A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton–Jacobi Equations

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In memory of Riichi Iino, my former adviser at Waseda University.

Abstract We present an introduction to the theory of viscosity solutions of first-order partial differential equations and a review on the optimal control/dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations, with the Neumann boundary condition. This article also includes some of basics of mathematical analysis related to the optimal control/dynamical approach for easy accessibility to the topics.

Introduction

This article is an attempt to present a brief introduction to viscosity solutions of first-order partial differential equations (PDE for short) and to review some aspects of the large time behavior of solutions of Hamilton–Jacobi equations with Neumann boundary conditions.

The notion of viscosity solution was introduced in [20] (see also [18]) by Crandall and Lions, and it has been widely accepted as the right notion of generalized solutions of the first-order PDE of the Hamilton–Jacobi type and fully nonlinear (possibly degenerate) elliptic or parabolic PDE. There have already been many nice contributions to overview of viscosity solutions of first-order and/or

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second-order partial differential equations. The following list touches just a few of them [2, 6, 15, 19, 29, 31, 41, 42].

This article is meant to serve as a quick introduction for graduate students or young researchers to viscosity solutions and is, of course, an outcome of the lectures delivered by the author at the CIME school as well as at Waseda University, Collège de France, Kumamoto University, King Abdulaziz University and University of Tokyo. For its easy readability, it contains some of very basics of mathematical analysis which are usually left aside to other textbooks.

The first section is an introduction to viscosity solutions of first-order partial differential equations. As a motivation to viscosity solutions we take up an optimal control problem and show that the value function of the control problem is characterized as a unique viscosity solution of the associated Bellman equation. This choice is essentially the same as used in the book [42] by Lions as well as in [2, 6, 29].

In Sects. 2–5, we develop the theory of viscosity solutions of Hamilton–Jacobi equations with the linear Neumann boundary condition together with the corresponding optimal control problems, which we follow [8,38,39]. In Sect. 6, following [38], we show the convergence of the solution of Hamilton–Jacobi equation of evolution type with the linear Neumann boundary condition to a solution of the stationary problem.

The approach here to the convergence result depends heavily on the variational formula for solutions, that is, the representation of solutions as the value function of the associated control problem. There is another approach, due to [3], based on the asymptotic monotonicity of a certain functional of the solutions as time goes to infinity, which is called the PDE approach. The PDE approach does not depend on the variational formula for the solutions and provides a very simple proof of the convergence with sharper hypotheses. The approach taken here may be called the dynamical or optimal control one. This approach requires the convexity of the Hamiltonian, so that one can associate it with an optimal control problem. Although it requires lots of steps before establishing the convergence result, its merit is that one can get an interpretation to the convergence result through the optimal control representation.

The topics covered in this article are very close to the ones discussed by Barles [4]. Both are to present an introduction to viscosity solutions and to discuss the large time asymptotics for solutions of Hamilton–Jacobi equations. This article has probably a more elementary flavor than [4] in the part of the introduction to viscosity solutions, and the paper [4] describes the PDE-viscosity approach to the large time asymptotics while this article concentrates on the dynamical or optimal control approach.

The reference list covers only those papers which the author more or less consulted while he was writing this article, and it is far from a complete list of those which have contributed to the developments of the subject.

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Notation:

- When $\mathscr{F}$ is a set of real-valued functions on X, $\sup\mathscr{F}$ and $\inf\mathscr{F}$ denote the functions on X given, respectively, by

$$(\sup\mathscr{F})(x) := \sup\{f(x)\,:\,f\in\mathscr{F}\} \quad\text{and}\quad (\inf\mathscr{F})(x) := \inf\{f(x)\,:\,f\in\mathscr{F}\}.$$

- For any $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Also, we write $a_+ = a \vee 0$ and $a_- = (-a)_+$.
- A function $\omega \in C([0, R))$, with $0 < R \le \infty$, is called a modulus if it is nondecreasing and satisfies $\omega(0) = 0$.
- For any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $x \cdot y$ denotes the Euclidean inner product $x_1y_1 + \cdots + x_ny_n$ of x and y.
- For any $x, y \in \mathbb{R}^n$ the line segment between x and y is denoted by $[x, y] := \{(1-t)x + ty\,:\,t \in [0, 1]\}$.
- For $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$, $C^k(\Omega, \mathbb{R}^m)$ (or simply, $C^k(\Omega, \mathbb{R}^m)$) denotes the collection of functions $f\,:\,\Omega \to \mathbb{R}^m$ (not necessarily open), each of which has an open neighborhood U of Ω and a function $g \in C^k(U)$ such that $f(x) = g(x)$ for all $x \in \Omega$.
- For $f \in C(\Omega, \mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^n$, the support of f is defined as the closure of $\{x \in \Omega\,:\,f(x) \neq 0\}$ and is denoted by $\operatorname{supp} f$.
- $\mathrm{UC}(X)$ (resp., $\mathrm{BUC}(X)$) denotes the space of all uniformly continuous (resp., bounded, uniformly continuous) functions in a metric space X.
- We write $\mathbf{1}_E$ for the characteristic function of the set E. That is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise.
- The sup-norm of function f on a set Ω is denoted by $\|f\|_{\infty,\Omega} = \|f\|_\infty := \sup_\Omega |f|$.
- We write $\mathbb{R}_+$ for the interval $(0, \infty)$.
- For any interval $J \subset \mathbb{R}$, $\mathrm{AC}(J, \mathbb{R}^m)$ denotes the space of all absolutely continuous functions in J with value in $\mathbb{R}^m$.
- Given a convex Hamiltonian $H \in C(\overline{\Omega} \times \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is an open set, we denote by L the Lagrangian given by

$$L(x, \xi) = \sup_{p\in\mathbb{R}^n} (\xi \cdot p - H(x, p)) \quad \text{for } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

- Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$, $g \in C(\partial\Omega, \mathbb{R})$, $t > 0$ and $(\eta, v, l) \in L^1([0, t], \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ such that $\eta(s) \in \overline{\Omega}$ for all $s \in [0, t]$ and $l(s) = 0$ whenever $\eta(s) \in \Omega$. We write

$$\mathscr{L}(t, \eta, v, l) = \int_0^t [L(\eta(s), -v(s)) + g(\eta(s))l(s)]\mathrm{d}s.$$

1 Introduction to Viscosity Solutions

We give the definition of viscosity solutions of first-order PDE and study their basic properties.

1.1 Hamilton–Jacobi Equations

Let Ω be an open subset of $\mathbb{R}^n$. Given a function $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$, we consider the PDE

$$H(x, Du(x)) = 0 \quad \text{in } \Omega, \tag{1}$$

where Du denotes the gradient of u, that is,

$$Du := (u_{x_1}, u_{x_2}, \ldots, u_{x_n}) \equiv (\partial u/\partial x_1, \ldots, \partial u/\partial x_n).$$

We also consider the PDE

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{in } \Omega \times (0, \infty). \tag{2}$$

Here the variable t may be regarded as the time variable and u_t denotes the time derivative $\partial u/\partial t$. The variable x is then regarded as the space variable and $D_x u$ (or, Du) denotes the gradient of u in the space variable x.

The PDE of the type of (1) or (2) are called Hamilton–Jacobi equations. A more concrete example of (1) is given by

$$|Du(x)| = k(x),$$

which appears in geometrical optics and describes the surface front of propagating waves. Hamilton–Jacobi equations arising in Mechanics have the form

$$|Du(x)|^2 + V(x) = 0,$$

where the terms $|Du(x)|^2$ and $V(x)$ correspond to the kinetic and potential energies, respectively.

More generally, the PDE of the form

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega \tag{3}$$

may be called Hamilton–Jacobi equations.

1.2 An Optimal Control Problem

We consider the function

$$X = X(t) = (X_1(t), X_2(t), \ldots, X_n(t)) \in \mathbb{R}^n$$

of time $t \in \mathbb{R}$, and

$$\dot{X} = \dot{X}(t) = \frac{\mathrm{d}X}{\mathrm{d}t}(t)$$

denotes its derivative. Let $A \subset \mathbb{R}^m$ be a given set, let $g : \mathbb{R}^n \times A \to \mathbb{R}^n$, $f : \mathbb{R}^n \times A \to \mathbb{R}$ be given functions and $\lambda > 0$ be a given constant. We denote by $\mathbf{A}$ the set of all Lebesgue measurable $\alpha : [0, \infty) \to A$.

Fix any $x \in \mathbb{R}^n$ and $\alpha \in \mathbf{A}$, and consider the initial value problem for the ordinary differential equation (for short, ODE)

$$\begin{cases} \dot{X}(t) = g(X(t), \alpha(t)) \quad \text{for a.e. } t > 0, \\ X(0) = x. \end{cases} \tag{4}$$

The solution of (4) will be denoted by $X = X(t) = X(t; x, \alpha)$. The solution $X(t)$ may depend significantly on choices of $\alpha \in \mathbf{A}$. Next we introduce the functional

$$J(x, \alpha) = \int_0^\infty f(X(t), \alpha(t)) \mathrm{e}^{-\lambda t} \, \mathrm{d}t, \tag{5}$$

a function of x and $\alpha \in \mathbf{A}$, which serves a criterion to decide which choice of α is better. The best value of the functional J is given by

$$V(x) = \inf_{\alpha \in \mathbf{A}} J(x, \alpha). \tag{6}$$

This is an optimization problem, and the main theme is to select a control $\alpha = \alpha_x \in \mathbf{A}$ so that

$$V(x) = J(x, \alpha).$$

Such a control α is called an *optimal control*. The ODE in (4) is called the *dynamics* or *state equation*, the functional J given by (5) is called the *cost functional*, and the function V given by (6) is called the value function. The function f or $t \mapsto \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))$ is called the *running cost* and λ is called the *discount rate*.

In what follows, we assume that f, g are bounded continuous functions on $\mathbb{R}^n \times A$ and moreover, they satisfy the Lipschitz condition, i.e., there exists a constant $M > 0$ such that

$$\begin{cases} |f(x, a)| \le M, \qquad |g(x, a)| \le M, \\ |f(x, a) - f(y, a)| \le M|x - y|, \\ |g(x, a) - g(y, a)| \le M|x - y|. \end{cases} \tag{7}$$

A basic result in ODE theory guarantees that the initial value problem (4) has a unique solution $X(t)$.

There are two basic approaches in optimal control theory:

1. Pontryagin's Maximum Principle Approach.
2. Bellman's Dynamic Programming Approach.

Both of approaches have been introduced and developed since 1950s.

Pontryagin's maximum principle gives a necessary condition for the optimality of controls and provides a powerful method to design an optimal control.

Bellman's approach associates the optimization problem with a PDE, called the *Bellman equation*. In the problem, where the value function V is given by (6), the corresponding Bellman equation is the following.

$$\lambda V(x) + H(x, DV(x)) = 0 \quad \text{in } \mathbb{R}^n, \tag{8}$$

where H is a function given by

$$H(x, p) = \sup_{a \in A}\{-g(x, a) \cdot p - f(x, a)\},$$

with $x \cdot y$ denoting the Euclidean inner product in $\mathbb{R}^n$. Bellman's idea is to characterize the value function V by the Bellman equation, to use the characterization to compute the value function and to design an optimal control. To see how it works, we assume that (8) has a smooth bounded solution V and compute formally as follows. First of all, we choose a function $a : \mathbb{R}^n \to A$ so that

$$H(x, DV(x)) = -g(x, a(x)) \cdot DV(x) - f(x, a(x)),$$

and solve the initial value problem

$$\dot{X}(t) = g(X(t), a(X(t))), \qquad X(0) = x,$$

where x is a fixed point in $\mathbb{R}^n$. Next, writing $\alpha(t) = a(X(t))$, we have

$$\begin{aligned}
0 &= \int_0^\infty \mathrm{e}^{-\lambda t}\big(\lambda V(X(t)) + H(X(t), DV(X(t)))\big)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\big(\lambda V(X(t)) - g(X(t), \alpha(t)) \cdot DV(X(t)) - f(X(t), \alpha(t))\big)\,\mathrm{d}t \\
&= \int_0^\infty \left(-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\lambda t} V(X(t)) - \mathrm{e}^{\lambda t} f(X(t), \alpha(t))\right)\,\mathrm{d}t \\
&= V(X(0)) - \int_0^\infty \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\,\mathrm{d}t.
\end{aligned}$$

Thus we have

$$V(x) = J(x,\alpha).$$

If PDE (8) characterizes the value function, that is, the solution V is the value function, then the above equality says that the control $\alpha(t) = a(X(t))$ is an optimal control, which we are looking for.

In Bellman's approach PDE plays a central role, and we discuss this approach in what follows. The first remark is that the value function may not be differentiable at some points. A simple example is as follows.

Example 1.1. We consider the case where $n = 1$, $A = [-1,1] \subset \mathbb{R}$, $f(x,a) = \mathrm{e}^{-x^2}$, $g(x,a) = a$ and $\lambda = 1$. Let $X(t)$ be the solution of (4) for some control $\alpha \in \mathbf{A}$, which means just to satisfy

$$|\dot{X}(t)| \le 1 \quad \text{a.e. } t > 0.$$

Let V be the value function given by (6). Then it is clear that $V(-x) = V(x)$ for all $x \in \mathbb{R}$ and that

$$V(x) = \int_0^\infty \mathrm{e}^{-t-(x+t)^2}\,\mathrm{d}t = \mathrm{e}^x \int_x^\infty \mathrm{e}^{-t-t^2}\,\mathrm{d}t \quad \text{if } x > 0.$$

For $x > 0$, one gets

$$V'(x) = \mathrm{e}^x \int_x^\infty \mathrm{e}^{-t-t^2}\,\mathrm{d}t - \mathrm{e}^{-x^2},$$

and

$$V'(0+) = \int_0^\infty \mathrm{e}^{-t-t^2}\,\mathrm{d}t - 1 < \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t - 1 = 0.$$

This together with the symmetry property, $V(-x) = V(x)$ for all $x \in \mathbb{R}$, shows that V is not differentiable at $x = 0$.

Value functions in optimal control do not have enough regularity to satisfy, in the classical sense, the corresponding Bellman equations in general as the above example shows.

We introduce the notion of viscosity solution of the first-order PDE

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{FE}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given continuous function.

Definition 1.1. (i) We call $u \in C(\Omega)$ a viscosity subsolution of (FE) if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \max_{\Omega}(u-\phi) = (u-\phi)(z) \\ \quad \Longrightarrow\ F(z, u(z), D\phi(z)) \le 0. \end{cases}$$

(ii) We call $u \in C(\Omega)$ a viscosity supersolution of (FE) if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \min_{\Omega}(u-\phi) = (u-\phi)(z) \\ \quad \Longrightarrow\ F(z, u(z), D\phi(z)) \geq 0. \end{cases}$$

(iii) We call $u \in C(\Omega)$ a viscosity solution of (FE) if u is both a viscosity subsolution and supersolution of (FE).

The viscosity subsolution or supersolution property is checked through smooth functions ϕ in the above definition, and such smooth functions ϕ are called test functions.

Remark 1.1. If we set $F^-(x,r,p) = -F(x,-r,-p)$, then it is obvious that $u \in C(\Omega)$ is a viscosity subsolution (resp., supersolution) of (FE) if and only if $u^-(x) := -u(x)$ is a viscosity supersolution (resp., subsolution) of

$$F^-(x, u^-(x), Du^-(x)) = 0 \quad \text{in } \Omega.$$

Note also that $(F^-)^- = F$ and $(u^-)^- = u$. With these observations, one property for viscosity subsolutions can be phrased as a property for viscosity supersolutions. In other words, every proposition concerning viscosity subsolutions has a counterpart for viscosity supersolutions.

Remark 1.2. It is easily seen by adding constants to test functions that $u \in C(\Omega)$ is a viscosity subsolution of (FE) if and only if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \max_{\Omega}(u-\phi) = (u-\phi)(z) = 0 \\ \quad \Longrightarrow\ F(z, \phi(z), D\phi(z)) \leq 0. \end{cases}$$

One can easily formulate a counterpart of this proposition for viscosity supersolutions.

Remark 1.3. It is easy to see by an argument based on a partition of unity (see Appendix A.1) that $u \in C(\Omega)$ is a viscosity subsolution of (FE) if and only if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ u-\phi \text{ attains a local maximum at } z \\ \quad \Longrightarrow\ F(z, \phi(z), D\phi(z)) \leq 0. \end{cases}$$

Remark 1.4. It is easily seen that $u \in C(\Omega)$ is a viscosity subsolution of (FE) if and only if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ u-\phi \text{ attains a strict maximum at } z \\ \quad \Longrightarrow\ F(z, \phi(z), D\phi(z)) \leq 0. \end{cases}$$

Similarly, one may replace "strict maximum" by "strict local maximum" in the statement. The idea to show these is to replace the function ϕ by $\phi(x) + |x - z|^2$ when needed.

Remark 1.5. The condition, $\phi \in C^1(\Omega)$, can be replaced by the condition, $\phi \in C^\infty(\Omega)$ in the above definition. The argument in the following example explains how to see this equivalence.

Example 1.2 (Vanishing viscosity method). The term "viscosity solution" originates to the vanishing viscosity method, which is one of classical methods to construct solutions of first-order PDE.

Consider the second-order PDE

$$-\varepsilon\Delta u^\varepsilon + F(x, u^\varepsilon(x), Du^\varepsilon(x)) = 0 \quad \text{in } \Omega, \tag{9}$$

where $\varepsilon > 0$ is a parameter to be sent to zero later on, Ω is an open subset of $\mathbb{R}^n$, F is a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and Δ denotes the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We assume that functions $u^\varepsilon \in C^2(\Omega)$, with $\varepsilon \in (0,1)$, and $u \in C(\Omega)$ are given and that

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = u(x) \quad \text{locally uniformly on } \Omega.$$

Then the claim is that u is a viscosity solution of

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega. \tag{FE}$$

In what follows, we just check that u is a viscosity subsolution of (FE). For this, we assume that

$$\phi \in C^1(\Omega), \quad \hat{x} \in \Omega, \quad \max_\Omega(u - \phi) = (u - \phi)(\hat{x}),$$

and moreover, this maximum is a strict maximum of $u - \phi$. We need to show that

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x})) \le 0. \tag{10}$$

First of all, we assume that $\phi \in C^2(\Omega)$, and show that (10) holds. Fix an $r > 0$ so that $\overline{B}_r(\hat{x}) \subset \Omega$. Let x_ε be a maximum point over $\overline{B}_r(\hat{x})$ of the function $u^\varepsilon - \phi$. We may choose a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ so that $\lim_{j\to\infty} \varepsilon_j = 0$ and $\lim_{j\to\infty} x_{\varepsilon_j} = y$ for some $y \in \overline{B}_r(\hat{x})$. Observe that

$$\begin{aligned}(u-\phi)(\hat{x}) &\le (u^{\varepsilon_j}-\phi)(\hat{x}) + \|u-u^{\varepsilon_j}\|_{\infty,B_r(\hat{x})}\\ &\le (u^{\varepsilon_j}-\phi)(x_{\varepsilon_j}) + \|u-u^{\varepsilon_j}\|_{\infty,B_r(\hat{x})}\\ &\le (u-\phi)(x_{\varepsilon_j}) + 2\|u^{\varepsilon_j}-u\|_{\infty,B_r(\hat{x})}\\ &\to (u-\phi)(y) \quad \text{as } j\to\infty.\end{aligned}$$

Accordingly, since $\hat{x}$ is a strict maximum point of $u-\phi$, we see that $y = \hat{x}$. Hence, if j is sufficiently large, then $x_{\varepsilon_j} \in B_r(\hat{x})$. By the maximum principle from Advanced Calculus, we find that

$$\frac{\partial}{\partial x_i}(u^{\varepsilon_j}-\phi)(x_{\varepsilon_j}) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x_i^2}(u^{\varepsilon_j}-\phi)(x_{\varepsilon_j}) \le 0 \quad \text{for all } i = 1,2,\ldots,n.$$

Hence, we get

$$Du^{\varepsilon_j}(x_{\varepsilon_j}) = D\phi(x_{\varepsilon_j}), \quad \Delta u^{\varepsilon_j}(x_{\varepsilon_j}) \le \Delta\phi(x_{\varepsilon_j}).$$

These together with (9) yield

$$-\varepsilon_j \Delta\phi(x_{\varepsilon_j}) + F(x_{\varepsilon_j}, u^{\varepsilon_j}(x_{\varepsilon_j}), D\phi(x_{\varepsilon_j})) \le 0.$$

Sending $j \to \infty$ now ensures that (10) holds.

Finally we show that the C^2 regularity of ϕ can be relaxed, so that (10) holds for all $\phi \in C^1(\Omega)$. Let $r > 0$ be the constant as above, and choose a sequence $\{\phi_k\} \subset C^\infty(\Omega)$ so that

$$\lim_{k\to\infty} \phi_k(x) = \phi(x) \quad \text{uniformly on } B_r(\hat{x}).$$

Let $\{y_k\} \subset \overline{B}_r(\hat{x})$ be a sequence consisting of a maximum point of $u-\phi_k$. An argument similar to the above yields

$$\lim_{k\to\infty} y_k = \hat{x}.$$

If k is sufficiently large, then we have $y_k \in B_r(\hat{x})$ and, due to (10) valid for C^2 test functions,

$$F(y_k, u(y_k), D\phi_k(y_k)) \le 0.$$

Sending $k \to \infty$ allows us to conclude that (10) holds.

1.3 Characterization of the Value Function

In this subsection we are concerned with the characterization of the value function V by the Bellman equation

$$\lambda V(x) + H(x, DV(x)) = 0 \quad \text{in } \mathbb{R}^n, \tag{11}$$

where λ is a positive constant and

$$H(x, p) = \sup_{a \in A}\{-g(x, a) \cdot p - f(x, a)\}.$$

Recall that

$$V(x) = \inf_{\alpha \in \mathbf{A}} J(x, \alpha),$$

and

$$J(x, \alpha) = \int_0^\infty f(X(t), \alpha(t))e^{-\lambda t}\, dt,$$

where $X(t) = X(t; x, \alpha)$ denotes the solution of the initial value problem

$$\begin{cases} \dot{X}(t) = g(X(t), \alpha(t)) & \text{for a.e. } t > 0,\\ X(0) = x. \end{cases}$$

Recall also that for all $(x, a) \in \mathbb{R}^n \times A$ and some constant $M > 0$,

$$\begin{cases} |f(x, a)| \le M, \qquad |g(x, a)| \le M,\\ |f(x, a) - f(y, a)| \le M|x - y|,\\ |g(x, a) - g(y, a)| \le M|x - y|. \end{cases} \tag{12}$$

The following lemma will be used without mentioning, the proof of which may be an easy exercise.

Lemma 1.1. *Let $h, k : A \to \mathbb{R}$ be bounded functions. Then*

$$\left|\sup_{a\in A} h(a) - \sup_{a\in A} k(a)\right| \vee \left|\inf_{a\in A} h(a) - \inf_{a\in A} k(a)\right| \le \sup_{a\in A} |h(a) - k(a)|.$$

In view of the above lemma, the following lemma is an easy consequence of (12), and the detail of the proof is left to the reader.

Lemma 1.2. *The Hamiltonian H satisfies the following inequalities:*

$$|H(x, p) - H(y, p)| \le M|x - y|(|p| + 1) \quad \text{for all } x, y, p \in \mathbb{R}^n,$$
$$|H(x, p) - H(x, q)| \le M|p - q| \quad \text{for all } x, p, q \in \mathbb{R}^n.$$

In particular, we have $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$.

Proposition 1.1. *The inequality*

$$|V(x)| \le \frac{M}{\lambda}$$

holds for all $x \in \mathbb{R}^n$. Hence, the value function V is bounded on $\mathbb{R}^n$.

Proof. For any $(x, \alpha) \in \mathbb{R}^n \times \mathbf{A}$, we have

$$|J(x,\alpha)| \le \int_0^\infty e^{-\lambda t} |f(X(t), \alpha(t))| \, dt \le M \int_0^\infty e^{-\lambda t} \, dt = \frac{M}{\lambda}.$$

Applying Lemma 1.1 yields

$$|V(x)| \le \sup_{\alpha \in \mathbf{A}} |J(x,\alpha)| \le \frac{M}{\lambda}. \qquad \square$$

Proposition 1.2. *The function V is Hölder continuous on $\mathbb{R}^n$.*

Proof. Fix any $x, y \in \mathbb{R}^n$. For any $\alpha \in \mathbf{A}$, we estimate the difference of $J(x,\alpha)$ and $J(y,\alpha)$. To begin with, we estimate the difference of $X(t) := X(t; x, \alpha)$ and $Y(t) := X(t; y, \alpha)$. Since

$$\begin{aligned} |\dot{X}(t) - \dot{Y}(t)| &= |g(X(t), \alpha(t)) - g(Y(t), \alpha(t))| \\ &\le M |X(t) - Y(t)| \quad \text{for a.e. } t \ge 0, \end{aligned}$$

we find that

$$\begin{aligned} |X(t) - Y(t)| &\le |X(0) - Y(0)| + \int_0^t |\dot{X}(s) - \dot{Y}(s)| \, ds \\ &\le |x - y| + M \int_0^t |X(s) - Y(s)| \, ds \quad \text{for all } t \ge 0. \end{aligned}$$

By applying Gronwall's inequality, we get

$$|X(t) - Y(t)| \le |x - y| \, e^{Mt} \quad \text{for all } t \ge 0.$$

Next, since

$$|J(x,\alpha) - J(y,\alpha)| \le \int_0^\infty e^{-\lambda s} |f(X(s), \alpha(s)) - f(Y(s), \alpha(s))| \, ds,$$

if $\lambda > M$, then we have

$$\begin{aligned} |J(x,\alpha) - J(y,\alpha)| &\le \int_0^\infty e^{-\lambda s} M |X(s) - Y(s)| \, ds \\ &\le M \int_0^\infty e^{-\lambda s} |x - y| e^{Ms} \, ds = \frac{M|x - y|}{\lambda - M}, \end{aligned}$$

and

$$|V(x) - V(y)| \leq \frac{M}{\lambda - M}|x - y|. \tag{13}$$

If $0 < \lambda < M$, then we select $0 < \theta < 1$ so that $\theta M < \lambda$, and calculate

$$\begin{aligned}|f(\xi, a) - f(\eta, a)| &\leq |f(\xi, a) - f(\eta, a)|^{\theta+(1-\theta)} \\ &\leq (M|\xi - \eta|)^{\theta}(2M)^{1-\theta} \quad \text{for all } \ \xi, \eta \in \mathbb{R}^n, a \in A,\end{aligned}$$

and

$$\begin{aligned}|J(x, \alpha) - J(y, \alpha)| &\leq (2M)^{1-\theta} \int_0^{\infty} \mathrm{e}^{-\lambda s}(M|X(s) - Y(s)|)^{\theta} \, \mathrm{d}s \\ &\leq (2M)^{1-\theta} \int_0^{\infty} \mathrm{e}^{-\lambda s}(M|x - y|)^{\theta} \mathrm{e}^{\theta M s} \, \mathrm{d}s \\ &\leq 2M|x - y|^{\theta} \int_0^{\infty} \mathrm{e}^{-(\lambda - \theta M)s} \, \mathrm{d}s = \frac{2M|x - y|^{\theta}}{\lambda - \theta M},\end{aligned}$$

which shows that

$$|V(x) - V(y)| \leq \frac{2M|x - y|^{\theta}}{\lambda - \theta M}. \tag{14}$$

Thus we conclude from (13) and (14) that V is Hölder continuous on $\mathbb{R}^n$. □

Proposition 1.3 (Dynamic programming principle). Let $0 < \tau < \infty$ and $x \in \mathbb{R}^n$. Then

$$V(x) = \inf_{\alpha \in \mathbf{A}} \Big(\int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t), \alpha(t)) \, \mathrm{d}t + \mathrm{e}^{-\lambda \tau} V(X(\tau)) \Big),$$

where $X(t)$ denotes $X(t; x, \alpha)$.

Proof. Let $0 < \tau < \infty$ and $x \in \mathbb{R}^n$. Fix $\gamma \in \mathbf{A}$. We have

$$\begin{aligned}J(x, \gamma) &= \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t), \gamma(t)) \, \mathrm{d}t + \int_{\tau}^{\infty} \mathrm{e}^{-\lambda t} f(X(t), \gamma(t)) \, \mathrm{d}t \\ &= \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t), \alpha(t)) \, \mathrm{d}t + \mathrm{e}^{-\lambda \tau} \int_0^{\infty} \mathrm{e}^{-\lambda t} f(Y(t), \beta(t)) \, \mathrm{d}t,\end{aligned} \tag{15}$$

where

$$\begin{aligned}&X(t) = X(t; x, \gamma), \quad \alpha(t) := \gamma(t), \quad \beta(t) := \gamma(t + \tau), \\ &Y(t) := X(t + \tau) = X(t; X(\tau), \beta).\end{aligned}$$

By (15), we get

$$J(x,\gamma) \geq \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)),$$

from which we have

$$J(x,\gamma) \geq \inf_{\alpha\in\mathbf{A}} \Big(\int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)) \Big).$$

Consequently,

$$V(x) \geq \inf_{\alpha\in\mathbf{A}} \Big(\int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)) \Big). \tag{16}$$

Now, let $\alpha, \beta \in \mathbf{A}$. Define $\gamma \in \mathbf{A}$ by

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if } 0 \leq t \leq \tau, \\ \beta(t-\tau) & \text{if } \tau < t. \end{cases}$$

Set

$$X(t) := X(t; x, \alpha) \quad \text{and} \quad Y(t) := X(t; X(\tau), \beta).$$

We have

$$\begin{cases} X(t) = X(t; x, \gamma) \ \text{ and } \ \alpha(t) = \gamma(t) \quad \text{for all } t \in [0, \tau], \\ \beta(t) = \gamma(t+\tau) \ \text{ and } \ Y(t) = X(t+\tau) \quad \text{for all } t \geq 0. \end{cases}$$

Hence, we have (15) and therefore,

$$V(x) \leq \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} J(X(\tau),\beta).$$

Moreover, we get

$$V(x) \leq \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)),$$

and

$$V(x) \leq \inf_{\alpha\in\mathbf{A}} \Big(\int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)) \Big). \tag{17}$$

Combining (16) and (17) completes the proof. □

Theorem 1.1. *The value function V is a viscosity solution of* (11).

Proof. (Subsolution property) Let $\phi \in C^1(\mathbb{R}^n)$ and $\hat{x} \in \mathbb{R}^n$, and assume that

$$(V-\phi)(\hat{x}) = \max_{\mathbb{R}^n}(V-\phi) = 0.$$

Fix any $a \in A$ and set $\alpha(t) := a$, $X(t) := X(t;\hat{x},\alpha)$. Let $0 < h < \infty$. Now, since $V \le \phi$, $V(\hat{x}) = \phi(\hat{x})$, by Proposition 1.3 we get

$$\begin{aligned}\phi(\hat{x}) = V(\hat{x}) &\le \int_0^h \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda h} V(X(h)) \\ &\le \int_0^h \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda h}\phi(X(h)).\end{aligned}$$

From this, we get

$$\begin{aligned}0 &\le \int_0^h \mathrm{e}^{-\lambda t} f(X(t),a)\,\mathrm{d}t + \int_0^h \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\lambda t}\phi(X(t)))\,\mathrm{d}t \\ &= \int_0^h \mathrm{e}^{-\lambda t}\left(f(X(t),a) - \lambda\phi(X(t)) + D\phi(X(t))\cdot\dot{X}(t)\right)\,\mathrm{d}t \\ &= \int_0^h \mathrm{e}^{-\lambda t}\left(f(X(t),a) - \lambda\phi(X(t)) + D\phi(X(t))\cdot g(X(t),a)\right)\mathrm{d}t.\end{aligned} \tag{18}$$

Noting that

$$|X(t)-\hat{x}| = \left|\int_0^t \dot{X}(s)\,\mathrm{d}s\right| \le \int_0^t |g(X(s),a)|\,\mathrm{d}s \le M\int_0^t \mathrm{d}s = Mt, \tag{19}$$

dividing (18) by h and sending $h \to 0$, we find that

$$0 \le -\lambda\phi(\hat{x}) + f(\hat{x},a) + g(\hat{x},a)\cdot D\phi(\hat{x}).$$

Since $a \in A$ is arbitrary, we have $\lambda\phi(\hat{x}) + H(\hat{x}, D\phi(\hat{x})) \le 0$.

(Supersolution property) Let $\phi \in C^1(\mathbb{R}^n)$ and $\hat{x} \in \mathbb{R}^n$, and assume that

$$(V-\phi)(\hat{x}) = \min_{\mathbb{R}^n}(V-\phi) = 0.$$

Fix $\varepsilon > 0$ and $h > 0$. By Proposition 1.3, we may choose $\alpha \in \mathbf{A}$ so that

$$V(\hat{x}) + \varepsilon h > \int_0^h \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda h} V(X(h)),$$

where $X(t) := X(t;\hat{x},\alpha)$. Since $V \geq \phi$ in $\mathbb{R}^n$ and $V(\hat{x}) = \phi(\hat{x})$, we get

$$\phi(\hat{x}) + \varepsilon h > \int_0^h e^{-\lambda t} f(X(t),\alpha(t))\,dt + e^{-\lambda h}\phi(X(h)).$$

Hence we get

$$\begin{aligned} 0 &\geq \int_0^h e^{-\lambda t} f(X(t),\alpha(t))\,dt + \int_0^h \frac{d}{dt}(e^{-\lambda t}\phi(X(t)))\,dt - \varepsilon h \\ &= \int_0^h e^{-\lambda t}\left(f(X(t),\alpha(t)) - \lambda\phi(X(t)) + D\phi(X(t))\cdot \dot{X}(t)\right)\,dt - \varepsilon h \\ &= \int_0^h e^{-\lambda t}\big(f(X(t),\alpha(t)) - \lambda\phi(X(t)) + D\phi(X(t))\cdot g(X(t),\alpha(t))\big)\,dt - \varepsilon h. \end{aligned}$$

By the definition of H, we get

$$\int_0^h e^{-\lambda t}(\lambda\phi(X(t)) + H(X(t), D\phi(t))\,dt + \varepsilon h > 0. \tag{20}$$

As in (19), we have

$$|X(t) - \hat{x}| \leq Mt.$$

Dividing (20) by h and sending $h \to 0$ yield

$$\lambda\phi(\hat{x}) + H(\hat{x}, D\phi(\hat{x})) + \varepsilon \geq 0,$$

from which we get $\lambda\phi(\hat{x}) + H(\hat{x}, D\phi(\hat{x})) \geq 0$. The proof is now complete. □

Theorem 1.2. *Let $u \in \mathrm{BUC}(\mathbb{R}^n)$ and $v \in \mathrm{BUC}(\mathbb{R}^n)$ be a viscosity subsolution and supersolution of (11), respectively. Then $u \leq v$ in $\mathbb{R}^n$.*

Proof. Let $\varepsilon > 0$, and define $u_\varepsilon \in C(\mathbb{R}^n)$ by $u_\varepsilon(x) = u(x) - \varepsilon(\langle x\rangle + M)$, where $\langle x\rangle = (|x|^2 + 1)^{1/2}$. A formal calculation

$$\begin{aligned} u_\varepsilon(x) + H(x, Du_\varepsilon(x)) &\leq u(x) - \varepsilon M + H(x, Du(x)) + \varepsilon M|D\langle x\rangle| \\ &\leq u(x) + H(x, Du(x)) \leq 0 \end{aligned}$$

reveals that u_ε is a viscosity subsolution of (11), which can be easily justified.

We show that the inequality $u_\varepsilon \leq v$ holds, from which we deduce that $u \leq v$ is valid. To do this, we assume that $\sup_{\mathbb{R}^n}(u_\varepsilon - v) > 0$ and will get a contradiction. Since

$$\lim_{|x|\to\infty}(u_\varepsilon - v)(x) = -\infty,$$

we may choose a constant $R > 0$ so that

$$\sup_{\mathbb{R}^n \setminus B_R} (u_\varepsilon - v) < 0.$$

The function $u_\varepsilon - v \in C(\overline{B}_R)$ then attains a maximum at a point in B_R, but not at any point in ∂B_R.

Let $\alpha > 1$ and consider the function

$$\Phi(x, y) = u_\varepsilon(x) - v(y) - \alpha |x - y|^2$$

on $K := \overline{B}_R \times \overline{B}_R$. Since $\Phi \in C(K)$, Φ attains a maximum at a point in K. Let $(x_\alpha, y_\alpha) \in K$ be its maximum point. Because K is compact, we may choose a sequence $\{\alpha_j\} \subset (1, \infty)$ diverging to infinity so that for some $(\hat{x}, \hat{y}) \in K$,

$$(x_{\alpha_j}, y_{\alpha_j}) \to (\hat{x}, \hat{y}) \quad \text{as } j \to \infty.$$

Note that

$$\begin{aligned} 0 < \max_{\overline{B}_R}(u_\varepsilon - v) &= \max_{x \in \overline{B}_R} \Phi(x, x) \le \Phi(x_\alpha, y_\alpha) \\ &= u_\varepsilon(x_\alpha) - v(y_\alpha) - \alpha |x_\alpha - y_\alpha|^2, \end{aligned} \tag{21}$$

from which we get

$$\alpha |x_\alpha - y_\alpha|^2 \le \sup_{\mathbb{R}^n} u_\varepsilon + \sup_{\mathbb{R}^n}(-v).$$

We infer from this that $\hat{x} = \hat{y}$. Once again by (21), we get

$$\max_{\overline{B}_R}(u_\varepsilon - v) \le u_\varepsilon(x_\alpha) - v(y_\alpha).$$

Setting $\alpha = \alpha_j$ and sending $j \to \infty$ in the above, since $u, v \in C(\mathbb{R}^n)$, we see that

$$\begin{aligned} \max_{\overline{B}_R}(u_\varepsilon - v) &\le \lim_{\alpha = \alpha_j, j \to \infty} (u_\varepsilon(x_\alpha) - v(y_\alpha)) \\ &= u_\varepsilon(\hat{x}) - v(\hat{x}). \end{aligned}$$

That is, the point $\hat{x}$ is a maximum point of $u_\varepsilon - v$. By (21), we have

$$\alpha |x_\alpha - y_\alpha|^2 \le u_\varepsilon(x_\alpha) - v(y_\alpha) - \max_{\overline{B}_R}(u - v),$$

and hence

$$\lim_{\alpha = \alpha_j, j \to \infty} \alpha |x_\alpha - y_\alpha|^2 = 0.$$

Since $\hat{x}$ is a maximum point of $u_\varepsilon - v$, by our choice of R we see that $\hat{x} \in B_R$. Accordingly, if $\alpha = \alpha_j$ and j is sufficiently large, then $x_\alpha, y_\alpha \in B_R$. By the viscosity property of u_ε and v, for $\alpha = \alpha_j$ and $j \in \mathbb{N}$ large enough, we have

$$u_\varepsilon(x_\alpha) + H(x_\alpha, 2\alpha(x_\alpha - y_\alpha)) \le 0, \qquad v(y_\alpha) + H(y_\alpha, 2\alpha(x_\alpha - y_\alpha)) \ge 0.$$

Subtracting one from the other yields

$$u_\varepsilon(x_\alpha) - v(y_\alpha) \le H(y_\alpha, 2\alpha(x_\alpha - y_\alpha)) - H(x_\alpha, 2\alpha(x_\alpha - y_\alpha)).$$

Using one of the properties of H from Lemma 1.2, we obtain

$$u_\varepsilon(x_\alpha) - v(y_\alpha) \le M|x_\alpha - y_\alpha|(2\alpha|x_\alpha - y_\alpha| + 1).$$

Sending $\alpha = \alpha_j \to \infty$, we get

$$u_\varepsilon(\hat{x}) - v(\hat{x}) \le 0,$$

which is a contradiction. □

1.4 Semicontinuous Viscosity Solutions and the Perron Method

Let $u, v \in C(\Omega)$ be a viscosity subsolutions of (FE) and set

$$w(x) = \max\{u(x), v(x)\} \quad \text{for } x \in \Omega.$$

It is easy to see that w is a viscosity subsolution of (FE). Indeed, if $\phi \in C^1(\Omega)$, $y \in \Omega$ and $w - \phi$ has a maximum at y, then we have either $w(y) = u(y)$ and $(u - \phi)(x) \le (w - \phi)(x) \le (w - \phi)(y) = (u - \phi)(y)$ for all $x \in \Omega$, or $w(y) = v(y)$ and $(v - \phi)(x) \le (v - \phi)(y)$, from which we get $F(y, w(y), D\phi(y)) \le 0$. If $\{u_k\}_{k\in\mathbb{N}} \subset C(\Omega)$ is a uniformly bounded sequence of viscosity subsolutions of (FE), then the function w given by $w(x) = \sup_k u_k(x)$ defines a bounded function on Ω but it may not be continuous, a situation that the notion of viscosity subsolution does not apply.

We are thus led to extend the notion of viscosity solution to that for discontinuous functions.

Let $U \subset \mathbb{R}^n$, and recall that a function $f : U \to \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$ is *upper semicontinuous* if

$$\limsup_{y \to x} f(y) \le f(x) \quad \text{for all } x \in U.$$

The totality of all such upper semicontinuous functions f will be denoted by $\mathrm{USC}(U)$. Similarly, we denote by $\mathrm{LSC}(U)$ the space of all *lower* semicontinuous functions on U. That is, $\mathrm{LSC}(U) := -\mathrm{USC}(U) = \{-f \,:\, f \in \mathrm{USC}(U)\}$.

Some basic observations regarding semicontinuity are the following three propositions.

Proposition 1.4. *Let $f : U \to [-\infty, \infty]$. Then, $f \in \mathrm{USC}(U)$ if and only if the set $\{x \in U \,:\, f(x) < a\}$ is a relatively open subset of U for any $a \in \mathbb{R}$.*

Proposition 1.5. *If $\mathscr{F} \subset \mathrm{LSC}(U)$, then $\sup \mathscr{F} \in \mathrm{LSC}(U)$. Similarly, if $\mathscr{F} \subset \mathrm{USC}(U)$, then $\inf \mathscr{F} \in \mathrm{USC}(U)$.*

Proposition 1.6. *Let K be a compact subset of $\mathbb{R}^n$ and $f \in \mathrm{USC}(K)$. Then f attains a maximum. Here the maximum value may be either $-\infty$ or ∞.*

Next, we define the upper (resp., lower) *semicontinuous envelopes* f^* (resp., f_*) of $f : U \to [-\infty, \infty]$ by

$$f^*(x) = \lim_{r \to 0+} \sup\{f(y) \,:\, y \in U \cap B_r(x)\}$$

(resp., $f_* = -(-f)^*$ or, equivalently, $f_*(x) = \lim_{r\to 0+} \inf\{f(y) \,:\, y \in U \cap B_r(x)\}$).

Proposition 1.7. *Let $f : U \to [-\infty, \infty]$. Then we have $f^* \in \mathrm{USC}(U)$, $f_* \in \mathrm{LSC}(U)$ and*

$$f^*(x) = \min\{g(x) \,:\, g \in \mathrm{USC}(U),\ g \ge f\} \quad \textit{for all} \quad x \in U.$$

A consequence of the above proposition is that if $f \in \mathrm{USC}(U)$, then $f^* = f$ in U. Similarly, $f_* = f$ in U if $f \in \mathrm{LSC}(U)$.

We go back to

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega. \tag{FE}$$

Here we assume neither that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous nor that $\Omega \subset \mathbb{R}^n$ is open. We just assume that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is locally bounded and that Ω is a subset of $\mathbb{R}^n$.

Definition 1.2. (i) A locally bounded function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution (resp., supersolution) of (FE) if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \max_{\Omega}(u^* - \phi) = (u^* - \phi)(z) \\ \implies F_*(z, u^*(z), D\phi(z)) \le 0 \end{cases}$$

$$\left(resp., \quad \begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \min_{\Omega}(u_* - \phi) = (u_* - \phi)(z) \\ \implies F^*(z, u_*(z), D\phi(z)) \ge 0 \end{cases} \right).$$

(ii) A locally bounded function $u : \Omega \to \mathbb{R}$ is a viscosity solution of (FE) if it is both a viscosity subsolution and supersolution of (FE).

We warn here that the envelopes F_* and F^* are taken in the full variables. For instance, if $\xi \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, then

$$F_*(\xi) = \lim_{r\to 0+} \inf\{F(\eta) : \eta \in \Omega \times \mathbb{R} \times \mathbb{R}^n,\ |\eta - \xi| < r\}.$$

We say conveniently that u is a viscosity solution (or subsolution) of $F(x, u(x), Du(x)) \le 0$ in Ω if u is a viscosity subsolution of (FE). Similarly, we say that u is a viscosity solution (or supersolution) of $F(x, u(x), Du(x)) \ge 0$ in Ω if u is a viscosity supersolution of (FE). Also, we say that u satisfies $F(x, u(x), Du(x)) \le 0$ in Ω (resp., $F(x, u(x), Du(x)) \ge 0$ in Ω) in the *viscosity sense* if u is a viscosity subsolution (resp., supersolution) of (FE).

Once we fix a PDE, like (FE), on a set Ω, we denote by $\mathscr{S}^-$ and $\mathscr{S}^+$ the sets of all its viscosity subsolutions and supersolutions, respectively.

The above definition differs from the one in [19]. As is explained in [19], the above one allows the following situation: let Ω be a nonempty open subset of $\mathbb{R}^n$ and suppose that the Hamilton–Jacobi equation (1) has a continuous solution $u \in C(\Omega)$. Choose two dense subsets U and V of Ω such that $U \cap V = \emptyset$ and $U \cup V \ne \Omega$. Select a function $v : \Omega \to \mathbb{R}$ so that $v(x) = u(x)$ if $x \in U$, $v(x) = u(x) + 1$ if $x \in V$ and $v(x) \in [u(x),\ u(x)+1]$ if $x \in \Omega \setminus (U \cup V)$. Then we have $v_*(x) = u(x)$ and $v^*(x) = u(x) + 1$ for all $x \in \Omega$. Consequently, v is a viscosity solution of (1). If $U \cup V \ne \Omega$, then there are infinitely many choices of such functions v.

The same remarks as Remarks 1.1–1.4 are valid for the above generalized definition.

Definition 1.3. Let $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. The subdifferential $D^-u(x)$ and superdifferential $D^+u(x)$ of the function u at $x \in \Omega$ are defined, respectively, by

$$D^-u(x) = \{p \in \mathbb{R}^n : u(x+h) \ge u(x) + p\cdot h + o(|h|) \text{ as } x+h \in \Omega,\ h \to 0\},$$

$$D^+u(x) = \{p \in \mathbb{R}^n : u(x+h) \le u(x) + p\cdot h + o(|h|) \text{ as } x+h \in \Omega,\ h \to 0\},$$

where $o(|h|)$ denotes a function on an interval $(0, \delta)$, with $\delta > 0$, having the property: $\lim_{h\to 0} o(|h|)/|h| = 0$.

We remark that $D^-u(x) = -D^+(-u)(x)$. If u is a convex function in $\mathbb{R}^n$ and $p \in D^-u(x)$ for some $x, p \in \mathbb{R}^n$, then

$$u(x+h) \ge u(x) + p\cdot h \quad \text{for all } h \in \mathbb{R}^n.$$

See Proposition B.1 for the above claim. In convex analysis, $D^-u(x)$ is usually denoted by $\partial u(x)$.

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be locally bounded. Let $x \in \Omega$. Then*

$$D^+u(x) = \{D\phi(x) \,:\, \phi \in C^1(\Omega),\ u - \phi \text{ attains a maximum at } x\}.$$

As a consequence of the above proposition, we have the following: if u is locally bounded in Ω, then

$$\begin{aligned} D^-u(x) &= -\,D^+(-u)(x) \\ &= -\,\{D\phi(x) \,:\, \phi \in C^1(\Omega),\ -u - \phi \text{ attains a maximum at } x\} \\ &= \{D\phi(x) \,:\, \phi \in C^1(\Omega),\ u - \phi \text{ attains a minimum at } x\}. \end{aligned}$$

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^n$. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be locally bounded. Then u is a viscosity subsolution (resp., supersolution) of* (FE) *if and only if*

$$F_*(x, u^*(x), p) \le 0 \quad \text{for all } x \in \Omega,\ p \in D^+u^*(x)$$
$$(\text{ resp., } \ F^*(x, u_*(x), p) \ge 0 \quad \text{for all } x \in \Omega,\ p \in D^-u_*(x)\ ).$$

This corollary (or Remark 1.3) says that the viscosity properties of a function, i.e., the properties that the function be a viscosity subsolution, supersolution, or solution are of local nature. For instance, under the hypotheses of Corollary 1.1, the function u is a viscosity subsolution of (FE) if and only if for each $x \in \Omega$ there exists an open neighborhood U_x, in $\mathbb{R}^n$, of x such that u is a viscosity subsolution of (FE) in $U_x \cap \Omega$.

