

for $\bar{\phi}$ defined in (10.2.17) and (10.2.18). Hence, (10.2.21) and (10.2.20) prove (10.2.16). □

We are now able to prove the *inf-sup* condition

$$\sup_{\underline{\underline{\tau}}\in V_0(\mathcal{T}_h)} \frac{b(\underline{\underline{\tau}},\phi)}{\|\underline{\underline{\tau}}\|_0\|\phi\|_{Q/\mathrm{Ker}B^t}} = \sup_{\underline{\underline{\tau}}\in V_0(\mathcal{T}_h)} \frac{(\underline{\underline{\tau}},\underline{\underline{D}}_2\phi)}{\|\underline{\underline{\tau}}\|_0\|\underline{\underline{D}}_2\bar{\phi}\|_0}$$
$$\geq \frac{(\underline{\underline{D}}_2\bar{\phi},\underline{\underline{D}}_2\phi)}{\|\underline{\underline{D}}_2\bar{\phi}\|_0^2} = 1 \tag{10.2.22}$$

because $\phi-\bar{\phi}$ is the projection (in Q) of ϕ onto $\mathrm{Ker}B^t$ so that $\bar{\phi}$ and $\phi-\bar{\phi}$ are orthogonal in Q.

Remark 10.2.1. A way of getting rid of $\mathrm{Ker}B^t$ (which is infinite dimensional) is to consider as a space of Lagrange multipliers the space

$$\tilde{Q} := \{\phi\,|\,\phi\in H_0^2(\Omega),\ \triangle^2\phi = 0 \text{ in each } T\}. \tag{10.2.23}$$

This is what has been done in [114, 127]. The drawback in the choice (10.2.23) is that the actual transversal displacement w does not belong to $\tilde{Q}$ so that, as a solution, we have the unique function $\bar{w}$ in $\tilde{Q}$ that coincides with w (with its first derivatives) at the inter-element boundaries (as in (10.2.17) and (10.2.18)). □

Let us continue our analysis of problem (10.2.8). We already noted that (4.2.12) is satisfied in our case. Hence, we have to check that the right-hand side of the second equation in (10.2.8) (that is $-b(\underline{\underline{\sigma}}^f, v)$) is in $\mathrm{Im}B$; this means that we have to find a particular solution of (10.2.8), which is obvious by taking $\underline{\underline{\sigma}}^f := \underline{\underline{D}}_2 w - \underline{\underline{\sigma}}^f$.

We can now go to the discretisation of (10.2.8); for this, we have to choose subspaces $V_h \subset V_0(\mathcal{T}_h)$ and $Q_h \subset Q$. For instance, for any triple (m,r,s) of integers, we may choose

$$V_h^m := (\mathcal{L}_m^0(\mathcal{T}_h))_s^{2\times 2} \cap V_0(\mathcal{T}_h), \tag{10.2.24}$$

$$Q_h^{r,s} := \{\phi \mid \phi \in H_0^2(\Omega),\ \phi|_{\partial T} \in T_r(\partial T),\ \frac{\partial \phi}{\partial n}|_{\partial T} \in R_s(\partial T) \quad \forall T \in \mathcal{T}_h\}. \tag{10.2.25}$$

Note that V_h is made of tensor-valued polynomials of degree $\leq m$ which are completely discontinuous from one element to another and verify $D_2^* \underline{\underline{\tau}} = 0$ in each T. On the other hand, Q_h is clearly infinite dimensional (which is quite unusual); however, this does not show up in the computations, where only the values of ϕ and $\partial\phi/\partial n$ on $\mathcal{E}_h$ are considered. To get coercivity, we now have to choose (m, r, s) in such a way that $\operatorname{Ker} B_h^t \subset \operatorname{Ker} B^t$. This means, in our case, that we have to show

$$\begin{cases} \text{if } \phi \in Q_h^{r,s} \text{ and } b(\underline{\underline{\tau}}, \phi) = 0 \qquad \forall \underline{\underline{\tau}} \in V_h^m \text{ (that is, if } \phi \in \operatorname{Ker} B_h^t), \\ \text{then } \phi = \underline{\operatorname{grad}}\, \phi = 0 \text{ on } \mathcal{E}_h, \text{ (that is, } \phi \in \operatorname{Ker} B^t). \end{cases} \tag{10.2.26}$$

The proof of (10.2.26) (or, rather, the finding of sufficient conditions on m for having (10.2.26)) will be easier with the following characterisation of V_h^m.

Lemma 10.2.1. *We have*

$$V_h^m \equiv \underline{\underline{S}}\,[(\mathcal{L}_{m+1}^0(\mathcal{T}_h))^2], \tag{10.2.27}$$

where $\underline{\underline{S}}$ *is defined, for* $\underline{q} = (\alpha, \beta)$,

$$\underline{\underline{S}} : (\underline{q}) \to \begin{pmatrix} \partial\alpha/\partial y & -\frac{1}{2}(\partial\alpha/\partial x + \partial\beta/\partial y) \\ -\frac{1}{2}(\partial\alpha/\partial x + \partial\beta/\partial y) & \partial\beta/\partial x \end{pmatrix}. \tag{10.2.28}$$

Proof. The inclusion $\underline{\underline{S}}\,[(\mathcal{L}_{m+1}^0(\mathcal{T}_h))^2] \subseteq V_h^m$ is trivial; the opposite inclusion is an exercise (see [127] for more details). □

We now notice that if $\underline{\underline{\tau}} = \underline{\underline{S}}(\underline{q})$, then

$$b(\underline{\underline{\tau}}, v) = \sum_K \int_{\partial K} \underline{\operatorname{grad}}\, v \cdot \frac{\partial}{\partial t} \underline{q}\, ds, \tag{10.2.29}$$

where $\underline{t}$ is the tangent to ∂T. We also notice that

$$\begin{cases} \phi \in H_0^2(\Omega) \text{ and } \underline{\operatorname{grad}}\, \phi = \text{ constant on } \mathcal{E}_h \\ \text{imply } \phi = 0 \text{ and } \underline{\operatorname{grad}}\, \phi = 0 \text{ on } \mathcal{E}_h. \end{cases} \tag{10.2.30}$$

We may now use (10.2.27)–(10.2.30) in (10.2.26) which becomes

$$\begin{cases} \text{if } \phi \in Q_h^{r,s} \text{ and } \displaystyle\sum_K \int_{\partial K} \underline{\operatorname{grad}}\, \phi \cdot \frac{\partial}{\partial t} \underline{q}\, ds = 0 \quad \forall \underline{q} \in (\mathcal{L}_{m+1}^0(\mathcal{T}_h))^2, \\ \text{then } \underline{\operatorname{grad}}\, \phi = \text{ constant on } \mathcal{E}_h. \end{cases} \tag{10.2.31}$$

Now, (10.2.31) is implied by

$$\begin{cases} \text{if } \phi \in Q_h^{r,s} \text{ and } \displaystyle\int_{\partial K} \underline{\text{grad}}\,\phi \cdot \frac{\partial}{\partial t}\underline{q}\, ds = 0 \quad \forall \underline{q} \in (P_{m+1}(K))^2 \\ \text{then } \underline{\text{grad}}\,\phi = \text{ constant on } \partial T \end{cases} \tag{10.2.32}$$

(but not vice-versa). Now let k be the degree of $\underline{\text{grad}}\,\phi$ on ∂T, that is,

$$k = \max(s, r-1). \tag{10.2.33}$$

The following technical lemma is proved in [127].

Lemma 10.2.2. *If $\phi \in H^1(K)$ and $\phi|_{e_i} \in P_k(e_i)$ $(i = 1,2,3)$, and if*

$$\int_{\partial K} \phi \frac{\partial q}{\partial t} ds = 0 \quad \forall q \in P_k(K), \tag{10.2.34}$$

then

$$\phi|_{e_i} = c\ell_k^i(s) + c_1, \ (i = 1,2,3), \tag{10.2.35}$$

where, on each e_i, we define ℓ_k^i as the kth Legendre polynomial (normalised with value 1 in the second endpoint in the anticlockwise order).

Formula (10.2.35), for k odd, directly implies that ϕ is constant on ∂K. We therefore have a first result.

Proposition 10.2.4. *If $m+1 = k = \max(r-1, s)$ and k is odd, then (10.2.32) holds.*

If $m+1$ is even, we can apply Lemma 10.2.2 to both $\partial\phi/\partial x$ and $\partial\phi/\partial y$ and get

$$\frac{\partial \phi}{\partial x} = c\ell_k^i + c_1, \ \frac{\partial \phi}{\partial y} = \gamma\ell_k^i + \gamma \tag{10.2.36}$$

on each e_i. If now $r-1 \neq s$, there must exist a combination of $\partial\phi/\partial x$ and $\partial\phi/\partial y$ on each e_i (to get $\partial\phi/\partial n$) which has degree lower than k. This easily implies that both $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are constants on ∂K. We therefore have the following result:

Proposition 10.2.5. *If $m+1 = k = \max(r-1, s)$ and $r-1 \neq s$, then (10.2.32) holds.*

We are finally left with the last and worst case in which $r-1 = s$ is *even*. We have several escapes. First, brutally, we may take $m+1 = k+1$. It is easy to see that, then, (10.2.32) always holds. As a second possibility, we may take $m+1 = k$ and *enrich* $(\mathcal{L}_{m+1}^0(\mathcal{T}_h))^2$ into $(\mathcal{L}_{m+1}^0(\mathcal{T}_h))_{enr}^2$ by adding, in each K, a pair of functions $\underline{q}$ in $(P_{m+1})^2$ such that $\partial q_j/\partial t|_{e_i} = \ell_k^i$ ($j = 1,2$ and $= 1,2,3$). Again, it is easy to check that (10.2.32) is satisfied if we take the enriched space $(\mathcal{L}_{m+1}^0(\mathcal{T}_h))_{enr}^2$ instead of the original one. Then, of course, we must consider $V_{h,enr}^m = \underline{\underline{S}}[(\mathcal{L}_{m+1}^0(\mathcal{T}_h))_{enr}^2]$

instead of V_h. Finally, we might give up (10.2.32) and go directly to (10.2.31). It is easy to check that in (10.2.36), the values of c, c_1, γ, and γ_1 must remain constants from one K to another due to the continuity of $\underline{\mathrm{grad}}\,\phi|_e$ across the edges. Hence, since $\phi \in H^2_0(\Omega)$, we must have $c = c_1 = \gamma = \gamma_1 = 0$ and, *actually*, (10.2.31) *holds for* $m + 1 = k = \max(r-1, s)$ *in any case*, that is, also for $r - 1 = s =$ even. However, we shall see in a moment that (10.2.32) has other basic advantages over (10.2.31) that we are not very willing to give up. We summarise the results in the following theorem.

Theorem 10.2.1. *The condition* $\mathrm{Ker}B^t_h \subset \mathrm{Ker}B^t$ *holds whenever*

$$m + 1 \geq k = \max(r-1, s). \tag{10.2.37}$$

Moreover, (10.2.32) *holds when* (10.2.37) *is satisfied, unless* $r - 1 = s =$ *even. In that case,* (10.2.32) *is satisfied by taking* $m + 1 > k$ *or by using an enriched* $V^{k-1}_{h,enr}$ *(between* V^{k-1}_h *and* V^k_h*) as described above.*

The condition $\mathrm{Ker}B^t_h = \mathrm{Ker}B^t$ implies, by Proposition 5.5.2, the existence of an operator Π_h from $V_0(\mathcal{T}_h)$ to V^m_h such that

$$b(\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}}, v) = 0 \qquad \forall v \in Q^{r,s}_h. \tag{10.2.38}$$

However, in view of the use of Proposition 5.4.3, we would also like to show that there exists a Π_h which satisfies (10.2.38) and

$$\|\Pi_h\underline{\underline{\tau}}\|_0 \leq c\,\|\underline{\underline{\tau}}\|_0, \qquad \forall \underline{\underline{\tau}} \in V_0(\mathcal{T}_h), \tag{10.2.39}$$

with c *independent of* h. Since V^m_h is finite dimensional, (10.2.39) will always hold, but the constant might depend on h. Now, if (10.2.32) holds, we see that Π_h can be defined *element by element*. Now, the dimension of $V^m_h|_K$ depends only on m, but not on h. A continuous dependence argument on the shape of the element can now prove (10.2.39) without major difficulty (but, to be honest, not quickly); we refer to [127] for a detailed proof of (10.2.39). Once we have (10.2.38) and (10.2.39), we apply Proposition 5.4.3 to prove the discrete *inf-sup* condition. Then, Theorem 5.2.5 immediately gives

$$\begin{aligned} \|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_0 &= \|\underline{\underline{D}}_2(w - \tilde{w}_h)\|_0 \\ &\leq c\Big\{\inf_{\underline{\underline{\tau}}\in V^m_h} \|\underline{\underline{\sigma}}^0 - \underline{\underline{\tau}}\|_0 + \inf_{\phi\in Q^{r,s}_h} \|\underline{\underline{D}}_2(w-\phi)\|_0\Big\}, \end{aligned} \tag{10.2.40}$$

where $\tilde{w}_h$ is the (unique) element in $Q^{r,s}_h$ that satisfies $\Delta^2\tilde{w}_h = f$ in each K and belongs to the set of discrete solutions.

Theorem 10.2.2. *If $m+1 \geq \max(r-1,s)$ (and $m+1 > s$ for $r-1 = s$ is even), we have*

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_0 + \|\underline{\underline{D}}_2(w - \tilde{w}_h)\|_0 \leq ch^t \left(\|w\|_{t+2} + \sum_K \|\underline{\underline{\sigma}}^f\|^2_{t,K}\right)^{\frac{1}{2}} \tag{10.2.41}$$

with $t = \min(m+1, r-1, s)$.

Proof. The proof is obvious from (10.2.40) and the standard approximation results. □

We end this section with a few computational remarks. First, we notice that our discretisation of (10.2.8) has obviously the matrix structure

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix}, \tag{10.2.42}$$

where A, corresponding to the approximation of the identity in V_h^m, is obviously block diagonal because V_h^m is made of discontinuous tensors. Hence, one usually makes an a priori inversion of A, to end with the matrix $BA^{-1}B^t$ which operates on the unknown w_h and is symmetric and positive definite. However, the computation of the right-hand side is, in general, a weak point in the use of dual hybrid methods, unless f is very special (zero, Dirac mass, constant, etc.) and allows the use of a simple $\underline{\underline{\sigma}}^f$. A few computational tricks for dealing with more general cases can be found in [127, 289, 290]. Here, we recall from [115] a simple method that works for low-order approximations (more precisely, when t in Theorem 10.2.2 is ≤ 2). We first define the operator $R :=$ orthogonal projection onto V_h. We then remark that the discretisations (10.2.24) and (10.2.25) of (10.2.8) may be written as

$$\begin{cases} (\underline{\underline{\sigma}}_h^0 + \underline{\underline{\sigma}}^f, \underline{\underline{\tau}}) = (\underline{\underline{D}}_2 w_h, \underline{\underline{\tau}}) & \forall \underline{\underline{\tau}} \in V_h, \\ (\underline{\underline{\sigma}}_h + \underline{\underline{\sigma}}^f, \underline{\underline{D}}_2 \phi) = (f, \phi) & \forall \phi \in Q_h. \end{cases} \tag{10.2.43}$$

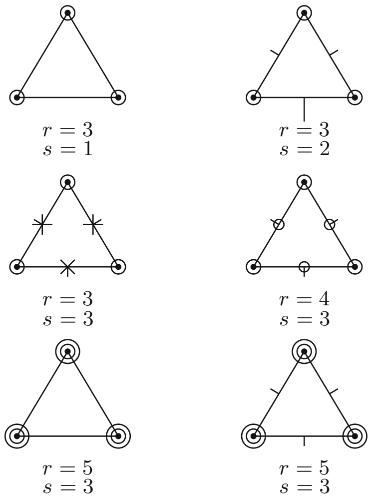
Solving a priori in $\underline{\underline{\sigma}}_h^0$ from the first equation and substituting into the second equation, we obtain

$$(R\underline{\underline{D}}_2 w_h, \underline{\underline{D}}_2 \phi) = (f, \phi) - (\underline{\underline{\sigma}}^f - R\underline{\underline{\sigma}}^f, \underline{\underline{D}}_2 \phi) \qquad \forall \phi \in Q_h. \tag{10.2.44}$$

Now, the left-hand side of (10.2.44) corresponds to the matrix $BA^{-1}B^t$ acting on the unknown w_h. The right-hand side is actually *computable* because both $(f, \phi) - (\underline{\underline{\sigma}}^f, \underline{\underline{D}}_2 \phi)$ and $(R\underline{\underline{\sigma}}^f, \underline{\underline{D}}_2 \phi)$ depend (looking carefully) only on the values of ϕ and its gradient at the inter-element boundaries. However, the computation, in general, is not easy. Therefore, in some cases, it can be convenient to use a rough approximation of it, for instance

$$(f, \phi) - (\underline{\underline{\sigma}}^f - R\underline{\underline{\sigma}}^f, \underline{\underline{D}}_2 \phi) \simeq \sum_K \frac{meas(K)}{3} \sum_{j=1}^{3} f(V_j)\phi(V_j), \tag{10.2.45}$$

Fig. 10.1 Some common choices for the space $Q_h^{r,s}$



Symbol	Values of
•	ϕ
⊙	$\underline{\text{grad}}\ \phi$
◎	$\partial^2\phi/\partial_n\partial_t$
—	$\underline{\underline{D}}$
×	$\partial\phi/\partial n$
⊸	$\partial\phi/\partial t$, $\partial\phi/\partial t$, $\partial^2\phi/\partial_n\partial_t$

where the V_j are the vertices of K. It can be shown (see [115]) that this involves an additional error of order $O(h^2)$ (essentially because V_h contains *all* piecewise linear stress functions and therefore $\|\underline{\underline{\sigma}}^f - R\underline{\underline{\sigma}}^f\|_0 \leq ch^2$) and hence this procedure is recommended whenever $t \leq 2$ in (10.2.41).

