Buckling of Nonlinearly Elastic Plates with Microstructure

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Abstract In the framework of the general nonlinear plate theory we consider a buckling problem for an elastic plate with incompatible plane strains generated by continuous distributions of edge dislocations and wedge disclinations as well as other sources of residual stress (non-elastic growth or plasticity). In contrast to the Föppl-von Kármán model the plane strains are not supposed to be small. To explore buckling transition of such kind of structures, the problem is reduced to a system of nonlinear partial differential equations with respect to the transverse deflection of the plate and the embedded metrics coefficients, which naturally leads to the non-trivial plate shapes that are seen even in the absence of any external forces. In the case of very thin plate (membrane) that doesn't resist bending we present several exact solutions for the axially-symmetric domains.

1 Introduction

The problem of plate buckling due to the presence of single dislocations of different types dates back to the work of Eshelby and Stroh [1]. Their study was continued in [2], with a single wedge disclination (removed or inserted sector in the terminology of [2]) being considered.

Later on Seung and Nelson [3] published a milestone paper on defects in crystalline membranes, where they generalized the continuum theory of dislocations to include buckling transition and solved exactly the disclination problem in the inextensional limit (Fig. 1). In order to do so, they used the Föppl-von Kármán plate model [4, 5] as was originally proposed in [1]. The work of Seung and Nelson is becoming more and more important in the contemporary graphene era.

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H. Altenbach et al. (eds.), *Generalized Continua as Models for Materials*, Advanced Structured Materials 22, DOI: 10.1007/978-3-642-36394-8_4,

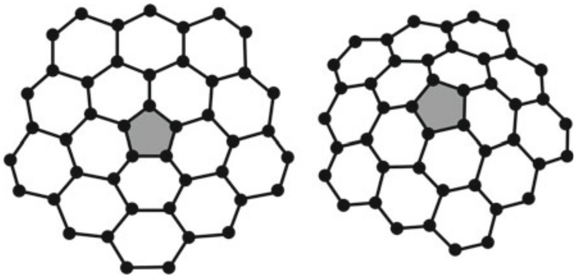

Fig. 1 Buckling of a *graphite membrane* with a single *positive* disclination

The question if it really makes sense to include torsional and couple-stress components in the equilibrium equations as in [6–9] is still open. In this chapter we deal only with the geometrical side of the problem, assuming that the equilibrium equations are satisfied identically, which means the complete relaxation of stresses (zero-stress in the classical setting or some special stress states in the framework of more advanced theory [6–9]).

A rigorous mathematical analysis of the Föppl-von Kármán plates containing incompatible strains was performed in [10]. An application of residual stresses in the Föppl-von Kármán plate to the problem of morphogenesis and biological growth was initiated in [11]. In [12] the Föppl-von Kármán model was applied to study fingerprint patterns as the result of a buckling instability in the basal cell layer of the fetal epidermis.

For modern applications of controllable buckling to thin-film electronics and residual stress measurement, see [13–16]. The strain-induced effects in the electronic structure of graphene are of great importance for the strain engineering. The buckling mechanism has been expected to be a new way to fabricate microscale devices or operate microstuctures.

The chapter is organized as follows. In Sect. 2 we present a planar nonlinear continuum theory of dislocations and disclinations following [17, 18]. In Sect. 3 we explain the buckling process in a nonlinear plate. In general, the problem can be reduced to a system of nonlinear partial differential equations with respect to the transverse deflection of the plate and the embedded metrics coefficients. In Sect. 4 we find explicitly the buckled form of nonlinear membrane with distributed edge dislocations using for this purpose the incompatibility conditions of the first order. In Sect. 5 we discuss in details the problem related to buckling of a membrane containing distributed wedge disclinations, which leads to the Monge-Ampère equation with a non-trivial right hand side. In Sect. 6 we draw some conclusions and give perspectives for the future work.

In the whole chapter we employ a version of tensor analysis used, for example, in [19, 20], where the first index always indicates differentiation, that allows to perform lengthy calculations.

2 Planar Nonlinear Continuum Theory of Dislocations and Disclinations

The mathematical theory of dislocations appeared for the first time in the work of Volterra [21]. He analyzed the behavior of linear elasticity solutions in multiply connected domains.

