Chapter 7
Three-Dimensional and Applied Electroelastic Problems

Abstract In this chapter, there are mainly two kinds of problems discussed. The first kind of problems is the 3D electroelastic problems: the potential function method, the solutions of the penny-shaped crack and elliptic inclusions. The second kind of problems is the applied electroelastic problems which are used in engineering: simple electroelastic problems, laminated piezoelectric plates containing classical and higher-order theories and piezoelectric composite shells. A unified first-order approximate theory of an electro-magneto-elastic thin plate derived by the physical variational principle is given when the electromagnetic induction effect can be neglected.

Keywords Penny-shaped crack • Laminated piezoelectric plate • Piezoelectric composite shell

7.1 Potential Function Methods in Transversely Isotropic Piezoelectric Materials

7.1.1 Governing Equations

The governing equations of transversely isotropic piezoelectric materials have been discussed in previous chapters. In the material principle coordinates, the constitutive equations are

$$\begin{aligned}
&\sigma_1 = C_{11}u_{1,1} + C_{12}u_{2,2} + C_{13}u_{3,3} + e_{31}\varphi_{,3}, && \sigma_2 = C_{12}u_{1,1} + C_{11}u_{2,2} + C_{13}u_{3,3} + e_{31}\varphi_{,3}\\
&\sigma_3 = C_{13}u_{1,1} + C_{13}u_{2,2} + C_{33}u_{3,3} + e_{33}\varphi_{,3}, && \sigma_4 = \sigma_{23} = C_{44}(u_{2,3} + u_{3,2}) + e_{15}\varphi_{,2}\\
&\sigma_5 = \sigma_{31} = C_{44}(u_{1,3} + u_{3,1}) + e_{15}\varphi_{,1}, && \sigma_6 = \sigma_{12} = C_{66}(u_{2,1} + u_{1,2})\\
&D_1 = e_{15}(u_{1,3} + u_{3,1}) - \epsilon_{11}\varphi_{,1}, && D_2 = e_{24}(u_{2,3} + u_{3,2}) - \epsilon_{11}\varphi_{,1}\\
&D_3 = e_{31}u_{1,1} + e_{31}u_{2,2} + e_{33}u_{3,3} - \epsilon_{33}\varphi_{,3}; && C_{66} = (C_{11} - C_{12})/2
\end{aligned} \tag{7.1}$$

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The generalized momentum equations are

$$\begin{aligned}&\sigma_{1,1}+\sigma_{6,2}+\sigma_{5,3}=\rho\ddot{u}_1,\quad \sigma_{6,1}+\sigma_{2,2}+\sigma_{4,3}=\rho\ddot{u}_2,\quad \sigma_{5,1}+\sigma_{4,2}+\sigma_{3,3}=\rho\ddot{u}_3\\&D_{1,1}+D_{2,2}+D_{3,3}=0\end{aligned}\tag{7.2}$$

where $i=1,2,3$. Substitution of Eq. (7.1) into Eq. (7.2) yields

$$\begin{aligned}&C_{11}u_{1,11}+C_{66}u_{1,22}+C_{44}u_{1,33}+(C_{12}+C_{66})u_{2,12}+(C_{13}+C_{44})u_{3,13}\\&\quad+(e_{15}+e_{31})\varphi_{,13}=\rho u_{1,tt}\\&(C_{12}+C_{66})u_{1,12}+C_{66}u_{2,11}+C_{11}u_{2,22}+C_{44}u_{2,33}+(C_{13}+C_{44})u_{3,23}\\&\quad+(e_{15}+e_{31})\varphi_{,23}=\rho u_{2,tt}\\&(C_{13}+C_{44})(u_{1,13}+u_{2,23})+C_{44}\nabla^2u_3+C_{33}u_{3,33}+e_{15}\nabla^2\varphi+e_{33}\varphi_{,33}=\rho u_{3,tt}\\&(e_{15}+e_{31})(u_{1,13}+u_{2,23})+e_{15}\nabla^2u_3+e_{33}u_{3,33}-\epsilon_{11}\nabla^2\varphi-\epsilon_{33}\varphi_{,33}=0\\&\nabla^2u=u_{,11}+u_{,22}\end{aligned}\tag{7.3}$$

7.1.2 General Solution of the Static Problem (I)

Wang and Zheng (1995) discussed the general solution of (7.3) for the static problem by introducing potential functions. They assume

$$u_1=\psi_{,1}-\chi_{,2},\quad u_2=\psi_{,2}+\chi_{,1},\quad u_3=k_1\psi_{,3},\quad \varphi=k_2\psi_{,3}\tag{7.4}$$

where k_1 and k_2 are undetermined constants and ψ and χ are potential functions. Substitution of Eq. (7.4) into Eq. (7.3) yields

$$\begin{aligned}&C_{66}\nabla^2\chi+C_{44}\chi_{,33}=0;\quad \text{or}\\&\nabla^2\chi+\partial^2\chi/\partial z_0^2=0,\quad z_0=s_0x_3,\quad s_0^2=C_{66}/C_{44}=1/\lambda_0\end{aligned}\tag{7.5}$$

$$\begin{aligned}&C_{11}\nabla^2\psi+[C_{44}+k_1(C_{13}+C_{44})+k_2(e_{15}+e_{31})]\psi_{,33}=0\\&[(C_{13}+C_{44})+k_1C_{44}+k_2e_{15}]\nabla^2\psi+(k_1C_{33}+k_2e_{33})\psi_{,33}=0\\&[(e_{15}+e_{31})+k_1e_{33}-k_2\epsilon_{11}]\nabla^2\psi+(k_1e_{33}-k_2\epsilon_{33})\psi_{,33}=0\end{aligned}\tag{7.6}$$

In order to have nontrivial solution of Eq. (7.6), the following relations must be held:

$$\begin{aligned}&\frac{C_{44}+k_1(C_{13}+C_{44})+k_2(e_{15}+e_{31})}{C_{11}}=\frac{k_1C_{33}+k_2e_{33}}{(C_{13}+C_{44})+k_1C_{44}+k_2e_{15}}\\&\quad=\frac{k_1e_{33}-k_2\epsilon_{33}}{(e_{15}+e_{31})+k_1e_{33}-k_2\epsilon_{11}}=\lambda\end{aligned}\tag{7.7}$$

Eliminating k_1 and k_2, a cubic algebra equation of λ is obtained:

$$
\begin{aligned}
&A\lambda^3 + B\lambda^2 + C\lambda + D = 0 \\
&A = e_{15}^2 + C_{44}\epsilon_{11}, \quad D = -C_{11}^{-1}\left(e_{33}^2 C_{44} + \epsilon_{33} C_{11} C_{33}\right) \\
&B = C_{11}^{-1}\{2e_{15}^2 C_{13} - e_{31}^2 C_{44} + 2e_{15}(e_{31}C_{13} - e_{33}C_{11}) + \epsilon_{11}\left(C_{13}^2 + 2C_{13}C_{44}\right) \\
&\qquad - \epsilon_{11}C_{11}C_{33} - \epsilon_{33}C_{11}C_{44}\} \\
&C = C_{11}^{-1}\Big\{(e_{15}+e_{31})^2 C_{33} - 2e_{33}(e_{15}+e_{31})(C_{13}+C_{44}) - (C_{13}+C_{44})^{-1}[(e_{15}+e_{31})C_{33} \\
&\qquad -(e_{15}+e_{11})C_{11}]C_{44}e_{15} + \epsilon_{11}C_{44}C_{33} + e_{33}^2 C_{11} - \epsilon_{33}(C_{13}+C_{44})^2 + \epsilon_{33}\left(C_{44}^2 + C_{11}C_{33}\right)\Big\}
\end{aligned}
\tag{7.8}
$$

Assume root λ_1 is positive real and λ_2 and λ_3 are either a pair of conjugate complex roots with positive real parts or positive real roots. Corresponding to each λ_i, a potential function ψ_j in Eq. (7.6) can be obtained:

$$
\nabla^2\psi_j + \lambda_j \frac{\partial^2\psi_j}{\partial x_3^2} = \nabla^2\psi_j + \frac{\partial^2\psi_j}{\partial z_j^2} = 0, \quad z_j = s_j x_3, \quad s_j^2 = 1/\lambda_j; \quad j = 1,2,3 \tag{7.9}
$$

Substituting λ_j into Eq. (7.7), k_{1j} and k_{2j} can be obtained. So the general solution of Eq. (7.3) can be expressed in potential functions:

$$
\begin{aligned}
&u_1 = (\psi_1 + \psi_2 + \psi_3)_{,1} - \chi_{,2}, \quad u_2 = (\psi_1 + \psi_2 + \psi_3)_{,2} + \chi_{,1}, \\
&u_3 = k_{11}\psi_{1,3} + k_{12}\psi_{2,3} + k_{13}\psi_{3,3}, \quad \varphi = k_{21}\psi_{1,3} + k_{22}\psi_{2,3} + k_{23}\psi_{3,3}
\end{aligned}
\tag{7.10}
$$

Usually the numerical method is used to solve λ_j in Eq. (7.8) due to its complex roots. As an example for material PZT-6B with material constants,

$$
\begin{aligned}
&C_{11} = 168(\text{MPa}), \quad C_{33} = 163, \quad C_{44} = 27.1, \quad C_{12} = 60, \quad C_{13} = 60 \\
&e_{31} = -0.9\left(\text{C/m}^2\right), \quad e_{33} = 7.1, \quad e_{15} = 4.6, \quad \epsilon_{11} = 36 \times 10^{-10}(\text{F/m}), \\
&\epsilon_{33} = 34 \times 10^{-10}
\end{aligned}
$$

The solved λ is $\lambda_1 = 3.92, \quad \lambda_2 = 0.73 + 0.87\text{i}, \quad \lambda_3 = 0.73 - 0.87\text{i}$.

7.1.3 General Solution of the Dynamic Problem

Ding et al. (1996) discussed the dynamic problem. Let

$$
u_1 = \psi_{,2} - \chi_{,1}, \quad u_2 = -\psi_{,1} - \chi_{,2} \tag{7.11}
$$

where ψ and χ are potential functions, but their meanings are different with that in Sect. 7.1.2. Substituting Eq. (7.11) into the first two equations in Eq. (7.3) yields

$$
\begin{aligned}
&B_{,2}-A_{,1}=0,\quad B_{,1}+A_{,2}=0;\quad B=C_{66}\nabla^2\psi+C_{44}\psi_{,33}-\rho\psi_{,tt}\\
&A=C_{11}\nabla^2\chi+C_{44}\chi_{,33}-\rho\chi_{,tt}-(C_{13}+C_{44})u_{3,3}-(e_{15}+e_{31})\varphi_{,3}
\end{aligned}
\tag{7.12}
$$

Let $A=H_{,2},\quad B=H_{,1}$,and Eq. (7.12) is reduced to $\nabla^2 H=0$. One particular solution is $H=$ constant. Adopt a particular solution

$$A=0,\ B=0 \tag{7.13}$$

Using this result, substituting Eq. (7.11) into the last two equations in Eq. (7.3) and listing the results with Eq. (7.13) together we get

$$C_{66}\nabla^2\psi+C_{44}\psi_{,33}-\rho\psi_{,tt}=0 \tag{7.14}$$

$$
\boldsymbol{D}\boldsymbol{G}=\boldsymbol{0},\quad \boldsymbol{G}=[\chi,u_3,\varphi]^T
$$

$$
\boldsymbol{D}=\begin{pmatrix}
C_{11}\nabla^2+C_{44}\dfrac{\partial^2}{\partial x_3^2}-\rho\dfrac{\partial^2}{\partial t^2} & -(C_{13}+C_{44})\dfrac{\partial}{\partial x_3} & -(e_{15}+e_{31})\dfrac{\partial}{\partial x_3}\\
-(C_{13}+C_{44})\nabla^2\dfrac{\partial}{\partial x_3} & C_{44}\nabla^2+C_{33}\dfrac{\partial^2}{\partial x_3^2}-\rho\dfrac{\partial^2}{\partial t^2} & e_{15}\nabla^2+e_{33}\dfrac{\partial^2}{\partial x_3^2}\\
(e_{15}+e_{31})\nabla^2\dfrac{\partial}{\partial x_3} & -\left(e_{15}\nabla^2+e_{33}\dfrac{\partial^2}{\partial x_3^2}\right) & \epsilon_{11}\nabla^2+\epsilon_{33}\dfrac{\partial^2}{\partial x_3^2}
\end{pmatrix}
\tag{7.15}
$$

Introduce a new function F and let $|\boldsymbol{D}|F=0$ or

$$
\begin{aligned}
|\boldsymbol{D}|F=\Big\{&a\frac{\partial^6}{\partial x_3^6}+b\nabla^2\frac{\partial^4}{\partial x_3^4}+c\nabla^4\frac{\partial^2}{\partial x_3^2}+d\nabla^6+g\nabla^2\frac{\partial^4}{\partial t^4}+h\nabla^4\frac{\partial^2}{\partial t^2}\\
&+k\nabla^2\frac{\partial^2}{\partial x_3^2}\frac{\partial^2}{\partial t^2}+l\frac{\partial^4}{\partial x_3^4}\frac{\partial^2}{\partial t^2}+m\frac{\partial^2}{\partial x_3^2}\frac{\partial^4}{\partial t^4}\Big\}F=0
\end{aligned}
\tag{7.16}
$$

where

$$
\begin{aligned}
&a=C_{44}\left(e_{33}^2+C_{33}\epsilon_{33}\right)\\
&b=C_{33}\left[C_{44}\epsilon_{11}+(e_{15}+e_{31})^2\right]+\epsilon_{33}\left[C_{11}C_{33}+C_{44}^2-(C_{13}+C_{44})^2\right]\\
&\qquad+e_{33}[2C_{44}e_{15}+C_{11}e_{33}-2(C_{13}+C_{44})(e_{15}+e_{31})]\\
&c=C_{44}\left[C_{11}\epsilon_{33}+(e_{15}+e_{31})^2\right]+\epsilon_{11}\left[C_{11}C_{33}+C_{44}^2-(C_{13}+C_{44})^2\right]\\
&\qquad+e_{15}[2C_{11}e_{33}+C_{44}e_{15}-2(C_{13}+C_{44})(e_{15}+e_{31})]\\
&d=C_{11}\left(e_{15}^2+C_{44}\epsilon_{11}\right),\quad g=\rho^2\epsilon_{11},\quad h=-\rho\left[e_{15}^2+(C_{11}+C_{44})\epsilon_{11}\right]\\
&k=-\rho\left[2e_{15}e_{33}+(C_{33}+C_{44})\epsilon_{11}+(C_{11}+C_{44})\epsilon_{33}+(e_{15}+e_{31})^2\right]\\
&l=-\rho\left[e_{33}^2+(C_{33}+C_{44})\epsilon_{33}\right],\quad m=\rho^2\epsilon_{33}
\end{aligned}
\tag{7.17}
$$

After solving F, it can be proved that the three group solutions of χ, u_3, φ are

$$\chi = A_{i1}F, \quad u_3 = A_{i2}F, \quad \varphi = A_{i3}F; \qquad i = 1,2,3 \tag{7.18}$$

where A_{ij} in Eq. (7.18) is the algebraic complement of $|\boldsymbol{D}|$, i.e.,

$$\begin{aligned}
A_{11} &= \left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right) + \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)^2 \\
A_{12} &= \left[\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13}+C_{44}) + \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(e_{15}+e_{31})\right]\nabla^2\frac{\partial}{\partial x_3} \\
A_{13} &= \left[\left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13}+C_{44}) - \left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)(e_{15}+e_{31})\right]\nabla^2\frac{\partial}{\partial x_3}
\end{aligned} \tag{7.19}$$

$$\begin{aligned}
A_{21} &= \left[\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13}+C_{44}) + \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(e_{15}+e_{31})\right]\frac{\partial}{\partial x_3} \\
A_{22} &= \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right) + (e_{15}+e_{31})^2\nabla^2\frac{\partial^2}{\partial x_3^2} \\
A_{23} &= \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right) - (C_{13}+C_{44})(e_{15}+e_{31})\nabla^2\frac{\partial^2}{\partial x_3^2}
\end{aligned} \tag{7.20}$$

$$\begin{aligned}
A_{31} &= \left[\left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)(e_{15}+e_{31}) - \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13}+C_{44})\right]\frac{\partial}{\partial x_3} \\
A_{32} &= (e_{15}+e_{31})(C_{13}+C_{44})\nabla^2\frac{\partial^2}{\partial x_3^2} - \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right) \\
A_{33} &= \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right) - (C_{13}+C_{44})^2\nabla^2\frac{\partial^2}{\partial x_3^2}
\end{aligned} \tag{7.21}$$

By substitution of Eq. (7.18) into Eq. (7.11), the general solution is

$$u_1 = \psi_{,2} - A_{i1}F_{,1}, \quad u_2 = -\psi_{,1} - A_{i1}F_{,2}, \quad u_3 = A_{i2}F, \quad \varphi = A_{i3}F; \quad i = 1,2,3 \tag{7.22}$$

The general solution of an axial-symmetric problem can be obtained from Eq. (7.22) if let $\psi = 0$ and F is independent of θ.