Finally, we provide a few remarks on the choice of the degrees of freedom in V_h^m and $Q_h^{r,s}$. As we have seen, the unknown $\underline{\underline{\sigma}}_h^0$ is usually eliminated a priori at the element level due to the complete discontinuity of V_h^m. As a consequence, the choice of the degrees of freedom in V_h^m is of little relevance. In general, it is more convenient to start from $(\mathcal{L}_{m+1}^0(\mathcal{T}_h))^2$ and to derive V_h through (10.2.27).

When m is "large" (say $m \geq 4$, to fix the ideas), however, the resulting matrix A can be severely ill-conditioned unless the degrees of freedom in V_h^m are chosen in a suitable way. We refer to [289,290] for a discussion of this point. On the other hand, the degrees of freedom in $Q_h^{r,s}$ are the ones that count in the final stiffness matrix, and, besides, they have to take into account the C^{-1} continuity requirements. We sketch in Fig. 10.1 some commonly used choices for different values of r and s.

Remark 10.2.2. It is impossible to say what is, in general, the best choice for r and s. Numerical evidence shows obviously that the accuracy/number of degrees of freedom ratio is improved for large r and s, at least when the solution is smooth.

However, it is clear that the simplest (and most widely used) choice $r = 3$, $s = 1$ allows a much easier implementation. Similar considerations also hold with the choice of m, in particular in the case of an even $r - 1 = s$, for instance for $r = 3$, $s = 2$. The use of the enriched $V^1_{h,enr}$ implies a smaller matrix to be inverted on each element than with the "brutal" choice V^2_h (11×11 instead of 17×17), but the latter may allow some simplification in writing the program. □

Remark 10.2.3. We have used, so far, homogeneous Dirichlet boundary conditions corresponding to a clamped plate. Nothing changes when considering non-homogeneous Dirichlet conditions. If, instead, a part of the plate is simply supported ($w =$ given; $M_{nn} = 0$) or free ($M_{nn} = 0$; $K_n = 0$), then we have two possibilities for dealing with them. Let us discuss a simple case: let $\partial\Omega = \Gamma_D \cup \Gamma_N$ and assume that $w = \partial w/\partial n = 0$ on Γ_D and $M_n = K_n = 0$ on Γ_N. One possibility is to choose $Q^{r,s}_h$ so that its elements vanish only on Γ_D, and to let V^m_h unchanged. In this case, the conditions $M_n = K_n = 0$ on Γ_N will be satisfied only in a *weak* sense. A second possibility is to choose V^m_h in such a way that its elements satisfy, a priori, the boundary condition $M_n = K_n = 0$ on Γ_N. However, care must be taken in this case to enrich conveniently the stress field in the boundary elements so that the *inf-sup* condition still holds. Otherwise, a loss in the order of convergence is likely to occur. □

Remark 10.2.4. One may think to use other discretisations of the dual hybrid formulations than the ones discussed here (see, for instance, the previous remarks). In any case, the *inf-sup* condition should be checked. Although this is not evident from our discussion (because we wanted to deal with many cases at the same time), nevertheless, it is true that to check the *inf-sup* condition in hybrid methods is basically an *easy* task. What is really needed is the following: for any element K, the only displacement modes with zero energy on K, that is, the only modes ϕ such that

$$\int_{\partial K} \Big(M_{nn}(\underline{\underline{\tau}})\phi/n - K_n(\underline{\underline{\tau}})\phi \Big)\, ds = 0 \qquad \forall \underline{\underline{\tau}} \in V_h, \tag{10.2.46}$$

must be the *rigid* modes (that is, $\underline{\mathrm{grad}}\,\phi = $ constant on T). If this condition is violated, one can expect trouble (minor or major, depending on the cases). □

10.3 Mixed Methods for Linear Thin Plates

We consider the variational formulation of a problem discussed in Chap. 1 which we recall here for the convenience of the reader. We had

$$L(\underline{\underline{\sigma}}, w) = \inf_{\underline{\underline{\tau}} \in (L^2(\Omega))^{2\times 2}_s} \ \sup_{\phi \in H^2_0(\Omega)} L(\underline{\underline{\tau}}, \phi) \tag{10.3.1}$$

where

$$L(\underline{\underline{\tau}}, \phi) := \frac{1}{2}\left(\frac{12}{Et^3}\right)\int_\Omega [(1+\nu)\underline{\underline{\tau}} : \underline{\underline{\tau}} - \nu(\mathrm{tr}(\underline{\underline{\tau}}))^2]\,dx \\ - \int_\Omega \underline{\underline{\tau}} : \underline{\underline{D}}_2\phi\,dx + \int_\Omega f\phi\,dx \tag{10.3.2}$$

$$E = \text{Young's modulus,} \tag{10.3.3}$$

$$t = \text{thickness of the plate,} \tag{10.3.4}$$

$$\nu = \text{Poisson's ratio,} \tag{10.3.5}$$

$$f = \text{transversal load / unit surface,} \tag{10.3.6}$$

$$w = \text{transversal displacement,} \tag{10.3.7}$$

$$\underline{\underline{\sigma}} = \text{stresses (in the Kirchoff assumption).} \tag{10.3.8}$$

In order to use a more compact notation, we set

$$C\underline{\underline{\tau}} := \frac{1}{2}Et^3((1+\nu)\underline{\underline{\tau}} - \nu\,\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{\delta}}) \tag{10.3.9}$$

and write $L(\underline{\underline{\tau}}, \phi)$ as

$$L(\underline{\underline{\tau}}, \phi) = \frac{1}{2}(C\underline{\underline{\tau}}, \underline{\underline{\tau}}) - (\underline{\underline{\tau}}, \underline{\underline{D}}_2\phi) + (f, \phi). \tag{10.3.10}$$

Assume that we are given a triangulation $\mathcal{T}_h$ of Ω and that we are willing to discretise the stress field $\underline{\underline{\sigma}}$ by means of piecewise polynomials for which the normal bending moment

$$M_{nn}(\underline{\underline{\sigma}}) = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} \tag{10.3.11}$$

is continuous from one element to another. We recall the following Green's formulae,

$$\int_K \underline{\underline{\tau}} : \underline{\underline{D}}_2\phi\,dx = -\int_K \underline{\mathrm{div}}\,\underline{\underline{\tau}} \cdot \underline{\mathrm{grad}}\,\phi\,dx + \int_{\partial K} M_{nn}(\underline{\underline{\tau}})\frac{\partial\phi}{\partial n}\,ds \\ + \int_{\partial K} M_{nt}(\underline{\underline{\tau}})\frac{\partial\phi}{\partial t}\,ds, \tag{10.3.12}$$

$$-\int_K \underline{\mathrm{div}}\,\underline{\underline{\tau}} \cdot \underline{\mathrm{grad}}\,\phi\,dx = \int_K D_2^*(\underline{\underline{\tau}})\,\phi\,dx - \int_{\partial K} Q_n(\underline{\underline{\tau}})\,\phi\,ds, \tag{10.3.13}$$

valid for all $\underline{\underline{\tau}}$ and ϕ smooth in K; we recall again that, here, $\underline{t}$ is the unit tangent (anticlockwise) vector and

$$M_{nt}(\underline{\underline{\tau}}) = (\underline{\underline{\tau}} \cdot \underline{n}) \cdot \underline{t}, \quad Q_n(\underline{\underline{\tau}}) = \underline{\mathrm{div}}(\underline{\underline{\tau}}) \cdot \underline{n}. \tag{10.3.14}$$

If $M_{nn}(\underline{\underline{\tau}})$ is continuous and ϕ is smooth, we can write

$$L(\underline{\underline{\tau}},\phi)=\frac{1}{2}(C\underline{\underline{\tau}},\underline{\underline{\tau}})+\sum_K\Big\{\int_K \underline{\mathrm{div}}(\underline{\underline{\tau}})\cdot\underline{\mathrm{grad}}\,\phi\,dx-\int_{\partial K}M_{nt}(\underline{\underline{\tau}})\frac{\partial\phi}{\partial t}ds\Big\}+(f,\phi). \tag{10.3.15}$$

A little functional analysis shows that every integral in (10.3.15) makes sense (at least as a suitable duality pairing), provided $\underline{\underline{\tau}}$ and ϕ are, respectively, in the following spaces:

$$V:=\{\underline{\underline{\tau}}\mid\underline{\underline{\tau}}|_K\in(H^1(\Omega))^{2\times 2}_s,\ M_{nn}(\underline{\underline{\tau}})\text{ continuous}\}, \tag{10.3.16}$$

$$Q:=W^{1,p}(\Omega),\ \ p>2. \tag{10.3.17}$$

Remark 10.3.1 (For mathematicians). We have to choose $p>2$ in (10.3.17) because for $\phi\in H^1(K)$ we have $\partial\phi/\partial t\in H^{-1/2}(\partial K)$ whereas $M_{nt}(\underline{\underline{\tau}})$ is in $\prod_{e_i}H^{1/2}(e_i)$ but *not* in $H^{1/2}(\partial K)$. On the other hand, for $\phi\in W^{1,p}$, we have $\partial\phi/\partial t\in W^{-1/p,p}(\partial K)$. Since $M_{nt}(\underline{\underline{\tau}})$ is in $H^s(\partial K)$ for all $s<1/2$ and since $W^{-1/p,p}(\partial K)\subset H^{-1}(\partial K)$ for $s>1/p$, the boundary integral which appears in (10.3.15) can now be interpreted as a duality pairing between $H^{-s}(\partial K)$ and $H^s(\partial K)$ for $1/p<s<1/2$ (which is possible since $p>2$). □

The Euler equations of (10.3.15) can now be written as:

$$(C\underline{\underline{\sigma}},\underline{\underline{\tau}})+\sum_K\Big\{\int_K \underline{\mathrm{div}}(\underline{\underline{\tau}})\cdot\underline{\mathrm{grad}}\,w\,dx-\int_{\partial K}M_{nt}(\underline{\underline{\tau}})\frac{\partial\phi}{\partial t}\,ds\Big\}=0\qquad\forall\underline{\underline{\tau}}\in V, \tag{10.3.18}$$

$$\sum_K\Big\{\int_K \underline{\mathrm{div}}(\underline{\underline{\tau}})\cdot\underline{\mathrm{grad}}\,\phi\,dx-\int_{\partial K}M_{nt}(\underline{\underline{\tau}})\frac{\partial\phi}{\partial t}ds\Big\}=(-f,\phi)\qquad\forall\phi\in Q, \tag{10.3.19}$$

which has the form (5.1.9) if we set

$$a(\underline{\underline{\sigma}},\underline{\underline{\tau}}):=(C\underline{\underline{\sigma}},\underline{\underline{\tau}}), \tag{10.3.20}$$

$$b(\underline{\underline{\sigma}},\phi):=\sum_K\Big\{\int_K \underline{\mathrm{div}}(\underline{\underline{\tau}})\cdot\underline{\mathrm{grad}}\,\phi\,dx-\int_{\partial K}M_{nt}(\underline{\underline{\tau}})\frac{\partial\phi}{\partial t}\,ds\Big\}. \tag{10.3.21}$$

Unfortunately, problem (10.3.18) and (10.3.19), as it stands, does not satisfy any of the conditions given in Chap. 4 in order to have a well posed problem. However, we know that the original problem (1.2.4) has a solution w. If $\underline{\underline{\sigma}}=C^{-1}(\underline{\underline{D}}_2w)$ is in $H^1(\Omega)$, that is if the solution w of (1.2.4) is smooth enough, it is easy to check that the pair $(\underline{\underline{\sigma}},w)$ solves (10.3.18) and (10.3.19). Hence, we only have to prove the *uniqueness* of the solution of (10.3.18) and (10.3.19).

Proposition 10.3.1. *Problem* (10.3.18) *and* (10.3.19) *has a unique solution.*

Proof. It is obvious that

$$a(\underline{\underline{\tau}},\underline{\underline{\tau}}) \geq \alpha\ \|\underline{\underline{\tau}}\|_0^2, \qquad \forall \underline{\underline{\tau}} \in V. \tag{10.3.22}$$

Let us now check a weaker *inf-sup* condition. For every ϕ in Q, we define $\underline{\underline{\tau}}(\phi)$ by

$$\tau_{11} = \tau_{22} = \phi,\ \tau_{12} = \tau_{21} = 0. \tag{10.3.23}$$

It is immediate to check that $M_{nt}(\underline{\underline{\tau}})$ is continuous across the inter-element boundaries, so that

$$\sum_K \int_{\partial K} M_{nt}(\underline{\underline{\tau}}(\phi))\frac{\partial \phi}{\partial t} ds = 0 \tag{10.3.24}$$

and therefore

$$b(\underline{\underline{\tau}}(\phi),\phi) = |\phi|_{1,\Omega}^2. \tag{10.3.25}$$

It is also easy to check, using (10.3.23) and the Poincaré's inequality (1.2.14), that

$$\|\underline{\underline{\tau}}(\phi)\|_V \leq c\ |\phi|_{1,\Omega}; \tag{10.3.26}$$

hence, we have from (10.3.25) and (10.3.26) that

$$\begin{aligned}\inf_{\phi\in H_0^1(\Omega)}\sup_{\underline{\underline{\tau}}\in V}\frac{b(\underline{\underline{\tau}},\phi)}{\|\underline{\underline{\tau}}\|_V|\phi|_{1,\Omega}} &\geq \inf_{\phi\in H_0^1(\Omega)}\frac{b(\underline{\underline{\tau}}(\phi),\phi)}{\|\underline{\underline{\tau}}(\phi)\|_V|\phi|_{1,\Omega}}\\ &\geq \frac{|\phi|_{1,\Omega}}{\|\underline{\underline{\tau}}(\phi)\|_V} \geq \frac{1}{c} > 0.\end{aligned} \tag{10.3.27}$$

Now using (10.3.22) and (10.3.27), we have the desired uniqueness by standard arguments. □

We are now ready to discretise our problem. Following [132] and [261], for any integer $k \geq 0$, we set

$$V_h = (\mathcal{L}_k^0)_s^{2\times 2} \cap V \tag{10.3.28}$$

$$Q_h = \mathcal{L}_{k+1}^1 \tag{10.3.29}$$

with the notation of Chap. 2. Note that the space V_h in (10.3.28) is made of tensors whose normal bending moment is continuous across the inter-element boundaries. The degrees of freedom for Q_h will be the usual ones (see Sect. 2.2). As degrees of freedom for V_h, we may choose, for instance, the following ones:

$$\int_e M_{nn}(\underline{\underline{\tau}})p(s)\ ds \qquad \forall p \in P_k(e), \quad \forall e \in \mathcal{E}_h, \tag{10.3.30}$$

$$\int_T \underline{\underline{\tau}} : \underline{\underline{p}}\ dx \qquad \forall \underline{\underline{p}} \in (P_{k-1}(T))_s^{2\times 2},\ \forall K \in \mathcal{T}_h,\ (k \geq 1). \tag{10.3.31}$$

The possibility of choosing (10.3.30) and (10.3.31) as degrees of freedom in V_h is shown by the following lemma and by a standard dimensional count.

Lemma 10.3.1. *Let* $\underline{\underline{\tau}} \in (P_{k-1}(T))_s^{2\times 2}$ *be such that*

$$\int_{e_i} M_{nn}(\underline{\underline{\tau}}) p(s)\, ds = 0 \qquad \forall p \in P_k(e_i),\ (i = 1, 2, 3), \tag{10.3.32}$$

$$\int_K \underline{\underline{\tau}} : \underline{\underline{p}}\, dx = 0 \qquad \forall \underline{\underline{p}} \in (P_{k-1}(T))_s^{2\times},\ (k \geq 1). \tag{10.3.33}$$

Then, $\underline{\underline{\tau}} \equiv 0$.