The continuum theory of translational dislocations was initiated mainly by the works of Kondo [22], Bilby et al. [23], and Kröner [24]. It is based on the notion of the elastic body as a differential-geometric manifold with definite properties. Modern expositions on continuum theory of dislocations in the framework of multiplicative elastoplasticity can be found in [25–27].

Later on [20, 28] there appeared the continuum theory of disclinations (rotational dislocations). Continuous distribution of disclinations in 3D is hampered by the fact of non-commutativity of finite rotations. In [29] continuum theory of disclinations was applied to model phase transformations.

Let ω be a 2D domain, $\mathbf{r}$ and $\mathbf{R}$ be the planar position vectors in the reference and actual configurations respectively, $\mathbf{F} = \nabla\mathbf{R} = \mathbf{r}^{\alpha}\mathbf{R}_{\alpha}$ be the planar deformation gradient (distortion tensor). $\nabla = \mathbf{r}^{\alpha}(\partial/\partial q^{\alpha})$ is the two-dimensional gradient operator with respect to some curvilinear coordinates q^1, q^2 in the reference configuration. Here and below the Greek indices take the values 1, 2. Consider the problem of determining the position $\mathbf{R}(\mathbf{r})$ by a given smooth and single valued field of $\mathbf{F}$. In a simply connected domain the solution may be written in terms of line integrals

$$\mathbf{R}(\mathcal{M}) = \int_{\mathcal{M}_0}^{\mathcal{M}} d\mathbf{r} \cdot \mathbf{F} + \mathbf{R}(\mathcal{M}_0). \tag{1}$$

The integral in (1) does not depend on the path of integration connecting an initial point $\mathcal{M}_0$ with a final point $\mathcal{M}$ iff the following compatibility condition (of the first order) is fulfilled

$$\nabla \cdot (\mathbf{e} \cdot \mathbf{F}) = 0. \tag{2}$$

Here, $\mathbf{e} = e_{\alpha\beta}\mathbf{r}^{\alpha}\mathbf{r}^{\beta}$ is the 2D permutation tensor.

In the case of multiply connected domain (Fig. 2) the position vector in (1) is determined, in general, not uniquely, which means that dislocations of translational type can exist in the body, each of these is characterized by the Burgers vector

$$\mathbf{b}_N = \oint_{\gamma_N} d\mathbf{r} \cdot \mathbf{F} \quad (N = 1, 2, \ldots, N_0). \tag{3}$$

Here, γ_N is a simple closed contour (the Burgers circuit) around the axis of the Nth dislocation. The total Burgers vector of a discrete set of N_0 dislocations is given by

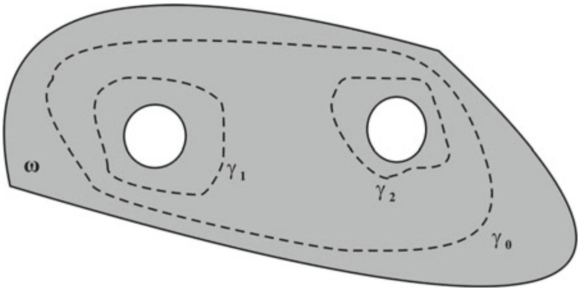

Fig. 2 Integration in a multiply connected *domain*

$$\mathbf{B} = \oint_{\gamma_0} d\mathbf{r} \cdot \mathbf{F}, \tag{4}$$

where γ_0 is a contour enclosing the lines of all N_0 dislocations.

In the case of plane deformation, only edge dislocations are possible and their axes are orthogonal to the plane $q^1 q^2$.

The discrete set of dislocations can be replaced by a continuous one if we again consider the domain ω as simply connected and allow the space inside the dislocation hole shrink to zero along with multiplication the number of dislocations. Then, the integral in (4) may be transformed using the Green formula

$$\mathbf{B} = \iint_{\omega_0} \nabla \cdot (\mathbf{e} \cdot \mathbf{F})\, ds. \tag{5}$$

Here, ω_0 is a planar domain bounded by the contour γ_0.