Proof. Let $\phi \in C^1(\Omega)$ and $y \in \Omega$, and assume that $u - \phi$ has a maximum at y. Then

$$(u - \phi)(y + h) \le (u - \phi)(y) \quad \text{if } y + h \in \Omega,$$

and hence, as $y + h \in \Omega,\ h \to 0$,

$$u(y + h) \le u(y) + \phi(y + h) - \phi(y) = u(y) + D\phi(y) \cdot h + o(|h|).$$

This shows that

$$\{D\phi(y) \,:\, \phi \in C^1(\Omega),\ u - \phi \text{ attains a maximum at } y\} \subset D^+u(y).$$

Next let $y \in \Omega$ and $p \in D^+u(y)$. Then we have

$$u(y + h) \le u(y) + p \cdot h + \omega(|h|)|h| \quad \text{if } y + h \in \Omega \text{ and } |h| \le \delta$$

for some constant $\delta > 0$ and a function $\omega \in C([0,\ \delta])$ satisfying $\omega(0) = 0$. We may choose ω to be nondecreasing in $[0,\ \delta]$. In the above inequality, we want to replace the term $\omega(|h|)|h|$ by a C^1 function $\psi(h)$ having the property: $\psi(h) = o(|h|)$. Following [23], we define the function $\gamma : [0, \delta/2] \to \mathbb{R}$ by

$$\gamma(r) = \int_0^{2r} \omega(t)\,\mathrm{d}t.$$

Noting that

$$\gamma(r) \geq \int_r^{2r} \omega(t)\,\mathrm{d}t \geq \omega(r)r \quad \text{for } r \in [0,\ \delta/2],$$

we see that

$$u(y+h) \leq u(y) + p \cdot h + \gamma(|h|) \quad \text{if } y+h \in \Omega \text{ and } |h| \leq \delta/2.$$

It immediate to see that $\gamma \in C^1([0,\ \delta/2])$ and $\gamma(0) = \gamma'(0) = 0$. We set $\psi(h) = \gamma(|h|)$ for $h \in B_{\delta/2}(0)$. Then $\psi \in C^1(B_{\delta/2}(0))$, $\psi(0) = 0$ and $D\psi(0) = 0$. It is now clear that if we set

$$\phi(x) = u(y) + p \cdot (x-y) + \psi(x-y) \quad \text{for } x \in B_{\delta/2}(y),$$

then the function $u - \phi$ attains a maximum over $\Omega \cap B_{\delta/2}(y)$ at y and $D\phi(y) = p$. □

Now, we discuss a couple of stability results concerning viscosity solutions.

Proposition 1.9. *Let $\{u_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathscr{S}^-$. Assume that Ω is locally compact and $\{u_\varepsilon\}$ converges locally uniformly to a function u in Ω as $\varepsilon \to 0$. Then $u \in \mathscr{S}^-$.*

Proof. Let $\phi \in C^1(\Omega)$. Assume that $u^* - \phi$ attains a strict maximum at $\hat{x} \in \Omega$. We choose a constant $r > 0$ so that $K := \overline{B_r(\hat{x})} \cap \Omega$ is compact. For each $\varepsilon \in (0,\ 1)$, we choose a maximum point (over K) x_ε of $u_\varepsilon^* - \phi$.

Next, we choose a sequence $\{\varepsilon_j\} \subset (0,\ 1)$ converging to zero such that $x_{\varepsilon_j} \to z$ for some $z \in K$ as $j \to \infty$. Next, observe in view of the choice of x_ε that

$$\begin{aligned}(u^* - \phi)(x_{\varepsilon_j}) &\geq (u_{\varepsilon_j}^* - \phi)(x_{\varepsilon_j}) - \|u^* - u_{\varepsilon_j}^*\|_{\infty,K} \\ &\geq (u^* - \phi)(x_{\varepsilon_j}) - 2\|u^* - u_{\varepsilon_j}^*\|_{\infty,K} \\ &\geq (u^* - \phi)(\hat{x}) - 2\|u^* - u_{\varepsilon_j}^*\|_{\infty,K}.\end{aligned}$$

Sending $j \to \infty$ yields

$$(u^* - \phi)(z) \geq \limsup(u_{\varepsilon_j}^* - \phi)(x_{\varepsilon_j}) \geq \liminf_{j \to \infty}(u_{\varepsilon_j}^* - \phi)(x_{\varepsilon_j}) \geq (u^* - \phi)(\hat{x}),$$

which shows that $z = \hat{x}$ and $\lim_{j\to\infty} u_{\varepsilon_j}^*(x_{\varepsilon_j}) = u^*(\hat{x})$. For $j \in \mathbb{N}$ sufficiently large, we have $x_{\varepsilon_j} \in B_r(\hat{x})$ and, since $u_{\varepsilon_j} \in \mathscr{S}^-$,

$$F_*(x_{\varepsilon_j}, u^*_{\varepsilon_j}(x_{\varepsilon_j}), D\phi(x_{\varepsilon_j})) \le 0.$$

If we send $j \to \infty$, we find that $u \in \mathscr{S}^-$. □

Proposition 1.10. *Let Ω be locally compact. Let $\mathscr{F} \subset \mathscr{S}^-$. That is, $\mathscr{F}$ is a family of viscosity subsolutions of* (FE). *Assume that* $\sup \mathscr{F}$ *is locally bounded in Ω. Then we have* $\sup \mathscr{F} \in \mathscr{S}^-$.

Remark 1.6. By definition, the set Ω is locally compact if for any $x \in \Omega$, there exists a constant $r > 0$ such that $\Omega \cap \overline{B}_r(x)$ is compact. For instance, every open subset and closed subset of $\mathbb{R}^n$ are locally compact. The set $A := (0, 1) \times [0, 1] \subset \mathbb{R}^2$ is locally compact, but the set $A \cup \{(0, 0)\}$ is not locally compact.

Remark 1.7. Similarly to Remark 1.5, if Ω is locally compact, then the C^1 regularity of the test functions in the Definition 1.2 can be replaced by the C^∞ regularity.

Proof. Set $u = \sup \mathscr{F}$. Let $\phi \in C^1(\Omega)$ and $\hat{x} \in \Omega$, and assume that

$$\max_\Omega (u^* - \phi) = (u^* - \phi)(\hat{x}) = 0.$$

We assume moreover that $\hat{x}$ is a strict maximum point of $u^* - \phi$. That is, we have $(u^* - \phi)(x) < 0$ for all $x \neq \hat{x}$. Choose a constant $r > 0$ so that $W := \Omega \cap \overline{B}_r(\hat{x})$ is compact.

By the definition of u^*, there are sequences $\{y_k\} \subset W$ and $\{v_k\} \subset \mathscr{F}$ such that

$$y_k \to \hat{x}, \qquad v_k(y_k) \to u^*(\hat{x}) \quad \text{as} \quad k \to \infty.$$

Since W is compact, for each $k \in \mathbb{N}$ we may choose a point $x_k \in W$ such that

$$\max_W (v_k^* - \phi) = (v_k^* - \phi)(x_k).$$

By passing to a subsequence if necessary, we may assume that $\{x_k\}$ converges to a point $z \in W$. We then have

$$\begin{aligned} 0 = (u^* - \phi)(\hat{x}) &\ge (u^* - \phi)(x_k) \ge (v_k^* - \phi)(x_k) \\ &\ge (v_k^* - \phi)(y_k) \ge (v_k - \phi)(y_k) \to (u^* - \phi)(\hat{x}) = 0, \end{aligned}$$

and consequently

$$\lim_{k\to\infty} u^*(x_k) = \lim_{k\to\infty} v_k^*(x_k) = u^*(\hat{x}).$$

In particular, we see that

$$(u^* - \phi)(z) \ge \lim_{k\to\infty} (u^* - \phi)(x_k) = 0,$$

which shows that $z = \hat{x}$. That is, $\lim_{k\to\infty} x_k = \hat{x}$.

Thus, we have $x_k \in B_r(\hat{x})$ for sufficiently large $k \in \mathbb{N}$. Since $v_k \in \mathscr{S}^-$, we get

$$F_*(x_k, v_k^*(x_k), D\phi(x_k)) \le 0$$

if k is large enough. Hence, sending $k \to \infty$ yields

$$F_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \le 0,$$

which proves that $u \in \mathscr{S}^-$. □

Theorem 1.3. *Let Ω be a locally compact subset of $\mathbb{R}^n$. Let $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{F_\varepsilon\}_{\varepsilon\in(0,1)}$ be locally uniformly bounded collections of functions on Ω and $\Omega \times \mathbb{R} \times \mathbb{R}^n$, respectively. Assume that for each $\varepsilon \in (0, 1)$, u_ε is a viscosity subsolution of*

$$F_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x)) \le 0 \quad \text{in } \Omega.$$

Set

$$\bar{u}(x) = \lim_{r\to 0+} \sup\{u_\varepsilon(y) \,:\, y \in B_r(x) \cap \Omega,\ \varepsilon \in (0, r)\},$$

$$\underline{F}(\xi) = \lim_{r\to 0+} \inf\{F_\varepsilon(\eta) \,:\, \eta \in \Omega \times \mathbb{R} \times \mathbb{R}^n,\ |\eta - \xi| < r,\ \varepsilon \in (0, r)\}.$$

Then $\bar{u}$ is a viscosity subsolution of

$$\underline{F}(x, \bar{u}(x), D\bar{u}(x)) \le 0 \quad \text{in } \Omega.$$

Remark 1.8. The function $\bar{u}$ is upper semicontinuous in Ω. Indeed, we have

$$\bar{u}(y) \le \sup\{u_\varepsilon(z) \,:\, z \in B_r(x) \cap \Omega,\ \varepsilon \in (0, r)\}$$

for all $x \in \Omega$ and $y \in B_r(x) \cap \Omega$. This yields

$$\limsup_{\Omega \ni y \to x} \bar{u}(y) \le \sup\{u_\varepsilon(z) \,:\, z \in B_r(x) \cap \Omega,\ \varepsilon \in (0, r)\}$$

for all $x \in \Omega$. Hence,

$$\limsup_{\Omega \ni y \to x} \bar{u}(y) \le \bar{u}(x) \quad \text{for all } x \in \Omega.$$

Similarly, the function $\underline{F}$ is lower semicontinuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Proof. It is easily seen that for all $x \in \Omega$, $r > 0$ and $y \in B_r(x) \cap \Omega$,

$$u_\varepsilon^*(y) \le \sup\{u_\varepsilon(z) \,:\, z \in B_r(x) \cap \Omega\}.$$

From this we deduce that

$$\bar{u}(x) = \lim_{r\to 0+} \sup\{u_\varepsilon^*(y) \,:\, y \in B_r(x) \cap \Omega,\ 0 < \varepsilon < r\} \quad \text{for all } x \in \Omega.$$

Hence, we may assume by replacing u_ε by u_ε^* if necessary that $u_\varepsilon \in \mathrm{USC}(\Omega)$. Similarly, we may assume that $F_\varepsilon \in \mathrm{LSC}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$.

Let $\phi \in C^1(\Omega)$ and $\hat{x} \in \Omega$. Assume that $\bar{u} - \phi$ has a strict maximum at $\hat{x}$. Let $r > 0$ be a constant such that $\overline{B}_r(\hat{x}) \cap \Omega$ is compact.

For each $k \in \mathbb{N}$ we choose $y_k \in B_{r/k}(\hat{x}) \cap \Omega$ and $\varepsilon_k \in (0,\ 1/k)$ so that

$$|\bar{u}(\hat{x}) - u_{\varepsilon_k}(y_k)| < 1/k,$$

and then choose a maximum point $x_k \in B_r(\hat{x}) \cap \Omega$ of $u_{\varepsilon_k} - \phi$ over $\overline{B}_r(\hat{x}) \cap \Omega$.

Since

$$(u_{\varepsilon_k} - \phi)(x_k) \geq (u_{\varepsilon_k} - \phi)(y_k),$$

we get

$$\limsup_{k\to 0} (u_{\varepsilon_k} - \phi)(x_k) \geq (\bar{u} - \phi)(\hat{x}),$$

which implies that

$$\lim_{k\to\infty} x_k = \hat{x} \quad \text{and} \quad \lim_{k\to\infty} u_{\varepsilon_k}(x_k) = \bar{u}(\hat{x}).$$

If $k \in \mathbb{N}$ is sufficiently large, we have $x_k \in B_r(\hat{x}) \cap \Omega$ and hence

$$F_{\varepsilon_k}(x_k, u_{\varepsilon_k}(x_k), D\phi(x_k)) \leq 0.$$

Thus, we get

$$\underline{F}(\hat{x}, \bar{u}(\hat{x}), D\phi(\hat{x})) \leq 0. \qquad \square$$

Proposition 1.9 can be seen now as a direct consequence of the above theorem. The following proposition is a consequence of the above theorem as well.

Proposition 1.11. *Let Ω be locally compact. Let $\{u_k\}$ be a sequence of viscosity subsolutions of (FE). Assume that $\{u_k\} \subset \mathrm{USC}(\Omega)$ and that $\{u_k\}$ is a nonincreasing sequence of functions on Ω, i.e., $u_k(x) \geq u_{k+1}(x)$ for all $x \in \Omega$ and $k \in \mathbb{N}$. Set*

$$u(x) = \lim_{k\to\infty} u_k(x) \quad \textit{for } x \in \Omega.$$

Assume that u is locally bounded on Ω. Then $u \in \mathscr{S}^-$.

Let us introduce the (outer) normal cone $N(z, \Omega)$ at $z \in \Omega$ by

$$N(z, \Omega) = \{p \in \mathbb{R}^n \,:\, 0 \geq p \cdot (x - z) + o(|x - z|) \ \text{as} \ \Omega \ni x \to z\}.$$

Another definition equivalent to the above is the following:

$$N(z,\Omega) = -D^{+}\mathbf{1}_{\Omega}(z),$$

where $\mathbf{1}_{\Omega}$ denotes the characteristic function of Ω. Note that if $z \in \Omega$ is an interior point of Ω, then $N(z,\Omega) = \{0\}$.

We say that (FE) or the pair (F,Ω) is *proper* if $F(x,r,p+q) \geq F(x,r,p)$ for all $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and all $q \in N(x,\Omega)$.

Proposition 1.12. *Assume that* (FE) *is proper. If $u \in C^1(\Omega)$ is a classical subsolution of* (FE)*, then $u \in \mathscr{S}^-$.*

Proof. Let $\phi \in C^1(\Omega)$ and assume that $u-\phi$ attains a maximum at $z \in \Omega$. We may assume by extending the domain of definition of u and ϕ that u and ϕ are defined and of class C^1 in $B_r(z)$ for some $r > 0$. By reselecting $r > 0$ small enough if needed, we may assume that

$$(u-\phi)(x) < (u-\phi)(z) + 1 \quad \text{for all } x \in B_r(z).$$

It is clear that the function $u - \phi + \mathbf{1}_{\Omega}$ attains a maximum over $B_r(z)$ at z, which shows that $D\phi(z) - Du(z) \in D^{+}\mathbf{1}_{\Omega}(z)$. Setting $q = -D\phi(z) + Du(z)$, we have $Du(z) = D\phi(z) + q$ and

$$0 \geq F(z,u(z),D\phi(z)+q) \geq F(z,u(z),D\phi(z)) \geq F_*(z,u(z),D\phi(z)),$$

which completes the proof. □

Proposition 1.13 (Perron method). *Let $\mathscr{F}$ be a nonempty subset of $\mathscr{S}^-$ having the properties:*

(P1) $\sup \mathscr{F} \in \mathscr{F}$.
(P2) *If $v \in \mathscr{F}$ and $v \notin \mathscr{S}^+$, then there exists a $w \in \mathscr{F}$ such that $w(y) > v(y)$ at some point $y \in \Omega$.*

Then $\sup \mathscr{F} \in \mathscr{S}$.

Proof. We have $\sup \mathscr{F} \in \mathscr{F} \subset \mathscr{S}^-$. That is, $\sup \mathscr{F} \in \mathscr{S}^-$. If we suppose that $\sup \mathscr{F} \notin \mathscr{S}^+$, then, by (P2), we have $w \in \mathscr{F}$ such that $w(y) > (\sup \mathscr{F})(y)$ for some $y \in \Omega$, which contradicts the definition of $\sup \mathscr{F}$. Hence, $\sup \mathscr{F} \in \mathscr{S}^+$. □

Theorem 1.4. *Assume that Ω is locally compact and that* (FE) *is proper. Let $f \in \mathrm{LSC}(\Omega) \cap \mathscr{S}^-$ and $g \in \mathrm{USC}(\Omega) \cap \mathscr{S}^+$. Assume that $f \leq g$ in Ω. Set*

$$\mathscr{F} = \{v \in \mathscr{S}^- \,:\, f \leq v \leq g \text{ in } \Omega\}.$$

Then $\sup \mathscr{F} \in \mathscr{S}$.

In the above theorem, the semicontinuity requirement on f, g is "opposite" in a sense: the lower (resp., upper) semicontinuity for the subsolution f (resp., supersolution g). This choice of semicontinuities is convenient in practice since in the construction of supersolution f, for instance, one often takes the infimum of a collection of continuous supersolutions and the resulting function is automatically upper semicontinuous.

Of course, under the same hypotheses of the above theorem, we have following conclusion as well: if we set $\mathscr{F}^+ = \{v \in \mathscr{S}^+ : f \le v \le g \text{ in } \Omega\}$, then $\inf \mathscr{F}^+ \in \mathscr{S}$.

Lemma 1.3. *Assume that Ω is locally compact and that (FE) is proper. Let $u \in \mathscr{S}^-$ and $y \in \Omega$, and assume that u is not a viscosity supersolution of (FE) at y, that is,*

$$F^*(y, u_*(y), p) < 0 \quad \textit{for some} \;\; p \in D^- u_*(y).$$

Let $\varepsilon > 0$ and U be a neighborhood of y. Then there exists a $v \in \mathscr{S}^-$ such that

$$\begin{cases} u(x) \le v(x) \le \max\{u(x), u_*(y) + \varepsilon\} & \textit{for all} \;\; x \in \Omega, \\ v = u & \textit{in} \;\; \Omega \setminus U, \\ v_*(y) > u_*(y). \end{cases} \tag{22}$$

Furthermore, if u is continuous at y, then there exist an open neighborhood V of y and a constant $\delta > 0$ such that v is a viscosity subsolution of

$$F(x, v(x), Dv(x)) = -\delta \quad \textit{in } V \cap \Omega. \tag{23}$$

Proof. By assumption, there exists a function $\phi \in C^1(\Omega)$ such that $u_*(y) = \phi(y)$, $u_*(x) > \phi(x)$ for all $x \ne y$ and

$$F^*(y, u_*(y), D\phi(y)) < 0.$$

Thanks to the upper semicontinuity of F^*, there exists a $\delta \in (0, \varepsilon)$ such that

$$F^*(x, \phi(x) + t, D\phi(x)) < -\delta \quad \text{for all} \;\; (x, t) \in (\overline{B}_\delta(y) \cap \Omega) \times [0, \delta], \tag{24}$$

and $\overline{B}_\delta(y) \cap \Omega$ is a compact subset of U.

By replacing $\delta > 0$ by a smaller number if needed, we may assume that

$$\phi(x) + \delta \le u_*(y) + \varepsilon \quad \text{for all} \;\; x \in \overline{B}_\delta(y) \cap \Omega. \tag{25}$$

Since $u_* - \phi$ attains a strict minimum at y and the minimum value is zero, if $(\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y) \ne \emptyset$, then the constant

$$m := \min_{(\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y)} (u_* - \phi)$$

is positive. Of course, in this case, we have

$$u_*(x) \geq \phi(x) + m \quad \text{for all } x \in (\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y).$$

Set $\lambda = \min\{m, \delta\}$ if $(\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y) \neq \emptyset$ and $\lambda = \delta$ otherwise, and observe that

$$u_*(x) \geq \phi(x) + \lambda \quad \text{for all } x \in (\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y). \tag{26}$$

We define $v : \Omega \to \mathbb{R}$ by

$$v(x) = \begin{cases} \max\{u(x), \phi(x) + \lambda\} & \text{if } x \in B_\delta(y), \\ u(x) & \text{if } x \notin B_\delta(y). \end{cases}$$

If we set $\psi(x) = \phi(x) + \lambda$ for $x \in B_\delta(y) \cap \Omega$, by (24), ψ is a classical subsolution of (FE) in $B_\delta(y) \cap \Omega$. Since (FE) is proper, ψ is a viscosity subsolution of (FE) in $B_\delta(y) \cap \Omega$. Hence, by Proposition 1.10, we see that v is a viscosity subsolution of (FE) in $B_\delta(y) \cap \Omega$.

According to (26) and the definition of v, we have

$$v(x) = u(x) \quad \text{for all } x \in \Omega \setminus B_{\delta/2}(y),$$

and, hence, v is a viscosity subsolution of (FE) in $\Omega \setminus \overline{B}_{\delta/2}(y)$ Thus, we find that $v \in \mathscr{S}^-$.

Since $v = u$ in $\Omega \setminus B_\delta(y)$ by the definition of v, it follows that $v = u$ in $\Omega \setminus U$. It is clear by the definition of v that $v \geq u$ in Ω. Moreover, by (25) we get

$$v(x) \leq \max\{u(x),\, u_*(y) + \varepsilon\} \quad \text{for all } x \in \Omega \cap B_\delta(y).$$

Also, observe that

$$v_*(y) = \max\{u_*(y),\, u_*(y) + \lambda\} = u_*(y) + \lambda > u_*(y).$$

Thus, (22) is valid.

Now, we assume that u is continuous at y. Then we find an open neighborhood $V \subset B_\delta(y)$ of y such that

$$u(x) < \phi(x) + \lambda \quad \text{for all } \ x \in V \cap \Omega,$$

and hence, we have $v(x) = \phi(x) + \lambda$ for all $x \in V \cap \Omega$. Now, by (24) we see that v is a classical (and hence viscosity) subsolution of (23). □

Proof (Theorem 1.4). We have $\mathscr{F} \neq \emptyset$ since $f \in \mathscr{F}$. In view of Proposition 1.13, we need only to show that the set $\mathscr{F}$ satisfies (P1) and (P2).

By Proposition 1.10, we see immediately that $\mathscr{F}$ satisfies (P1).

To check property (P2), let $v \in \mathscr{F}$ be not a viscosity supersolution of (FE). There is a point $y \in \Omega$ where v is not a viscosity supersolution of (FE). That is, for some $p \in D^- v_*(y)$, we have

$$F^*(y, v_*(y), p) < 0. \tag{27}$$

Noting $v_* \le g_*$ in Ω, there are two possibilities: $v_*(y) = g_*(y)$ or $v_*(y) < g_*(y)$. If $v_*(y) = g_*(y)$, then $p \in D^- g_*(y)$. Since $g \in \mathscr{S}^+$, we have

$$F^*(y, g_*(y), p) \ge 0,$$

which contradicts (27). If $v_*(y) < g_*(y)$, then we choose a constant $\varepsilon > 0$ and a neighborhood V of y so that

$$v_*(y) + \varepsilon < g_*(x) \quad \text{for all } x \in V \cap \Omega. \tag{28}$$

Now, Lemma 1.3 guarantees that there exist $w \in \mathscr{S}^-$ such that $v \le w \le \max\{v, v_*(y) + \varepsilon\}$ in Ω, $v = w$ in $\Omega \setminus V$ and $w_*(y) > v_*(y)$. For any $x \in \Omega \cap V$, by (28) we have

$$w(x) \le \max\{v(x), g_*(x)\} \le g(x).$$

For any $x \in \Omega \setminus V$, we have

$$w(x) = v(x) \le g(x).$$

Thus, we find that $w \in \mathscr{F}$. Since $w_*(y) > v_*(y)$, it is clear that $w(z) > v(z)$ at some point $z \in \Omega$. Hence, $\mathscr{F}$ satisfies (P2). □

1.5 An Example

We illustrate the use of the stability properties established in the previous subsection by studying the solvability of the Dirichlet problem for the eikonal equation

$$|Du(x)| = k(x) \qquad \text{in } \Omega, \tag{29}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega, \tag{30}$$

where Ω is a bounded, open, connected subset of $\mathbb{R}^n$ and $k \in C(\overline{\Omega})$ is a positive function in $\overline{\Omega}$, i.e., $k(x) > 0$ for all $x \in \overline{\Omega}$.

Note that the constant function $f(x) := 0$ is a classical subsolution of (29). Set $M = \max_{\overline{\Omega}} k$. We observe that for each $y \in \partial\Omega$ the function $g_y(x) := M|x - y|$ is a classical supersolution of (29). We set

$$g(x) = \inf\{g_y(x) \,:\, y \in \partial\Omega\} \quad \text{for } x \in \overline{\Omega}.$$

By Proposition 1.10 (its version for supersolutions), we see that g is a viscosity supersolution of (29). Also, by applying Lemma 1.1, we find that g is Lipschitz continuous in $\overline{\Omega}$.

An application of Theorem 1.4 ensures that there is a viscosity solution $u : \Omega \to \mathbb{R}$ of (29) such that $f \le u \le g$ in Ω. Since $f(x) = g(x) = 0$ on $\partial\Omega$, if we set $u(x) = 0$ for $x \in \partial\Omega$, then the resulting function u is continuous at points on the boundary $\partial\Omega$ and satisfies the Dirichlet condition (30) in the classical sense.

Note that $u^* \le g$ in Ω, which clearly implies that $u = u^* \in \mathrm{USC}(\Omega)$. Now, if we use the next proposition, we find that u is locally Lipschitz continuous in Ω and conclude that $u \in C(\overline{\Omega})$. Thus, the Dirichlet problem (29)–(30) has a viscosity solution $u \in C(\overline{\Omega})$ which satisfies (30) in the classical (or pointwise) sense.

Proposition 1.14. *Let $R > 0$, $C > 0$ and $u \in \mathrm{USC}(B_R)$. Assume that u is a viscosity solution of*

$$|Du(x)| \le C \quad \text{in } B_R.$$

Then u is Lipschitz continuous in B_R with C being a Lipschitz bound. That is, $|u(x) - u(y)| \le C|x - y|$ for all $x, y \in B_R$.

Proof. Fix any $z \in B_R$ and set $r = (R - |z|)/4$. Fix any $y \in B_r(z)$. Note that $B_{3r}(y) \subset B_R$. Choose a function $f \in C^1([0, 3r))$ so that $f(t) = t$ for all $t \in [0, 2r]$, $f'(t) \ge 1$ for all $t \in [0, 3r)$ and $\lim_{t\to 3r-} f(t) = \infty$. Fix any $\varepsilon > 0$, and we claim that

$$u(x) \le v(x) := u(y) + (C + \varepsilon) f(|x - y|) \quad \text{for all } x \in B_{3r}(y). \tag{31}$$

Indeed, if this were not the case, we would find a point $\xi \in B_{3r}(y) \setminus \{y\}$ such that $u - v$ attains a maximum at ξ, which yields together with the viscosity property of u

$$C \ge |Dv(\xi)| = (C + \varepsilon) f'(|\xi - y|) \ge C + \varepsilon.$$

This is a contradiction. Thus we have (31).

Note that if $x \in B_r(z)$, then $x \in B_{2r}(y)$ and $f(|x - y|) = |x - y|$. Hence, from (31), we get

$$u(x) - u(y) \le (C + \varepsilon)|x - y| \quad \text{for all } x, y \in B_r(z).$$

By symmetry, we see that

$$|u(x) - u(y)| \le (C + \varepsilon)|x - y| \quad \text{for all } x, y \in B_r(z),$$

from which we deduce that

$$|u(x) - u(y)| \le C|x - y| \quad \text{for all } x, y \in B_r(z), \tag{32}$$

Now, let $x, y \in B_R$ be arbitrary points. Set $r = \frac{1}{4}\min\{R - |x|,\ R - |y|\}$, and choose a finite sequence $\{z_i\}_{i=0}^N$ of points on the line segment $[x,\ y]$ so that $z_0 = x$, $z_N = y$, $|z_i - z_{i-1}| < r$ for all $i = 1, \ldots, N$ and $\sum_{i=1}^N |z_i - z_{i-1}| = |x - y|$. By (32), we get

$$|u(z_i) - u(z_{i-1})| \le C|z_i - z_{i-1}| \quad \text{for all } i = 1, \ldots, N.$$

Summing these over $i = 1, \ldots, N$ yields the desired inequality. □

1.6 Sup-convolutions

Sup-convolutions and inf-convolutions are basic and important tools for regularizing or analyzing viscosity solutions. In this subsection, we recall some properties of sup-convolutions.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function and $\varepsilon \in \mathbb{R}_+$. The standard sup-convolution $u^\varepsilon : \mathbb{R}^n \to \mathbb{R}$ and inf-convolution $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ are defined, respectively, by

$$u^\varepsilon(x) = \sup_{y\in\mathbb{R}^n}\left(u(y) - \frac{1}{2\varepsilon}|y-x|^2\right) \quad \text{and} \quad u_\varepsilon(x) = \inf_{y\in\mathbb{R}^n}\left(u(y) + \frac{1}{2\varepsilon}|y-x|^2\right).$$

Note that

$$u_\varepsilon(x) = -\sup\left(-u(y) - \frac{1}{2\varepsilon}|y-x|^2\right) = -(-u)^\varepsilon(x).$$

This relation immediately allows us to interpret a property of sup-convolutions into the corresponding property of inf-convolutions.

In what follows we assume that u is bounded and upper semicontinuous in $\mathbb{R}^n$. Let $M > 0$ be a constant such that $|u(x)| \le M$ for all $x \in \mathbb{R}^n$.

Proposition 1.15. (i) *We have*

$$-M \le u(x) \le u^\varepsilon(x) \le M \quad \textit{for all } x \in \mathbb{R}^n.$$

(ii) *Let $x \in \mathbb{R}^n$ and $p \in D^+u^\varepsilon(x)$. Then*

$$|p| \le 2\sqrt{\frac{M}{\varepsilon}} \quad \textit{and} \quad p \in D^+u(x + \varepsilon p).$$

Another important property of sup-convolutions is that the sup-convolution u^ε is semiconvex in $\mathbb{R}^n$. More precisely, the function

$$u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2 = \sup_{y\in\mathbb{R}^n}\left(u(y) - \frac{1}{2\varepsilon}|y|^2 + \frac{1}{\varepsilon}y\cdot x\right)$$

is convex in $\mathbb{R}^n$ (see Appendix A.2) as is clear from the form of the right hand side of the above identity.

Proof. To show assertion (i), we just check that for all $x \in \mathbb{R}^n$,

$$u^\varepsilon(x) \le \sup_{y\in\mathbb{R}^n} u(y) \le M,$$

and

$$u^\varepsilon(x) \ge u(x) \ge -M.$$

Next, we prove assertion (ii). Let $\hat{x} \in \mathbb{R}^n$ and $\hat{p} \in D^+u^\varepsilon(\hat{x})$. Choose a point $\hat{y} \in \mathbb{R}^n$ so that

$$u^\varepsilon(\hat{x}) = u(\hat{y}) - \frac{1}{2\varepsilon}|\hat{y} - \hat{x}|^2.$$

(Such a point $\hat{y}$ always exists under our assumptions on u.) It is immediate to see that

$$\frac{1}{2\varepsilon}|\hat{y} - \hat{x}|^2 = u(\hat{y}) - u^\varepsilon(\hat{x}) \le 2M. \tag{33}$$

We may choose a function $\phi \in C^1(\mathbb{R}^n)$ so that $D\phi(\hat{x}) = \hat{p}$ and $\max_{\mathbb{R}^n}(u^\varepsilon - \phi) = (u^\varepsilon - \phi)(\hat{x})$. Observe that the function

$$\mathbb{R}^{2n} \ni (x, y) \mapsto u(y) - \frac{1}{2\varepsilon}|y - x|^2 - \phi(x)$$

attains a maximum at $(\hat{x}, \hat{y})$. Hence, both the functions

$$\mathbb{R}^n \ni x \mapsto -\frac{1}{2\varepsilon}|\hat{y} - x|^2 - \phi(x)$$

and

$$\mathbb{R}^n \ni x \mapsto u(x + \hat{y} - \hat{x}) - \phi(x)$$

attain maximum values at $\hat{x}$. Therefore, we find that

$$\frac{1}{\varepsilon}(\hat{x} - \hat{y}) + D\phi(\hat{x}) = 0 \quad \text{and} \quad D\phi(\hat{x}) \in D^+u(\hat{y}),$$

which shows that

$$\hat{p} = \frac{1}{\varepsilon}(\hat{y} - \hat{x}) \in D^+u(\hat{y}).$$

From this, we get $\hat{y} = \hat{x} + \varepsilon\hat{p}$, and, moreover, $\hat{p} \in D^+u(\hat{x} + \varepsilon\hat{p})$. Also, using (33), we get $|\hat{p}| \le 2\sqrt{M/\varepsilon}$. Thus we see that (ii) holds. □

The following observations illustrate a typical use of the above proposition. Let Ω be an open subset of $\mathbb{R}^n$. Let $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be bounded and upper semicontinuous. Let $M > 0$ be a constant such that $|u(x)| \le M$ for all $x \in \Omega$. Let $\varepsilon > 0$. Set $\delta = 2\sqrt{\varepsilon M}$ and $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$.

Define u^ε as above with u extended to $\mathbb{R}^n$ by setting $u(x) = -M$ for $x \in \mathbb{R}^n \setminus \Omega$. (Or, in a slightly different and more standard way, one may define u^ε by

$$u^\varepsilon(x) = \sup_{y\in\Omega}\left(u(y) - \frac{1}{2\varepsilon}|x-y|^2\right).)$$

By applying Proposition 1.15, we deduce that if u is a viscosity subsolution of

$$H(x, Du(x)) \le 0 \quad \text{in } \Omega,$$

then u^ε is a viscosity subsolution of both

$$H(x + \varepsilon Du^\varepsilon(x),\, Du^\varepsilon(x)) \le 0 \quad \text{in } \Omega_\delta, \tag{34}$$

and

$$|Du^\varepsilon(x)| \le 2\sqrt{\frac{M}{\varepsilon}} \quad \text{in } \Omega_\delta. \tag{35}$$

If we set

$$G(x, p) = \inf_{y\in B_\delta} H(x+y, p) \quad \text{for } x \in \Omega_\delta,$$

then (34) implies that u^ε is a viscosity subsolution of

$$G(x, Du^\varepsilon(x)) \le 0 \quad \text{in } \Omega_\delta.$$

If we apply Proposition 1.14 to u^ε, we see from (35) that u^ε is locally Lipschitz in Ω_δ.

2 Neumann Boundary Value Problems

We assume throughout this section and the rest of this article that $\Omega \subset \mathbb{R}^n$ is *open*.

We will be concerned with the initial value problem for the Hamilton–Jacobi equation of evolution type

$$\frac{\partial u}{\partial t}(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{in } \Omega \times (0, \infty),$$

and the asymptotic behavior of its solutions $u(x,t)$ as $t \to \infty$.

The stationary problem associated with the above Hamilton–Jacobi equation is stated as

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega, \\ \text{boundary condition on } \partial\Omega. \end{cases} \tag{36}$$

In this article we will be focused on the Neumann boundary value problem among other possible choices of boundary conditions like periodic, Dirichlet, state-constraints boundary conditions.

We are thus given two functions $\gamma \in C(\partial\Omega, \mathbb{R}^n)$ and $g \in C(\partial\Omega, \mathbb{R})$ which satisfy

$$\nu(x) \cdot \gamma(x) > 0 \quad \text{for all } x \in \partial\Omega, \tag{37}$$

where $\nu(x)$ denotes the outer unit normal vector at x, and the boundary condition posed on the unknown function u is stated as

$$\gamma(x) \cdot Du(x) = g(x) \quad \text{for } x \in \partial\Omega.$$

This condition is called the (inhomogeneous, linear) *Neumann boundary condition*. We remark that if $u \in C^1(\overline{\Omega})$, then the directional derivative $\partial u/\partial\gamma$ of u in the direction of γ is given by

$$\frac{\partial u}{\partial \gamma}(x) = \gamma(x) \cdot Du(x) = \lim_{t\to 0} \frac{u(x + t\gamma(x)) - u(x)}{t} \quad \text{for } x \in \partial\Omega.$$

(Note here that u is assumed to be defined in a neighborhood of x.)

Our boundary value problem (36) is now stated precisely as

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{SNP}$$

Let U be an open subset of $\mathbb{R}^n$ such that $U \cap \Omega \neq \emptyset$. At this stage we briefly explain the *viscosity formulation* of a more general boundary value problem

$$\begin{cases} F(x, u(x), Du(x)) = 0 & \text{in } U \cap \Omega, \\ B(x, u(x), Du(x)) = 0 & \text{on } U \cap \partial\Omega, \end{cases} \tag{38}$$

where the functions $F : (U \cap \overline{\Omega}) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $B : (U \cap \partial\Omega) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $u : (U \cap \overline{\Omega}) \to \mathbb{R}$ are assumed to be locally bounded in their domains of definition. The function u is said to be a *viscosity subsolution* of (38) if the following requirements are fulfilled:

$$\begin{cases} \phi \in C^1(\overline{\Omega}), \quad \hat{x} \in \overline{\Omega}, \quad \max_{\overline{\Omega}}(u^* - \phi) = (u^* - \phi)(\hat{x}) \\ \Longrightarrow \\ \text{(i)} \quad F_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \leq 0 & \text{if } \hat{x} \in U \cap \Omega, \\ \text{(ii)} \quad F_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \wedge B_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \leq 0 & \text{if } \hat{x} \in U \cap \partial\Omega. \end{cases}$$

The upper and lower semicontinuous envelopes are taken in all the variables. That is, for $x \in U \cap \Omega$, $\xi \in (U \cap \overline{\Omega}) \times \mathbb{R} \times \mathbb{R}^n$ and $\eta \in (U \cap \partial\Omega) \times \mathbb{R} \times \mathbb{R}^n$,

$$u^*(x) = \lim_{r \to 0+} \sup\{u(y) : y \in B_r(x) \cap U \cap \overline{\Omega})\},$$

$$F_*(\xi) = \lim_{r \to 0+} \inf\{F(X) : X \in (U \cap \Omega) \times \mathbb{R} \times \mathbb{R}^n, |X - \xi| < r\},$$

$$B_*(\eta) = \lim_{r \to 0+} \inf\{B(Y) : Y \in (U \cap \partial\Omega) \times \mathbb{R} \times \mathbb{R}^n, |Y - \eta| < r\}.$$

The definition of viscosity supersolutions of the boundary value problem (38) is given by reversing the upper and lower positions of $*$, the inequalities, and "sup" and "inf" (including $\wedge$ and $\vee$), respectively. Then viscosity solutions of (38) are defined as those functions which are both viscosity subsolution and supersolution of (38).

Here, regarding the above definition of boundary value problems, we point out the following: define the function $G : (U \cap \overline{\Omega}) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$G(x, u, p) = \begin{cases} F(x, u, p) & \text{if } x \in \Omega, \\ B(x, u, p) & \text{if } x \in \partial\Omega, \end{cases} \tag{39}$$

and note that the lower (resp., upper) semicontinuous envelope G_* (resp., G^*) of G is given by

$$G_*(x, u, p) = \begin{cases} F_*(x, u, p) & \text{if } x \in \Omega, \\ F_*(x, u, p) \wedge B_*(x, u, p) & \text{if } x \in \partial\Omega \end{cases}$$

$$\left(\text{resp., } G^*(x, u, p) = \begin{cases} F^*(x, u, p) & \text{if } x \in \Omega, \\ F^*(x, u, p) \vee B^*(x, u, p) & \text{if } x \in \partial\Omega \end{cases}\right).$$

Thus, the above definition of viscosity subsolutions, supersolutions and solutions of (38) is the same as that of Definition 1.2 with F and Ω replaced by G defined by (39) and $U \cap \overline{\Omega}$, respectively. Therefore, the propositions in Sect. 1.4 are valid as well to viscosity subsolutions, supersolutions and solutions of (38). In order to apply the above definition to (SNP), one may take $\mathbb{R}^n$ as U or any open neighborhood of $\overline{\Omega}$.

In Sect. 1.4 we have introduced the notion of properness of PDE (FE). The following example concerns this property.

Example 2.1. Consider the boundary value problem (38) in the case where $n = 1$, $\Omega = (0, 1)$, $U = \mathbb{R}$, $F(x, p) = p - 1$ and $B(x, p) = p - 1$. The function $u(x) = x$ on $[0, 1]$ is a classical solution of (38). But this function u is not a viscosity subsolution of (38). Indeed, if we take the test function $\phi(x) = 2x$, then $u - \phi$ takes a maximum at $x = 0$ while we have $B(0, \phi'(0)) = F(0, \phi'(0)) = 2 - 1 = 1 > 0$.

However, if we reverse the direction of derivative at 0 by replacing the above B by the function

$$B(x,p) = \begin{cases} p-1 & \text{for } x=1, \\ -p+1 & \text{for } x=0, \end{cases}$$

then the function u is a classical solution of (38) as well as a viscosity solution of (38).

Definition 2.1. The domain Ω is said to be of class C^1 (or simply $\Omega \in C^1$) if there is a function $\rho \in C^1(\mathbb{R}^n)$ which satisfies

$$\Omega = \{x \in \mathbb{R}^n \,:\, \rho(x) < 0\},$$
$$D\rho(x) \neq 0 \quad \text{for all } x \in \partial\Omega.$$

The functions ρ having the above properties are called defining functions of Ω.

Remark 2.1. If ρ is chosen as in the above definition, then the outer unit normal vector $\nu(x)$ at $x \in \partial\Omega$ is given by

$$\nu(x) = \frac{D\rho(x)}{|D\rho(x)|}.$$

Indeed, we have

$$N(x,\Omega) = \{t\nu(x) \,:\, t \geq 0\} \quad \text{for all } x \in \partial\Omega.$$

To see this, observe that if $t \geq 0$, then $\mathbf{1}_{\overline{\Omega}} + t\rho$ as a function in $\mathbb{R}^n$ attains a local maximum at any point $x \in \partial\Omega$, which shows that

$$t|D\rho(x)|\nu(x) \in -D^+\mathbf{1}_{\overline{\Omega}}(x) = N(x,\overline{\Omega}).$$

Next, let $z \in \partial\Omega$ and $\phi \in C^1(\mathbb{R}^n)$ be such that $\mathbf{1}_{\overline{\Omega}} - \phi$ attains a strict maximum over $\mathbb{R}^n$ at z. Observe that $-\phi$ attains a strict maximum over $\overline{\Omega}$ at x. Fix a constant $r > 0$ and, for each $k \in \mathbb{N}$, choose a maximum (over $\overline{B}_r(z)$) point $x_k \in \overline{B}_r(z)$ of $-\phi - k\rho^2$, and observe that $-(\phi + k\rho^2)(x_k) \geq -(\phi + k\rho^2)(z) = -\phi(z)$ for all $k \in \mathbb{N}$ and that $x_k \to z$ as $k \to \infty$. For $k \in \mathbb{N}$ sufficiently large we have

$$D(\phi + k\rho^2)(x_k) = 0,$$

and hence

$$D\phi(x_k) = -2k\rho(x_k)D\rho(x_k),$$

which shows in the limit as $k \to \infty$ that

$$D\phi(z) = -tD\rho(z) = -t|D\rho(z)|\nu(z),$$

where $t = \lim_{k\to\infty} 2k\rho(x_k) \in \mathbb{R}$. Noting that $-(\phi + k\rho^2)(x) < -\phi(x) \le -\phi(z)$ for all $x \in \Omega$, we find that $x_k \notin \overline{B}_r(z) \setminus \Omega$ for all $k \in \mathbb{N}$. Hence, we have $t \ge 0$. Thus, we see that $N(z, \overline{\Omega}) \subset \{t\nu(z) \,:\, t \ge 0\}$ and conclude that $N(z, \overline{\Omega}) = \{t\nu(z) \,:\, t \ge 0\}$

Henceforth in this section we assume that Ω is of class C^1.

Proposition 2.1. *If $u \in C^1(\overline{\Omega})$ is a classical solution (resp., subsolution, or supersolution) of (SNP), then it is a viscosity solution (resp., subsolution, or supersolution) of (SNP).*

Proof. Let G be the function given by (39), with $B(x, u, p) = \gamma(x) \cdot p - g(x)$. According to the above discussion on the equivalence between the notion of viscosity solution for (SNP) and that for PDE $G(x, Du(x)) = 0$ in $\overline{\Omega}$ and Proposition 1.12, it is enough to show that the pair $(G, \overline{\Omega})$ is proper. From the above remark, we know that for any $x \in \partial\Omega$ we have $N(x, \overline{\Omega}) = \{t\nu(x) \,:\, t \ge 0\}$ and

$$G(x, p + t\nu(x)) = \gamma(x) \cdot (p + t\nu(x)) \ge \gamma(x) \cdot p = G(x, p) \quad \text{for all } t \ge 0.$$

As we noted before, we have $N(x, \overline{\Omega}) = \{0\}$ if $x \in \Omega$. Thus, we have for all $x \in \overline{\Omega}$ and all $q \in N(x, \overline{\Omega})$,

$$G(x, p + q) \ge G(x, p).$$

□

We may treat in the same way the evolution problem

$$\begin{cases} u_t(x,t) + H(x, t, D_x u(x,t)) = 0 & \text{in } \Omega \times J, \\ \dfrac{\partial u}{\partial \gamma}(x,t) = g(x,t) & \text{on } \partial\Omega \times J, \end{cases} \tag{40}$$

where J is an open interval in $\mathbb{R}$, $H : \overline{\Omega} \times J \times \mathbb{R}^n \to \mathbb{R}$ and $g : \partial\Omega \times J \to \mathbb{R}$. If we set $\widetilde{\Omega} = \Omega \times \mathbb{R}$, $U = \mathbb{R}^n \times J$,

$$F(x, t, p, q) = q + H(x, p) \quad \text{for } (x, t, p, q) \in \overline{\Omega} \times J \times \mathbb{R}^n \times \mathbb{R},$$

and

$$B(x, t, p, q) = \gamma(x) \cdot p - g(x, t) \quad \text{for } (x, t, p, q) \in \partial\Omega \times J \times \mathbb{R}^n \times \mathbb{R},$$

then the viscosity formulation for (38) applies to (40), with Ω replaced by $\widetilde{\Omega}$.

We note here that if ρ is a defining function of Ω, then it, as a function of (x, t), is also a defining function of the "cylinder" $\Omega \times \mathbb{R}$. Hence, if we set $\tilde{\gamma}(x, t) = (\gamma(x), 0)$ and $\tilde{\nu}(x, t) = (\nu(x), 0)$ for $(x, t) \in \partial(\Omega \times \mathbb{R}) = \partial\Omega \times \mathbb{R}$, then $\tilde{\nu}(x, t)$ is the outer unit normal vector at $(x, t) \in \partial\Omega \times \mathbb{R}$. Moreover, if γ satisfies (37), then we have $\tilde{\gamma}(x, t) \cdot \tilde{\nu}(x, t) = \gamma(x) \cdot \nu(x) > 0$ for all $(x, t) \in \partial\Omega \times \mathbb{R}$. Thus, as Proposition 2.1 says, if (37) holds, then any classical solution (resp., subsolution or supersolution) of (40) is a viscosity solution (resp., subsolution or supersolution) of (40).

Before closing this subsection, we add two lemmas concerning C^1 domains.

Lemma 2.1. *Let Ω be a bounded, open, connected subset of $\mathbb{R}^n$. Assume that Ω is of class C^1. Then there exists a constant $C > 0$ and, for each $x, y \in \overline{\Omega}$ with $x \neq y$, a curve $\eta \in \mathrm{AC}([0,\, t(x,y)])$, with $t(x,y) > 0$, such that $t(x,y) \leq C|x-y|$, $\eta(s) \in \Omega$ for all $s \in (0,\, t(x,y))$, and $|\dot{\eta}(s)| \leq 1$ for a.e. $s \in [0,\, t(x,y)]$.*

Lemma 2.2. *Let Ω be a bounded, open, connected subset of $\mathbb{R}^n$. Assume that Ω is of class C^1. Let $M > 0$ and $u \in C(\Omega)$ be a viscosity subsolution of $|Du(x)| \leq M$ in Ω. Then the function u is Lipschitz continuous in Ω.*

The proof of these lemmas is given in Appendix A.3.

3 Initial-Boundary Value Problem for Hamilton–Jacobi Equations

We study the initial value problem for Hamilton–Jacobi equations with the Neumann boundary condition.

To make the situation clear, we collect our assumptions on Ω, γ and H.

(A1) Ω is bounded open connected subset of $\mathbb{R}^n$.
(A2) Ω is of class C^1.
(A3) $\gamma \in C(\partial\Omega, \mathbb{R}^n)$ and $g \in C(\partial\Omega, \mathbb{R})$.
(A4) $\gamma(x) \cdot \nu(x) > 0$ for all $x \in \partial\Omega$D.
(A5) $H \in C(\overline{\Omega} \times \mathbb{R}^n)$.
(A6) H is coercive, i.e.,

$$\lim_{R\to\infty} \inf\{H(x,p) \,:\, (x,p) \in \overline{\Omega} \times \mathbb{R}^n,\ |p| \geq R\} = \infty.$$

In what follows, we assume always that (A1)–(A6) hold.

3.1 Initial-Boundary Value Problems

Given a function $u_0 \in C(\overline{\Omega})$, we consider the problem of evolution type

$$\begin{cases} u_t + H(x, D_x u) = 0 & \text{in } \Omega \times (0,\infty), \\ \gamma(x) \cdot D_x u = g(x) & \text{on } \partial\Omega \times (0,\,\infty), \end{cases} \tag{ENP}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \overline{\Omega}. \tag{ID}$$

Here $u = u(x,t)$ is a function of $(x,t) \in \overline{\Omega} \times [0,\infty)$ and represents the unknown function.

When we say u is a (viscosity) solution of (ENP)–(ID), u is assumed to satisfy the initial condition (ID) in the pointwise (classical) sense.

Henceforth Q denotes the set $\overline{\Omega} \times (0, \infty)$.