Proof. We only give a hint of the proof. From (10.3.32), we get $M_{nn}(\underline{\underline{\tau}}) = 0$. We first show that $D_2^*(\underline{\underline{\tau}}) = 0$. This is trivial for $k \leq 1$; for $k > 1$, take $\underline{\underline{p}} = \underline{\underline{D}}_2 b$ with $b = b_3 D_2^* \underline{\underline{\tau}}$ in (10.3.33) to get $\int_K b_3 (D_2^*(\underline{\underline{\tau}}))^2\, dx = 0$ and hence $D_2^*(\underline{\underline{\tau}}) = 0$. Now use the formula (see Sect. 10.2)

$$\int_K \underline{\underline{\tau}} : \underline{\underline{D}}_2 \phi = \int_K D_2^*(\underline{\underline{\tau}})\, \phi + \int_{\partial K} [M_{nn}(\underline{\underline{\tau}}) \frac{\partial \phi}{\partial n} - \mathcal{K}_n(\underline{\underline{\tau}}) \phi]\, ds \tag{10.3.34}$$

for $\phi \in P_{k+1}(T)$; thus, we get

$$\int_{\partial K} \mathcal{K}_n(\underline{\underline{\tau}})\, ds = 0 \qquad \forall \phi \in P_{k+1}(T), \tag{10.3.35}$$

and easily obtain that $\mathcal{K}_n(\underline{\underline{\tau}}) = 0$. It is now simple to show that $\underline{\underline{\tau}} = \underline{\underline{S}}(\underline{q})$ (see (10.2.27) for the definition of $\underline{\underline{S}}$) for some $\underline{q} \in (P_{k+1}(K))^2$ with $\underline{q} = 0$ on ∂K. Therefore, q_1 (for instance) has the form $b_3 z$ with $z \in P_{k-2}(K)$. Let us now choose, in (10.3.33), p_{11} such that $\partial p_{11}/\partial y$ and $p_{12} = p_{22} = 0$. We then get

$$0 = \int_K \tau_{11} p_{11}\, dx = \int_K \frac{\partial q_1}{\partial y} p_{11}\, dx = -\int_K q_1 z\, dx = -\int_K b_3 z^2\, dx \tag{10.3.36}$$

so that $z = 0$ and $q_1 = 0$. Similarly, one proves that $q_2 = 0$. □

We are now able to define the operator Π_h. We set, for $\underline{\underline{\tau}} \in V$,

$$\int_e M_{nn}(\Pi_h \underline{\underline{\tau}} - \underline{\underline{\tau}})\ p(s)\, ds = 0 \qquad \forall p \in P_k(e), \qquad \forall e \in \mathcal{E}_h, \tag{10.3.37}$$

$$\int_K (\Pi_h \underline{\underline{\tau}} - \underline{\underline{\tau}}) : \underline{\underline{p}}\, ds = 0 \qquad \forall \underline{\underline{p}} \in (P_{k-1}(K))_s^{2\times 2}, \qquad \forall K \in \mathcal{T}_h. \tag{10.3.38}$$

Lemma 10.3.2. *Let* Π_h *be defined by* (10.3.37) *and* (10.3.38). *Then, we have*

$$\|\Pi_h \underline{\underline{\tau}}\|_V \leq c\ \|\underline{\underline{\tau}}\|_V \qquad \forall \underline{\underline{\tau}} \in V \tag{10.3.39}$$

and

$$b(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}, \phi_h) = 0 \qquad \forall \underline{\underline{\tau}} \in V \qquad \forall \phi_h \in Q_h. \tag{10.3.40}$$

Proof. Formula (10.3.39) is easy to check. Let us prove (10.3.40). From (10.3.12) and (10.3.21), we have

$$b(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}, \phi) = -\sum_K \Big\{ \int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{\underline{D}}_2 \phi \, dx - \int_{\partial K} M_{nn}(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \frac{\partial \phi}{\partial n} ds \Big\} \tag{10.3.41}$$

and from (10.3.41), (10.3.37), and (10.3.38), we get (10.3.40). □

Lemma 10.3.3. *If* $\underline{\underline{\tau}}_h \in V_h$ *is such that*

$$b(\underline{\underline{\tau}}_h, \phi_h) = 0, \qquad \forall \phi_h \in Q_h, \tag{10.3.42}$$

then

$$b(\underline{\underline{\tau}}_h, \phi) = 0, \qquad \forall \phi \in Q. \tag{10.3.43}$$

Proof. We have, from (10.3.13) and (10.3.21),

$$b(\underline{\underline{\tau}}_h, \phi) = -\sum_K \Big\{ \int_K D_2^*(\underline{\underline{\tau}}_h) \phi \, dx + \int_{\partial K} [M_{nt}(\underline{\underline{\tau}}_h) \frac{\partial \phi}{\partial t} - Q_n(\underline{\underline{\tau}}_h) \phi] \, ds \Big\}. \tag{10.3.44}$$

Integrating $\int_{\partial K} M_{nt} \frac{\partial \phi}{\partial t} ds$ by parts and recalling the definition of $\mathcal{K}_n$ in (10.2.3), we then have

$$b(\underline{\underline{\tau}}_h, \phi) = -\sum_K \Big\{ \int_K D_2^*(\underline{\underline{\tau}}_h) \; \phi \, dx - \int_{\partial K} \mathcal{K}_n(\underline{\underline{\tau}}_h) \phi \, ds \Big\}. \tag{10.3.45}$$

Note that (10.3.45) holds for any $\underline{\underline{\tau}}_h$ and ϕ piecewise smooth. If now (10.3.42) holds, we first have $D_2^*(\underline{\underline{\tau}}_h) = 0$ by choosing $\phi|_K = b_3 D_2^*(\underline{\underline{\tau}}_h)$ (for $k \geq 2$, otherwise the property is trivial). Hence, we are left with

$$\sum_K \int_{\partial K} \mathcal{K}_n(\underline{\underline{\tau}}_h) \phi_h \, ds = 0 \qquad \forall \phi \in Q_h. \tag{10.3.46}$$

Since $\mathcal{K}_n$ is made of Dirac measures at the vertices and of polynomials of degree less or equal to $k-1$ on each edge, it is easy to see that (10.3.46) implies $\mathcal{K}_n(\underline{\underline{\tau}}_h) = 0$. Therefore, we have proved that if $\underline{\underline{\tau}}_h \in V_h$ satisfies (10.3.42), then $D_2^*(\underline{\underline{\tau}}_h) = 0$ and $\mathcal{K}_n(\underline{\underline{\tau}}_h) = 0$. We now insert those two equations into (10.3.45) and we get (10.3.43). □

This last property was denoted, in Chap. 5, as $Z_h(0) \subset Z(0)$. We have seen that, together with the existence of the operator Π_h, this property is so important that it can provide optimal error estimates even in desperate situations (no ellipticity, no *inf-sup* condition) like ours.

Actually, we first remark that (10.3.27) and Lemma 10.3.2 provide, through Proposition 5.4.3, the following *inf-sup* type condition:

$$\inf_{\phi_h \in Q_h} \sup_{\underline{\underline{\tau}}_h \in V_h} \frac{b(\underline{\underline{\tau}}_h, \phi_h)}{\|\underline{\underline{\tau}}_h\|_v |\phi_h|_1} \geq c > 0 \text{ (c independent of h)}. \tag{10.3.47}$$

On the other hand, since Q_h and V_h are finite dimensional, (10.3.22) and (10.3.47) ensure that the discrete problem has a unique solution. We are now ready for error estimates.

Proposition 10.3.2. *If* $(\underline{\underline{\sigma}}, w)$ *is the solution of* (10.3.18) *and* (10.3.19) *and* $(\underline{\underline{\sigma}}_h, w_h)$ *is the discrete solution of* (10.3.18) *and* (10.3.19)*, then, through* (10.3.28) *and* (10.3.29)*, we have*

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_0 \leq c\ \|\underline{\underline{\sigma}} - \Pi_h \underline{\underline{\sigma}}\|_0. \tag{10.3.48}$$

□

The proof is immediate from the standard theory of Chap. 5.

From (10.3.48) and standard approximation results, we then have

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_0 \leq ch^{k+1} \|\underline{\underline{\sigma}}\|_{k+1}. \tag{10.3.49}$$

Proposition 10.3.3. *With the notation of Proposition 10.3.2, we have*

$$\|w - w_h\|_1 \leq c\ \{h^{k+1} \|\underline{\underline{\sigma}}\|_{k+1} + h^{k+1} \|w\|_{k+2}\}. \tag{10.3.50}$$

Proof. Let $\phi_h \in Q_h$ to be chosen. From (10.3.47), we have for some $\underline{\underline{\tau}}_h \in V_h$

$$\begin{aligned} c\|\phi_h - w_h\|_1 \|\underline{\underline{\tau}}_h\|_V &\leq b(\underline{\underline{\tau}}_h, \phi_h - w_h) \\ &= b(\underline{\underline{\tau}}_h, \phi_h - w) + b(\underline{\underline{\tau}}_h, w - w_h) \\ &= b(\underline{\underline{\tau}}_h, \phi_h - w) + a(\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h). \end{aligned} \tag{10.3.51}$$

It is now elementary to see that ϕ_h can be chosen in such a way that

$$\int_e p \frac{\partial}{\partial t}(w - \phi_h)\, ds = 0 \qquad \forall p \in P_k(e), \quad \forall e \in \mathcal{E}_h, \tag{10.3.52}$$

$$\|w - \phi_h\|_1 \leq ch^{k+1} \|w\|_{k+2}. \tag{10.3.53}$$

With such a choice, we have

$$\begin{aligned} b(\underline{\underline{\tau}}_h, w-\phi_h) &= \sum_K \int_K \underline{\mathrm{div}}(\underline{\underline{\tau}}_h)\cdot \underline{\mathrm{grad}}(w-\phi_h)dx \\ &\le \|\underline{\underline{\tau}}_h\|_V \|w-\phi_h\|_1 \\ &\le ch^{k+1}\ \|\underline{\underline{\tau}}_h\|_V \|w\|_{k+2}, \end{aligned} \tag{10.3.54}$$

so that from (10.3.51), (10.3.54) and (10.3.49) we get (10.3.50). □

Remark 10.3.2. Result (10.3.50) is not optimal as far as the regularity of w is involved. Actually, it says

$$\|w-w_h\|_1 \le ch^s\ \|w\|_{s+2}\ (s \le k+1), \tag{10.3.55}$$

while an $(s+1)$-norm on w should be enough for optimality. Furthermore, a more sophisticated analysis [44, 192] shows that

$$\|w-w_h\|_r \le ch^{s-r}\|w\|_s\ (s \le k+2,\ 0 \le r \le 1) \tag{10.3.56}$$

for $k \ge 1$ and

$$\|w-w_h\|_0 \le ch^2\ \|w\|_4 \text{ for } k=0. \tag{10.3.57}$$

In particular, the approach of [44] has a special interest because, by a suitable use of mesh-dependent norms in V_h and Q_h, they can show that the discretised problem (in the new norms) satisfy the abstract assumptions (5.2.33) and (5.2.34) so that optimal error estimates (in the new norms) can be directly obtained by Theorem 5.2.5. Their approach also works for other fourth-order mixed methods, like those analysed in Sects. 10.1 and 10.2. □

Remark 10.3.3. For the actual solution of the discretised problem, the most convenient method is to disconnect the continuity of $\underline{\underline{\sigma}}_h \cdot \underline{n}$ and to enforce it back via Lagrange multipliers λ_h. Then, one eliminates $\underline{\underline{\sigma}}_h$ at the element level and one solves a symmetric and positive definite system for the unknowns λ_h and w_h. The procedure is identical to the one described in Sect. 7.2 and we refer to it for a detailed description. As far as the error estimates for the Lagrange multipliers λ_h are concerned, recent results have been obtained in [158]. □

Remark 10.3.4. It is interesting to analyse the relationship between the mixed methods described here and some nonconforming methods for fourth-order problems. For instance, the following result is proved in [23]. Let us consider the space built by means of the Morley element $\mathcal{L}_2^{2,NC}$ described in Example 2.2.6 and let us define

$$a_h(\psi_h,\phi_h) := \frac{Et^3}{12(1-\nu^2)}\sum_K \int_K [(1-\nu)\underline{\underline{D}}_2\psi_h : \underline{\underline{D}}_2\phi_h + \nu\triangle\psi_h\triangle\phi_h]\, dx. \tag{10.3.58}$$

For every $\phi_h \in \mathcal{L}_2^{2,NC}$, let ϕ_h^I be the piecewise linear interpolant of ϕ_h (that is $\phi_h^I \in \mathcal{L}_1^1$ and $\phi_h^I = \phi_h$ at the vertices). Consider now the modified Morley problem: *find* $\psi_h \in \mathcal{L}_2^{2,NC}$ *such that*

$$a_h(\psi_h, \phi_h) = (f, \phi_h^I) \qquad \forall \phi_h \in \mathcal{L}_2^{2,NC}. \tag{10.3.59}$$

Then, we have

$$\underline{\underline{D}}_2 \psi_h = \underline{\underline{\sigma}}_h, \quad \psi_h^I = w^h, \tag{10.3.60}$$

where $(\underline{\underline{\sigma}}_h, w_h)$ is the discrete solution of the mixed problem (10.3.18) and (10.3.19) through (10.3.28) and (10.3.29) for $k = 0$. We note explicitly that, in the case of variable coefficients, the equivalence is more complicated. Also note that $\partial\psi_h/\partial n|_e = \lambda_h|_e$ for all $e \in \mathcal{E}_h$, where λ_h is the Lagrange multiplier introduced in the previous remark. Notice that we have, from [23],

$$\|\psi_h - w\|_{1,h} \leq ch^2 \, \|w\|_3, \tag{10.3.61}$$

which improves (10.3.50) and (10.3.57) since it requires only H^3-regularity on w. This is particularly striking since the cost for computing ψ_h is cheaper (or equal, using λ_h) than the cost for computing $(\underline{\underline{\sigma}}_h, w_h)$. □

10.4 Moderately Thick Plates

10.4.1 Generalities

We end this chapter with a hint on the theory for the so-called "Mindlin–Reissner plates". The corresponding model stands somehow in between the standard three-dimensional linear elasticity and the two-dimensional Kirchhoff theory for thin plates. Let us recall it briefly. Assume that we are given a three-dimensional elastic body that, in absence of forces, occupies the region $\Omega \times]-t, t[$, where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and $t > 0$ is "small" (but not "too small") with respect to diam(Ω). This is what we call a "moderately thick" plate. We shall assume, for the sake of simplicity, that the plate is clamped along the entire boundary $\partial\Omega \times]-t, t[$ and that a vertical load $\underline{f} = (0, 0, f_3)$ is imposed.

Here below, we present the "Mindlin-Reissner" model following the classical engineering "derivation". Such derivation is questionable, from the mathematical point of view, at some points, but it has the clear merit of being short and simple. From the mathematical point of view, the derivation of [35] is much more convincing, but it is surely longer and more complicated. As the aim of this book is mainly concentrated on the mathematical properties of models and on their discretisations rather than on the modelling aspects, we decided to stick to the simpler choice.

The Mindlin model assumes that the "in plane" displacements u_1 and u_2 have the form

$$u_1(x, y, z) = -z\,\theta_1(x, y),\ u_2(x, y, z) = -z\,\theta_2(x, y) \tag{10.4.1}$$

and that the "transversal" displacement u_3 has the form

$$u_3(x, y, z) = w(x, y). \tag{10.4.2}$$

The corresponding strain field therefore takes the form:

$$\begin{cases} \varepsilon_{11} = -z\,\partial\theta_1/\partial x;\ \varepsilon_{22} = -z\,\partial\theta_2/\partial y;\ \varepsilon_{33} = 0; \\ 2\varepsilon_{12} = -z(\partial\theta_1/\partial y + \partial\theta_2/\partial x);\ 2\varepsilon_{13} = \partial w/\partial x - \theta_1;\ 2\varepsilon_{23} = \partial w/\partial y - \theta_2; \end{cases} \tag{10.4.3}$$

and assuming a linear elastic material, the stress field is

$$\begin{cases} \sigma_{11} = (\varepsilon_{11} + \nu\varepsilon_{22})\ E/(1-\nu^2);\ \sigma_{22} = (\varepsilon_{22} + \nu_{11})E/(1-\nu^2); \\ \sigma_{ij} = \varepsilon_{ij}E/(1+\nu);\ i, j = 1, 2, 3,\ i \neq j. \end{cases} \tag{10.4.4}$$

If we now write the total potential energy

$$\Pi = \frac{1}{2}\int_{\Omega\times]-t,t[} (\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - 2\ \underline{f}\cdot\underline{u})\,dx\,dy\,dz \tag{10.4.5}$$

in terms of θ and w through (10.4.1)–(10.4.4), we obtain (after some calculations)

$$\Pi = \frac{t^3}{2}(a(\underline{\theta}, \underline{\theta}) + \frac{\lambda t}{2}\int_\Omega |\,\underline{\mathrm{grad}}\,w - \underline{\theta}|^2\,dx\,dy - \int_{\Omega\times]-t,t[} f_3\ w\,dx\,dy\,dz, \tag{10.4.6}$$

where the symmetric bilinear form a is identified by

$$\begin{aligned} a(\underline{\theta}, \underline{\eta}) := \frac{E}{12(1-\nu^2)}\int_\Omega \Big[\Big(\frac{\partial\theta_1}{\partial x} + \frac{\nu\partial\theta_2}{\partial y}\Big)\frac{\partial\eta_1}{\partial x} + \Big(\frac{\nu\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y}\Big)\frac{\partial\eta_2}{\partial y} \\ + \frac{(1-\nu)}{2}\Big(\frac{\partial\theta_1}{\partial y} + \frac{\partial\theta_2}{\partial x}\Big)\Big(\frac{\partial\eta_1}{\partial y} + \frac{\partial\eta_2}{\partial x}\Big)\Big]\,dx\,dy, \end{aligned} \tag{10.4.7}$$

where

$$\lambda := \frac{E\ k}{2(1+\nu)} \tag{10.4.8}$$

and k is a correction factor which is often used to account for the "nonconformity" of (10.4.4). Indeed, from (10.4.1)–(10.4.4), we deduce that σ_{13} and σ_{23} are *constants* in z, whereas the physical problem has $\sigma_{13} = \sigma_{23} = 0$ on the upper and lower face of the plate: $\Omega \times \{t\}$ and $\Omega \times \{-t\}$; hence, (10.4.4) is often corrected by assuming that σ_{13} and σ_{23} behave parabolically in z, vanishing for $z = \pm t$ and assuming the value (10.4.4) for $z = 0$. For a mathematically more convincing justification of the classical $5/6$ factor, we refer again to [35]. Actually, for the sake of simplicity, we shall assume, from now on, that

$$\lambda = 1.$$

In fact, as far as we do not expect the true value (10.4.8) to go to zero or to $+\infty$, assuming $\lambda = 1$ will just change the numerical value of the constants appearing in the stability estimates or in the a priori error estimates, but it will not change the behaviour in function of the thickness t or the mesh-size h.