Relationship (5) makes it possible to introduce the density of continuously distributed edge dislocations $\boldsymbol{\alpha}$ (a vectorial quantity in 2D).

$$\mathbf{B} = \iint_{\omega_0} \boldsymbol{\alpha}\, ds. \tag{6}$$

It means that the compatibility condition (2) is now replaced by

$$\nabla \cdot (\mathbf{e} \cdot \mathbf{F}) = \boldsymbol{\alpha}. \tag{7}$$

Equation (7) may be treated as the incompatibility condition of the first order. In Cartesian coordinates $q^1 = x_1$, $q^2 = x_2$ it has, for example, the following form

$$\frac{\partial F_{21}}{\partial x_1} - \frac{\partial F_{11}}{\partial x_2} = \alpha_1, \quad \frac{\partial F_{22}}{\partial x_1} - \frac{\partial F_{12}}{\partial x_2} = \alpha_2. \tag{8}$$

Let us now introduce the metric tensor of the deformed configuration $\mathbf{G}$ and the connection coefficients $\Gamma^{\nu}_{\alpha\beta}$ making use of the formulae

$$\mathbf{G} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = G_{\alpha\beta}\mathbf{r}^{\alpha}\mathbf{r}^{\beta} = G^{\alpha\beta}\mathbf{r}_{\alpha}\mathbf{r}_{\beta}, \tag{9}$$

$$\frac{\partial \mathbf{R}_{\beta}}{\partial q^{\alpha}} = \Gamma^{\nu}_{\alpha\beta}\mathbf{R}_{\nu}, \quad \frac{\partial \mathbf{R}^{\beta}}{\partial q^{\alpha}} = -\Gamma^{\beta}_{\alpha\nu}\mathbf{R}^{\nu}. \tag{10}$$

From (9), (10) it follows that Ricci's lemma holds

$$\nabla_{\mu} G_{\alpha\beta} = \frac{\partial G_{\alpha\beta}}{\partial q^{\mu}} - \Gamma^{\nu}_{\mu\alpha} G_{\nu\beta} - \Gamma^{\nu}_{\mu\beta} G_{\alpha\nu} = 0. \tag{11}$$

Here, ∇_{μ} is the covariant derivative with respect to $\Gamma^{\nu}_{\alpha\beta}$.

The formulae (9)–(11) give us possibility to interprete the deformed configuration of an elastic body with dislocations as a metrically connected space $\mathcal{V}_2$ [30].

Incompatibilty equation (7) now reads as

$$S^{\cdot\cdot\nu}_{\alpha\beta} = \frac{1}{2} e_{\alpha\beta} \alpha^{\nu}, \tag{12}$$

where $S^{\cdot\cdot\nu}_{\alpha\beta}$ is the torsion tensor of É. Cartan [31]

$$S^{\cdot\cdot\nu}_{\alpha\beta} = \Gamma^{\nu}_{[\alpha\beta]} = \frac{1}{2}(\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha}). \tag{13}$$

Let us consider the problem of specification the distortion $\mathbf{F}$ with the metric tensor $\mathbf{G}$ and the dislocation density vector $\boldsymbol{\alpha}$ being given. We use for this purpose the polar decomposition of $\mathbf{F}$

$$\mathbf{F} = \mathbf{U} \cdot \mathbf{A}, \quad \mathbf{U} = \mathbf{G}^{1/2}, \tag{14}$$

where $\mathbf{U}$ is the left stretch tensor (symmetric, positive-definite), $\mathbf{A}$ is the rotation tensor (properly orthogonal).

Under conditions of plane deformation $\mathbf{A}$ has the following representation

$$\mathbf{A} = (\mathbf{E} - \mathbf{i}_3\mathbf{i}_3)\cos\chi + \mathbf{e}\sin\chi + \mathbf{i}_3\mathbf{i}_3, \tag{15}$$

where $\mathbf{E}$ denotes the 3D identity tensor. Equation (15) means that the fibers rotate around the axis of $\mathbf{i}_3$, orthogonal to the plane q^1q^2, and χ is the angle of rotation. According to [17, 18] χ satisfies the following equation

$$\nabla\chi = \boldsymbol{\psi}, \quad \boldsymbol{\psi} = I_2^{-1/2}\left[\mathbf{U}\cdot(\nabla\cdot\mathbf{e}\cdot\mathbf{U}) - \boldsymbol{\alpha}_0\cdot\mathbf{G}\right], \tag{16}$$

$$I_2 = \det\mathbf{U} = \frac{1}{2}(\mathrm{tr}^2\mathbf{G} - \mathrm{tr}\mathbf{G}^2), \quad \boldsymbol{\alpha}_0 = \alpha^{\mu}\mathbf{r}_{\mu}.$$