Theorem 3.1 (Comparison). *Let $u \in \mathrm{USC}(\overline{Q})$ and $v \in \mathrm{LSC}(\overline{Q})$ be a viscosity subsolution and supersolution of* (ENP)*, respectively. Assume furthermore that $u(x,0) \le v(x,0)$ for all $x \in \overline{\Omega}$. Then $u \le v$ in Q.*

To proceed, we concede the validity of the above theorem and will come back to its proof in Sect. 3.3.

Remark 3.1. The above theorem guarantees that if u is a viscosity solution of (ENP)–(ID) and continuous for $t = 0$, then it is unique.

Theorem 3.2 (Existence). *There exists a viscosity solution u of* (ENP)–(ID) *in the space $C(\overline{Q})$.*

Proof. Fix any $\varepsilon \in (0, 1)$. Choose a function $u_{0,\varepsilon} \in C^1(\overline{\Omega})$ so that

$$|u_{0,\varepsilon}(x) - u_0(x)| < \varepsilon \quad \text{for all } \ x \in \overline{\Omega}.$$

Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. Since

$$D\rho(x) = |D\rho(x)|\nu(x) \quad \text{for } x \in \partial\Omega,$$

we may choose a constant $M_\varepsilon > 0$ so large that

$$M_\varepsilon \gamma(x) \cdot D\rho(x) \ge \max_{\partial\Omega}(|g| + |\gamma \cdot Du_{0,\varepsilon}|) \quad \text{for all } \ x \in \partial\Omega.$$

Next choose a function $\zeta \in C^1(\mathbb{R})$ so that

$$\begin{cases} \zeta'(0) = 1, & \\ -1 \le \zeta(r) \le 0 & \text{for } \ r \le 0, \\ \ 0 \le \zeta'(r) \le 1 & \text{for } \ r \le 0. \end{cases}$$

Setting

$$\chi_\varepsilon(x) = \varepsilon\zeta(M_\varepsilon\rho(x)/\varepsilon),$$

we have

$$\begin{cases} -\varepsilon \le \chi_\varepsilon(x) \le 0 & \text{for all } \ x \in \overline{\Omega}, \\ \gamma(x) \cdot D\chi_\varepsilon(x) \ge |g(x)| + |\gamma(x) \cdot Du_{0,\varepsilon}(x)| & \text{for all } \ x \in \partial\Omega, \end{cases}$$

and we may choose a constant $C_\varepsilon > 0$ such that

$$|D\chi_\varepsilon(x)| \le C_\varepsilon \quad \text{for all } \ x \in \overline{\Omega}.$$

Then define the functions $f_\varepsilon^\pm \in C^1(\overline{\Omega})$ by

$$f_\varepsilon^\pm(x) = u_{0,\varepsilon}(x) \pm (\chi_\varepsilon(x) + 2\varepsilon),$$

and observe that

$$\begin{cases} u_0(x) \le f_\varepsilon^+(x) \le u_0(x) + 3\varepsilon & \text{for all } x \in \overline{\Omega}, \\ u_0(x) \ge f_\varepsilon^-(x) \ge u_0(x) - 3\varepsilon & \text{for all } x \in \overline{\Omega}, \\ \gamma(x) \cdot Df_\varepsilon^+(x) \ge g(x) & \text{for all } x \in \partial\Omega, \\ \gamma(x) \cdot Df_\varepsilon^-(x) \le g(x) & \text{for all } x \in \partial\Omega. \end{cases}$$

Now, we choose a constant $A_\varepsilon > 0$ large enough so that

$$|H(x, Df_\varepsilon^\pm(x))| \le A_\varepsilon \quad \text{for all } x \in \overline{\Omega},$$

and set

$$g_\varepsilon^\pm(x,t) = f_\varepsilon^\pm(x) \pm A_\varepsilon t \quad \text{for } (x,t) \in \overline{Q}.$$

The functions g_ε^+, $g_\varepsilon^- \in C^1(\overline{Q})$ are a viscosity supersolution and subsolution of (ENP), respectively.

Setting

$$h^+(x,t) = \inf\{g_\varepsilon^+(x,t) \,:\, \varepsilon \in (0,\, 1)\},$$

$$h^-(x,t) = \sup\{g_\varepsilon^-(x,t) \,:\, \varepsilon \in (0,\, 1)\},$$

we observe that $h^+ \in \mathrm{USC}(\overline{Q})$ and $h^- \in \mathrm{LSC}(\overline{Q})$ are, respectively, a viscosity supersolution and subsolution of (ENP). Moreover we have

$$u_0(x) = h^\pm(x,0) \quad \text{for all } x \in \overline{\Omega},$$

$$h^-(x,t) \le u_0(x) \le h^+(x,t) \quad \text{for all } (x,t) \in \overline{Q}.$$

By Theorem 1.4, we find that there exists a viscosity solution u of (ENP) which satisfies

$$h^-(x,t) \le u(x,t) \le h^+(x,t) \quad \text{for all } (x,t) \in \overline{Q}.$$

Applying Theorem 3.1 to u^* and u_* yields

$$u^* \le u_* \quad \text{for all } (x,t) \in \overline{Q},$$

while $u_* \le u^*$ in $\overline{Q}$ by definition, which in particular implies that $u \in C(\overline{Q})$. The proof is complete. □

Theorem 3.3 (Uniform continuity). *The viscosity solution $u \in C(\overline{Q})$ of* (ENP)–(ID) *is uniformly continuous in $\overline{Q}$. Furthermore, if $u_0 \in \mathrm{Lip}(\overline{\Omega})$, then $u \in \mathrm{Lip}(\overline{Q})$.*

Lemma 3.1. *Let* $u_0 \in \mathrm{Lip}(\overline{\Omega})$*. Then there is a constant* $C > 0$ *such that the functions* $u_0(x) + Ct$ *and* $u_0(x) - Ct$ *are, respectively, a viscosity supersolution and subsolution of* (ENP)–(ID).

Proof. Let ρ and ζ be the functions which are used in the proof of Theorem 3.2. Choose the collection $\{u_{0,\varepsilon}\}_{\varepsilon\in(0,1)} \subset C^1(\overline{\Omega})$ of functions so that

$$\begin{cases} \lim_{\varepsilon\to 0} \|u_{0,\varepsilon} - u_0\|_{\infty,\Omega} = 0, \\ \sup_{\varepsilon\in(0,1)} \|Du_{0,\varepsilon}\|_{\infty,\Omega} < \infty. \end{cases}$$

As in the proof of Theorem 3.2, we may fix a constant $M > 0$ so that

$$\begin{aligned} M\gamma(x)\cdot D\rho(x) &= M|D\rho(x)|\nu(x)\cdot\gamma(x) \\ &\geq |g(x)| + |\gamma(x)\cdot Du_{0,\varepsilon}(x)| \quad \text{for all } x \in \partial\Omega. \end{aligned}$$

Next set

$$R = \sup_{\varepsilon\in(0,1)} \|Du_{0,\varepsilon}\|_{\infty,\Omega} + M\|D\rho\|_{\infty,\Omega},$$

and choose $C > 0$ so that

$$\max_{\overline{\Omega}\times\overline{B}_R} |H| \leq C.$$

Now, we put

$$v_\varepsilon^\pm(x,t) = u_{0,\varepsilon}(x) \pm (M\varepsilon\zeta(\rho(x)/\varepsilon) + Ct) \quad \text{for } (x,t) \in \overline{Q},$$

and note that v_ε^+ and v_ε^- are a classical supersolution and subsolution of (ENP). Sending $\varepsilon \to 0+$, we conclude by Proposition 1.9 that the functions $u_0(x) + Ct$ and $u_0(x) - Ct$ are a viscosity supersolution and subsolution of (ENP), respectively. □

Proof (Theorem 3.3). We first assume that $u_0 \in \mathrm{Lip}(\overline{\Omega})$, and show that $u \in \mathrm{Lip}(\overline{Q})$. According to Lemma 3.1, there exists a constant $C > 0$ such that the function $u_0(x) - Ct$ is a viscosity subsolution of (ENP). By Theorem 3.1, we get

$$u(x,t) \geq u_0(x) - Ct \quad \text{for all } (x,t) \in \overline{Q}.$$

Fix any $t > 0$, and apply Theorem 3.1 to the functions $u(x, t+s)$ and $u(x,s) - Ct$ of (x,s), both of which are viscosity solutions of (ENP), to get

$$u(x, t+s) \geq u(x,s) - Ct \quad \text{for all } (x,s) \in \overline{Q}.$$

Hence, if $(p,q) \in D^+u(x,s)$, then we find that as $t \to 0+$,

$$u(x,s) \le u(x,s+t) + Ct \le u(x,s) + qt + Ct + o(t),$$

and consequently, $q \ge -C$. Moreover, if $x \in \Omega$, we have

$$0 \ge q + H(x,p) \ge H(x,p) - C.$$

Due to the coercivity of H, there exists a constant $R > 0$ such that

$$p \in B_R.$$

Therefore, we get

$$q \le -H(x,p) \le \max_{\overline{\Omega}\times\overline{B}_R} |H|.$$

Thus, if $(x,s) \in \Omega \times (0,\infty)$ and $(p,q) \in D^+u(x,s)$, then we have

$$|p| + |q| \le M := R + C + \max_{\overline{\Omega}\times\overline{B}_R} |H|.$$

Thanks to Proposition 1.14, we conclude that u is Lipschitz continuous in Q.

Next, we show in the general case that $u \in \mathrm{UC}(\overline{Q})$. Let $\varepsilon \in (0,\ 1)$, and choose a function $u_{0,\varepsilon} \in \mathrm{Lip}(\overline{\Omega})$ so that

$$\|u_{0,\varepsilon} - u_0\|_\infty \le \varepsilon.$$

Let u_ε be the viscosity solution of (ENP) satisfying the initial condition

$$u_\varepsilon(x,0) = u_{0,\varepsilon}(x) \quad \text{for all } \ x \in \overline{\Omega}.$$

As we have shown above, we know that $u_\varepsilon \in \mathrm{Lip}(\overline{Q})$. Moreover, by Theorem 3.1 we have

$$\|u_\varepsilon - u\|_{\infty,\overline{Q}} \le \varepsilon.$$

It is now obvious that $u \in \mathrm{UC}(\overline{Q})$. □

3.2 Additive Eigenvalue Problems

Under our hypotheses (A1)–(A6), the boundary value problem

$$\begin{cases} H(x,Du) = 0 & \text{in } \Omega, \\ \gamma(x)\cdot Du = g(x) & \text{on } \partial\Omega \end{cases} \tag{SNP}$$

may not have a viscosity solution. For instance, the Hamiltonian $H(x, p) = |p|^2 + 1$ satisfies (A5) and (A6), but, since $H(x, p) > 0$, (SNP) does not have any viscosity subsolution.

Instead of (SNP), we consider the *additive eigenvalue* problem

$$\begin{cases} H(x, Dv) = a & \text{in } \Omega, \\ \gamma(x) \cdot Dv = g(x) & \text{on } \partial\Omega. \end{cases} \tag{EVP}$$

This is a problem to seek for a pair $(a, v) \in \mathbb{R} \times C(\overline{\Omega})$ such that v is a viscosity solution of the stationary problem (EVP). If $(a, v) \in \mathbb{R} \times C(\overline{\Omega})$ is such a pair, then a and v are called an (additive) *eigenvalue* and *eigenfunction* of (EVP), respectively. This problem is often called the *ergodic problem* in the viewpoint of ergodic optimal control.

Theorem 3.4. (i) *There exists a solution* $(a, v) \in \mathbb{R} \times \mathrm{Lip}(\overline{\Omega})$ *of* (EVP).
(ii) *The eigenvalue of* (EVP) *is unique. That is, if* $(a, v), (b, w) \in \mathbb{R} \times C(\overline{\Omega})$ *are solutions of* (EVP)*, then* $a = b$.

The above result has been obtained by Lions et al., Homogenization of Hamilton-Jacobi equations, unpublished.

In what follows we write $c^\#$ for the unique eigenvalue a of (EVP).

Corollary 3.1. *Let* $u \in C(\overline{Q})$ *be the solution of* (ENP)–(ID)*. Then the function* $u(x, t) + c^\# t$ *is bounded on* $\overline{Q}$.

Corollary 3.2. *We have*

$$c^\# = \inf\{a \in \mathbb{R} : (\text{EVP}) \textit{ has a viscosity subsolution } v\}.$$

Lemma 3.2. *Let* $b, c \in \mathbb{R}$ *and* $v, w \in C(\overline{\Omega})$*. Assume that* v *(resp.,* w*) is a viscosity supersolution (resp., subsolution) of* (EVP) *with* $a = b$ *(resp.,* $a = c$*). Then* $b \le c$.

Remark 3.2. As the following proof shows, the assertion of the above lemma is valid even if one replaces the continuity of v and w by the boundedness.

Proof. By adding a constant to v if needed, we may assume that $v \ge w$ in $\overline{\Omega}$. Since the functions $v(x) - bt$ and $w(x) - ct$ are a viscosity supersolution and subsolution of (ENP), by Theorem 3.1 we get

$$v(x) - bt \ge w(x) - ct \quad \text{for all } (x, t) \in \overline{Q},$$

from which we conclude that $b \le c$. □

Proof (Theorem 3.4). Assertion (ii) is a direct consequence of Lemma 3.2.

We prove assertion (i). Consider the boundary value problem

$$\begin{cases} \lambda v + H(x, Dv) = 0 & \text{in } \Omega, \\ \gamma(x) \cdot Dv = g & \text{on } \partial\Omega, \end{cases} \tag{41}$$

where $\lambda > 0$ is a given constant. We will take the limit as $\lambda \to 0$ later on.

We fix $\lambda \in (0, 1)$. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of the domain Ω. Select a constant $A > 0$ so large that $A\gamma(x) \cdot D\rho(x) \geq |g(x)|$ for all $x \in \partial\Omega$, and then $B > 0$ so large that $B \geq A|\rho(x)| + |H(x, \pm AD\rho(x))|$ for all $x \in \overline{\Omega}$. Observe that the functions $A\rho(x) + B/\lambda$ and $-A\rho(x) - B/\lambda$ are a classical supersolution and subsolution of (41), respectively.

The Perron method (Theorem 1.4) guarantees that there is a viscosity solution v_λ of (41) which satisfies

$$|v_\lambda(x)| \leq A\rho(x) + B/\lambda \leq B/\lambda \quad \text{for all } x \in \overline{\Omega}.$$

Now, since

$$-\lambda v_\lambda(x) \leq B \quad \text{for all } x \in \overline{\Omega},$$

v_λ satisfies in the viscosity sense

$$H(x, Dv_\lambda(x)) \leq B \quad \text{for all } x \in \Omega,$$

which implies, together with the coercivity of H, the equi-Lipschitz continuity of $\{v_\lambda\}_{\lambda \in (0,1)}$. Thus the collections $\{v_\lambda - \inf_\Omega v_\lambda\}_{\lambda \in (0,1)}$ and $\{\lambda v_\lambda\}_{\lambda \in (0,1)}$ of functions on $\overline{\Omega}$ are relatively compact in $C(\overline{\Omega})$. We may select a sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset (0, 1)$ such that

$$\begin{aligned} &\lambda_j \to 0, \\ &v_{\lambda_j}(x) - \inf_\Omega v_{\lambda_j} \to v(x), \\ &\lambda_j v_{\lambda_j}(x) \to w(x) \end{aligned}$$

for some functions $v, w \in C(\overline{\Omega})$ as $j \to \infty$, where the convergences to v and w are uniform on $\overline{\Omega}$. Observe that for all $x \in \overline{\Omega}$,

$$\begin{aligned} w(x) &= \lim_{j\to\infty} \lambda_j v_{\lambda_j}(x) \\ &= \lim_{j\to\infty} \lambda_j \big[(v_{\lambda_j}(x) - \inf_\Omega v_{\lambda_j}) + \inf_\Omega v_{\lambda_j}\big] \\ &= \lim_{j\to\infty} \lambda_j \inf_\Omega v_{\lambda_j}, \end{aligned}$$

which shows that w is constant on $\overline{\Omega}$. If we write this constant as a, then we see by Proposition 1.9 that v is a viscosity solution of (EVP). This completes the proof of (i). □

Proof (Corollary 3.1). Let $v \in C(\overline{\Omega})$ be an eigenfunction of (EVP). That is, v is a viscosity solution of (EVP), with $a = c^{\#}$. Then, for any constant $C \in \mathbb{R}$, the function $w(x,t) := v(x) - c^{\#}t + C$ is a viscosity solution of (ENP). We may choose constants C_i, $i = 1, 2$, so that $v(x) + C_1 \le u_0(x) \le v(x) + C_2$ for all $x \in \overline{\Omega}$. By Theorem 3.1, we see that

$$v(x) - c^{\#}t + C_1 \le u(x,t) \le v(x) - c^{\#}t + C_2 \quad \text{for all } (x,t) \in \overline{Q},$$

which shows that the function $u(x,t) + c^{\#}t$ is bounded on $\overline{Q}$. □

Proof (Corollary 3.2). It is clear that

$$c^{\#} \ge c^{\star} := \inf\{a \in \mathbb{R} \, : \, \text{(EVP) has a viscosity subsolution } v\}.$$

To show that $c^{\#} \le c^{\star}$, we suppose by contradiction that $c^{\#} > c^{\star}$. By the definition of $c^{\star}$, there is a $b \in [c^{\star}, c^{\#})$ such that (EVP), with $a = b$, has a viscosity subsolution ψ. Let v be a viscosity solution of (EVP), with $a = c^{\#}$. Since $b < c^{\#}$, v is a viscosity supersolution of (EVP), with $a = b$. We may assume that $\psi \le v$ in $\overline{\Omega}$. Theorem 1.4 now guarantees the existence of a viscosity solution of (EVP), which contradicts Theorem 3.4, (ii) (see Remark 3.2). □

Example 3.1. We consider the case where $n = 1$, $\Omega = (-1, 1)$, $H(x,p) = H(p) := |p|$ and $\gamma(\pm 1) = \pm 1$, respectively, and evaluate the eigenvalue $c^{\#}$. We set $g_{\min} = \min\{g(-1), g(1)\}$. Assume first that $g_{\min} \ge 0$. In this case, the function $v(x) = 0$ is a classical subsolution of (SNP) and, hence, $c^{\#} \le 0$. On the other hand, since $H(p) \ge 0$ for all $p \in \mathbb{R}$, we have $c^{\#} \ge 0$. Thus, $c^{\#} = 0$. We next assume that $g_{\min} < 0$. It is easily checked that if $g(1) = g_{\min}$, then the function $v(x) = g_{\min}x$ is a viscosity solution of (EVP), with $a = |g_{\min}|$. (Notice that

$$-D^{+}v(-1) = (-\infty, -|g_{\min}|\,] \cup [-|g_{\min}|, |g_{\min}|\,],$$
$$-D^{-}v(-1) = [\,|g_{\min}|, \infty).)$$

Similarly, if $g(-1) = g_{\min}$, then the function $v(x) = |g_{\min}|x$ is a viscosity solution of (EVP), with $a = |g_{\min}|$. These observations show that $c^{\#} = |g_{\min}|$.

3.3 *Proof of Comparison Theorem*

This subsection will be devoted to the proof of Theorem 3.1.

We begin with the following two lemmas.

Lemma 3.3. *Let u be the function from Theorem 3.1. Set $P = \Omega \times (0, \infty)$. Then, for every $(x,t) \in \partial\Omega \times (0, \infty)$, we have*

$$u(x,t) = \limsup_{P \ni (y,s) \to (x,t)} u(y,s). \tag{42}$$

Proof. Fix any $(x,t) \in \partial\Omega \times (0, \infty)$. To prove (42), we argue by contradiction, and suppose that

$$\limsup_{P \ni (y,s) \to (x,t)} u(y,s) < u(x,t).$$

We may choose a constant $r \in (0, t)$ so that

$$u(y,s) + r < u(x,t) \quad \text{for all } (y,s) \in P \cap (\overline{B}_r(x) \times [t-r, t+r]). \tag{43}$$

Note that

$$P \cap (\overline{B}_r(x) \times [t-r, t+r]) = (\Omega \cap \overline{B}_r(x)) \times [t-r, t+r].$$

Since u is bounded on $\overline{\Omega} \times [t-r, t+r]$, we may choose a constant $\alpha > 0$ so that for all $(y,s) \in \overline{\Omega} \times [t-r, t+r]$,

$$u(y,s) + r - \alpha(|y-x|^2 + (s-t)^2) < u(x,t) \quad \text{if } |y-x| \geq r/2 \text{ or } |s-t| \geq r/2. \tag{44}$$

Let ρ be a defining function of Ω. Let ζ be the function on $\mathbb{R}$ introduced in the proof of Theorem 3.2. For $k \in \mathbb{N}$ we define the function $\psi \in C^1(\mathbb{R}^{n+1})$ by

$$\psi(y,s) = k^{-1}\zeta(k^2\rho(y)) + \alpha(|y-x|^2 + (s-t)^2).$$

Consider the function

$$u(y,s) - \psi(y,s)$$

on the set $\big(\overline{\Omega} \cap \overline{B}_r(x)\big) \times [t-r, t+r]$. Let $(y_k, s_k) \in \big(\overline{\Omega} \cap \overline{B}_r(x)\big) \times [t-r, t+r]$ be a maximum point of the above function. Assume that $k > r^{-1}$.

Using (43) and (44), we observe that for all $(y,s) \in (\overline{\Omega} \cap \overline{B}_r(x)) \times [t-r, t+r]$,

$$u(y,s) - \psi(y,s) < u(x,t) = u(x,t) - \psi(x,t)$$

if either $y \in \Omega$, $|y-x| \geq r/2$, or $|s-t| \geq r/2$. Accordingly, we have

$$(y_k, s_k) \in \big(\partial\Omega \cap \overline{B}_{r/2}(x)\big) \times (t-r/2, t+r/2).$$

Hence, setting

$$p_k = kD\rho(y_k) + 2\alpha(y_k - x) \quad \text{and} \quad q_k = 2\alpha(s_k - t),$$

we have

$$\min\{q_k + H(y_k, p_k), \gamma(y_k) \cdot p_k - g(y_k)\} \leq 0.$$

If we note that

$$\gamma(y_k) \cdot D\rho(y_k) \geq \min_{\partial\Omega} \gamma \cdot D\rho > 0,$$

then, by sending $k \to \infty$, we get a contradiction. □

Lemma 3.4. *Let $y, z \in \mathbb{R}^n$, and assume that $y \cdot z > 0$. Then there exists a quadratic function ζ in $\mathbb{R}^n$ which satisfies:*

$$\begin{cases} \zeta(tx) = t^2\zeta(x) & \textit{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \zeta(x) > 0 & \textit{if } x \neq 0, \\ z \cdot D\zeta(x) = 2(y \cdot z)(y \cdot x) & \textit{for all } x \in \mathbb{R}^n. \end{cases}$$

Proof. We define the function ζ by

$$\zeta(x) = \left| x - \frac{y \cdot x}{y \cdot z} z \right|^2 + (y \cdot x)^2.$$

We observe that for any $t \in \mathbb{R}$,

$$\begin{aligned} \zeta(x + tz) &= \left| x + tz - \frac{y \cdot (x + tz)}{y \cdot z} z \right|^2 + (y \cdot (x + tz))^2 \\ &= \left| x - \frac{y \cdot x}{y \cdot z} z \right|^2 + (y \cdot x)^2 + 2t(y \cdot x)(y \cdot z) + t^2(y, z)^2, \end{aligned}$$

from which we find that

$$z \cdot D\zeta(x) = 2(y \cdot z)(y \cdot x).$$

If $\zeta(x) = 0$, then $y \cdot x = 0$ and

$$0 = \zeta(x) = \left| x - \frac{y \cdot x}{y \cdot z} z \right|^2 = |x|^2.$$

Hence, we have $x = 0$ if $\zeta(x) = 0$, which shows that $\zeta(x) > 0$ if $x \neq 0$. It is obvious that the function ζ is homogeneous of degree two. The function ζ has the required properties. □

For the proof of Theorem 3.1, we argue by contradiction: we suppose that

$$\sup_{\overline{\Omega} \times [0,\infty)} (u - v) > 0,$$

and, to conclude the proof, we will get a contradiction.

Reduction 1: We may assume that there exist a constant $\delta > 0$ and a finite open interval $J \subset (0, \infty)$ such that

$$u \text{ is a viscosity subsolution of}$$
$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le -\delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x u(x,t) \le g(x) & \text{on } \partial\Omega \times J, \end{cases} \tag{45}$$

$$\max_{\overline{\Omega} \times \overline{J}} (u - v) > 0 > \max_{\overline{\Omega} \times \partial J} (u - v), \tag{46}$$

and

$$u \text{ and } v \text{ are bounded on } \overline{\Omega} \times \overline{J}. \tag{47}$$

Proof. We choose a $T > 0$ so that $\sup_{\overline{\Omega} \times (0,\, T)} (u - v) > 0$ and set

$$u_\varepsilon(x,t) = u(x,t) - \frac{\varepsilon}{T - t} \quad \text{for } (x,t) \in \overline{\Omega} \times [0,\, T),$$

where $\varepsilon > 0$ is a constant. It is then easy to check that u_ε is a viscosity subsolution of

$$\begin{cases} u_{\varepsilon,t} + H(x, D_x u_\varepsilon(x,t)) \le -\dfrac{\varepsilon}{T^2} & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u_\varepsilon}{\partial \gamma}(x,t) \le g(x) & \text{on } \partial\Omega \times (0,\, T). \end{cases}$$

Choosing $\varepsilon > 0$ sufficiently small, we have

$$\sup_{\overline{\Omega} \times [0,T)} (u_\varepsilon - v) > 0 > \max_{\overline{\Omega} \times \{0\}} (u_\varepsilon - v).$$

If we choose $\alpha > 0$ sufficiently small, then

$$\max_{\overline{\Omega} \times [0,T-\alpha]} (u_\varepsilon - v) > 0 > \max_{\overline{\Omega} \times \partial[0,T-\alpha]} (u_\varepsilon - v).$$

Thus, if we set $J = (0,\, T - \alpha)$ and replace u by u_ε, then we are in the situation of (45)–(47). □

We may assume furthermore that $u \in \mathrm{Lip}(\overline{\Omega} \times \overline{J})$ as follows.

Reduction 2: We may assume that there exist a constant $\delta > 0$ and a finite open interval $J \subset (0, \infty)$ such that

u is a viscosity subsolution of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le -\delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x u(x,t) \le g(x) & \text{on } \partial\Omega \times J, \end{cases} \tag{48}$$

$$\max_{\overline{\Omega}\times\overline{J}}(u-v) > 0 > \max_{\overline{\Omega}\times\partial J}(u-v), \tag{49}$$

and

$$u \in \mathrm{Lip}(\overline{\Omega}\times\overline{J}) \ \text{ and } \ v \ \text{ is bounded on } \ \overline{\Omega}\times\overline{J}. \tag{50}$$

Proof. Let J be as in Reduction 1. We set $J = (a,b)$. Let $M > 0$ be a bound of $|u|$ on $\overline{\Omega} \times [a,\, b]$.

For each $\varepsilon > 0$ we define the sup-convolution in the t-variable

$$u_\varepsilon(x,t) = \max_{s\in[a,b]} \Big(u(x,s) - \frac{(t-s)^2}{2\varepsilon}\Big).$$

We note as in Sect. 1.6 that

$$M \ge u_\varepsilon(x,t) \ge u(x,t) \ge -M \quad \text{for all } \ (x,t) \in \overline{\Omega}\times[a,\, b].$$

Noting that

$$\frac{1}{2\varepsilon}(t-s)^2 \le 2M \quad \Longleftrightarrow \quad |t-s| \le 2\sqrt{\varepsilon M} \tag{51}$$

and setting $m_\varepsilon = 2\sqrt{\varepsilon M}$, we find that

$$u_\varepsilon(x,t) = \max_{a<s<b} \Big(u(x,s) - \frac{(t-s)^2}{2\varepsilon}\Big) \quad \text{for all } \ (x,t) \in \overline{\Omega}\times(a+m_\varepsilon,\, b-m_\varepsilon).$$

Let $(x,t) \in \overline{\Omega}\times(a+m_\varepsilon,\, b-m_\varepsilon)$. Choose an $s \in (a,\, b)$ so that

$$u_\varepsilon(x,t) = u(x,s) - \frac{(t-s)^2}{2\varepsilon}.$$

Note by (51) that

$$|t-s| \le m_\varepsilon.$$

Let $(p,q) \in D^+u_\varepsilon(x,t)$ and choose a function $\phi \in C^1(\overline{\Omega}\times(a,\, b))$ so that $D\phi(x,t) = (p,q)$ and $\max(u_\varepsilon - \phi) = (u_\varepsilon - \phi)(x,t)$. Observe as in Sect. 1.6 that

$$(p, (s-t)/\varepsilon) \in D^+u(x,s) \quad \text{and} \quad \frac{(t-s)}{\varepsilon} + q = 0.$$

Hence,

$$(p, q) \in D^+u(x, s).$$

Therefore, we have

$$\begin{cases} q + H(x, p) + \delta \le 0 & \text{if } x \in \Omega, \\ \min\{q + H(x, p) + \delta, \gamma(x) \cdot p - g(x)\} \le 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{52}$$

Moreover, we see that

$$|q| = \frac{|t - s|}{\varepsilon} \le \frac{m_\varepsilon}{\varepsilon},$$

and

$$H(x, p) \le -q \le \frac{m_\varepsilon}{\varepsilon} \quad \text{if } x \in \Omega.$$

Hence, by the coercivity of H, we have

$$|q| + |p| \le R(\varepsilon) \quad \text{if } x \in \Omega, \tag{53}$$

for some constant $R(\varepsilon) > 0$.

Thus, we conclude from (52) that u_ε is a viscosity subsolution of

$$\begin{cases} u_t + H(x, D_x u) \le -\delta & \text{in } \overline{\Omega} \times (a + m_\varepsilon, b - m_\varepsilon), \\ \gamma \cdot D_x u \le g & \text{on } \partial\Omega \times (a + m_\varepsilon, b - m_\varepsilon), \end{cases}$$

and from (53) that u_ε is Lipschitz continuous in $\Omega \times (a + m_\varepsilon, b - m_\varepsilon)$. By Lemma 3.3, we have

$$u_\varepsilon(x, t) = \limsup_{\Omega \times (a + m_\varepsilon, b - m_\varepsilon) \ni (y,s) \to (x,t)} u_\varepsilon(y, s) \quad \text{for all } (x, t) \in \partial\Omega \times (a + m_\varepsilon, b - m_\varepsilon).$$

Since $u_\varepsilon \in \mathrm{Lip}(\Omega \times (a + m_\varepsilon, b - m_\varepsilon))$, the limsup operation in the above formula can be replaced by the limit operation. Hence,

$$u_\varepsilon \in C(\overline{\Omega} \times (a + m_\varepsilon, b - m_\varepsilon)),$$

which guarantees that u_ε is Lipschitz continuous in $\overline{\Omega} \times (a + m_\varepsilon, b - m_\varepsilon)$.

Finally, if we replace u and J by u_ε and $(a + 2m_\varepsilon, b - 2m_\varepsilon)$, respectively, and select $\varepsilon > 0$ small enough so that

$$\max_{\overline{\Omega} \times [a + 2m_\varepsilon, b - 2m_\varepsilon]} (u_\varepsilon - v) > 0 > \max_{\overline{\Omega} \times \partial[a + 2m_\varepsilon, b - 2m_\varepsilon]} (u_\varepsilon - v),$$

then conditions (48)–(50) are satisfied. □

Reduction 3: We may assume that there exist a constant $\delta > 0$ and a finite open interval $J \subset (0, \infty)$ such that

u is a viscosity subsolution of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \leq -\delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x u(x,t) \leq g(x) - \delta & \text{on } \partial\Omega \times J, \end{cases} \tag{54}$$

v is a viscosity supersolution of

$$\begin{cases} v_t(x,t) + H(x, D_x v(x,t)) \geq \delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x v(x,t) \geq g(x) + \delta & \text{on } \partial\Omega \times J, \end{cases} \tag{55}$$

$$\max_{\overline{\Omega}\times\overline{J}}(u - v) > 0 > \max_{\overline{\Omega}\times\partial J}(u - v), \tag{56}$$

and

$$u \in \text{Lip}(\overline{\Omega} \times \overline{J}) \;\text{ and }\; v \;\text{ is bounded on }\; \overline{\Omega} \times \overline{J}. \tag{57}$$

Proof. Let u, v, J be as in Reduction 2. Set $J = (a, b)$. Let ρ be a defining function of Ω as before. Let $0 < \varepsilon < 1$. We set

$$u_\varepsilon(x,t) = u(x,t) - \varepsilon\rho(x) \;\text{ and }\; v_\varepsilon(x,t) = v(x,t) + \varepsilon\rho(x) \;\text{ for }\; (x,t) \in \overline{\Omega} \times \overline{J},$$

and

$$H_\varepsilon(x, p) = H(x, p - \varepsilon D\rho(x)) + \varepsilon \quad \text{for } (x, p) \in \overline{\Omega} \times \mathbb{R}^n.$$

Let $(x,t) \in \overline{\Omega} \times J$ and $(p,q) \in D^- v_\varepsilon(x,t)$. Then we have

$$(p - \varepsilon D\rho(x), q) \in D^- v(x,t).$$

Since v is a viscosity supersolution of (ENP), if $x \in \Omega$, then

$$q + H(x, p - \varepsilon D\rho(x)) \geq 0.$$

If $x \in \partial\Omega$, then either

$$q + H(x, p - \varepsilon D\rho(x)) \geq 0,$$

or

$$\begin{aligned} \gamma(x) \cdot p &= \gamma(x) \cdot (p - \varepsilon D\rho(x)) + \varepsilon\gamma(x) \cdot D\rho(x) \\ &\geq g(x) + \varepsilon\gamma(x) \cdot D\rho(x) \geq g(x) + \lambda\varepsilon, \end{aligned}$$

where

$$\lambda = \min_{\partial\Omega} \gamma \cdot D\rho \; (> 0).$$

Now let $(p, q) \in D^+u_\varepsilon(x, t)$. Note that $(p + \varepsilon D\rho(x), q) \in D^+u(x, t)$. Since $u \in \mathrm{Lip}(\overline{\Omega} \times [a, b])$, we have a bound $C_0 > 0$ such that

$$|q| \le C_0.$$

If $x \in \Omega$, then

$$\begin{aligned} q + H(x, p - \varepsilon D\rho(x)) &\le q + H(x, p + \varepsilon D\rho(x)) + \omega(2\varepsilon|D\rho(x)|) \\ &\le -\delta + \omega(2\varepsilon C_1), \end{aligned}$$

where

$$C_1 = \max_{\overline{\Omega}} |D\rho|,$$

and ω denotes the modulus of continuity of H on the set $\overline{\Omega} \times \overline{B}_{R+2C_1}$, with $R > 0$ being chosen so that

$$\min_{\overline{\Omega}\times(\mathbb{R}^n\setminus B_R)} H > C_0.$$

(Here we have used the fact that $H(x, p + \varepsilon D\rho(x)) \le C_0$, which implies that $|p + \varepsilon D\rho(x)| \le R$.)

If $x \in \partial\Omega$, then either

$$q + H(x, p - \varepsilon D\rho(x)) \le -\delta + \omega(2\varepsilon C_1),$$

or

$$\gamma(x) \cdot p \le \gamma(x) \cdot (p + \varepsilon D\rho(x)) - \varepsilon\gamma(x) \cdot D\rho(x) \le g(x) - \lambda\varepsilon.$$

Thus we see that v_ε is a viscosity supersolution of

$$\begin{cases} v_{\varepsilon,t} + H_\varepsilon(x, D_x v_\varepsilon) \ge \varepsilon & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x v_\varepsilon(x, t) \ge g(x) + \lambda\varepsilon & \text{on } \partial\Omega \times J, \end{cases}$$

and u_ε is a viscosity subsolution of

$$\begin{cases} u_{\varepsilon,t} + H_\varepsilon(x, D_x u_\varepsilon) \le -\delta + \omega(2C_1\varepsilon) + \varepsilon & \text{in } \Omega \times J, \\ \gamma \cdot Du_\varepsilon \le g(x) - \lambda\varepsilon & \text{on } \partial\Omega \times J, \end{cases}$$

If we replace u, v, H and δ by u_ε, v_ε, H_ε and

$$\min\{\varepsilon, \lambda\varepsilon, \delta - \omega(2C_1\varepsilon) - \varepsilon\},$$

respectively, and choose $\varepsilon > 0$ sufficiently small, then conditions (54)–(57) are satisfied. □

Final step: Let u, v, J and δ be as in Reduction 3. We choose a maximum point $(z,\tau) \in \overline{\Omega} \times \overline{J}$ of the function $u - v$. Note that $\tau \in J$, that is, $\tau \notin \partial J$.

By replacing u, if necessary, by the function

$$u(x,t) - \varepsilon|x-z|^2 - \varepsilon(t-\tau)^2,$$

where $\varepsilon > 0$ is a small constant, we may assume that (z,τ) is a strict maximum point of $u - v$.

By making a change of variables, we may assume that $z = 0$ and

$$\Omega \cap B_{2r} = \{x = (x_1, \ldots, x_n) \in B_{2r} \,:\, x_n < 0\},$$

while we may assume as well that $[\tau - r,\, \tau + r] \subset J$.

We set $\hat{\gamma} = \gamma(0)$ and apply Lemma 3.4, with $y = (0, \ldots, 0, 1) \in \mathbb{R}^n$ and $z = \hat{\gamma}$, to find a quadratic function ζ so that

$$\begin{cases} \zeta(t\xi) = t^2\zeta(\xi) & \text{for all } (\xi, t) \in \mathbb{R}^n \times \mathbb{R}, \\ \zeta(\xi) > 0 & \text{if } \xi \neq 0, \\ \hat{\gamma} \cdot D\zeta(\xi) = 2\hat{\gamma}_n \xi_n & \text{for all } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \end{cases}$$

where $\hat{\gamma}_n$ denotes the n-th component of the n-tuple $\hat{\gamma}$.

By replacing ζ by a constant multiple of ζ, we may assume that

$$\zeta(\xi) \geq |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n,$$
$$|D\zeta(\xi)| \leq C_0|\xi| \quad \text{for all } \xi \in \mathbb{R}^n,$$
$$\hat{\gamma} \cdot D\zeta(\xi) \begin{cases} \geq 0 & \text{if } \xi_n \geq 0, \\ \leq 0 & \text{if } \xi_n \leq 0, \end{cases}$$

where $C_0 > 0$ is a constant.

Let $M > 0$ be a Lipschitz bound of the function u. Set

$$\hat{g} = g(0), \quad \mu = \hat{g}\frac{\hat{\gamma}}{|\hat{\gamma}|^2} \quad \text{and} \quad M_1 = M + |\mu|.$$

We may assume by replacing r by a smaller positive constant if needed that for all $x \in B_r \cap \partial\Omega$,

$$|\gamma(x) - \hat{\gamma}| < \frac{\delta}{2(|\mu| + C_0 M_1)} \quad \text{and} \quad |g(x) - \hat{g}| < \frac{\delta}{2}. \tag{58}$$

For $\alpha > 1$ we consider the function

$$\Phi(x,t,y,s) = u(x,t) - v(y,s) - \mu \cdot (x-y) - \alpha\zeta(x-y) - \alpha(t-s)^2$$

on $K := \big((\overline{\Omega} \cap \overline{B}_r(0,\tau) \times [\tau - r, \tau + r]\big)^2$. Let $(x_\alpha, t_\alpha, y_\alpha, s_\alpha)$ be a maximum point of the function Φ. By the inequality $\Phi(y_\alpha, s_\alpha, y_\alpha, s_\alpha) \le \Phi(x_\alpha, t_\alpha, y_\alpha, s_\alpha)$, we get

$$\begin{aligned}\alpha(|x_\alpha - y_\alpha|^2 + (t_\alpha - s_\alpha)^2) &\le \alpha(\zeta(x_\alpha - y_\alpha) + (t_\alpha - s_\alpha)^2)\\ &\le u(x_\alpha, t_\alpha) - u(y_\alpha, s_\alpha) + |\mu||x_\alpha - y_\alpha|\\ &\le M_1(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2},\end{aligned}$$

and hence

$$\alpha(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2} \le M_1. \tag{59}$$

As usual we may deduce that as $\alpha \to \infty$,

$$\begin{cases}(x_\alpha, \tau_\alpha),\ (y_\alpha, s_\alpha) \to (0, \tau),\\ u(x_\alpha, t_\alpha) \to u(0, \tau),\\ v(y_\alpha, s_\alpha) \to v(0, \tau).\end{cases}$$

Let $\alpha > 1$ be so large that

$$(x_\alpha, t_\alpha), (y_\alpha, s_\alpha) \in (\overline{\Omega} \cap B_r) \times (\tau - r,\ \tau + r).$$

Accordingly, we have

$$\begin{aligned}(\mu + \alpha D\zeta(x_\alpha - y_\alpha),\ 2\alpha(t_\alpha - s_\alpha)) &\in D^+u(x_\alpha, t_\alpha),\\ (\mu + \alpha D\zeta(x_\alpha - y_\alpha),\ 2\alpha(t_\alpha - s_\alpha)) &\in D^-v(y_\alpha, s_\alpha).\end{aligned}$$

Using (59), we have

$$\alpha|D\zeta(x_\alpha - y_\alpha)| \le C_0\alpha|x_\alpha - y_\alpha| \le C_0 M_1. \tag{60}$$

If $x_\alpha \in \partial\Omega$, then $x_{\alpha,n} = 0$ and $(x_\alpha - y_\alpha)_n \ge 0$. Hence, in this case, we have

$$\hat{\gamma} \cdot D\zeta(x_\alpha - y_\alpha) \ge 0,$$

and moreover, in view of (58) and (60),

$$\begin{aligned}\gamma(x_\alpha) \cdot (\mu + \alpha D\zeta(x_\alpha - y_\alpha)) &\ge \hat{\gamma} \cdot (\mu + \alpha D\zeta(x_\alpha - y_\alpha))\\ &\quad - |\gamma(x_\alpha) - \hat{\gamma}|(|\mu| + C_0 M_1)\\ &> g(x_\alpha) - |\hat{g} - g(x_\alpha)| - \frac{\delta}{2} > g(x_\alpha) - \delta.\end{aligned}$$

Now, by the viscosity property of u, we obtain

$$2\alpha(t_\alpha - s_\alpha) + H(x_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \le -\delta,$$

which we certainly have when $x_\alpha \in \Omega$.

If $y_\alpha \in \partial\Omega$, then $(x_\alpha - y_\alpha)_n \le 0$ and

$$\hat{\gamma} \cdot D\zeta(x_\alpha - y_\alpha) \le 0.$$

As above, we find that if $y_\alpha \in \partial\Omega$, then

$$\gamma(y_\alpha) \cdot (\mu + \alpha D\zeta(x_\alpha - y_\alpha)) < \delta,$$

and hence, by the viscosity property of v,

$$2(t_\alpha - s_\alpha) + H(y_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \ge \delta,$$

which is also valid in case when $y_\alpha \in \Omega$.

Thus, we always have

$$\begin{cases} 2\alpha(t_\alpha - s_\alpha) + H(x_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \le -\delta, \\ 2(t_\alpha - s_\alpha) + H(y_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \ge \delta. \end{cases}$$

Sending $\alpha \to \infty$ along a sequence, we obtain

$$q + H(0, \mu + p) \le -\delta \quad \text{and} \quad q + H(0, \mu + p) \ge \delta$$

for some $p \in \overline{B}_{C_0 M_1}$ and $q \in [-2M_1, 2M_1]$, which is a contradiction. This completes the proof of Theorem 3.1. □

4 Stationary Problem: Weak KAM Aspects

In this section we discuss some aspects of weak KAM theory for Hamilton–Jacobi equations with the Neumann boundary condition. We refer to Fathi [25, 27], E [22] and Evans [24] for origins and developments of weak KAM theory.

Throughout this section we assume that (A1)–(A6) and the following (A7) hold:

(A7) The Hamiltonian H is convex. That is, the function $p \mapsto H(x, p)$ is convex in $\mathbb{R}^n$ for any $x \in \overline{\Omega}$.

As in Sect. 2 we consider the stationary problem

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{SNP}$$

As remarked before this boundary value problem may have no solution in general, but, due to Theorem 3.4, if we replace H by $H - a$ with the right choice of $a \in \mathbb{R}$, the problem (SNP) has a viscosity solution. Furthermore, if we replace H by $H - a$ with a sufficiently large $a \in \mathbb{R}$, the problem (SNP) has a viscosity subsolution. With a change of Hamiltonians of this kind in mind, we make the following hypothesis throughout this section:

(A8) The problem (SNP) has a viscosity subsolution.

4.1 Aubry Sets and Representation of Solutions

We start this subsection by the following Lemma.

Lemma 4.1. *Let $u \in \mathrm{USC}(\overline{\Omega})$ be a viscosity subsolution of (SNP). Then $u \in \mathrm{Lip}(\overline{\Omega})$. Moreover, u has a Lipschitz bound which depends only on H and Ω.*

Proof. By the coercivity of H, there exists a constant $M > 0$ such that $H(x, p) > 0$ for all $(x, p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_M)$. Fix such a constant $M > 0$ and note that u is a viscosity subsolution of $|Du(x)| \leq M$ in Ω. Accordingly, we see by Lemma 2.2 that $u \in \mathrm{Lip}(\Omega)$. Furthermore, if $C > 0$ is the constant from Lemma 2.1, then we have $|u(x) - u(y)| \leq CM|x - y|$ for all $x, y \in \Omega$. (See also Appendix A.3.)

Since the function $u(x)$, as a function of (x, t), is a viscosity subsolution of (ENP), Lemma 3.3 guarantees that u is continuous up to the boundary $\partial\Omega$. Thus, we get $|u(x) - u(y)| \leq CM|x - y|$ for all $x, y \in \overline{\Omega}$, which completes the proof. □

We introduce the distance-like function $d : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ by

$$d(x, y) = \sup\{v(x) - v(y) : v \in \mathrm{USC}(\overline{\Omega}) \cap \mathscr{S}^-\},$$

where $\mathscr{S}^- = \mathscr{S}^-(\overline{\Omega})$ has been defined as the set of all viscosity subsolutions of (SNP). By (A8), we have $\mathscr{S}^- \neq \emptyset$ and hence $d(x, x) = 0$ for all $x \in \overline{\Omega}$. Since $\mathrm{USC}(\overline{\Omega}) \cap \mathscr{S}^-$ is equi-Lipschitz continuous on $\overline{\Omega}$ by Lemma 4.1, we see that the functions $(x, y) \mapsto v(x) - v(y)$, with $v \in \mathrm{USC}(\overline{\Omega}) \cap \mathscr{S}^-$, are equi-Lipschitz continuous and d is Lipschitz continuous on $\overline{\Omega} \times \overline{\Omega}$. By Proposition 1.10, the functions $x \mapsto d(x, y)$, with $y \in \overline{\Omega}$, are viscosity subsolutions of (SNP). Hence, by the definition of $d(x, z)$ we get

$$d(x, y) - d(z, y) \leq d(x, z) \quad \text{for all } x, y, z \in \overline{\Omega}.$$

We set

$$\mathscr{F}_y = \{v(x) - v(y) \,:\, v \in \mathscr{S}^-\}, \quad \text{with } \ y \in \overline{\Omega},$$

and observe by using Proposition 1.10 and Lemma 1.3 that $\mathscr{F}_y$ satisfies (P1) and (P2), with Ω replaced by $\overline{\Omega} \setminus \{y\}$, of Proposition 1.13. Hence, by Proposition 1.13, the function $d(\cdot, y) = \sup \mathscr{F}_y$ is a viscosity solution of (SNP) in $\overline{\Omega} \setminus \{y\}$.

The following proposition collects these observations.

Proposition 4.1. *We have:*

(i) $d(x, x) = 0$ *for all* $x \in \overline{\Omega}$.
(ii) $d(x, y) \le d(x, z) + d(z, y)$ *for all* $x, y, z \in \overline{\Omega}$.
(iii) $d(\cdot, y) \in \mathscr{S}^-(\overline{\Omega})$ *for all* $y \in \overline{\Omega}$.
(iv) $d(\cdot, y) \in \mathscr{S}(\overline{\Omega} \setminus \{y\})$ *for all* $y \in \overline{\Omega}$.