7.1.4 General Solution of the Static Problem (II)

Let all the potential functions be independent of the time the general solution of the static problem can be obtained from the results in Sect. 7.1.3. Equation (7.14) yields

$$(\nabla^2 + \partial^2/\partial z_0^2)\psi_0 = 0 \quad z_0^2 = s_0^2 x_3^2, \quad s_0^2 = C_{66}/C_{44} \tag{7.23}$$

Equation (7.16) can be reduced to

$$\left(\nabla^2+\frac{\partial^2}{\partial z_1^2}\right)\left(\nabla^2+\frac{\partial^2}{\partial z_2^2}\right)\left(\nabla^2+\frac{\partial^2}{\partial z_3^2}\right)F=0 \quad z_i^2=s_i^2x_3^2, \quad i=1,2,3 \tag{7.24}$$

where s_i^2 is the root of the following equation:

$$as^6-bs^4+cs^2-d=0 \tag{7.25}$$

No loss of generality let s_1 be real and assume $\mathrm{Re}(s_i)>0$. It is easy to prove that F_i satisfying the following equation is the solution of Eq. (7.24):

$$\left(\nabla^2+\frac{\partial^2}{\partial z_i^2}\right)F_i=0, \quad z_i^2=s_i^2x_3^2, \quad i=1,2,3 \tag{7.26}$$

The general solutions of Eq. (7.24) are

1. $s_1^2\neq s_2^2\neq s_3^2;\quad F=F_1+F_2+F_3$ (7.27)
2. $s_1^2\neq s_2^2=s_3^2\quad F=F_1+F_2+x_3F_3$ (7.28a)
3. $s_1^2=s_2^2=s_3^2;\quad F=F_1+x_3F_2+x_3^2F_3;$ (7.28b)

From Eq. (7.26), it is obtained that $\nabla^2=-\partial^2/\partial z_i^2,\ \partial/\partial x_3=s_i\partial/\partial z_i$. Substituting these results into Eq. (7.20) yields

$$\begin{aligned}
&A_{21}=\left(\beta_1\nabla^2+\beta_2\frac{\partial^2}{\partial x_3^2}\right)\frac{\partial}{\partial x_3}=(\beta_2 s_i^2-\beta_1)s_i\frac{\partial^3}{\partial z_i^3}\\
&A_{22}=C_{11}\,\epsilon_{11}\nabla^4+\beta_3\nabla^2\frac{\partial^2}{\partial x_3^2}+C_{44}\,\epsilon_{33}\frac{\partial^4}{\partial x_3^4}=\left(C_{44}\,\epsilon_{33}s_i^4-\beta_3 s_i^2+C_{11}\,\epsilon_{11}\right)\frac{\partial^4}{\partial z_i^4}\\
&A_{23}=C_{11}e_{15}\nabla^4+\beta_4\nabla^2\frac{\partial^2}{\partial x_3^2}+C_{44}e_{33}\frac{\partial^4}{\partial z_3^4}=\left(C_{44}e_{33}s_i^4-\beta_4 s_i^2+C_{11}e_{15}\right)\frac{\partial^4}{\partial z_i^4}\\
&\beta_1=\epsilon_{11}(C_{13}+C_{44})+e_{15}(e_{15}+e_{31}),\quad \beta_2=\epsilon_{33}(C_{13}+C_{44})+e_{33}(e_{15}+e_{31})\\
&\beta_3=C_{11}\epsilon_{33}+C_{44}\epsilon_{11}+(e_{15}+e_{31})^2,\quad \beta_4=C_{11}e_{33}+C_{44}e_{15}-(C_{13}+C_{44})(e_{15}+e_{31})
\end{aligned} \tag{7.29}$$

The general solution Eq. (7.22) can be rewritten as

$$\begin{aligned}
&u_1=\frac{\partial\psi}{\partial x_2}+\sum_{i=1}^{3}\alpha_{i1}s_i\frac{\partial^4F_i}{\partial x_1\partial z_i^3},\quad u_2=-\frac{\partial\psi}{\partial x_1}+\sum_{i=1}^{3}\alpha_{i1}s_i\frac{\partial^4F_i}{\partial x_2\partial z_1^3},\\
&u_3=\sum_{i=1}^{3}\alpha_{i2}\frac{\partial^4F_i}{\partial z_i^4},\quad \varphi=\sum_{i=1}^{3}\alpha_{i3}\frac{\partial^4F_i}{\partial z_i^4}\\
&\alpha_{i1}=\beta_1-\beta_2s_i^2,\quad \alpha_{i2}=C_{11}\,\epsilon_{11}-\beta_3s_i^2+C_{44}\,\epsilon_{33}s_i^4,\quad \alpha_{i3}=C_{11}e_{15}-\beta_4s_i^2+C_{44}e_{33}s_i^4
\end{aligned} \tag{7.30}$$

If let $\alpha_{1i}s_i\partial^3 F_i/\partial z_i^3 = \psi_i,\ \psi_0 = -\psi$, Eq. (7.30) can be reduced to

$$\begin{aligned} &u_1 = -\frac{\partial\psi_0}{\partial x_2} + \sum_{i=1}^{3}\frac{\partial\psi_i}{\partial x_1}, \quad u_2 = \frac{\partial\psi_0}{\partial x_1} + \sum_{i=1}^{3}\frac{\partial\psi_i}{\partial x_2}, \quad u_3 = \sum_{i=1}^{3}k_{i1}\frac{\partial\psi_i}{\partial z_i} \\ &\varphi = \sum_{i=1}^{3}k_{i2}\frac{\partial\psi_i}{\partial z_i}; \quad k_{i1} = \alpha_{i2}/\alpha_{i1}s_i, \quad k_{i2} = \alpha_{i3}/\alpha_{i1}s_i \end{aligned} \tag{7.31}$$

Equations (7.31) and (7.10) are formally the same.

7.2 A Penny-Shaped Crack in Transversely Isotropic Material

7.2.1 Governing Equations

Consider a transversely isotropic piezoelectric material weakened by a flat impermeable crack of radius a occupied region S in the plane $x_3 = 0$, subjected to distributed pressure $-p(x_1, x_2)$ and surface electric charge $q(x_1, x_2)$ (Fig. 7.1). In Fig. 7.1 only a pair of concentrated force and electric charge is shown. The Cartesian coordinates (x_1, x_2, x_3) and cylindrical coordinates (r, θ, z) are adopted simultaneously. Using the symmetry with respect to the crack surface, this problem can be reduced to a mixed boundary value problem for a half space subjected to the following boundary conditions:

$$\begin{aligned} &\sigma_{33} = -p(x_1, x_2), \quad D_3 = q(x_1, x_2), \quad \text{when} \quad (x_1, x_2) \in S \\ &u_3 = \varphi = 0, \quad \text{when} \quad (x_1, x_2) \notin S; \quad \sigma_{31} = \sigma_{32} = 0, \quad -\infty < (x_1, x_2) < \infty \end{aligned} \tag{7.32}$$

Chen and Shioya (1999) extended the method proposed by Fabrikant (1989) in the elasticity, to solve above problem. Introduce notation $\Lambda = \partial/\partial x_1 + \mathrm{i}\partial/\partial x_2$ and complex displacement $U = u_1 + \mathrm{i}u_2$. Let $w = x_3$, the generalized momentum equations in complex displacement is

$$\begin{aligned} &(1/2)(C_{11} + C_{66})\nabla^2 U + C_{44}U_{,33} + (1/2)(C_{11} - C_{66})\Lambda^2\bar{U} + (C_{13} + C_{44})\Lambda w_{,3} \\ &\quad + (e_{15} + e_{31})\Lambda\varphi_{,3} = 0 \\ &(1/2)(C_{13} + C_{44})(\bar{\Lambda}U + \Lambda\bar{U})_{,3} + C_{44}\nabla^2 w + C_{33}w_{,33} + e_{15}\nabla^2\varphi + e_{33}\varphi_{,33} = 0 \\ &(1/2)(e_{15} + e_{31})(\bar{\Lambda}U + \Lambda\bar{U})_{,3} + e_{15}\nabla^2 w + e_{33}w_{,33} - \epsilon_{11}\nabla^2\varphi - \epsilon_{33}\varphi_{,33} = 0 \end{aligned} \tag{7.33}$$

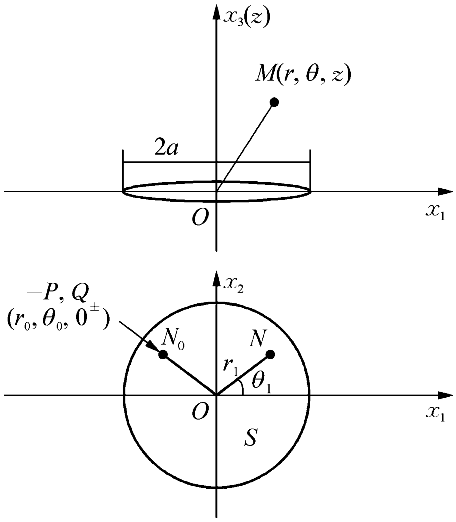


Fig. 7.1 A penny-shaped crack in a transversely isotropic piezoelectric material

where $\bar{U}, \bar{\Lambda}$ mean the conjugate value of U, Λ. By using the complex displacement, the general solution in potential functions, Eq. (7.31), become

$$U = \Lambda\left(\sum_{i=1}^{3}\psi_i + \mathrm{i}\psi_0\right), \quad w = \sum_{i=1}^{3}k_{i1}\frac{\partial\psi_i}{\partial z_i}, \quad \varphi = \sum_{i=1}^{3}k_{i2}\frac{\partial\psi_i}{\partial z_i} \tag{7.34}$$

The generalized stresses become

$$\begin{aligned}
&\sigma_{11} + \sigma_{22} = 2\nabla^2\sum_{i=1}^{3}(C_{11} - C_{66} - C_{13}s_ik_{i1} - e_{31}s_ik_{i2})\psi_i \\
&\sigma_{11} - \sigma_{22} + 2\mathrm{i}\sigma_{12} = 2C_{66}\Lambda^2(\psi_1 + \psi_2 + \psi_3 + \mathrm{i}\psi_0) \\
&\sigma_{31} + \mathrm{i}\sigma_{32} = \Lambda\left\{\sum_{i=1}^{3}[C_{44}(k_{i1} + s_i) + e_{15}k_{i2}]\frac{\partial\psi_i}{\partial z_i} + \mathrm{i}s_4C_{44}\frac{\partial\psi_0}{\partial z_0}\right\} \\
&D_1 + \mathrm{i}D_2 = \Lambda\left\{\sum_{i=1}^{3}[e_{15}(k_{i1} + s_i) - \epsilon_{11}k_{i2}]\frac{\partial\psi_i}{\partial z_i} + \mathrm{i}s_4e_{15}\frac{\partial\psi_0}{\partial z_0}\right\} \\
&\sigma_{33} = -\nabla^2\sum_{i=1}^{3}\gamma_{1i}\psi_i, \quad D_3 = -\nabla^2\sum_{i=1}^{3}\gamma_{2i}\psi_i, \quad \nabla^2 = -\partial^2/\partial z_i^2 \\
&\gamma_{1i} = -C_{13} + C_{33}s_ik_{i1} + e_{33}s_ik_{i2}, \quad \gamma_{2i} = -e_{31} + e_{33}s_ik_{i1} - \epsilon_{33}s_ik_{i2}
\end{aligned} \tag{7.35}$$

where s_i is the root of the Eq. (7.25).

7.2.2 Potential Theory Method of Crack Problem

The solution satisfying the boundary conditions in Eq. (7.32) can be expressed by two harmonic functions G and H:

$$\psi_i(z_i) = c_i G(z_i) + d_i H(z_i), \quad i = 1, 2, 3; \quad \psi_0(z_0) = 0 \tag{7.36}$$

where c_i, d_i are undetermined constants. G and H can be expressed by two potentials of a simple layer:

$$G(r,\theta,z) = \int_S \frac{\hat{u}(N)}{\rho(M,N)} \mathrm{d}S, \quad H(r,\theta,z) = \int_S \frac{\hat{\varphi}(N)}{\rho(M,N)} \mathrm{d}S \tag{7.37}$$

where $\hat{u}(N) = w(x_1, x_2, 0)$ and $\hat{\varphi}(N) = \hat{\varphi}(x_1, x_2, 0)$ are undetermined displacement and electric potential on the crack surface, respectively. $N(r_1, \theta_1, 0)$ is a point on S, $M(r,\theta,z)$ is a certain point in the material, and $\rho(M,N)$ is the distance between N and M. According to the property of the potential of a simple layer, the boundary conditions $w = \varphi = 0$ outside the crack in Eq. (7.37) are satisfied automatically. Inside the crack we have

$$(\partial G/\partial z)_{z=0} = -2\pi\hat{u}, \quad (\partial H/\partial z)_{z=0} = -2\pi\hat{\varphi} \tag{7.38}$$

Equations (7.34), (7.36), and (7.38) yield

$$\sum_{i=1}^{3} c_i k_{i1} = -\frac{1}{2\pi}, \quad \sum_{i=1}^{3} d_i k_{i1} = 0, \quad \sum_{i=1}^{3} c_i k_{i2} = 0, \quad \sum_{i=1}^{3} d_i k_{i2} = -\frac{1}{2\pi} \tag{7.39}$$

The boundary conditions of σ_{31}, σ_{32} in Eq. (7.32) demand

$$\sum_{i=1}^{3} c_i [C_{44}(k_{i1} + s_i) + e_{15} k_{i2}] = 0, \quad \sum_{i=1}^{3} d_i [C_{44}(k_{i1} + s_i) + e_{15} k_{i2}] = 0 \tag{7.40}$$

Combining Eqs. (7.39) and (7.40) yields

$$\begin{gathered}
\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{pmatrix} s_1 & s_2 & s_3 \\ k_{11} & k_{21} & k_{31} \\ k_{12} & k_{22} & k_{32} \end{pmatrix}^{-1} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}, \\
\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{pmatrix} s_1 & s_2 & s_3 \\ k_{11} & k_{21} & k_{31} \\ k_{12} & k_{22} & k_{32} \end{pmatrix}^{-1} \begin{Bmatrix} e_{15}/C_{44} \\ 0 \\ -1 \end{Bmatrix}
\end{gathered} \tag{7.41}$$

$\hat{u}(N)$ and $\hat{\varphi}(N)$ can be determined from the first two boundary conditions in Eq. (7.32):

$$\begin{aligned}
p(N_0) &= -\eta_1 \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0,N)} \mathrm{d}S - \eta_2 \nabla^2 \iint_S \frac{\hat{\varphi}(N)}{\rho(N_0,N)} \mathrm{d}S, \\
q(N_0) &= -\eta_3 \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0,N)} \mathrm{d}S - \eta_4 \nabla^2 \iint_S \frac{\hat{\varphi}(N)}{\rho(N_0,N)} \mathrm{d}S \\
\eta_1 &= -\sum_{i=1}^{3} c_i \gamma_{1i}, \quad \eta_2 = -\sum_{i=1}^{3} d_i \gamma_{1i}, \quad \eta_3 = \sum_{i=1}^{3} c_i \gamma_{2i}, \quad \eta_4 = \sum_{i=1}^{3} d_i \gamma_{2i}
\end{aligned} \tag{7.42}$$

where $N_0(r_0, \theta_0, 0), N(r_1, \theta_1, 0) \in S$ and the integral is over all points on S. Equation (7.42) yields

$$\begin{aligned}
\eta_4 p(N_0) - \eta_2 q(N_0) &= -\frac{1}{4\pi^2 A} \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0,N)} \mathrm{d}S \\
\eta_1 q(N_0) - \eta_3 p(N_0) &= -\frac{1}{4\pi^2 A} \nabla^2 \iint_S \frac{\hat{\varphi}(N)}{\rho(N_0,N)} \mathrm{d}S, \quad A = \frac{1}{4\pi^2}(\eta_1\eta_4 - \eta_2\eta_3)
\end{aligned} \tag{7.43}$$

Equation (7.43) can be applied for a crack with any shape.

7.2.3 The Solution of a Circular Penny-Shaped Crack

For a circular penny-shaped crack of diameter $2a$, the solution of Eq. (7.43) is

$$\begin{aligned}
\hat{u} &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a \frac{1}{\rho} \arctan\left(\frac{\xi}{\rho}\right) [\eta_4 p(r_0,\theta_0) - \eta_2 q(r_0,\theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \\
\hat{\varphi} &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a \frac{1}{\rho} \arctan\left(\frac{\xi}{\rho}\right) [\eta_1 q(r_0,\theta_0) - \eta_3 p(r_0,\theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \\
\rho &= \sqrt{r_1^2 + r_0^2 - 2r_1 r_0 \cos(\theta - \theta_0)}, \quad \xi = \sqrt{(a^2 - r_1^2)(a^2 - r_0^2)} \Big/ a
\end{aligned} \tag{7.44}$$

Substituting Eq. (7.44) into (7.37) yields

$$\begin{aligned}
G(r,\theta,z) &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a K(r,\theta,z;r_0,\theta_0)[\eta_4 p(r_0,\theta_0) - \eta_2 q(r_0,\theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \\
H(r,\theta,z) &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a K(r,\theta,z;r_0,\theta_0)[\eta_1 q(r_0,\theta_0) - \eta_3 p(r_0,\theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0
\end{aligned} \tag{7.45}$$

The kernel function K in Eq. (7.45) is

$$K(M,N_0) = \int_0^{2\pi}\int_0^a \frac{1}{\rho(N,N_0)} \arctan\left[\frac{\sqrt{(a^2-r^2)(a^2-r_0^2)}}{a\rho(N,N_0)}\right]\frac{r_1 \mathrm{d}r_1 \mathrm{d}\theta_1}{\rho(M,N)} \tag{7.46}$$
$$K(M,N_0) = K(r,\theta,z;r_0,\theta_0)$$

Using the relation $\partial K/\partial z = -[2\pi/\rho(M,N_0)]\arctan[h/\rho(M,N_0)]$ (Fabrikant 1989), the derivatives of G and H in Eq. (7.45) are

$$\frac{\partial G}{\partial z} = -4A\int_0^{2\pi}\int_0^a \frac{1}{\rho(M,N_0)}\arctan\left[\frac{h}{\rho(M,N_0)}\right][\eta_4 p(r_0,\theta_0) - \eta_2 q(r_0,\theta_0)]r_0 \mathrm{d}r_0 \mathrm{d}\theta_0$$
$$\frac{\partial H}{\partial z} = -4A\int_0^{2\pi}\int_0^a \frac{1}{\rho(M,N_0)}\arctan\left[\frac{h}{\rho(M,N_0)}\right][\eta_1 q(r_0,\theta_0) - \eta_3 p(r_0,\theta_0)]r_0 \mathrm{d}r_0 \mathrm{d}\theta_0$$
$$h = \sqrt{(a^2-l^2)(a^2-r_0^2)}/a, \quad l = \left\{\sqrt{(r+a)^2+z^2} - \sqrt{(r-a)^2+z^2}\right\}\Big/2 \tag{7.47}$$

So for arbitrary polynomial distributed loadings p and q, all the generalized stresses can be expressed by elementary functions.