Once again, the solution to (16) may be given in terms of line integrals

$$\chi(\mathcal{M}) = \int_{\mathcal{M}_0}^{\mathcal{M}} \boldsymbol{\psi}\cdot d\mathbf{r} + \chi(\mathcal{M}_0). \tag{17}$$

The integral in (17) does not depend on the path (in a simply connected domain) iff

$$\nabla\cdot\left[I_2^{-1/2}\mathbf{e}\cdot\mathbf{U}\cdot(\nabla\cdot\mathbf{e}\cdot\mathbf{U})\right] - \nabla\cdot\left(I_2^{-1/2}\mathbf{e}\cdot\mathbf{G}\cdot\boldsymbol{\alpha}_0\right) = 0. \tag{18}$$

Geometrically (18) is equivalent to the demand that the Gaussian curvature R of the Riemann-Cartan manifold $\mathcal{V}_2$ should vanish.

In a multiply connected domain (17) provides a single valued solution χ only up to a cyclic integral defined by

$$\theta_M = \oint_{\gamma_M} \boldsymbol{\psi}\cdot d\mathbf{r}. \tag{19}$$

The quantity θ_M is called the Frank vector of Mth disclination. In the case of plane deformation only wedge disclinations are possible.

The total Frank vector of a set of M_0 disclinations

$$\Theta = \sum_{M=1}^{M_0} \oint_{\gamma_M} \boldsymbol{\psi}\cdot d\mathbf{r} = \oint_{\gamma_0} \boldsymbol{\psi}\cdot d\mathbf{r} \tag{20}$$

can be transformed into a surface integral in the case of continuous distribution of these

$$\Theta = \iint_{\omega_0} \nabla\cdot(\mathbf{e}\cdot\boldsymbol{\psi})\, d\sigma. \tag{21}$$

Formula (21) serves as a definition for the density of wedge disclinations β

$$\Theta = \iint_{\omega_0} \beta\, d\sigma. \tag{22}$$

The disclination density (a scalar quantity in 2D) satisfies the incompatibility condition of the second order

$$\nabla \cdot (\mathbf{e} \cdot \boldsymbol{\psi}) = \beta \tag{23}$$

or

$$\nabla \cdot \left[I_2^{-1/2} \mathbf{e} \cdot \mathbf{U} \cdot (\nabla \cdot \mathbf{e} \cdot \mathbf{U}) \right] - \nabla \cdot \left(I_2^{-1/2} \mathbf{e} \cdot \mathbf{G} \cdot \boldsymbol{\alpha}_0 \right) = \beta. \tag{24}$$

Geometrically (24) means that in the presence of distributed dislinations the Gaussian curvature R is proportional to the density of wedge disclinations

$$R = I_2^{-1/2} \beta. \tag{25}$$

In the linear elasticity this fact is known [20, 32] in the form

$$\mathbf{i}_3 \cdot \mathrm{Ink}\, \boldsymbol{\varepsilon} \cdot \mathbf{i}_3 = \nabla \cdot \mathbf{e} \cdot \boldsymbol{\alpha} + \beta \tag{26}$$

or in Cartesian coordinates

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2} + \beta. \tag{27}$$

Here, $\varepsilon_{\alpha\beta}$ are the components of the linear strain tensor $\boldsymbol{\varepsilon}$, $\mathrm{Ink}\, \boldsymbol{\varepsilon} = \nabla \times (\nabla \times \boldsymbol{\varepsilon})^T$ is the incompatibility tensor.

3 Escape in the Third Dimension

The stresses due to distributed defects in the case of plane deformation of nonlinearly elastic material were found in [18]. The particular advantage of slender bodies makes it possible to consider relaxation of stresses by the escape in the third dimension (Fig. 3).