10.4.2 The Mathematical Formulation

The assumed boundary conditions lead to the kinematic constraints

$$\theta_1 = \theta_2 = w = 0 \text{ on } \partial\Omega. \tag{10.4.9}$$

Hence, we define the spaces

$$\Theta := (H^1_0(\Omega))^2;\ Z := H^1_0(\Omega);\ V := \Theta \times Z \tag{10.4.10}$$

with the norm

$$\|(\underline{\eta}, \zeta)\|^2_V := \|\underline{\eta}\|^2_1 + \|\zeta\|^2_1. \tag{10.4.11}$$

When convenient, the generic element of V will be denoted $\underline{v} = (\underline{\eta}, \zeta)$ with $\underline{\eta} = (\eta_1, \eta_2) \in \Theta$ and $\zeta \in Z$. We finally recall the Korn inequality

$$\exists\, \alpha_{Korn} > 0 \text{ such that } a(\underline{\eta}, \underline{\eta}) \ge \alpha_{Korn}\, \|\underline{\eta}\|^2_1 \quad \forall \underline{\eta} \in \Theta, \tag{10.4.12}$$

where, from now on in this section, the symmetric bilinear form a will be the one given in (10.4.7).

It is easy to check that, for any fixed $t > 0$, functional (10.4.5) has a unique minimiser $(\underline{\theta}, w)$ on V which satisfies

$$t^3\, a(\underline{\theta}, \underline{\eta}) + t \int_\Omega (\underline{\text{grad}}\, w - \underline{\theta}) \cdot \underline{\eta}\, dx\, dy = 0 \quad \forall \underline{\eta} \in \Theta, \tag{10.4.13}$$

$$t \int_\Omega (\underline{\text{grad}}\, w - \underline{\theta}) \cdot \underline{\text{grad}}\, \zeta\, dx\, dy = \int_{\Omega\times]-t,t[} f_3\, \zeta\, dx\, dy\, dz \quad \forall \zeta \in Z. \tag{10.4.14}$$

In particular, we have

$$\frac{t^3}{2} a(\underline{\eta}, \underline{\eta}) + \frac{t}{2} \int_\Omega |\, \underline{\text{grad}}\, \zeta - \underline{\eta}|^2\, dx\, dy \ge c(t)\ (\|\underline{\eta}\|^2_1 + \|\zeta\|^2_1), \tag{10.4.15}$$

for any $\underline{v} = (\eta, \zeta) \in V$. Note that for fixed t, (10.4.15) always guarantees that (10.4.13), (10.4.14) is a nice linear elliptic problem so that, for instance, any reasonable conforming approximation of V will have optimal order of convergence.

The troubles start when we take a *small* t; then, the constant in (10.4.15) deteriorates and so does the constant in front of the optimal error bound. In practice, it is well known that if we use "any reasonable conforming approximation of V", we will get pretty bad answers for small t. Here, we shall make an analysis of the nature of the trouble. We shall also give some sufficient conditions on the discretisation so that it stays good for t smaller and smaller. The one-dimensional case was treated in [15], but the two-dimensional case, as we shall see, is more complicated.

The first thing that we have to do is to construct a *sequence* of physical problems $\mathcal{P}_t$ (for $t > 0$ and, say $t < T_0$) that fulfil the following requirements:

(1) Each $\mathcal{P}_t$ is of type (10.4.13) and (10.4.14) and so has a unique solution $\underline{\theta}(t)$, $w(t)$;
(2) There exists two constants c_1, c_2 with $0 < c_1 < c_2$ such that

$$c_1 \le \|\underline{\theta}(t)\|_1 + \|w(t)\|_1 \le c_2 \qquad \forall\, t \in]0, T_0[. \tag{10.4.16}$$

A possible answer is to fix Ω, E, and ν, and to choose, for each $t > 0$, the load $f_3(x, y, z)$ of the form

$$f_3(x, y, z) := \frac{t^2}{2} f(x, y), \tag{10.4.17}$$

with $g(x, y)$ fixed (once and for all) independent of t. It is clear that (10.4.17) implies

$$\int_{\Omega\times]-t,t[} f_3\, w\, dx\, dy\, dz = t^3 \int_\Omega f\, w\, dx\, dy = t^3 (f, w), \tag{10.4.18}$$

where as usual (f, w) denotes the $L^2(\Omega)$ inner product or (with an abuse of notation) whenever f is assumed to be only in $H^{-1}(\Omega)$, the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Hence, dividing (10.4.6) by t^3, each problem $\mathcal{P}_t$ will amount to minimise, in V,

$$\Pi_t(\underline{\theta}, w) = \frac{1}{2} a(\underline{\theta}, \underline{\theta}) + \frac{t^{-2}}{2} \int |\,\underline{\mathrm{grad}}\, w - \underline{\theta}|^2\, dx\, dy - (f, w). \tag{10.4.19}$$

Proposition 10.4.1. *Let* $\underline{\theta}(t)$, $w(t)$ *be the minimiser of* (10.4.19) *in* V. *Then,* (10.4.16) *holds with* c_1 *and* c_2 *independent of* t.

Proof. We obviously have

$$a(\underline{\theta}, \underline{\theta}) + t^{-2} \|\,\underline{\mathrm{grad}}\, w - \underline{\theta}\|_0^2 = (f, w). \tag{10.4.20}$$

Using (10.4.12) and a little algebra, we deduce from (10.4.20) that

$$\|\underline{\theta}\|_1^2 + \|w\|_1^2 \leq c(\alpha_{Korn})\|f\|_{-1}\|w\|_1, \tag{10.4.21}$$

which implies the boundedness of $\|\underline{\theta}\|_1 + \|w\|_1$ from above. Then, one observes that the minimum of Π_t over all V is surely smaller than the minimum of Π_t over $V_0 = \{(\underline{\eta}, \zeta) \mid \underline{\eta} = \underline{\text{grad}}\, \zeta\}$ (which is clearly independent of t and negative). Hence,

$$\frac{1}{2}a(\underline{\theta}, \underline{\theta}) + \frac{t^{-2}}{2}\| \underline{\text{grad}}\, w - \underline{\theta}\|_0^2 - (f, w) \leq -c < 0 \tag{10.4.22}$$

for some positive c independent of t, which immediately gives

$$(f, w) \geq c > 0 \tag{10.4.23}$$

which implies that $\|w\|_0$ (and hence $\|\underline{\theta}\|_1 + \|w\|_1$) is bounded from below by a positive constant. This completes the proof. □

According to Proposition 10.4.1, we have now *a sequence* of problems, indexed by the thickness t, whose solutions are bounded uniformly (in t) and also bounded uniformly away from zero.

For the convenience of the reader, we repeat explicitly the general problem of our sequence.

The sequence of minimum problems. Given a bounded domain $\Omega \subset \mathbb{R}^2$ with diameter $T := diam(\Omega)$ and an element $f \in L^2(\Omega)$, for every thickness $t \in]0, T[$, we consider the problem: *find* $(\underline{\theta}(t), w(t))$ *in* $V := (H_0^1(\Omega))^2 \times H_0^1(\Omega)$ *such that*

$$\Pi_t(\underline{\theta}, w) \leq \Pi_t(\underline{\eta}, z) \qquad \forall(\underline{\eta}, z) \in V, \tag{10.4.24}$$

where Π_t is given by (10.4.19).

The sequence (10.4.24) is what we need to analyse the performance of numerical methods. Indeed, we expect a "good and reliable" numerical method to perform *uniformly well* on all the problems of our sequence, regardless of the possible smallness of t. We therefore look for error bounds (in terms of powers of the mesh-size h) which hold uniformly in t.

10.4.3 Mixed Formulation of the Mindlin-Reissner Model

It will be convenient, in order to carry on the analysis, to introduce the auxiliary variable

$$\underline{\gamma}(t) := t^{-2}(\underline{\text{grad}}\, w(t) - \underline{\theta}(t)) \tag{10.4.25}$$

which is related to the shear stresses but does not go to zero with t (and could be considered as a sort of *normalised shear stress*). We can now write the Euler equations for Π_t in the form

$$a(\underline{\theta},\underline{\eta}) + (\underline{\gamma}, \underline{\mathrm{grad}}\,\zeta - \underline{\eta}) = (f,\zeta), \qquad \forall(\underline{\eta},\zeta)\in V, \tag{10.4.26}$$

$$\underline{\gamma} = t^{-2}(\underline{\mathrm{grad}}\,w - \underline{\theta}). \tag{10.4.27}$$

This is now taking the form of the abstract problems studied in Chap. 4, especially in Sect. 4.3. In particular, we can define the bilinear forms

$$\mathcal{A}((\underline{\theta},w),(\underline{\eta},z)) := a(\underline{\theta},\underline{\eta}), \tag{10.4.28}$$

where a is defined in (10.4.7), and

$$\mathcal{B}((\underline{\eta},\zeta),\underline{\delta}) := (\underline{\mathrm{grad}}\,\zeta - \underline{\eta},\underline{\delta}) \tag{10.4.29}$$

corresponding to the operator

$$\mathbb{B} : (\underline{\eta},\zeta) \longrightarrow (\underline{\mathrm{grad}}\,\zeta - \underline{\eta}), \tag{10.4.30}$$

and finally the functional

$$(\mathbb{F},(\underline{\eta},\zeta)) := (f,\zeta). \tag{10.4.31}$$

With this notation, Eqs. (10.4.26) and (10.4.27) can be written as

$$\mathcal{A}((\underline{\theta},w),(\underline{\eta},\zeta)) + \mathcal{B}((\underline{\eta},\zeta),\underline{\gamma}) = (\mathbb{F},(\underline{\eta},\zeta)) \qquad \forall\,(\underline{\eta},\zeta)\in V, \tag{10.4.32}$$

$$\mathcal{B}((\underline{\theta},w),\underline{\delta}) - t^2(\underline{\gamma},\underline{\delta}) = 0 \qquad \forall\,\underline{\delta}. \tag{10.4.33}$$

As we have already seen on several other examples, it is convenient, from many aspects, to consider (10.4.32) and (10.4.33) as a perturbation of the "limit problem" that we have for $t = 0$, namely

$$\mathcal{A}((\underline{\theta}_0,w_0),(\underline{\eta},\zeta)) + \mathcal{B}((\underline{\eta},\zeta),\underline{\gamma}_0) = (\mathbb{F},(\underline{\eta},\zeta)) \qquad \forall\,(\underline{\eta},\zeta)\in V, \tag{10.4.34}$$

$$\mathcal{B}((\underline{\theta}_0,w_0),\underline{\delta}) = 0 \qquad \forall\,\underline{\delta}. \tag{10.4.35}$$

It is easy to check that the kernel $K := \mathrm{Ker}\mathbb{B}$ is given by

$$K = \{(\underline{\eta},\zeta)\mid(\underline{\eta},\zeta)\in V \text{ such that } \underline{\eta} = \underline{\mathrm{grad}}\,\zeta\}. \tag{10.4.36}$$

It is then clear that the Korn inequality (10.4.12) implies that *the bilinear form $\mathcal{A}$, defined in* (10.4.28)*, is elliptic in the kernel K of* $\mathbb{B}$:

$$\mathcal{A}((\underline{\eta},\zeta),(\underline{\eta},\zeta)) \geq \alpha_0\|(\underline{\eta},\zeta)\|_V^2 \qquad \forall\,(\underline{\eta},\zeta)\in K, \tag{10.4.37}$$

with α_0 depending only on the Korn constant α_{Korn} appearing in (10.4.12).

On the other hand, we note that we *did not* decide yet what the space Q should be, and hence where $\underline{\delta}$ is allowed to vary in (10.4.33) or in (10.4.35). Recalling the general theory of Chap. 4, we observe that the space Q should be defined in such a way that the operator $\mathbb{B}$, associated with the bilinear form $\mathcal{B}$, is *surjective* from V to Q' (or, at least, that its image is a *closed subspace of* Q'). It is therefore clear that the next, crucial, step has to be the characterisation of the *image of* $\mathbb{B}$, that is $\mathbb{B}(V)$ with V given in (10.4.10).

In what follows, we are going to use the notation introduced in Chap. 2 for the two-dimensional operators

$$\begin{aligned} \underline{\mathrm{curl}} : \phi \longrightarrow \underline{\mathrm{curl}}\, \phi &= \left\{ \frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x} \right\}, \\ \mathrm{curl} : \underline{\chi} \longrightarrow \mathrm{curl}\, \underline{\chi} &= -\frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial x}. \end{aligned} \tag{10.4.38}$$

Note as well that (for the same reason) we are using here (x, y, z) instead of (x_1, x_2, x_3).

Proposition 10.4.2. *The mapping $\mathbb{B}$ is surjective from V onto the space $\Gamma = H_0(curl, \Omega)$ defined by*

$$H_0(curl; \Omega) = \{\underline{\chi} \mid \underline{\chi} \in (L^2(\Omega))^2,\ curl\, \underline{\chi} \in L^2(\Omega),\ \underline{\chi} \cdot \underline{t} = 0 \text{ on } \partial\Omega\} \tag{10.4.39}$$

$$\|\underline{\chi}\|^2_{H_0(curl;\Omega)} := \|\underline{\chi}\|^2_0 + \|\, curl\, \underline{\chi}\|^2_0 \tag{10.4.40}$$

(where $\underline{t}$ is the unit tangent to $\partial\Omega$) and admits a continuous lifting.