7.2.4 A Circular Penny-Shaped Crack Subjected to Generalized Concentrate Loading

Assume the penny-shaped crack is subjected to a pair of normal concentrated generalized loading $(-P,Q)$ at point $(r_0,\theta_0,0^{\pm}), r_0 < a$ (Fig. 7.1). By using the general solution in the above section, after some manipulation, the generalized displacements and stresses can be obtained as

$$\begin{aligned}
u &= 4A\sum_{i=1}^{3}[\tau_{i1}f_1(z_i)P + \tau_{i2}f_1(z_i)Q] \\
u_3 &= -4A\sum_{i=1}^{3}k_{i1}[\tau_{i1}f_2(z_i)P + \tau_{i2}f_2(z_i)Q] \\
\varphi &= -4A\sum_{i=1}^{3}k_{i2}[\tau_{i1}f_2(z_i)P + \tau_{i2}f_2(z_i)Q] \\
\sigma_{33} &= 4A\sum_{i=1}^{3}\gamma_{1i}[\tau_{i1}f_3(z_i)P + \tau_{i2}f_3(z_i)Q] \\
D_3 &= 4A\sum_{i=1}^{3}\gamma_{2i}[\tau_{i1}f_3(z_i)P + \tau_{i2}f_3(z_i)Q]
\end{aligned} \tag{7.48a}$$

where

$$f_1(z_i)=\frac{1}{t_0}\left\{\frac{z_i}{R_0}\arctan\frac{h_0}{R_0}-\frac{\sqrt{a^2-r_0^2}}{\alpha_0}\arctan\left(\frac{\bar{\alpha}_0}{\sqrt{m^2-a^2}}\right)\right\},\quad f_2(z_i)=\frac{1}{R_0}\arctan\frac{h_0}{R_0}$$
$$f_3(z_i)=\frac{1}{R_0}\arctan\frac{h_0}{R_0}-\frac{h_0}{z_i\left(R_0^2+h_0^2\right)}\left(\frac{r^2-l^2}{m^2-l^2}-\frac{z_j^2}{R_0^2}\right),\quad \bar{\alpha}_0=\sqrt{a^2-rr_0e^{-\mathrm{i}(\theta-\theta_0)}}$$
$$R_0=\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+z^2},\quad m=\frac{1}{2}\left\{\sqrt{(r+a)^2+z^2}+\sqrt{(r-a)^2+z^2}\right\}$$
$$\tau_{i1}=c_i\eta_4-d_i\eta_3,\ \ \tau_{i2}=d_i\eta_1-c_i\eta_2,\ \ h_0=\sqrt{(a^2-l^2)\left(a^2-r_0^2\right)}\Big/a,\ \ t_0=re^{-\mathrm{i}\theta}-r_0e^{-\mathrm{i}\theta_0}\tag{7.48b}$$

The generalized stress intensity factors are

$$\begin{aligned}K_{\mathrm{I}}&=\frac{P}{\pi^{3/2}\sqrt{a}}\frac{\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)},\\ K_D&=\frac{Q}{\pi^{3/2}\sqrt{a}}\frac{\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}\end{aligned}\tag{7.49}$$

For the distributed loadings $(-p,q)$, the generalized stress intensity factors are

$$\begin{aligned}K_{\mathrm{I}}&=\frac{\sqrt{2\pi}}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a\frac{p(r_0,\theta_0)\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}r_0\mathrm{d}r_0\mathrm{d}\theta_0\\ K_{\mathrm{D}}&=\frac{\sqrt{2\pi}}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a\frac{q(r_0,\theta_0)\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}r_0\mathrm{d}r_0\mathrm{d}\theta_0\end{aligned}\tag{7.50}$$

For homogeneous distributed loadings $(-p_0,D_{30})$, the generalized stress intensity factors are

$$K_{\mathrm{I}}=2p_0\sqrt{a/\pi},\quad K_{\mathrm{D}}=2D_{30}\sqrt{a/\pi}\tag{7.51}$$

From Eq. (7.51), it is seen that the stress intensity factor is determined independently by the mechanical loading and the electric displacement intensity factor is determined independently by the electric loading. Equation (7.51) obviously can be used to the case where homogeneous generalized stresses σ_{33} and $-D_3$ are applied at infinity.

There are many papers to discuss the penny-shaped crack, such as Huang (1997) and Wang (1992).

7.2.5 A Conducting Penny-Shaped Crack

Chen and Lim (2005) discussed the conducting penny-shaped crack. For a conducting crack, it adopts $\varphi=0$ instead of $D_3=q(x_1,x_2)$ in Eq. (7.32). The boundary conditions are

$$\begin{aligned} &\sigma_{33} = -p(x_1, x_2), \quad \text{when} \quad (x_1, x_2) \in S; \qquad w = 0, \quad \text{when} \quad (x_1, x_2) \notin S \\ &\varphi = 0, \qquad \sigma_{31} = \sigma_{32} = 0, \qquad -\infty < (x_1, x_2) < \infty \end{aligned} \tag{7.52}$$

According to $\varphi = 0, -\infty < (x_1, x_2) < \infty$ in Eq. (7.52), it is concluded that $H = 0$ in Eqs. (7.36) and (7.37). Equation (7.42) becomes

$$p(N_0) = -\eta_1 \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0, N)} \mathrm{d}S, \quad \eta_1 = -\sum_{i=1}^{3} c_i \gamma_{1i} \tag{7.53}$$

Equation (7.45) is reduced to

$$G(r, \theta, z) = \frac{1}{2\pi^3 \eta_1} \int_0^{2\pi} \int_0^a K(r, \theta, z; r_0, \theta_0) p(r_0, \theta_0) r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \tag{7.54}$$

where $K(r, \theta, z; r_0, \theta_0)$ is still expressed by Eq. (7.46). The generalized displacements and stresses for a circular penny-shaped crack subjected to a concentrated force $-P$ are

$$\begin{aligned} &U = \frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} c_i f_1(z_i), \quad u_3 = -\frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} k_{i1} c_i f_2(z_i), \quad \varphi = -\frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} k_{i2} c_i f_2(z_i) \\ &\sigma_{33} = \frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} \gamma_{1i} c_i f_3(z_i), \quad D_3 = \frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} \gamma_{2i} c_i f_3(z_i) \end{aligned} \tag{7.55}$$

The functions in Eq. (7.55) are still expressed by Eq. (7.48b). The generalized stress intensity factors are

$$\begin{aligned} K_{\mathrm{I}} &= \frac{P}{\pi^{3/2} \sqrt{a}} \frac{\sqrt{a^2 - r_0^2}}{a^2 + r_0^2 - 2ar_0 \cos(\theta - \theta_0)}, \\ K_{\mathrm{D}} &= \frac{\beta}{\pi^{3/2} \sqrt{a}} \frac{\sqrt{a^2 - r_0^2}}{a^2 + r_0^2 - 2ar_0 \cos(\theta - \theta_0)} \end{aligned} \tag{7.56}$$

where $\beta = -\sum_{i=1}^{3} \gamma_{i1} c_i / \eta_1$. Above results are assumed that Eq. (7.27) or Eq. (7.24) has three different roots, $s_i, i = 1, 2, 3$. Chen and Lim (2005) also discussed other cases.

7.2.6 *Solve an Impermeable Penny-Shaped Crack by Hankel Transform*

In order to discuss the results of the Vickers indentation cracking of experiments, Jiang and Sun (2001) gave a solution of an impermeable penny-shaped crack with boundary conditions as shown in Eq. (7.32). They discussed an axisymmetric piezoelectric body under axisymmetric loading by using the Hankel transform method in cylindrical coordinates. In cylindrical coordinates, the constitutive equations are

$$\begin{Bmatrix} \sigma_r \\ \sigma_\vartheta \\ \sigma_z \\ \tau_{rz} \\ D_r \\ D_z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & -e_{31} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -e_{31} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & 0 & C_{44} & -e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & \epsilon_{11} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\vartheta \\ \varepsilon_z \\ \gamma_{rz} \\ E_r \\ E_z \end{Bmatrix} \tag{7.57}$$

The generalized geometric equations are

$$\varepsilon_r = u_{,r}, \quad \varepsilon_\vartheta = u/r, \quad \varepsilon_z = w_{,z}, \quad \gamma_{rz} = u_{,z} + w_{,r}; \quad E_r = -\varphi_{,r}, \quad E_z = -\varphi_{,z} \tag{7.58}$$

where u, w are the displacements along r, z directions, respectively.

The generalized equilibrium equations are

$$\begin{aligned} &\sigma_{r,r} + \tau_{rz,z} + (\sigma_r - \sigma_\vartheta)/r = 0, \quad \tau_{rz,r} + \sigma_{z,z} + \tau_{rz}/r = 0, \\ &\partial(rD_r)/\partial r + r\,\partial(D_z)/\partial z = 0 \end{aligned} \tag{7.59}$$

or

$$\begin{aligned} &C_{11}\left(u_{,rr} + \frac{1}{r}u_{,r} - \frac{u}{r^2}\right) + C_{44}u_{,zz} + (C_{13}+C_{44})w_{,rz} + (e_{15}+e_{31})\varphi_{,rz} = 0 \\ &(C_{13}+C_{44})\frac{1}{r}\left(ru_{,z}\right)_{,r} + C_{44}\left(w_{,rr} + \frac{1}{r}w_{,r}\right) + C_{33}w_{,zz} + e_{15}\left(\varphi_{,rr} + \frac{1}{r}\varphi_{,r}\right) + e_{33}\varphi_{,zz} = 0 \\ &(e_{15}+e_{31})\frac{1}{r}\left(ru_{,z}\right)_{,r} + e_{15}\left(w_{,rr} + \frac{1}{r}w_{,r}\right) + e_{33}w_{,zz} - \epsilon_{11}\left(\varphi_{,rr} + \frac{1}{r}\varphi_{,r}\right) - \epsilon_{33}\varphi_{,zz} = 0 \end{aligned} \tag{7.60}$$

Equation (7.32) becomes

$$\begin{aligned} &\sigma_z(r,0) = -p(r), \quad D_z(r,0) = q(r), \quad \text{when} \quad r < a \\ &w(r,0) = \varphi(r,0) = 0, \quad \text{when} \quad r > a; \quad \tau_{rz} = 0, \quad -\infty < (x_1, x_2) < \infty \end{aligned} \tag{7.61}$$

Introduce the Hankel transform pair with J_k; J_k is the Bessel's function of the first kind of order k:

$$\bar{F}(\xi, z) = \int_0^\infty F(r,z) r J_k(\xi r)\mathrm{d}r, \quad F(r,z) = \int_0^\infty \bar{F}(\xi,z)\xi J_k(\xi r)\mathrm{d}\xi \tag{7.62}$$

Applying the Hankel transform, Eq. (7,62) to Eq. (7,60) with $k = 1$ for u and $k = 0$ for w and φ yields

$$\begin{aligned}
&C_{44}\bar{u}'' - C_{11}\xi^2\bar{u} - (C_{13} + C_{44})\xi\bar{w}' - (e_{15} + e_{31})\xi\bar{\varphi}' = 0\\
&(C_{13} + C_{44})\xi\bar{u}' + C_{33}\bar{w}'' - C_{44}\xi^2\bar{w} + e_{33}\bar{\varphi}'' - e_{15}\xi^2\bar{\varphi} = 0\\
&(e_{15} + e_{31})\xi\bar{u}' + e_{33}\bar{w}'' - e_{15}\xi^2\bar{w} - \epsilon_{33}\bar{\varphi}'' + \epsilon_{11}\xi^2\bar{\varphi} = 0
\end{aligned} \tag{7.63}$$

where a prime indicates the derivative with respect to z. The solutions are assumed in the forms

$$\bar{u}(\xi, z) = \hat{u}(\xi)\mathrm{e}^{-\eta\,\xi\,z}, \quad \bar{w}(\xi, z) = \hat{w}(\xi)\mathrm{e}^{-\eta\,\xi\,z}, \quad \bar{\varphi} = \hat{\varphi}\mathrm{e}^{-\eta\,\xi\,z} \tag{7.64}$$

Substituting Eq. (7,64) into Eq. (7,63) yields

$$\begin{bmatrix} C_{44}\eta^2 - C_{11} & (C_{13} + C_{44})\eta & (e_{15} + e_{31})\eta \\ -(C_{13} + C_{44})\eta & C_{33}\eta^2 - C_{44} & e_{33}\eta^2 - e_{15} \\ -(e_{15} + e_{31})\eta & e_{33}\eta^2 - e_{15} & -\epsilon_{33}\eta^2 + \epsilon_{11} \end{bmatrix} \begin{Bmatrix} \hat{u}(\xi) \\ \hat{w}(\xi) \\ \hat{\varphi}(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7.65}$$

In order to have nontrivial solutions for $\hat{u}$, $\hat{w}$ and $\hat{\varphi}$, it should obtain the characteristic equation

$$\eta^6 + B_1\eta^4 + B_2\eta^2 + B_3 = 0 \tag{7.66}$$

where B_1, B_2 and B_3 are coefficients constituted of material constants. Since coefficients in Eq. (7.66) are real, in general it has six roots. Because the upper half space ($z \geq 0$) is discussed, without loss of generality, we take eigenvalue η_1 is a real positive number and η_2, η_3 are, in general, a pair of complex conjugates with positive real part. One form of the corresponding eigenvectors is

$$\begin{aligned}
&\hat{u}_i(\xi) = \alpha_i\eta_i\hat{w}_i(\xi), \quad \alpha_i = \frac{(e_{15} + e_{31})\left(C_{33}\eta_i^2 - C_{44}\right) - \left(e_{33}\eta_i^2 - e_{15}\right)(C_{13} + C_{44})}{\left(e_{33}\eta_i^2 - e_{15}\right)\left(C_{44}\eta_i^2 - C_{11}\right) + (e_{15} + e_{31})(C_{13} + C_{44})\eta_i^2}\\
&\hat{\varphi}_i(\xi) = \gamma_i\eta_i\hat{w}_i(\xi), \quad \gamma_i = -\frac{\left(C_{44}\eta_i^2 - C_{11}\right)\alpha + (C_{13} + C_{44})}{(e_{15} + e_{31})\eta_i}, \quad i = 1, 2, 3
\end{aligned} \tag{7.67}$$

Let $\hat{w}_i(\xi) = (1/\eta_i\xi)f_i(\xi)$. Through the inverse transform, the general solutions are

$$\begin{aligned}
u(r, z) &= \sum_{i=1}^{3}\alpha_i\int_0^\infty f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_1(\xi r)\mathrm{d}\xi\\
\varphi(r, z) &= \sum_{i=1}^{3}\gamma_i\int_0^\infty f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi\\
w(r, z) &= \sum_{i=1}^{3}(1/\eta_i)\int_0^\infty f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi
\end{aligned} \tag{7.68}$$

Using relations $[J_1(\xi r)]_{,r} = \xi[J_0(\xi r) - (1/2)J_1(\xi r)],\ [J_0(\xi r)]_{,r} = -\xi J_1(\xi r)$, the generalized stresses are

$$\begin{aligned}
\sigma_z(r,z) &= \sum_{i=1}^{3}(C_{13}\alpha_i - C_{33} - e_{33}\eta_i\gamma_i)\int_0^\infty \xi f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi \\
\tau_{rz}(r,z) &= \sum_{i=1}^{3}(-C_{44}/\eta_i - C_{44}\eta_i\alpha_i - e_{15}\gamma_i)\int_0^\infty \xi f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_1(\xi r)\mathrm{d}\xi \\
D_z(r,z) &= \sum_{i=1}^{3}(e_{31}\alpha_i - e_{33} + \epsilon_{33}\eta_i\gamma_i)\int_0^\infty \xi f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi
\end{aligned} \tag{7.69}$$

For discussed penny-shaped crack (Fig. 7.1), since $\tau_{rz}(r,z)0 = 0$ due to symmetry, Eq. (7.69) yields

$$\sum_{i=1}^{3}(-C_{44}/\eta_i - C_{44}\eta_i\alpha_i - e_{15}\gamma_i)f_i = 0 \tag{7.70}$$

Eliminating f_3 from Eq. (7.68), (7.69), and (7.70) and then substituting the results into Eq. (7.61), two pairs of integral equations are obtained:

$$\begin{aligned}
&\int_0^\infty f_i(\xi)\xi J_0(\rho\xi)\mathrm{d}\xi = t_{i1}p + t_{i2}q, \quad \text{for} \quad \rho < 1; \quad \rho = r/a \\
&\int_0^\infty f_i(\xi)\xi J_0(\rho\xi)\mathrm{d}\xi = 0, \qquad\qquad \text{for} \quad \rho > 1; \quad i = 1,2
\end{aligned} \tag{7.71}$$

where

$$\begin{aligned}
&t_{11} = k_{22}a^2/H, \quad t_{12} = -k_{12}a^2/H, \quad t_{21} = -k_{21}a^2/H, \quad t_{22} = k_{11}a^2/H \\
&k_{11} = C_{13}\alpha_1 - C_{33} - \gamma_1\eta_1 e_{33} + \Delta_1(C_{13}\alpha_3 - C_{33} - \gamma_3\eta_3 e_{33}) \\
&k_{12} = C_{13}\alpha_2 - C_{33} - \gamma_2\eta_2 e_{33} + \Delta_2(C_{13}\alpha_3 - C_{33} - \gamma_3\eta_3 e_{33}) \\
&k_{21} = e_{31}\alpha_1 - e_{33} + \gamma_1\eta_1\,\epsilon_{33} + \Delta_1(e_{13}\alpha_3 - e_{33} + \gamma_3\eta_3 e_{33}) \\
&k_{22} = e_{31}\alpha_2 - e_{33} + \gamma_2\eta_2\,\epsilon_{33} + \Delta_2(e_{13}\alpha_3 - e_{33} + \gamma_3\eta_3 e_{33}) \\
&H = k_{11}k_{22} - k_{12}k_{21}, \quad \Delta_1 = -\frac{C_{44}(1/\eta_1 + \eta_1\alpha_1) + e_{15}\gamma_1}{C_{44}(1/\eta_3 + \eta_3\alpha_3) + e_{15}\gamma_3}, \\
&\Delta_2 = -\frac{C_{44}(1/\eta_2 + \eta_2\alpha_2) + e_{15}\gamma_2}{C_{44}(1/\eta_3 + \eta_3\alpha_3) + e_{15}\gamma_3}
\end{aligned} \tag{7.72}$$

The solution of the dual integral equations in Eq. (7.71) is

$$f_i(\xi) = \frac{2}{\pi}\int_0^1 \mu\sin(\mu\xi)\mathrm{d}\mu\int_0^1 \frac{\rho[t_{i1}p(\rho) + t_{i2}q(\rho)]}{\sqrt{1-\rho^2}}\mathrm{d}\rho \tag{7.73}$$

When uniform pressure p_0 is applied over the area of radius $r = c < a$ and uniform charge q_0 applied over the area of $r = b < a$, we get

$$\sigma_z(\rho, 0) = -\frac{2p_0}{\pi a}\left(1 - \sqrt{1 - (c/a)^2}\right)\left[\arcsin(1/\rho) - \frac{1}{\sqrt{\rho^2 - 1}}\right], \quad \rho \geq 1$$
$$D_z(\rho, 0) = -\frac{2q_0}{\pi a}\left(1 - \sqrt{1 - (b/a)^2}\right)\left[\arcsin(1/\rho) - \frac{1}{\sqrt{\rho^2 - 1}}\right], \quad \rho \geq 1 \tag{7.74}$$

The solution for a point force $-P_0$ and point charge Q_0 acted at the crack center $(r = 0)$ can be obtained by using the limiting procedure $\lim_{c \to 0} \pi c^2 p_0 = P_0$ and $\lim_{b \to 0} \pi b^2 q_0 = Q_0$.