The *Aubry set* (or Aubry–Mather set) $\mathscr{A}$ associated with (SNP) is defined by

$$\mathscr{A} = \{y \in \overline{\Omega} \,:\, d(\cdot, y) \in \mathscr{S}(\overline{\Omega})\}.$$

Example 4.1. Let $n = 1$, $\Omega = (-1, 1)$, $H(x, p) = |p| - f(x)$, $f(x) = 1 - |x|$, $\gamma(\pm 1) = \pm 1$ and $g(\pm 1) = 0$. The function $v \in C^1([-1, 1])$ given by

$$v(x) = \begin{cases} 1 - \frac{1}{2}(x+1)^2 & \text{if } x \le 0, \\ \frac{1}{2}(x-1)^2 & \text{if } x \ge 0 \end{cases}$$

is a classical solution of (SNP). We show that $d(x, 1) = v(x)$ for all $x \in [-1, 1]$. It is enough to show that $d(x, 1) \le v(x)$ for all $x \in [-1, 1]$. To prove this, we suppose by contradiction that $\max_{x \in [-1,1]}(d(x, 1) - v(x)) > 0$. We may choose a constant $\varepsilon > 0$ so small that $\max_{x \in [-1,1]}(d(x, 1) - v(x) - \varepsilon(1 - x)) > 0$. Let $x_\varepsilon \in [-1, 1]$ be a maximum point of the function $d(x, 1) - v(x) - \varepsilon(1 - x)$. Since this function vanishes at $x = 1$, we have $x_\varepsilon \in [-1, 1)$. If $x_\varepsilon > -1$, then we have

$$0 \ge H(x_\varepsilon, v'(x_\varepsilon) - \varepsilon) = |v'(x_\varepsilon)| + \varepsilon - f(x_\varepsilon) = \varepsilon > 0,$$

which is impossible. Here we have used the fact that $v'(x) = |x| - 1 \le 0$ for all $x \in [-1, 1]$. On the other hand, if $x_\varepsilon = -1$, then we have

$$0 \ge \min\{H(-1, v'(-1) - \varepsilon), -(v'(-1) - \varepsilon)\} = \min\{\varepsilon, \varepsilon\} = \varepsilon > 0,$$

which is again impossible. Thus we get a contradiction. That is, we have $d(x, 1) \le v(x)$ and hence $d(x, 1) = v(x)$ for all $x \in [-1, 1]$. Arguments similar to the above show moreover that

$$d(x, -1) = \begin{cases} \frac{1}{2}(x+1)^2 & \text{if } x \le 0, \\ 1 - \frac{1}{2}(x-1)^2 & \text{if } x \ge 0, \end{cases}$$

and

$$d(x,y) = \begin{cases} d(x,1) - d(y,1) & \text{if } x \le y, \\ d(x,-1) - d(y,-1) & \text{if } x \ge y. \end{cases}$$

Since two functions $d(x, \pm 1)$ are classical solutions of (SNP), we see that $\pm 1 \in \mathscr{A}$. Noting that $d(x,y) \ge 0$ and $d(x,x) = 0$ for all $x, y \in [-1, 1]$, we find that for each fixed $y \in [-1, 1]$ the function $x \mapsto d(x,y)$ has a minimum at $x = y$. If $y \in (-1, 1)$, then $H(y,0) = -f(y) < 0$. Hence, we see that the interval $(-1, 1)$ does not intersect $\mathscr{A}$. Thus, we conclude that $\mathscr{A} = \{-1, 1\}$.

A basic observation on $\mathscr{A}$ is the following:

Proposition 4.2. *The Aubry set $\mathscr{A}$ is compact.*

Proof. It is enough to show that $\mathscr{A}$ is a closed subset of $\overline{\Omega}$. Note that the function d is Lipschitz continuous in $\overline{\Omega} \times \overline{\Omega}$. Therefore, if $\{y_k\}_{k\in\mathbb{N}} \subset \mathscr{A}$ converges to $y \in \overline{\Omega}$, then the sequence $\{d(\cdot, y_k)\}_{k\in\mathbb{N}}$ converges to the function $d(\cdot, y)$ in $C(\overline{\Omega})$. By the stability of the viscosity property under the uniform convergence, we see that $d(\cdot, y) \in \mathscr{S}$. Hence, we have $y \in \mathscr{A}$. □

The main assertion in this section is the following and will be proved at the end of the section.

Theorem 4.1. *Let $u \in C(\overline{\Omega})$ be a viscosity solution of* (SNP). *Then*

$$u(x) = \inf\{u(y) + d(x,y) \,:\, y \in \mathscr{A}\} \quad \textit{for all } x \in \overline{\Omega}. \tag{61}$$

We state the following approximation result on viscosity subsolutions of (SNP).

Theorem 4.2. *Let $u \in C(\overline{\Omega})$ be a viscosity subsolution of* (SNP). *There exists a collection $\{u^\varepsilon\}_{\varepsilon\in(0,\,1)} \subset C^1(\overline{\Omega})$ such that for any $\varepsilon \in (0,\, 1)$,*

$$\begin{cases} H(x, Du^\varepsilon(x)) \le \varepsilon & \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x) \le g(x) & \text{on } \partial\Omega, \end{cases}$$

and

$$\|u^\varepsilon - u\|_{\infty,\Omega} < \varepsilon.$$

A localized version of the above theorem is in [39] (see also Appendix A.4 and [8]) and the above theorem seems to be new in the global nature.

As a corollary, we get the following theorem.

Theorem 4.3. *Let $f_1, f_2 \in C(\overline{\Omega})$ and $g_1, g_2 \in C(\partial\Omega)$. Let $u, v \in C(\overline{\Omega})$ be viscosity solutions of*

$$\begin{cases} H(x, Du) \le f_1 & \textit{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} \le g_1 & \textit{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} H(x, Dv) \le f_2 & \textit{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma} \le g_2 & \textit{on } \partial\Omega, \end{cases}$$

respectively. Let $0 < \lambda < 1$ *and set* $w = (1-\lambda)u + \lambda v$. *Then* w *is a viscosity subsolution of*

$$\begin{cases} H(x, Dw) \le (1-\lambda) f_1 + \lambda f_2 & \textit{in } \Omega, \\ \dfrac{\partial w}{\partial \gamma} \le (1-\lambda) g_1 + \lambda g_2 & \textit{on } \partial\Omega, \end{cases} \tag{62}$$

Proof. By Theorem 4.2, for each $\varepsilon \in (0, 1)$ there are functions $u^\varepsilon, v^\varepsilon \in C^1(\overline{\Omega})$ such that

$$\|u^\varepsilon - u\|_{\infty,\Omega} + \|v^\varepsilon - v\|_{\infty,\Omega} < \varepsilon,$$

$$\begin{cases} H(x, Du^\varepsilon(x)) \le f_1(x) + \varepsilon & \text{in } \overline{\Omega}, \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x) \le g_1(x) & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} H(x, Dv^\varepsilon(x)) \le f_2(x) + \varepsilon & \text{in } \overline{\Omega}, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le g_2(x) & \text{on } \partial\Omega. \end{cases}$$

If we set $w^\varepsilon = (1-\lambda)u^\varepsilon + \lambda v^\varepsilon$, then we get with use of (A7)

$$\begin{cases} H(x, Dw^\varepsilon(x)) \le (1-\lambda) f_1(x) + \lambda f_2(x) + \varepsilon & \text{in } \overline{\Omega}, \\ \dfrac{\partial w^\varepsilon}{\partial \gamma}(x) \le (1-\lambda) g_1(x) + \lambda g_2(x) & \text{on } \partial\Omega. \end{cases}$$

Thus, in view of the stability property (Proposition 1.9), we see in the limit as $\varepsilon \to 0$ that w is a viscosity subsolution of (62). □

The following theorem is also a consequence of (A7), the convexity of H, and Theorem 4.2.

Theorem 4.4. *Let* $\mathscr{F} \subset USC(\overline{\Omega})$ *be a nonempty collection of viscosity subsolutions of* (SNP). *Assume that* $u(x) := \inf \mathscr{F}(x) > -\infty$ *for all* $x \in \overline{\Omega}$. *Then* $u \in \mathrm{Lip}(\overline{\Omega})$ *and it is a viscosity subsolution of* (SNP).

This theorem may be regarded as part of the theory of Barron–Jensen's lower semicontinuous viscosity solutions. There are at least two approaches to this theory: the original one by Barron–Jensen [11] and the other due to Barles [5]. The following proof is close to Barles' approach.

Proof. By Lemma 4.1, the collection $\mathscr{F}$ is equi-Lipschitz in $\overline{\Omega}$. Hence, u is a Lipschitz continuous function in $\overline{\Omega}$. For each $x \in \overline{\Omega}$ there is a sequence $\{u_{x,k}\}_{k\in\mathbb{N}} \subset \mathscr{F}$ such that $\lim_{k\to\infty} u_{x,k}(x) = u(x)$. Fix such sequences $\{u_{x,k}\}_{k\in\mathbb{N}}$, with $x \in \overline{\Omega}$ and select a countable dense subset $Y \subset \overline{\Omega}$. Observe that $Y \times \mathbb{N}$ is a countable set and

$$u(x) = \inf\{u_{y,k}(x) \,:\, (y,k) \in Y \times \mathbb{N}\} \quad \text{for all } \ x \in \overline{\Omega}.$$

Thus we may assume that $\mathscr{F}$ is a sequence.

Let $\mathscr{F} = \{u_k\}_{k\in\mathbb{N}}$. Then we have

$$u(x) = \lim_{k\to\infty} (u_1 \wedge u_2 \wedge \cdots \wedge u_k)(x) \quad \text{for all } \ x \in \overline{\Omega}.$$

We show that $u_1 \wedge u_2 \wedge \cdots \wedge u_k$ is a viscosity subsolution of (SNP) for every $k \in \mathbb{N}$. It is enough to show that if v and w are viscosity subsolutions of (SNP), then so is the function $v \wedge w$.

Let v and w be viscosity subsolutions of (SNP). Fix any $\varepsilon > 0$. In view of Theorem 4.2, we may select functions $v_\varepsilon,\ w_\varepsilon \in C^1(\overline{\Omega})$ so that both for $(\phi_\varepsilon, \phi) = (v_\varepsilon, v)$ and $(\phi_\varepsilon, \phi) = (w_\varepsilon, w)$, we have

$$\begin{cases} H(x, D\phi_\varepsilon(x)) \le \varepsilon & \text{for all } \ x \in \Omega, \\ \dfrac{\partial \phi_\varepsilon}{\partial \gamma}(x) \le g(x) & \text{for all } \ x \in \partial\Omega, \\ \|\phi_\varepsilon - \phi\|_{\infty,\Omega} < \varepsilon. \end{cases}$$

Note that $(v_\varepsilon \wedge w_\varepsilon)(x) = v_\varepsilon(x) - (v_\varepsilon - w_\varepsilon)_+(x)$. Let $\{\eta_k\}_{k\in\mathbb{N}} \subset C^1(\mathbb{R})$ be such that

$$\begin{cases} \eta_k(r) \to r_+ & \text{uniformly on } \ \mathbb{R} \ \text{ as } \ k \to \infty, \\ 0 \le \eta_k'(r) \le 1 & \text{for all } \ r \in \mathbb{R},\ k \in \mathbb{N}. \end{cases}$$

We set $z_{\varepsilon,k} = v_\varepsilon - \eta_k \circ (v_\varepsilon - w_\varepsilon)$ and observe that

$$Dz_{\varepsilon,k}(x) = \big(1 - \eta_k'(v_\varepsilon(x) - w_\varepsilon(x))\big)\, Dv_\varepsilon(x) + \eta_k'(v_\varepsilon(x) - w_\varepsilon(x)) Dw_\varepsilon(x).$$

By the convexity of H, we see easily that $z_{\varepsilon,k}$ satisfies

$$\begin{cases} H(x, Dz_{\varepsilon,k}(x)) \le \varepsilon & \text{for all } \ x \in \Omega, \\ \dfrac{\partial z_{\varepsilon,k}}{\partial \gamma}(x) \le g(x) & \text{for all } \ x \in \partial\Omega. \end{cases}$$

Since $v \wedge w$ is a uniform limit of $z_{\varepsilon,k}$ in $\overline{\Omega}$ as $k \to \infty$ and $\varepsilon \to 0$, we see that $v \wedge w$ is a viscosity subsolution of (SNP).

By the Ascoli–Arzela theorem or Dini's lemma, we deduce that the convergence

$$u(x) = \lim_{k\to\infty} (u_1 \wedge \cdots \wedge u_k)(x)$$

is uniform in $\overline{\Omega}$. Thus we conclude that u is a viscosity subsolution of (SNP). □

Remark 4.1. Theorem 4.2 has its localized version which concerns viscosity subsolutions of

$$\begin{cases} H(x, Du(x)) \le 0 & \text{in } U \cap \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) \le g(x) & \text{on } U \cap \partial\Omega, \end{cases}$$

where U is an open subset of $\mathbb{R}^n$ having nonempty intersection with Ω. More importantly, it has a version for the Neumann problem for Hamilton–Jacobi equations of evolution type, which concerns solutions of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le 0 & \text{in } U \cap (\Omega \times \mathbb{R}_+), \\ \dfrac{\partial u}{\partial \gamma}(x,t) \le g(x) & \text{on } U \cap (\partial\Omega \times \mathbb{R}_+), \end{cases}$$

where U is an open subset of $\mathbb{R}^n \times \mathbb{R}_+$, with $U \cap (\Omega \times \mathbb{R}_+) \ne \emptyset$. Consequently, Theorems 4.3 and 4.4 are valid for these problems with trivial modifications. For these, see Appendix A.4.

Theorem 4.5. *We have*

$$c^\# = \inf \left\{ \max_{x\in\overline{\Omega}} H(x, D\psi(x)) \;:\; \psi \in C^1(\overline{\Omega}),\ \partial\psi/\partial\gamma \le g \text{ on } \partial\Omega \right\}.$$

Remark 4.2. A natural question here is if there is a function $\psi \in C^1(\overline{\Omega})$ which attains the infimum in the above formula. See [12, 28].

Proof. Let $c^\star$ denote the right hand side of the above minimax formula. By the definition of $c^\star$, it is clear that for any $a > c^\star$, there is a classical subsolution of (EVP). Hence, by Corollary 3.2, we see that $c^\# \le c^\star$.

On the other hand, by Theorem 3.4, there is a viscosity solution v of (EVP), with $a = c^\#$. By Theorem 4.2, for any $a > c^\#$ there is a classical subsolution of (EVP). That is, we have $c^\star \le c^\#$. Thus we conclude that $c^\# = c^\star$. □

Theorem 4.6 (Comparison). *Let $v, w \in C(\overline{\Omega})$ be a viscosity subsolution and supersolution of* (SNP)*, respectively. Assume that $v \le w$ on $\mathscr{A}$. Then $v \le w$ in $\overline{\Omega}$.*

For the proof of the above theorem, we need the following lemma.

Lemma 4.2. *Let K be a compact subset of $\overline{\Omega} \setminus \mathscr{A}$. Then there exists a function $\psi \in C^1(U \cap \overline{\Omega})$, where U is an open neighborhood of K in $\mathbb{R}^n$, and a positive constant $\delta > 0$ such that*

$$\begin{cases} H(x, D\psi(x)) \le -\delta & \text{in } U \cap \Omega, \\ \dfrac{\partial \psi}{\partial \gamma}(x) \le g(x) - \delta & \text{on } U \cap \partial\Omega. \end{cases} \tag{63}$$

We assume temporarily the validity of the above lemma and complete the proof of Theorems 4.6 and 4.1. The proof of the above lemma will be given in the sequel.

Proof (Theorem 4.6). By contradiction, we suppose that $M := \sup_{\overline{\Omega}}(v - w) > 0$. Let

$$K = \{x \in \overline{\Omega} \,:\, (v - w)(x) = M\},$$

which is a compact subset of $\overline{\Omega} \setminus \mathscr{A}$. According to Lemma 4.2, there are $\delta > 0$ and $\psi \in C^1(U \cap \overline{\Omega})$, where U is an open neighborhood of K such that ψ is a subsolution of (63).

According to Theorem 4.2, for each $\varepsilon \in (0,\ 1)$ there is a function $v^\varepsilon \in C^1(\overline{\Omega})$ such that

$$\begin{cases} H(x, Dv^\varepsilon(x)) \le \varepsilon & \text{in } \ \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le g(x) & \text{on } \ \partial\Omega, \end{cases}$$

and

$$\|v^\varepsilon - v\|_{\infty,\Omega} < \varepsilon.$$

We fix a $\lambda \in (0,\ 1)$ so that $\delta_\varepsilon := -(1-\lambda)\varepsilon + \delta\lambda > 0$ and set

$$u_\varepsilon(x) = (1-\lambda)v^\varepsilon(x) + \lambda\psi(x).$$

This function satisfies

$$\begin{cases} H(x, Du_\varepsilon(x)) \le -\delta_\varepsilon & \text{in } \ U \cap \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \gamma}(x) \le g(x) - \delta_\varepsilon & \text{on } \ U \cap \partial\Omega. \end{cases}$$

This contradicts the viscosity property of the function w if $u_\varepsilon - w$ attains a maximum at a point $z \in U \cap \overline{\Omega}$. Hence, we have

$$\max_{\overline{U} \cap \overline{\Omega}}(u_\varepsilon - w) = \max_{\partial U \cap \overline{\Omega}}(u_\varepsilon - w).$$

Sending $\varepsilon \to 0$ and then $\lambda \to 0$ yields

$$\max_{\overline{U} \cap \overline{\Omega}}(v - w) = \max_{\partial U \cap \overline{\Omega}}(v - w),$$

that is,

$$M = \max_{\partial U \cap \overline{\Omega}} (v - w).$$

This is a contradiction. □

Remark 4.3. Obviously, the continuity assumption on v, w in the above lemma can be replaced by the assumption that $v \in \mathrm{USC}(\overline{\Omega})$ and $w \in \mathrm{LSC}(\overline{\Omega})$.

Proof (Theorem 4.1). We write $w(x)$ for the right hand side of (61) in this proof. By the definition of d, we have

$$u(x) - u(y) \le d(x, y) \quad \text{for all } x, y \in \overline{\Omega},$$

from which we see that $u(x) \le w(x)$.

By the definition of w, for every $x \in \mathscr{A}$, we have

$$w(x) \le u(x) + d(x, x) = u(x).$$

Hence, we have $w = u$ on $\mathscr{A}$.

Now, by Proposition 1.10 (its version for supersolutions), we see that w is a viscosity supersolution of (SNP) while Theorem 4.4 guarantees that w is a viscosity subsolution of (SNP). We invoke here Theorem 4.6, to see that $u = w$ in $\overline{\Omega}$. □

Proof (Lemma 4.2). In view of Theorem 4.2, it is enough to show that there exist functions $w \in \mathrm{Lip}(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ such that

$$\begin{cases} f(x) \ge 0 & \text{in } \Omega, \\ f(x) > 0 & \text{in } K, \end{cases}$$

and w is a viscosity subsolution of

$$\begin{cases} H(x, Dw(x)) \le -f(x) & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \gamma}(x) \le g(x) & \text{on } \partial\Omega. \end{cases}$$

For any $z \in \overline{\Omega} \setminus \mathscr{A}$, the function $x \mapsto d(x, z)$ is not a viscosity supersolution of (SNP) at z while it is a viscosity subsolution of (SNP). Hence, according to Lemma 1.3, there exist a function $\psi_z \in \mathrm{Lip}(\overline{\Omega})$, a neighborhood U_z of z in $\mathbb{R}^n$ and a constant $\delta_z > 0$ such that ψ_z is a viscosity subsolution of (SNP) and it is moreover a viscosity subsolution of

$$\begin{cases} H(x, D\psi_z(x)) \le -\delta_z & \text{in } U_z \cap \Omega, \\ \dfrac{\partial \psi_z}{\partial \gamma}(x) \le g(x) - \delta_z & \text{on } U_z \cap \partial\Omega. \end{cases}$$

We choose a function $f_z \in C(\overline{\Omega})$ so that $0 < f_z(x) \le \delta$ for all $x \in \overline{\Omega} \cap U_z$ and $f_z(x) = 0$ for all $x \in \overline{\Omega} \setminus U_z$, and note that ψ_z is a viscosity subsolution of

$$\begin{cases} H(x, D\psi_z(x)) \le -f_z(x) & \text{in } \Omega, \\ \dfrac{\partial \psi_z}{\partial \gamma}(x) \le g(x) - f_z(x) & \text{on } \partial\Omega. \end{cases}$$

We select a finite number of points $z_1, \ldots, z_k$ of K so that $\{U_{z_i}\}_{i=1}^k$ covers K.

Now, we define the function $\psi \in \mathrm{Lip}(\overline{\Omega})$ by

$$\psi(x) = \frac{1}{k} \sum_{i=1}^{k} \psi_{z_i}(x),$$

and observe by Theorem 4.3 that ψ is a viscosity subsolution of

$$\begin{cases} H(x, D\psi(x)) \le -f(x) & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial \gamma}(x) \le g(x) - f(x) & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\overline{\Omega})$ is given by

$$f(x) = \frac{1}{k} \sum_{i=1}^{k} f_{z_i}(x).$$

Finally, we note that $\inf_K f > 0$. □

4.2 Proof of Theorem 4.2

We give a proof of Theorem 4.2 in this subsection.

We begin by choosing continuous functions on $\mathbb{R}^n$ which extend the functions g, γ and ν. We denote them again by the same symbols g, γ and ν.

The following proposition guarantees the existence of test functions which are convenient to prove Theorem 4.2.

Theorem 4.7. *Let $\varepsilon > 0$ and $M > 0$. Then there exist a constant $\Lambda > 0$ and moreover, for each $R > 0$, a neighborhood U of $\partial\Omega$, a function $\chi \in C^1((\Omega \cup U) \times \mathbb{R}^n)$ and a constant $\delta > 0$ such that for all $(x, \xi) \in (\Omega \cup U) \times \mathbb{R}^n$,*

$$M|\xi| \le \chi(x, \xi) \le \Lambda(|\xi| + 1),$$

and for all $(x, \xi) \in U \times B_R$,

$$\gamma(x) \cdot D_\xi \chi(x,\xi) \begin{cases} \le g(x) + 2\varepsilon & \textit{if } \nu(x)\cdot\xi \le \delta, \\ \ge g(x) + \dfrac{\varepsilon}{2} & \textit{if } \nu(x)\cdot\xi \ge -\delta. \end{cases}$$

It should be noted that the constant Λ in the above statement does not depend on R while U, χ and δ do.

We begin the proof with two Lemmas.

We fix $r > 1$ and set

$$\mathbb{R}^{2n}_r = \{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n \; : \; y \cdot z \ge r^{-1}, \, \max\{|y|, |z|\} \le r\}.$$

We define the function $\zeta \in C^\infty(\mathbb{R}^{2n}_r \times \mathbb{R}^n)$ by

$$\zeta(y,z,\xi) = \left|\xi - \frac{y\cdot\xi}{y\cdot z} z\right|^2 + (y\cdot\xi)^2.$$

Lemma 4.3. *The function ζ has the properties:*

$$\begin{cases} \zeta(y,z,t\xi) = t^2\zeta(y,z,\xi) & \textit{for all } (y,z,\xi,t) \in \mathbb{R}^{2n}_r \times \mathbb{R}^n \times \mathbb{R}, \\ \zeta(y,z,\xi) > 0 & \textit{for all } (y,z,\xi) \in \mathbb{R}^{2n}_r \times (\mathbb{R}^n \setminus \{0\}), \\ z \cdot D_\xi\zeta(y,z,\xi) = 2(y\cdot z)(y\cdot\xi) & \textit{for all } (y,z,\xi) \in \mathbb{R}^{2n}_r \times \mathbb{R}^n. \end{cases}$$

This is a version of Lemma 3.4, the proof of which is easily adapted to the present case.

We define the function $\phi \; : \; \mathbb{R}^{2n}_r \times \mathbb{R}^n \to \mathbb{R}$ by

$$\phi(y,z,\xi) = (\zeta(y,z,\xi) + 1)^{1/2}.$$

Lemma 4.4. *There exists a constant $\Lambda > 1$, which depends only on r, such that for all $(y,z,\xi) \in \mathbb{R}^{2n}_r \times \mathbb{R}^n$,*

$$\begin{cases} z \cdot D_\xi\phi(y,z,\xi) = \phi(y,z,\xi)^{-1}(y\cdot z)(y\cdot\xi), \\ \max\{\Lambda^{-1}|\xi|, \, 1\} \le \phi(y,z,\xi) \le \Lambda(|\xi| + 1), \\ \max\{|D_y\phi(y,z,\xi)|, |D_z\phi(y,z,\xi)|\} \le \Lambda, \\ |D_\xi\phi(y,z,\xi)| \le \Lambda. \end{cases}$$

Proof. It is clear by the definition of ϕ that

$$\phi(y,z,\xi) \ge 1.$$

We may choose a constant $C > 1$ so that for all $(y, z, \xi) \in \mathbb{R}_r^{2n} \times S^{n-1}$,

$$\max\{\zeta(y,z,\xi), \zeta(y,z,\xi)^{-1}, |D_y\zeta(y,z,\xi)|, |D_z\zeta(y,z,\xi)|, |D_\xi\zeta(y,z,\xi)|\} \le C,$$

where $S^{n-1} := \{x \in \mathbb{R}^n \,:\, |x| = 1\}$. By the homogeneity of the function $\zeta(y,z,\xi)$ in ξ, we have

$$\begin{aligned} \max\{\zeta(y,z,\xi), |D_y\zeta(y,z,\xi)|, |D_z\zeta(y,z,\xi)|\} &\le C|\xi|^2, \\ |D_\xi\zeta(y,z,\xi)| &\le C|\xi|, \\ \zeta(y,z,\xi) &\ge C^{-1}|\xi|^2 \end{aligned} \tag{64}$$

for all $(y, z, \xi) \in \mathbb{R}_r^{2n} \times \mathbb{R}^n$. From this it follows that

$$C^{-1/2}|\xi| \le \phi(y,z,\xi) \le C^{1/2}(|\xi| + 1).$$

By a direct computation, we get

$$D_x\phi(y,z,\xi) = \frac{D_x\zeta(y,z,\xi)}{2\phi(y,z,\xi)} \quad \text{for } x = y, z, \xi.$$

Hence, using (64), we get

$$|D_y\phi(y,z,\xi)| \le \frac{C|\xi|^2}{2\phi(y,z,\xi)} \le C^{3/2}|\xi|.$$

In the same way, we get

$$|D_z\phi(y,z,\xi)| \le C^{3/2}|\xi|.$$

Also, we get

$$|D_\xi\phi(y,z,\xi)| \le \frac{C|\xi|^2}{2\phi(y,z,\xi)} \le C^{3/2}|\xi|.$$

We observe that

$$z \cdot D_\xi\phi(y,z,\xi) = \frac{z \cdot D_\xi\zeta(y,z,\xi)}{2\phi(y,z,\xi)} = \frac{(y \cdot z)(y \cdot \xi)}{\phi(y,z,\xi)}.$$

By setting $\Lambda = C^{3/2}$, we conclude the proof. □

Let $\alpha > 0$. For any $W \subset \mathbb{R}^n$ we denote by W^α the α–neighborhood of W, that is,

$$W^\alpha = \{x \in \mathbb{R}^n \,:\, \operatorname{dist}(x, W) < \alpha\}.$$

For each $\delta \in (0, 1)$ we select $\nu_\delta \in C^1(\Omega^1, \mathbb{R}^n)$, $\gamma_\delta \in C^1(\Omega^1, \mathbb{R}^n)$ and $g_\delta \in C^1(\Omega^1, \mathbb{R})$ so that for all $x \in \Omega^1$,

$$\max\{|\nu_\delta(x) - \nu(x)|, \ |\gamma_\delta(x) - \gamma(x)|, \ |g_\delta(x) - g(x)|\} < \delta. \tag{65}$$

(Just to be sure, note that $\Omega^1 = \{x \in \mathbb{R}^n \ : \ \text{dist}(x, \Omega) < 1\}$.)

By assumption, we have

$$\nu(x) \cdot \gamma(x) > 0 \quad \text{for all } x \in \partial\Omega.$$

Hence, we may fix $\delta_0 \in (0, 1)$ so that

$$\inf\{\nu_\delta(x) \cdot \gamma_\delta(x) \ : \ x \in (\partial\Omega)^{\delta_0}, \ \delta \in (0, \delta_0)\} > 0.$$

We choose a constant $r > 1$ so that if $\delta \in (0, \delta_0)$, then

$$\begin{cases} \min\{\nu_\delta(x) \cdot \gamma_\delta(x), \ |\gamma_\delta(x)|\} \geq r^{-1}, \\ \max\{|\nu_\delta(x)|, \ |\gamma_\delta(x)|\} \leq r, \\ |g_\delta(x)| + 1 < r. \end{cases} \tag{66}$$

for all $x \in (\partial\Omega)^{\delta_0}$. In particular, we have

$$(\nu_\delta(x), \ \gamma_\delta(x)) \in \mathbb{R}^{2n}_r \quad \text{for all } \ x \in (\partial\Omega)^{\delta_0} \ \text{ and } \ \delta \in (0, \delta_0). \tag{67}$$

To proceed, we fix any $\varepsilon \in (0, 1)$, $M > 0$ and $R > 0$. For each $\delta \in (0, \delta_0)$ we define the function $\psi_\delta \in C^1((\partial\Omega)^{\delta_0} \times \mathbb{R}^n)$ by

$$\psi_\delta(x, \xi) = (g_\delta(x) + \varepsilon) \frac{\gamma_\delta(x) \cdot \xi}{|\gamma_\delta(x)|^2},$$

choose a cut-off function $\eta_\delta \in C^1_0(\mathbb{R}^n)$ so that

$$\begin{cases} \operatorname{supp} \eta_\delta \subset (\partial\Omega)^\delta, \\ 0 \leq \eta_\delta(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^n, \\ \eta_\delta(x) = 1 \quad \text{for all } x \in (\partial\Omega)^{\delta/2}, \end{cases}$$

and define the function $\chi_\delta \in C^1(\Omega^{\delta_0})$ by

$$\chi_\delta(x, \xi) = M\langle\xi\rangle(1 - \eta_\delta(x)) + \eta_\delta(x)\left[\psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi)\right],$$

where Λ and ϕ are the constant and function from Lemma 4.4, $\langle\xi\rangle := (|\xi|^2 + 1)^{1/2}$ and $\phi_\delta(x, \xi) := \phi(\nu_\delta(x), \gamma_\delta(x), \xi)$. Since $\operatorname{supp} \eta_\delta \subset (\partial\Omega)^{\delta_0}$ for all $\delta \in (0, \delta_0)$, in view of (67) we see that χ_δ is well-defined.

Proof (Theorem 4.7). Let $\delta_0 \in (0,\ 1)$ and $\psi_\delta,\ \phi_\delta,\ \chi_\delta \in C^1(\Omega^{\delta_0} \times \mathbb{R}^n)$ be as above.

Let $\delta \in (0, \delta_0)$, which will be fixed later on. It is obvious that for all $(x, \xi) \in (\Omega)^{\delta_0} \times \mathbb{R}^n$,

$$\begin{cases} \gamma_\delta(x) \cdot D_\xi \psi_\delta(x, \xi) = g_\delta(x) + \varepsilon, \\ |\psi_\delta(x, \xi)| \le r^2|\xi|. \end{cases} \tag{68}$$

For any $(x, \xi) \in (\partial\Omega)^\delta \times \mathbb{R}^n$, using (66), (68) and Lemma 4.4, we get

$$\psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi) \ge -r^2|\xi| + (r^2 + M)|\xi| \ge M|\xi|,$$

and

$$\begin{aligned} \psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi) &\le r^2|\xi| + (r^2 + M)\Lambda^2(|\xi| + 1) \\ &\le (2r^2 + M)\Lambda^2(|\xi| + 1). \end{aligned}$$

Thus, we have

$$M|\xi| \le \chi_\delta(x, \xi) \le (2r^2 + M)\Lambda^2(|\xi| + 1) \quad \text{for all } (x, \xi) \in \Omega^\delta \times \mathbb{R}^n. \tag{69}$$

Now, note that if $(x, \xi) \in (\partial\Omega)^{\delta/2} \times \mathbb{R}^n$, then

$$\chi_\delta(x, \xi) = \psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi).$$

Hence, by Lemma 4.4 and (68), we get

$$\gamma_\delta(x) \cdot D_\xi \chi_\delta(x, \xi) = g_\delta(x) + \varepsilon + (r^2 + M)\Lambda \frac{(\nu_\delta(x) \cdot \gamma_\delta(x))(\nu_\delta(x) \cdot \xi)}{\phi_\delta(x, \xi)}$$

for all $(x, \xi) \in (\partial\Omega)^{\delta/2} \times \mathbb{R}^n$.

Next, let $(x, \xi) \in \Omega^\delta \times \mathbb{R}^n$. Since

$$D_\xi \chi_\delta(x, \xi) = M(1 - \eta_\delta(x))D\langle\xi\rangle + \eta_\delta(x)\left[D_\xi \psi_\delta(x, \xi) + (r^2 + M)\Lambda D_\xi \phi_\delta(x, \xi)\right],$$

using Lemma 4.4, we get

$$\begin{aligned} |D_\xi \chi_\delta(x, \xi)| &\le \max\left\{M|D\langle\xi\rangle|,\ \frac{|g_\delta(x) + \varepsilon|}{|\gamma_\delta(x)|} + (r^2 + M)\Lambda|D_\xi \phi_\delta(x, \xi)|\right\} \\ &\le \max\{M,\ r^2 + (r^2 + M)\Lambda^2\} = (2r^2 + M)\Lambda^2. \end{aligned} \tag{70}$$

Let $(x, \xi) \in (\partial\Omega)^{\delta/2} \times B_R$. Note by (65) and (70) that

$$\left|(\gamma_\delta(x) - \gamma(x)) \cdot D_\xi \chi_\delta(x, \xi)\right| \le \delta(2r^2 + M)\Lambda^2.$$

Note also that if $\nu(x)\cdot\xi \le \delta$, then

$$\begin{aligned}(r^2+M)\Lambda &\frac{(\nu_\delta(x)\cdot\gamma_\delta(x))(\nu_\delta(x)\cdot\xi)}{\phi_\delta(x,\xi)}\\ &\le (r^2+M)\Lambda\frac{(\nu_\delta(x)\cdot\gamma_\delta(x))(\nu(x)\cdot\xi)}{\phi_\delta(x,\xi)} + (r^2+M)\Lambda r^2 R\delta\\ &\le (r^2+M)\Lambda r^2\delta(1+R).\end{aligned}$$

Hence, if $\nu(x)\cdot\xi \le \delta$, then

$$\begin{aligned}&\gamma(x)\cdot D_\xi\chi_\delta(x,\xi) \le \gamma_\delta(x)\cdot D_\xi\chi_\delta(x,\xi) + \delta(2r^2+M)\Lambda^2\\ &\le \delta(2r^2+M)\Lambda^2 + g_\delta(x) + \varepsilon + (r^2+M)\Lambda\frac{(\nu_\delta(x)\cdot\gamma_\delta(x))(\nu_\delta(x)\cdot\xi)}{\phi_\delta(x,\xi)}\\ &\le g(x)+\varepsilon+\delta\left[1+(2r^2+M)\Lambda^2 r^2 + (r^2+M)\Lambda r^2(1+R)\right].\end{aligned}$$

Similarly, we see that if $\nu(x)\cdot\xi \ge -\delta$, then

$$\gamma(x)\cdot D_\xi\chi_\delta(x,\xi) \ge g(x)+\varepsilon-\delta\left[1+(2r^2+M)\Lambda^2 r^2 + (r^2+M)\Lambda r^2(1+R)\right].$$

If we select $\delta \in (0,\delta_0)$ so that

$$\delta\left[1+(2r^2+M)\Lambda^2 r^2 + (r^2+M)\Lambda r^2(1+R)\right] \le \frac{\varepsilon}{2},$$

then we have for all $(x,\xi)\in(\partial\Omega)^{\delta/2}\times B_R$,

$$\gamma(x)\cdot D_\xi\chi_\delta(x,\xi)\begin{cases}\le g(x)+2\varepsilon & \text{if } \nu(x)\cdot\xi\le\delta,\\ \ge g(x)+\frac{\varepsilon}{2} & \text{if } \nu(x)\cdot\xi\ge-\delta.\end{cases}$$

Thus, the function $\chi=\chi_\delta$ has the required properties, with $(\partial\Omega)^{\delta/2}$ and $(2r^2+M)\Lambda^2$ in place of U and Λ, respectively. □

We are ready to prove the following theorem.

Theorem 4.8. *Let $\varepsilon>0$ and $u\in \mathrm{Lip}(\overline{\Omega})$ be a viscosity subsolution of* (SNP). *Then there exist a neighborhood U of $\partial\Omega$ and a function $u_\varepsilon \in C^1(\Omega\cup U)$ such that*

$$\begin{cases}H(x,Du_\varepsilon(x))\le\varepsilon & \text{for all } x\in\Omega\cup U,\\ \gamma(x)\cdot Du_\varepsilon(x)\le g(x)+\varepsilon & \text{for all } x\in U,\\ \|u_\varepsilon-u\|_{\infty,\Omega}\le\varepsilon.\end{cases}\tag{71}$$

Proof. Fix any $\varepsilon > 0$ and a constant $M > 1$ so that $M - 1$ is a Lipschitz bound of the function u. With these constants ε and M, let $\Lambda > 0$ be the constant from Theorem 4.7. Set $R = M + 2\Lambda$, and let U, χ and δ be as in Theorem 4.7.

Let $\alpha > 0$. We define the sup-convolution $u_\alpha \in C(\Omega \cup U)$ by

$$u_\alpha(x) = \max_{y\in\overline{\Omega}}(u(y) - \alpha\chi(x, (y-x)/\alpha)).$$

Let $x \in \Omega \cup U$, $p \in D^+u_\alpha(x)$ and $y \in \overline{\Omega}$ be a maximum point in the definition of u_α, that is,

$$u_\alpha(x) = u(y) - \alpha\chi(x, (y-x)/\alpha). \tag{72}$$

It is easily seen that

$$\begin{cases} D_\xi\chi(x, (y-x)/\alpha) \in D^+u(y), \\ p = D_\xi\chi(x, (y-x)/\alpha) - \alpha D_x\chi(x, (y-x)/\alpha). \end{cases} \tag{73}$$

Fix an $\alpha_0 \in (0,\ 1)$ so that

$$\overline{(\partial\Omega)^{\alpha_0^2}} \subset U.$$

Here, of course, $\overline{V}$ denotes the closure of V. For $\alpha \in (0,\ \alpha_0)$ we set $U_\alpha = (\partial\Omega)^{\alpha^2}$ and $V_\alpha = \Omega \cup U_\alpha = \Omega^{\alpha^2}$. Note that $\chi \in C^1(\overline{V}_\alpha \times \mathbb{R}^n)$. We set $W_\alpha = \{(x, y) \in V_\alpha \times \overline{\Omega}\ :\ (72)\ \text{holds}\}$.

Now, we fix any $\alpha \in (0,\ \alpha_0)$. Let $(x, y) \in W_\alpha$. We may choose a point $z \in \overline{\Omega}$ so that $|x - z| < \alpha^2$. Note that

$$u(y) - \alpha\chi(x, (y-x)/\alpha) = u_\alpha(x) \ge u(z) - \alpha\chi(x, (z-x)/\alpha).$$

Hence,

$$\alpha\chi(x, (y-x)/\alpha) \le (M-1)|z-y| + \alpha\chi(x, (z-x)/\alpha).$$

Now, since $M|\xi| \le \chi(x, \xi) \le \Lambda(|\xi| + 1))$ for all $(x, \xi) \in V_\alpha \times \mathbb{R}^n$ and $|x - z| \le \alpha^2 < \alpha$, we get

$$\begin{aligned} M|x-y| &\le (M-1)(|x-y| + \alpha^2) + \alpha\Lambda(|z-x|/\alpha + 1) \\ &\le (M-1)|x-y| + \alpha(M + 2\Lambda). \end{aligned}$$

Consequently,

$$|y - x| \le \alpha(M + 2\Lambda) = R\alpha \quad \text{for all } (x, y) \in W_\alpha. \tag{74}$$

Next, we choose a constant $C > 0$ so that

$$|D_x\chi(x, \xi)| + |D_\xi\chi(x, \xi)| \le C \quad \text{for all } (x, \xi) \in V_{\alpha_0} \times B_R.$$

Let $(x, y) \in W_\alpha$ and $z \in B_{R\alpha}(x) \cap V_{\alpha_0}$. Assume moreover that $x \in U$. In view of (74) and the choice of χ and δ, we have

$$\gamma(x) \cdot D_\xi \chi(x, (y-x)/\alpha) \begin{cases} \leq g(x) + 2\varepsilon & \text{if } \nu(x) \cdot (y-x) \leq \alpha\delta, \\ \geq g(x) + \dfrac{\varepsilon}{2} & \text{if } \nu(x) \cdot (y-x) \geq -\alpha\delta. \end{cases}$$

We observe that

$$\nu(x) \cdot (y-x) \begin{cases} \leq \dfrac{\alpha\delta}{2} + \omega_\nu(R\alpha)R\alpha & \text{if } \nu(z) \cdot (y-x) \leq \dfrac{\alpha\delta}{2}, \\ \geq \dfrac{\alpha\delta}{2} - \omega_\nu(R\alpha)R\alpha & \text{if } \nu(z) \cdot (y-x) \geq -\dfrac{\alpha\delta}{2}, \end{cases}$$

where ω_ν denotes the modulus of continuity of the function ν on V_{α_0}. Observe as well that

$$\begin{aligned} \left|\gamma(z) \cdot D_\xi \chi(x, (y-x)/\alpha) - \gamma(x) \cdot D_\xi \chi(x, (y-x)/\alpha)\right| &\leq C\omega_\gamma(R\alpha), \\ |g(z) - g(x)| &\leq \omega_g(R\alpha), \end{aligned}$$

where ω_γ and ω_g denote the moduli of continuity of the functions γ and g on the set V_{α_0}, respectively.

We may choose an $\alpha_1 \in (0, \alpha_0)$ so that

$$\omega_\nu(R\alpha_1)R < \frac{\delta}{2} \quad \text{and} \quad C\omega_\gamma(R\alpha_1) + \omega_g(R\alpha_1) < \frac{\varepsilon}{4},$$

and conclude from the above observations that for all $(x, y) \in W_\alpha$ and $z_i \in B_{R\alpha}(x) \cap V_{\alpha_0}$, with $i = 1, 2, 3$, if $x \in U$ and $\alpha < \alpha_1$, then

$$\gamma(z_1) \cdot D_\xi \chi(x, (y-x)/\alpha) \begin{cases} \leq g(z_2) + 3\varepsilon & \text{if } \nu(z_3) \cdot (y-x) \leq \alpha\delta/2, \\ \geq g(z_2) + \dfrac{\varepsilon}{4} & \text{if } \nu(z_3) \cdot (y-x) \geq -\alpha\delta/2. \end{cases} \tag{75}$$

We may assume, by reselecting $\alpha_1 > 0$ small enough if necessary, that

$$(\partial\Omega)^{R\alpha_1} \subset U. \tag{76}$$

In what follows we assume that $\alpha \in (0, \alpha_1)$. Let $(x, y) \in W_\alpha$ and $p \in D^+u_\alpha(x)$. By (73) and (74), we have

$$\max\{|p|, |D_\xi \chi(x, (y-x)/\alpha)|\} \leq C(1+\alpha). \tag{77}$$

Let ω_H denote the modulus of continuity of H on $V_{\alpha_0} \times B_{C(1+\alpha_0)}$.

We now assume that $y \in \partial\Omega$. By (74) and (76), we have $x \in U$. Let ρ be a defining function of Ω. We may assume that $|D\rho| \le 1$ in V_{α_0} and $\rho_0 := \inf_{U_{\alpha_0}} |D\rho| > 0$. Observe that

$$\alpha^2 > \rho(x) = \rho(x) - \rho(y) = D\rho(z) \cdot (x - y) = |D\rho(z)|\nu(z) \cdot (x - y)$$

for some point z on the line segment $[x, y]$. Hence, we get

$$\nu(z) \cdot (x - y) \le \rho_0^{-1}\alpha^2.$$

If $\alpha \le \rho_0\delta/2$, then

$$\nu(z) \cdot (y - x) \ge -\alpha\delta/2.$$

Hence, noting that $|z - x| \le |x - y| < R\alpha$, by (75), we get

$$\gamma(y) \cdot D_\xi \chi(x, (y - x)/\alpha) \ge g(y) + \frac{\varepsilon}{4},$$

and, by the viscosity property of u,

$$0 \ge H(y, D_\xi \chi(x, (y - x)/\alpha)) \ge H(x, p) - \omega_H((R + C)\alpha).$$

Thus, if $\omega_H((R + C)\alpha) < \varepsilon$ and $\alpha \le \rho_0\delta/2$, then we have

$$H(x, p) \le \varepsilon.$$

On the other hand, if $y \in \Omega$, then, by the viscosity property of u, we have

$$0 \ge H(y, D_\xi \chi(x, (y - x)/\alpha)).$$

Therefore, if $\omega_H((R + C)\alpha) < \varepsilon$, then

$$H(x, p) \le \varepsilon.$$

We may henceforth assume by selecting $\alpha_1 > 0$ small enough that

$$\omega_H((R + C)\alpha_1) < \varepsilon \quad \text{and} \quad \alpha_1 \le \rho_0\delta/2,$$

and we conclude that u_α is a viscosity subsolution of

$$H(x, Du_\alpha(x)) \le \varepsilon \quad \text{in } V_\alpha. \tag{78}$$

As above, let $(x, y) \in W_\alpha$ and $p \in D^+u_\alpha(x)$. We assume that $x \in U_\alpha$. Then

$$-\alpha^2 < \rho(x) \le \rho(x) - \rho(y) \le D\rho(z) \cdot (x - y)$$

for some $z \in [x, y]$, which yields

$$\nu(z) \cdot (y - x) < |D\rho(z)|^{-1}\alpha^2 \le \rho_0^{-1}\alpha^2.$$

Hence, if $\alpha \le \rho_0\delta/2$, then

$$\nu(z) \cdot (y - x) \le \frac{\delta\alpha}{2},$$

and, by (75), we get

$$\gamma(x) \cdot D_\xi \chi(x, (y - x)/\alpha) \le g(x) + 3\varepsilon.$$

Furthermore,

$$\begin{aligned}\gamma(x) \cdot p &\le \gamma(x) \cdot D_\xi \chi(x, (y - x)/\alpha) + \alpha C \|\gamma\|_{\infty, U_{\alpha_0}} \\ &\le g(x) + 3\varepsilon + \alpha C \|\gamma\|_{\infty, U_{\alpha_0}}.\end{aligned}$$

We may assume again by selecting $\alpha_1 > 0$ small enough that

$$\alpha_1 C \|\gamma\|_{\infty, U_{\alpha_0}} < \varepsilon.$$

Thus, u_α is a viscosity subsolution of

$$\gamma(x) \cdot Du_\alpha(x) \le g(x) + 4\varepsilon \quad \text{in } U_\alpha. \tag{79}$$

Let $(x, y) \in W_\alpha$ and observe by using (74) that if $x \in \overline{\Omega}$, then

$$|u(x) - u_\alpha(x)| \le |u(x) - u(y)| + \alpha|\chi(x, (y - x)/\alpha)| \le \alpha(MR + C).$$

We fix $\alpha \in (0, \alpha_1)$ so that $\alpha_1(MR + C) < \varepsilon$, and conclude that u_α is a viscosity subsolution of (78) and (79) and satisfies

$$\|u_\alpha - u\|_{\infty, \Omega} \le \varepsilon.$$

The final step is to mollify the function u_α. Let $\{k_\lambda\}_{\lambda>0}$ be a collection of standard mollification kernels.

We note by (77) or (78) that u_α is Lipschitz continuous on any compact subset of V_α. Fix any $\lambda \in (0, \alpha^2/4)$. We note that the closure of $V_{\alpha/2} + B_\lambda$ is a compact subset of V_α. Let $M_1 > 0$ be a Lipschitz bound of the function u_α on $V_{\alpha/2} + B_\lambda$.

We set

$$u^\lambda(x) = u_\alpha * k_\lambda(x) \quad \text{for } x \in V_{\alpha/2}.$$

In view of Rademacher's theorem (see Appendix A.6), we have

$$\begin{aligned}&H(x, Du_\alpha(x)) \le \varepsilon && \text{for a.e. } x \in V_\alpha, \\ &\gamma(x) \cdot Du_\alpha(x) \le g(x) + 4\varepsilon && \text{for a.e. } x \in U_\alpha.\end{aligned}$$

Here Du_α denotes the distributional derivative of u_α, and we have

$$Du^\lambda = k_\lambda * Du_\alpha \quad \text{in } V_{\alpha/2}.$$

By Jensen's inequality, we get

$$\begin{aligned} H(x, Du^\lambda(x)) &\le \int_{B_\lambda} H(x, Du_\alpha(x-y))k_\lambda(y)\,\mathrm{d}y \\ &\le \int_{B_\lambda} H(x-y, Du_\alpha(x-y))k_\lambda(y)\,\mathrm{d}y + \omega_H(\lambda) \\ &\le \varepsilon + \omega_H(\lambda), \end{aligned}$$

where ω_H is the modulus of continuity of H on the set $V_\alpha \times B_{M_1}$. Similarly, we get

$$\gamma(x)\cdot Du^\lambda(x) \le g(x) + 4\varepsilon + \omega_g(\lambda) + M_1\omega_\gamma(\lambda),$$

where ω_g and ω_γ are the moduli of continuity of the functions g and γ on V_α, respectively. If we choose $\lambda > 0$ small enough, then (71) holds with $u^\lambda \in C^1(V_{\alpha/2})$, $U_{\alpha/2}$ and 5ε in place of u_ε, U and ε, respectively. The proof is complete. □

Proof (Theorem 4.2). Let $\varepsilon > 0$ and $u \in \mathrm{Lip}(\overline{\Omega})$ be a viscosity subsolution of (SNP). Let ρ be a defining function of Ω. We may assume that

$$D\rho(x)\cdot\gamma(x) \ge 1 \quad \text{for all } x \in \partial\Omega.$$

For $\delta > 0$ we set

$$u^\delta(x) = u(x) - \delta\rho(x) \quad \text{for } x \in \overline{\Omega}.$$

It is easily seen that if $\delta > 0$ is small enough, then u^δ is a viscosity subsolution of

$$\begin{cases} H(x, Du^\delta(x)) \le \varepsilon & \text{in } \Omega, \\ \gamma(x)\cdot Du^\delta(x) \le g(x) - \delta & \text{on } \partial\Omega, \end{cases}$$

and the following inequality holds:

$$\|u^\delta - u\|_{\infty,\Omega} \le \varepsilon.$$

Then, Theorem 4.8, with $\min\{\varepsilon,\delta\}$, u^δ, $H-\varepsilon$ and $g-\delta$ in place of ε, u, H and g, respectively, ensures that there are a neighborhood U of $\partial\Omega$ and a function $u_\varepsilon \in C^1(\Omega \cup U)$ such that

$$\begin{cases} H(x, Du_\varepsilon(x)) \le 2\varepsilon & \text{in } \Omega \cup U, \\ \gamma(x) \cdot Du_\varepsilon(x) \le g(x) & \text{in } U, \\ \|u_\varepsilon - u\|_{\infty,\Omega} \le 2\varepsilon, \end{cases}$$

which concludes the proof. □

5 Optimal Control Problem Associated with (ENP)–(ID)

In this section we introduce an optimal control problem associated with the initial-boundary value problem (ENP)–(ID),

5.1 *Skorokhod Problem*

In this section, following [39, 44], we study the Skorokhod problem. We recall that $\mathbb{R}_+$ denotes the interval $(0, \infty)$, so that $\overline{\mathbb{R}}_+ = [0, \infty)$. We denote by $L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^k)$ (resp., $\mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^k)$) the space of functions $v : \overline{\mathbb{R}}_+ \to \mathbb{R}^k$ which are integrable (resp., absolutely continuous) on any bounded interval $J \subset \overline{\mathbb{R}}_+$.