Proof. We shall show that there exists a $\beta_{RM} > 0$ such that: for every $\underline{\chi} \in H_0(\mathrm{rot}; \Omega)$ there exists $(\underline{\eta}, \zeta) \in V$ verifying

$$\underline{\chi} = \underline{\mathrm{grad}}\, \zeta - \underline{\eta} \equiv \mathbb{B}(\underline{\eta}, \zeta), \tag{10.4.41}$$

and

$$\|\zeta\|_1 + \|\underline{\eta}\|_1 \le \frac{1}{\beta_{RM}}\ \|\underline{\chi}\|_{H_0(\mathrm{curl};\Omega)}. \tag{10.4.42}$$

For this, we first choose $\underline{v} \in (H^1_0)^2$ such that

$$\mathrm{div}\, \underline{v} = -\,\mathrm{curl}\, \underline{\chi}, \tag{10.4.43}$$

$$\|\underline{v}\|_1 \le c\|\, \mathrm{curl}\, \underline{\chi}\|_0; \tag{10.4.44}$$

this is obviously possible because

$$\int_\Omega \mathrm{curl}\, \underline{\chi}\, dx\, dy = \int_{\partial\Omega} \underline{\chi} \cdot \underline{t}\, ds = 0. \tag{10.4.45}$$

Then, we set

$$\underline{\eta} = (\eta_1, \eta_2) := (-v_2, v_1) \tag{10.4.46}$$

so that from (10.4.43) and (10.4.44) we have

$$\operatorname{curl} \underline{\eta} = -\operatorname{curl} \underline{\chi}, \tag{10.4.47}$$

$$\|\underline{\eta}\|_1 \le \|\operatorname{curl} \underline{\chi}\|_0. \tag{10.4.48}$$

Now choose ζ as the unique solution in $H_0^1(\Omega)$ of

$$\triangle \zeta = \operatorname{div} \underline{\chi} + \operatorname{div} \underline{\eta} \in H^{-1}(\Omega); \tag{10.4.49}$$

we have, using (10.4.48) and (10.4.49),

$$\|\zeta\|_1 \le c\ (\|\operatorname{div} \underline{\chi}\|_{-1} + \|\operatorname{div} \underline{\eta}\|_{-1}) \le c\ (\|\underline{\chi}\|_0 + \|\operatorname{curl} \underline{\chi}\|_0). \tag{10.4.50}$$

We now have

$$\begin{cases} \operatorname{div}(\underline{\operatorname{grad}}\, \zeta - \underline{\eta}) = \operatorname{div} \underline{\chi} \text{ in } \Omega, \\ \operatorname{curl}(\underline{\operatorname{grad}}\, \zeta - \underline{\eta}) = \operatorname{curl} \underline{\chi} \text{ in } \Omega, \\ (\underline{\operatorname{grad}}\, \zeta - \underline{\eta}) \cdot \underline{t} = \underline{\chi} \cdot \underline{t} = 0 \text{ on } \partial\Omega, \end{cases} \tag{10.4.51}$$

which easily implies (10.4.41). On the other hand, (10.4.42) follows from (10.4.48) and (10.4.50). □

Proposition 10.4.2 tells us *how to choose* Q in order to have that $\mathbb{B}$ is surjective from V to Q'. Actually, we have little choice: Q' must be equal to the space $\Gamma = H_0(\text{curl}, \Omega)$ defined in (10.4.39). As we are dealing with Hilbert space, this implies that Q has to be the dual space of Γ:

$$Q \equiv \Gamma' := (H_0(\text{curl}; \Omega))'. \tag{10.4.52}$$

On the other hand, a little functional analysis allows us to characterise Γ' as follows:

$$\begin{aligned} \Gamma' &:= (H_0(\text{curl}; \Omega))' \\ &= H^{-1}(\text{div}; \Omega) \\ &= \{\underline{\gamma} \mid \underline{\gamma} \in (H^{-1}(\Omega))^2,\ \operatorname{div}\ \underline{\gamma} \in H^{-1}(\Omega)\} \end{aligned} \tag{10.4.53}$$

with the norm

$$\|\underline{\gamma}\|_Q^2 \equiv \|\underline{\gamma}\|_{\Gamma'}^2 := \|\underline{\gamma}\|_{-1}^2 + \|\operatorname{div}\ \underline{\gamma}\|_{-1}^2\ . \tag{10.4.54}$$

Then, the Closed Range Theorem (see Sect. 4.2.2) tells us that Proposition 10.4.2 can be written in the form of an *inf-sup* condition:

$$\exists \beta_{RM} > 0 \text{ such that } \inf_{\underline{\chi} \in Q} \sup_{(\underline{\eta}, \zeta) \in V} \frac{\int_\Omega (\underline{\text{grad}}\, \zeta - \underline{\eta}) \cdot \underline{\chi}\, dx\, dy}{\|(\underline{\eta}, \zeta)\|_V \|\underline{\chi}\|_Q} \geq \beta_{RM}. \tag{10.4.55}$$

Hence, to start with, we can make precise the limit problem (10.4.34) and (10.4.35) as follows

$$\begin{cases} \text{find } (\underline{\theta}_0, w_0) \in V \text{ and } \underline{\gamma}_0 \in Q \text{ such that} \\ \mathcal{A}((\underline{\theta}, w), (\underline{\eta}, \zeta)) + \mathcal{B}((\underline{\eta}, \zeta), \underline{\gamma}_0) = (f, \zeta) \quad \forall\, (\underline{\eta}, \zeta) \in V, \\ \mathcal{B}((\underline{\theta}_0, w_0), \underline{\delta}) = 0 \quad \forall\, \underline{\delta} \in Q. \end{cases} \tag{10.4.56}$$

From (10.4.37) and (10.4.55), using Theorem 4.2.3, we then have the following result on the *limit problem* (10.4.34) and (10.4.35) in the form (10.4.56).

Proposition 10.4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be defined as* (10.4.28) *and* (10.4.29)*, respectively. Then, for every $f \in L^2(\Omega)$, the limit problem* (10.4.56) *has a unique solution $(\underline{\theta}_0, w_0, \underline{\gamma_0})$ and we have*

$$\|\underline{\theta}_0\|_1 + \|w_0\|_1 + \|\underline{\gamma}_0\|_{\Gamma'} \leq c\|f\|_{-1}. \tag{10.4.57}$$

□

Remark 10.4.1. Actually, the abstract theory of Chap. 4 tells us that we could take any framework that is much more general than the one used for problem (10.4.56). For instance, we could have allowed a general $\mathbb{F} \in V'$ (not necessarily of the form (10.4.31)) in the right-hand side of the first equation. Besides, we did not need to assume $f \in L^2(\Omega)$, as $f \in H^{-1}(\Omega)$ would clearly have been sufficient. Moreover, a right-hand side in $Q' = \Gamma$ would also be allowed (instead of zero) in the second equation. We decided, however, to present the result in the framework of our original plate problem. □

Remark 10.4.2. It is not difficult to check that the unique solution of (10.4.56) is related to the solution of the Kirchhoff model: *find $w_K \in H^2_0(\Omega)$ such that*

$$\frac{E}{12(1-\nu^2)} \Delta^2 w_K = f \tag{10.4.58}$$

by the relations

$$w_0 = w_K, \qquad \underline{\theta}_0 = \underline{\text{grad}}\, w_K. \tag{10.4.59}$$

□

Remark 10.4.3. In the case of beam problems, the space Γ' is replaced by L^2, which makes things much easier. □

Remark 10.4.4. We now remark that, with our choice, we have $Q' \equiv H_0(\text{curl}; \Omega) \hookrightarrow (L^2(\Omega))^2$. As Q' is clearly dense in $(L^2(\Omega))^2$, we also have (identifying, as usual, $(L^2(\Omega))^2$ with its dual space) $(L^2(\Omega))^2 \hookrightarrow Q$. This implies that the perturbation introduced, for positive t, in the full problem (10.4.32) and (10.4.33) has to be regarded as a **singular perturbation** of the limit problem (10.4.34) and (10.4.35). Hence, it has to be dealt with using the instruments of Sect. 4.3.2. □

In view of the previous remark, we introduce the space

$$W := (L^2(\Omega))^2 \tag{10.4.60}$$

and set the mathematical framework for the Mindlin-Reissner problem (10.4.32) and (10.4.33) as follows

$$\begin{cases} \text{find } (\underline{\theta}(t), w(t)) \in V \text{ and } \underline{\gamma}(t) \in W \text{ such that} \\ \mathcal{A}((\underline{\theta}(t), w(t)), (\underline{\eta}, \zeta)) + \mathcal{B}((\underline{\eta}, \zeta), \underline{\gamma}(t)) = (f, \zeta) \quad \forall\, (\underline{\eta}, \zeta) \in V, \\ \mathcal{B}((\underline{\theta}(t), w(t)), \underline{\chi}) = t^2(\underline{\gamma}, \underline{\chi})_W \quad \forall\, \underline{\chi} \in W = (L^2(\Omega))^2. \end{cases} \tag{10.4.61}$$

Having chosen W as well as Q, we can now prove the following result.

Proposition 10.4.4. *Let the spaces V, Q, and W be defined as in* (10.4.10)–(10.4.60)*, respectively, and let the bilinear forms $\mathcal{A}$ and $\mathcal{B}$ and the operator* (10.4.30) *be defined in* (10.4.28)*,* (10.4.29) *and* (10.4.30)*, respectively. Then, there exists an $\tilde{\alpha} > 0$ such that*

$$\tilde{\alpha}\|(\underline{\eta}, \zeta)\|_V^2 \le \mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + \|\mathbb{B}(\underline{\eta}, \zeta)\|_W^2. \tag{10.4.62}$$

Proof. The result is essentially trivial. Indeed, using (10.4.11), the triangle inequality, and the Poincaré inequality (1.2.14), we have first

$$\|(\underline{\eta}, \zeta)\|_V^2 \le \|\underline{\eta}\|_1^2 + C_1\|\,\underline{\text{grad}}\,\zeta\|_0^2 \le C_2(\|\underline{\eta}\|_1^2 + \|\,\underline{\text{grad}}\,\zeta - \underline{\eta}\|_0^2),$$

where C_1 and C_2 depend only on the Poincaré constant. Then, we can use the Korn inequality (10.4.12) and the definition of $\mathcal{A}$ and $\mathbb{B}$ to obtain

$$\|\underline{\eta}\|_1^2 + \|\,\underline{\text{grad}}\,\zeta\|_0^2 \le \frac{1}{\alpha_{Korn}}\mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + \|\mathbb{B}(\underline{\eta}, \zeta)\|_0^2,$$

and the result follows. □

We can now apply Theorem 4.3.4 (with $g = 0$) and obtain the following result.

Theorem 10.4.1. *With the same assumptions as in Proposition 10.4.4, for every $f \in V'$ and for every $t \in]0,1[$, problem (10.4.61) has a unique solution $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$. Moreover, there exists a constant c, depending only on Ω, such that*

$$\|\underline{\theta}(t)\|_1 + \|w(t)\|_1 + \|\underline{\gamma}(t)\|_{\Gamma'} + t\|\underline{\gamma}(t)\|_0 \leq c\|f\|_{V'}. \tag{10.4.63}$$

□

We can now study the behaviour of the solutions of problem (10.4.61) when $t \to 0$.

Proposition 10.4.5. *With the same assumptions as in Theorem 10.4.1, we have*

$$\begin{aligned} &\underline{\theta}(t) \rightharpoonup \underline{\theta}_0 \text{ in } (H_0^1(\Omega))^2, \\ &w(t) \rightharpoonup w_0 \text{ in } H_0^1(\Omega), \\ &\underline{\gamma}(t) \rightharpoonup \underline{\gamma}_0 \text{ in } \Gamma', \end{aligned} \tag{10.4.64}$$

where $(\underline{\theta}_0, w_0, \underline{\gamma}_0)$ is the solution of the limit problem (10.4.56).

Proof. The weak convergence (a priori, up to a subsequence) in (10.4.64) just follows from (10.4.16) and (10.4.57). A passage to the limit in (10.4.61) gives (10.4.56). □

Remark 10.4.5. Additional results in this direction can be found in [171]. □

We can now apply the results of Proposition 4.3.5 and of Remarks 4.3.12 and 4.3.14 to estimate the convergence rate as a function of t^2 which plays here the role of λ. This leads us to a convergence rate in $\sqrt{\lambda} = t$. In order to improve this bound and also to enable us later to get sharper error estimates, we now introduce a decomposition principle for (10.4.26) and (10.4.27).

10.4.4 A Decomposition Principle and the Stokes Connection

We shall first prove the following decomposition principle for vector-valued functions in $\Gamma' = H_0(\text{curl}; \Omega)$.

Proposition 10.4.6. *Every element $\underline{\gamma} \in \Gamma'$ can be written in a unique way as*

$$\underline{\gamma} = \underline{grad}\,\psi + \underline{curl}\;p, \tag{10.4.65}$$

with $\psi \in H_0^1(\Omega)$, $p \in L^2(\Omega)/\mathbb{R}$, and $\underline{curl}\,p = \{-\partial p/\partial y, \partial p/\partial x\}$. Moreover, we may use

$$\|\underline{\gamma}\|^2_{\Gamma'} = \|\psi\|^2_{H_0^1(\Omega)} + \|p\|^2_{L^2(\Omega)/\mathbb{R}} \tag{10.4.66}$$

as a norm on Γ'.

Proof. Set $\xi := \operatorname{div} \underline{\gamma} \in H^{-1}(\Omega)$. We define ψ to be the unique solution of $-\Delta\psi = \xi$, $\psi \in H_0^1(\Omega)$ and we set $\underline{\alpha} = \underline{\gamma} - \underline{\operatorname{grad}}\,\psi$. One has $\operatorname{div}\underline{\alpha} = 0$ so that $\underline{\alpha} = \underline{\operatorname{curl}}\, p$ and p is determined up to a constant in $L^2(\Omega)$. Condition (10.4.66) is then immediate. □

Remark 10.4.6. The decomposition introduced in Proposition 10.4.6 also holds for $(L^2(\Omega))^2$ and $H(\operatorname{curl};\Omega)$. The difference between these spaces lies in the regularity of the p component. Indeed, taking $\underline{\gamma} = \underline{\operatorname{grad}}\,\psi + \underline{\operatorname{curl}}\,p$ with $\psi \in H_0^1(\Omega)$, we have

$$\underline{\gamma} \in (L^2(\Omega))^2 \quad \Leftrightarrow \; p \in H^1(\Omega)/\mathbb{R}, \tag{10.4.67}$$

$$\underline{\gamma} \in H(\operatorname{rot};\Omega) \quad \Leftrightarrow \; p \in H^2(\Omega)/\mathbb{R}, \tag{10.4.68}$$

$$\underline{\gamma} \in H_0(\operatorname{rot};\Omega) \Leftrightarrow p \in H^2(\Omega)/\mathbb{R} \text{ and } \frac{\partial p}{\partial n} = 0 \text{ on } \partial\Omega. \tag{10.4.69}$$

□

It is now a simple exercise to transform problem (10.4.61) in terms of the new unknowns $\underline{\theta}(t)$, $w(t)$, $\psi(t)$, and $p(t)$. We have indeed the following basic theorem, which is of considerable help in understanding the nature of the Mindlin-Reissner equations.

Theorem 10.4.2. *Any solution of (10.4.61) is a solution of the following problem (and conversely) through the change of variables (10.4.65): find $(\underline{\theta}(t), w(t), \psi(t), p(t))$ in $\underline{\Theta} \times Z \times H_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ such that*

$$(\underline{grad}\,\psi, \underline{grad}\,\xi) = (f, \xi) \qquad \forall \xi \in H_0^1(\Omega), \tag{10.4.70}$$

$$\begin{cases} a(\underline{\theta}(t), \underline{\eta}) - (\underline{curl}\,p(t), \underline{\eta}) = (\underline{grad}\,\psi, \underline{\eta}) & \forall \underline{\eta} \in (H_0^1(\Omega))^2, \\ -(\underline{\theta}(t), \underline{curl}\,q) - t^2(\underline{curl}\,p(t), \underline{curl}\,q) = 0 & \forall q \in H^1(\Omega)_{/\mathbb{R}}, \end{cases} \tag{10.4.71}$$

$$(\underline{grad}\,w(t), \underline{grad}\,\chi) = (\underline{\theta}(t), \underline{grad}\,\chi) + t^2(\underline{grad}\,\psi, \underline{grad}\,\chi) \quad \forall \chi \in H_0^1(\Omega). \tag{10.4.72}$$

Proof. The proof is immediate: it is enough to make the substitution (10.4.65), and observe that both (10.4.61) and (10.4.70)–(10.4.72) have a unique solution. □

Remark 10.4.7. It must be noted that (10.4.71) implies $\partial p/\partial n|_{\partial\Omega} = 0$ and $p \in H^2(\Omega)$ so that $\underline{\gamma} = \underline{\operatorname{grad}}\,\psi + \underline{\operatorname{curl}}\,p$ is indeed an element of $\Gamma = H_0(\operatorname{curl};\Omega)$. Note also that $\psi(t)$ is actually *independent* of t. □

Remark 10.4.8. It is important to note that although (10.4.70)–(10.4.72) seems, at first sight, a system of four equations, it actually decomposes immediately into equation (10.4.70) (which allows to compute ψ directly from f), plus equations (10.4.71) (which allow to compute $\underline{\theta}(t)$ and $p(t)$ once we know ψ) plus equation (10.4.72) (which allows to compute $w(t)$ once we know $\underline{\theta}(t)$ and ψ). We have thus reduced, through Theorem 10.4.2, our original problem into the following sequence

– A Dirichlet problem (10.4.70) that is independent of t,
– A "Stokes-like" problem (10.4.71),
– A Dirichlet problem (10.4.72).

□

The decomposition provided by Theorem 10.4.2 shows us that it is the p component of $\underline{\gamma}$ which depends on t. Before coming back to the quantification of this dependency, we rapidly develop the analogy between (10.4.71) and a Stokes problem. Let us set $\underline{\eta}^{\perp} = \{-\eta_2, \eta_1\}$. We can write (10.4.71) in the form

$$\begin{cases} a(\underline{\theta}^{\perp}, \underline{\eta}^{\perp}) + (p, \operatorname{div}\ \underline{\eta}^{\perp}) = \ (\underline{\operatorname{grad}}\, \psi, \underline{\eta}^{\perp}) \qquad \forall \underline{\eta}^{\perp} \in (H_0^1(\Omega))^2, \\ (\operatorname{div}\ \underline{\theta}^{\perp}, q) = t^2\ (\underline{\operatorname{grad}}\, p, \underline{\operatorname{grad}}\, q) \qquad \forall q \in H^1(\Omega)/\mathbb{R}. \end{cases} \tag{10.4.73}$$

The limit problem ($t = 0$) is thus a standard Stokes problem and we shall be able to rely on results of Chap. 8 to build approximations. We shall not analyse here the case $t \neq 0$ in too much detail. However, it is important to see the behaviour of p as $t \to 0$.