The generalized stress intensity factors for uniform loadings p_0 and q_0 applied over $r = a$ are the same as shown in Eq. (7.51). For the point loadings are

$$K_{\mathrm{I}} = P_0/(\pi a)^{3/2}, \quad K_{\mathrm{D}} = Q_0/(\pi a)^{3/2} \tag{7.75}$$

A modified stress intensity factor K_{I}^* for a semicircular surface crack in a homogeneous isotropic elastic material given by Cherepanov (1979) is

$$K_{\mathrm{I}}^* = \kappa(\theta) K_{\mathrm{I}}, \quad \kappa(\theta) = 1 + 0.2[(\pi - 2\theta)/\pi]^2 \tag{7.76}$$

In the Vicker's indentation cracking of experiments, a semicircular surface crack is located in an isotropic plane (x_1, x_2), the stress intensity factor can approximately adopt Eq. (7.76), but it should take $2P_0$ instead of P_0 in K_{I}.

There were many literatures discussing the penny-shaped crack, such as Ueda (2007) which discussed a penny-shaped crack in a functionally graded piezoelectric strip under thermal loading.

7.3 Ellipsoidal Inclusion and Inhomogeneity

7.3.1 Basic Concept of Electroelastic Green Functions

A subdomain with prescribed eigenstrain in a matrix is usually called the inclusion, and material properties of the inclusion are the same with the matrix. A subdomain with different material properties from the matrix is usually called the inhomogeneity. The eigenstrain may be introduced by many physical phenomena, such as phase transformation strains, thermal strains, and plastic strains. The eigenstrain will produce self-equilibrium stresses in a constrained matrix. Eshelby's theory (1957)

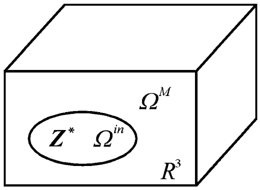


Fig. 7.2 An elliptic inclusion

of the inclusion and inhomogeneity problems is important in the analyses of piezoelectric composite materials.

For convenience the notations given in Eq. (3.8) in Sect. 3.1 are adopted. A subscript in upper case takes the value 1,2,3, and 4 and a subscript in lower case takes the value 1,2, and 3. Figure 7.2 shows an ellipsoid inclusion occupied region Ω^{in} in a 3D space R^3. In Ω^{in} there is generalized eigenstrain $\boldsymbol{Z}^*\left(Z_{ij} = \varepsilon^*_{ij}, Z_{4j} = -E^*_j;\ i,j = 1,2,3;\ \text{without } Z_{44}\right)$. The constitutive equations with eigenstrain are

$$\Sigma_{iJ} = E_{iJKl}(Z_{Kl} - Z^*_{Kl}); \quad Z^*_{Kl}(\boldsymbol{x}) = Z^*_{Kl}, \quad \boldsymbol{x} \in \Omega; \quad Z^*_{Kl}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \notin \Omega \tag{7.77}$$

where $Z_{Kl} = U_{K,l}, \boldsymbol{U} = [u_k, \varphi]^{\mathrm{T}}$. The generalized equilibrium equations are

$$\Sigma_{iJ,i} = -f_J; \qquad E_{iJKl}U_{K,li} = E_{iJKl}Z^*_{Kl,i}(\boldsymbol{x}) - f_J \tag{7.78}$$

where f_1, f_2, f_3 are components of the body force and $f_4 = -\rho_{\mathrm{e}}$ is the body electric charge density. From Eq. (7.78), it is seen that the role of $E_{iJKl}Z^*_{Kl,i}(\boldsymbol{x})$ is analogous to the body force.

The Green function $G_{KR,il}(\boldsymbol{x} - \boldsymbol{x}')$ in an infinite body is defined as

$$E_{iJKl}G_{KR,il}(\boldsymbol{x} - \boldsymbol{x}') + \delta_{JR}\delta(\boldsymbol{x} - \boldsymbol{x}') = \boldsymbol{0} \tag{7.79}$$

where δ_{JR} is the generalized Kronecker delta and $\delta(\boldsymbol{x} - \boldsymbol{x}')$ is the three-dimensional Dirac delta function. Except $\boldsymbol{x} = \boldsymbol{x}'$, $\delta(\boldsymbol{x} - \boldsymbol{x}') = 0$, and for a regular function $f(\boldsymbol{x})$,

$$\int_{-\infty}^{\infty} f(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')\mathrm{d}\boldsymbol{x}' = f(\boldsymbol{x}) \tag{7.80}$$

The Green function defined in Eq. (7.79) satisfies the generalized equilibrium equation. $G_{IJ}(\boldsymbol{x} - \boldsymbol{x}')$ and its derivative approach zero when $|\boldsymbol{x} - \boldsymbol{x}'| \to \infty$.

The electroelastic Green function $G_{IJ}(\boldsymbol{x} - \boldsymbol{x}')$ is extensively applied to study the inclusion and inhomogeneity problems in piezoelectric materials. $G_{ij}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the elastic displacement at $\boldsymbol{x}$ in the x_i direction due to a unit point force at $\boldsymbol{x}'$ in the x_j direction; $G_{i4}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the elastic displacement at $\boldsymbol{x}$ in the x_i direction due to a unit point charge at $\boldsymbol{x}'$; $G_{4j}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the electric potential at $\boldsymbol{x}$ due to a unit point force at $\boldsymbol{x}'$ in the x_j direction; and $G_{44}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the electric potential at x due to a unit point charge at $\boldsymbol{x}'$.

7.3.2 Fourier Transform Method

Assume that the generalized eigenstrains and displacements can be expressed as (Mura 1987)

$$Z_{Kl}^{*}(\boldsymbol{x}) = \bar{Z}_{Kl}^{*}(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\xi\cdot x}, \quad U_K(\boldsymbol{x}) = \bar{U}_K(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\xi\cdot x} \tag{7.81}$$

Analogous to elasticity, substituting Eq. (7.81) into (7.78) and neglecting the body force yield

$$\begin{aligned} &\Pi_{JK}(\boldsymbol{\xi})\bar{U}_K = \Xi_J(\boldsymbol{\xi});, \qquad \Pi_{JK}(\boldsymbol{\xi}) = E_{iJKl}\xi_i\xi_l, \quad \Xi_J(\boldsymbol{\xi}) = -\mathrm{i}E_{iJKl}\bar{Z}_{Kl}^{*}\xi_i \\ &\bar{U}_K = \Xi_J N_{JK}/D; \quad N_{MJ}(\boldsymbol{\xi}) = \frac{1}{2}\varpi_{IKL}\varpi_{JMN}\Pi_{KM}\Pi_{LN}, \quad D = |\Pi_{KM}| \end{aligned} \tag{7.82}$$

where ϖ_{JMN} is the permutation tensor and N_{MJ} is the algebraic complement of Π_{MJ} in the matrix $\boldsymbol{\Pi}$. For the general case, the Fourier integral transform is used (Mura 1987; Wang 1992):

$$\begin{aligned} Z_{Kl}^{*}(\boldsymbol{x}) &= \int_{-\infty}^{\infty} \bar{Z}_{Kl}^{*}(\boldsymbol{\xi})\ \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\xi, \quad \bar{Z}_{Kl}^{*}(\boldsymbol{\xi}) = \frac{1}{2\pi^3}\int_{-\infty}^{\infty} Z_{Kl}^{*}(\boldsymbol{x})\ \mathrm{e}^{-\mathrm{i}\xi\cdot x}\mathrm{d}\boldsymbol{x} \\ U_K(\boldsymbol{x}) &= \int_{-\infty}^{\infty} \bar{U}_K(\boldsymbol{\xi})\ \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\xi, \quad \bar{U}_K(\boldsymbol{\xi}) = \frac{1}{2\pi^3}\int_{-\infty}^{\infty} U_K(\boldsymbol{x})\ \mathrm{e}^{-\mathrm{i}\xi\cdot x}\mathrm{d}\boldsymbol{x} \end{aligned} \tag{7.83}$$

Analogous to elasticity, substituting Eq. (7.83) into (7.78) and using (7.82) yield

$$\begin{aligned} U_M(\boldsymbol{x}) &= -\mathrm{i}\int_{-\infty}^{\infty} E_{iJKl}\bar{Z}_{Kl}^{*}(\boldsymbol{\xi})\xi_i N_{MJ}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\ \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot x}\mathrm{d}\boldsymbol{\xi} \quad \text{or} \\ U_M(\boldsymbol{x}) &= -\int_{-\infty}^{\infty} E_{iJKl}Z_{Kl}^{*}(\boldsymbol{x}')G_{MJ,i}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}\boldsymbol{x}' \\ G_{MJ}(\boldsymbol{x}-\boldsymbol{x}') &= \frac{1}{2\pi^3}\int_{-\infty}^{\infty} N_{MJ}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\ \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot(x-x')}\mathrm{d}\boldsymbol{\xi} \end{aligned} \tag{7.84}$$

where $G_{MJ,i}(\boldsymbol{x}-\boldsymbol{x}') = \partial G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')/\partial x_i = -\partial G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')/\partial x_i'$. If $G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')$ is a Green function in a finite region, then

$$U_M(\boldsymbol{x}) = \int_a E_{iJKl}Z_{Kl}^{*}(\boldsymbol{x}')n_i G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}a(x') - \int_V E_{iJKl}Z_{Kl,i}^{*}(\boldsymbol{x}')G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x}') \tag{7.85}$$

7.3.3 Radon Transform Method

The Radon integral transform plays a fundamental role in the tomography, such as CT scanning in medicine. At first the basic concepts are introduced as follows (Deans 1983):

An arbitrary function $f(x_1, x_2)$ is defined on some domain Ω of a 2D plane $\boldsymbol{R}^2$. Let L be any line in the plane, then the mapping defined by the projection or line integral of f along all possible lines L is the 2D Radon transform of f, i.e.,

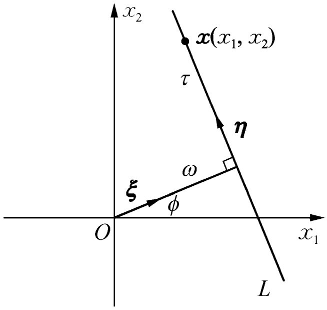


Fig. 7.3 A sketch of Radon transform

$$\begin{aligned}\bar{f}(\omega,\phi) &= Rf = \int_L f(x_1,x_2)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(\omega\boldsymbol{\xi}+\tau\boldsymbol{\eta})\mathrm{d}\tau, \quad \text{or}\\ \bar{f}(\omega,\boldsymbol{\xi}) &= \int_{R^2} f(\boldsymbol{x})\delta(\omega-\boldsymbol{\xi}\cdot\boldsymbol{x})\,\mathrm{d}a(\boldsymbol{x})\end{aligned} \tag{7.86}$$

where $\omega = \boldsymbol{\xi}\cdot\boldsymbol{x} = x_1\cos\phi + x_2\sin\phi$ is the perpendicular distance from the origin to L, $\boldsymbol{\xi} = (\cos\phi, \sin\phi)$ is a unit vector along $\boldsymbol{\omega}$ which defines the orientation of L, $\boldsymbol{\eta} = (-\sin\phi, \cos\phi)\perp\boldsymbol{\xi}$ is along L, τ is determined by $\boldsymbol{x} = \omega\boldsymbol{\xi} + \tau\boldsymbol{\eta}$, and ϕ is the angle between $\boldsymbol{\xi}$ and positive x_1-axis (Fig. 7.3). The last equation in Eq. (7.86) is easy extended to three- and higher-dimensional space. If $\bar{f}(\omega,\boldsymbol{\xi})$ is known for all ω and ϕ, then $\bar{f}(\omega,\boldsymbol{\xi})$ is a 2D Radon transform of $f(x_1,x_2)$. In an n-dimensional space, L represents $(n-1)$-dimensional hyperplane. Especially for a 3D space, L represents a plane. For n-dimensional space, the inversion Radon transform is

$$f(\boldsymbol{x}) = \frac{1}{2(2\pi\mathrm{i})^{n-1}}\boldsymbol{\Delta}_x^{(n-1)/2}\int_{|\boldsymbol{\xi}|=1} Rf(\boldsymbol{\xi},\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}a(\boldsymbol{\xi}) \tag{7.87}$$

where $\boldsymbol{\Delta}_x$ is the Laplacian operator. Especially for 3D space, Eq. (7.87) is reduced to

$$f(\boldsymbol{x}) = -\frac{1}{8\pi^2}\boldsymbol{\Delta}_3\int_{|\boldsymbol{\xi}|=1} Rf(\boldsymbol{\xi},\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{\xi} = -\frac{1}{8\pi^2}\int_{|\boldsymbol{\xi}|=1}\left[\partial^2\bar{f}(\boldsymbol{\xi},\boldsymbol{\xi}\cdot\boldsymbol{x})/\partial\omega^2\right]_{\omega=\boldsymbol{\xi}\cdot x}\mathrm{d}a(\boldsymbol{\xi}) \tag{7.88}$$

Deeg (1980), Dunn and Taya (1993), and Dunn (1994) pointed out that the Radon transform can also be used to the electroelastic Green function $G_{IJ}(\boldsymbol{x}-\boldsymbol{x}')$, i.e.,

$$\begin{aligned}\bar{G}_{IJ}(\boldsymbol{\xi},\omega-\boldsymbol{\xi}\cdot\boldsymbol{x}') &= \iint_{\boldsymbol{\xi}\cdot\boldsymbol{x}=\omega} G_{IJ}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}a(\boldsymbol{x})\\ G_{IJ}(\boldsymbol{x}-\boldsymbol{x}') &= \frac{1}{8\pi^2}\iint_{|\boldsymbol{\xi}|=1}\left[\partial^2\bar{G}_{IJ}(\boldsymbol{\xi},\omega-\boldsymbol{\xi}\cdot\boldsymbol{x}')/\partial\omega^2\right]_{\boldsymbol{\xi}\cdot\boldsymbol{x}=\omega}\mathrm{d}a(\boldsymbol{\xi})\end{aligned} \tag{7.89}$$

where $\boldsymbol{\xi}$, ω are variables in the transform space and the integral domain is a 2D plane $\boldsymbol{\xi} \cdot \boldsymbol{x} = \omega$. In the inverse Radon transform, the integral domain is the surface of a unit sphere $|\boldsymbol{\xi}| = 1$. Using the Radon transform to Eq. (7.79), after some manipulation, yields

$$K_{JM}(\boldsymbol{\xi})(\partial^2/\partial x_k \partial x_k)\bar{G}_{MR}(\boldsymbol{\xi}, \omega - \boldsymbol{\xi} \cdot \boldsymbol{x}') + \delta_{JR}\delta(\omega - \boldsymbol{\xi} \cdot \boldsymbol{x}') = \mathbf{0} \tag{7.90}$$

where $K_{JM}(\boldsymbol{\xi}) = E_{iJMn}\xi_i\xi_n$. Using the inversion Radon transform from Eq. (7.90) yields

$$G_{MR}(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x} - \boldsymbol{x}'|} \iint_{|\boldsymbol{\xi}|=1} K_{MR}^{-1}(\boldsymbol{\xi})\delta(\boldsymbol{\xi} \cdot \boldsymbol{t})\mathrm{d}a(\boldsymbol{\xi}) \tag{7.91}$$

where $K_{JM}(\boldsymbol{\xi})K_{MR}^{-1}(\boldsymbol{\xi}) = \delta_{JR}$ and $\boldsymbol{t}$ is the unit vector along $\boldsymbol{x} - \boldsymbol{x}'$. Using the property of the Dirac delta, in an orthogonal coordinate system $\boldsymbol{t}$-$\boldsymbol{m}$-$\boldsymbol{n}$, Deeg (1980) reduced the integral in Eq. (7.91) to the following contour integral:

$$G_{MR}(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x} - \boldsymbol{x}'|} \int_C K_{MR}^{-1}(\boldsymbol{\xi})\delta(\boldsymbol{\xi} \cdot \boldsymbol{t})\mathrm{d}a(\boldsymbol{\xi}) \tag{7.92}$$

where C is the contour produced by $|\boldsymbol{\xi}| = 1$ in $\boldsymbol{m}$-$\boldsymbol{n}$ plane normal to $\boldsymbol{x} - \boldsymbol{x}'$. Compared to Fourier transform method, Eq. (7.92) is a simpler effective method to seek the Green function. The second partial derivatives of the electroelastic Green function is

$$G_{MR,kl}(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x} - \boldsymbol{x}'|} \frac{\partial^2}{\partial x_n \partial x_n} \int_{|\xi|=1} \xi_k\xi_l K_{MR}^{-1}(\boldsymbol{\xi})\delta[\boldsymbol{\xi} \cdot (\boldsymbol{x} - \boldsymbol{x}')]\mathrm{d}a(\boldsymbol{\xi}) \tag{7.93}$$