Given $x \in \overline{\Omega}$ and $v \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$, the Skorokhod problem is to seek for a pair of functions, $(\eta, l) \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R})$, such that

$$\begin{cases} \eta(0) = x, & \\ \eta(t) \in \overline{\Omega} & \text{for all } t \in \overline{\mathbb{R}}_+, \\ \dot{\eta}(t) + l(t)\gamma(\eta(t)) = v(t) & \text{for a.e. } t \in \overline{\mathbb{R}}_+, \\ l(t) \ge 0 & \text{for a.e. } t \in \overline{\mathbb{R}}_+, \\ l(t) = 0 \ \text{ if } \eta(t) \in \Omega & \text{for a.e. } t \in \overline{\mathbb{R}}_+. \end{cases} \tag{80}$$

Regarding the solvability of the Skorokhod problem, our main claim is the following.

Theorem 5.1. *Let $v \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ and $x \in \overline{\Omega}$. Then there exits a pair $(\eta, l) \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R})$ such that (80) holds.*

We refer to [44] and references therein for more general viewpoints (especially, for applications to stochastic differential equations with reflection) on the Skorokhod problem.

A natural question arises whether uniqueness of the solution (η, l) holds or not in the above theorem. On this issue we just give the following counterexample and do not discuss it further.

Example 5.1. Let $n = 2$ and $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$. (For simplicity of presentation, we consider the case where Ω is unbounded.) Define $\gamma \in C(\partial\Omega, \mathbb{R}^2)$ and $v \in L^\infty(\overline{\mathbb{R}}_+, \mathbb{R}^2)$ by

$$\gamma(0, x_2) = (-1, -3|x_2|^{-1/3}x_2) \quad \text{and} \quad v(t) = (-1, 0).$$

Set

$$\eta^\pm(t) = (0, \pm t^3) \quad \text{for all } t \geq 0.$$

Then the pairs $(\eta^+, 1)$ and $(\eta^-, 1)$ are both solutions of (80), with $\eta^\pm(0) = (0, 0)$.

We first establish the following assertion.

Theorem 5.2. *Let $v \in L^\infty(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ and $x \in \overline{\Omega}$. Then there exits a pair $(\eta, l) \in$* $\mathrm{Lip}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^\infty(\overline{\mathbb{R}}_+, \mathbb{R})$ *such that* (80) *holds.*

Proof. We may assume that γ is defined and continuous on $\mathbb{R}^n$. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. We may assume that $\liminf_{|x|\to\infty} \rho(x) > 0$ and that $D\rho$ is bounded on $\mathbb{R}^n$. We may select a constant $\delta > 0$ so that for all $x \in \mathbb{R}^n$,

$$\gamma(x) \cdot D\rho(x) \geq \delta|D\rho(x)| \quad \text{and} \quad |D\rho(x)| \geq \delta \quad \text{if } 0 \leq \rho(x) \leq \delta.$$

We set $q(x) = (\rho(x) \vee 0) \wedge \delta$ for $x \in \mathbb{R}^n$ and observe that $q(x) = 0$ for all $x \in \overline{\Omega}$ and $q(x) > 0$ for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$.

Fix $\varepsilon > 0$ and $x \in \overline{\Omega}$. We consider the initial value problem for the ODE

$$\dot{\xi}(t) + \frac{1}{\varepsilon}q(\xi(t))\gamma(\xi(t)) = v(t) \ \text{ for a.e. } t \in \mathbb{R}_+, \quad \xi(0) = x. \tag{81}$$

By the standard ODE theory, there is a solution $\xi \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$ of (81). Fix such a solution $\xi \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ in what follows.

Note that $(\mathrm{d}\, q \circ \xi/\mathrm{d}t)(t) = D\rho(\xi(t)) \cdot \dot{\xi}(t)$ a.e. in the set $\{t \in \mathbb{R}_+ : \rho \circ \xi(t) \in (0, \delta)\}$. Moreover, noting that $q \circ \xi \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \mathbb{R})$ and hence it is differentiable a.e., we deduce that $(\mathrm{d}\, q \circ \xi/\mathrm{d}t)(t) = 0$ a.e. in the set $\{t \in \mathbb{R}_+ : \rho \circ \xi(t) \in \{0, \delta\}\}$.

Let $m \geq 2$. We multiply the ODE of (81) by $mq(\xi(t))^{m-1}D\rho(\xi(t))$, to get

$$\frac{\mathrm{d}}{\mathrm{d}t}q(\xi(t))^m + \frac{m}{\varepsilon}q(\xi(t))^m Dq(\xi(t)) \cdot \gamma(\xi(t)) = mq(\xi(t))^{m-1}Dq(\xi(t)) \cdot v(t)$$

a.e. in the set $\{t \in \mathbb{R}_+ : \rho \circ \xi(t) \in (0, \delta)\}$. For any $T \in \mathbb{R}_+$, integration over $E_T := \{t \in [0, T] : \rho \circ \xi(t) \in (0, \delta)\}$ yields

$$\begin{aligned} q(\xi(T))^m - q(\xi(0))^m + \frac{m}{\varepsilon}\int_{E_T} q(\xi(s))^m \gamma(\xi(s)) \cdot D\rho(\xi(s))\mathrm{d}s \\ = m\int_{E_T} q(\xi(s))^{m-1}D\rho(\xi(s)) \cdot v(s)\mathrm{d}s. \end{aligned}$$

Here we note

$$\int_{E_T} q(\xi(s))^m \gamma(\xi(s)) \cdot D\rho(\xi(s))\mathrm{d}s \geq \delta \int_{E_T} q(\xi(s))^m |D\rho(\xi(s))|\mathrm{d}s,$$

and

$$\begin{aligned}\int_{E_T} q(\xi(s))^{m-1} D\rho(\xi(s)) \cdot v(s)\mathrm{d}s \\ \leq \left(\int_{E_T} q(\xi(s))^m |D\rho(\xi(s)|\mathrm{d}s\right)^{1-\frac{1}{m}} \left(\int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}}.\end{aligned}$$

Combining these, we get

$$\begin{aligned}q(\xi(T))^m + \frac{m\delta}{\varepsilon}\int_{E_T} q(\xi(s))^m |D\rho(\xi(s))|\mathrm{d}s \\ \leq m\left(\int_{E_T} q(\xi(s))^m |D\rho(\xi(s)|\mathrm{d}s\right)^{1-\frac{1}{m}} \left(\int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}}.\end{aligned}$$

Hence,

$$\frac{\delta}{\varepsilon}\left(\int_{E_T} q(\xi(s))^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}} \leq \left(\int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}}$$

and

$$q(\xi(T))^m \leq \left(\frac{\varepsilon}{\delta}\right)^{m-1} m \int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s.$$

Thus, setting $C_0 = \|D\rho\|_{L^\infty(\mathbb{R}^n)}$, we find that for any $T \in \mathbb{R}_+$,

$$q(\xi(t))^m \leq \left(\frac{\varepsilon}{\delta}\right)^{m-1} mC_0T\|v\|^m_{L^\infty(0,T)} \quad \text{for all } t \in [0,\ T]. \tag{82}$$

We henceforth write ξ_ε for ξ, in order to indicate the dependence on ε of ξ, and observe from (82) that for any $T > 0$,

$$\lim_{\varepsilon \to 0+} \max_{t \in [0,\ T]} \mathrm{dist}(\xi_\varepsilon(t),\ \Omega) = 0. \tag{83}$$

Also, (82) ensures that for any $T > 0$,

$$\frac{\delta}{\varepsilon}\|q \circ \xi_\varepsilon\|_{L^\infty(0,T)} \leq \left(\frac{\delta m C_0 T}{\varepsilon}\right)^{\frac{1}{m}} \|v\|_{L^\infty(0,T)}.$$

Sending $m \to \infty$, we find that $(\delta/\varepsilon)\|q \circ \xi_\varepsilon\|_{L^\infty(0,T)} \le \|v\|_{L^\infty(0,T)}$, and moreover

$$\frac{\delta}{\varepsilon}\|q \circ \xi_\varepsilon\|_{L^\infty(\mathbb{R}_+)} \le \|v\|_{L^\infty(\mathbb{R}_+)}. \tag{84}$$

We set $l_\varepsilon = (1/\varepsilon)q \circ \xi_\varepsilon$. Thanks to (84), we may choose a sequence $\varepsilon_j \to 0+$ (see Lemma E.1) so that $l_{\varepsilon_j} \to l$ weakly-star in $L^\infty(\mathbb{R}_+)$ as $j \to \infty$ for a function $l \in L^\infty(\mathbb{R}_+)$. It is clear that $l(s) \ge 0$ for a.e. $s \in \mathbb{R}_+$.

ODE (81) together with (84) guarantees that $\{\dot\xi_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}_+)$. Hence, we may assume as well that ξ_{ε_j} converges locally uniformly on $\overline{\mathbb{R}}_+$ to a function $\eta \in \mathrm{Lip}(\mathbb{R}_+)$ as $j \to \infty$. It is then obvious that $\eta(0) = x$ and the pair (η, l) satisfies

$$\eta(t) + \int_0^t \big(l(s)\gamma(\eta(s)) - v(s)\big)\mathrm{d}s = x \quad \text{for all } t \in \mathbb{R}_+,$$

from which we get

$$\dot\eta(t) + l(t)\gamma(\eta(t)) = v(t) \quad \text{for a.e. } t \in \mathbb{R}_+.$$

It follows from (83) that $\eta(t) \in \overline{\Omega}$ for $t \ge 0$.

In order to show that the pair (η, l) is a solution of (80), we need only to prove that for a.e. $t \in \mathbb{R}_+$, $l(t) = 0$ if $\eta(t) \in \Omega$. Set $A = \{t \ge 0 \,:\, \eta(t) \in \Omega\}$. It is clear that A is an open subset of $[0, \infty)$. We can choose a sequence $\{I_k\}_{k\in\mathbb{N}}$ of closed finite intervals of A such that $A = \bigcup_{k\in\mathbb{N}} I_k$. Note that for each $k \in \mathbb{N}$, the set $\eta(I_k)$ is a compact subset of Ω and the convergence of $\{\xi_{\varepsilon_j}\}$ to η is uniform on I_k. Hence, for any fixed $k \in \mathbb{N}$, we may choose $J \in \mathbb{N}$ so that $\xi_{\varepsilon_j}(t) \in \Omega$ for all $t \in I_k$ and $j \ge J$. From this, we have $q(\xi_{\varepsilon_j}(t)) = 0$ for $t \in I_k$ and $j \ge J$. Moreover, in view of the weak-star convergence of $\{l_{\varepsilon_j}\}$, we find that for any $k \in \mathbb{N}$,

$$\int_{I_k} l(t)\mathrm{d}t = \lim_{j\to\infty}\int_{I_k}\frac{1}{\varepsilon_j}q(\xi_j(t))\mathrm{d}t = 0,$$

which yields $l(t) = 0$ for a.e. $t \in I_k$. Since $A = \bigcup_{k\in\mathbb{N}} I_k$, we see that $l(t) = 0$ a.e. in A. The proof is now complete. □

For $x \in \overline{\Omega}$, let $\mathrm{SP}(x)$ denote the set of all triples

$$(\eta, v, l) \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$$

which satisfies (80). We set $\mathrm{SP} = \bigcup_{x\in\overline{\Omega}} \mathrm{SP}(x)$.

We remark that for any $x, y \in \overline{\Omega}$ and $T \in \mathbb{R}_+$, there exists a triple $(\eta, v, l) \in \mathrm{SP}(x)$ such that $\eta(T) = y$. Indeed, given $x, y \in \overline{\Omega}$ and $T \in \mathbb{R}_+$, we choose a curve $\eta \in \mathrm{Lip}([0,\, T], \overline{\Omega})$ (see Lemma 2.1) so that $\eta(0) = x$, $\eta(T) = y$ and $\eta(t) \in \overline{\Omega}$ for all $t \in [0, T]$. We extend the domain of definition of η to $\overline{\mathbb{R}}_+$ by setting $\eta(t) = y$

for $t > T$. If we set $v(t) = \dot\eta(t)$ and $l(t) = 0$ for $t \geq 0$, we have $(\eta, v, l) \in \mathrm{SP}(x)$, which has the property, $\eta(T) = y$.

We note also that problem (80) has the following *semi-group* property: for any $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$ and (η_1, v_1, l_1), $(\eta_2, v_2, l_2) \in \mathrm{SP}$, if $\eta_1(0) = x$ and $\eta_2(0) = \eta_1(t)$ hold and if (η, v, l) is defined on $\overline{\mathbb{R}}_+$ by

$$(\eta(s), v(s), l(s)) = \begin{cases} (\eta_1(s), v_1(s), l_1(s)) & \text{for } s \in [0, t), \\ (\eta_2(s-t), v_2(s-t), l_2(s-t)) & \text{for } s \in [t, \infty), \end{cases}$$

then $(\eta, v, l) \in \mathrm{SP}(x)$.

The following proposition concerns a stability property of sequences of points in SP.

Proposition 5.1. *Let* $\{(\eta_k, v_k, l_k)\}_{k\in\mathbb{N}} \subset \mathrm{SP}$. *Let* $x \in \overline{\Omega}$ *and* $(w, v, l) \in L_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^{2n+1})$. *Assume that as* $k \to \infty$*,*

$$\eta_k(0) \to x,$$
$$(\dot\eta_k, v_k, l_k) \to (w, v, l) \quad \textit{weakly in } L^1([0, T], \mathbb{R}^{2n+1})$$

for every $T \in \mathbb{R}_+$. *Set*

$$\eta(s) = x + \int_0^s w(r)\mathrm{d}r \quad \textit{for } s \geq 0.$$

Then $(\eta, v, l) \in \mathrm{SP}(x)$.

Proof. For all $t > 0$ and $k \in \mathbb{N}$, we have

$$\eta_k(t) = \eta_k(0) + \int_0^t \dot\eta_k(s)\mathrm{d}s = \eta_k(0) + \int_0^t (v_k(s) - l_k(s)\gamma(\eta_k(s)))\,\mathrm{d}s.$$

First, we observe that as $k \to \infty$,

$$\eta_k(t) \to \eta(t) \quad \text{locally uniformly on } \overline{\mathbb{R}}_+,$$

and then we get in the limit as $k \to \infty$,

$$\eta(t) = x + \int_0^t (v(s) - l(s)\gamma(\eta(s)))\,\mathrm{d}s \quad \text{for all } t > 0.$$

This shows that $\eta \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ and

$$\dot\eta(s) + l(s)\gamma(\eta(s)) = v(s) \quad \text{for a.e. } s \in \mathbb{R}_+.$$

It is clear that $\eta(0) = x$, $\eta(s) \in \overline{\Omega}$ for all $s \in \mathbb{R}_+$ and $l(s) \geq 0$ for a.e. $s \in \mathbb{R}_+$.

To show that $(\eta, v, l) \in \mathrm{SP}(x)$, it remains to prove that for a.e. $t \in \mathbb{R}_+$, $l(t) = 0$ if $\eta(t) \in \Omega$. As in the last part of the proof of Theorem 5.2, we set $A = \{t \ge 0 : \eta(t) \in \Omega\}$ and choose a sequence $\{I_j\}_{j\in\mathbb{N}}$ of closed finite intervals of A such that $A = \bigcup_{j\in\mathbb{N}} I_j$. Fix any $j \in \mathbb{N}$ and choose $K \in \mathbb{N}$ so that $\eta_k(t) \in \Omega$ for all $t \in I_j$ and $k \ge K$. From this, we have $l_k(t) = 0$ for a.e. $t \in I_j$ and $k \ge K$. Moreover, in view of the weak convergence of $\{l_k\}$, we find that

$$\int_{I_j} l(t)\mathrm{d}t = \lim_{k\to\infty} \int_{I_j} l_k(t)\mathrm{d}t = 0,$$

which yields $l(t) = 0$ for a.e. $t \in I_j$. Since j is arbitrary, we see that $l(t) = 0$ a.e. in $A = \bigcup_{j\in\mathbb{N}} I_j$. □

Proposition 5.2. *There is a constant $C > 0$, depending only on Ω and γ, such that for all $(\eta, v, l) \in \mathrm{SP}$,*

$$|\dot\eta(s)| \vee l(s) \le C|v(s)| \quad \textit{for a.e. } s \ge 0.$$

An immediate consequence of the above proposition is that for $(\eta, v, l) \in \mathrm{SP}$, if $v \in L^p(\mathbb{R}_+, \mathbb{R}^n)$ (resp., $v \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$), with $1 \le p \le \infty$, then $(\dot\eta, l) \in L^p(\mathbb{R}_+, R^{n+1})$ (resp., $(\dot\eta, l) \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^{n+1})$).

Proof. Thanks to hypothesis (A4), there is a constant $\delta_0 > 0$ such that $\nu(x)\cdot\gamma(x) \ge \delta_0$ for $x \in \partial\Omega$. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω.

Let $s \in \mathbb{R}_+$ be such that $\eta(s) \in \partial\Omega$, η is differentiable at s, $l(s) \ge 0$ and $\dot\eta(s) + l(s)\gamma(\eta(s)) = v(s)$. Observe that the function $\rho\circ\eta$ attains a maximum at s. Hence,

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}s}\rho(\eta(s)) &= D\rho(\eta(s))\cdot\dot\eta(s) = |D\rho(\eta(s))|\nu(\eta(s))\cdot\dot\eta(s) \\ &= |D\rho(\eta(s))|\nu(\eta(s))\cdot\big(v(s) - l(s)\gamma(\eta(s))\big) \\ &\le |D\rho(\eta(s))|\big(\nu(\eta(s))\cdot v(s) - l(s)\delta_0\big). \end{aligned}$$

Thus, we get

$$l(s) \le \delta_0^{-1}\nu(\eta(s))\cdot v(s) \le \delta_0^{-1}|v(s)|$$

and

$$\begin{aligned} |\dot\eta(s)| &= |v(s) - l(s)\gamma(\eta(s))| \le |v(s)| + l(s)|\|\gamma\|_{\infty,\partial\Omega} \\ &\le (1 + \delta_0^{-1}\|\gamma\|_{\infty,\partial\Omega})|v(s)|, \end{aligned}$$

which completes the proof. □

5.2 Value Function I

We define the function $L \in \mathrm{LSC}(\overline{\Omega} \times \mathbb{R}^n,\ (-\infty, \infty])$, called the *Lagrangian* of H, by

$$L(x, \xi) = \sup_{p \in \mathbb{R}^n} \big(\xi \cdot p - H(x, p)\big).$$

For each x the function $\xi \mapsto L(x, \xi)$ is the convex conjugate of the function $p \mapsto H(x, p)$. See Appendix A.2 for properties of conjugate convex functions.

We consider the optimal control with the dynamics given by (80), the running cost $(L,\ g)$ and the pay-off u_0, and its value function V on $\overline{Q}$, where $Q = \overline{\Omega} \times \mathbb{R}_+$, is given by

$$\begin{aligned} V(x, t) = \inf \Big\{ \int_0^t \big(L(\eta(s), -v(s)) + g(\eta(s))l(s)\big)\mathrm{d}s \\ + u_0(\eta(t))\ :\ (\eta, v, l) \in \mathrm{SP}(x) \Big\} \quad \text{for } (x, t) \in Q, \end{aligned} \tag{85}$$

and $V(x, 0) = u_0(x)$ for all $x \in \overline{\Omega}$.

For $t > 0$ and $(\eta, v, l) \in \mathrm{SP} = \bigcup_{x \in \overline{\Omega}} \mathrm{SP}(x)$, we write

$$\mathscr{L}(t, \eta, v, l) = \int_0^t \big(L(\eta(s), -v(s)) + g(\eta(s))l(s)\big)\mathrm{d}s$$

for notational simplicity, and then formula (85) reads

$$V(x, t) = \inf \big\{ \mathscr{L}(t, \eta, v, l) + u_0(\eta(t))\ :\ (\eta, v, l) \in \mathrm{SP}(x) \big\}.$$

Under our hypotheses, the Lagrangian L may take the value ∞ and, on the other hand, if we set $C = \min_{x \in \overline{\Omega}}(-H(x, 0))$, then we have

$$L(x, \xi) \geq C \quad \text{for all } \ (x,\ \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Thus, it is reasonable to interpret

$$\int_0^t L(\eta(s), -v(s))\mathrm{d}s = \infty$$

if the function: $s \mapsto L(\eta(s), -v(s))$ is not integrable, which we adopt here.

It is easily checked as in the proof of Proposition 1.3 that the value function V satisfies the dynamic programming principle: given a point $(x, t) \in Q$ and a nonanticipating mapping $\tau\ :\ \mathrm{SP}(x) \to [0,\ t]$, we have

$$V(x, t) = \inf \big\{ \mathscr{L}(\tau(\alpha), \alpha) + V(\eta(\tau(\alpha)),\ t - \tau(\alpha))\ :\ \alpha = (\eta, v, l) \in \mathrm{SP}(x) \big\}. \tag{86}$$

Here a mapping $\tau : \mathrm{SP}(x) \to [0,\, t]$ is called *nonanticipating* if $\tau(\alpha) = \tau(\beta)$ whenever $\alpha(s) = \beta(s)$ a.e. in the interval $[0,\, \tau(\alpha)]$.

We here digress to recall the state-constraint problem, whose Bellman equation is given by the Hamilton–Jacobi equation

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

and to make a comparison between (ENP) and the state-constraint problem. For $x \in \overline{\Omega}$ let $\mathrm{SC}(x)$ denote the collection of all $\eta \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ such that $\eta(0) = x$ and $\eta(s) \in \overline{\Omega}$ for all $s \in \overline{\mathbb{R}}_+$. The value function $\hat{V} : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ of the state-constraint problem is given by

$$\hat{V}(x,t) = \inf\Big\{ \int_0^t L(\eta(s), -\dot{\eta}(s))\mathrm{d}s + u_0(\eta(t)) \;:\; \eta \in \mathrm{SC}(x) \Big\}.$$

Observe that if $\eta \in \mathrm{SC}(x)$, with $x \in \overline{\Omega}$, then $(\eta, \dot{\eta}, 0) \in \mathrm{SP}(x)$. Hence, we have

$$\begin{aligned} \hat{V}(x,t) &= \inf\big\{ \mathscr{L}(t, \eta, \dot{\eta}, 0) + u_0(\eta(t)) \;:\; \eta \in \mathrm{SC}(x) \big\} \\ &\geq V(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_+. \end{aligned}$$

Heuristically it is obvious that if $g(x) \approx \infty$, then

$$V(x,t) \approx \hat{V}(x,t).$$

In terms of PDE the above state-constraint problem is formulated as follows: the value function $\hat{V}$ is a unique viscosity solution of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \leq 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u_t(x,t) + H(x, D_x u(x,t)) \geq 0 & \text{in } \overline{\Omega} \times \mathbb{R}_+. \end{cases}$$

See [48] for a proof of this result in this generality. We refer to [17, 55] for state-constraint problems. The corresponding additive eigenvalue problem is to find $(a, v) \in \mathbb{R} \times C(\overline{\Omega})$ such that v is a viscosity solution of

$$\begin{cases} H(x, Dv(x)) \leq a & \text{in } \Omega, \\ H(x, Dv(x)) \geq a & \text{in } \overline{\Omega}. \end{cases} \tag{87}$$

We refer to [17, 40, 48] for this eigenvalue problem.

Example 5.2. We recall (see [48]) that the additive eigenvalue $\hat{c}$ for (87) is given by

$$\hat{c} = \inf\{a \in \mathbb{R} \;:\; (87) \text{ has a viscosity subsolution } v\},$$

For a comparison between the Neumann problem and the state-constraint problem, we go back to the situation of Example 3.1. Then it is easy to see that $\hat{c} = 0$. Thus, we have $c^\# = \hat{c} = 0$ if and only if $\min\{g(-1), g(1)\} \geq 0$.

We here continue the above example with some more generality. Let $c^\#$ and $\hat{c}$ denote, as above, the eigenvalues of (EVP) and (87), respectively. It is easily seen that if $\psi \in C(\bar{\Omega})$ is a subsolution of (EVP) with $a = c^\#$, then it is also a subsolution of (87) with $a = c^\#$, which ensures that $\hat{c} \leq c^\#$.

Next, note that the subsolutions of (87) with $a = \hat{c}$ are equi-Lipschitz continuous on $\bar{\Omega}$. That is, there exists a constant $M > 0$ such that for any subsolution ψ of (87) with $a = \hat{c}$, $|\psi(x) - \psi(y)| \leq M|x - y|$ for all $x, y \in \bar{\Omega}$. Let ψ be any subsolution of (87) with $a = \hat{c}$, $y \in \partial\Omega$ and $p \in D^+\psi(y)$. Choose a $\phi \in C^1(\bar{\Omega})$ so that $D\phi(y) = p$ and $\psi - \phi$ has a maximum at y. If $t > 0$ is sufficiently small, then we have $y - t\gamma(y) \in \Omega$ and, moreover, $\psi(y - t\gamma(y)) - \psi(y) \leq \phi(y - t\gamma(y)) - \phi(y)$. By the last inequality, we deduce that $\gamma(y) \cdot p \leq M|\gamma(y)|$. Accordingly, we have $\gamma(y) \cdot p \leq M|\gamma(y)|$ for all $p \in D^+\psi(y)$. Thus, we see that if $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$, then any subsolution ψ of (87) with $a = \hat{c}$ is a subsolution of (EVP) with $a = \hat{c}$. This shows that if $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$, then $c^\# \leq \hat{c}$. As we have already seen above, we have $\hat{c} \leq c^\#$, and, therefore, $c^\# = \hat{c}$, provided that $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$.

Now, assume that $c^\# = \hat{c}$ and let $a = c^\# = \hat{c}$. It is easily seen that

$$\{\psi \ : \ \psi \text{ is a subsolution of (EVP)}\} \subset \{\psi \ : \ \psi \text{ is a subsolution of (87)}\},$$

which guarantees that $d_N \leq d_S$ on $\overline{\Omega}^2$, where $d_N(\cdot, y) = \sup \mathscr{F}_y^N$, $d_S(\cdot, y) = \sup \mathscr{F}_y^S$, and

$$\mathscr{F}_y^N \text{ (resp., } \mathscr{F}_y^S) = \{\psi - \psi(y) \ : \ \psi \text{ is a subsolution of (EVP) (resp., (87))}\}.$$

Let $\mathscr{A}_N$ and $\mathscr{A}_S$ denote the Aubry sets associated with (EVP) and (87), respectively. That is,

$$\mathscr{A}_N = \{y \in \overline{\Omega} \ : \ d_N(\cdot, y) \text{ is a solution of (EVP)}\},$$

$$\mathscr{A}_S = \{y \in \overline{\Omega} \ : \ d_S(\cdot, y) \text{ is a solution of (87)}\}.$$

The above inequality and the fact that $d_N(y, y) = d_S(y, y) = 0$ for all $y \in \overline{\Omega}$ imply that $D_x^- d_N(x, y)|_{x=y} \subset D_x^- d_S(x, y)|_{x=y}$. From this inclusion, we easily deduce that $\mathscr{A}_S \subset \mathscr{A}_N$.

Thus the following proposition holds.

Proposition 5.3. *With the above notation, we have:*

(i) $\hat{c} \leq c^\#$.
(ii) *If $M > 0$ is a Lipschitz bound of the subsolutions of* (87) *with $a = \hat{c}$ and $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$, then $\hat{c} = c^\#$.*
(iii) *If $\hat{c} = c^\#$, then $d_N \leq d_S$ on $\bar{\Omega}^2$ and $\mathscr{A}_S \subset \mathscr{A}_N$.*

5.3 Basic Lemmas

In this subsection we present a proof of the sequential lower semicontinuity of the functional $(\eta, v, l) \mapsto \mathscr{L}(T, \eta, v, l)$ (see Theorem 5.3 below). We will prove an existence result (Theorem 5.6) for the variational problem involving the functional $\mathscr{L}$ in Sect. 5.4. These results are variations of Tonelli's theorem in variational problems. For a detailed description of the theory of one-dimensional variational problems, with a central focus on Tonelli's theorem, we refer to [14].

Lemma 5.1. *For each $A > 0$ there exists a constant $C_A \geq 0$ such that*

$$L(x,\ \xi) \geq A|\xi| - C_A \quad \textit{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Proof. Fix any $A > 0$ and observe that

$$\begin{aligned} L(x, \xi) &\geq \max_{p \in \overline{B}_A} (\xi \cdot p - H(x, p)) \\ &\geq A|\xi| + \min_{p \in \overline{B}_A} (-H(x, p)) \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \end{aligned}$$

Hence, setting $C_A \geq \max_{\overline{\Omega} \times \overline{B}_A} |H|$, we get

$$L(x, \xi) \geq A|\xi| - C_A \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \qquad \square$$

Lemma 5.2. *There exist constants $\delta > 0$ and $C_0 > 0$ such that*

$$L(x, \xi) \leq C_0 \quad \textit{for all } (x, \xi) \in \overline{\Omega} \times B_\delta.$$

Proof. By the continuity of H, there exists a constant $M > 0$ such that $H(x, 0) \leq M$ for all $x \in \overline{\Omega}$. Also, by the coercivity of H, there exists a constant $R > 0$ such that $H(x, p) > M + 1$ for all $(x, p) \in \overline{\Omega} \times \partial B_R$. We set $\delta = R^{-1}$. Let $(x, \xi) \in \overline{\Omega} \times B_\delta$. Let $q \in \overline{B}_R$ be the minimum point of the function $f(p) := H(x, p) - \xi \cdot p$ on $\overline{B}_R$. Noting that $f(0) = H(x, 0) \leq M$ and $f(p) > -\delta R + M + 1 = M$ for all $p \in \partial B_R$, we see that $q \in B_R$ and hence $\xi \in D_p^- H(x, q)$, where $D_p^- H(x, q)$ denotes the subdifferential at q of the function $p \mapsto H(x, p)$. Thanks to the convexity of H, this implies (see Theorem B.2) that $L(x, \xi) = \xi \cdot q - H(x, q)$. Consequently, we get

$$L(x, \xi) \leq \delta R + \max_{\overline{\Omega} \times \overline{B}_R} |H|.$$

Thus we have the desired inequality with $C_0 = \delta R + \max_{\overline{\Omega} \times \overline{B}_R} |H|$. $\square$

For later convenience, we formulate the following lemma, whose proof is left to the reader.

Lemma 5.3. *For each* $i \in \mathbb{N}$ *define the function* L_i *on* $\overline{\Omega} \times \mathbb{R}^n$ *by*

$$L_i(x, \xi) = \max_{p \in \overline{B}_i} (\xi \cdot p - H(x, p)).$$

Then $L_i \in \mathrm{UC}(\overline{\Omega} \times \mathbb{R}^n)$*,*

$$L_i(x, \xi) \le L_{i+1}(x, \xi) \le L(x, \xi) \quad \text{for all} \ \ (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n \ \ \text{and} \ \ i \in \mathbb{N},$$

and for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$*,*

$$L_i(x, \xi) \to L(x, \xi) \quad \text{as} \ \ i \to \infty.$$

The following lemma is a consequence of the Dunford–Pettis theorem.

Lemma 5.4. *Let* $J = [a, b]$*, with* $-\infty < a < b < \infty$*. Let* $\{f_j\}_{j \in \mathbb{N}} \subset L^1(J, \mathbb{R}^m)$ *be uniformly integrable in* J*. That is, for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any measurable* $E \subset J$ *and* $j \in \mathbb{N}$*, we have*

$$\int_E |f_j(t)| \mathrm{d}t < \varepsilon \quad \text{if} \ |E| < \delta,$$

where $|E|$ *denotes the Lebesgue measure of* E*. Then* $\{f_j\}$ *has a subsequence which converges weakly in* $L^1(J, \mathbb{R}^m)$*.*

See Appendix A.5 for a proof of the above lemma.

Lemma 5.5. *Let* $J = [0, T]$ *with* $T \in \mathbb{R}_+$*,* $(\eta, v) \in L^\infty(J, \mathbb{R}^n) \times L^1(J, \mathbb{R}^n)$*,* $i \in \mathbb{N}$ *and* $\varepsilon > 0$*. Let* $L_i \in \mathrm{UC}(\overline{\Omega} \times \mathbb{R}^n)$ *be the function defined in Lemma 5.3. Assume that* $\eta(s) \in \overline{\Omega}$ *for all* $s \in J$*. Then there exists a function* $q \in L^\infty(J, \mathbb{R}^n)$ *such that for a.e.* $s \in J$*,*

$$q(s) \in \overline{B}_i \quad \text{and} \quad H(\eta(s), q(s)) + L_i(\eta(s), -v(s)) \le -v(s) \cdot q(s) + \varepsilon.$$

Proof. Note that for each $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$ there is a point $q = q(x, \xi) \in \overline{B}_i$ such that $L_i(x, \xi) = \xi \cdot q - H(x, q)$. By the continuity of the functions H and L_i, there exists a constant $r = r(x, \xi) > 0$ such that

$$L_i(y, z) + H(y, q) \le z \cdot q + \varepsilon \quad \text{for all} \ \ (y, z) \in (\overline{\Omega} \cap B_r(x)) \times B_r(\xi).$$

Hence, as $\overline{\Omega} \times \mathbb{R}^n$ is σ-compact, we may choose a sequence $\{(x_k, \xi_k, q_k, r_k)\}_{k \in \mathbb{N}} \subset \overline{\Omega} \times \mathbb{R}^n \times \overline{B}_i \times \mathbb{R}_+$ such that

$$\overline{\Omega} \times \mathbb{R}^n \subset \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k) \times B_{r_k}(\xi_k)$$

and for all $k \in \mathbb{N}$,

$$L_i(y, z) + H(y, q_k) \le z \cdot q_k + \varepsilon \quad \text{for all } \ (y, z) \in B_{r_k}(x_k) \times B_{r_k}(\xi_k).$$

Now we set $U_k = (\overline{\Omega} \cap B_{r_k}(x_k)) \times B_{r_k}(\xi_k)$ for $k \in \mathbb{N}$ and define the function $P : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(x, \xi) = q_k \quad \text{for all } \ (x, \xi) \in U_k \setminus \bigcup_{j<k} U_j \ \text{ and all } \ k \in \mathbb{N}.$$

It is clear that P is Borel measurable in $\overline{\Omega} \times \mathbb{R}^n$. Moreover we have $P(x, \xi) \in \overline{B}_i$ for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$ and

$$L_i(x, \xi) + H(x, P(x, \xi)) \le \xi \cdot P(x, \xi) + \varepsilon \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \tag{88}$$

We define the function $q \in L^\infty(J, \mathbb{R}^n)$ by setting $q(s) = P(\eta(s), -v(s))$. From (88), we see that $q(s) \in \overline{B}_i$ and

$$L_i(\eta(s), -v(s)) + H(\eta(s), q(s)) \le -v(s) \cdot q(s) + \varepsilon \quad \text{for a.e. } s \in J. \qquad \square$$

Lemma 5.6. *Let $J = [0, T]$ with $T \in \mathbb{R}_+$, $\varepsilon > 0$, $i \in \mathbb{N}$, $q \in L^\infty(J, \mathbb{R}^n)$ and $\eta \in C(J, \mathbb{R}^n)$ such that $\eta(s) \in \overline{\Omega}$ for all $s \in J$. Assume that $\|q\|_{L^\infty(J)} < i$. Let L_i be the function defined in Lemma 5.3. Then there exists a function $v \in L^\infty([0, T], \mathbb{R}^n)$ such that*

$$H(\eta(s), q(s)) + L_i(\eta(s), -v(s)) < -v(s) \cdot q(s) + \varepsilon \quad \textit{for a.e. } s \in [0, T]. \tag{89}$$

Before going into the proof we remark that for any $x \in \overline{\Omega}$ the function $L_i(x, \cdot)$ is the convex conjugate of the function $\tilde{H}(x, \cdot)$ given by $\tilde{H}(x, p) = H(x, p)$ if $p \in \overline{B}_i$ and $\tilde{H}(x, p) = \infty$ otherwise.

Proof. The same construction as in the proof of Lemma 5.5, with the roles of H and L_i being exchanged, yields a measurable function $v : [0, T] \to \mathbb{R}^n$ for which (89) holds. Set $C = \max_{\overline{\Omega} \times \overline{B}_i} |H|$ and observe that

$$L_i(x, \xi) \ge i|\xi| - C \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

We combine this with (89), to get

$$\varepsilon + \|q\|_{L^\infty(J)}|v(s)| > i|v(s)| - 2C \quad \text{for a.e. } s \in J.$$

Hence,

$$\|v\|_{L^\infty(J)} \le \frac{\varepsilon + 2C}{i - \|q\|_{L^\infty(J)}}. \qquad \square$$

The following proposition concerns the lower semicontinuity of the functional

$$(\eta, v) \mapsto \int_0^T L(\eta(s), -v(s))\mathrm{d}s.$$

Theorem 5.3. *Let* $J = [0, T]$ *with* $T \in \mathbb{R}_+$, $\{(\eta_k, v_k)\}_{k\in\mathbb{N}} \subset L^\infty(J, \mathbb{R}^n) \times L^1(J, \mathbb{R}^n)$ *and* $(\eta, v) \in L^\infty(J, \mathbb{R}^n) \times L^1(J, \mathbb{R}^n)$. *Assume that* $\eta_k(s) \in \overline{\Omega}$ *for all* $(s, k) \in J \times \mathbb{N}$ *and that as* $k \to \infty$,

$$\begin{aligned} \eta_k(s) &\to \eta(s) \quad \text{uniformly for } s \in J, \\ v_k &\to v \quad \text{weakly in } L^1(J, \mathbb{R}^n). \end{aligned}$$

Let ψ *be a function in* $L^\infty(J, \mathbb{R})$ *such that* $\psi(s) \geq 0$ *for a.e.* $s \in J$. *Then*

$$\int_J \psi(s)L(\eta(s), -v(s))\mathrm{d}s \leq \liminf_{k\to\infty} \int_J \psi(s)L(\eta_k(s), -v_k(s))\mathrm{d}s. \tag{90}$$

Proof. Fix any $i \in \mathbb{N}$. Due to Lemma 5.5, there is a function $q \in L^\infty(J, \mathbb{R}^n)$ such that $q(s) \in \overline{B}_i$ and

$$H(\eta(s), q(s)) + L_i(\eta(s), -v(s)) < -v(s) \cdot q(s) + \frac{1}{i} \quad \text{for a.e. } s \in J. \tag{91}$$

Note that for all $k \in \mathbb{N}$,

$$\begin{aligned} \int_J \psi(s)L(\eta_k(s), -v_k(s))\mathrm{d}s &\geq \int_J \psi(s)L_i(\eta_k(s), -v_k(s))\mathrm{d}s \\ &\geq \int_J \psi(s)[-v_k(s) \cdot q(s) - H(\eta_k(s), q(s))]\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} \lim_{k\to\infty} &\int_J \psi(s)[-v_k(s) \cdot q(s) - H(\eta_k(s), q(s))]\mathrm{d}s \\ &= \int_J \psi(s)[-v(s) \cdot q(s) - H(\eta(s), q(s))]\mathrm{d}s. \end{aligned}$$

Hence, using (91), we get

$$\begin{aligned} \liminf_{k\to\infty} \int_J \psi(s)L(\eta_k(s), -v_k(s))\mathrm{d}s &\geq \int_J \psi(s)[-v(s) \cdot q(s) - H(\eta(s), q(s))]\mathrm{d}s \\ &\geq \int_J \psi(s)[L_i(\eta(s), -v(s)) - 1/i]\mathrm{d}s. \end{aligned}$$

By the monotone convergence theorem, we conclude that (90) holds. □

Corollary 5.1. *Under the hypotheses of the above theorem, let* $\{f_k\} \subset L^1(J, \mathbb{R})$ *be a sequence of functions converging weakly in* $L^1(J, \mathbb{R})$ *to* f*. Assume furthermore that for all* $k \in \mathbb{N}$*,*

$$L(\eta_k(s), -v_k(s)) \le f_k(s) \quad \textit{for a.e. } s \in J.$$

Then

$$L(\eta(s), -v(s)) \le f(s) \quad \textit{for a.e. } s \in J.$$

Proof. Set $E = \{s \in J : L(\eta(s), -v(s)) > f(s)\}$. By Theorem 5.3, we deduce that

$$\begin{aligned} 0 &\ge \liminf_{k\to\infty} \int_J \mathbf{1}_E(s)[L(\eta_k(s), -v_k(s)) - f_k(s)]\mathrm{d}s \\ &\ge \int_J \mathbf{1}_E(s)[L(\eta(s), -v(s)) - f(s)]\mathrm{d}s \\ &= \int_J [L(\eta(s), -v(s)) - f(s)]_+ \mathrm{d}s, \end{aligned}$$

where $[\cdots]_+$ denotes the positive part of $[\cdots]$. Thus we see that $L(\eta(s), -v(s)) \le f(s)$ for a.e. $s \in J$. □

Lemma 5.7. *Let* $J = [0, T]$*, with* $T \in \mathbb{R}_+$*, and* $q \in C(\overline{\Omega} \times J, \mathbb{R}^n)$*. Let* $x \in \overline{\Omega}$*. Then there exists a triple* $(\eta, v, l) \in \mathrm{SP}(x)$ *such that*

$$H(\eta(s), q(\eta(s), s)) + L(\eta(s), -v(s)) = -v(s) \cdot q(\eta(s), s) \quad \textit{for a.e. } s \in J.$$

Proof. Fix $k \in \mathbb{N}$. Set $\delta = T/k$ and $s_j = (j-1)\delta$ for $j = 1, 2, \ldots, k+1$. We define inductively a sequence $\{(x_j, \eta_j, v_j, l_j)\}_{j=1}^k \subset \overline{\Omega} \times \mathrm{SP}$. We set $x_1 = x$ and choose a $\xi_1 \in \mathbb{R}^n$ so that

$$H(x_1, q(x_1, 0)) + L(x_1, -\xi_1) \le -\xi_1 \cdot q(x_1, 0) + 1/k.$$

Set $v_1(s) = \xi_1$ for $s \ge 0$ and choose a pair $(\eta_1, l_1) \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \overline{\Omega}) \times L^\infty(\mathbb{R}_+, \mathbb{R})$ so that $(\eta_1, v_1, l_1) \in \mathrm{SP}(x_1)$. In fact, Theorem 5.2 guarantees the existence of such a pair.

We argue by induction and now suppose that $k \ge 2$ and we are given (x_i, η_i, v_i, l_i) for all $i = 1, \ldots, j-1$ and some $2 \le j \le k$. Then set $x_j = \eta_{j-1}(\delta)$, choose a $\xi_j \in \mathbb{R}^n$ so that

$$H(x_j, q(x_j, s_j)) + L(x_j, -\xi_j) \le -\xi_j \cdot q(x_j, s_j) + 1/k, \tag{92}$$

set $v_j(s) = \xi_j$ for $s \ge 0$, and select a pair $(\eta_j, l_j) \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \overline{\Omega}) \times L^\infty(\mathbb{R}_+, \mathbb{R})$ so that $(\eta_j, v_j, l_j) \in \mathrm{SP}(x_j)$. Thus, by induction, we can select a sequence

$\{(x_j, \eta_j, v_j, l_j)\}_{j=1}^{k} \subset \overline{\Omega} \times \mathrm{SP}$ such that $x_1 = \eta_1(0)$, $x_j = \eta_{j-1}(\delta) = \eta_j(0)$ for $j = 2, \ldots, k$ and for each $j = 1, 2, \ldots, k$, (92) holds with $\xi_j = v_j(s)$ for all $s \geq 0$. We set $\alpha_j = (\eta_j, v_j, l_j)$ for $j = 1, \ldots, k$.

Note that the choice of x_j, η_j, v_j, l_j, with $j = 1, \ldots, k$, depends on k, which is not explicit in our notation. We define $\bar{\alpha}_k = (\bar{\eta}_k, \bar{v}_k, \bar{l}_k) \in \mathrm{SP}(x)$ by setting

$$\bar{\alpha}_k(s) = \alpha_j(s - s_j) \quad \text{for } s \in [s_j, s_{j+1}) \text{ and } j = 1, \ldots, k.$$

and

$$\bar{\alpha}_k(s) = (\eta_k(\delta), 0, 0) \quad \text{for } s \geq s_{k+1} = T.$$

Also, we define $\bar{x}_k, \bar{q}_k \in L^\infty(J, \mathbb{R}^n)$ by

$$\bar{x}_k(s) = x_j \quad \text{and} \quad \bar{q}_k(s) = q(x_j, s_j) \quad \text{for } s \in [s_j, s_{j+1}) \text{ and } j = 1, \ldots, k.$$

Now we observe by (92) that for all $j = 1, \ldots, k$,

$$L(x_j, -\xi_j) \leq |\xi_j| R + \max_{\overline{\Omega} \times \overline{B}_R} |H| + 1,$$

where $R > 0$ is such a constant that $R \geq \max_{\overline{\Omega} \times J} |q|$. Combining this estimate with Lemma 5.1, we see that there is a constant $C_1 > 0$, independent of k, such that

$$\max_{s \geq 0} |\bar{v}_k(s)| = \max_{1 \leq j \leq k} |\xi_j| \leq C_1.$$

By Proposition 5.2, we find a constant $C_2 > 0$, independent of k, such that $\|\dot{\bar{\eta}}_k\|_{L^\infty(\mathbb{R}_+)} \vee \|\bar{l}_k\|_{L^\infty(\mathbb{R}_+)} \leq C_2$.

We may invoke standard compactness theorems, to find a triple $(\eta, v, l) \in \mathrm{Lip}(J, \mathbb{R}^n) \times L^\infty(J, \mathbb{R}^{n+1})$ and a subsequence of $\{(\bar{\eta}_k, \bar{v}_k, \bar{l}_k)\}_{k \in \mathbb{N}}$, which will be denoted again by the same symbol, so that for every $0 < S < \infty$, as $k \to \infty$,

$$\bar{\eta}_k \to \eta \quad \text{uniformly on } [0, S],$$
$$(\dot{\bar{\eta}}_k, \bar{v}_k, \bar{l}_k) \to (\dot{\eta}, v, l) \quad \text{weakly-star in } L^\infty([0, S], \mathbb{R}^{2n+1}).$$

By Proposition 5.1, we see that $(\eta, v, l) \in \mathrm{SP}(x)$. It follows as well that $\bar{x}_k(s) \to \eta(s)$ and $\bar{q}_k(s) \to q(\eta(s), s)$ uniformly for $s \in J$ as $k \to \infty$.

Now, the inequalities (92), $1 \leq j \leq k$, can be rewritten as

$$L(\bar{x}_k(s), -\bar{v}_k(s)) \leq -\bar{v}_k(s) \cdot \bar{q}_k(s) - H(\bar{x}_k(s), \bar{q}_k(s)) + 1/k \quad \text{for all } s \in [0, T).$$

It is obvious to see that the sequence of functions

$$-\bar{v}_k(s) \cdot q_k(s) + 1/k - H(\bar{x}_k(s), \bar{q}_k(s))$$

on J converges weakly-star in $L^\infty(J,\mathbb{R})$ to the function

$$-v(s)\cdot q(\eta(s),s)-H(\eta(s),q(\eta(s),s)).$$

Hence, by Corollary 5.1, we conclude that

$$H(\eta(s),q(\eta(s),s))+L(\eta(s),-v(s))\le -v(s)\cdot q(\eta(s),s)\quad \text{for a.e. } s\in J,$$

which implies the desired equality. □

Theorem 5.4. *Let* $J=[0,\,T]$, *with* $T\in\mathbb{R}_+$, *and* $\{(\eta_k,\,v_k,\,l_k)\}_{k\in\mathbb{N}}\subset \mathrm{SP}$. *Assume that there is a constant* $C>0$, *independent of* $k\in\mathbb{N}$, *such that*

$$\mathscr{L}(T,\eta_k,v_k,l_k)\le C\quad \textit{for all }\ k\in\mathbb{N}.$$

Then there exists a triple $(\eta,\,v,\,l)\in\mathrm{SP}$ *such that*

$$\mathscr{L}(T,\eta,v,l)\le\liminf_{k\to\infty}\mathscr{L}(T,\eta_k,v_k,l_k).$$

Moreover, there is a subsequence $\{(\eta_{k_j},\,v_{k_j},\,l_{k_j})\}_{j\in\mathbb{N}}$ *of* $\{(\eta_k,\,v_k,\,l_k)\}$ *such that as* $j\to\infty$,

$$\eta_{k_j}(s)\to\eta(s)\quad \textit{uniformly on}\ \ J,$$
$$(\dot\eta_{k_j},\,v_{k_j},\,l_{k_j})\to(\dot\eta,\,v,\,l)\quad \textit{weakly in}\ \ L^1(J,\mathbb{R}^{2n+1}).$$

Proof. We may assume without loss of generality that $\eta_k(t)=\eta_k(T)$, $v_k(t)=0$ and $l_k(t)=0$ for all $t\ge T$ and all $k\in\mathbb{N}$.