Proposition 10.4.7. *Let $\underline{\theta}(t)$, $w(t)$, $p(t)$, and ψ be the solution of* (10.4.70)–(10.4.72)*. We then have*

$$\|\underline{\theta}(t)\|_2 + \|w(t)\|_2 + \|\psi(t)\|_2 + \|p(t)\|_1 + t\ \|p(t)\|_2 \le c\ \|f\|_0 \tag{10.4.74}$$

where the constant c is independent of t. □

We refer to [122] for the proof of this result which is based essentially on the regularity properties of the Dirichlet problem and the Stokes problem.

An important point is that (10.4.74) does not improve too much for a more regular f (even in a smooth domain). It is not possible to bound $\|p(t)\|_2$ uniformly in t. The reason is that the normal derivative of $p(t)$ vanishes although this is not the case for the solution $p(0)$ of the limit problem. We thus have a boundary layer effect which has been studied in [29]. This analysis shows that an analogue of (10.4.74) exists for $\|\underline{\theta}\|_{\frac{5}{2}}$ and $\|p\|_{\frac{3}{2}}$ but not for more regular spaces.

Remark 10.4.9. We can now try to apply Remarks 4.3.12 and 4.3.14 to our problem. Denoting $W_+ := \{p \mid p \in H^2(\Omega)/\mathbb{R},\ \partial p/\partial n|_{\partial\Omega} = 0\}$, it is clear that we have

$$|(\underline{\operatorname{curl}}\, p, \underline{\operatorname{curl}}\, q)| \le c\|p\|_{W_+}\|q\|_{L^2(\Omega)/\mathbb{R}}. \tag{10.4.75}$$

Whenever the solution p_0 of the limit problem is regular enough (this is the case for smooth data and a smooth domain), we shall have

$$p_0 \in [L^2(\Omega), W_+]_\theta \quad \forall\, \theta < \frac{3}{4}. \tag{10.4.76}$$

No improvement is possible because of the fact that $\partial p(0)/\partial n \neq 0$. We can thus apply Remark 4.3.14 to get for $\theta < \frac{3}{4}$

$$\|\underline{\Theta}(t) - \underline{\theta}_0\|_1 + \|p(t) - p_0\|_0 + \|w(t) - w_0\|_1 \leq ct^{2\theta}\, \|p_0\|_\theta, \tag{10.4.77}$$

where $\|p_0\|_\theta$ is the norm of p_0 in $[L^2(\Omega), W_+]_\theta$. We can summarise (10.4.77) by saying that we have an $O(t^{3/2-\varepsilon})$ convergence. This requires, however, a smooth domain. In the case where $\partial\Omega$ is only Lipschitz continuous, the best we can get is $O(t)$. □

10.4.5 Discretisation of the Problem

We now turn our attention to the discretisation of our problem (10.4.26) and (10.4.27). Let us thus assume that we are given finite-dimensional subspaces Θ_h and Z_h of Θ and Z and use $V_h = \Theta_h \times Z_h$ as a subspace of V. We also discretise the space $W = (L^2(\Omega))^2$ by Γ_h and we consider the discretised problem: *find* $(\underline{\theta}^h, w_h, \underline{\gamma}_h)$ *such that*

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}_h) + (\underline{\gamma}_h, \underline{\mathrm{grad}}\,\zeta_h - \underline{\eta}_h) = (f, \zeta_h) \quad \forall(\underline{\eta}_h, \zeta_h) \in V_h, \\ (\underline{\mathrm{grad}}\, w_h - \underline{\theta}_h, \underline{\chi}_h) - t^2(\underline{\gamma}_h, \underline{\chi}_h) = 0 \quad \forall \underline{\chi}_h \in \Gamma_h. \end{cases} \tag{10.4.78}$$

This could also be written with the notation of Sect. 10.4.3, that is, in particular, making use of the bilinear form $\mathcal{A}$ and $\mathcal{B}$ defined in (10.4.28) and (10.4.29). The discrete problem (10.4.78) becomes: *find* $((\underline{\theta}_h, w_h), \underline{\gamma}_h) \in V_h \times Q_h$ *such that*

$$\begin{cases} \mathcal{A}((\underline{\theta}_h, w_h), (\underline{\eta}, \zeta)) + \mathcal{B}((\underline{\eta}, \zeta), \underline{\gamma}_h) = (\mathbb{F}, (\underline{\eta}, \zeta)) \quad \forall\, (\underline{\eta}, \zeta) \in V_h, \\ \mathcal{B}((\underline{\theta}, w), \underline{\chi}) - t^2(\underline{\gamma}, \underline{\chi}) = 0 \quad \forall\, \underline{\chi} \in Q_h \equiv \Gamma_h. \end{cases} \tag{10.4.79}$$

In what follows, we shall use either the form (10.4.78) or the form (10.4.79), according to the notational convenience.

Remark 10.4.10. Note that from the second equation of (10.4.78), we *do not have* in general $\underline{\gamma}_h = \lambda t^{-2}(\underline{\mathrm{grad}}\, w_h - \underline{\theta}_h)$ unless we take Θ_h, Z_h, and Γ_H such that $\underline{\mathrm{grad}}\, Z_h - \Theta_h \subseteq \Gamma_h$. This, as we shall see, could be a problem regarding the actual *implementation* of the method. Indeed, in the common engineering practice, one prefers to solve the discrete problems in terms of $\underline{\theta}_h$ and w_h alone. In this case, the use of the mixed formulation (and the introduction of the variable $\underline{\gamma}_h$) should be regarded as a *mathematical artefact* used in order to have *a better understanding of* *the mathematical structure of the discretised problem*. We will come back several times to this important point. □

It is easy to check that, now, the **discrete kernel** $K_h := \mathrm{Ker}\mathbb{B}_h$ is given by

$$K_h = \{(\underline{\eta}, \zeta) \mid \underline{\eta}, \zeta) \in V_h, (\underline{\eta} - \underline{\mathrm{grad}}\, \zeta, \underline{\delta}) = 0 \quad \forall \underline{\delta} \in \Gamma_h\}, \tag{10.4.80}$$

and we consider the problem of having, for our discrete problem, the *ellipticity in the discrete kernel*;

$$\mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) \geq \alpha_0^h \|(\underline{\eta}, \zeta)\|_V^2 \qquad \forall\, (\underline{\eta}, \zeta) \in K_h. \tag{10.4.81}$$

For the continuous case, the Korn inequality (10.4.12) implied that the bilinear form $\mathcal{A}$ is elliptic in the kernel K (see (10.4.37)). As the variable ζ does not appear in the actual expression of $\mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta))$, we deduce that the only possibility in order to have the ellipticity in K_h is that the following property holds

$$\exists \kappa > 0 \text{ s. t. } \{(\underline{\eta}, \zeta) \in K_h\} \Rightarrow \{\|\zeta\|_1 \leq \kappa \|\underline{\eta}\|_1\} \tag{10.4.82}$$

and a simple *necessary* condition for it is that

$$\{(\underline{\mathrm{grad}}\, \zeta, \underline{\delta}) = 0 \quad \forall \underline{\delta} \in \Gamma_h\} \Rightarrow \{\underline{\mathrm{grad}}\, \zeta = \underline{0}\}. \tag{10.4.83}$$

This can easily be satisfied assuming for instance that

$$\underline{\mathrm{grad}}(Z_h) \subseteq \Gamma_h. \tag{10.4.84}$$

As we shall see, the above condition (10.4.84) is not difficult to enforce when choosing the finite element spaces and the vast majority of the good and reliable methods will satisfy it. On the other hand, the discrete *inf-sup* condition

$$\exists \beta_{RM} > 0 \text{ such that } \inf_{\underline{\chi} \in Q_h} \sup_{(\underline{\eta}, \zeta) \in V_h} \frac{\int_\Omega (\underline{\mathrm{grad}}\, \zeta - \underline{\eta}) \cdot \underline{\chi}\, dx\, dy}{\|(\underline{\eta}, \zeta)\|_V \|\underline{\chi}\|_Q} \geq \beta_{RM} \tag{10.4.85}$$

is *a major difficulty*, and most methods will be designed in order to *get around it*. For this, the first methods that we are going to consider are those based on the decomposition principle given in Proposition 10.4.6 and on the re-formulation of the problem given in Theorem 10.4.2.

Remark 10.4.11. It will often be convenient to look as well at the limit problem: *find* $(\underline{\theta}_{0h}, w_{0h}, \underline{\gamma}_{0h}) \in \Theta_h \times Z_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}_{0h}, \underline{\eta}_h) + (\underline{\gamma}_{0h}, \underline{\mathrm{grad}}\, \zeta_h - \underline{\eta}_h) = (f, \zeta_h) & \forall (\underline{\eta}_h, \zeta_h) \in V_h, \\ (\underline{\chi}_h, \underline{\mathrm{grad}}\, w_{0h} - \underline{\theta}_{0h}) = 0 & \forall \underline{\chi}_h \in \Gamma_h, \end{cases} \tag{10.4.86}$$

that could also be expressed in the form (10.4.79) with $t = 0$. It also comes from the results of Sects. 4.3.2 and 5.5.3 that to get a good approximation of (10.4.61) by (10.4.79) (that is, with convergence properties independent of t), it is necessary for (10.4.86) to be a good approximation of (10.4.34) and (10.4.35). □

We shall first consider the most "naive" case.

Example 10.4.1 (The direct approach). Let us suppose that we are given $\Theta_h \subset \Theta$ and $Z_h \subset Z$, and let us choose

$$\Gamma_h = \underline{\mathrm{grad}}(Z_h) - \Theta_h. \tag{10.4.87}$$

This choice implies that

$$\mathrm{Ker}\mathbb{B}_h = \{(\underline{\eta}_h, \zeta_h) \mid \underline{\eta}_h = \underline{\mathrm{grad}}\,\zeta_h\} \subset \mathrm{Ker}\mathbb{B}, \tag{10.4.88}$$

so that the ellipticity in K_h (10.4.81) evidently holds. It is important to note that the choice (10.4.87) is very easy to use on the computer, as it actually corresponds to minimising the energy functional Π_t given by (10.4.19) on $V_h = \Theta_h \times Z_h$ and that you *do not even see* $\underline{\gamma}_h$ (nor Γ_h). The choice (10.4.87) is then one of the most widely used choices for Γ_h although, in general, one does not realise it.

However, in the limit $t \to 0$, one is lead to minimise

$$\Pi_t \equiv a(\underline{\eta}_h, \underline{\eta}_h) - (f, \zeta_h) \tag{10.4.89}$$

on $\mathrm{Ker}\mathbb{B}_h$. Now, a quick glance to $\mathrm{Ker}B_h$ will make us understand that we have a long way to go. Consider $\underline{\eta}_h^{\perp} = \{-\eta_{2h}, \eta_{1h}\}$, that is, a rotation of $\pi/2$ of $\underline{\eta}_h$. It is clear that if $(\underline{\eta}_h, \zeta_h)$ belongs to $\mathrm{Ker}B_h$, we then have, by (10.4.88),

$$\mathrm{div}\ \underline{\eta}_h^{\perp} = \mathrm{curl}\,\underline{\eta}_h = 0. \tag{10.4.90}$$

Therefore, with choice (10.4.87), we are minimising Π_t in (10.4.89) on a subset of functions $\underline{\eta}_h$ satisfying (10.4.90). However, we have already seen in Chap. 8, for the linear Stokes problem, that it is not recommended to work with velocity fields which are exactly incompressible (because there are too few of them in general). A direct application of (10.4.87) is likely to lead to bad results (e.g. locking) unless a very special choice of Θ_h and Z_h has been made. □

In what follows, we shall mainly concentrate on two groups of finite element approaches: the Methods based on the decomposition principle, and the Methods based on a nonconforming approximation of the original minimisation problem (10.4.24).

10.4.5.1 Methods Based on the Decomposition Principle

The first group of methods that we present is directly guided by the decomposition principle of Propositions 10.4.6 and 10.4.2 in which a Stokes-like problem explicitly appears. For the sake of simplicity, we shall describe *one* possible method in this group, based on the MINI element for Stokes. However, it will be clear that starting from every finite element stable approximation for the Stokes problem using continuous pressures, one can derive a Reissner-Mindlin method belonging to the present group.

The basic idea is to give up a direct approximation of $\underline{\gamma}$ and to approximate instead each component of its decomposition into $\underline{\text{grad}}\,\psi_h + \underline{\text{curl}}\,p_h$. Moreover, as (10.4.71) shows us that θ_h and p_h are analogous to a velocity field and a pressure field in a Stokes problem, we shall try to use some results of Chap. 8 to build a suitable approximation.

We assume that Ω is a convex polygon and that we are given a sequence $\{\mathcal{T}_h\}$ of partitions of Ω into triangles. Let Θ_h be built by employing the MINI element of Chap. 8 , that is, in the notations of Chap. 2,

$$\begin{cases} \Theta_h & = (\mathcal{L}_1^1 \cap H_0^1(\Omega))^2 \oplus B_3, \\ Z_h & = \mathcal{L}_1^1 \cap H_0^1(\Omega). \end{cases} \tag{10.4.91}$$

These are spaces of piecewise linear polynomials enriched by a bubble function in the case of Θ_h. We also introduce

$$\Gamma_h := \underline{\text{grad}}(\mathcal{L}_1^1 \cap H_0^1(\Omega)) \oplus \underline{\text{curl}}\,\mathcal{L}_1^1 \equiv \underline{\text{grad}}\,Z_h \oplus \underline{\text{curl}}\,\mathcal{L}_1^1. \tag{10.4.92}$$

This space is then a strict subspace of piecewise constant vector functions constructed by discretising the ingredients of the decomposition principle of Proposition 10.4.6 and Remark 10.4.6.

It is straightforward to check that $\text{Ker}\mathbb{B}_h$ is made of the pairs $(\underline{\eta}_h, \zeta_h)$ in $\Theta_h \times Z_h$ such that

$$(\underline{\eta}_h, \underline{\text{curl}}\,q_h) = 0 \qquad \forall q_h \in \mathcal{L}_1^1, \tag{10.4.93}$$

$$(\underline{\text{grad}}\,\zeta_h, \underline{\text{grad}}\,\phi_h) = (\underline{\eta}_h, \underline{\text{grad}}\,\phi_h) \qquad \forall \phi_h \in Z_h \equiv \mathcal{L}_1^1 \cap H_0^1(\Omega). \tag{10.4.94}$$

Now, condition (10.4.94) is especially nice as it implies

$$\|\zeta_h\|_1 \le c\,\|\underline{\eta}_h\|_1, \qquad \forall (\underline{\eta}_h, \zeta_h) \in \text{Ker}B_h, \tag{10.4.95}$$

and hence, (10.4.82) holds and we have the ellipticity in the kernel (10.4.81). We still have to check the *inf-sup* condition (10.4.85) and we can do it using Proposition 5.4.3: given $(\underline{\eta}, \zeta)$, we must then be able to build $(\underline{\eta}_h, \zeta_h) = \Pi_h(\underline{\eta}, \zeta)$ such that

$$\mathcal{B}((\underline{\eta}, \zeta) - (\underline{\eta}_h, \zeta_h), \underline{\delta}_h) = 0 \qquad \forall \underline{\delta}_h, \tag{10.4.96}$$

with

$$\|\underline{\eta}_h\|_1 + \|\zeta_h\|_1 \leq c\ (\|\underline{\eta}\|_1 + \|\zeta\|_1). \tag{10.4.97}$$

Using the structure $\underline{\delta} = \underline{\text{grad}}\,\phi_h + \underline{\text{curl}}\,q_h$, condition (10.4.96) becomes:

$$\begin{cases} (\underline{\text{grad}}\,\phi_h, \underline{\eta}_h - \underline{\text{grad}}\,\zeta_h) - (\underline{\text{grad}}\,\phi_h, \underline{\eta} - \underline{\text{grad}}\,\zeta) = 0 & \forall \phi_h \in Z_h, \\ (\underline{\text{curl}}\,q_h, \underline{\eta}_h - \underline{\text{grad}}\,\zeta_h) - (\underline{\text{curl}}\,q_h, \underline{\eta} - \underline{\text{grad}}\,\zeta) = 0 & \forall q_h \in \mathcal{L}^1_1. \end{cases} \tag{10.4.98}$$

In order to construct the operator Π_h, we use the result already obtained in Chap. 8 to deal with the *inf-sup* condition for the MINI element. In particular, we proved that there exists an operator Π_S, from $\Theta = (H^1_0(\Omega))^2$ into Θ_h, such that

$$(\underline{\text{grad}}\,q_h, \underline{\eta} - \Pi_S(\underline{\eta})) = 0 \qquad \forall q_h \in \mathcal{L}^1_1, \tag{10.4.99}$$

with $\|\Pi_S(\underline{\eta})\|_1 \leq C\ \|\underline{\eta}\|_1$ and C independent of h. With the same arguments, we can obviously prove that there exists an operator Π_R from $\Theta = (H^1_0(\Omega))^2$ into Θ_h such that

$$(\text{curl}\,q_h, \underline{\eta} - \Pi_R(\underline{\eta})) = 0 \qquad \forall q_h \in \mathcal{L}^1 - 1, \tag{10.4.100}$$

with

$$\|\Pi_R(\underline{\eta})\|_1 \leq C\ \|\underline{\eta}\|_1, \tag{10.4.101}$$

with C independent of h. Condition (10.4.100), taking into account the fact that $(\underline{\text{curl}} q_h, \underline{\text{grad}}\,\zeta_h) = (\underline{\text{curl}}\,q_h, \underline{\text{grad}}\,\zeta) \equiv 0$ (by Green's formula), tells us that the second equation of (10.4.98) is satisfied if we take $\underline{\eta}_h = \Pi_r(\underline{\eta})$. We now observe that the first equation of (10.4.98) reduces to

$$(\underline{\text{grad}}\,\phi_h,\ \underline{\text{grad}}\,\zeta_h) = (\ \underline{\text{grad}}\,\phi_h,\ \underline{\text{grad}}\,\zeta - \underline{\eta} + \Pi_R(\underline{\eta})) \qquad \forall \phi_h \in Z_h, \tag{10.4.102}$$

and this is a discrete Dirichlet problem for the Laplace operator for which we have easily $\|\zeta_h\|_1 \leq c\ (\|\zeta\|_1 + \|\underline{\eta}\|_1)$, yielding the second part of (10.4.97).