7.3.4 A Single Ellipsoidal Inclusion with Uniform Eigenstrains

Let the coordinates coincide with the principle axes of the material. For an ellipsoidal inclusion with uniform eigenstrains, the Eshelby's method is used. At first the inclusion is cut off from the matrix, so the inclusion and matrix are all free. The eigenstrains of the inclusion in the free state is denoted by $\boldsymbol{Z}^*$. The following generalized stresses are applied on the boundary of the inclusion:

$$T_J = -\Sigma_{iJ}^* \cdot n_i, \quad \Sigma_{iJ}^* = E_{iJMl}Z_{Ml}^* \tag{7.94}$$

Then put the inclusion subjected to above generalized stresses into the matrix. After this procedure, the original problem is transformed to a homogeneous material with a force $-T_J = E_{iJMl}Z^*_{Ml}$ acting on the counter which originally is the boundary of the inclusion. If Z^* is uniform, Eq. (7.94) yields (Deeg 1980)

$$U_{M,n}(\boldsymbol{\xi}) = -E_{iJKl}Z^*_{Kl}\iiint_{\Omega} G_{MJ,in}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x}') \tag{7.95}$$

Using Eq. (7.93) from Eq. (7.95) yields (Deeg 1980; Dunn and Taya 1993)

$$U_{M,n}(\boldsymbol{\xi}) = \frac{a_1a_2a_3}{4\pi}E_{iJKl}Z^*_{Kl}\int_{|\boldsymbol{\xi}|=1}\frac{1}{\alpha^3}\xi_k\xi_l K^{-1}_{MR}(\boldsymbol{\xi})\mathrm{d}a(\boldsymbol{\xi}), \quad \alpha = \sqrt{a_1^2\xi_1^2 + a_2^2\xi_2^2 + a_3^2\xi_3^2} \tag{7.96}$$

Analogous to the elastic inclusion problem, the generalized strains inside the ellipsoid induced by the uniform eigenstrains are also uniform and

$$\begin{aligned} &Z_{Mn}(\boldsymbol{\xi}) = S_{MnKl}Z^*_{Kl} \\ &S_{MnKl} = \begin{cases} (1/8\pi)E_{iJKl}(I_{MJin}+I_{nJiM}), & M=1,2,3 \\ (1/4\pi)E_{iJKl}I_{4Jin}, & M=4 \end{cases} \\ &I_{MJin} = a_1a_2a_3\int_{|\boldsymbol{\xi}|=1}(1/\alpha^3)G_{MJin}(\boldsymbol{\xi})\mathrm{d}a(\boldsymbol{\xi}), \quad G_{MJin}(\boldsymbol{\xi}) = \xi_i\xi_n K^{-1}_{MJ}(\boldsymbol{\xi}) \end{aligned} \tag{7.97}$$

where S_{MnKl} is called the "electroelastic Eshelby tensor," but it is not a tensor, i.e., it does not obey the tensor transform rule under a coordinate transformation. The key point to solve S_{MnKl} is to calculate $I_{MJin}(\boldsymbol{\xi})$. $I_{MJin}(\boldsymbol{\xi})$ can be transformed to the following form (Mikata 2000):

$$\begin{aligned} I_{MJin}(\boldsymbol{\xi}) &= \int_{|\xi|=1} G_{MJin}(y_1/a_1, y_2/a_{12}, y_3/a_3)\mathrm{d}a(\boldsymbol{\xi}) \\ &= \int_{-1}^{1}\mathrm{d}t\int_0^{2\pi} G_{MJin}(y_1/a_1, y_2/a_{12}, y_3/a_3)\mathrm{d}\phi \\ y_1 = \sqrt{1-t^2}\cos\phi, &\quad y_2 = \sqrt{1-t^2}\sin\phi, \quad y_3 = t \end{aligned} \tag{7.98}$$

For transversely isotropic piezoelectric materials, Mikata (2000) pointed out that in $I_{MJin}(\boldsymbol{\xi})$ only $I_{1212}, I_{1313}, I_{1314}, I_{2323}, I_{2324}$ and $I_{11MJ}, I_{22MJ}, I_{33MJ}, MJ = 11, 22, 33, 44, 34$ are not zero, and the Eshelby tensor only has 36 components:

$$\begin{aligned} &S_{1111}, S_{1122}, S_{1133}, S_{1143}, S_{1212} = S_{1221} = S_{2112} = S_{2121}, S_{1313} = S_{1331} = S_{3113} = S_{3131}, \\ &S_{1341} = S_{3141}, S_{2211}, S_{2222}, S_{2233}, S_{2243}, S_{2323} = S_{2332} = S_{3223} = S_{3232}, S_{2342} = S_{3242}, \\ &S_{3311}, S_{3322}, S_{3333}, S_{3343}, S_{4113}, S_{4141}, S_{4223}, S_{4242}, S_{4311}, S_{4322}, S_{4333}, S_{4343}. \end{aligned}$$

For an elliptic cylindrical inclusion along the x_3-axis, the Eshelby tensor with 22 components is

$$S_{1111} = \frac{\alpha}{2(1+\alpha)^2}\left(3 + \frac{C_{12}}{C_{11}} + \frac{2}{\alpha}\right), \quad S_{1122} = \frac{\alpha}{2(1+\alpha)^2}\left[\frac{(2+\alpha)C_{12}}{\alpha C_{11}} + \frac{2}{\alpha}\right] - 1,$$
$$S_{1133} = \frac{C_{13}}{(1+\alpha)C_{11}}, \quad S_{1143} = \frac{e_{31}}{(1+\alpha)C_{11}}, \quad S_{1212} = \frac{\alpha}{2(1+\alpha)^2}\left(\frac{1+\alpha+\alpha^2}{\alpha} - \frac{C_{12}}{C_{11}}\right),$$
$$S_{1313} = \frac{1}{2(1+\alpha)}, \quad S_{2233} = \frac{\alpha C_{13}}{(1+\alpha)C_{11}}, \quad S_{2243} = \frac{\alpha e_{13}}{(1+\alpha)C_{11}}, \quad S_{2323} = \frac{\alpha}{2(1+\alpha)},$$
$$S_{2211} = \frac{\alpha}{2(1+\alpha)^2}\left[(1+2\alpha)\frac{C_{12}}{C_{11}} - 1\right], \quad S_{2222} = \frac{\alpha}{2(1+\alpha)^2}\left(3 + \frac{C_{12}}{C_{11}} + 2\alpha\right),$$
$$S_{4141} = \frac{1}{(1+\alpha)}, \quad S_{4242} = \frac{\alpha}{(1+\alpha)}, \quad \text{other } S_{MnKl} = 0$$
$$S_{1212} = S_{1221} = S_{2112} = S_{2121}, S_{1313} = S_{1331} = S_{3113} = S_{3131}, S_{2323} = S_{2332} = S_{3223} = S_{3232} \tag{7.99a}$$

For a penny-shaped crack perpendicular to x_3-axis, the Eshelby tensor with 18 components is

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232} = 1/2$$
$$S_{1341} = S_{3141} = S_{2342} = S_{3242} = \frac{e_{15}}{2C_{44}}, \quad S_{3311} = S_{3322} = \frac{C_{13}\,\epsilon_{33} + e_{31}e_{33}}{C_{33}\,\epsilon_{33} + e_{33}^2}$$
$$S_{3333} = S_{4343} = 1, \quad S_{4311} = S_{4322} = \frac{C_{13}e_{33} - C_{33}e_{31}}{C_{33}\,\epsilon_{33} + e_{33}^2}, \quad \text{other } S_{MJin} = 0 \tag{7.99b}$$

7.3.5 Ellipsoid Inhomogeneity

Analogous to elastic inhomogeneity problem, the electroelastic inhomogeneity problem can be handled by the equivalent inclusion method. Let E^{M}_{iJKl} and E^{in}_{iJKl} denote the material constants in the matrix and inhomogeneity, respectively. The generalized stress Σ^0_{Mn} is applied at infinity. Obviously for a homogeneous material, the generalized stress in material is also Σ^0_{Mn} and the corresponding strain is $Z^0_{Mn} = \left(E^M_{iJMn}\right)^{-1}\Sigma^0_{Mn}$. Assume the strain due to the inhomogeneity is Z_{Mn} and then the stress in the matrix and inhomogeneity are, respectively,

$$\Sigma^{\mathrm{M}}_{iJ}(\boldsymbol{x}) = E^{\mathrm{M}}_{iJMn}\left[Z^0_{Mn} + Z_{Mn}(\boldsymbol{x})\right], \quad \Sigma^{\mathrm{in}}_{iJ}(\boldsymbol{x}) = E^{\mathrm{in}}_{iJMn}\left[Z^0_{Mn} + Z_{Mn}(\boldsymbol{x})\right] \tag{7.100}$$

The key point of the equivalent inclusion method is that let the inhomogeneity possess the same material constants with the matrix, but the artificial eigenstrain Z^*_{Mn} is added. Then let

$$\Sigma^{\mathrm{in}}_{iJ}(\boldsymbol{x}) = E^{\mathrm{in}}_{iJMn}\left[Z^0_{Mn} + Z_{Mn}(\boldsymbol{x})\right] = E^{\mathrm{M}}_{iJMn}\left[Z^0_{Mn} + Z_{Mn}(\boldsymbol{x}) + Z^*_{Mn}\right] \tag{7.101}$$

Using $Z_{Mn}(\boldsymbol{x}) = S_{MnKl}Z^*_{Kl}$ in the inhomogeneity, so Eq. (7.101) yields

$$Z^*_{Kl} = \left[\left(E^{\mathrm{M}}_{iJMn} - E^{\mathrm{in}}_{iJMn}\right)S_{MnKl} + E^{\mathrm{M}}_{iJKl}\right]^{-1}\left(E^{\mathrm{in}}_{iJMn} - E^{\mathrm{M}}_{iJMn}\right)Z^0_{Mn} \tag{7.102}$$

Solving Z^*_{Kl}, the problem can be solved.

As an example an ellipsoidal piezoelectric sensor embedded in an elastic material is discussed (Fan and Qin 1995). The constitutive equations of the sensor (as an elliptic inclusion) and matrix are, respectively,

$$\sigma_{ij} = C^{\mathrm{in}}_{ijkl}\varepsilon_{kl} - e^{\mathrm{in}}_{kij}E_k, \quad D_i = e^{\mathrm{in}}_{ikl}\varepsilon_{kl} + \epsilon^{\mathrm{in}}_{ik}E_k; \quad \text{in } \Omega^{\mathrm{in}} \tag{7.103}$$

$$\sigma_{ij} = C^{\mathrm{M}}_{ijkl}\varepsilon_{kl}, \quad D_i = \epsilon^{\mathrm{M}}_{ik}E_k; \quad \text{in } \Omega^{\mathrm{M}} \tag{7.104}$$

Comparing Eqs. (7.103) and (7.104), one can consider the terms $e^{\mathrm{in}}_{kij}E_k, e^{\mathrm{in}}_{ikl}\varepsilon_{kl}$ in Eq. (7.103) produced by some kind of eigenstrains. When the matrix is subjected to uniform generalized stresses $\left(\sigma^0_{ij}, E^0_i\right)$ at infinity, the generalized stresses in the sensor are changed to

$$\sigma^{\mathrm{in}}_{ij} = \sigma^0_{ij} + \sigma'^{\mathrm{in}}_{ij} = C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + \varepsilon_{kl} - \varepsilon^{\mathrm{E}}_{kl}\right), \quad C^{\mathrm{in}}_{ijkl}\varepsilon^{\mathrm{E}}_{kl} = e^{\mathrm{in}}_{kij}\left(E_k + E^0_k\right) \tag{7.105}$$

$$D^{\mathrm{in}}_i = D^0_i + D'^{\mathrm{in}}_i = \epsilon^{\mathrm{in}}_{ik}\left(E^0_k + E_k - E^{\varepsilon}_k\right), \quad -\epsilon^{\mathrm{in}}_{ik}E^{\varepsilon}_k = e^{\mathrm{in}}_{ikl}\left(\varepsilon_{kl} + \varepsilon^0_{kl}\right) \tag{7.106}$$

By using the equivalent inclusion method, the original inhomogeneity problem subjected to uniform generalized stresses at infinity can be decoupled into two equivalent inclusion problems:

1. The elastic equivalent inclusion problem. Equation (7.105) can also be written as

$$\sigma^{\mathrm{in}}_{ij} = \sigma^0_{ij} + \sigma'^{\mathrm{in}}_{ij} = C^{\mathrm{M}}_{ijkl}\left(\varepsilon^0_{kl} + \varepsilon_{kl} - \varepsilon^{\mathrm{E}}_{kl} - \varepsilon^*_{kl}\right) \tag{7.107}$$

where ε^*_{kl} is the virtual eigenstrain. Using the Eshelby inclusion theory yields

$$\varepsilon_{kl} = S_{klmn}\varepsilon^{**}_{kl}, \quad \varepsilon^{**}_{kl} = \varepsilon^{\mathrm{E}}_{kl} + \varepsilon^*_{kl} \tag{7.108}$$

2. The dielectric equivalent inclusion problem. Equation (7.106) can also be written as

$$D^{\mathrm{in}}_i = D^0_i + D'^{\mathrm{in}}_i = \epsilon^{\mathrm{M}}_{ik}\left(E^0_k + E_k - E^{\varepsilon}_k - E^*_k\right) \tag{7.109}$$

where E^*_k is the virtual eigen electric field. Using the Eshelby inclusion theory yields

$$E_k = s_{kl}E^{**}_l, \quad E^{**}_k = E^{\varepsilon}_k + E^*_k \tag{7.110}$$

where $s_{kl} = S_{4k4l}$. Comparing Eqs. (7.105) and (7.107), it is concluded that

$$\begin{aligned} C^{\mathrm{M}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl} - \varepsilon^{**}_{kl}\right) &= C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl} - \varepsilon^{\mathrm{E}}_{kl}\right) \\ &= C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl}\right) - e^{\mathrm{in}}_{kij}\left(s_{kl}E^{**}_l + E^0_k\right) \end{aligned} \tag{7.111}$$

Comparing Eqs. (7.106) and (7.110), it is concluded that

$$\begin{aligned} &\epsilon^{\mathrm{in}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{\varepsilon}_k\right) = \epsilon^{\mathrm{M}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{**}_k\right), \qquad \text{or} \\ &E^{**}_l = \left[s_{ml}\left(\epsilon^{\mathrm{M}}_{im} - \epsilon^{\mathrm{in}}_{im}\right) - \epsilon^{\mathrm{M}}_{ik}\right]^{-1}\left[\left(\epsilon^{\mathrm{in}}_{im} - \epsilon^{\mathrm{M}}_{im}\right)E^0_k + e^{\mathrm{in}}_{imn}S_{mnpq}\varepsilon^{**}_{pq} + e^{\mathrm{in}}_{ikl}\varepsilon^0_{kl}\right] \end{aligned} \tag{7.112}$$

Solving ε^{**}_{kl} and E^{**}_k, the stress and electric fields are obtained in the sensor, i.e.,

$$\begin{aligned} \sigma^{\mathrm{in}}_{ij} &= C^{\mathrm{M}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl} - \varepsilon^{**}_{kl}\right) = C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl}\right) - e^{\mathrm{in}}_{kij}\left(s_{kl}E^{**}_l + E^0_k\right) \\ D^{\mathrm{in}}_i &= \epsilon^{\mathrm{M}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{**}_k\right) = \epsilon^{\mathrm{in}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{\varepsilon}_k\right) \end{aligned} \tag{7.113}$$

The stress $\sigma^{\mathrm{out}}_{ij}$ and electric fields E^{out} in the matrix can be solved as follows:

$$\begin{aligned} &\sigma^{\mathrm{out}}_{ij} = \sigma^0_{ij} + \sigma'^{\mathrm{out}}_{ij} = \sigma^0_{ij} + [\![\sigma_{ij}]\!] + \sigma'^{\mathrm{in}}_{ij}, \quad [\![\sigma_{ij}]\!] = \sigma'^{\mathrm{out}}_{ij} - \sigma'^{\mathrm{in}}_{ij} = \sigma^{\mathrm{out}}_{ij} - \sigma^{\mathrm{in}}_{ij} \\ &E^{\mathrm{out}}_i = E^0_i + E'^{\mathrm{out}}_i = E^0_i + [\![E_i]\!] + E'^{\mathrm{in}}_i, \quad [\![E_i]\!] = E'^{\mathrm{out}}_i - E'^{\mathrm{in}}_i = E^{\mathrm{out}}_i - E^{\mathrm{in}}_i \end{aligned} \tag{7.114}$$

The displacement $\boldsymbol{u}$ and the surface traction $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0}$ across the interface of inclusion and matrix must be continuous, and jump of the displacement gradient $\nabla \otimes \boldsymbol{u}$ must be normal to the interface. The continuous conditions of the electric field and electric displacement $\boldsymbol{E} \times \boldsymbol{n}$ and $\boldsymbol{D} \cdot \boldsymbol{n}$ across the interface demand that the jump of $\boldsymbol{E}$ must be normal to the interface. So

$$\begin{aligned} &[\![u_i]\!] = u^{\mathrm{out}}_i - u^{\mathrm{in}}_i = 0, \quad [\![u_{i,j}]\!] = u^{\mathrm{out}}_{i,j} - u^{\mathrm{in}}_{i,j} = \lambda_i n_j; \quad [\![\sigma_{ij}]\!]n_j = 0 \\ &[\![E_i]\!] = \eta n_i, \quad [\![D_i]\!]n_i = 0 \end{aligned} \tag{7.115}$$

where $\boldsymbol{\lambda}, \eta$ are proportional constants. Substituting the constitutive equations into Eq. (7.115), it is obtained:

$$C_{ijkl}\lambda_k n_l n_j = -C_{ijkl}\varepsilon^{**}_{kl} n_j, \quad \eta\epsilon_{ik}n_i n_k = -\epsilon_{ik}n_i E^{**}_k \tag{7.116}$$

Therefore, the stress $\sigma^{\mathrm{out}}_{ij}$ and electric fields E^{out} in the matrix are

$$\sigma^{\mathrm{out}}_{ij} = \sigma^0_{ij} + C_{ijkl}\left(\lambda_k n_l + \varepsilon^{**}_{kl}\right) + \sigma'^{\mathrm{in}}_{ij}, \quad E^{\mathrm{out}}_i = E^0_i + \epsilon_{ik}\left(\eta n_k + E^{**}_k\right) + E'^{\mathrm{in}}_i \tag{7.117}$$

7.4 Some Simpler Practical Problems

7.4.1 Extension of a Rod

Figure 7.4 shows a transversely isotropic piezoelectric long cylindrical rod with polarized x_3-axis. The two silver-coated end faces are used as electrodes and subjected to uniform normal traction p. Using the first kind of constitutive equation, the solutions of this problem are as follows:

1. For shorted electrodes,

$$\begin{aligned}&\sigma_{33} = p, \quad \text{all other } \sigma_{ij} = 0; \quad \varepsilon_{33} = s_{33}p, \quad \varepsilon_{11} = \varepsilon_{22} = s_{13}p;\\ &E_3 = E_1 = E_2 = 0, \quad D_3 = d_{33}p, \quad D_1 = D_2 = 0; \quad \mathfrak{A}_s = \sigma_{33}\varepsilon_{33}/2 = s_{33}p^2/2\end{aligned} \tag{7.118}$$

2. For open electrodes,

$$\begin{aligned}&\sigma_{33} = p, \quad \text{all other } \sigma_{ij} = 0; \quad \varepsilon_{33} = s_{33}\left(1 - d_{33}^2/\epsilon_{33}s_{33}\right)p, \quad \varepsilon_{11} = \varepsilon_{22} = s_{13}p + d_{31}E_3;\\ &D_3 = D_1 = D_2 = 0, \; E_3 = -(d_{33}/\epsilon_{33})p, \; E_1 = E_2 = 0; \; U_o = (s_{33}/2)\left(1 - d_{33}^2/\epsilon_{33}s_{33}\right)p^2\end{aligned} \tag{7.119}$$

   The rod appears to be stiffer for the open electrodes than sorted electrodes due to $d_{33}^2/\epsilon_{33}s_{33} > 0$.