Here w is used to denote the transverse deflection of the plate. Then

$$\mathbf{R}^* = \mathbf{R} + w\mathbf{i}_3, \tag{28}$$

and

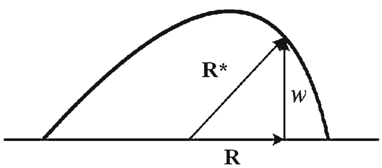

Fig. 3 Buckling of the *plate* due to the relaxation process

$$\mathbf{G}^* = \nabla\mathbf{R}^* \cdot \nabla\mathbf{R}^{*\mathsf{T}} = \nabla\mathbf{R} \cdot \nabla\mathbf{R}^{\mathsf{T}} + \nabla w \nabla w = \mathbf{G} + \nabla w \nabla w. \tag{29}$$

From (29) we have

$$\mathbf{G} = \mathbf{G}^* - \nabla w \nabla w. \tag{30}$$

Substituting now (30) into (24) written in terms of the embedded metric coefficients $G^*_{\alpha\beta}$ and the transverse deflection w we get a general form of geometrical equation that determine the bent form of the plate. It is quite lengthy and will not be presented here. In Sect. 5 a particular case of nonlinear membrane with distributed wedge disclinations will be studied in details.

4 Buckling of a Flat Membrane with Distributed Dislocations

The interesting fact about dislocations is that it is possible to solve problems directly appealing to the first order incompatibility condition (7). Mathematically it means that, when the disclination density vanishes, the incompatibility condition of the first order (7) may be considered in a way as a first integral for the incompatibility condition of the second order (24).

Let us introduce the polar coordinates r, φ and the corresponding vector basis $\mathbf{e}_r$, $\mathbf{e}_\varphi$ in the plane of a circular membrane of radius r_0 and assume that the edge dislocations are distributed with the density $\boldsymbol{\alpha}_0 = \alpha_\varphi(r)\mathbf{e}_\varphi$, which contains only the azimuthal component. The distortion tensor will be sought in the form that corresponds to the axisymmetric bending of the membrane [33]

$$\mathbf{F} = F_1(r)\mathbf{e}_r\mathbf{e}_r + F_2(r)\mathbf{e}_\varphi\mathbf{e}_\varphi + F_3(r)\mathbf{e}_r\mathbf{i}_3. \tag{31}$$

The incompatibility condition of the first order (7) reads then as

$$\frac{dF_2}{dr} + \frac{F_2 - F_1}{r} = \alpha_\varphi(r). \tag{32}$$

This equation has the following solution (under the condition $\mathbf{F} \cdot \mathbf{F}^{\mathsf{T}} = \mathbf{E}$)

$$F_1 = \cos\eta(r), \quad F_2 = 1, \quad F_3 = -\sin\eta(r), \tag{33}$$

where $\eta(r)$ satisfies

$$\cos\eta(r) = 1 - r\alpha_\varphi(r). \tag{34}$$

Solution (33) describes the membrane buckling which results in the release of residual stresses caused by dislocations. Although there is no displacement field with the distributed dislocations in the general case, in this special case, when $\boldsymbol{\alpha} \cdot \mathbf{i}_3 = 0$,

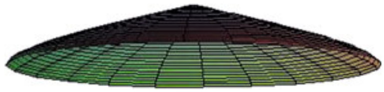

Fig. 4 Buckling of a *membrane* with distributed dislocations

it is possible to find the normal membrane deflection $w = \mathbf{R}^* \cdot \mathbf{i}_3$ from the relation $\mathbf{F}^* \cdot \mathbf{i}_3 = \nabla \mathbf{R}^* \cdot \mathbf{i}_3$. Using (33) and taking that $w(r_0) = 0$, we get

$$w(r) = \int_r^{r_0} \sqrt{2\rho\alpha_\varphi(\rho) - \rho^2\alpha_\varphi^2(\rho)}\, d\rho. \tag{35}$$

If $\alpha_\varphi(r) = \alpha$ is a constant function, then (35) gives the following exact solution

$$w(r) = \frac{1}{2\alpha}\left[(1 - \alpha r)\sqrt{2\alpha r - \alpha^2 r^2} - \arcsin(\alpha r - 1)\right] + C, \tag{36}$$

where C corresponds to the boundary condition $w(r_0) = 0$. The buckled form of the membrane is presented in Fig. 4.