Remark 10.4.12. It should be clear from our construction that the crucial step is to have an operator Π_R satisfying (10.4.100) and (10.4.101). This, always changing as we did the *div* operator into *rot*, essentially means that we could take, instead of the MINI element, any other finite element pair that is stable for the Stokes problem and which uses continuous pressures. □

Having proved the *inf-sup* condition (10.4.85), we can therefore apply to the limit problem (10.4.86) the basic results of Chap. 5. We can summarise this in the following proposition.

Proposition 10.4.8. *Problem* (10.4.86) *with the choice* (10.4.91) *and* (10.4.92) *has a unique solution. Moreover, if* $(\underline{\theta}_0, w_0, \underline{\gamma}_0)$ *is the solution of* (10.4.34) *and* (10.4.35)*, we have*

$$\|\underline{\theta}_0-\underline{\theta}_{0h}\|_1 + \|w_0-w_{0h}\|_1 + \|\underline{\gamma}_0-\underline{\gamma}_{0h}\|_{\Gamma'} \\ \leq ch\,\{\|w_0\|_3 + \|\underline{\gamma}_0\|_{H(\mathrm{div};\Omega)}\}. \qquad (10.4.103)$$

□

Remark 10.4.13. The result of Proposition 10.4.2 can be applied to the discrete problem in the present case. Indeed, we built, a priori, Γ_h in order to obtain a decomposition principle. Problem (10.4.78) can be written in the form: *find* $(\underline{\theta}_h(t), w_h(t), \psi_h(t), p_h(t))$ in $\underline{\Theta}_h \times Z_h \times Z_h \times \mathcal{L}^1_1/\mathbb{R}$ *such that*

$$(\underline{\mathrm{grad}}\,\psi_h, \underline{\mathrm{grad}}\,\xi) = (f, \xi) \qquad \forall \xi \in Z_h, \qquad (10.4.104)$$

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}) - (\underline{\mathrm{curl}}\,p_h, \underline{\eta}) = (\,\underline{\mathrm{grad}}\,\psi_h, \underline{\eta}) & \forall \underline{\eta} \in \Theta_h, \\ -(\underline{\theta}_h, \underline{\mathrm{curl}}\,q) - t^2(\underline{\mathrm{curl}}\,p_h, \underline{\mathrm{curl}}\,q) = 0 & \forall q \in \mathcal{L}^1_{1/\mathbb{R}}, \end{cases} \qquad (10.4.105)$$

$$(\underline{\mathrm{grad}}\,w_h,\ \underline{\mathrm{grad}}\,\chi) = (\underline{\theta}_h,\ \underline{\mathrm{grad}}\,\chi) + t^2(\underline{\mathrm{grad}}\,\psi_h,\ \underline{\mathrm{grad}}\,\chi) \quad \forall \chi \in Z_h. \qquad (10.4.106)$$

These problems can be solved sequentially and (10.4.105) is a Stokes-like problem using the MINI element of Chap. 8. This approximation has been introduced and studied for $t \neq 0$ in [122]. Using this decomposition and Proposition 10.4.8, recalling that

$$\|\underline{\gamma}\|_{\Gamma'} = \|\psi\|_1 + \|p\|_{0/\mathbb{R}}, \qquad (10.4.107)$$

and bringing in the regularity result of Proposition 10.4.7, we have, for $t = 0$, the following estimate:

$$\|\psi_{0h} - \psi_0\|_1 + \|p_0 - p_{0h}\|_{0/\mathbb{R}} \leq ch\ \{\|w_0\|_2 + \|\psi_0\|_2 + \|p_0\|_1\} \leq ch\ \|f\|_0. \qquad (10.4.108)$$

From a numerical point of view, (10.4.104)–(10.4.106) can lead to an efficient method, provided one has a Stokes solver available. □

Remark 10.4.14. An easy duality argument would also show that we have the estimate

$$\|\underline{\theta}_0 - \underline{\theta}_{0h}\|_0 + \|w_0 - w_{0h}\|_0 \leq ch^2\{\|w_0\|_3 + \|\underline{\gamma}_0\|_{H(\,\mathrm{div};\Omega)}\}. \qquad (10.4.109)$$

□

To end the discussion on this group of methods, we rapidly show how the results of Sect. 5.5 can be applied to the case $t \neq 0$. We consider the error estimate (5.5.52) from Remark 5.5.5, where we denote $V = (H^1_0(\Omega))^2 \times H^1_0(\Omega)$, $Q = \Gamma'$ and $W = (L^2(\Omega))^2$. The parameter λ is, of course, t^2 in the present case. It is easily verified that all conditions are satisfied and that we have, taking into account regularity properties of Remark 10.4.6,

$$\|\underline{\theta}(t)-\underline{\theta}_h(t)\|_1^2 + \|w(t)-w_h(t)\|_1^2 + \|\underline{\gamma}(t)-\underline{\gamma}_h(t)\|_{\Gamma'}^2 \\ + t^2\,\|\underline{\gamma}(t)-\underline{\gamma}_h(t)\|_0^2 \le C\left(\inf_{\underline{\eta}_h}\|\underline{\theta}(t)-\underline{\eta}_h\|_1^2 + \inf_{q_h}\|w(t)-q_h\|_1^2 \right. \\ \left. + \inf_{\underline{\chi}_h}\{\|\underline{\gamma}(t)-\underline{\chi}_h\|_{\Gamma'}^2 + t^2\|\underline{\gamma}(t)-\underline{\chi}_h\|_0^2\}\right). \qquad (10.4.110)$$

Using the decomposition principle and the estimate (10.4.74), we can recover the following result of [122].

Theorem 10.4.3. *For every* $t \in]0,T[$*, problem* (10.4.104)–(10.4.106) *with the choices* (10.4.91) *and* (10.4.92) *has a unique solution* $(\underline{\theta}_h(t), w_h(t), \psi_h(t), p_h(t))$*. If moreover* $(\underline{\theta}(t), w(t), \psi(t), p(t))$ *is the solution of* (10.4.70)–(10.4.72)*, then we have*

$$\|\underline{\theta}(t)-\underline{\theta}_h(t)\|_1^2 + \|w(t)-w_h(t)\|_1^2 \\ + \|\psi(t)-\psi_h(t)\|_1^2 + |p(t)-p_h(t)|_0^2 + t^2\,\|p(t)-p_h(t)\|_1^2 \\ \le c\,h^2\{\|\underline{\theta}(t)\|_2^2 + \|w(t)\|_2^2 + \|\psi(t)\|_2^2 + |p(t)|_1^2 + t^2\|p(t)\|_2^2\}, \qquad (10.4.111)$$

with c independent of h and t.

We therefore have an $O(h)$ convergence uniform in t. This result cannot be (much) improved because of the boundary layer effect already described.

10.4.5.2 Nonconforming Approximations of the Minimum Problem

The previous class of methods is, although interesting, rather remote from the actual engineering practice in which one tries to stick as closely as possible to the original formulation. In particular, as already pointed out in Remark 10.4.10, what is preferred in the engineering practice is to work only in terms of the original unknowns $\underline{\theta}$ and w, and, possibly, having their degrees of freedom at the same nodes (in particular if one wants to extend the methods to *shell problems*).

As we have seen, however, in Example 10.4.1, working directly on the minimisation problem (10.4.24) would require approximations $\underline{\theta}_h(t)$ and $w_h(t)$ that, in the limit for $t \to 0$, satisfy $\underline{\theta}_h(0) = \underline{\mathrm{grad}}\, w_h(0)$, and if we want to use a *conforming approximation* $\Theta_h \subset \Theta$ this would require $w_h \in Z_h$ to belong to $H_0^2(\Omega)$, which is not so easy to obtain, in particular for low degree elements.

The most common escape to the troubles that we are facing is to use some kind of numerical integration (or a nonconforming approximation) for the term

$t^{-2}\| \underline{\mathrm{grad}}\, w - \underline{\theta}\|^2$ which appears in (10.4.19), thus weakening condition (10.4.90). A way of formalising it is the following. We assume that we are given a linear operator r which maps $\Theta_h \times Z_h$ into (for instance) $L^2(\Omega)$. To see an example, consider for instance the possible, but not necessarily recommended, choices:

$$r(\underline{\eta}, \zeta) \in \mathcal{L}_0^0 \text{ and } r(\underline{\eta}, \zeta)|_K = \text{ mean value of } (\underline{\mathrm{grad}}\, \zeta - \underline{\eta}) \text{ on } K \qquad (10.4.112)$$

or

$$r(\underline{\eta}, \zeta) \in \mathcal{L}_0^0 \text{ and } r(\underline{\eta}, \zeta)|_K = \text{ value of } (\underline{\mathrm{grad}}\, \zeta - \underline{\eta}) \qquad (10.4.113)$$

at the barycentre of K. Then, one minimises, instead of Π_t (as in (10.4.24)), the functional

$$M_t^r := \frac{1}{2} a(\underline{\theta}, \underline{\theta}) + \frac{t^{-2}}{2} \|r(\underline{\theta}, w)\|_0^2 - (f, w) \qquad (10.4.114)$$

on $\Theta_h \times Z_h$. This can be regarded as obtained from the problem *find* $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times Z_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}_h) + (\underline{\gamma}_h, r(\underline{\eta}_h, \zeta_h)) - (f, \zeta_h) = 0 & \forall (\underline{\eta}_h, \zeta_h) \in V_h, \\ (\underline{\chi}_h, r(\underline{\theta}_h, w_h)) - t^2(\underline{\gamma}_h, \underline{\chi}_h) = 0 & \forall \underline{\chi}_h \in \Gamma_h, \end{cases} \qquad (10.4.115)$$

whenever its second equation is equivalent to

$$\underline{\gamma}_h = t^{-2} r(\underline{\theta}_h, w_h). \qquad (10.4.116)$$

This will always be the case for choices of Γ_h that verify

$$r(\Theta_h, Z_h) \subseteq \Gamma_h. \qquad (10.4.117)$$

In this case, the limit problem (for $t = 0$) will be: *find* $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times Z_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}_h) + (\underline{\gamma}_h, r(\underline{\eta}_h, \zeta_h)) - (f, \zeta_h) = 0 & \forall (\underline{\eta}_h, \zeta_h) \in V_h, \\ (\underline{\chi}_h, r(\underline{\theta}_h, w_h)) = 0 & \forall \underline{\chi}_h \in \Gamma_h. \end{cases} \qquad (10.4.118)$$

With the notation (10.4.28) for $\mathcal{A}$ and setting

$$\tilde{\mathcal{B}}_h((\underline{\eta}, \zeta), \underline{\chi}) = (r(\underline{\eta}, \zeta), \underline{\chi})_{(L^2(\Omega))^2} \qquad \forall (\underline{\eta}, \zeta) \in V_h \quad \forall \underline{\chi} \in \Gamma_h, \qquad (10.4.119)$$

we can write the problem (10.4.115) as

$$\begin{cases} \text{find } (\underline{\theta}_h(t), w_h(t)) \in V_h \text{ and } \underline{\gamma}_h(t) \in W_h \equiv \Gamma_h \text{ such that} \\ \mathcal{A}((\underline{\theta}_h(t), w_h(t)), (\underline{\eta}, \zeta)) + \tilde{\mathcal{B}}_h((\underline{\eta}, \zeta), \underline{\gamma}_h(t)) = (f, \zeta) \quad \forall\, (\underline{\eta}, \zeta) \in V_h, \\ \tilde{\mathcal{B}}_h((\underline{\theta}_h(t), w_h(t)), \underline{\chi}) = t^2(\underline{\gamma}, \underline{\chi})_W \quad \forall\, \underline{\chi} \in W = (L^2(\Omega))^2. \end{cases} \tag{10.4.120}$$

The kernel of the operator $\tilde{\mathbb{B}}_h$ associated with $\tilde{\mathcal{B}}$ will then be

$$\mathrm{Ker}\tilde{\mathbb{B}}_h = \{(\underline{\eta}_h, \zeta_h) \in V_h \text{ such that } (r(\underline{\eta}_h, \zeta_h), \underline{\chi}) = 0 \qquad \forall\, \underline{\chi} \in \Gamma_h\}, \tag{10.4.121}$$

which, assuming that (10.4.117) is satisfied, can also be written as

$$\mathrm{Ker}\tilde{\mathbb{B}}_h = \{(\underline{\eta}_h, \zeta_h) \in V_h \text{ such that } r(\underline{\eta}_h, \zeta_h) = 0\}. \tag{10.4.122}$$

All this should be connected to the *ellipticity in the kernel*, or, better, to the following (more powerful) property, strongly related to (5.5.46)

$$\begin{aligned} \exists\, \tilde{\alpha}_{RM} > 0 \text{ such that } & \mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + t^{-2}\|\mathbb{B}_h(\underline{\eta}, \zeta)\|_W^2 \\ &\equiv \mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + t^{-2}\|r(\underline{\eta}, \zeta)\|_0^2 \\ &\geq \tilde{\alpha}_{RM}\|(\underline{\eta}, \zeta)\|_V^2 \quad \forall\, (\underline{\eta}, \zeta) \in V_h \quad \forall t \in]0, T[, \end{aligned} \tag{10.4.123}$$

where T is always the diameter of Ω as in (10.4.19).

We have for this the following result.

Proposition 10.4.9. *Let $\mathcal{A}$ and $\tilde{\mathcal{B}}_h$ be defined as in* (10.4.28) *and* (10.4.119) *for an r that satisfies* (10.4.117). *If moreover we have*

$$\exists c_r \text{ and } C_r > 0 \text{ such that } \|r(\underline{\eta}, \zeta\|_0^2 \geq C_r\|\underline{\mathrm{grad}}\,\zeta\|_0^2 - c_r\|\underline{\eta}\|_1^2 \qquad \forall\, (\underline{\eta}, \zeta) \in V_h, \tag{10.4.124}$$

then (10.4.123) *holds.*

Proof. The proof is almost immediate using the Korn inequality (10.4.12). It is sufficient to combine the two inequalities

$$\mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + t^{-2}\|r(\underline{\eta}, \zeta)\|_0^2 \geq \mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) \geq \alpha_{Korn}\|\underline{\eta}\|_1^2$$

and

$$\begin{aligned} &\mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + t^{-2}\|r(\underline{\eta}, \zeta)\|_0^2 \\ &\qquad \geq T^{-2}\|r(\underline{\eta}, \zeta)\|_0^2 \geq T^{-2}\Big(C_r\|\underline{\mathrm{grad}}\,\zeta\|_0^2 - c_r\|\underline{\eta}\|_1^2\Big). \end{aligned}$$

Condition (10.4.124) might look cumbersome. We have, however, a simple sufficient condition for that.

Proposition 10.4.10. *Assume that* $r(\underline{\eta},\zeta)$ *has the form*

$$r(\underline{\eta},\zeta) := R_h(\underline{\eta}) - \underline{grad}\,\zeta, \tag{10.4.125}$$

where R *is a mapping from* Θ_h *to* Γ_h *such that*

$$\|R_h(\underline{\eta})\|_0 \le C_R\|\underline{\eta}\|_1, \tag{10.4.126}$$

for some constant C_R *independent of* h*. Then,* (10.4.124) *holds.*

The proof is an easy exercise.