3. The longitudinal electromechanical coupling factor k_{33}:

$$k_{33}^2 = (U_\mathrm{s} - U_\mathrm{o})/U_\mathrm{s} = d_{33}^2/\epsilon_{33}s_{33} \tag{7.120}$$

7.4.2 Torsion of a Piezoelectric Circular Cylinder

Figure 7.5 shows a transversely isotropic piezoelectric circular cylinder of length L, inner radius a, and outer radius b with polarized θ-axis. The two silver-coated end faces are used as electrodes and subjected to a torque M and charge Q_e. Using the second kind of constitutive equation, the general solution of this problem is

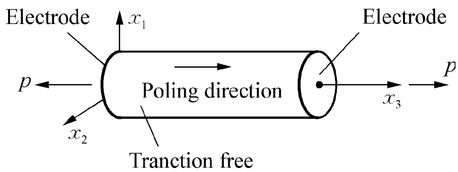


Fig. 7.4 An axial poled rod

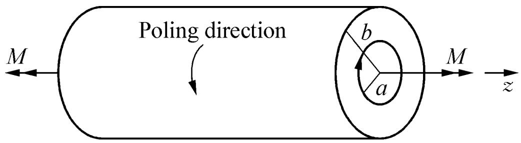


Fig. 7.5 A circular cylinder in torsion

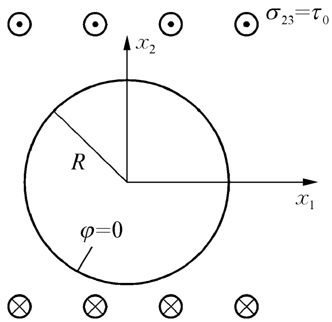


Fig. 7.6 An infinite plate with a *circular cylindrical hole* under longitudinal shear

$$
\begin{aligned}
&u_\theta = Arz, \quad u_r = u_z = 0; \quad \gamma_{\theta z} = Ar, \quad \sigma_{\theta z} = C_{44}Ar - e_{15}B; \\
&\varphi = -Bz, \quad E_z = B; \quad D_z = e_{15}Ar + \epsilon_{11}B \\
&M = \int_a^b \sigma_{\theta z}(2\pi r\mathrm{d}r)r = C_{44}AI_p - e_{15}2\pi B\left(b^3 - a^3\right)/3, \quad I_\mathrm{p} = \pi\left(b^4 - a^4\right)/2 \\
&Q_\mathrm{e} = \int_a^b D_z(2\pi r\mathrm{d}r) = e_{15}2\pi A\left(b^3 - a^3\right)/3 + \epsilon_{11}B\pi\left(b^2 - a^2\right)
\end{aligned}
\tag{7.121}
$$

1. Shorted electrodes : $B = 0, \quad A = M/C_{44}I_\mathrm{p}$ (7.122)
2. Open electrodes : $Q_\mathrm{e} = 0,$

$$
A = M\left[C_{44}I_\mathrm{p} + \frac{e_{15}^2}{\epsilon_{11}}\left(\frac{2\pi}{3}\right)^2\frac{\left(b^3 - a^3\right)}{\pi\left(b^2 - a^2\right)}\right]^{-1}
\tag{7.123}
$$

7.4.3 A Circular Hole Under Longitudinal Shear in an Infinite Piezoelectric Plate

Figure 7.6 shows a circular cylindrical hole of radius R in an unbounded transversely isotropic piezoelectric material with polarized x_3-axis under a uniform longitudinal shear stress $\sigma_{23} = \tau_0$ at $x_2 = \pm\infty$. The hole surface is a grounded electrode. The governing equations and boundary conditions for an electrically open case are

$$
\begin{aligned}
&\nabla^2 u_3 = 0, \quad \nabla^2\varphi = 0; \quad r > R \\
&\sigma_{rz} = 0, \quad \varphi = 0, \quad r = R; \quad \sigma_{23} = \tau_0, \quad x_2 = \pm\infty \\
&D_2 = 0, \quad x_2 = \pm\infty \text{ (electrically open)}; \quad E_2 = 0, \quad x_2 = \pm\infty \text{ (electrically shorted)}
\end{aligned}
\tag{7.124}
$$

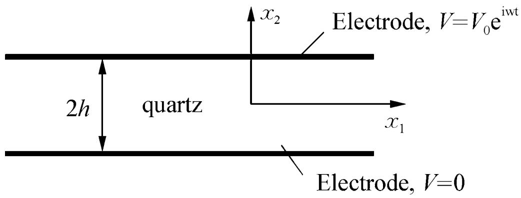


Fig. 7.7 An electrode quartz plate

For the electrically open case, the solution is

$$u_3 = \frac{\tau_0}{C_{44}(1+k^2)}\left[r + \left(1+2k^2\right)\frac{R^2}{r}\right]\sin\theta, \quad \varphi = \frac{\tau_0}{C_{44}\epsilon_{11}(1+k^2)}\left(r - \frac{R^2}{r}\right)\sin\theta \tag{7.125}$$

where $k^2 = e_{15}^2/C_{44}\epsilon_{11}$. For the electrically shorted case, the solution can also be obtained.

7.4.4 Thickness-Shear Vibration of a Quartz Plate

Figure 7.7 shows the sketch of a widely used piezoelectric resonator manufactured by rotated Y-cut quartz plate. Surfaces at $x_2 = \pm h$ are traction-free and electroded, with a driving voltage $V_0 e^{\mathrm{i}\omega t}$. Let $u_3 = 0$ and $\partial u_2/\partial x_2 \approx 0$ due to the plate is thin enough. So the displacement and potential fields can be assumed in the following forms:

$$u_1 = U_1(x_2)e^{\mathrm{i}\omega t}, \quad u_2(x_2) \approx 0, \quad u_3 = 0; \quad \varphi = \Phi(x_2)e^{\mathrm{i}\omega t} \tag{7.126}$$

Under above assumptions, the constitutive equations are

$$\sigma_6 = C_{66}u_{1,2} + e_{26}\varphi_{,2}, \quad D_2 = e_{26}u_{1,2} - \epsilon_{22}\varphi_{,2} \tag{7.127}$$

The stresses $\sigma_5 = C_{56}u_{1,2} + e_{25}\varphi_{,2}$, $D_3 = e_{36}u_{1,2} - \epsilon_{23}\varphi_{,2}$ are omitted because they are not used. The generalized momentum equation and boundary condition are

$$\begin{aligned} &\sigma_{6,2} = -\rho\omega^2 u_1, \quad D_{2,2} = 0 \\ &\sigma_6 = 0, \quad \text{at } x_2 = \pm h; \quad \Phi(h) - \Phi(-h) = V_0 \end{aligned} \tag{7.128}$$

The general solutions are

$$\begin{aligned} &U_1 = A_1\sin\lambda x_2 + A_2\cos\lambda x_2; \quad \lambda = \sqrt{\rho/C_{66}^*}\,\omega, \quad C_{66}^* = C_{66}\left(1+k^2\right), \quad k = e_{26}\Big/\sqrt{C_{66}\epsilon_{22}} \\ &\Phi = (e_{26}/\epsilon_{22})(A_1\sin\lambda x_2 + A_2\cos\lambda x_2) + Bx_2 \end{aligned} \tag{7.129}$$

From the boundary conditions, we can get two group equations:

$$C_{66}^{*}A_1\lambda\cos\lambda h+e_{26}B=0,\quad 2(e_{26}/\epsilon_{22})A_1\sin\lambda h+2Bh=V_0 \tag{7.130}$$

$$C_{66}^{*}A_2\lambda\sin\lambda h=0 \tag{7.131}$$

1. *Free vibration $V_0=0$, symmetric modes* From Eq. (7.131) it is obtained

$$\sin\lambda h=0;\quad \text{or}\quad \lambda_n h=n\pi/2,\quad \omega_n=(n\pi/2h)\sqrt{C_{66}^{*}/\rho},\quad n=0,2,4,6,\ldots, \tag{7.132}$$

where ω_n is the nth order resonance frequency. In the same time $A_2\neq 0,\ A_1=B=0$. The corresponding symmetric modes are

$$U_1=\cos\lambda_n x_2;\quad \Phi=(e_{26}/\epsilon_{22})\cos\lambda_n x_2 \tag{7.133}$$

2. *Free vibration $V_0=0$, antisymmetric modes* In this case $A_1\neq 0, B\neq 0, A_2=0$. Nontrivial solutions may exist in Eq. (7.130) if

$$C_{66}^{*}A_1\lambda h\cos\lambda h-\left(e_{26}^2/\epsilon_{22}\right)\sin\lambda h=0,\quad \text{or}\tan\lambda_\nu h=\lambda_\nu h(1+k^2)/k^2,\quad \omega_\nu=\lambda_\nu\sqrt{C_{66}^{*}/\rho} \tag{7.134}$$

From Eq. (7.130), it is obtained $B_\nu=-\left(C_{66}^{*}/e_{26}\right)A_1\lambda_\nu\cos\lambda_\nu h$. In this case $\sin\lambda h\neq 0$, so $A_2=0$. The corresponding antisymmetric modes are

$$U_1=\sin\lambda x_2;\quad \Phi=(e_{26}/\epsilon_{22})\sin\lambda x_2-\left[\left(C_{66}^{*}/e_{26}\right)\lambda_\nu\cos\lambda_\nu h\right]x_2 \tag{7.135}$$

3. *Forced vibration* From Eq. (7.131) we have $A_2=0$. From Eq. (7.130) we get

$$A_1=-\frac{V_0}{2}\frac{e_{26}V_0}{C_{66}^{*}\lambda h\cos\lambda h-\left(e_{26}^2/\epsilon_{22}\right)\sin\lambda h},\quad B=\frac{V_0}{2}\frac{C_{66}^{*}\lambda\cos\lambda h}{C_{66}^{*}\lambda h\cos\lambda h-\left(e_{26}^2/\epsilon_{22}\right)\sin\lambda h} \tag{7.136}$$

Yang (2005) gave also some other interest problems except the above examples in this section.

7.5 Laminated Piezoelectric Plates

7.5.1 Basic Concepts and Governing Equations

In the earlier work, the piezoelectric actuator structure constituted of an elastic substrate (beam or bar), electroded piezoelectric elements, and finite-thickness bonding layers. For a pair actuators fixed on the upper and lower surfaces, if the

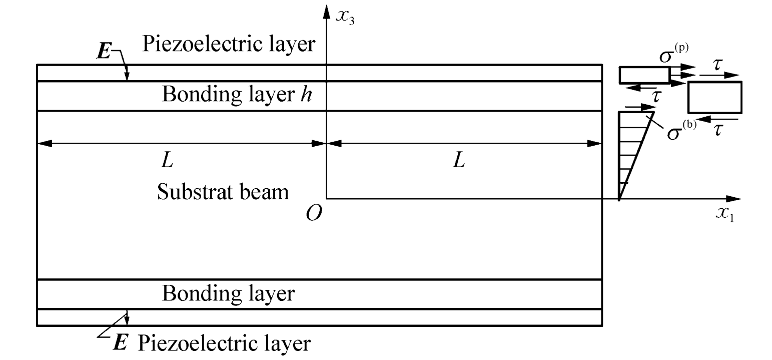


Fig. 7.8 A sketch of a simple intelligent beam

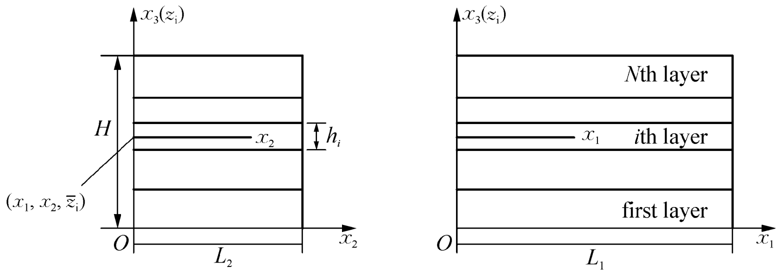


Fig. 7.9 A multiply laminated piezoelectric plate

same voltage is applied to both actuators, it results in pure extension, and if the opposite voltage is applied to both actuators, it results in bending (Fig. 7.8). In the present time, the "intelligent structure" may be a laminated piezoelectric beam, plate, shell and distributed actuator, sensor, and processor networks. In engineering the classical beam, plate and shell theory are commonly used, and sometimes will use the higher-order theories.

Consider an N-layer laminated piezoelectric plate of dimensions L_1 and L_2 in x_1 and x_2 directions and total thickness H in x_3-direction, and the ith layer has thickness h_i. The plate is polarized along x_3-axis. Except the global coordinate system, a local coordinate system (x_1, x_2, z_i) in the middle plane of ith layer is also adopted (Fig. 7.9). Layer 1 is the bottom layer and the layer N is the top layer. At each interface with perfect bonding between layers, continuity conditions of generalize displacements and tractions must be satisfied. For the ith interface between ith and $(i+1)$th layers in the local coordinate system, the continuity conditions are

$$
\begin{aligned}
&\boldsymbol{U}^{(i)}(x_1, x_2, h_i/2) = \boldsymbol{U}^{(i+1)}(x_1, x_2, -h_{i+1}/2), \\
&\boldsymbol{\Sigma}^{(i)}\boldsymbol{n}^{(i)}(x_1, x_2, h_i/2) = \boldsymbol{\Sigma}^{(i+1)}\boldsymbol{n}^{(i)}(x_1, x_2, -h_{i+1}/2), \quad i = 1 - (N-1)
\end{aligned}
\tag{7.137}
$$

It is also noted that sometimes a bond-line may be simulated by a layer of small thickness. For each interface, there are eight continuity conditions, so for a laminate plate with N layers, there are $8(N-1)$ continuity conditions; on the lower surface $(z=-h_1/2)$ of the first layer and on the upper surface $(z=h_N/2)$ of the Nth layer, there are four boundary conditions, respectively. Therefore, there are total $8N$ boundary conditions to determine $8N$ unknowns. For an orthotropic material layer, the constitutive equation is shown in Eq. (3.69). Analogous to Eq. (7.3), the motion equations in terms of generalized displacements are

$$\begin{aligned}
&C_{11}u_{1,11}+C_{66}u_{1,22}+C_{55}u_{1,33}+(C_{12}+C_{66})u_{2,12}+(C_{13}+C_{55})u_{3,13}+(e_{15}+e_{31})\varphi_{,13}=\rho u_{1,tt}\\
&(C_{12}+C_{66})u_{1,12}+C_{66}u_{2,11}+C_{22}u_{2,22}+C_{44}u_{2,33}+(C_{23}+C_{44})u_{3,23}+(e_{24}+e_{32})\varphi_{,23}=\rho u_{2,tt}\\
&(C_{13}+C_{55})u_{1,13}+(C_{23}+C_{44})u_{2,23}+C_{55}u_{3,11}+C_{44}u_{3,22}+C_{33}u_{3,33}+e_{24}\varphi_{,22}+e_{33}\varphi_{,33}=\rho u_{3,tt}\\
&(e_{31}+e_{15})u_{1,13}+(e_{32}+e_{24})u_{2,23}+e_{15}u_{3,11}+e_{24}u_{3,22}+e_{33}u_{3,33}\\
&\quad-\epsilon_{11}\varphi_{,11}-\epsilon_{22}\varphi_{,22}-\epsilon_{33}\varphi_{,33}=0
\end{aligned} \tag{7.138}$$

7.5.2 *Bending in Simply Supported Orthotropic Laminated Rectangular Plate*

It is assumed that the bottom and lateral surfaces are free, and known normal traction and potential are imposed on the top surface:

$$\begin{aligned}
&q(x_1,x_2)=q_0\sin p_1x_1\sin p_2x_2,\quad \varphi(x_1,x_2)=\Phi_0\sin p_1x_1\sin p_2x_2\\
&p_1=(n\pi/L_1),\quad p_2=(m\pi/L_2);\quad \text{when}\quad x_3=H
\end{aligned} \tag{7.139}$$

For the simply supported orthotropic laminated rectangular plate, the solution in each layer is assumed (Heyliger 1997):

$$\begin{aligned}
&u_1=u_{10}e^{sx_3}\cos p_1x_1\sin p_2x_2,\quad u_2=u_{20}e^{sx_3}\sin p_1x_1\cos p_2x_2\\
&u_3=u_{30}e^{sx_3}\sin p_1x_1\sin p_2x_2,\quad \varphi=\varphi_0e^{sx_3}\sin p_1x_1\sin p_2x_2
\end{aligned} \tag{7.140}$$

where u_{i0},φ_0,s are undetermined constants and the superscript (i) is omitted. Substituting Eq. (7.140) into (7.138) yields

$$\boldsymbol{\Lambda}\boldsymbol{U}_0=\mathbf{0},\quad \boldsymbol{U}_0=[u_{10},u_{20},u_{30},\varphi_0]^{\mathrm{T}},\quad \boldsymbol{\Lambda}=$$
$$\begin{bmatrix}
C_{11}p_1^2+C_{66}p_2^2-C_{55}s^2 & (C_{12}+C_{66})p_1p_2 & -(C_{13}+C_{55})p_1s & -(e_{15}+e_{31})p_1s\\
(C_{12}+C_{66})p_1p_2 & C_{66}p_1^2+C_{22}p_2^2-C_{44}s^2 & -(C_{23}+C_{44})p_2s & -(e_{24}+e_{32})p_2s\\
(C_{13}+C_{55})p_1s & (C_{23}+C_{44})p_2s & C_{55}p_1^2+C_{44}p_2^2-C_{33}s^2 & e_{15}p_1^2+e_{24}p_2^2-e_{33}s^2\\
(e_{15}+e_{31})p_1s & (e_{24}+e_{32})p_2s & e_{15}p_1^2+e_{24}p_2^2-e_{33}s^2 & -\epsilon_{11}p_1^2-\epsilon_{22}p_2^2+\epsilon_{33}s^2
\end{bmatrix} \tag{7.141}$$

Equation (7.141) will have nontrivial solution if $|\boldsymbol{\Lambda}|=0$. From which we can obtained eight eigenvector s_i, $s_i, i=1-8$. Corresponding each s_i, an eigenvector

$\boldsymbol{U}_{0j}$ $(u_{10i}, u_{20i}, u_{30i}, \varphi_{0i})$ with one unknown u_{10j} is obtained. As shown in Sect. 7.5.1, the problem can be solved uniquely.