According to Proposition 5.2, there is a constant $C_0>0$ such that for any $(\eta,\,v,\,l)\in\mathrm{SP}$, $|\dot\eta(t)|\vee|l(t)|\le C_0|v(t)|$ for a.e. $t\ge 0$. Note by Lemma 5.1 that for each $A>0$ there is a constant $C_A>0$ such that $L(x,\xi)\ge A|\xi|-C_A$ for all $(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$. From this lower bound of L, it is obvious that for all $(x,\xi,r)\in\partial\Omega\times\mathbb{R}^n\times\overline{\mathbb{R}}_+$, if $r\le C_0|\xi|$, then

$$L(x,\,\xi)+g(x)r\ge\left(A-C_0\max_{\partial\Omega}|g|\right)|\xi|-C_A, \tag{93}$$

which ensures that there is a constant $C_1>0$ such that for $(\eta,\,v,\,l)\in\mathrm{SP}$,

$$L(\eta(s),-v(s))+g(\eta(s))l(s)+C_1\ge 0\quad \text{for a.e. } s\ge 0. \tag{94}$$

Set

$$\Lambda=\liminf_{k\to\infty}\mathscr{L}(T,\eta_k,v_k,l_k),$$

and note by (94) that $-C_1 T \le \Lambda \le C$. We may choose a subsequence $\{(\eta_{k_j}, v_{k_j}, l_{k_j})\}_{j\in\mathbb{N}}$ of $\{(\eta_k, v_k, l_k)\}$ so that

$$\Lambda = \lim_{j\to\infty} \mathscr{L}(T, \eta_{k_j}, v_{k_j}, l_{k_j}).$$

Using (94), we obtain for any measurable $E \subset [0,\ T]$,

$$\int_E \big(L(\eta_k(s), -v_k(s)) + g(\eta_k(s))l_k(s) + C_1\big)\,\mathrm{d}s$$
$$\le \int_0^T \big(L(\eta_k(s), -v_k(s)) + g(\eta_k(s))l_k(s) + C_1\big)\,\mathrm{d}s \le C + C_1 T.$$

This together with (93) yields

$$\Big(A - C_0 \max_{\partial\Omega} |g|\Big) \int_E |v_k(s)|\,\mathrm{d}s \le C_A|E| + C + C_1 T \quad \text{for all } A > 0.$$

This shows that the sequence $\{v_k\}$ is uniformly integrable on $[0,\ T]$. Since $|\dot\eta_k(s)| \vee |l_k(s)| \le C_0|v_k(s)|$ for a.e. $s \ge 0$ and $v_k(s) = 0$ for all $s > T$, we see easily that the sequence $\{(\dot\eta_k, v_k, l_k)\}$ is uniformly integrable on $\overline{\mathbb{R}}_+$.

Due to Lemma 5.4, we may assume by reselecting the subsequence $\{(\eta_{k_j}, v_{k_j}, l_{k_j})\}$ if necessary that as $j \to \infty$,

$$(\dot\eta_{k_j}, v_{k_j}, l_{k_j}) \to (w, v, l) \quad \text{weakly in } \ L^1([0,\ S], \mathbb{R}^{2n+1})$$

for every $S > 0$ and some $(w, v, l) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+,\ \mathbb{R}^{2n+1})$. We may also assume that $\eta_{k_j}(0) \to x$ as $j \to \infty$ for some $x \in \overline{\Omega}$. By Proposition 5.1, if we set $\eta(s) = x + \int_0^s w(r)\mathrm{d}r$ for $s \ge 0$, then $(\eta, v, l) \in \mathrm{SP}(x)$ and, as $j \to \infty$,

$$\eta_{k_j}(s) \to \eta(s) \quad \text{locally uniformly on } \ \overline{\mathbb{R}}_+.$$

We apply Theorem 5.3, with the function $\psi(s) \equiv 1$, to find that

$$\int_J L(\eta(s), -v(s))\mathrm{d}s \le \liminf_{j\to\infty} \int_J L(\eta_{k_j}(s), -v_{k_j}(s))\mathrm{d}s.$$

Consequently, we have

$$\mathscr{L}(T, \eta, v, l) \le \liminf_{j\to\infty} \mathscr{L}(T, \eta_{k_j}, v_{k_j}, l_{k_j}) = \Lambda,$$

which completes the proof. □

5.4 Value Function II

Theorem 5.5. *Let $u \in \mathrm{UC}(\overline{\Omega} \times \overline{\mathbb{R}}_+)$ be the viscosity solution of* (ENP)–(ID). *Then $V = u$ in $\overline{\Omega} \times \mathbb{R}_+$.*

This is a version of classical observations on the value functions in optimal control, and, in this regard, we refer for instance to [43, 45]. The above theorem has been established in [39]. The above theorem gives a variational formula for the unique solution of (ENP)–(ID). This variational formula is sometimes called the Lax–Oleinik formula.

For the proof of Theorem 5.5, we need the following three lemmas.

Lemma 5.8. *Let $U \subset \mathbb{R}^n$ be an open set and $J = [a,\ b]$ a finite subinterval of $\overline{\mathbb{R}}_+$. Let $\psi \in C^1((\overline{U} \cap \overline{\Omega}) \times J)$ and assume that*

$$\psi_t(x,t) + H(x, D_x\psi(x,t)) \le 0 \qquad \text{for all } (x,t) \in (U \cap \Omega) \times J, \tag{95}$$

$$\frac{\partial \psi}{\partial \gamma}(x,t) \le g(x) \qquad \text{for all } (x,t) \in (U \cap \partial\Omega) \times J, \tag{96}$$

$$\psi(x,t) \le V(x,t) \qquad \text{for all } (x,t) \in (\partial U \cap \overline{\Omega}) \times J, \tag{97}$$

$$\psi(x,a) \le V(x,a) \qquad \text{for all } x \in \overline{U} \cap \overline{\Omega}. \tag{98}$$

Then $\psi \le V$ in $(U \cap \overline{\Omega}) \times J$.

We note that the following inclusion holds: $\partial(U \cap \overline{\Omega}) \subset [\partial U \cap \overline{\Omega}] \cup (U \cap \partial\Omega)$.

Proof. Let $(x,t) \in (U \cap \overline{\Omega}) \times J$. Define the mapping $\tau\ :\ \mathrm{SP}(x) \to [0,\ t-a]$ by

$$\tau(\eta, v, l) = \inf\{s \ge 0\ :\ \eta(s) \notin U\} \wedge (t-a).$$

It is clear that τ is nonanticipating. Let $\alpha = (\eta, v, l) \in \mathrm{SP}(x)$, and observe that $\eta(s) \in U$ for all $s \in [0,\ \tau(\alpha))$ and that $\eta(\tau(\alpha)) \in \partial U$ if $\tau(\alpha) < t-a$. In particular, we find from (97) and (98) that

$$\psi(\eta(\tau(\alpha)), t-\tau(\alpha)) \le V(\eta(\tau(\alpha)), t-\tau(\alpha)). \tag{99}$$

Fix any $\alpha = (\eta,\ v,\ l) \in \mathrm{SP}(x)$. Note that

$$\begin{aligned}
&\psi(\eta(\tau(\alpha)), t-\tau(\alpha)) - \psi(x,t) \\
&\quad = \int_0^{\tau(\alpha)} \frac{\mathrm{d}}{\mathrm{d}s}\psi(\eta(s), t-s)\mathrm{d}s \\
&\quad = \int_0^{\tau(\alpha)} \big(D_x\psi(\eta(s), t-s)\cdot\dot{\eta}(s) - \psi_t(\eta(s), t-s)\big)\mathrm{d}s \\
&\quad = \int_0^{\tau(\alpha)} \big(D_x\psi(\eta(s), t-s)\cdot(v(s) - l(s)\gamma(\eta(s))) - \psi_t(\eta(s), t-s)\big)\mathrm{d}s.
\end{aligned}$$

Now, using (95), (96) and (99), we get

$$
\begin{aligned}
&\psi(x,t) - V(\eta(\tau(\alpha)), t-\tau(\alpha)) \\
&\quad \le \int_0^{\tau(\alpha)} \big(- D_x\psi(\eta(s), t-s)\cdot v(s) + l(s) D_x\psi(\eta(s))\cdot\gamma(\eta(s)) \\
&\qquad + \psi_t(\eta(s), t-s)\big)\mathrm{d}s \\
&\quad \le \int_0^{\tau(\alpha)} \big(H(\eta(s), D_x\psi(\eta(s), t-s)) + L(\eta(s), -v(s)) + l(s) g(\eta(s)) \\
&\qquad + \psi_t(\eta(s), t-s)\big)\mathrm{d}s \\
&\le \mathscr{L}(\tau(\alpha), \eta, v, l),
\end{aligned}
$$

which immediately shows that

$$\psi(x,t) \le \inf\left(\mathscr{L}(\tau(\alpha), \eta, v, l) + V(\eta(\tau(\alpha)), t-\tau(\alpha))\right),$$

where the infimum is taken over all $\alpha = (\eta, v, l) \in \mathrm{SP}(x)$. Thus, by (86), we get $\psi(x,t) \le V(x,t)$. □

Lemma 5.9. *For any* $\varepsilon > 0$ *there is a constant* $C_\varepsilon > 0$ *such that* $V(x,t) \ge u_0(x) - \varepsilon - C_\varepsilon t$ *for* $(x,t) \in \overline{Q}$.

Proof. Fix any $\varepsilon > 0$. According to the proof of Theorem 3.2, there are a function $f \in C^1(\overline{\Omega})$ and a constant $C > 0$ such that if we set $\psi(x,t) = f(x) - Ct$ for $(x,t) \in \overline{Q}$, then ψ is a classical subsolution of (ENP) and $u_0(x) \ge f(x) \ge u_0(x) - \varepsilon$ for all $x \in \overline{\Omega}$.

We apply Lemma 5.8, with $U = \mathbb{R}^n$, $a = 0$, arbitrary $b > 0$, to obtain

$$V(x,t) \ge \psi(x,t) \ge -\varepsilon + u_0(x) - Ct \quad \text{for all } (x,t) \in Q,$$

which completes the proof. □

Lemma 5.10. *There is a constant* $C > 0$ *such that* $V(x,t) \le u_0(x) + Ct$ *for* $(x,t) \in Q$.

Proof. Let $(x,t) \in Q$. Set $\eta(s) = x$, $v(s) = 0$ and $l(s) = 0$ for $s \ge 0$. Then $(\eta, v, l) \in \mathrm{SP}(x)$. Hence, we have

$$V(x,t) \le u_0(x) + \int_0^t L(x,0)\mathrm{d}s = u_0(x) + tL(x,0) \le u_0(x) - t\min_{p\in\mathbb{R}^n} H(x,p).$$

Setting $C = -\min_{\overline{\Omega}\times\mathbb{R}^n} H$, we get $V(x,t) \le u_0(x) + Ct$. □

Proof (Theorem 5.5). By Lemmas 5.9 and 5.10, there is a constant $C > 0$ and for each $\varepsilon > 0$ a constant $C_\varepsilon > 0$ such that

$$-\varepsilon - C_\varepsilon t \le V(x,t) - u_0(x) \le Ct \quad \text{for all } (x,t) \in Q.$$

This shows that V is locally bounded on $\overline{Q}$ and that

$$\lim_{t \to 0+} V(x,t) = u_0(x) \quad \text{uniformly for } x \in \overline{\Omega}.$$

In particular, we have $V_*(x,0) = V^*(x,0) = u_0(x)$ for all $x \in \overline{\Omega}$.

We next prove that V is a subsolution of (ENP). Let $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^1(\overline{Q})$. Assume that $V^* - \phi$ attains a strict maximum at $(\hat{x},\hat{t})$. We want to show that if $\hat{x} \in \Omega$, then

$$\phi_t(\hat{x},\hat{t}) + H(\hat{x}, D_x\phi(\hat{x},\hat{t})) \le 0,$$

and if $\hat{x} \in \partial\Omega$, then either

$$\phi_t(\hat{x},\hat{t}) + H(\hat{x}, D_x\phi(\hat{x},\hat{t})) \le 0 \quad \text{or} \quad \gamma(\hat{x}) \cdot D_x\phi(\hat{x},\hat{t}) \le g(\hat{x}).$$

We argue by contradiction and thus suppose that

$$\phi_t(\hat{x},\hat{t}) + H(\hat{x}, D_x\phi(\hat{x},\hat{t})) > 0$$

and furthermore

$$\gamma(\hat{x}) \cdot D_x\phi(\hat{x},\hat{t}) > g(\hat{x}) \quad \text{if } \hat{x} \in \partial\Omega.$$

By continuity, we may choose a constant $r \in (0,\hat{t}\,)$ so that

$$\phi_t(x,t) + H(x, D_x\phi(x,t)) > 0 \quad \text{for all } (x,t) \in (\overline{B}_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}, \tag{100}$$

where $\hat{J} = [\hat{t} - r,\ \hat{t} + r]$, and

$$\gamma(x) \cdot D_x\phi(x,t) > g(x) \quad \text{for all } (x,t) \in (\overline{B}_r(\hat{x}) \cap \partial\Omega) \times \hat{J}. \tag{101}$$

(Of course, if $\hat{x} \in \Omega$, we can choose r so that $\overline{B}_r(\hat{x}) \cap \partial\Omega = \emptyset$.)

We may assume that $(V^* - \phi)(\hat{x},\hat{t}) = 0$. Set

$$B = \Big(\big(\partial B_r(\hat{x}) \cap \overline{\Omega}\big) \times \hat{J} \Big) \cup \Big(\big(B_r(\hat{x}) \cap \overline{\Omega}\big) \times \{\hat{t} - r\} \Big),$$

and $m = -\max_B(V^* - \phi)$. Note that $m > 0$ and $V(x,t) \le \phi(x,t) - m$ for $(x,t) \in B$.

We set $\varepsilon = r/2$. In view of the definition of V^*, we may choose a point $(\bar{x},\bar{t}) \in \overline{\Omega} \cap B_\varepsilon(\hat{x}) \times (\hat{t} - \varepsilon,\ \hat{t} + \varepsilon)$ so that $(V - \phi)(\bar{x},\bar{t}) > -m$. Set $a = \bar{t} - \hat{t} + r$, and note that $a > \varepsilon$ and $\mathrm{dist}(\bar{x}, \partial B_r(\hat{x})) > \varepsilon$. For each $\alpha = (\eta, v, l) \in \mathrm{SP}(\bar{x})$ we set

$$S(\alpha) = \{s \ge 0 \,:\, \eta(s) \in \partial B_r(\hat{x})\} \quad \text{and} \quad \tau = a \wedge \inf S(\alpha).$$

Clearly, the mapping $\tau : \mathrm{SP}(\bar{x}) \to [0, a]$ is nonanticipating. Observe also that if $\tau(\alpha) < a$, then $\eta(\tau(\alpha)) \in \partial B_r(\hat{x})$ or, otherwise, $\bar{t} - \tau(\alpha) = \bar{t} - a = \hat{t} - r$. That is, we have

$$(\eta(\tau(\alpha)), \bar{t} - \tau(\alpha)) \in B \quad \text{for all } \alpha = (\eta, v, l) \in \mathrm{SP}(\bar{x}). \tag{102}$$

Note as well that $(\eta(s), \bar{t} - s) \in \overline{B}_r(\hat{x}) \times \hat{J}$ for all $s \in [0, \tau(\alpha)]$.

We apply Lemma 5.7, with $J = [0, a]$ and the function $q(x, s) = D\phi(x, \bar{t} - s)$, to find a triple $\alpha = (\eta, v, l) \in \mathrm{SP}(\bar{x})$ such that for a.e. $s \in [0, a]$,

$$H(\eta(s), D_x\phi(\eta(s), \bar{t} - s)) + L(\eta(s), -v(s)) \le -v(s) \cdot D_x\phi(\eta(s), \bar{t} - s) \tag{103}$$

For this α, we write $\tau = \tau(\alpha)$ for simplicity of notation. Using (102), by the dynamic programming principle, we have

$$\begin{aligned} \phi(\bar{x}, \bar{t}) &< V(\bar{x}, \bar{t}) + m \\ &\le \mathscr{L}(\tau, \eta, v, l) + V(\tau, \bar{t} - \tau) + m \\ &\le \mathscr{L}(\tau, \eta, v, l) + \phi(\eta(\tau), \bar{t} - \tau). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} 0 &< \int_0^\tau \big(L(\eta(s), -v(s)) + g(\eta(s))l(s) + \frac{\mathrm{d}}{\mathrm{d}s}\phi(\eta(s), \bar{t} - s)\big)\mathrm{d}s \\ &\le \int_0^\tau \big(L(\eta(s), -v(s)) + g(\eta(s))l(s) + D_x\phi(\eta(s), \bar{t} - s) \cdot \dot{\eta}(s) - \phi_t(\eta(s), \bar{t} - s)\big)\mathrm{d}s \\ &\le \int_0^\tau \big(L(\eta(s), -v(s)) + g(\eta(s))l(s) \\ &\quad + D_x\phi(\eta(s), \bar{t} - s) \cdot (v(s) - l(s)\gamma(\eta(s)) - \phi_t(\eta(s), \bar{t} - s)\big)\mathrm{d}s. \end{aligned}$$

Now, using (103), (100) and (101), we get

$$\begin{aligned} 0 &< \int_0^\tau \big(- H(\eta(s), D_x\phi(\eta(s), \bar{t} - s)) + g(\eta(s))l(s) \\ &\quad - l(s)D_x\phi(\eta(s), \bar{t} - s) \cdot \gamma(\eta(s)) - \phi_t(\eta(s), \bar{t} - s)\big)\mathrm{d}s \\ &< \int_0^\tau l(s)\big(g(\eta(s)) - \gamma(\eta(s)) \cdot D_x\phi(\eta(s), \bar{t} - s)\big)\mathrm{d}s \le 0, \end{aligned}$$

which is a contradiction. We thus conclude that V is a viscosity subsolution of (ENP).

Now, we turn to the proof of the supersolution property of V. Let $\phi \in C^1(\overline{Q})$ and $(\hat{x}, \hat{t}) \in \overline{\Omega} \times \mathbb{R}_+$. Assume that $V_* - \phi$ attains a strict minimum at $(\hat{x}, \hat{t})$. As usual, we assume furthermore that $\min_{\overline{Q}}(V_* - \phi) = 0$.

We need to show that if $\hat{x} \in \Omega$, then

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x\phi(\hat{x}, \hat{t})) \geq 0,$$

and if $\hat{x} \in \partial\Omega$, then

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x\phi(\hat{x}, \hat{t})) \geq 0 \quad \text{or} \quad \gamma(\hat{x}) \cdot D_x\phi(\hat{x}, \hat{t}) \geq g(\hat{x}).$$

We argue by contradiction and hence suppose that this were not the case. That is, we suppose that

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x\phi(\hat{x}, \hat{t})) < 0,$$

and moreover

$$\gamma(\hat{x}) \cdot D_x\phi(\hat{x}, \hat{t}) < g(\hat{x}) \quad \text{if } \hat{x} \in \partial\Omega.$$

We may choose a constant $r \in (0, \hat{t})$ so that

$$\phi_t(x, t) + H(x, D_x\phi(x, t)) < 0 \quad \text{for all } (x, t) \in (B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J},$$

where $\hat{J} = [\hat{t} - r, \hat{t} + r]$, and

$$\gamma(x) \cdot D_x\phi(x, t) < g(x) \quad \text{for all } (x, t) \in (B_r(\hat{x}) \cap \partial\Omega) \times \hat{J}. \tag{104}$$

We set

$$R = \Big((\partial B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}\Big) \cup \big((B_r(\hat{x}) \cap \overline{\Omega}) \times \{\hat{t} - r\}\big) \text{ and } m = \min_R (V_* - \phi),$$

and define the function $\psi \in C^1((\overline{B}_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J})$ by $\psi(x, t) = \phi(x, t) + m$. Note that $m > 0$, $\inf_{(B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}} (V_* - \psi) = -m < 0$ and $V(x, t) \geq \psi(x, t)$ for all $(x, t) \in R$. Observe moreover that

$$\begin{aligned} &\psi_t(x, t) + H(x, D_x\psi(x, t)) < 0 && \text{for all } (x, t) \in (B_r(\hat{x}) \cap \Omega) \times \hat{J} \\ &\frac{\partial\psi}{\partial\gamma}(x, t) < g(x) && \text{for all } (x, t) \in (B_r(\hat{x}) \cap \partial\Omega) \times \hat{J}. \end{aligned}$$

We invoke Lemma 5.8, to find that $\psi \leq V$ in $(B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}$. This means that $\inf_{(B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}} (V_* - \psi) \geq 0$. This contradiction shows that V is a viscosity supersolution of (ENP).

We apply Theorem 3.1 to V_*, u and V^*, to obtain $V^* \leq u \leq V_*$ in $\overline{Q}$, from which we conclude that $u = V$ in $\overline{Q}$. □

Our control problem always has an optimal "control" in SP:

Theorem 5.6. *Let* $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$. *Then there exists a triple* $(\eta, v, l) \in \mathrm{SP}(x)$ *such that*

$$V(x,t) = \mathscr{L}(t,\eta,v,l) + u_0(\eta(t)).$$

If, in addition, $V \in \mathrm{Lip}(\overline{\Omega} \times J, \mathbb{R})$*, with* J *being an interval of* $[0,\ t]$*, then the triple* $(\eta,\ v,\ l)$*, restricted to* $\tilde{J}_t := \{s \in [0,\ t]\ :\ t - s \in J\}$*, belongs to* $\mathrm{Lip}(\tilde{J}_t, \mathbb{R}^n) \times L^\infty(\tilde{J}_t, \mathbb{R}^{n+1})$.

Proof. We may choose a sequence $\{(\eta_k,\ v_k,\ l_k)\} \subset \mathrm{SP}(x)$ such that

$$V(x,t) = \lim_{k\to\infty} \mathscr{L}(t,\eta_k,v_k,l_k) + u_0(\eta_k(t)).$$

In view of Theorem 5.4, we may assume by replacing the sequence $\{(\eta_k,\ v_k,\ l_k)\}$ by a subsequence if needed that for some $(\eta, v, l) \in \mathrm{SP}(x)$, $\eta_k(s) \to \eta(s)$ uniformly on $[0,\ t]$ as $k \to \infty$ and

$$\mathscr{L}(t,\eta,v,l) \le \liminf_{k\to\infty} \mathscr{L}(t,\eta_k,v_k,l_k).$$

It is then easy to see that

$$V(x,t) = \mathscr{L}(r,\eta,v,l) + u_0(\eta(t)). \tag{105}$$

Note by (105) that for all $r \in (0,\ t)$,

$$V(x,t) \ge \mathscr{L}(r,\eta,v,l) + V(\eta(r),t-r),$$

which yields together with the dynamic programming principle

$$V(x,t) = \mathscr{L}(r,\eta,v,l) + V(\eta(r),t-r) \tag{106}$$

for all $r \in (0,\ t)$.

Now, we assume that $V \in \mathrm{Lip}(\overline{\Omega} \times J)$, where $J \subset [0,t]$ is an interval. Observe by (106) that for a.e. $r \in \tilde{J}_t$,

$$\begin{aligned} L(\eta(r),-v(r)) + l(r)g(\eta(r)) &= \lim_{\varepsilon\to 0} \frac{V(\eta(r),t-r) - V(\eta(r+\varepsilon),t-r-\varepsilon)}{\varepsilon} \\ &\le M(|\dot\eta(r)|^2+1)^{1/2} \le M(|\dot\eta(r)|+1), \end{aligned}$$

where $M > 0$ is a Lipschitz bound of the function V on $\overline{\Omega} \times J$. Let $C > 0$ be the constant from Proposition 5.2, so that $|\dot\eta(s)| \vee l(s) \le C|v(s)|$ for a.e. $s \ge 0$. By Lemma 5.1, for each $A > 0$, we may choose a constant $C_A > 0$ so that $L(y,\xi) \ge A|\xi| - C_A$ for $(y,\ \xi) \in \overline{\Omega} \times \mathbb{R}^n$. Accordingly, for any $A > 0$, we get

$$\begin{aligned} A|v(r)| &\le L(\eta(r),-v(r)) + C_A \le -l(r)g(\eta(r)) + M(|\dot\eta(r)|+1) + C_A \\ &\le C(\|g\|_{\infty,\partial\Omega} + M)|v(r)| + M + C_A \quad \text{for a.e. } r \in \tilde{J}_t. \end{aligned}$$

This implies that $v \in L^\infty(\tilde{J}_t, \mathbb{R}^n)$ and moreover that $\eta \in \mathrm{Lip}(\tilde{J}_t, \mathbb{R}^n)$ and $l \in L^\infty(\tilde{J}_t, \mathbb{R})$. The proof is complete. □

Corollary 5.2. *Let $u \in \mathrm{Lip}(\overline{\Omega})$ be a viscosity solution of (SNP) and $x \in \overline{\Omega}$. Then there exists a $(\eta, v, l) \in \mathrm{SP}(x)$ such that for all $t > 0$,*

$$u(x) - u(\eta(t)) = \mathscr{L}(t, \eta, v, l). \tag{107}$$

Proof. Note that the function $u(x)$, as a function of (x,t), is a viscosity solution of (ENP). In view of Theorem 5.6, we may choose a sequence $\{(\eta_j, v_j, l_j)\}_{j\in\mathbb{N}}$ so that $\eta_1(0) = x$, $\eta_{j+1}(0) = \eta_j(1)$ for all $j \in \mathbb{N}$ and

$$u(\eta_j(0)) - u(\eta_j(1)) = \mathscr{L}(1, \eta_j, v_j, l_j) \quad \text{for all} \ \ j \in \mathbb{N}.$$

We define $(\eta, v, l) \in \mathrm{SP}(x)$ by

$$(\eta(s), v(s), l(s)) = (\eta_j(s-j+1), v_j(s-j+1), l_j(s-j+1))$$

for all $s \in [j-1, j)$ and $j \in \mathbb{N}$. By using the dynamic programming principle, we see that (107) holds for all $t > 0$. □

5.5 Distance-Like Function d

We assume throughout this subsection that (A8) holds, and discuss a few aspects of weak KAM theory related to (SNP).

Proposition 5.4. *We have the variational formula for the function d introduced in Sect. 4.1: for all $x, y \in \overline{\Omega}$,*

$$d(x,y) = \inf\left\{\mathscr{L}(t, \eta, v, l) \ : \ t > 0, \ (\eta, v, l) \in \mathrm{SP}(x) \quad \textit{such that} \quad \eta(t) = y\right\}. \tag{108}$$

We use the following lemma for the proof of the above proposition.

Lemma 5.11. *Let $u_0 \in C(\overline{\Omega})$ and $u \in \mathrm{UC}(\overline{Q})$ be the viscosity solution of (ENP)–(ID). Set*

$$v(x,t) = \inf_{r>0} u(x, t+r) \quad \textit{for} \ \ x \in \overline{Q}.$$

Then $v \in \mathrm{UC}(\overline{Q})$ and it is a viscosity solution of (ENP). Moreover, for each $t > 0$, the function $v(\cdot, t)$ is a viscosity subsolution of (SNP).

Proof. By assumption (A8), there is a viscosity subsolution ψ of (SNP). Note that the function $(x,t) \mapsto \psi(x)$ is a viscosity subsolution of (ENP) as well.

We may assume by adding a constant to ψ if needed that $\psi \le u_0$ in $\overline{\Omega}$. By Theorem 3.1, we have $u(x,t) \ge \psi(x) > -\infty$ for all $(x,t) \in \overline{Q}$. Since $u \in \mathrm{UC}(\overline{Q})$, we see immediately that $v \in \mathrm{UC}(\overline{Q})$. Applying a version for (ENP) of Theorem 4.4, which can be proved based on Theorem D.2, to the collection of viscosity solutions

$(x,t) \mapsto u(x,t+r)$, with $r>0$, of (ENP), we find that v is a viscosity subsolution of (ENP). Also, by Proposition 1.10 (its version for supersolutions), we see that v is a viscosity supersolution of (ENP). Thus, the function v is a viscosity solution of (ENP).

Next, note that for each $x \in \overline{\Omega}$, the function $v(x,\cdot)$ is nondecreasing in $\mathbb{R}_+$. Let $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^1(\overline{\Omega})$. Assume that the function $\overline{\Omega} \ni x \mapsto v(x,\hat{t}) - \phi(x)$ attains a strict maximum at $\hat{x}$. Let $\alpha > 0$ and consider the function

$$v(x,t) - \phi(x) - \alpha(t-\hat{t})^2 \quad \text{on } \overline{\Omega} \times [0,\, \hat{t}+1].$$

Let (x_α, t_α) be a maximum point of this function. It is easily seen that $(x_\alpha, t_\alpha) \to (\hat{x},\hat{t})$ as $\alpha \to \infty$. For sufficiently large α, we have $t_\alpha > 0$ and either

$$x_\alpha \in \partial\Omega \quad \text{and} \quad \gamma(x_\alpha)\cdot D\phi(x_\alpha) \le g(x_\alpha),$$

or

$$2\alpha(t_\alpha - \hat{t}) + H(x_\alpha, D\phi(x_\alpha)) \le 0.$$

By the monotonicity of $v(x,t)$ in t, we see easily that $2\alpha(t_\alpha - \hat{t}) \ge 0$. Hence, sending $\alpha \to \infty$, we conclude that the function $v(\cdot,\hat{t})$ is a viscosity subsolution of (SNP). □

Proof (Proposition 5.4). We write $W(x,y)$ for the right hand side of (108).

Fix any $y \in \overline{\Omega}$. For each $k \in \mathbb{N}$ let $u_k \in \mathrm{Lip}(\overline{Q})$ be the unique viscosity solution of (ENP)–(ID), with u_0 defined by $u_0(x) = k|x-y|$. By Theorem 5.5, we have the formula:

$$u_k(x,t) = \inf\bigl\{\mathscr{L}(t,\eta,v,l) + k|\eta(t)-y| \,:\, (\eta,v,l) \in \mathrm{SP}(x)\bigr\}.$$

It is then easy to see that

$$\inf_{t>0} u_k(x,t) \le W(x,y) \quad \text{for all } (x,k) \in \overline{\Omega}\times\mathbb{N}. \tag{109}$$

Since $d(\cdot,y) \in \mathrm{Lip}(\overline{\Omega})$, if k is sufficiently large, say $k \ge K$, we have $d(\cdot,y) \le k|x-y|$ for all $x \in \overline{\Omega}$. Noting that the function $(x,t) \mapsto d(x,y)$ is a viscosity subsolution of (ENP) and applying Theorem 3.1, we get $d(x,y) \le u_k(x,t)$ for all $(x,t) \in \overline{Q}$ if $k \ge K$. Combining this and (109), we find that $d(x,y) \le W(x,y)$ for all $x \in \overline{\Omega}$.

Next, we give an upper bound on W. According to Lemma 2.1, there exist a constant $C_1 > 0$ and a function $\tau : \overline{\Omega} \to \overline{\mathbb{R}}_+$ such that $\tau(x) \le C_1|x-y|$ for all $x \in \overline{\Omega}$ and, for each $x \in \overline{\Omega}$, there is a curve $\eta_x \in \mathrm{Lip}([0,\,\tau(x)])$ having the properties: $\eta_x(0) = x$, $\eta_x(\tau(x)) = y$, $\eta_x(s) \in \overline{\Omega}$ for all $s \in [0,\,\tau(x)]$ and $|\dot{\eta}_x(s)| \le 1$ for a.e. $s \in [0,\tau(x)]$. We fix such a function τ and a collection $\{\eta_x\}$ of curves. Thanks to Lemma 5.2, we may choose constants $\delta > 0$ and $C_0 > 0$ such that

$$L(x,\xi) \le C_0 \quad \text{for all } (x,\xi) \in \overline{\Omega}\times\overline{B}_\delta.$$

Fix any $x \in \overline{\Omega} \setminus \{y\}$ and define $(\eta, v, l) \in \mathrm{SP}(x)$ by setting $\eta(s) = \eta_x(\delta s)$ for $s \in [0,\ \tau(x)/\delta]$, $\eta(s) = y$ for $s > \tau(x)/\delta$ and $(v(s),\ l(s)) = (\dot\eta(s), 0)$ for $s \in \overline{\mathbb{R}}_+$. Observe that

$$\begin{aligned}\mathscr{L}(\tau(x)/\delta, \eta, v, l) &= \int_0^{\tau(x)/\delta} L(\eta_x(\delta s),\ \delta\dot\eta_x(\delta s))\mathrm{d}s \\ &= \delta^{-1}\int_0^{\tau(x)} L(\eta_x(s), -\delta\dot\eta_x(s))\mathrm{d}s \\ &\le \delta^{-1}C_0\tau(x) \le \delta^{-1}C_0C_1|x-y|,\end{aligned}$$

which yields

$$W(x, y) \le \delta^{-1}C_0C_1|x-y|. \tag{110}$$

We define the function $w : \overline{Q} \to \mathbb{R}$ by

$$w(x,t) = \inf\big\{\mathscr{L}(r, \eta, v, l)\ :\ r > t,\ (\eta, v, l) \in \mathrm{SP}(x) \quad \text{such that} \quad \eta(r) = y\big\}.$$

It is clear by the above definition that

$$W(x, y) = \inf_{t>0} w(x,t) \quad \text{for all } x \in \overline{\Omega}. \tag{111}$$

Also, the dynamic programming principle yields

$$w(x,t) = \inf\big\{\mathscr{L}(t, \eta, v, l) + W(\eta(t), y)\ :\ (\eta, v, l) \in \mathrm{SP}(x)\big\}.$$

(We leave it to the reader to prove this identity.) In view of (110), we fix a $k \in \mathbb{N}$ so that $\delta^{-1}C_0C_1 \le k$ and note that for all $(x,t) \in Q$,

$$w(x,t) \le \inf\big\{\mathscr{L}(t, \eta, v, l) + k|\eta(t) - y|\ :\ (\eta, v, l) \in \mathrm{SP}(x)\big\} = u_k(x,t).$$

Consequently, we have

$$\inf_{t>0} w(x,t) \le \inf_{t>0} u_k(x,t) \quad \text{for all } \ x \in \overline{\Omega},$$

which together with (111) yields

$$W(x, y) \le \inf_{t>0} u_k(x,t) \quad \text{for all } \ x \in \overline{\Omega}.$$

By Lemma 5.11, if we set $v(x) = \inf_{t>0} u_k(x,t)$ for $x \in \overline{\Omega}$, then $v \in C(\overline{\Omega})$ is a viscosity subsolution of (SNP). Moreover, since $v(x) \le u_k(x,0) = k|x-y|$ for all $x \in \overline{\Omega}$, we have $v(y) \le 0$. Thus, we find that $v(x) \le v(y) + d(x,y) \le d(x,y)$ for all $x \in \overline{\Omega}$. We now conclude that $W(x,y) \le v(x) \le d(x,y)$ for all $x \in \overline{\Omega}$. The proof is complete. □

Proposition 5.5. *Let* $y \in \overline{\Omega}$ *and* $\delta > 0$. *Then we have* $y \in \mathscr{A}$ *if and only if*

$$\inf\left\{\mathscr{L}(t,\eta,v,l) \,:\, t > \delta,\ (\eta,v,l) \in \mathrm{SP}(y) \quad \textit{such that} \quad \eta(t) = y\right\} = 0. \qquad (112)$$

Proof. First of all, we define the function $u \in \mathrm{UC}(\overline{Q})$ as the viscosity solution of (ENP)–(ID), with $u_0 = d(\cdot, y)$. By Theorem 5.5, we have

$$u(x,t) = \inf\left\{\mathscr{L}(t,\eta,v,l) + d(\eta(t),y) \,:\, (\eta,v,l) \in \mathrm{SP}(x)\right\} \quad \text{for all } (x,t) \in Q.$$

We combine this formula and Proposition 5.4, to get

$$u(x,t) = \inf\left\{\mathscr{L}(r,\eta,v,l) \,: r > t,\ (\eta,v,l) \in \mathrm{SP}(x) \ \text{ such that } \ \eta(r) = y\right\} \qquad (113)$$
$$\text{for all } (x,t) \in Q.$$

Now, we assume that $y \in \mathscr{A}$. The function $d(\cdot, y)$ is then a viscosity solution of (SNP) and u is a viscosity solution of (ENP)–(ID), with $u_0 = d(\cdot, y)$. Hence, by Theorem 3.1, we have $d(x,y) = u(x,t)$ for all $(x,t) \in \overline{Q}$. Thus,

$$0 = d(y,y) = \inf\left\{\mathscr{L}(r,\eta,v,l) \,:\, r > t, (\eta,v,l) \in \mathrm{SP}(y) \quad \text{such that} \quad \eta(r) = y\right\}$$
$$\text{for all } t > 0.$$

This shows that (112) is valid.

Now, we assume that (112) holds. This assumption and (113) show that $u(y,\delta) = 0$. Formula (113) shows as well that for each $x \in \overline{\Omega}$, the function $u(x,\cdot)$ is nondecreasing in $\overline{\mathbb{R}}_+$. In particular, we have $d(x,y) \le u(x,t)$ for all $(x,t) \in Q$. Let $p \in D_x^- d(x,y)|_{x=y}$. Then we have $(p,0) \in D^- u(y,\delta)$ and

$$\begin{cases} H(y,p) \ge 0 & \text{if } y \in \Omega, \\ \max\{H(y,p),\ \gamma(y)\cdot p - g(y)\} \ge 0 & \text{if } y \in \partial\Omega. \end{cases}$$

This shows that $d(\cdot, y)$ is a viscosity solution of (SNP). Hence, we have $y \in \mathscr{A}$. □

6 Large-Time Asymptotic Solutions

We discuss the large-time behavior of solutions of (ENP)–(ID) following [8, 38, 39].

There has been much interest in the large time behavior of solutions of Hamilton–Jacobi equations since Namah and Roquejoffre in [53] have first established a general convergence result for solutions of

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{in } (x,t) \in \Omega \times \mathbb{R}_+ \qquad (1.2)$$

under (A5), (A6) and the assumptions

$$\begin{aligned} &H(x,p) \ge H(x,0) \quad \text{for all } (x,p) \in \Omega \times \mathbb{R}^n, \\ &\max_{\Omega} H(x,0) = 0, \end{aligned} \tag{114}$$

where Ω is a smooth compact n-dimensional manifold without boundary. Fathi in [26] has then established a similar convergence result but under different type hypotheses, where (114) replaced by a strict convexity of the Hamiltonian $H(x,p)$ in p, by the dynamical approach based on weak KAM theory [25, 27]. Barles and Souganidis have obtained in [3] more general results in the periodic setting (i.e., in the case where Ω is n-dimensional torus), for possibly non-convex Hamiltonians, by using a PDE-viscosity solutions approach, which does not depend on the variational formula for the solutions like the one in Theorem 5.5. We refer to [7] for a recent view on this approach.

The approach of Fathi has been later modified and refined by Roquejoffre [54], Davini and Siconolfi in [21], and others. The same asymptotic problem in the whole domain $\mathbb{R}^n$ has been investigated by Barles and Roquejoffre in [10], Fujita et al., Ichihara and the author in [30, 34–37] in various situations.

There has been as well a considerable interest in the large time asymptotic behavior of solutions of Hamilton–Jacobi equation with boundary conditions. The investigations in this direction are papers: Mitake [48] (the state-constraint boundary condition), Roquejoffre [54] (the Dirichlet boundary condition in the classical sense), Mitake [49, 50] (the Dirichlet boundary condition in the viscosity framework). More recent studies are due to Barles, Mitake and the author in [8, 9, 38], where the Neumann boundary conditions including the dynamical boundary conditions are treated. In [8, 9], the PDE-viscosity solutions approach of Barles–Souganidis is adapted to problems with boundary conditions.

Yokoyama et al. in [58] and Giga et al. in [32, 33] have obtained some results on the large time behavior of solutions of Hamilton–Jacobi equations with noncoercive Hamiltonian which is motivated by a crystal growth model.

We also refer to the articles [13, 54] and to [16, 51, 52] for the large time behavior of solutions, respectively, of time-dependent Hamilton–Jacobi equations and of weakly coupled systems of Hamilton–Jacobi equations.

As before, we assume throughout this section that hypotheses (A1)–(A7) hold and that $u_0 \in C(\overline{\Omega})$. Moreover, we assume that $c^{\#} = 0$. Throughout this section $u = u(x,t)$ denotes the viscosity solution of (ENP)–(ID).

We set

$$Z = \{(x,p) \in \overline{\Omega} \times \mathbb{R}^n \,:\, H(x,p) = 0\}.$$

(A9)$_\pm$ There exists a function $\omega_0 \in C([0,\infty))$ satisfying $\omega_0(r) > 0$ for all $r > 0$ such that if $(x,p) \in Z$, $\xi \in D_p^- H(x,p)$ and $q \in \mathbb{R}^n$, then

$$H(x, p+q) \ge \xi \cdot q + \omega_0((\xi \cdot q)_\pm).$$

The following proposition describes the long time behavior of solutions of (ENP)–(ID).

Theorem 6.1. *Assume that either* (A9)$_+$ *or* (A9)$_-$ *holds. Then there exists a viscosity solution* $w \in \mathrm{Lip}(\overline{\Omega})$ *of* (SNP) *for which*

$$\lim_{t\to\infty} u(x,t) = w(x) \quad \textit{uniformly on } \overline{\Omega}. \tag{115}$$

The following example is an adaptation of the one from Barles–Souganidis to the Neumann problem, which shows the necessity of a stronger condition like (A9)$_\pm$ beyond the convexity assumption (A7) in order to have the asymptotic behavior described in the above theorem.

Example 6.1. Let $n = 2$ and $\Omega = B_4$. Let $\eta, \zeta \in C^1(\overline{\mathbb{R}}_+)$ be functions such that $0 \le \eta(r) \le 1$ for all $r \in \overline{\mathbb{R}}_+$, $\eta(r) = 1$ for all $r \in [0, 1]$, $\eta(r) = 0$ for all $r \in [2,\infty)$, $\zeta(r) \ge 0$ for all $r \in \overline{\mathbb{R}}_+$, $\zeta(r) = 0$ for all $r \in [0,2] \cup [3,\infty)$ and $\zeta(r) > 0$ for all $r \in (2, 3)$. Fix a constant $M > 0$ so that $M \ge \|\zeta'\|_{\infty,\mathbb{R}_+}$. We consider the Hamiltonian $H : \overline{\Omega} \times \mathbb{R}^2$ given by

$$\begin{aligned} H(x,y,p,q) &= |-yp + xq + \zeta(r)| - \zeta(r) \\ &\quad + \eta(r)\sqrt{p^2+q^2} + (1-\eta(r))\left(\left|\frac{x}{r}p + \frac{y}{r}q\right| - M\right)_+, \end{aligned}$$

where $r = r(x,y) := \sqrt{x^2+y^2}$. Let $u \in C^1(\overline{\Omega} \times \overline{\mathbb{R}}_+)$ be the function given by

$$u(x,y,t) = \zeta(r)\left(\frac{y}{r}\cos t - \frac{x}{r}\sin t\right),$$

where, as above, $r = \sqrt{x^2+y^2}$. It is easily checked that u is a classical solution of

$$\begin{cases} u_t(x,y,t) + H(x,y,u_x(x,y,t),u_y(x,y,t)) = 0 & \text{in } B_4 \times \mathbb{R}_+, \\ \nu(x,y)\cdot(u_x(x,y,t),u_y(x,y,t)) = 0 & \text{on } \partial B_4 \times \mathbb{R}_+, \end{cases}$$

where $\nu(x,y)$ denotes the outer unit normal at $(x,y) \in \partial B_4$. Note here that if we introduce the polar coordinate system

$$x = r\cos\theta, \quad y = r\sin\theta$$

and the new function

$$v(r,\theta,t) = u(r\cos\theta, r\sin\theta, t) \quad \text{for } (r,\theta,t) \in \mathbb{R}_+ \times \mathbb{R} \times \overline{\mathbb{R}}_+,$$

then the above Hamilton–Jacobi equation reads

$$v_t + \widetilde{H}(r,\theta,v_r,v_\theta) = 0,$$

where

$$\widetilde{H}(r,\theta,p_r,p_\theta)$$
$$= |p_\theta + \zeta(r)| - \zeta(r) + \eta(r)\sqrt{p_r^2 + \left(\frac{p_\theta}{r}\right)^2} + (1-\eta(r))\,(|p_r| - M)_+ ,$$

while the definition of u reads

$$v(r,\theta,t) = \zeta(r)\sin(\theta - t).$$

Note also that any constant function w on $\overline{B}_4$ is a classical solution of

$$\begin{cases} H(x,y,w_x(x,y,t),w_y(x,y,t)) = 0 & \text{in } B_4, \\ \nu(x,y)\cdot(w_x(x,y,t),w_y(x,y,t)) = 0 & \text{on } \partial B_4, \end{cases}$$

which implies that the eigenvalue $c^\#$ is zero.

It is clear that u does not have the asymptotic behavior (115). As is easily seen, the Hamiltonian H satisfies (A5)–(A7), but neither of (A9)$_\pm$.

6.1 Preliminaries to Asymptotic Solutions

According to Theorem 3.3 and Corollary 3.1, we know that $u \in \mathrm{BUC}(\overline{Q})$. We set

$$u_\infty(x) = \liminf_{t\to\infty} u(x,t) \quad \text{for all } x \in \overline{\Omega}.$$

Lemma 6.1. *The function u_∞ is a viscosity solution of* (SNP) *and* $u_\infty \in \mathrm{Lip}(\overline{\Omega})$.

Proof. Note that

$$u_\infty(x) = \lim_{t\to\infty} \inf\{u(x,t+r) \,:\, r>0\} \quad \text{for all } x \in \overline{\Omega}. \tag{116}$$

By Lemma 5.11, if we set

$$v(x,t) = \inf\{u(x,t+r) \,:\, r>0\} \quad \text{for } (x,t) \in \overline{Q},$$

then $v \in \mathrm{BUC}(\overline{Q})$ and it is a viscosity solution of (ENP). For each $x \in \overline{\Omega}$, the function $v(x,\cdot)$ is nondecreasing in $\overline{\mathbb{R}}_+$. Hence, by the Ascoli–Arzela theorem or Dini's lemma, we see that the convergence in (116) is uniform in $\overline{\Omega}$. By Proposition 1.9, we see that the function $u_\infty(x)$, as a function of (x,t), is a viscosity solution of (ENP), which means that u_∞ is a viscosity solution of (SNP). Finally, Proposition 1.14 guarantees that $u_\infty \in \mathrm{Lip}(\overline{\Omega})$. □

We introduce the following notation:

$$S = \{(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \ : \ \xi \in D_p^- H(x,p) \ \text{ for some } \ (x,p) \in Z\},$$

$$P(x,\xi) = \{p \in \mathbb{R}^n \ : \ \xi \in D_p^- H(x,p)\} \quad \text{for } \ (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Lemma 6.2. (i) $Z,\ S \subset \overline{\Omega} \times B_{R_0}$ *for some* $R_0 > 0$.

(ii) *Assume that* (A9)$_+$ *holds. Then there exist constants* $\delta > 0$ *and* $R_1 > 0$ *such that for any* $(x,\xi) \in S$ *and any* $\varepsilon \in (0,\ \delta)$, *we have* $P(x,(1+\varepsilon)\xi) \neq \emptyset$ *and* $P(x,(1+\varepsilon)\xi) \subset B_{R_1}$.

(iii) *Assume that* (A9)$_-$ *holds. Then there exist constants* $\delta > 0$ *and* $R_1 > 0$ *such that for any* $(x,\xi) \in S$ *and any* $\varepsilon \in (0,\ \delta)$, *we have* $P(x,(1-\varepsilon)\xi) \neq \emptyset$ *and* $P(x,(1-\varepsilon)\xi) \subset B_{R_1}$.

Proof. (i) It follows from coercivity (A6) that there exists a constant $R_1 > 0$ such that $Z \subset \mathbb{R}^n \times B_{R_1}$. Next, fix any $(x,\xi) \in S$. Then, by the definition of S, we may choose a point $p \in P(x,\xi)$ such that $(x,p) \in Z$. Note that $|p| < R_1$. By convexity (A7), we have

$$H(x,p') \geq H(x,p) + \xi \cdot (p' - p) \qquad \text{for all } p' \in \mathbb{R}^n.$$

Assuming that $\xi \neq 0$ and setting $p' = p + \xi/|\xi|$ in the above, we get

$$|\xi| = \xi \cdot (p' - p) \leq H(x,p') - H(x,p) < \sup_{\overline{\Omega} \times B_{R_1+1}} H - \inf_{\overline{\Omega} \times B_{R_1}} H.$$

We may choose a constant $R_2 > 0$ so that the right-hand side is less than R_2, and therefore $\xi \in B_{R_2}$. Setting $R_0 = \max\{R_1, R_2\}$, we conclude that $Z,\ S \subset \mathbb{R}^n \times B_{R_0}$.