We can now use Theorem 5.5.5 and obtain the following abstract error bound.

Theorem 10.4.4. *Assume that* R *is an operator from* Θ_h *to* Γ_h *satisfying* (10.4.126)*, and assume that the bilinear form* $\tilde{\mathcal{B}}$ *is defined through* (10.4.125) *and* (10.4.119)*. For every* $t \in]0,T[$*, let* $(\underline{\theta}(t), w(t), \underline{\gamma}(t)$ *be the solution of Problem* (10.4.61) *and let* $(\underline{\theta}_h(t), w_h(t), \underline{\gamma}_h(t)$ *be the solution of* (10.4.120)*. Then, for every* $(\underline{\theta}_I(t), w_I(t), \underline{\gamma}_I(t)$ *in* $\Theta_h \times Z_h \times \Gamma_h$ *such that*

$$R_h(\underline{\theta}_I) - \underline{grad}\,w_I = t^2\underline{\gamma}_I, \tag{10.4.127}$$

we have

$$\begin{aligned}\|\underline{\theta}_h(t) - \underline{\theta}_I(t)\|_1 + \|w_h(t) - w_I(t)\|_1 + t\|\underline{\gamma}_h(t) - \underline{\gamma}_I(t)\|_0 \\ \le C\Big(\|\underline{\theta}(t) - \underline{\theta}_I(t)\|_1 + \|w(t) - w_I(t)\|_1 + \|\underline{\gamma}(t) - \underline{\gamma}_I(t)\|_Q \\ + \sup_{\underline{\eta}\in\Theta_h}\frac{(R_h\underline{\eta},\underline{\gamma}) - (\underline{\eta},\underline{\gamma}_I)}{\|\underline{\eta}\|_1}\Big)\end{aligned} \tag{10.4.128}$$

where C *is a constant independent of* t *and* h*.*

Proof. The proof is elementary: using (10.4.126) and Proposition 10.4.10, we obtain (10.4.124). Using Proposition 10.4.9, we obtain (10.4.123), which is the crucial assumption needed to apply Theorem 5.5.5. □

Remark 10.4.15. In many cases, the last term in the right-hand side of (10.4.128) can be better estimated by

$$\sup_{\underline{\eta}\in\Theta_h}\frac{(R_h\underline{\eta}_I - \underline{\eta},\underline{\gamma})}{\|\underline{\eta}\|_1} + \sup_{\underline{\eta}\in\Theta_h}\frac{(\underline{\eta},\underline{\gamma} - \underline{\gamma}_I)}{\|\underline{\eta}\|_1} \tag{10.4.129}$$

which, in a sense, separates the errors $\|\underline{\gamma}(t) - \underline{\gamma}_I(t)\|_{-1}$ and $\|R_h - Identity\|$. It has to be pointed out that, in most cases, the difference $R_h\underline{\eta}_I - \underline{\eta}$ will be orthogonal to all (vector-valued) polynomial of a certain degree ℓ so that

$$\sup_{\underline{\eta}\in\Theta_h} \frac{(R_h\underline{\eta}_I - \underline{\eta}, \underline{\gamma})}{\|\underline{\eta}\|_1} \leq C\, h\|\underline{\gamma} - \pi_\ell\underline{\gamma}\|_0 \qquad (10.4.130)$$

where π_ℓ is the projection operator on polynomials of degree ℓ. □

As we did for the previous class of methods (the ones based on the decomposition), we will not present here a list of all methods of this type available on the market. We will instead present a single method, as an example, in order to show the general guidelines that rule their construction.

We assume again that Ω is a convex polygon and then we are given a sequence $\{\mathcal{T}_h\}$ of partitions of Ω into triangles. We set, with the notation of Chap. 2,

$$\Theta_h := (\mathcal{L}_2^1 + B_3)^2 \cap (H_0^1(\Omega))^2, \qquad Z_h := (\mathcal{L}_2^1 + B_3) \cap H_0^1(\Omega), \quad (10.4.131)$$

$$\Gamma_h := \{\underline{\chi} \in (\mathcal{L}_2^0)^2 \text{ s. t. } \underline{\chi} \mid \underline{\chi}\cdot\underline{t} \in P_1(e)\ \forall \text{ edge } e\} \cap H_0(\text{curl}; \Omega). \quad (10.4.132)$$

Note that this is the *rotated* $\mathcal{BDFM}_2$, following Remark 2.3.2. Together with the spaces (10.4.131), we consider the operator Π_h from, say, $(H^1(\Omega))^2$ into Γ_h defined in each triangle K by

$$\int_e (\Pi_h\underline{\eta} - \underline{\eta})\cdot\underline{t}\mu_1\, ds = 0 \qquad \forall e \in \partial K \qquad \forall \mu_1 \in P_1(e), \qquad (10.4.133)$$

$$\int_K (\Pi_h\underline{\eta} - \underline{\eta})\cdot\underline{q}\, dx = 0 \qquad \forall\underline{q} \in \mathcal{RT}_0(K), \qquad (10.4.134)$$

where $\mathcal{RT}_0(K)$ is the lowest order Raviart-Thomas space (see Chap. 2).

We can now define the operator r. Following the structure (10.4.125), we set

$$r(\underline{\eta}_h, \zeta_h) = \underline{\text{grad}}\,\zeta_h - \Pi_h\underline{\eta}_h \in \Gamma_h. \qquad (10.4.135)$$

The kernel of $\mathbb{B}_h$ as defined in (10.4.121) is now easily characterised as the set of $(\underline{\eta}_h, \zeta_h)$ such that

$$\Pi_h\underline{\eta}_h = \underline{\text{grad}}\,\zeta_h. \qquad (10.4.136)$$

Since $\|\Pi_h\underline{\eta}_h\|_0 \leq c\|\underline{\eta}_h\|_1$ for some constant c independent of h, we can apply Proposition 10.4.10 and then Proposition (10.4.9) to get

$$\mathcal{A}((\underline{\eta}, \zeta), (\underline{\eta}, \zeta)) + t^{-2}\|r(\underline{\eta}, \zeta)\|_0^2 \geq \tilde{\alpha}_{RM}\|(\underline{\eta}, \zeta)\|_V^2 \qquad (10.4.137)$$

that is, more precisely, condition (10.4.137). In order to apply Theorem 10.4.4, we just need to check that condition (10.4.127) holds for suitable $\underline{\theta}_I, w_I, \underline{\gamma_I}$ having optimal approximation properties. For the construction of $\underline{\theta}_I, w_I, \underline{\gamma_I}$, we can use the following lemma.

Lemma 10.4.1. *Assume that*

$$\{\underline{\chi} \in \Gamma_h \text{ such that } curl\, \underline{\chi} = 0\} \subseteq \underline{grad}(Z_h). \tag{10.4.138}$$

Set $\underline{\gamma}_I := \Pi_h \underline{\gamma}$ *and assume that we can find* $\underline{\theta}^i$ *and* w^i *verifying*

$$curl\, \Pi_h(\underline{\theta}^i - \underline{\theta}) = 0 \qquad \Pi_h(\underline{grad}\, w^i - \underline{grad}\, w) = 0. \tag{10.4.139}$$

Then, from (10.4.138) *and* (10.4.139)*, one obviously has*

$$\Pi_h(\underline{\theta}^i - \underline{\theta}) = \underline{grad}\, \zeta_h \tag{10.4.140}$$

for some $\zeta_h \in Z_h$*. Then setting*

$$\underline{\theta}_I := \underline{\theta}^i \qquad w_I = w^i - \zeta_h, \tag{10.4.141}$$

one has (10.4.127) *as well as*

$$\|\Pi_h(\underline{\theta}_I - \underline{\theta})\|_1 + \|w_I - w\|_1 \leq 2\, \|\Pi_h(\underline{\theta}^i - \underline{\theta})\|_1 + \|w_i - w\|_1. \tag{10.4.142}$$

Note that, in other words, inequality (10.4.142) tells us that we can "arrange" (10.4.127) without losing accuracy. The proof is simple: first we check that

$$\begin{aligned} \Pi_h\underline{\theta}_I - \underline{\mathrm{grad}}\, w_I & \\ &= \Pi_h\underline{\theta}^i - \underline{\mathrm{grad}}\, w_I = \Pi_h\underline{\theta} + (\Pi_h\underline{\theta}_i - \Pi_h\underline{\theta}) - \underline{\mathrm{grad}}\, w_I \\ &= \Pi_h\underline{\theta} + \underline{\mathrm{grad}}\, \zeta_h - \underline{\mathrm{grad}}\, w_I = \Pi_h\underline{\theta} - \underline{\mathrm{grad}}\, w^i \\ &= \Pi_h\underline{\theta} - \Pi_h\, \underline{\mathrm{grad}}\, w^i = \Pi_h(\underline{\theta} - \underline{\mathrm{grad}}\, w^i) = \Pi_h(t^2\underline{\gamma}) \\ &= t^2\underline{\gamma}_I, \end{aligned} \tag{10.4.143}$$

giving us (10.4.127). Inequality (10.4.142) then follows immediately from

$$\|w_I - w\|_1 \leq \|w_i - w\|_1 + \|\zeta_h\|_1 \leq \|w_i - w\|_1 + \|\Pi_h(\underline{\theta}^i - \underline{\theta})\|_1. \tag{10.4.144}$$

□

Then, we just have to construct $\underline{\theta}^i$ and w^i satisfying (10.4.139). The construction of $\underline{\theta}^i$ is easy. Indeed, denoting Π_{CR} the B-compatible operator for the Couzeix-Raviart element for the Stokes problem and by π_1 the projection onto $\mathcal{L}_1^0$, one has from Example 8.6.1

$$\pi_1\, \mathrm{curl}(\underline{\theta} - \mathrm{curl}\, \Pi_{CR}\underline{\theta}) = 0 \tag{10.4.145}$$

and similarly, from the properties of the $\mathcal{BDFM}$ element,

$$\pi_1\, \mathrm{curl}\, \underline{\theta} - \mathrm{curl}\, \Pi_h\underline{\theta} = 0. \tag{10.4.146}$$

We deduce that

$$\operatorname{curl} \Pi_h \underline{\theta} = \pi_1 \operatorname{curl} \underline{\theta} = \pi_1 \operatorname{curl} \Pi_{CR} \underline{\theta} = \operatorname{curl} \Pi_h \Pi_{CR} \underline{\theta}. \tag{10.4.147}$$

This says that the choice

$$\underline{\theta}^i := \Pi_{CR} \underline{\theta} \tag{10.4.148}$$

will satisfy the first condition of (10.4.139). On the other hand, taking

$$w^i(P) = w(P) \qquad \text{for all node } P \text{ in } \mathcal{T}_h, \tag{10.4.149}$$

$$\int_e (w^i - w)\, ds = 0 \qquad \text{for all edge } e \text{ in } \mathcal{T}_h, \tag{10.4.150}$$

and

$$\int_K (w^i - w)\, dx = 0 \qquad \text{for all triangle } K \text{ in } \mathcal{T}_h, \tag{10.4.151}$$

we easily have that

$$\int_e \underline{\operatorname{grad}}(w^i - w) \cdot \underline{t}\, \mu_1, ds = 0 \qquad \forall \text{ edge } e \text{ in } \mathcal{T}_h, \quad \forall\, \mu_1 \in P_1(e) \tag{10.4.152}$$

and

$$\int_T \underline{\operatorname{grad}}(w^i - w) \cdot \underline{q}\, dx = 0 \qquad \forall \text{ triangle } T \text{ in } \mathcal{T}_h, \quad \forall \underline{q} \in \mathcal{RT}_{01}(T), \tag{10.4.153}$$

implying the second condition of (10.4.139).

We can therefore use Theorem 10.4.2 and standard interpolation estimates (together with Remark 10.4.15) to obtain the following result.

Theorem 10.4.5. *Consider the discretised problem* (10.4.120) *with the choices* (10.4.131)–(10.4.135)*. Then, for every* $t \in]0, T[$*, it has a unique solution* $(\underline{\theta}_h(t), w_h(t), \underline{\gamma}_h(t)$*. Let moreover* $(\underline{\theta}(t), w(t), \underline{\gamma}(t)$ *be the solution of Problem* (10.4.61)*. Then we have*

$$\begin{aligned} &\|\underline{\theta}_h(t) - \underline{\theta}_I(t)\|_1 + \|w_h(t) - w_I(t)\|_1 + t\, \|\underline{\gamma}_h(t) - \underline{\gamma}_I(t)\|_0 \\ &\qquad \le C\, h^2 \Big(\|\underline{\theta}(t)\|_3 + \|w(t)\|_3 + t \|\underline{\gamma}(t)\|_2 + \|\gamma\|_{H^1(\mathrm{div};\Omega)} \Big) \end{aligned} \tag{10.4.154}$$

where C *is a constant independent of* t *and* h*.*

As we already noted, this estimate is overoptimistic because it ignores the boundary layer effects. From the results of [29], an $O(h^{3/2})$ convergence rate should be expected.

Remark 10.4.16. Similar estimates have been obtained in [117] for the presently discussed element and related ones, including elements defined on quadrilaterals. More refined estimates can be found in [126]. A recent review of different Mindlin-Reissner approximations, including the *Linked interpolation* techniques (that have not been considered here), can be found in [190]. □

Remark 10.4.17. The choice of second-order accuracy has been made only for the sake of simplicity. Higher-order elements are possible and we shall indicate at the end of this chapter a general framework within which they could be built. On the contrary, lower-order elements are more difficult to get; see for instance [54] for the convergence analysis of a similar method, which is only $O(h)$ accurate [55, 258]. We also refer to [28, 54, 122, 126, 181, 182, 323] for other examples. □

Remark 10.4.18. It is possible to use a duality argument to get an $O(h^3)$ estimate for $\|\underline{\theta} - \underline{\theta}_h\|_0$ and $\|w - w_h\|_0$. See [126]. □

Now to end this lengthy section, we are in a position to present general guidelines for the discretisation of Mindlin–Reissner problems.

We must emphasise again that the decomposition principle makes apparent a direct link with the Stokes problem. Indeed, all examples for which a satisfactory analysis could be achieved contained an already proven Stokes element. If we distinguish the case of continuous pressure approximation and the case of discontinuous pressure element, we get two types of strategies.

10.4.6 Continuous Pressure Approximations

- Suppose one knows $\Theta_h \times Q_h$ to be a good approximation of the Stokes problem with $Q_h \subset H^1(\Omega)$.
- Choose Z_h an approximation of $H_0^1(\Omega)$ of the same order of accuracy.
- Write $\Gamma_h = \underline{\mathrm{grad}}\, Z_h + \underline{\mathrm{curl}}\, Q_h$.

In this context, the definition of Γ_h does not lead, in general, to a standard space and the decomposition principle of Theorem 10.4.2 and Remark 10.4.5 is the only way to handle things from a computational point of view. It may, however, happen, for a clever choice of Z_h and Q_h, that Γ_h turns out to be a standard polynomial space. Such a situation has been encountered in [28] where, using for $\Theta_h \times Q_h$ the MINI element, but taking Z_h to be $\mathcal{L}_1^{1,NC}$, that is, a nonconforming P_1 approximation of $H_0^1(\Omega)$, Γ_h comes to be the whole space $(\mathcal{L}_0^0)^2$ and not a proper subspace. For an extension of the Arnold-Falk element to higher degree, see [14, 26].

10.4.7 Discontinuous Pressure Elements

This second class of approximations to the Stokes problem has been the basis for the "reduced integration" method of the last subsection. Here, we shall try to outline the principal features of this strategy in order to provide a guide for possible extensions, some of which can be found in [117].

1. Here again, our starting point is an approximation of the Stokes problem $\Theta_h \times Q_h$, Q_h being a space of discontinuous polynomial functions. This approximation should, of course, satisfy the *inf-sup* condition.
2. We need to match this with an approximation Γ_h of $H_0(\text{curl}, \Omega)$. More precisely, we need a couple of spaces (Γ_h, Q_h) (where Q_h is the same as before) and a uniformly bounded linear operator $\Pi_h \to \Gamma_h$ such that we have the commuting diagram:

$$\begin{array}{ccc} H & \xrightarrow{\text{curl}} & L^2(\Omega) \\ \Pi_h \downarrow & & P_h \downarrow \\ \Gamma_h & \xrightarrow{\text{curl}} & Q_h \end{array} \tag{10.4.155}$$

   where $\Theta = (H^1(\Omega))^2 \cap H_0\,(\text{curl}, \Omega)$ and P_h is the L^2-projection operator.
3. We finally need a space $Z_h \subset H_0^1(\Omega)$ such that

$$\underline{\text{grad}}\, Z_h = \{\underline{\delta}_h \in \Gamma_h,\ \text{curl}\, \underline{\delta}_h = 0\}. \tag{10.4.156}$$

Ingredients (1), (2), (3) will produce a plate element for which one can essentially repeat the proof of Theorem 10.4.5 and obtain optimal error estimates for $\underline{\theta}$ and w.