7.5.3 Free Vibration of Laminates in Cylindrical Bending

Let $L_1 \to \infty$ in x_1 direction and all variables be independent with x_1. Assuming $u_1 = 0$, from Eq. (7.138), the generalized motion equations are

$$\begin{aligned} &C_{22}u_{2,22} + C_{44}u_{2,33} + (C_{23} + C_{44})u_{3,23} + (e_{24} + e_{32})\varphi_{,23} = \rho u_{2,tt} \\ &(C_{23} + C_{44})u_{2,23} + C_{44}u_{3,22} + C_{33}u_{3,33} + e_{24}\varphi_{,22} + e_{33}\varphi_{,33} = \rho u_{3,tt} \\ &(e_{32} + e_{24})u_{2,23} + e_{24}u_{3,22} + e_{33}u_{3,33} - \epsilon_{22}\varphi_{,22} - \epsilon_{33}\varphi_{,33} = 0 \end{aligned} \tag{7.142}$$

The continuity conditions on interface are shown in Eq. (7.137). For the free vibration, the mechanical boundary conditions on the top and bottom surfaces are

$$\sigma_{33}(x_2, h_N/2) = \sigma_{33}(x_2, -h_1/2) = \sigma_{23}(x_2, h_N/2) = \sigma_{23}(x_2, -h_1/2) = 0 \tag{7.143}$$

The electrical boundary conditions have two different kinds:

$$(1)\ \ \varphi(x_2, h_N/2) = \varphi(x_2, -h_1/2) = 0; \quad \text{or} \quad (2)\ \ D_3(x_2, h_N/2) = D_3(x_2, -h_1/2) = 0 \tag{7.144}$$

For the cylindrical bending vibration in (x_2, x_3) plane, the boundary conditions on the lateral surfaces are

$$\sigma_{22}(0, x_3) = \sigma_{22}(0, L_1) = 0, \quad u_3(0, x_3) = u_3(0, L_1) = 0, \quad \varphi(0, x_3) = \varphi(0, L_1) = 0 \tag{7.145}$$

For each layer, there are six unknowns and six continuity conditions due to $u_1 = 0$. There are also total six boundary conditions on the top and bottom surfaces. In order to satisfy Eq. (7.145) automatically, Heyliger and Brooks (1995) took the solution in the following form:

$$(u_2, u_3, \varphi) = (u_{20}\cos px_2, u_{30}\sin px_2, \varphi_0 \sin px_2)\mathrm{e}^{sx_3}\mathrm{e}^{\mathrm{i}\omega t}, \quad p = n\pi/L_2 \tag{7.146}$$

Substituting Eq. (7.146) into (7.142) yields

$$\begin{bmatrix} -C_{22}p^2 + C_{44}s^2 + \rho\omega^2 & (C_{23} + C_{44})ps & (e_{24} + e_{32})ps \\ -(C_{23} + C_{44})ps & -C_{44}p^2 + C_{33}s^2 + \rho\omega^2 & -e_{24}p^2 + e_{33}s^2 \\ -(e_{32} + e_{24})ps & -e_{24}p^2 + e_{33}s^2 & \epsilon_{22}p^2 - \epsilon_{33}s^2 \end{bmatrix} \begin{Bmatrix} u_{20} \\ u_{30} \\ \varphi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7.147}$$

Setting the determinant of above matrix to zero for a nontrivial solution yields an eigen-equation. From the eigen-equation, we get six eigenvalues $s_i, i = 1 - 6$ for s. Corresponding each s_i, an eigenvector $\boldsymbol{U}_{0j}$ $(u_{10i}, u_{20i}, u_{30i}, \varphi_{0i})$ with one unknown u_{10j} can be obtained. As shown in Sect. 7.5.1, the problem can be solved uniquely.

7.5.4 A Mindlin-Type Plate Bending Theory

Consider an orthotropic piezoelectric plate of moderate thickness. Let (x_1, x_2) be located on the middle surface. The basic assumptions of the Mindlin bending theory are:

1. Straight lines normal to the $x_1 - x_2$ plane before deformation remain straight with unchanged length after deformation, but not compulsory normal to the mid-surface, i.e.,

$$\begin{aligned} &\varepsilon_{11} = u^0_{1,1} + x_3\psi_{1,1}, \quad \varepsilon_{22} = u^0_{2,2} + x_3\psi_{2,2}, \quad \gamma_{23} = u^0_{3,2} + \psi_2 \\ &\gamma_{13} = u^0_{3,1} + \psi_1, \quad \gamma_{12} = u^0_{1,2} + u^0_{2,1} + x_3(\psi_{1,2} + \psi_{2,1}) \end{aligned} \tag{7.148}$$

where $\boldsymbol{u}^0$ is the displacement in the mid-surface and ψ_1 and ψ_2 are the absolute cross-sectional rotations.

2. Stress σ_{33} can be neglected. So the constitutive equation can be written as

$$\begin{aligned} &\sigma_1 = \sigma_{11} = \bar{C}_{11}u_{1,1} + \bar{C}_{12}u_{2,2} + \bar{e}_{31}\varphi_{,3}, \quad \sigma_2 = \sigma_{22} = \bar{C}_{12}u_{1,1} + \bar{C}_{22}u_{2,2} + \bar{e}_{32}\varphi_{,3} \\ &\sigma_4 = \sigma_{23} = \bar{C}_{44}(u_{2,3} + u_{3,2}) + \bar{e}_{24}\varphi_{,2}, \quad \sigma_5 = \sigma_{31} = \bar{C}_{55}(u_{1,3} + u_{3,1}) + \bar{e}_{15}\varphi_{,1} \\ &\sigma_6 = \sigma_{12} = \bar{C}_{66}(u_{2,1} + u_{1,2}), \quad D_1 = \bar{e}_{15}(u_{1,3} + u_{3,1}) - \bar{\epsilon}_{11}\varphi_{,1} \\ &D_2 = \bar{e}_{24}(u_{2,3} + u_{3,2}) - \bar{\epsilon}_{22}\varphi_{,1}, \quad D_3 = \bar{e}_{31}u_{1,1} + \bar{e}_{32}u_{2,2} - \bar{\epsilon}_{33}\varphi_{,3} \end{aligned} \tag{7.149}$$

where $u_{3,3}$ has been eliminated by using $\sigma_3 = 0$, and

$$\bar{C}_{ij} = C_{ij} - C_{i3}C_{j3}/C_{33}, \quad \bar{e}_{ij} = e_{ij} - e_{33}C_{ji}/C_{33}, \quad \bar{\epsilon}_{ij} = \bar{\epsilon}_{ij} + e^2_{33}\delta_{i3}\delta_{j3}/C_{33} \tag{7.150}$$

are the reduced material coefficients.

Wang and Yang (2000) reviewed the higher-order theories of the piezoelectric plates. The equivalent single-layer models for the multiply layer plate (Krommer and Irschik 2000) are adopted here. Substituting Eq. (7.148) into (7.149) and integrating through the thickness yield the constitutive equation in the stress resultants and moments:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D \end{bmatrix} \begin{Bmatrix} u^0_{1,1} \\ u^0_{2,2} \\ u^0_{1,2} + u^0_{2,1} \\ \psi_{1,1} \\ \psi_{2,2} \\ \psi_{1,2} + \psi_{2,1} \end{Bmatrix} - \begin{Bmatrix} N_{1e} \\ N_{2e} \\ N_{12e} \\ M_{1e} \\ M_{2e} \\ M_{12e} \end{Bmatrix}$$

$$\begin{Bmatrix} q_2 \\ q_1 \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} - \begin{Bmatrix} q_{2e} \\ q_{1e} \end{Bmatrix} \tag{7.151}$$

where $N_1, N_2, N_{12} = N_6$ are the membrane forces, $M_1, M_2, M_{12} = M_6$ are the bending moments, and q_1, q_2 are the shear forces per unit length. The generalized stiffness in Eq. (7.151) are

$$(N_i, M_i) = \sum_{k=1}^{N} \int_{h_k} \sigma_i^{(k)}(1, x_3,)\mathrm{d}x_3,\ (i = 1,2,6); \quad (q_2, q_1) = \sum_{k=1}^{N} \int_{h_k} (\sigma_4^{(k)}, \sigma_5^{(k)})\mathrm{d}x_3$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_k} \bar{C}_{ij}^{(k)}(1, x_3, x_3^2)\mathrm{d}x_3, \quad S_{ij} = \sum_{k=1}^{N} \int_{h_k} \Phi_i \Phi_j \bar{C}_{ij}^{(k)}\mathrm{d}x_3$$

$$\begin{Bmatrix} N_{1e} & M_{1e} \\ N_{2e} & M_{2e} \\ N_{12e} & M_{12e} \end{Bmatrix} = \sum_{k=1}^{N} \int_{h_k} \begin{Bmatrix} \bar{e}_{31}^{(k)} \\ \bar{e}_{32}^{(k)} \\ 0 \end{Bmatrix} E_3^{(k)}(1, x_3)\mathrm{d}x_3, \quad \begin{bmatrix} q_{1e} \\ q_{2e} \end{bmatrix} = \sum_{k=1}^{N} \int_{h_k} \begin{Bmatrix} \Phi_5^2 \bar{e}_{24}^{(k)} E_1^{(k)} \\ \Phi_4^2 \bar{e}_{15}^{(k)} E_2^{(k)} \end{Bmatrix} \mathrm{d}x_3 \tag{7.152}$$

where Φ_i, Φ_j are shear factors which are determined by the shear stress distribution on the cross section and $N_{1e}, M_{1e}, N_{2e}, M_{2e}$, $N_{12e}, M_{12e}, q_{1e}, q_{2e}$ are introduced by piezoelectric effect. The electric variables will be studied layer by layer, and for each layer, there are two-type boundary conditions:

1. Electrically open Given the electric charge density $\sigma^{(i)}$ on the upper and lower surfaces of the layer. Usually $D_1^{(i)}$ and $D_2^{(i)}$ are neglected and $D_3^{(i)}$ is reserved. So $D_3^{(i)}$ is constant along the thickness direction due to Gauss equation $D_{3,3}^{(i)} = 0$:

$$D_j^{(i)} n_j = D_3^{(i)} = -\sigma^{(i)}, \quad E_3^{(i)} = -\sigma^{(i)} \Big/ \bar{\epsilon}_{33}^{(i)} \tag{7.153}$$

2. Electrically shorted Given the potential $V^{(i)}$ on the upper and lower surfaces of the layer. For convenience, assume the variation of the potential is linear along the thickness direction, so

$$E_3^{(i)} = V^{(i)}/h_i, \quad D_3^{(i)} = \bar{\epsilon}_{33}^{(i)} V^{(i)}/h_i; \quad E_1^{(i)} = E_2^{(i)} = 0 \tag{7.154}$$

In Mindlin theory, the motion equations are

$$\begin{aligned}&\rho_0\ddot{u}_1^0+\rho_1\ddot{\psi}_1=N_{1,1}+N_{12,2},\quad \rho_0\ddot{u}_2^0+\rho_1\ddot{\psi}_2=N_{2,2}+N_{12,1},\quad \rho_0\ddot{u}_3^0=q_{1,1}+q_{2,2}+p\\&\rho_1\ddot{u}_1^0+\rho_2\ddot{\psi}_1=M_{1,1}+M_{12,2}-q_1,\quad \rho_1\ddot{u}_2^0+\rho_2\ddot{\psi}_2=M_{2,2}+M_{12,1}-q_2\end{aligned}\tag{7.155}$$

where p is the transverse loading and

$$(\rho_0,\rho_1,\rho_2)=\sum_{k=1}^{N}\int_{h_k}\rho_0^{(k)}\left(1,x_3,x_3^2\right)\mathrm{d}x_3\tag{7.156}$$

Equations (7.148), (7.149), (7.150), (7.151), (7.152), (7.153), (7.154), (7.155), and (7.156) are the complete governing equations.

For a symmetrically laminated, transversely isotropic and simply supported plate, bending and extension are decouple, and the following relations are held:

$$\begin{aligned}&\bar{C}_{11}=\bar{C}_{22},\quad \bar{C}_{44}=\bar{C}_{55},\quad \bar{C}_{66}=\tfrac{1}{2}(\bar{C}_{11}-\bar{C}_{12});\quad \bar{e}_{31}=\bar{e}_{32},\quad \bar{e}_{15}=\bar{e}_{24};\quad \bar{\epsilon}_{11}=\bar{\epsilon}_{22}\\&D_{11}=D_{22},\quad D_{66}=(D_{11}-D_{12})/2,\quad S_{44}=S_{55},\quad M_{1e}=M_{2e}\end{aligned}\tag{7.157}$$

Eliminating the cross-sectional rotations ψ from Eqs. (7.151) and (7.155) for the bending vibration, a fourth-order partial differential equation for u_3^0 is obtained (Krommer and Irschik, 2000):

$$\begin{aligned}&D_{11}\nabla^2\nabla^2u_3^0-[(D_{11}/S_{44})\rho_0+\rho_2]\nabla^2\ddot{u}_3^0+\rho_0\ddot{u}_3^0+(\rho_0\rho_2/S_{44})\,\ddddot{u}_3{}^0\\&\quad=p-(D_{11}/S_{44})\nabla^2p+(\rho_2/S_{44})\ddot{p}-\nabla^2M_{1e}\end{aligned}\tag{7.158}$$

where $\nabla^2u_3^0=u_{3,11}^0+u_{3,22}^0$ and $\ddddot{u}_3{}^0=\partial^4u_3^0/\partial t^4$. When the external loadings are $p=p_0\mathrm{e}^{\mathrm{i}\omega t}$ and $M_{1e}=M_{10e}\mathrm{e}^{\mathrm{i}\omega t}$, the frequency equation of the bending vibration is

$$\begin{aligned}&D_{11}\nabla^2\nabla^2U_3^0+[(D_{11}/S_{44})\rho_0+\rho_2]\omega^2\nabla^2U_3^0-\rho_0\omega^2\left[1-(\rho_2/S_{44})\omega^2\right]U_3^0\\&\quad=p_0\left[1-(\rho_2/S_{44})\omega^2\right]-\nabla^2[(D_{11}/S_{44})p_0+M_{10e}]\end{aligned}\tag{7.159}$$

where the common factor $\mathrm{e}^{\mathrm{i}\omega t}$ is omitted and $u_3^0=U_3^0\mathrm{e}^{\mathrm{i}\omega t}$.

7.5.5 Third-Order Shear Deformation Theory of Laminate Plate

Consider the linear piezoelectric material, so the Maxwell stress and the environment need not be considered. The variational principle $\delta\Pi=0$ in Eq. (2.7) is becomes

$$\delta\Pi = \int_V \sigma_{ik}\delta u_{i,k}\mathrm{d}V + \int_V D_k \delta\varphi_{,k}\mathrm{d}V + \int_V \rho\ddot{u}_k \delta u_k \mathrm{d}V - \int_{a_\sigma} T_k^* \delta u_k \mathrm{d}a + \int_{a_\rho} \sigma^* \delta\varphi \mathrm{d}a = 0 \tag{7.160}$$

where the body force and body electric charge are neglected. For the orthotropic material of moderate thickness, Mitchell and Reddy (1995) adopted the displacements of the equivalent single-layer plate as

$$\begin{aligned} u_1 &= u_1^0(x_1,x_2,t) + \eta_1(x_3)\psi_1(x_1,x_2,t) - \eta_2(x_3)u_{3,1}^0(x_1,x_2,t) \\ u_2 &= u_2^0(x_1,x_2,t) + \eta_1(x_3)\psi_2(x_1,x_2,t) - \eta_2(x_3)u_{3,2}^0 \\ u_3 &= u_3^0(x_1,x_2,t) \\ \eta_1(x_3) &= x_3 - cx_3^3, \quad \eta_2(x_3) = cx_3^3, \quad c = 4/3h^2 \end{aligned} \tag{7.161}$$

where $\left(u_1^0, u_2^0, u_3^0\right)$ are the displacements of a point on the midplane and (ψ_1, ψ_2) are the rotations of a transverse normal at $x_3 = 0$ on the midplane about the x_2 and $-x_1$ axes, respectively. h is the total thickness of the plate. The strains corresponding to Eq. (7.161) are

$$\begin{aligned} &\varepsilon_{11} = u_{1,1}^0 + \eta_1\psi_{1,1} - \eta_2 u_{3,11}^0; \quad \varepsilon_{22} = u_{2,2}^0 + \eta_1\psi_{2,2} - \eta_2 u_{3,22}^0; \quad \varepsilon_{33} = 0 \\ &\gamma_{23} = u_{2,3} + u_{3,2} = \eta_{1,3}\psi_2 - \eta_{2,3}u_{3,2}^0 + u_{3,2}^0; \quad \gamma_{31} = u_{1,3} + u_{3,1} = \eta_{1,3}\psi_1 - \eta_{2,3}u_{3,1}^0 + u_{3,1}^0 \\ &\gamma_{12} = u_{1,2} + u_{2,1} = \left(u_{1,2}^0 + u_{2,1}^0\right) + \eta_1(\psi_{1,2} + \psi_{2,1}) - \eta_2\left(u_{3,12}^0 + u_{3,21}^0\right) \end{aligned} \tag{7.162}$$

From Eqs. (7.161) and (7.162), it is known that for $\eta_1(x_3), \eta_2(x_3)$ given in Eq. (7.161), the transverse shear strains γ_{13}, γ_{23} are zeros on the upper and lower surfaces and vary quadratically through the thickness. It is also without the normal strain. The potential is modeled on a discrete layer approximation as

$$\varphi(x_1,x_2,x_3,t) = \sum_{k=1}^{N}\sum_{j=1}^{m} f_k(x_3)\varphi^{(k,j)}(x_1,x_2,t) \tag{7.163}$$

where N is the layer number of the laminate plate, m is the number of interpolation points in a layer, and $\varphi^{(k,j)}$ is the potential at jth interpolation point of kth layer. $f_k(x_3)$ is the Lagrange interpolation function. It is noted that the potential on the upper surface of the $k-1$ layer must be equal to that on the lower surface of the k layer.