5 Buckling of a Flat Membrane with Distributed Disclinations

Substituting (30) into (24) and assuming $\boldsymbol{\alpha}_0 = 0$ we obtain the following equation in the membrane limit (when $\mathbf{G}$ equals the 2D identity tensor)

$$[w, w] = \left[1 - (\nabla w)^2\right]^{\frac{3}{2}} \beta, \tag{37}$$

$$[w, w] = (\Delta w)^2 - \mathrm{tr}(\nabla\nabla w \cdot \nabla\nabla w). \tag{38}$$

Here, $[w, w]$ is the Monge-Ampère operator. In Cartesian coordinates x_1, x_2 it has a usual representation

$$[w, w] = \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2.$$

In the case of the Föppl-von Kármán theory [3, 34] we have in the membrane limit[1]

$$[w, w] = \beta. \tag{39}$$

[1] Seung and Nelson [3] deduced this equation in the *inextensional* limit.

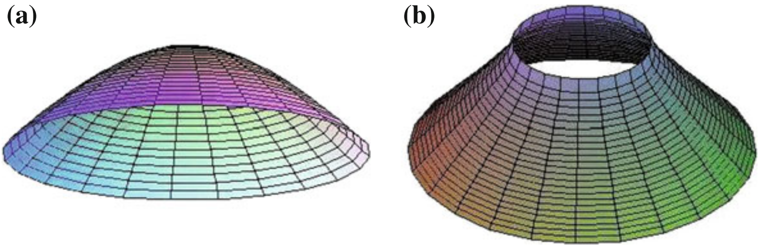

Fig. 5 **a** *Positive* disclinations; **b** *Negative* disclinations

This type of equation according to the general theory [35] gives no direct way of taking into account negative β.

Let the domain occupied by the plate, the distribution of wedge disclinations β and the transverse deflection w be axially-symmetric. Then Eq. (37) admits exact integration. We assume in addition the usual zero-slope condition in the center of the membrane: $w'(r)|_{r=0} = 0$.

Under such conditions for a constant positive β we obtain

$$w(r) = \frac{1}{\beta}\sqrt{\beta^2 r^2 + 4\beta} - \frac{1}{\sqrt{2\beta}} \log\left|\frac{\sqrt{2\beta} + \sqrt{\beta^2 r^2 + 4\beta}}{\sqrt{2\beta} - \sqrt{\beta^2 r^2 + 4\beta}}\right| - C, \tag{40}$$

whereas for a constant negative β

$$w(r) = \frac{1}{\beta}\sqrt{\beta^2 r^2 + 4\beta} + \frac{1}{\sqrt{-2\beta}} \arctan \frac{\sqrt{\beta^2 r^2 + 4\beta}}{\sqrt{-2\beta}} - C, \tag{41}$$

the constant C in both cases is furnished by vanishing $w(r)$ on the outer radius $r = r_0$ of the plate.

For negative β the solution exists only in some part of the circular disk, where $r \geqslant \sqrt{-4/\beta}$ (Fig. 5). The reason lies deeply in the topology of surfaces with negative Gaussian curvature. Being fixed on the outer radius the circular membrane can't be anymore a simply connected surface with everywhere negative Gaussian curvature (like a saddle point surface).

6 Conclusions

In the present chapter we gave some theoretical background and presented a few exact solutions for the buckling problem of a thin nonlinear plate containing continuously distributed fields of edge dislocations and wedge disclinations. As a particular application we have chosen the membrane model, because it allows to explore the geometrical side of the problem. The difficult challenge that still remains is to find how non-axially-symmetric buckled regimes appear. The first attempt was done in [36] for the case of a single negative disclination. In the case of the Föppl-von Kármán plate model a rigorous stability analysis of the influence of distributed disclinations was performed numerically in [37].

It is interesting also to take into account couple-stress and strain gradient effects. For this purpose one can use, for example, a nonlinear model from [38], where a simple version of strain gradient elasticity, proposed in [39], was combined with the Föppl-von Kármán approach.

Acknowledgments The author thanks Prof. L.M. Zubov for fruitful discussions.

This work was supported by the Russian Foundation for Basic Research (via the grants 12-01-00038 and 12-01-91270) and the Federal target programme "Research and Pedagogical Cadre for Innovative Russia" for 2009–2013 years (state contract N P596).

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