(ii) By (i), there is a constant $R_0 > 0$ such that $Z, S \subset \overline{\Omega} \times B_{R_0}$. We set $\delta = \omega_0(1)$, where ω_0 is from (A9)$_+$. In view of coercivity (A6), replacing $R_0 > 0$ by a larger constant if necessary, we may assume that $H(x,p) \geq 1 + \omega_0(1)$ for all $(x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_{R_0})$.

Fix any $(x,\xi) \in S$, $p \in P(x,\xi)$ and $\varepsilon \in (0,\delta)$. Note that $\xi, p \in B_{R_0}$. By (A9)$_+$, for all $x \in \mathbb{R}^n$ we have

$$H(x,q) \geq \xi \cdot (q-p) + \omega_0\left((\xi \cdot (q-p))_+\right).$$

We set $V := \{q \in \overline{B}_{2R_0}(p) \ : \ |\xi \cdot (q-p)| \leq 1\}$. Let $q \in V$ and observe the following: if $q \in \partial B_{2R_0}(p)$, which implies that $|q| \geq R_0$, then $H(x,q) \geq 1 + \omega_0(1) > 1 + \varepsilon \geq (1+\varepsilon)\xi \cdot (q-p)$. If $\xi \cdot (q-p) = 1$, then $H(x,q) \geq 1 + \omega_0(1) > 1 + \varepsilon = (1+\varepsilon)\xi \cdot (q-p)$. Also, if $\xi \cdot (q-p) = -1$, then $H(x,q) \geq \xi \cdot (q-p) > (1+\varepsilon)\xi \cdot (q-p)$. Accordingly, the function $G(q) := H(x,q) - (1+\varepsilon)\xi \cdot (q-p)$ on $\mathbb{R}^n$ is positive on ∂V while it vanishes at $q = p \in V$, and hence it attains a minimum over the set V at an interior point of V. Thus, $P(x,(1+\varepsilon)\xi) \neq \emptyset$. By the convexity of G, we see easily that $G(q) > 0$ for all $q \in \mathbb{R}^n \setminus V$ and conclude that $P(x, (1+\varepsilon)\xi) \subset B_{2R_0}$.

(iii) Let ω_0 be the function from (A9)$_-$. As before, we choose $R_0 > 0$ so that $Z, S \subset \overline{\Omega} \times B_{R_0}$ and $H(x, p) \geq 1 + \omega_0(1)$ for all $(x, p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_{R_0})$, and set $\delta = \omega_0(1) \wedge 1$. Note that for all $x \in \mathbb{R}^n$,

$$H(x, q) \geq \xi \cdot (q - p) + \omega_0\left((\xi \cdot (q - p))_-\right).$$

Fix any $(x, \xi) \in S$, $p \in P(x, \xi)$ and $\varepsilon \in (0, \delta)$. Set $V := \{q \in \overline{B}_{2R_0}(p) : |\xi \cdot (q - p)| \leq 1\}$. Let $q \in V$ and observe the following: if $q \in \partial B_{2R_0}(p)$, then $H(x, q) \geq 1 + \omega_0(1) > 1 + \varepsilon \geq (1 - \varepsilon)\xi \cdot (q - p)$. If $\xi \cdot (q - p) = -1$, then $H(x, q) \geq -1 + \omega_0(1) > -1 + \varepsilon = (1 - \varepsilon)\xi \cdot (q - p)$. If $\xi \cdot (q - p) = 1$, then $H(x, q) \geq \xi \cdot (q - p) > (1 - \varepsilon)\xi \cdot (q - p)$. As before, the function $G(q) := H(x, q) - (1 - \varepsilon)\xi \cdot (q - p)$ attains a minimum over V at an interior point of V. Consequently, $P(x, (1 - \varepsilon)\xi) \neq \emptyset$. Moreover, we get $P(x, (1 - \varepsilon)\xi) \subset B_{2R_0}$. □

Lemma 6.3. *Assume that* (A9)$_+$ *(resp.,* (A9)$_-$*) holds. Then there exist a constant* $\delta_1 > 0$ *and a modulus* ω_1 *such that for any* $\varepsilon \in [0, \delta_1]$ *and* $(x, \xi) \in S$,

$$L(x, (1 + \varepsilon)\,\xi) \leq (1 + \varepsilon)L(x, \xi) + \varepsilon\,\omega_1(\varepsilon) \tag{117}$$

(resp.,

$$L(x, (1 - \varepsilon)\,\xi) \leq (1 - \varepsilon)L(x, \xi) + \varepsilon\,\omega_1(\varepsilon)\,). \tag{118}$$

Before going into the proof, we make the following observation: under the assumption that H, L are smooth, for any $(x, \xi) \in S$, if we set $p := D_\xi L(x, \xi)$, then

$$H(x, p) = 0,$$
$$p \cdot \xi = H(x, p) + L(x, \xi) = L(x, \xi),$$

and, as $\varepsilon \to 0$,

$$\begin{aligned} L(x, (1 + \varepsilon)\xi) &= L(x, \xi) + \varepsilon p \cdot \xi + o(\varepsilon) \\ &= L(x, \xi) + \varepsilon L(x, \xi) + o(\varepsilon) = (1 + \varepsilon)L(x, \xi) + o(\varepsilon). \end{aligned}$$

Proof. Assume that (A9)$_+$ holds. Let $R_0 > 0$, $R_1 > 0$ and $\delta > 0$ be the constants from Lemma 6.2. Fix any $(x, \xi) \in S$ and $\varepsilon \in [0, \delta)$. In view of Lemma 6.2, we may choose a $p_\varepsilon \in P(x, (1 + \varepsilon)\xi)$. Then we have $|p_\varepsilon - p_0| < 2R_1$, $|\xi| < R_0$ and $|\xi \cdot (p_\varepsilon - p_0)| < 2R_0R_1$.

Note by (A9)$_+$ that

$$H(x, p_\varepsilon) \geq \xi \cdot (p_\varepsilon - p_0) + \omega_0\left((\xi \cdot (p_\varepsilon - p_0))_+\right).$$

Hence, we obtain

$$\begin{aligned}L(x,(1+\varepsilon)\,\xi) &= (1+\varepsilon)\,\xi\cdot p_\varepsilon - H(x,p_\varepsilon) \le (1+\varepsilon)\,\xi\cdot p_\varepsilon\\ &\quad -\xi\cdot(p_\varepsilon-p_0)-\omega_0\left((\xi\cdot(p_\varepsilon-p_0))_+\right)\\ &\le (1+\varepsilon)[\xi\cdot p_0 - H(x,p_0)]\\ &\quad +\varepsilon\,\xi\cdot(p_\varepsilon-p_0)-\omega_0\left((\xi\cdot(p_\varepsilon-p_0))_+\right)\\ &\le (1+\varepsilon)L(x,\xi)+\varepsilon\max_{0\le r\le 2R_0R_1}\left(r-\frac{1}{\varepsilon}\omega_0(r)\right).\end{aligned}$$

We define the function ω_1 on $[0,\infty)$ by setting $\omega_1(s)=\max_{0\le r\le 2R_0R_1}(r-\omega_0(r)/s)$ for $s>0$ and $\omega_1(0)=0$ and observe that $\omega_1\in C([0,\infty))$. We have also $L(x,(1+\varepsilon)\xi)\le(1+\varepsilon)L(x,\xi)+\varepsilon\omega_1(\varepsilon)$ for all $\varepsilon\in(0,\delta)$. Thus (117) holds with $\delta_1:=\delta/2$.

Next, assume that (A9)_ holds. Let $R_0>0$, $R_1>0$ and $\delta>0$ be the constants from Lemma 6.2. Fix any $(x,\xi)\in S$ and $\varepsilon\in[0,\delta)$.

As before, we may choose a $p_\varepsilon\in P(x,(1-\varepsilon)\xi)$, and observe that $|p_\varepsilon-p_0|<2R_1$, $|\xi|<R_0$ and $|\xi\cdot(p_\varepsilon-p_0)|<2R_0R_1$. Noting that

$$H(x,p_\varepsilon)\ge\xi\cdot(p_\varepsilon-p_0)+\omega_0\left((\xi\cdot(p_\varepsilon-p_0))_-\right),$$

we obtain

$$\begin{aligned}L(x,(1-\varepsilon)\,\xi) &= (1-\varepsilon)\,\xi\cdot p_\varepsilon - H(x,p_\varepsilon) \le (1-\varepsilon)\,\xi\cdot p_\varepsilon\\ &\quad -\xi\cdot(p_\varepsilon-p_0)-\omega_0\left((\xi\cdot(p_\varepsilon-p_0))_-\right)\\ &\le (1-\varepsilon)[\xi\cdot p_0 - H(x,p_0)]\\ &\quad -\varepsilon\,\xi\cdot(p_\varepsilon-p_0)-\omega_0\left((\xi\cdot(p_\varepsilon-p_0))_-\right)\\ &\le (1+\varepsilon)L(x,\xi)+\varepsilon\max_{0\le r\le 2R_0R_1}\left(r-\frac{1}{\varepsilon}\omega_0(r)\right).\end{aligned}$$

Setting $\omega_1(s)=\max_{0\le r\le 2R_0R_1}(r-\omega_0(r)/s)$ for $s>0$ and $\omega_1(0)=0$, we find a function $\omega_1\in C([0,\infty))$ vanishing at the origin for which $L(x,(1-\varepsilon)\xi)\le(1-\varepsilon)L(x,\xi)+\varepsilon\omega_1(\varepsilon)$ for all $\varepsilon\in(0,\delta)$. Thus (118) holds with $\delta_1:=\delta/2$. □

Theorem 6.2. *Let $u\in\mathrm{Lip}(\overline{\Omega})$ be a subsolution of* (SNP). *Let $\eta\in\mathrm{AC}(\mathbb{R}_+,\mathbb{R}^n)$ be such that $\eta(t)\in\overline{\Omega}$ for all $t\in\mathbb{R}_+$. Set $\mathbb{R}_{+,b}=\{t\in\mathbb{R}_+\,:\,\eta(t)\in\partial\Omega\}$. Then there exists a function $p\in L^\infty(\mathbb{R}_+,\mathbb{R}^n)$ such that*

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}u\circ\eta(t)=p(t)\cdot\dot\eta(t) & \text{for a.e. } t\in\mathbb{R}_+,\\ H(\eta(t),\,p(t))\le 0 & \text{for a.e. } t\in\mathbb{R}_+,\\ \gamma(\eta(t))\cdot p(t)\le g(\eta(t)) & \text{for a.e. } t\in\mathbb{R}_{+,b}.\end{cases}$$

Proof. According to Theorem 4.2, there is a collection $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(\overline{\Omega})$ such that

$$\begin{cases} H(x, Du_\varepsilon(x)) \le \varepsilon & \text{for all } x \in \overline{\Omega}, \\ \dfrac{\partial u_\varepsilon}{\partial \gamma}(x) \le g(x) & \text{for all } x \in \partial\Omega, \\ \|u_\varepsilon - u\|_{\infty,\Omega} < \varepsilon, \\ \sup\limits_{0<\varepsilon<1} \|Du_\varepsilon\|_{L^\infty(\Omega)} < \infty. \end{cases}$$

If we set $p_\varepsilon(t) = Du_\varepsilon \circ \eta(t)$ for all $t \in \overline{\mathbb{R}}_+$, then we have

$$\begin{cases} u_\varepsilon \circ \eta(t) - u_\varepsilon \circ \eta(0) = \displaystyle\int_0^t p_\varepsilon(s) \cdot \dot{\eta}(s)\mathrm{d}s & \text{for all } t \in \mathbb{R}_+, \\ H(\eta(t),\ p_\varepsilon(t)) \le \varepsilon \quad \text{for all } t \in \mathbb{R}_+, \\ \gamma(\eta(t)) \cdot p_\varepsilon(t) \le g(\eta(t)) \quad \text{for all } t \in \mathbb{R}_{+,\mathrm{b}}. \end{cases} \tag{119}$$

Since $\{p_\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\mathbb{R}_+)$, there is a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ converging to zero such that, as $j \to \infty$, the sequence $\{p_{\varepsilon_j}\}$ converges weakly-star in $L^\infty(\mathbb{R}_+)$ to some function $p \in L^\infty(\mathbb{R}_+)$. It is clear from (119) that

$$\begin{cases} u \circ \eta(t) - u \circ \eta(0) = \displaystyle\int_0^t p(s) \cdot \dot{\eta}(s)\mathrm{d}s & \text{for all } t \in \mathbb{R}_+, \\ \gamma(\eta(t)) \cdot p(t) \le g(\eta(t)) \quad \text{for a.e. } t \in \mathbb{R}_{+,\mathrm{b}}. \end{cases}$$

Now, we fix an $i \in \mathbb{N}$ so that $i > \|p\|_{L^\infty(\mathbb{R}_+)}$ and any $0 < T < \infty$, and set $J = [0,\ T]$. Using Lemma 5.6, for each $m \in \mathbb{N}$, we find a function $v_m \in L^\infty(J, \mathbb{R}^n)$ so that

$$H(\eta(s), p(s)) + L_i(\eta(s), -v_m(s)) < -v_m(s) \cdot p(s) + 1/m \quad \text{for a.e. } s \in J. \tag{120}$$

By the convex duality, we have

$$H(x, q) = \sup_{\xi\in\mathbb{R}^n} (\xi \cdot q - L_i(x, \xi)) \quad \text{for all } (x, q) \in \overline{\Omega} \times B_i.$$

(Note that $L_i(x, \cdot)$ is the convex conjugate of the function $H(x, \cdot) + \delta_{\overline{B}_i}$, where $\delta_{\overline{B}_i}(p) = 0$ if $p \in \overline{B}_i$ and $= \infty$ otherwise.) Hence, for any nonnegative function $\psi \in L^\infty(J, \mathbb{R})$ and any $(j, m) \in \mathbb{N}^2$, by (119) we get

$$\begin{aligned} \varepsilon_j \int_J \psi(s)\mathrm{d}s &\ge \int_J \psi(s) H(\eta(s), p_{\varepsilon_j}(s))\mathrm{d}s \\ &\ge \int_J \psi(s)[-v_m(s) \cdot p_{\varepsilon_j}(s) - L_i(\eta(s), -v_m(s))]\mathrm{d}s. \end{aligned}$$

Combining this observation with (120), after sending $j \to \infty$, we obtain

$$0 \geq \int_J \psi(s)(H(\eta(s), p(s)) - 1/m)\mathrm{d}s,$$

which implies that $H(\eta(s), p(s)) \leq 0$ for a.e. $s \in [0, T]$. Since $T > 0$ is arbitrary, we see that

$$H(\eta(s), p(s)) \leq 0 \quad \text{for a.e. } s \in \mathbb{R}_+.$$

The proof is complete. □

6.2 Proof of Convergence

This subsection is devoted to the proof of Theorem 6.1.

Proof (Theorem 6.1). It is enough to show that

$$\limsup_{t\to\infty} u(x,t) \leq u_\infty(x) \quad \text{for all } x \in \overline{\Omega}. \tag{121}$$

Indeed, once this is proved, it is obvious that $\lim_{t\to\infty} u(x,t) = u_\infty(x)$ for all $x \in \overline{\Omega}$, and moreover, since $u \in \mathrm{BUC}(Q)$, by the Ascoli–Arzela theorem, it follows that the convergence, $\lim_{t\to\infty} u(x,t) = u_\infty(x)$, is uniform in $\overline{\Omega}$.

Fix any $z \in \overline{\Omega}$. According to Lemma 6.1 and Corollary 5.2, we may choose a $(\eta, v, l) \in \mathrm{SP}(z)$ such that for all $t > 0$,

$$u_\infty(z) - u_\infty(\eta(t)) = \mathscr{L}(t, \eta, v, l). \tag{122}$$

Due to Theorem 6.2, there exists a function $q \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} u_\infty(\eta(s)) = q(s) \cdot \dot{\eta}(s) & \text{for a.e. } s \in \mathbb{R}_+, \\ H(\eta(s), q(s)) \leq 0 & \text{for a.e. } s \in \mathbb{R}_+, \\ \gamma(\eta(s)) \cdot q(s) \leq g(\eta(s)) & \text{for a.e. } s \in \mathbb{R}_{+,\mathrm{b}}, \end{cases} \tag{123}$$

where $\mathbb{R}_{+,\mathrm{b}} := \{s \in \mathbb{R}_+ \,:\, \eta(s) \in \partial\Omega\}$.

We now show that

$$\begin{cases} H(\eta(s), q(s)) = 0 & \text{for a.e. s } \in \mathbb{R}_+, \\ l(s)\gamma(\eta(s)) \cdot q(s) = l(s)g(\eta(s)) & \text{for a.e. } s \in \mathbb{R}_{+,\mathrm{b}}, \\ -q(s) \cdot v(s) = H(\eta(s), q(s)) + L(\eta(s), -v(s)) & \text{for a.e. s } \in \mathbb{R}_+. \end{cases} \tag{124}$$

We remark here that the last equality in (124) is equivalent to saying that

$$-v(s) \in D_p^- H(\eta(s),\, q(s)) \quad \text{for a.e. } s \in \mathbb{R}_+,$$

(or

$$q(s) \in D_\xi^- L(\eta(s),\, -v(s)) \quad \text{for a.e. } s \in \mathbb{R}_+.)$$

By differentiating (122), we get

$$-\frac{\mathrm{d}}{\mathrm{d}s} u_\infty(\eta(s)) = L(\eta(s), -v(s)) + l(s)g(\eta(s)) \quad \text{for a.e. } s \in \mathbb{R}_+.$$

Combining this with (123), we calculate

$$\begin{aligned} 0 &= q(s)\cdot\dot\eta(s) + L(\eta(s),-v(s)) + l(s)g(\eta(s)) \\ &= q(s)\cdot(v(s) - l(s)\gamma(\eta(s))) + L(\eta(s),-v(s)) + l(s)g(\eta(s)) \\ &\ge -H(\eta(s),q(s)) - l(s)(q(s)\cdot\gamma(\eta(s)) - g(\eta(s))) \ge 0 \end{aligned}$$

for a.e. $s \in \mathbb{R}_+$, which guarantees that (124) holds.

Fix any $\varepsilon > 0$. We prove that there is a constant $\tau > 0$ and for each $x \in \overline{\Omega}$ a number $\sigma(x) \in [0,\, \tau]$ for which

$$u_\infty(x) + \varepsilon > u(x, \sigma(x)). \tag{125}$$

In view of the definition of u_∞, for each $x \in \overline{\Omega}$ there is a constant $t(x) > 0$ such that

$$u_\infty(x) + \varepsilon > u(x,\, t(x)).$$

By continuity, for each fixed $x \in \overline{\Omega}$, we can choose a constant $r(x) > 0$ so that

$$u_\infty(y) + \varepsilon > u(y,\, t(x)) \quad \text{for } y \in \overline{\Omega} \cap B_{r(x)}(x),$$

where $B_\rho(x) := \{y \in \mathbb{R}^n \,:\, |y - x| < \rho\}$. By the compactness of $\overline{\Omega}$, there is a finite sequence x_i, $i = 1, 2, \ldots, N$, such that

$$\overline{\Omega} \subset \bigcup_{1\le i\le N} B_{r(x_i)}(x_i),$$

That is, for any $y \in \overline{\Omega}$ there exists x_i, with $1 \le i \le N$, such that $y \in B_{r(x_i)}(x_i)$, which implies

$$u_\infty(y) + \varepsilon > u(y,\, t(x_i)).$$

Thus, setting

$$\tau = \max_{1\le i\le N} t(x_i),$$

we find that for each $x \in \overline{\Omega}$ there is a constant $\sigma(x) \in [0, \tau]$ such that (125) holds.

In what follows we fix $\tau > 0$ and $\sigma(x) \in [0, \tau]$ as above. Also, we choose a constant $\delta_1 > 0$ and a modulus ω_1 as in Lemma 6.3.

We divide our argument into two cases according to which hypothesis is valid, $(A9)_+$ or $(A9)_-$. We first argue under hypothesis $(A9)_+$. Choose a constant $T > \tau$ so that $\tau/(T-\tau) \le \delta_1$. Fix any $t \ge T$, and set $\theta = \sigma(\eta(t)) \in [0, \tau]$. We set $\delta = \theta/(t-\theta)$ and note that $\delta \le \tau/(t-\tau) \le \delta_1$. We define functions $\eta_\delta, v_\delta, l_\delta$ on $\overline{\mathbb{R}}_+$ by

$$\eta_\delta(s) = \eta((1+\delta)s),$$
$$v_\delta(s) = (1+\delta)v((1+\delta)s),$$
$$l_\delta(s) = (1+\delta)l((1+\delta)s),$$

and note that $(\eta_\delta, v_\delta, l_\delta) \in \mathrm{SP}(z)$.

By (124) together with the remark after (124), we know that $H(\eta(s), q(s)) = 0$ and $-v(s) \in D_p^- H(\eta(s), q(s))$ for a.e. $s \in \mathbb{R}_+$. That is, $(\eta(s), -v(s)) \in S$ for a.e. $s \in \mathbb{R}_+$. Therefore, by (117), we get for a.e. $s \in \mathbb{R}_+$,

$$L(\eta_\delta(s), -v_\delta(s)) \le (1+\delta)L\big(\eta((1+\delta)s), -v((1+\delta)s)\big) + \delta\omega_1(\delta).$$

Integrating this over $(0, t-\theta)$, making a change of variables in the integral and noting that $(1+\delta)(t-\theta) = t$, we get

$$\begin{aligned}\int_0^{t-\theta} L(\eta_\delta(s), -v_\delta(s))\mathrm{d}s &\le \int_0^t L(\eta(s), -v(s))\mathrm{d}s + (t-\theta)\delta\omega_1(\delta)\\ &= \int_0^t L(\eta(s), -v(s))\mathrm{d}s + \theta\omega_1(\delta),\end{aligned}$$

as well as
$$\int_0^{t-\theta} l_\delta(s)g(\eta_\delta(s))\mathrm{d}s = \int_0^t l(s)g(\eta(s))\mathrm{d}s.$$

Moreover,

$$\begin{aligned}u(z,t) &\le \mathscr{L}(t-\theta, \eta_\delta, v_\delta, l_\delta) + u(\eta_\delta(t-\theta), \theta)\\ &\le \int_0^t \big(L(\eta(s), -v(s)) + l(s)g(\eta(s))\big)\mathrm{d}s + \theta\omega_1(\delta) + u\big(\eta(t), \sigma(\eta(t))\big)\\ &< u_\infty(z) - u_\infty(\eta(t)) + \tau\omega_1(\delta) + u_\infty(\eta(t)) + \varepsilon\\ &= u_\infty(z) + \tau\omega_1(\delta) + \varepsilon.\end{aligned}$$

Thus, recalling that $\delta \le \tau/(t-\tau)$, we get

$$u(z,t) \le u_\infty(z) + \tau\omega_1\Big(\frac{\tau}{t-\tau}\Big) + \varepsilon. \tag{126}$$

Next, we assume that (A9)$_-$ holds. We choose $T > \tau$ as before, and fix $t \geq T$. Set $\theta = \sigma(\eta(t-\tau)) \in [0, \tau]$ and $\delta = (\tau-\theta)/(t-\theta)$. Observe that $(1-\delta)(t-\theta) = t-\tau$ and $\delta \leq \tau/(t-\tau) \leq \delta_1$.

We set $\eta_\delta(s) = \eta((1-\delta)s)$, $v_\delta(s) = (1-\delta)v((1-\delta)s)$ and $l_\delta(s) = (1-\delta)l((1-\delta)s)$ for $s \in \overline{\mathbb{R}}_+$ and observe that $(\eta_\delta, v_\delta, l_\delta) \in \mathrm{SP}(z)$. As before, thanks to (118), we have

$$L(\eta_\delta(s), -v_\delta(s)) \leq (1-\delta)L(\eta((1-\delta)s), -v(1-\delta)s)) + \delta\omega_1(\delta) \quad \text{for a.e. } s \in \mathbb{R}_+.$$

Hence, we get

$$\begin{aligned}\int_0^{t-\theta} L(\eta_\delta(s), -v_\delta(s))\mathrm{d}s &\leq \int_0^{t-\tau} L(\eta(s), -v(s))\mathrm{d}s + (t-\theta)\delta\omega_1(\delta) \\ &= \int_0^{t-\tau} L(\eta(s), -v(s))\mathrm{d}s + (\tau-\theta)\omega_1(\delta),\end{aligned}$$

and

$$\int_0^{t-\theta} l_\delta(s)g(\eta_\delta(s))\mathrm{d}s = \int_0^{t-\tau} l(s)g(\eta(s))\mathrm{d}s.$$

Furthermore, we calculate

$$\begin{aligned}u(z,t) &\leq \mathscr{L}(t-\theta, \eta_\delta, v_\delta, l_\delta) + u(\eta_\delta(t-\theta), \theta) \\ &\leq \mathscr{L}(t-\tau, \eta, v, l) + \tau\omega_1(\delta) + u(\eta(t-\tau), \sigma(\eta(t-\tau))) \\ &< u_\infty(z) + \tau\omega_1(\delta) + \varepsilon.\end{aligned}$$

Thus, we get

$$u(z,t) \leq u_\infty(z) + \tau\omega_1\Big(\frac{\tau}{t-\tau}\Big) + \varepsilon,$$

From the above inequality and (126) we see that (121) is valid. □

6.3 Representation of the Asymptotic Solution u_∞

According to Theorem 6.1, if either (A9)$_+$ or (A9)$_-$ holds, then the solution $u(x,t)$ of (ENP)–(ID) converges to the function $u_\infty(x)$ in $C(\overline{\Omega})$ as $t \to \infty$, where the function u_∞ is given by

$$u_\infty(x) = \liminf_{t\to\infty} u(x,t) \quad \text{for } x \in \overline{\Omega}.$$

In this subsection, we *do not* assume (A9)$_+$ or (A9)$_-$ and give two characterizations of the function u_∞.

Let $\mathscr{S}^-$ and $\mathscr{S}$ denote the sets of all viscosity subsolutions of (SNP) and of all viscosity solutions of (SNP), respectively.

Theorem 6.3. *Set*

$$\mathscr{F}_1 = \{v \in \mathscr{S}^- \ : \ v \le u_0 \ in\ \overline{\Omega}\},$$
$$u_0^- = \sup \mathscr{F}_1,$$
$$\mathscr{F}_2 = \{w \in \mathscr{S} \ : \ w \ge u_0^- \ in\ \overline{\Omega}\}.$$

Then $u_\infty = \inf \mathscr{F}_2$.

Proof. By Proposition 1.10, we have $u_0^- \in \mathscr{S}^-$. It is clear that $u_0^- \le u_0$ in $\overline{\Omega}$. Hence, by Theorem 3.1 applied to the functions u_0^- and u, we get $u_0^-(x) \le u(x,t)$ for all $(x,t) \in Q$, which implies that $u_0^- \le u_\infty$ in $\overline{\Omega}$. This together with Lemma 6.1 ensures that $u_\infty \in \mathscr{F}_2$, which shows that $\inf \mathscr{F}_2 \le u_\infty$ in $\overline{\Omega}$.

Next, we set

$$u^-(x,t) = \inf_{r>0} u(x,t+r) \quad \text{for all } (x,t) \in \overline{Q}.$$

By Lemma 5.11, the function u^- is a solution of (ENP) and the function $u^-(\cdot, 0)$ is a viscosity subsolution of (SNP). Also, it is clear that $u^-(x,0) \le u_0(x)$ for all $x \in \overline{\Omega}$, which implies that $u^-(\cdot,0) \le u_0^- \le \inf \mathscr{F}_2$ in $\overline{\Omega}$. We apply Theorem 3.1 to the functions u^- and $\inf \mathscr{F}_2$, to obtain $u^-(x,t) \le \inf \mathscr{F}_2(x)$ for all $(x,t) \in Q$, from which we get $u_\infty \le \inf \mathscr{F}_2$ in $\overline{\Omega}$, and conclude the proof. □

Let $d : \overline{\Omega}^2 \to \mathbb{R}$ and $\mathscr{A}$ denote the distance-like function and the Aubry set, respectively, as in Sect. 4.

Theorem 6.4. *We have the formula:*

$$u_\infty(x) = \inf\{d(x,y) + d(y,z) + u_0(z) \ : \ z \in \overline{\Omega},\ y \in \mathscr{A}\} \quad \textit{for all } x \in \overline{\Omega}.$$

Proof. We first show that

$$u_0^-(x) = \inf\{u_0(y) + d(x,y) \ : \ y \in \overline{\Omega}\} \quad \text{for all } x \in \overline{\Omega},$$

where u_0^- is the function defined in Theorem 6.3.

Let u_d^- denote the function given by the right hand side of the above formula. Since $u_0^- \in \mathscr{S}^-$, we have

$$u_0^-(x) - u_0^-(y) \le d(x,y) \quad \text{for all } x,y \in \overline{\Omega},$$

which ensures that $u_0^- \le u_d^-$ in $\overline{\Omega}$.

By Theorem 4.4, we have $u_d^- \in \mathscr{S}^-$. Also, by the definition of u_d^-, we have $u_d^-(x) \le u_0(x) + d(x,x) = u_0(x)$ for all $x \in \overline{\Omega}$. Hence, by the definition of u_0^-, we find that $u_0^- \ge u_d^-$ in $\overline{\Omega}$. Thus, we have $u_0^- = u_d^-$ in $\overline{\Omega}$.

It is now enough to show that

$$u_\infty(x) = \inf_{y \in \mathscr{A}} (u_0^-(y) + d(x,y)).$$

Let ϕ denote the function defined by the right hand side of the above formula. The version of Proposition 1.10 for supersolutions ensures that $\phi \in \mathscr{S}^+$, while Theorem 4.4 guarantees that $\phi \in \mathscr{S}^-$. Hence, we have $\phi \in \mathscr{S}$. Observe also that

$$u_0^-(x) \le u_0^-(y) + d(x, y) \quad \text{for all } x, y \in \overline{\Omega},$$

which yields $u_0^- \le \phi$ in $\overline{\Omega}$. Thus, we see by Theorem 6.3 that $u_\infty \le \phi$ in $\overline{\Omega}$.

Now, applying Theorem 4.1 to u_∞, we observe that for all $x \in \overline{\Omega}$,

$$\begin{aligned} u_\infty(x) &= \inf\{u_\infty(y) + d(x, y) \,:\, y \in \mathscr{A}\} \\ &\ge \inf\{u_0^-(y) + d(x, y) \,:\, y \in \mathscr{A}\} = \phi(x). \end{aligned}$$

Thus we find that $u_\infty = \phi$ in $\overline{\Omega}$. The proof is complete. □

Combining the above theorem and Proposition 5.4, we obtain another representation formula for u_∞.

Corollary 6.1. *The following formula holds:*

$$\begin{aligned} u_\infty(x) = \inf\big\{&\mathscr{L}(T, \eta, v, l) + u_0(\eta(T)) \,:\, T > 0,\ (\eta, v, l) \in \mathrm{SP}(x) \\ &\textit{such that } \eta(t) \in \mathscr{A} \textit{ for some } t \in (0, T)\big\}. \end{aligned}$$

Example 6.2. As in Example 3.1, let $n = 1$, $\Omega = (-1, 1)$ and $\gamma = \nu$ on $\partial\Omega$ (i.e., $\gamma(\pm 1) = \pm 1$). Let $H = H(p) = |p|^2$ and $g : \partial\Omega \to \mathbb{R}$ be the function given by $g(-1) = -1$ and $g(1) = 0$. As in Example 3.1, we see that $c^\# = 1$. We set $\tilde{H}(p) = H(p) - c^\# = |p|^2 - 1$. Note that $\tilde{H}$ satisfies both (A9)$_\pm$, and consider the Neumann problem

$$\tilde{H}(v'(x)) = 0 \quad \text{in } \Omega, \qquad \gamma(x) \cdot v'(x) = g(x) \quad \text{on } \partial\Omega. \tag{127}$$

It is easily seen that the distance-like function $d : \overline{\Omega}^2 \to \mathbb{R}$ for this problem is given by $d(x, y) = |x - y|$. Let $\mathscr{A}$ denote the Aubry set for problem (127). By examining the function d, we see that $\mathscr{A} = \{-1\}$. For instance, by observing that

$$D_x^- d(x, -1) = \begin{cases} \{1\} & \text{if } x \in \Omega, \\ (-\infty,\ 1] & \text{if } x = -1, \\ [1,\ \infty) & \text{if } x = 1, \end{cases}$$

we find that $-1 \in \mathscr{A}$. Let $u_0(x) = 0$. Consider the problem

$$\begin{cases} u_t(x, t) + H(u_x(x, t)) = 0 & \text{for } (x, t) \in \Omega \times \mathbb{R}_+, \\ \gamma(x) u_x(x, t) = g(x) & \text{for } (x, t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x, 0) = u_0(x) & \text{for } x \in \overline{\Omega}. \end{cases}$$

If u is the viscosity solution of this problem and the function v is given by $v(x,t) = u(x,t) + c^{\#}t = u(x,t) + t$, then v solves in the viscosity sense

$$\begin{cases} v_t(x,t) + \tilde{H}(v_x(x,t)) = 0 & \text{for } (x,t) \in \Omega \times \mathbb{R}_+, \\ \gamma(x)v_x(x,t) = g(x) & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ v(x,0) = u_0(x) & \text{for } x \in \overline{\Omega}. \end{cases}$$

Setting

$$u_\infty(x) = \min\{d(x,y) + d(y,z) + u_0(z) \,:\, y \in \mathscr{A},\, z \in \overline{\Omega}\} \quad \text{for } x \in \overline{\Omega},$$

we note that $u_\infty(x) = |x+1|$ for all $x \in \overline{\Omega}$. Thanks to Theorems 6.1 and 6.4, we have

$$\lim_{t\to\infty} v(x,t) = u_\infty(x) \quad \text{uniformly on } \overline{\Omega},$$

which reads

$$\lim_{t\to\infty} (u(x,t) + t - |x+1|) = 0 \quad \text{uniformly on } \overline{\Omega}.$$

That is, we have $u(x,t) \approx -t + |x+1|$ as $t \to \infty$. If we replace $u_0(x) = 0$ by the function $u_0(x) = -3x$, then

$$u_\infty(x) = \min_{y\in\overline{\Omega}}\{|x+1| + |1+y| - 3y\} = |x+1| - 1 \quad \text{for all } x \in \overline{\Omega},$$

and $u(x,t) \approx -t + |x+1| - 1$ as $t \to \infty$.

In some cases the variational formula in Corollary 6.1 is useful to see the convergence assertion of Theorem 6.1.

Under the hypothesis that $c^{\#} = 0$, which is our case, we call a point $y \in \overline{\Omega}$ an *equilibrium point* if $L(y,0) = 0$. This condition, $L(y,0) = 0$, is equivalent to $\min_{p\in\mathbb{R}^n} H(y,p) = 0$.

Let $y \in \overline{\Omega}$ be an equilibrium point. If we define $(\eta, v, l) \in \mathrm{SP}(y)$ by setting $(\eta, v, l)(s) = (y, 0, 0)$, then $\mathscr{L}(t, \eta, v, l) = 0$ for all $t \in \mathbb{R}_+$, and Propositions 5.4 and 5.5 guarantee that $y \in \mathscr{A}$.

We now assume that $\mathscr{A}$ consists of only equilibrium points. Fix any $\varepsilon > 0$ and $x \in \overline{\Omega}$. According to Corollary 6.1, we can choose $\tau, \sigma \in \mathbb{R}_+$ and $(\eta, v, l) \in \mathrm{SP}(x)$ so that $\eta(\tau) \in \mathscr{A}$ and

$$\mathscr{L}(\tau + \sigma, \eta, v, l) + u_0(\eta(\tau + \sigma)) < u_\infty(x) + \varepsilon. \tag{128}$$

Fix any $t > \tau + \sigma$. We define $(\tilde{\eta}, \tilde{v}, \tilde{l}) \in \mathrm{SP}(x)$ by

$$(\tilde{\eta}, \tilde{v}, \tilde{l})(s) = \begin{cases} (\eta, v, l)(s) & \text{for } s \in [0, \tau), \\ (\eta(\tau), 0, 0) & \text{for } s \in [\tau, \tau + \theta), \\ (\eta, v, l)(s - \theta) & \text{for } s \in [\tau + \theta, \infty), \end{cases}$$

where $\theta = t - (\tau + \sigma)$. Using (128), we get

$$u_\infty(x) + \varepsilon > \mathscr{L}(t, \tilde{\eta}, \tilde{v}, \tilde{l}) + u_0(\tilde{\eta}(t)) \geq u(x, t).$$

Therefore, recalling that $\liminf_{t\to\infty} u(x, t) = u_\infty(x)$, we see that $\lim_{t\to\infty} u(x, t) = u_\infty(x)$ for all $x \in \overline{\Omega}$.

6.4 Localization of Conditions (A9)$_\pm$

In this subsection we explain briefly that the following versions of (A9)$_\pm$ localized to the Aubry set $\mathscr{A}$ may replace the role of (A9)$_\pm$ in Theorem 6.1.

(A10)$_\pm$ Let

$$Z_{\mathscr{A}} = \{(x, p) \in \mathscr{A} \times \mathbb{R}^n \;:\; H(x, p) = 0\}.$$

There exists a function $\omega_0 \in C([0, \infty))$ satisfying $\omega_0(r) > 0$ for all $r > 0$ such that if $(x, p) \in Z_{\mathscr{A}}$, $\xi \in D_p^- H(x, p)$ and $q \in \mathbb{R}^n$, then

$$H(x, p + q) \geq \xi \cdot q + \omega_0((\xi \cdot q)_\pm).$$

As before, assume that $c^\# = 0$ and let u be the solution of (ENP)–(ID) and $u_\infty(x) := \liminf_{t\to\infty} u(x, t)$.

Theorem 6.5. *Assume that either* (A10)$_+$ *or* (A10)$_-$ *holds. Then*

$$\lim_{t\to\infty} u(x, t) = u_\infty(x) \quad \textit{uniformly on } \overline{\Omega}. \tag{129}$$

If we set

$$u_\infty^+(x) = \limsup_{t\to\infty} u(x, t) \quad \text{for } x \in \overline{\Omega},$$

we see by Theorem 1.3 that the function $u_\infty^+(x)$ is a subsolution of (ENP), as a function of (x, t), and hence a subsolution of (SNP). That is, $u_\infty^+ \in \mathscr{S}^-$. Since $u_\infty \in \mathscr{S}^+$, once we have shown that $u_\infty^+ \leq u_\infty$ on $\mathscr{A}$, then, by Theorem 4.6, we get

$$u_\infty^+ \leq u_\infty \quad \text{in } \overline{\Omega},$$

which shows that the uniform convergence (129) is valid. Thus we only need to show that $u_\infty^+ \leq u_\infty$ on $\mathscr{A}$.

Following [21] (see also [39]), one can prove the following lemma.

Lemma 6.4. *For any $z \in \mathscr{A}$ there exists an $\alpha = (\eta, v, l) \in \mathrm{SP}(z)$ such that*

$$d(z, \eta(t)) = \mathscr{L}(t, \alpha) = -d(\eta(t), z) \quad \textit{for all} \ \ t > 0.$$

Proof. By Proposition 5.5, for each $k \in \mathbb{N}$ there are an $\alpha_k = (\eta_k, v_k, l_k) \in \mathrm{SP}(z)$ and $\tau_k \geq k$ such that

$$\mathscr{L}(\tau_k, \alpha_k) < \frac{1}{k} \quad \text{and} \quad \eta_k(\tau_k) = z.$$

Observe that for any $j, k \in \mathbb{N}$ with $j < k$,

$$\begin{aligned}\frac{1}{k} &> \mathscr{L}(j, \alpha_k) + \int_j^{\tau_k} [L(\eta_k(s), -v_k(s)) + l_k(s) g(\eta_k(s))] \mathrm{d}s \\ &\geq \mathscr{L}(j, \alpha_k) + d(\eta_k(j), \eta_k(\tau_k)),\end{aligned} \tag{130}$$

and hence

$$\sup_{k \in \mathbb{N}} \mathscr{L}(j, \alpha_k) < \infty \quad \text{for all} \ \ j \in \mathbb{N}.$$

We apply Theorem 5.4, with $T = j \in \mathbb{N}$, and use the diagonal argument, to conclude from (130) that there is an $\alpha = (\eta, v, l) \in \mathrm{SP}(z)$ such that for all $j \in \mathbb{N}$,

$$\mathscr{L}(j, \alpha) \leq \liminf_{k \to \infty} \mathscr{L}(j, \alpha_k) \leq -d(\eta(j), z).$$

Let $0 < t < \infty$, and choose a $j \in \mathbb{N}$ such that $t < j$. Using Propositions 5.4 and 4.1 (ii) (the triangle inequality for d), we compute that

$$\begin{aligned}d(z, \eta(t)) &\leq \mathscr{L}(t, \alpha) = \mathscr{L}(j, \alpha) - \int_t^j [L(\eta(s), -v(s)) + l(s) g(\eta(s))] \mathrm{d}s \\ &\leq \mathscr{L}(j, \alpha) - d(\eta(t), \eta(j)) \leq -d(\eta(j), z) - d(\eta(t), \eta(j)) \\ &\leq -d(\eta(t), z).\end{aligned}$$

Moreover, by the triangle inequality, we get

$$-d(\eta(t), z) \leq d(z, \eta(t)).$$

These together yield

$$d(z, \eta(t)) = \mathscr{L}(t, \alpha) = -d(\eta(t), z) \quad \text{for all } t > 0,$$

which completes the proof. □

The above assertion is somehow related to the idea of the quotient Aubry set (see [46, 47]). Indeed, if we introduce the equivalence relation $\equiv$ on $\mathscr{A}$ by

$$x \equiv y \quad \Longleftrightarrow \quad d(x,y) + d(y,x) = 0,$$

and consider the quotient space $\hat{\mathscr{A}}$ consisting of the equivalence classes

$$[x] = \{y \in \mathscr{A} \ : \ y \equiv x\}, \quad \text{with } \ x \in \mathscr{A},$$

then the space $\hat{\mathscr{A}}$ is a metric space with its distance given by

$$\hat{d}([x],[y]) = d(x,y) + d(y,x).$$

The property of the curve η in the above lemma that $d(z,\eta(t)) = -d(\eta(t),z)$ is now stated as: $\eta(t) \equiv \eta(0)$.

Lemma 6.5. *Let $\psi \in \mathscr{S}^-$ and $x, y \in \mathscr{A}$. If $x \equiv y$, then*

$$\psi(x) - \psi(y) = d(x,y).$$

Proof. By the definition of d, we have

$$\psi(x) - \psi(y) \le d(x,y) \quad \text{and} \quad \psi(y) - \psi(x) \le d(y,x).$$

Hence,

$$\psi(x) - \psi(y) \le d(x,y) = -d(y,x) \le \psi(x) - \psi(y),$$

which shows that $\psi(x) - \psi(y) = d(x,y) = -d(y,x)$. □

Proof (Theorem 6.5). As we have noted above, we need only to show that

$$u_\infty^+(x) \le u_\infty(x) \quad \text{for all } \ x \in \mathscr{A}.$$

To this end, we fix any $z \in \mathscr{A}$. Let $\alpha = (\eta, v, l) \in \mathrm{SP}(z)$ be as in Lemma 6.4. In view of Lemma 6.5, we have

$$u_\infty(z) - u_\infty(\eta(t)) = d(z,\eta(t)) = \mathscr{L}(t,\alpha) \quad \text{for all } \ t > 0.$$

It is obvious that the same assertion as Lemma 6.3 holds if we replace S by

$$S_{\mathscr{A}} := \{(x,\xi) \in \mathscr{A} \times \mathbb{R}^n \ : \ \xi \in D_p^- H(x,p) \quad \text{for some } (x,p) \in Z_{\mathscr{A}}\}.$$

We now just need to follow the arguments in Sect. 6.2, to conclude that

$$u_\infty^+(z) \le u_\infty(z).$$

The details are left to the interested reader. □

Appendix

A.1 Local maxima to global maxima

We recall a proposition from [56] which is about partition of unity.

Proposition A.1. *Let $\mathscr{O}$ be a collection of open subsets of $\mathbb{R}^n$. Set $W := \bigcup_{U\in\mathscr{O}} U$. Then there is a collection $\mathscr{F}$ of C^∞ functions in $\mathbb{R}^n$ having the following properties:*

(i) $0 \le f(x) \le 1$ *for all $x \in W$ and $f \in \mathscr{F}$.*
(ii) *For each $x \in W$ there is a neighborhood V of x such that all but finitely many $f \in \mathscr{F}$ vanish in V.*
(iii) $\sum_{f\in\mathscr{F}} f(x) = 1$ *for all $x \in W$.*
(iv) *For each $f \in \mathscr{F}$ there is a set $U \in \mathscr{O}$ such that* $\operatorname{supp} f \subset U$.

Proposition A.2. *Let Ω be any subset of $\mathbb{R}^n$, $u \in \mathrm{USC}(\Omega,\mathbb{R})$ and $\phi \in C^1(\Omega)$. Assume that $u-\phi$ attains a local maximum at $y \in \Omega$. Then there is a function $\psi \in C^1(\Omega)$ such that $u-\psi$ attains a global maximum at y and $\psi = \phi$ in a neighborhood of y.*

Proof. As usual it is enough to prove the above proposition in the case when $(u-\phi)(y) = 0$.

By the definition of the space $C^1(\Omega)$, there is an open neighborhood W_0 of Ω such that ϕ is defined in W_0 and $\phi \in C^1(W_0)$.

There is an open subset $U_y \subset W_0$ of $\mathbb{R}^n$ containing y such that $\max_{U_y\cap\Omega}(u-\phi) = (u-\phi)(y)$. Since $u \in \mathrm{USC}(\Omega,\mathbb{R})$, for each $x \in \Omega\setminus\{y\}$ we may choose an open subset U_x of $\mathbb{R}^n$ so that $x \in U_x$, $y \notin U_x$ and $\sup_{U_x\cap\Omega} u < \infty$. Set $a_x = \sup_{U_x\cap\Omega} u$ for every $x \in \Omega\setminus\{y\}$.

We set $\mathscr{O} = \{U_z \,:\, z \in \Omega\}$ and $W = \bigcup_{U\in\mathscr{O}} U$. Note that W is an open neighborhood of Ω. By Proposition A.1, there exists a collection $\mathscr{F}$ of functions $f \in C^\infty(\mathbb{R}^n)$ satisfying the conditions (i)–(iv) of the proposition. According to the condition (iv), for each $f \in \mathscr{F}$ there is a point $z \in \Omega$ such that $\operatorname{supp} f \subset U_z$. For each $f \in \mathscr{F}$ we fix such a point $z \in \Omega$ and define the mapping $p : \mathscr{F} \to \Omega$ by $p(f) = z$. We set

$$\psi(x) = \sum_{f\in\mathscr{F},\ p(f)\ne y} a_{p(f)} f(x) + \sum_{f\in\mathscr{F},\ p(f)=y} \phi(x) f(x) \quad \text{for } x \in W.$$

By the condition (ii), we see that $\psi \in C^1(W)$. Fix any $x \in \Omega$ and $f \in \mathscr{F}$, and observe that if $f(x) > 0$ and $p(f) \ne y$, then we have $x \in \operatorname{supp} f \subset U_{p(f)}$ and, therefore, $a_{p(f)} = \sup_{U_{p(f)}\cap\Omega} u \ge u(x)$. Observe also that if $f(x) > 0$ and $p(f) = y$, then we have $x \in U_y$ and $\phi(x) \ge u(x)$. Thus we see that for all $x \in \Omega$,

$$\psi(x) \ge \sum_{f\in\mathscr{F},\ p(f)\ne y} u(x) f(x) + \sum_{f\in\mathscr{F},\ p(f)=y} u(x) f(x) = u(x) \sum_{f\in\mathscr{F}} f(x) = u(x).$$

Thanks to the condition (ii), we may choose a neighborhood $V \subset W$ of y and a finite subset $\{f_j\}_{j=1}^N$ of $\mathscr{F}$ so that

$$\sum_{j=1}^N f_j(x) = 1 \quad \text{for all } x \in V.$$

If $p(f_j) \neq y$ for some $j = 1, \ldots, N$, then $U_{p(f_j)} \cap \{y\} = \emptyset$ and hence $y \notin \operatorname{supp} f_j$. Therefore, by replacing V by a smaller one we may assume that $p(f_j) = y$ for all $j = 1, \ldots, N$. Since $f = 0$ in V for all $f \in \mathscr{F} \setminus \{f_1, \ldots, f_N\}$, we see that

$$\psi(x) = \sum_{j=1}^N \phi(x) f_j(x) = \phi(x) \quad \text{for all } x \in V.$$

It is now easy to see that $u - \psi$ has a global maximum at y. □

A.2 A Quick Review of Convex Analysis

We discuss here basic properties of convex functions on $\mathbb{R}^n$.

By definition, a subset C of $\mathbb{R}^n$ is convex if and only if

$$(1-t)x + ty \in C \quad \text{for all } \; x, y \in C, \; 0 < t < 1.$$

For a given function $f : U \subset \mathbb{R}^n \to [-\infty, \infty]$, its epigraph $\operatorname{epi}(f)$ is defined as

$$\operatorname{epi}(f) = \{(x, y) \in U \times \mathbb{R} \, : \, y \geq f(x)\}.$$

A function $f : U \to [-\infty, \infty]$ is said to be convex if $\operatorname{epi}(f)$ is a convex subset of $\mathbb{R}^{n+1}$.

We are henceforth concerned with functions defined on $\mathbb{R}^n$. When we are given a function f on U with U being a proper subset of $\mathbb{R}^n$, we may think of f as a function defined on $\mathbb{R}^n$ having value ∞ on the set $\mathbb{R}^n \setminus U$.

It is easily checked that a function $f : \mathbb{R}^n \to [-\infty, \infty]$ is convex if and only if for all $x, y \in \mathbb{R}^n$, $t, s \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)t + \lambda s \quad \text{if } \; t \geq f(x) \; \text{ and } \; s \geq f(y).$$

From this, we see that a function $f : \mathbb{R}^n \to (-\infty, \infty]$ is convex if and only if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y).$$

Here we use the convention for extended real numbers, i.e., for any $x \in \mathbb{R}$, $-\infty < x < \infty$, $x \pm \infty = \pm\infty$, $x \cdot (\pm\infty) = \pm\infty$ if $x > 0$, $0 \cdot (\pm\infty) = 0$, etc.

Any affine function $f(x) = a \cdot x + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is a convex function on $\mathbb{R}^n$. Moreover, if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}$ are nonempty sets, then the function on $\mathbb{R}^n$ given by

$$f(x) = \sup\{a \cdot x + b \,:\, (a, b) \in A \times B\}$$

is a convex function. Note that this function f is lower semicontinuous on $\mathbb{R}^n$. We restrict our attention to those functions which take values only in $(-\infty, \infty]$.

Proposition B.1. *Let* $f : \mathbb{R}^n \to (-\infty, \infty]$ *be a convex function. Assume that* $p \in D^- f(y)$ *for some* $y, p \in \mathbb{R}^n$. *Then*

$$f(x) \geq f(y) + p \cdot (x - y) \quad \textit{for all} \;\; x \in \mathbb{R}^n.$$

Proof. By the definition of $D^- f(y)$, we have

$$f(x) \geq f(y) + p \cdot (x - y) + o(|x - y|) \quad \text{as} \;\; x \to y.$$

Hence, fixing $x \in \mathbb{R}^n$, we get

$$f(y) \leq f(tx + (1 - t)y) - tp \cdot (x - y) + o(t) \quad \text{as} \;\; t \to 0+.$$

Using the convexity of f, we rearrange the above inequality and divide by $t > 0$, to get

$$f(y) \leq f(x) - p \cdot (x - y) + o(1) \quad \text{as} \;\; t \to 0+.$$

Sending $t \to 0+$ yields

$$f(x) \geq f(y) + p \cdot (x - y) \quad \text{for all} \;\; x \in \mathbb{R}^n.$$

□

Proposition B.2. *Let* $\mathscr{F}$ *be a nonempty set of convex functions on* $\mathbb{R}^n$ *with values in* $(-\infty, \infty]$. *Then* $\sup \mathscr{F}$ *is a convex function on* $\mathbb{R}^n$ *having values in* $(-\infty, \infty]$.

Proof. It is clear that $(\sup \mathscr{F})(x) \in (-\infty, \infty]$ for all $x \in \mathbb{R}^n$. If $f \in \mathscr{F}$, $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, then we have

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) \leq (1 - t)(\sup \mathscr{F})(x) + t(\sup \mathscr{F})(y)$$

and hence

$$(\sup \mathscr{F})((1 - t)x + ty) \leq (1 - t)(\sup \mathscr{F})(x) + t(\sup \mathscr{F})(y),$$

which proves the convexity of $\sup \mathscr{F}$. □

We call a function $f : \mathbb{R}^n \to (-\infty, \infty]$ *proper convex* if the following three conditions hold:

(a) f is convex on $\mathbb{R}^n$.
(b) $f \in \mathrm{LSC}(\mathbb{R}^n)$.
(c) $f(x) \not\equiv \infty$.

Let $f : \mathbb{R}^n \to [-\infty, \infty]$. The conjugate convex function (or the Legendre–Fenchel transform) of f is the function $f^\star : \mathbb{R}^n \to [-\infty, \infty]$ given by

$$f^\star(x) = \sup_{y\in\mathbb{R}^n} (x \cdot y - f(y)).$$

Proposition B.3. *If f is a proper convex function, then so is $f^\star$.*

Lemma B.1. *If f is a proper convex function on $\mathbb{R}^n$, then $D^- f(y) \neq \emptyset$ for some $y \in \mathbb{R}^n$.*

Proof. We choose a point $x_0 \in \mathbb{R}^n$ so that $f(x_0) \in \mathbb{R}$. Let $k \in \mathbb{N}$, and define the function g_k on $\bar{B}_1(x_0)$ by the formula $g_k(x) = f(x) + k|x - x_0|^2$. Since $g_k \in \mathrm{LSC}(\overline{B}_1(x_0))$, and $g_k(x_0) = g(x_0) \in \mathbb{R}$, the function g_k has a finite minimum at a point $x_k \in \overline{B}_1(x_0)$. Note that if k is sufficiently large, then

$$\min_{\partial B_1(x_0)} g_k = \min_{\partial B_1(x_0)} f + k > f(x_0).$$

Fix such a large k, and observe that $x_k \in B_1(x_0)$ and, therefore, $-2k(x_k - x_0) \in D^- f(x_k)$. □

Proof (Proposition B.3). The function $x \mapsto x \cdot y - f(y)$ is an affine function for any $y \in \mathbb{R}^n$. By Proposition B.2, the function $f^\star$ is convex on $\mathbb{R}^n$. Also, since the function $x \mapsto x \cdot y - f(y)$ is continuous on $\mathbb{R}^n$ for any $y \in \mathbb{R}^n$, as stated in Proposition 1.5, the function $f^\star$ is lower semicontinuous on $\mathbb{R}^n$.

Since f is proper convex on $\mathbb{R}^n$, there is a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. Hence, we have

$$f^\star(y) \geq y \cdot x_0 - f(x_0) > -\infty \quad \text{for all } y \in \mathbb{R}^n.$$

By Lemma B.1, there exist points $y, p \in \mathbb{R}^n$ such that $p \in D^- f(y)$. By Proposition B.1, we have

$$f(x) \geq f(y) + p \cdot (x - y) \quad \text{for all } x \in \mathbb{R}^n.$$

That is,

$$p \cdot y - f(y) \geq p \cdot x - f(x) \quad \text{for all } x \in \mathbb{R}^n,$$

which implies that $f^\star(p) = p \cdot y - f(y) \in \mathbb{R}$. Thus, we conclude that $f^\star : \mathbb{R}^n \to (-\infty, \infty]$, $f^\star$ is convex on $\mathbb{R}^n$, $f^\star \in \mathrm{LSC}(\mathbb{R}^n)$ and $f^\star(x) \not\equiv \infty$. □

The following duality (called convex duality or Legendre–Fenchel duality) holds.

Theorem B.1. *Let* $f : \mathbb{R}^n \to (-\infty, \infty]$ *be a proper convex function. Then*

$$f^{\star\star} = f.$$

Proof. By the definition of $f^\star$, we have

$$f^\star(x) \geq x \cdot y - f(y) \quad \text{for all } \ x, y \in \mathbb{R}^n,$$

which reads

$$f(y) \geq y \cdot x - f^\star(x) \quad \text{for all } \ x, y \in \mathbb{R}^n.$$

Hence,

$$f(y) \geq f^{\star\star}(y) \quad \text{for all } \ y \in \mathbb{R}^n.$$

Next, we show that

$$f^{\star\star}(x) \geq f(x) \quad \text{for all } \ x \in \mathbb{R}^n.$$

We fix any $a \in \mathbb{R}^n$ and choose a point $y \in \mathbb{R}^n$ so that $f(y) \in \mathbb{R}$. We fix a number $R > 0$ so that $|y - a| < R$. Let $k \in \mathbb{N}$, and consider the function $g_k \in \mathrm{LSC}(\overline{B}_R(a))$ defined by $g_k(x) = f(x) + k|x - a|^2$. Let $x_k \in \overline{B}_R(a)$ be a minimum point of the function g_k. Noting that if k is sufficiently large, then

$$g_k(x_k) \leq f(y) + k|y - a|^2 < \min_{\partial B_R(a)} f + kR^2 = \min_{\partial B_R(a)} g_k,$$

we see that $x_k \in B_R(a)$ for k sufficiently large. We henceforth assume that k is large enough so that $x_k \in B_R(a)$. We have

$$D^- g_k(x_k) = D^- f(x_k) + 2k(x_k - a) \ni 0.$$

Accordingly, if we set $\xi_k = -2k(x_k - a)$, then we have $\xi_k \in D^- f(x_k)$. By Proposition B.1, we get

$$f(x) \geq f(x_k) + \xi_k \cdot (x - x_k) \quad \text{for all } \ x \in \mathbb{R}^n,$$

or, equivalently,

$$\xi_k \cdot x_k - f(x_k) \geq \xi_k \cdot x - f(x) \quad \text{for all } \ x \in \mathbb{R}^n.$$

Hence,

$$\xi_k \cdot x_k - f(x_k) = f^\star(\xi_k).$$

Using this, we compute that

$$\begin{aligned} f^{\star\star}(a) &\geq a \cdot \xi_k - f^\star(\xi_k) = \xi_k \cdot a - \xi_k \cdot x_k + f(x_k) \\ &= 2k|x_k - a|^2 + f(x_k). \end{aligned}$$

We divide our argument into the following cases, (a) and (b).

Case (a): $\lim_{k\to\infty} k|x_k - a|^2 = \infty$. In this case, if we set $m = \min_{\bar{B}_R(a)} f$, then we have

$$f^{\star\star}(a) \geq \liminf_{k\to\infty} 2k|x_k - a|^2 + m = \infty,$$

and, therefore, $f^{\star\star}(a) \geq f(a)$.

Case (b): $\liminf_{k\to\infty} k|x_k - a|^2 < \infty$. We may choose a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{x_k\}$ so that $\lim_{j\to\infty} x_{k_j} = a$. Then we have

$$f^{\star\star}(a) \geq \liminf_{j\to\infty} \big(2k_j|x_{k_j} - a|^2 + f(x_{k_j})\big) \geq \liminf_{j\to\infty} f(x_{k_j}) \geq f(a).$$

Thus, in both cases we have $f^{\star\star}(a) \geq f(a)$, which completes the proof. □

Theorem B.2. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be proper convex and $x, \xi \in \mathbb{R}^n$. Then the following three conditions are equivalent each other.*

(i) $\xi \in D^- f(x)$.
(ii) $x \in D^- f^\star(\xi)$.
(iii) $x \cdot \xi = f(x) + f^\star(\xi)$.

Proof. Assume first that (i) holds. By Proposition B.1, we have

$$f(y) \geq f(x) + \xi \cdot (y - x) \quad \text{for all} \;\; y \in \mathbb{R}^n,$$

which reads

$$\xi \cdot x - f(x) \geq \xi \cdot y - f(y) \quad \text{for all } y \in \mathbb{R}^n.$$

Hence,

$$\xi \cdot x - f(x) = \max_{y\in\mathbb{R}^n}(\xi \cdot y - f(y)) = f^\star(\xi).$$

Thus, (iii) is valid.

Next, we assume that (iii) holds. Then the function $y \mapsto \xi \cdot y - f(y)$ attains a maximum at x. Therefore, $\xi \in D^- f(x)$. That is, (i) is valid.

Now, by the convex duality (Theorem B.1), (iii) reads

$$x \cdot \xi = f^{\star\star}(x) + f^\star(\xi).$$

The equivalence between (i) and (iii), with f replaced by $f^\star$, is exactly the equivalence between (ii) and (iii). The proof is complete. □

Finally, we give a Lipschitz regularity estimate for convex functions.

Theorem B.3. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function. Assume that there are constants $M > 0$ and $R > 0$ such that*

$$|f(x)| \leq M \quad \textit{for all} \;\; x \in B_{3R}.$$

Then

$$|f(x) - f(y)| \le \frac{M}{R}|x - y| \quad \text{for all } \ x, y \in B_R.$$

Proof. Let $x, y \in B_R$ and note that $|x - y| < 2R$. We may assume that $x \neq y$. Setting $\xi = (x - y)/|x - y|$ and $z = y + 2R\xi$ and noting that $z \in B_{3R}$,

$$x - y = \frac{|x - y|}{2R}(z - y),$$

and

$$x = y + \frac{|x - y|}{2R}(z - y) = \frac{|x - y|}{2R} z + \left(1 - \frac{|x - y|}{2R}\right) y,$$

we obtain

$$f(x) \le \frac{|x - y|}{2R} f(z) + \left(1 - \frac{|x - y|}{2R}\right) f(y),$$

and

$$f(x) - f(y) \le \frac{|x - y|}{2R}(f(z) - f(y)) \le \frac{|x - y|}{2R}(|f(z)| + |f(y)|) \le \frac{M|x - y|}{R}.$$

In view of the symmetry in x and y, we see that

$$|f(x) - f(y)| \le \frac{M}{R}|x - y| \quad \text{for all } \ x, y \in B_R. \qquad \square$$

A.3 Global Lipschitz Regularity

We give here a proof of Lemmas 2.1 and 2.2.

Proof (Lemma 2.1). We first show that there is a constant $C > 0$, for each $z \in \overline{\Omega}$ a ball $B_r(z)$ centered at z, and for each $x, y \in B_r(z) \cap \overline{\Omega}$, a curve $\eta \in \mathrm{AC}([0, T], \mathbb{R}^n)$, with $T \in \overline{\mathbb{R}}_+$, such that $\eta(s) \in \Omega$ for all $s \in (0, T)$, $|\dot{\eta}(s)| \le 1$ for a.e. $s \in (0, T)$ and $T \le C|x - y|$.

Let ρ be a defining function of Ω. We may assume that $\|D\rho\|_{\infty,\mathbb{R}^n} \le 1$ and $|D\rho(x)| \ge \delta$ for all $x \in (\partial\Omega)^\delta := \{y \in \mathbb{R}^n \ : \ \mathrm{dist}(y, \partial\Omega) < \delta\}$ and some constant $\delta \in (0, 1)$.

Let $z \in \Omega$. We can choose $r > 0$ so that $B_r(z) \subset \Omega$. Then, for each $x, y \in B_r(z)$, with $x \neq y$, the line $\eta(s) = x + s(y - x)/|y - x|$, with $s \in [0, |x - y|]$, connects two points x and y and lies inside Ω. Note as well that $\dot{\eta}(s) = (y - x)/|y - x| \in \partial B_1$ for all $s \in [0, |x - y|]$.

Let $z \in \partial\Omega$. Since $|D\rho(z)|^2 \ge \delta^2$, by continuity, we may choose $r \in (0, \delta^3/4)$ so that $D\rho(x) \cdot D\rho(z) \ge \delta^2/2$ for all $x \in B_{4\delta^{-2}r}(z)$. Fix any $x, y \in B_r(z) \cap \overline{\Omega}$.

Consider the curve $\xi(t) = x + t(y-x) - t(1-t)6\delta^{-2}|x-y|D\rho(z)$, with $t \in [0, 1]$, which connects the points x and y. Note that

$$\begin{aligned}|\xi(t) - z| &\le (1-t)|x-z| + t|y-z| + 6t(1-t)\delta^{-2}|x-y||D\rho(z)| \\ &< (1 + 3\delta^{-2})r < 4\delta^{-2}r\end{aligned}$$

and $4\delta^{-2}r < \delta$. Hence, we have $\xi(t) \in B_{4\delta^{-2}r}(z) \cap (\partial\Omega)^{\delta}$ for all $t \in [0, 1]$. If $t \in (0, 1/2]$, then we have

$$\begin{aligned}\rho(\xi(t)) &\le \rho(x) + tD\rho(\theta\xi(t) + (1-\theta)x) \cdot (y - x - 6(1-t)\delta^{-2}|x-y|D\rho(z)) \\ &\le t|x-y|(1 - 3(1-t)) < 0\end{aligned}$$

for some $\theta \in (0, 1)$. Similarly, if $t \in [1/2, 1)$, we have

$$\rho(\xi(t)) \le \rho(y) + (1-t)|x-y|(1-3t) < 0.$$

Hence, $\xi(t) \in \Omega$ for all $t \in (0, 1)$. Note that

$$|\dot{\xi}(t)| \le |y-x|(1 + 6\delta^{-2}).$$

If $x = y$, then we just set $\eta(s) = x = y$ for $s = 0$ and the curve $\eta : [0, 0] \to \mathbb{R}^n$ has the required properties. Now let $x \ne y$. We set $t(x, y) = (1 + 6\delta^{-2})|x-y|$ and $\eta(s) = \xi(s/t(x, y))$ for $s \in [0, t(x, y)]$. Then the curve $\eta : [0, t(x, y)] \to \mathbb{R}^n$ has the required properties with $C = 1 + 6\delta^{-2}$.

Thus, by the compactness of $\overline{\Omega}$, we may choose a constant $C > 0$ and a finite covering $\{B^i\}_{i=1}^N$ of $\overline{\Omega}$ consisting of open balls with the properties: for each $x, y \in \hat{B}_i \cap \overline{\Omega}$, where $\hat{B}_i$ denotes the concentric open ball of B_i with radius twice that of B_i, there exists a curve $\eta \in \mathrm{AC}([0, t(x, y)], \mathbb{R}^n)$ such that $\eta(s) \in \Omega$ for all $s \in (0, t(x, y))$, $|\dot{\eta}(s)| \le 1$ for a.e. $s \in [0, t(x, y)]$ and $t(x, y) \le C|x-y|$.

Let r_i be the radius of the ball B_i and set $r = \min r_i$ and $R = \sum r_i$, where i ranges all over $i = 1, \ldots, N$.

Let $x, y \in \overline{\Omega}$. If $|x-y| < r$, then $x, y \in \hat{B}_i$ for some i and there is a curve $\eta \in \mathrm{AC}([0, t(x, y)], \mathbb{R}^n)$ such that $\eta(s) \in \Omega$ for all $s \in (0, t(x, y))$, $|\dot{\eta}(s)| \le 1$ for a.e. $s \in [0, t(x, y)]$ and $t(x, y) \le C|x-y|$. Next, we assume that $|x-y| \ge r$. By the connectedness of Ω, we infer that there is a sequence $\{B_{i_j} : j = 1, \ldots, J\} \subset \{B_i : i = 1, \ldots, N\}$ such that $x \in B_{i_1}$, $y \in B_{i_J}$, $B_{i_j} \cap B_{i_{j+1}} \cap \Omega \ne \emptyset$ for all $1 \le j < J$, and $B_{i_j} \ne B_{i_k}$ if $j \ne k$. It is clear that $J \le N$. If $J = 1$, then we may choose a curve η with the required properties as in the case where $|x-y| < r$. If $J > 1$, then we may choose a curve $\eta \in \mathrm{AC}([0, t(x, y)], \mathbb{R}^n)$ joining x and y as follows. First, we choose a sequence $\{x_j : j = 1, \ldots, J-1\}$ of points in Ω so that $x_j \in B_{i_j} \cap B_{i_{j+1}} \cap \Omega$ for all $1 \le j < J$. Next, setting $x_0 = x$, $x_J = y$ and $t_0 = 0$, since $x_{j-1}, x_{i_j} \in B_j \cap \overline{\Omega}$ for all $1 \le j \le J$, we may select $\eta_j \in \mathrm{AC}([t_{j-1}, t_j], \mathbb{R}^n)$, with $1 \le j \le J$, inductively so that $\eta_j(t_{j-1}) = x_{j-1}$,

$\eta_j(t_j) = x_j$, $\eta_j(s) \in \Omega$ for all $s \in (t_{j-1}, t_j)$ and $t_j \le t_{j-1} + C|x_j - x_{j-1}|$. Finally, we define $\eta \in \mathrm{AC}([0,\, t(x,y)], \mathbb{R}^n)$, with $t(x,y) = t_J$, by setting $\eta(s) = \eta_i(s)$ for $s \in [t_j - 1, t_j]$ and $1 \le j \le J$. Noting that

$$T \le C \sum_{j=1}^{J} |x_j - x_{j-1}| \le C \sum_{j=1}^{J} r_{i_j} \le CR \le CRr^{-1}|x-y|,$$

we see that the curve $\eta \in \mathrm{AC}([0,\, t(x,y)],\, \mathbb{R}^n)$ has all the required properties with C replaced by CRr^{-1}. □

Remark C.1. (i) A standard argument, different from the above one, to prove the local Lipschitz continuity near the boundary points is to flatten the boundary by a local change of variables. (ii) One can easily modify the above proof to prove the proposition same as Lemma 2.1, except that Ω is a Lipschitz domain.

Proof (Lemma 2.2). Let $C > 0$ be the constant from Lemma 2.1. We show that $|u(x) - u(y)| \le CM|x-y|$ for all $x, y \in \Omega$.

To show this, we fix any $x, y \in \Omega$ such that $x \ne y$. By Lemma 2.1, there is a curve $\eta \in \mathrm{AC}([0,\, t(x,y)],\, \mathbb{R}^n)$ such that $\eta(0) = x$, $\eta(t(x,y)) = y$, $t(x,y) \le C|x-y|$, $\eta(s) \in \Omega$ for all $s \in [0,\, t(x,y)]$ and $|\dot\eta(s)| \le 1$ for a.e. $s \in [0,\, t(x,y)]$.

By the compactness of the image $\eta([0,\, t(x,y)])$ of interval $[0,\, t(x,y)]$ by η, we may choose a finite sequence $\{B_i\}_{i=1}^N$ of open balls contained in Ω which covers $\eta([0,\, t(x,y)])$. We may assume by rearranging the label i if needed that $x \in B_1$, $y \in B_N$ and $B_i \cap B_{i+1} \ne \emptyset$ for all $1 \le i < N$. We may choose a sequence $0 = t_0 < t_1 < \cdots < t_N = t(x,y)$ of real numbers so that the line segment $[\eta(t_{i-1}),\, \eta(t_i)]$ joining $\eta(t_{i-1})$ and $\eta(t_i)$ lies in B_i for any $i = 1, \ldots, N$.

Thanks to Proposition 1.14, we have

$$|u(\eta(t_i)) - u(\eta(t_{i-1}))| \le M|\eta(t_i) - \eta(t_{i-1})| \quad \text{for all } i = 1, \ldots, N.$$

Using this, we compute that

$$\begin{aligned} |u(y) - u(x)| &= |u(\eta(t_N)) - u(\eta(t_0))| \le \sum_{i=1}^{N} |u(\eta(t_i)) - u(\eta(t_{i-1}))| \\ &\le M \sum_{i=1}^{N} |\eta(t_i) - \eta(t_{i-1})| \le M \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |\dot\eta(s)|\mathrm{d}s \\ &= M \int_{t_0}^{t_N} |\dot\eta(s)|\mathrm{d}s \le M(t_N - t_0) = Mt(x,y) \le CM|x-y|. \end{aligned}$$

This completes the proof. □

A.4 Localized Versions of Lemma 4.2

Theorem D.1. *Let U, V be open subsets of $\mathbb{R}^n$ with the properties: $\overline{V} \subset U$ and $V \cap \Omega \neq \emptyset$. Let $u \in C(U \cap \overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} H(x, Du(x)) \leq 0 & \text{in } U \cap \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) \leq g(x) & \text{on } U \cap \partial\Omega. \end{cases} \tag{131}$$

Then, for each $\varepsilon \in (0,\ 1)$, there exists a function $u^\varepsilon \in C^1(V \cap \overline{\Omega})$ such that

$$\begin{cases} H(x, Du^\varepsilon(x)) \leq \varepsilon & \text{in } V \cap \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x) \leq g(x) & \text{on } V \cap \partial\Omega, \\ \|u^\varepsilon - u\|_{\infty, V\cap\Omega} \leq \varepsilon. \end{cases}$$

Proof. We choose functions ζ, $\eta \in C^1(\mathbb{R}^n)$ so that $0 \leq \zeta(x) \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$, $\zeta(x) = 1$ for all $x \in V$, $\eta(x) = 1$ for all $x \in \operatorname{supp}\zeta$ and $\operatorname{supp}\eta \subset U$.

We define the function $v \in C(\overline{\Omega})$ by setting $v(x) = \eta(x)u(x)$ for $x \in U \cap \overline{\Omega}$ and $v(x) = 0$ otherwise. By the coercivity of H, u is locally Lipschitz continuous in $U \cap \overline{\Omega}$, and hence, v is Lipschitz continuous in $\overline{\Omega}$. Let $L > 0$ be a Lipschitz bound of v in $\overline{\Omega}$. Then v is a viscosity solution of

$$\begin{cases} |Dv(x)| \leq L & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma}(x) \leq M & \text{in } \partial\Omega, \end{cases}$$

where $M := L\|\gamma\|_{\infty,\partial\Omega}$. In fact, we have a stronger assertion that for any $x \in \overline{\Omega}$ and any $p \in D^+v(x)$,

$$\begin{cases} |p| \leq L & \text{if } x \in \Omega, \\ \gamma(x) \cdot p \leq M & \text{if } x \in \partial\Omega. \end{cases} \tag{132}$$

To check this, let $\phi \in C^1(\overline{\Omega})$ and assume that $v - \phi$ attains a maximum at $x \in \overline{\Omega}$. Observe that if $x \in \Omega$, then $|D\phi(x)| \leq L$ and that if $x \in \partial\Omega$, then

$$\begin{aligned} 0 &\leq \liminf_{t\to 0+} \frac{(v-\phi)(x - t\gamma(x)) - (v-\phi)(x)}{-t} \\ &= \liminf_{t\to 0+} \frac{v(x - t\gamma(x)) - v(x)}{-t} - \frac{\partial \phi}{\partial \gamma}(x), \end{aligned}$$

which yields

$$\gamma(x)\cdot D\phi(x) \le L|\gamma(x)| \le M.$$

Thus, (132) is valid.

We set

$$h(x) = \zeta(x)g(x) + (1-\zeta(x))M \quad \text{for } \ x \in \partial\Omega,$$
$$G(x,p) = \zeta(x)H(x,p) + (1-\zeta(x))(|p|-L) \quad \text{for } \ (x,p) \in \overline{\Omega}\times\mathbb{R}^n.$$

It is clear that $h \in C(\partial\Omega)$ and G satisfies (A5)–(A7), with H replaced by G

In view of the coercivity of H, we may assume by reselecting L if necessary that for all $(x,p) \in \overline{\Omega}\times\mathbb{R}^n$, if $|p| > L$, then $H(x,p) > 0$. We now show that v is a viscosity solution of

$$\begin{cases} G(x,Dv(x)) \le 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma}(x) \le h(x) & \text{on } \partial\Omega. \end{cases} \tag{133}$$

To do this, let $\hat{x} \in \overline{\Omega}$ and $\hat{p} \in D^+v(\hat{x})$. Consider the case where $\zeta(\hat{x}) > 0$, which implies that $\hat{x} \in U$. We have $\eta(x) = 1$ near the point $\hat{x}$, which implies that $\hat{p} \in D^+u(\hat{x})$. As u is a viscosity subsolution of (131), we have $H(\hat{x},\hat{p}) \le 0$ if $\hat{x} \in \Omega$ and $\min\{H(\hat{x},\hat{p}),\ \gamma(\hat{x})\cdot\hat{p} - h(\hat{x})\} \le 0$ if $\hat{x} \in \partial\Omega$. Assume in addition that $\hat{x} \in \partial\Omega$. By (132), we have $\gamma(\hat{x})\cdot\hat{p} \le M$. If $|\hat{p}| > L$, we have both

$$\gamma(\hat{x})\cdot\hat{p} \le g(\hat{x}) \quad \text{and} \quad \gamma(\hat{x})\cdot\hat{p} \le M.$$

Hence, if $|\hat{p}| > L$, then $\gamma(\hat{x})\cdot\hat{p} \le h(\hat{x})$. On the other hand, if $|\hat{p}| \le L$, we have two cases: in one case we have $H(\hat{x},\hat{p}) \le 0$ and hence, $G(\hat{x},\hat{p}) \le 0$. In the other case, we have $\gamma(\hat{x})\cdot\hat{p} \le g(\hat{x})$ and then $\gamma(\hat{x})\cdot\hat{p} \le h(\hat{x})$. These observations together show that

$$\min\{G(\hat{x},\hat{p}),\ \gamma(\hat{x})\cdot\hat{p} - h(\hat{x})\} \le 0.$$

We next assume that $\hat{x} \in \Omega$. In this case, we easily see that $G(\hat{x},\hat{p}) \le 0$.

Next, consider the case where $\zeta(\hat{x}) = 0$, which implies that $G(\hat{x},\hat{p}) = |\hat{p}| - L$ and $h(\hat{x}) = M$. By (132), we immediately see that $G(\hat{x},\hat{p}) \le 0$ if $\hat{x} \in \Omega$ and $\min\{G(\hat{x},\hat{p}), \gamma(\hat{x})\cdot\hat{p} - h(\hat{x})\} \le 0$ if $\hat{x} \in \partial\Omega$. We thus conclude that v is a viscosity solution of (133).

We may invoke Theorem 4.2, to find a collection $\{v^\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(\overline{\Omega})$ such that

$$\begin{cases} G(x,Dv^\varepsilon(x)) \le \varepsilon & \text{for all } x \in \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le h(x) & \text{for all } x \in \partial\Omega, \\ \|v^\varepsilon - v\|_{\infty,\Omega} \le \varepsilon. \end{cases}$$

But, this yields

$$\begin{cases} H(x, v^\varepsilon(x)) \le \varepsilon & \text{for all } x \in V \cap \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le g(x) & \text{for all } x \in V \cap \partial\Omega, \\ \|v^\varepsilon - u\|_{\infty, V\cap\Omega} \le \varepsilon. \end{cases}$$

The functions v^ε have all the required properties. □

The above theorem has a version for Hamilton–Jacobi equations of evolution type.

Theorem D.2. *Let U, V be bounded open subsets of $\mathbb{R}^n \times \mathbb{R}_+$ with the properties: $\overline{V} \subset U$, $\overline{U} \subset \mathbb{R}^n \times \mathbb{R}_+$ and $V \cap Q \neq \emptyset$. Let $u \in \mathrm{Lip}(U \cap Q)$ be a viscosity solution of*

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le 0 & \text{in } U \cap (\Omega \times \mathbb{R}_+), \\ \dfrac{\partial u}{\partial \gamma}(x,t) \le g(x) & \text{on } U \cap (\partial\Omega \times \mathbb{R}_+). \end{cases}$$

Then, for each $\varepsilon \in (0,\ 1)$, there exists a function $u^\varepsilon \in C^1(V \cap Q)$ such that

$$\begin{cases} u_t^\varepsilon(x,t) + H(x, D_x u^\varepsilon(x,t)) \le \varepsilon & \text{in } V \cap (\Omega \times \mathbb{R}_+), \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x,t) \le g(x) & \text{on } V \cap (\partial\Omega \times \mathbb{R}_+), \\ \|u^\varepsilon - u\|_{\infty, V\cap Q} \le \varepsilon. \end{cases} \tag{134}$$

Proof. Choose constants $a, b \in \mathbb{R}_+$ so that $U \subset \mathbb{R}^n \times (a, b)$ and let ρ be a defining function of Ω. We may assume that ρ is bounded in $\mathbb{R}^n$. We choose a function $\zeta \in C^1(\mathbb{R})$ so that $\zeta(t) = 0$ for all $t \in [a,\ b]$, $\zeta'(t) > 0$ for all $t > b$, $\zeta'(t) < 0$ for all $t < a$ and $\min\{\zeta(a/2), \zeta(2b)\} > \|\rho\|_{\infty,\Omega}$.

We set

$$\tilde\rho(x,t) = \rho(x) + \zeta(t) \quad \text{for } (x,t) \in \mathbb{R}^{n+1},$$
$$\tilde\Omega = \{(x,t) \in \mathbb{R}^{n+1} \ :\ \tilde\rho(x,t) < 0\}.$$

It is easily seen that

$$\tilde\Omega \subset \Omega \times (a/2,\ 2b) \quad \text{and} \quad \tilde\Omega \cap (\mathbb{R}^n \times [a,\ b]) = \Omega \times [a,\ b].$$

Let $(x,t) \in \mathbb{R}^{n+1}$ be such that $\tilde\rho(x,t) = 0$. It is obvious that $(x,t) \in \overline{\Omega} \times [a/2,\ 2b]$. If $a \le t \le b$, then $\rho(x) = 0$ and thus $D\rho(x) \neq 0$. If either $t > b$ or $t < a$, then $|\zeta'(t)| > 0$. Hence, we have $D\tilde\rho(x,t) \neq 0$. Thus, $\tilde\rho$ is a defining function of $\tilde\Omega$.

Let $M > 0$ and define $\tilde{\gamma} \in C(\partial\tilde{\Omega}, \mathbb{R}^{n+1})$ by

$$\tilde{\gamma}(x,t) = \big((1 + M\rho(x))_+\gamma(x),\ \zeta'(t)\big),$$

where we may assume that γ is defined and continuous in $\overline{\Omega}$. We note that for any $(x,t) \in \partial\tilde{\Omega}$,

$$\tilde{\gamma}(x,t) \cdot D\tilde{\rho}(x,t) = (1 + M\rho(x))_+\gamma(x) \cdot D\rho(x) + \zeta'(t)^2.$$

Note as well that $(1 + M\rho(x))_+ = 1$ for all $x \in \partial\Omega$ and

$$\lim_{M\to\infty} (1 + M\rho(x))_+ = 0 \quad \text{locally uniformly in } \Omega.$$

Thus we can fix $M > 0$ so that for all $(x,t) \in \partial\tilde{\Omega}$,

$$\tilde{\gamma}(x,t) \cdot D\tilde{\rho}(x,t) = (1 + M\rho(x))_+\gamma(x) \cdot D\rho(x) + \zeta'(t)^2 > 0.$$

Noting that for each $x \in \Omega$, the x-section $\{t \in \mathbb{R} \,:\, (x,t) \in \tilde{\Omega}\}$ of $\tilde{\Omega}$ is an open interval (or, line segment), we deduce that $\tilde{\Omega}$ is a connected set. We may assume that g is defined and continuous in $\overline{\Omega}$. We define $\tilde{g} \in C(\partial\tilde{\Omega})$ by $\tilde{g}(x,t) = g(x)$. Thus, assumptions (A1)–(A4) hold with $n+1$, $\tilde{\Omega}$, $\tilde{\gamma}$ and $\tilde{g}$ in place of n, Ω, γ and g.

Let $L > 0$ be a Lipschitz bound of the function u in $U \cap Q$. Set

$$\tilde{H}(x,t,p,q) = H(x,p) + q + 2(|q| - L)_+ \quad \text{for } (x,t,p,q) \in \overline{\tilde{\Omega}} \times \mathbb{R}^{n+1},$$

and note that $\tilde{H} \in C(\overline{\tilde{\Omega}} \times \mathbb{R}^{n+1})$ satisfies (A5)–(A7), with Ω replaced by $\tilde{\Omega}$.

We now claim that u is a viscosity solution of

$$\begin{cases} \tilde{H}(x,t,Du(x,t)) \le 0 & \text{in } U \cap \tilde{\Omega}, \\ \tilde{\gamma}(x,t) \cdot Du(x,t) \le \tilde{g}(x,t) & \text{on } U \cap \partial\tilde{\Omega}. \end{cases}$$

Indeed, since $U \cap \tilde{\Omega} = U \cap Q$ and $U \cap \partial\tilde{\Omega} = U \cap \partial Q$, if $(x,t) \in U \cap \overline{\tilde{\Omega}}$ and $(p,q) \in D^+u(x,t)$, then we get $|q| \le L$ by the cylindrical geometry of Q and, by the viscosity property of u,

$$\begin{cases} q + H(x,p) + 2(|q| - L)_+ \le 0 & \text{if } (x,t) \in \tilde{\Omega}, \\ \min\{q + H(x,p) + 2(|q| - L)_+, \gamma(x) \cdot p - g(x)\} \le 0 & \text{if } (x,t) \in \partial\tilde{\Omega}. \end{cases}$$

We apply Theorem D.1, to find a collection $\{u^\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(V \cap \overline{\tilde{\Omega}})$ such that

$$\begin{cases} \tilde{H}(x,t,Du^\varepsilon(x,t)) \le \varepsilon & \text{in } V \cap \tilde{\Omega}, \\ \tilde{\gamma}(x,t)\cdot Du^\varepsilon(x,t) \le \tilde{g}(x,t) & \text{on } U \cap \tilde{\Omega}, \\ \|u^\varepsilon - u\|_{\infty, V\cap\tilde{\Omega}} \le \varepsilon. \end{cases}$$

It is straightforward to see that the collection $\{u^\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(V \cap Q)$ satisfies (134). □

A.5 A Proof of Lemma 5.4

This subsection is mostly devoted to the proof of Lemma 5.4, a version of the Dunford–Pettis theorem. We also give a proof of the weak-star compactness of bounded sequences in $L^\infty(J,\mathbb{R}^m)$, where $J = [a, b]$ is a finite interval in $\mathbb{R}$.

Proof (Lemma 5.4). We define the functions $F_j \in C(J,\mathbb{R}^m)$ by

$$F_j(x) = \int_a^x f_j(t)\mathrm{d}t.$$

By the uniform integrability of $\{f_j\}$, the sequence $\{F_j\}_{j\in\mathbb{N}}$ is uniformly bounded and equi-continuous in J. Hence, the Ascoli–Arzela theorem ensures that it has a subsequence converging to a function F uniformly in J. We fix such a subsequence and denote it again by the same symbol $\{F_j\}$. Because of the uniform integrability assumption, the sequence $\{F_j\}$ is equi-absolutely continuous in J. That is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le b, \qquad \sum_{i=1}^n (b_i - a_i) < \delta,$$

$$\Longrightarrow \sum_{i=1}^n |f_j(b_i) - f_j(a_i)| < \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

An immediate consequence of this is that $F \in \mathrm{AC}(J,\mathbb{R}^m)$. Hence, for some $f \in L^1(J,\mathbb{R}^m)$, we have

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \quad \text{for all } x \in J.$$

Next, let $\phi \in C^1(J)$, and we show that

$$\lim_{j\to\infty} \int_a^b f_j(x)\phi(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x. \tag{135}$$

Integrating by parts, we observe that as $j \to \infty$,

$$\int_a^b f_j(x)\phi(x)\,\mathrm{d}x = \big[F_j\phi\big]_a^b - \int_a^b F_j(x)\phi'(x)\,\mathrm{d}x$$
$$\to \big[F\phi\big]_a^b - \int_a^b F(x)\phi'(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x.$$

Hence, (135) is valid.

Now, let $\phi \in L^\infty(J)$. We regard the functions f_j, f, ϕ as functions defined in $\mathbb{R}$ by setting $f_j(x) = f(x) = \phi(x) = 0$ for $x < a$ or $x > b$. Let $\{k_\varepsilon\}_{\varepsilon>0}$ be a collection of standard mollification kernels. We recall that

$$\lim_{\varepsilon\to 0} \|k_\varepsilon * \phi - \phi\|_{L^1(J)} = 0, \tag{136}$$

$$|k_\varepsilon * \phi(x)| \le \|\phi\|_{L^\infty(J)} \quad \text{for all } x \in J,\ \varepsilon > 0. \tag{137}$$

Fix any $\delta > 0$. By the uniform integrability assumption, we have

$$M := \sup_{j\in\mathbb{N}} \|f_j - f\|_{L^1(J)} < \infty.$$

Let $\alpha > 0$ and set

$$E_j := \{x \in J \,:\, |(f_j - f)(x)| > \alpha\}.$$

By the Chebychev inequality, we get

$$|E_j| \le \frac{M}{\alpha}.$$

By the uniform integrability assumption, if $\alpha > 0$ is sufficiently large, then

$$\int_{E_j} |(f_j - f)(x)|\,\mathrm{d}x < \delta. \tag{138}$$

In what follows we fix $\alpha > 0$ large enough so that (138) holds. We write $f_j - f = g_j + b_j$, where $g_j = (f_j - f)(1 - \mathbf{1}_{E_j})$ and $b_j = (f_j - f)\mathbf{1}_{E_j}$. Then,

$$|g_j(x)| \le \alpha \quad \text{for all } x \in J \quad \text{and} \quad \|b_j\|_{L^1(J)} < \delta.$$

Observe that

$$I_j := \int_J f_j(x)\phi(x)\,\mathrm{d}x - \int_J f(x)\phi(x)\,\mathrm{d}x$$
$$= \int_J (f_j - f)(x)\,k_\varepsilon * \phi(x)\,\mathrm{d}x + \int_J (f_j - f)(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x$$

and

$$\Big|\int_J (f_j - f)(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x\Big|$$
$$\le \Big|\int_J g_j(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x\Big| + \Big|\int_J b_j(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x\Big|$$
$$\le \alpha\|k_\varepsilon * \phi - \phi\|_{L^1(J)} + 2\delta\|\phi\|_{L^\infty(J)}.$$

Hence, in view of (135) and (136), we get $\limsup_{j\to\infty}|I_j| \le 2\delta\|\phi\|_{L^\infty(J)}$. As $\delta > 0$ is arbitrary, we get $\lim_{j\to\infty} I_j = 0$, which completes the proof. □

As a corollary of Lemma 5.4, we deduce that the weak-star compactness of bounded sequences in $L^\infty(J, \mathbb{R}^m)$:

Lemma E.1. *Let $J = [a, b]$, with $-\infty < a < b < \infty$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence of functions in $L^\infty(J, \mathbb{R}^m)$. Then $\{f_k\}$ has a subsequence which converges weakly-star in $L^\infty(J, \mathbb{R}^m)$.*

Proof. Set $M = \sup_{k\in\mathbb{N}} \|f_k\|_{L^\infty(J)}$. Let $E \subset J$ be a measurable set, and observe that

$$\int_E |f_k(t)|\mathrm{d}t \le M|E| \quad \text{for all } k \in \mathbb{N},$$

which shows that the sequence $\{f_k\}$ is uniformly integrable in J. Thanks to Lemma 5.4, there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ of $\{f_k\}$ which converges to a function f weakly in $L^1(J, R^m)$.

Let $i \in \mathbb{N}$ and set $E_i = \{t \in J : |f(t)| > M + 1/i\}$ and $g_i(t) = \mathbf{1}_{E_i}(t)f(t)/|f(t)|$ for $t \in J$. Since $g_i \in L^\infty(J, \mathbb{R}^m)$, we get

$$\int_J f_{k_j}(t)\cdot g_i(t)\mathrm{d}t \to \int_J |f(t)|\mathbf{1}_{E_i}(t)\mathrm{d}t \quad \text{as } j \to \infty.$$

Hence, using the Chebychev inequality, we obtain

$$\Big(M + \frac{1}{i}\Big)|E_i| \le \int_J |f(t)|\mathbf{1}_{E_i}(t)\mathrm{d}t \le \int_J M\mathbf{1}_{E_i}(t)\mathrm{d}t = M|E_i|,$$

which ensures that $|E_i| = 0$. Thus, we find that $|f(t)| \le M$ a.e. in J.

Now, fix any $\phi \in L^1(J, \mathbb{R}^m)$. We select a sequence $\{\phi_i\}_{i\in\mathbb{N}} \subset L^\infty(J, \mathbb{R}^m)$ so that, as $i \to \infty$, $\phi_i \to \phi$ in $L^1(J, \mathbb{R}^m)$. For each $i \in \mathbb{N}$, we have

$$\lim_{j\to\infty}\int_J f_{k_j}(t)\cdot\phi_i(t)\mathrm{d}t = \int_J f(t)\cdot\phi_i(t)\mathrm{d}t.$$

On the other hand, we have

$$\Big|\int_J f_{k_j}(t)\cdot\phi(t)\mathrm{d}t - \int_J f_{k_j}(t)\cdot\phi_i(t)\mathrm{d}t\Big| \le M\|\phi - \phi_i\|_{L^1(J)} \quad \text{for all } j \in \mathbb{N}$$

and

$$\left| \int_J f(t) \cdot \phi(t) \mathrm{d}t - \int_J f(t) \cdot \phi_i(t) \mathrm{d}t \right| \le M \|\phi - \phi_i\|_{L^1(J)}.$$

These together yield

$$\lim_{j \to \infty} \int_J f_{k_j}(t) \cdot \phi(t) \mathrm{d}t = \int_J f(t) \cdot \phi(t) \mathrm{d}t. \qquad \square$$

A.6 Rademacher's Theorem

We give here a proof of Rademacher's theorem.

Theorem F.1 (Rademacher). *Let $B = B_1 \subset \mathbb{R}^n$ and $f \in \mathrm{Lip}(B)$. Then f is differentiable almost everywhere in B.*

To prove the above theorem, we mainly follow the proof given in [1].

Proof. We first show that f has a distributional gradient $Df \in L^\infty(B)$.

Let $L > 0$ be a Lipschitz bound of the function f. Let $i \in \{1, 2, \ldots, n\}$ and e_i denote the unit vector in $\mathbb{R}^n$ with unity as the i-th entry. Fix any $\phi \in C_0^1(B)$ and observe that

$$\begin{aligned} \int_B f(x)\phi_{x_i}(x)\mathrm{d}x &= \lim_{r \to 0+} \int_B f(x) \frac{\phi(x + re_i) - \phi(x)}{r} \mathrm{d}x \\ &= \lim_{r \to 0+} \int_B \frac{f(x - re_i) - f(x)}{r} \phi(x) \mathrm{d}x \end{aligned}$$

and

$$\left| \int_B f(x)\phi_{x_i}(x)\mathrm{d}x \right| \le L \int_B |\phi(x)|\mathrm{d}x \le L|B|^{1/2}\|\phi\|_{L^2(B)}.$$

Thus, the map

$$C_0^1(B) \ni \phi \mapsto -\int_B f(x)\phi_{x_i}(x)\mathrm{d}x \in \mathbb{R}$$

extends uniquely to a bounded linear functional G_i on $L^2(B)$. By the Riesz representation theorem, there is a function $g_i \in L^2(B)$ such that

$$G_i(\phi) = \int_B g_i(x)\phi(x)\mathrm{d}x \quad \text{for all } \phi \in L^2(B).$$

This shows that $g = (g_1, \ldots, g_n)$ is the distributional gradient of f.

We plug the function $\phi \in L^2(B)$ given by $\phi(x) = (g_i(x)/|g_i(x)|)\mathbf{1}_{E_k}(x)$, where $k \in \mathbb{N}$ and $E_k = \{x \in B \, : \, |g_i(x)| > L + 1/k\}$, into the inequality $|G_i(\phi)| \le L\|\phi\|_{L^1(B)}$, to obtain

$$\int_B |g_i(x)|\mathbf{1}_{E_k}(x)\mathrm{d}x \le L \int_B \mathbf{1}_{E_k}(x)\mathrm{d}x = L|E_k|,$$

which yields

$$(L + 1/k)|E_k| \le L|E_k|.$$

Hence, we get $|E_k| = 0$ for all $k \in \mathbb{N}$ and $|\{x \in B : |g_i(x)| > L\}| = 0$. That is, $g_i \in L^\infty(B)$ and $|g_i(x)| \le L$ a.e. in B.

The Lebesgue differentiation theorem (see [57]) states that for a.e. $x \in B$, we have $g(x) \in \mathbb{R}^n$ and

$$\lim_{r\to 0+} \frac{1}{r^n} \int_{B_r} |g(x+y) - g(x)|\mathrm{d}y = 0. \tag{139}$$

Now, we fix such a point $x \in B$ and show that f is differentiable at x. Fix an $r > 0$ so that $B_r(x) \subset B$. For $\delta \in (0, r)$, consider the function $h_\delta \in C(\overline{B})$ given by

$$h_\delta(y) = \frac{f(x+\delta y) - f(x)}{\delta}.$$

We claim that

$$\lim_{\delta\to 0} h_\delta(y) = g(x)\cdot y \quad \text{uniformly for } \ y \in \overline{B}. \tag{140}$$

Note that $h_\delta(0) = 0$ and h_δ is Lipschitz continuous with Lipschitz bound L. By the Ascoli–Arzela theorem, for any sequence $\{\delta_j\} \subset (0, r)$ converging to zero, there exist a subsequence $\{\delta_{j_k}\}_{k\in\mathbb{N}}$ of $\{\delta_j\}$ and a function $h_0 \in C(\overline{B})$ such that

$$\lim_{k\to\infty} h_{\delta_{j_k}}(x) = h_0(y) \quad \text{uniformly for } \ y \in \overline{B}.$$

In order to prove (140), we need only to show that $h_0(y) = g(x)\cdot y$ for all $y \in B$.

Since $h_\delta(0) = 0$ for all $\delta \in (0, r)$, we have $h_0(0) = 0$. We observe from (139) that

$$\int_B |g(x+\delta y) - g(x)|\mathrm{d}y = \int_{B_\delta} |g(x+y) - g(x)|\delta^{-n}\mathrm{d}y \ \to \ 0 \ \text{ as } \ \delta \to 0.$$

Using this, we compute that for all $\phi \in C_0^1(B)$,

$$\begin{aligned}\int_B h_0(y)\phi_{y_i}(y)\mathrm{d}y &= \lim_{k\to\infty} \int_B h_{\delta_{j_k}}(y)\phi_{y_i}(y)\mathrm{d}y \\ &= -\lim_{k\to\infty} \int_B g_i(x+\delta_{j_k} y)\phi(y)\mathrm{d}y \\ &= -\int_B g_i(x)\phi(y)\mathrm{d}y = \int_B g(x)\cdot y\phi_{y_i}(y)\mathrm{d}y.\end{aligned}$$

This guarantees that $h_0(y) - g(x) \cdot y$ is constant for all $y \in B$ while $h_0(0) = 0$. Thus, we see that $h_0(y) = g(x) \cdot y$ for all $y \in B$, which proves (140).

Finally, we note that (140) yields

$$f(x + y) = f(x) + g(x) \cdot y + o(|y|) \quad \text{as} \quad y \to 0. \qquad \square$$

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