The middle plane is denoted by A and its boundary is L. Substituting Eqs. (7.161) and (7.163) into (7.160) and neglecting σ_{33} yield the following equations:

The variation of the mechanical energy:

$$
\begin{aligned}
\int_V \sigma_{ik}\delta u_{i,k}\mathrm{d}V &= \int_A \Big\{ \Big(N_1\delta u^0_{1,1} + M_1\delta\psi_{1,1} - P_1\delta u^0_{3,11}\Big) + \Big(N_2\delta u^0_{2,2} + M_2\delta\psi_{2,2} - P_2\delta u^0_{3,22}\Big) \\
&\quad + \Big[N_6\Big(\delta u^0_{1,2} + \delta u^0_{2,1}\Big) + M_6\big(\delta\psi_{1,2} + \delta\psi_{2,1}\big) - 2P_6\delta u^0_{3,12}\Big] \\
&\quad + Q_4\Big(\delta\psi_2 + \delta u^0_{3,2}\Big) + Q_5\Big(\delta\psi_1 + \delta u^0_{3,1}\Big)\Big\}\mathrm{d}A \\
&= -\int_A \big\{ \big(N_{1,1}\delta u^0_1 + M_{1,1}\delta\psi_1 - P_{1,11}\delta u^0_3\big) + \big(N_{2,2}\delta u^0_2 + M_{2,2}\delta\psi_2 - P_{2,22}\delta u^0_3\big) \\
&\quad + \big(N_{6,2}\delta u^0_1 + N_{6,1}\delta u^0_2 + M_{6,2}\delta\psi_1 + M_{6,1}\delta\psi_2 - 2P_{6,12}\delta u^0_3\big) + \big(Q_4\delta\psi_2 - Q_{4,2}\delta u^0_3\big) \\
&\quad + \big(Q_5\delta\psi_1 - Q_{5,1}\delta u^0_3\big)\big\}\mathrm{d}A + \int_L \Big\{\Big[N_1\delta u^0_1 + M_1\delta\psi_1 - \Big(P_1\delta u^0_{3,1} - P_{1,1}\delta u^0_3\Big)\Big]n_1 \\
&\quad + \Big[N_2\delta u^0_2 + M_2\delta\psi_2 - \Big(P_2\delta u^0_{3,2} - P_{2,2}\delta u^0_3\Big)\Big]n_2 + \big(N_6\delta u^0_2 + M_6\delta\psi_2\big)n_1 \\
&\quad + \big(N_6\delta u^0_1 + M_6\delta\psi_1\big)n_2 - \Big[P_6\delta u^0_{3,2}n_1 + P_6\delta u^0_{3,1}n_2 - \big(P_{6,1}n_2 + P_{6,2}n_1\big)\delta u^0_3\Big] \\
&\quad + (Q_4 n_2 + Q_5 n_1)\delta u^0_3\Big\}\mathrm{d}L
\end{aligned}
\tag{7.164}
$$

where the Voigt notation has been used and

$$
\begin{aligned}
&N_i = \int_{-h/2}^{h/2} \sigma_i \mathrm{d}x_3, \quad M_i = \int_{-h/2}^{h/2} \sigma_i\eta_1 \mathrm{d}x_3, \quad P_i = \int_{-h/2}^{h/2} \sigma_i\eta_2 \mathrm{d}x_3, \quad i = 1,2,6 \\
&Q_i = \int_{-h/2}^{h/2} \sigma_i\Big[1 - 4(z/h)^2\Big]\mathrm{d}x_3, \quad i = 4,5
\end{aligned}
\tag{7.165}
$$

The variation of the electric energy:

$$
\begin{aligned}
\int_V D_i\delta\varphi_{,i}\mathrm{d}V &= \sum_{k=1}^{N}\left\{\int_V \sum_{j=1}^{m}\Big(D^{(k,j)}_\alpha \delta\varphi^{(k,j)}_{,\alpha} + D^{(k,j)}_3\delta\varphi^{(k,j)}_{,3}\Big)\mathrm{d}V\right\} \\
&= \sum_{k=1}^{N}\left\{\int_A -\sum_{j=1}^{m}\Big(P^{(k,j)}_{\alpha,\alpha} - G^{(k,j)}_3\Big)\delta\varphi^{(k,j)}\mathrm{d}A + \int_L \sum_{j=1}^{m} P^{(k,j)}_\alpha n_\alpha \delta\varphi^{(k,j)}\mathrm{d}L\right\} \\
P^{(k,j)}_\alpha &= \int_{h_{j-1}}^{h_j} D^{(k,j)}_\alpha f_k(x_3)\mathrm{d}x_3, \quad G^{(k,j)}_3 = \int_{h_{j-1}}^{h_j} D^{(k,j)}_3 f_{k,3}(x_3)\mathrm{d}x_3
\end{aligned}
\tag{7.166}
$$

The variation of the kinetic energy or inertial energy:

$$
\begin{aligned}
\int_V \rho\ddot{u}_i\delta u_i\mathrm{d}V &= \int_A \Big\{\Big(I_1\ddot{u}^0_1 + I_2\ddot{\psi}_1 - I_3\ddot{u}^0_{3,1}\Big)\delta u^0_1 + \Big(I_2\ddot{u}^0_1 + I_4\ddot{\psi}_1 - I_5\ddot{u}^0_{3,1}\Big)\delta\psi_1 \\
&\quad + \Big[\Big(I_3\ddot{u}^0_1 + I_5\ddot{\psi}_1 - I_6\ddot{u}^0_{3,1}\Big)_{,1} + \Big(I_3\ddot{u}^0_2 + I_5\ddot{\psi}_2 - I_6\ddot{u}^0_{3,2}\Big)_{,2} + I_1\ddot{u}^0_3\Big]\delta u^0_3 \\
&\quad + \Big[\Big(I_1\ddot{u}^0_2 + I_2\ddot{\psi}_2 - I_3\ddot{u}^0_{3,2}\Big)\delta u^0_2 + \Big(I_2\ddot{u}^0_2 + I_4\ddot{\psi}_2 - I_5\ddot{u}^0_{3,2}\Big)\delta\psi_2\Big]\Big\}\mathrm{d}A
\end{aligned}
\tag{7.167}
$$

where

$$
\begin{aligned}
I_1 &= \int_{-h/2}^{h/2} \rho \mathrm{d}x_3, \quad I_3 = \int_{-h/2}^{h/2} \rho\eta_2(x_3)\mathrm{d}x_3, \quad I_5 = \int_{-h/2}^{h/2} \rho\eta_1(x_3)\eta_2(x_3)\mathrm{d}x_3 \\
I_2 &= \int_{-h/2}^{h/2} \rho\eta_1(x_3)\mathrm{d}x_3, \quad I_4 = \int_{-h/2}^{h/2} \rho\eta_1^2(x_3)\mathrm{d}x_3, \quad I_6 = \int_{-h/2}^{h/2} \rho\eta_2^2(x_3)\mathrm{d}x_3
\end{aligned} \tag{7.168}
$$

In Eq. (7.167) term $-\int_L \left[\left(I_3\ddot{u}_1^0 + I_5\ddot{\psi}_1 - I_6\ddot{u}_{3,1}^0\right)n_1 + \left(I_3\ddot{u}_2^0 + I_5\ddot{\psi}_2 - I_6\ddot{u}_{3,2}^0\right)n_2\right]$ δu_3^0 has been neglected. This term is difficult to explanation.

The variation of the work of the generalized external force is as follows: The mechanical force acted on the equivalent single-layer plate should be distinguished two parts—T_i^* is the equivalent traction on the midplane A, and t_i^* is the equivalent resultant force on the lateral surface a. The electric charge acted on the kth layer should also be distinguished two parts: $\sigma^{(k)*}$ is the surface electric charge density of the kth layer, and $q^{(k)*}$ is the surface electric charge on the lateral surface. Neglecting some secondary terms we can get

$$
\begin{aligned}
-\int_{a_\sigma} T_k^*\delta u_k \mathrm{d}a + \int_{a_\rho} \sigma^*\delta\varphi \mathrm{d}a = &-\int_A \left\{T_1^*\delta u_1^0 + T_2^*\delta u_2^0 + T_3^*\delta u_3^0\right\}\mathrm{d}A \\
&-\int_L \left\{t_1^*\delta u_1^0 + t_2^*\delta u_2^0 + t_3^*\delta u_3^0\right\}\mathrm{d}L + \sum_{k=1}^{N}\left[\int_A \sigma^{(k)*} f_k \delta\varphi_j \mathrm{d}A + \int_L q^{(k)*} f_j \delta\varphi_j \mathrm{d}L\right]
\end{aligned} \tag{7.169}
$$

Substitution of Eqs. (7.163), (7.164), (7.165), (7.166), (7.167), (7.168), and (7.169) into Eq. (7.160) yields

$$
\begin{aligned}
&\delta u_{10}: N_{1,1} + N_{6,2} + T_1^* = I_1\ddot{u}_1^0 + I_2\ddot{\psi}_1 - I_3\ddot{u}_{3,1}^0 \\
&\delta u_{20}: N_{2,2} + N_{6,1} + T_2^* = I_1\ddot{u}_2^0 + I_2\ddot{\psi}_2 - I_3\ddot{u}_{3,2}^0 0 \\
&\delta u_{30}: P_{1,11} + P_{2,22} + 2P_{6,12} + Q_{4,2} + Q_{5,1} - T_3^* \\
&\qquad = \left(I_3\ddot{u}_1^0 + I_5\ddot{\psi}_1 - I_6\ddot{u}_{3,1}^0\right)_{,1} + \left(I_3\ddot{u}_2^0 + I_5\ddot{\psi}_2 - I_6\ddot{u}_{3,2}^0\right)_{,2} + I_1\ddot{u}_3^0 \\
&\delta\psi_1: M_{1,1} + M_{6,2} - Q_5 = I_2\ddot{u}_1^0 + I_4\ddot{\psi}_1 - I_5\ddot{u}_{3,1}^0 \\
&\delta\psi_2: M_{2,2} + M_{6,1} - Q_4 = I_2\ddot{u}_2^0 + I_4\ddot{\psi}_2 - I_5\ddot{u}_{3,2}^0 \\
&\delta\varphi^{(k,j)}: \sum_{j=1}^{m}\left(P_{\alpha,\alpha}^{(k,j)} - G_3^{(k,j)}\right) = \sum_{j=1}^{m}\sigma^{(j)*} f_j
\end{aligned} \tag{7.170}
$$

and the natural boundary conditions are

$$\begin{aligned}
&\delta u_1^0 : N_1 n_1 + N_6 n_2 + t_1^* = 0, \quad \delta u_2^0 : N_2 n_2 + N_6 n_1 + t_2^* = 0 \\
&\delta u_3^0 : P_{1,1} n_1 + P_{2,2} n_2 + P_{6,1} n_2 + P_{6,2} n + Q_4 n_2 + Q_5 n_1 + t_3^* = 0 \\
&\delta \psi_1 : M_1 n_1 + M_6 n_2 = 0, \qquad \delta \psi_2 : M_2 n_2 + M_6 n_1 = 0 \\
&\delta u_{3,1}^0 : P_1 n_1 + P_6 n_2 = 0, \qquad \delta u_{3,2}^0 : P_2 n_2 + P_6 n_1 = 0 \\
&\delta \varphi_j^k : \sum_{j=1}^{m} P_\alpha^{(j,k)} n_\alpha - q^{(k)*} f_j(x_3) = 0
\end{aligned} \tag{7.171}$$

It is also noted that on the boundaries given generalized displacements, we have

$$\delta u_1^0 = \delta u_2^0 = \delta u_3^0 = \delta u_{3,1}^0 = \delta u_{3,2}^0 = \delta \psi_1 = \delta \psi_2 = 0 \tag{7.172}$$

Mitchell and Reddy (1995) adopted Hamilton principle, $\delta \Pi_{\mathrm{H}} = \delta \Pi_{\mathrm{H1}}$ in Eq. (2.32); their results are slightly different. It can be seen that this complex approximate theory is difficult to exactly discuss and give some new simplified postulations are needed.

The generalized forces can be obtained from Eqs. (7.165) and (7.149). Let

$$N_i = \bar{N}_i + N_i^{\mathrm{p}}, \quad M_i = \bar{M}_i + M_i^{\mathrm{p}}, \quad P_i = \bar{P}_i + P_i^{\mathrm{p}}, \quad Q_i = \bar{Q}_i + Q_i^{\mathrm{p}} \tag{7.173a}$$

where the elastic variables are denoted by an over-bar and variables related to piezoelectric effect are denoted by a superscript "p," and

$$\begin{Bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_3 \\ \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{11}^* & A_{12}^* & 0 & A_{11}^{**} & A_{12}^{**} & 0 \\ A_{12} & A_{22} & 0 & A_{12}^* & A_{22}^* & 0 & A_{12}^{**} & A_{22}^{**} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & A_{66}^* & 0 & 0 & A_6^{**} \\ A_{11}^* & A_{12}^* & 0 & B_{11} & B_{12} & 0 & B_{11}^* & B_{12}^* & 0 \\ A_{12}^* & A_{22}^* & 0 & B_{12} & B_{22} & 0 & B_{12}^* & B_{22}^* & 0 \\ 0 & 0 & A_{66}^* & 0 & 0 & B_{66} & 0 & 0 & B_{66}^* \\ A_{11}^{**} & A_{12}^{**} & 0 & B_{11}^* & B_{12}^* & 0 & D_{11} & D_{12} & 0 \\ A_{12}^{**} & A_{22}^{**} & 0 & B_{12}^* & B_{22}^* & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & A_{66}^{**} & 0 & 0 & B_{66}^* & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1}^0 \\ u_{2,2}^0 \\ u_{1,2}^0 + u_{2,1}^0 \\ \psi_{1,1} \\ \psi_{2,2} \\ \psi_{1,2} + \psi_{2,1} \\ -u_{3,11}^0 \\ -u_{3,22}^0 \\ -2u_{3,12}^0 \end{Bmatrix} \tag{7.173b}$$

$$\begin{Bmatrix} \bar{Q}_4 \\ \bar{Q}_5 \end{Bmatrix} = \begin{pmatrix} F_{44} & 0 \\ 0 & F_{55} \end{pmatrix} \begin{Bmatrix} \psi_2 + u_{30,2} \\ \psi_1 + u_{30,1} \end{Bmatrix} \tag{7.173c}$$

$$
\begin{aligned}
N_1^{\mathrm{p}} &= \sum_{k=1}^{N} e_{31}^{(k)} \sum_{j=1}^{m} \beta_1^{(k,j)} \varphi^{(k,j)}, \quad N_2^{\mathrm{p}} = \sum_{k=1}^{N} e_{32}^{(k)} \sum_{j=1}^{m} \beta_1^{(k,j)} \varphi^{(k,j)}, \quad N_6^{\mathrm{p}} = 0 \\
M_1^{\mathrm{p}} &= \sum_{k=1}^{N} e_{31}^{(k)} \sum_{j=1}^{m} \beta_2^{(k,j)} \varphi^{(k,j)}, \quad M_2^{\mathrm{p}} = \sum_{k=1}^{N} e_{32}^{(k)} \sum_{j=1}^{m} \beta_2^{(k,j)} \varphi^{(k,j)}, \quad M_6^{\mathrm{p}} = 0 \\
P_1^{\mathrm{p}} &= \sum_{k=1}^{N} e_{31}^{(k)} \sum_{j=1}^{m} \beta_3^{(k,j)} \varphi^{(k,j)}, \quad P_2^{\mathrm{p}} = \sum_{k=1}^{N} e_{32}^{(k)} \sum_{j=1}^{m} \beta_3^{(k,j)} \varphi^{(k,j)}, \quad P_6^{\mathrm{p}} = 0 \\
Q_4^{\mathrm{p}} &= \sum_{k=1}^{N} e_{24}^{(k)} \sum_{j=1}^{m} \beta_4^{(k,j)} \partial \varphi^{(k,j)} / \partial x_2, \quad Q_5^{p} = \sum_{k=1}^{N} e_{15}^{(k)} \sum_{j=1}^{m} \beta_4^{(k,j)} \partial \varphi^{(k,j)} / \partial x_1
\end{aligned}
\tag{7.173d}
$$

where

$$
\begin{aligned}
&\mathbf{A}^{(k)} = \int_{-h_k/2}^{h_k/2} \mathbf{C} \mathrm{d}x_3, \quad \mathbf{A}^{(k)*} = \int_{-h_k/2}^{h_k/2} \mathbf{C} \mathrm{g}_1 \mathrm{d}x_3, \quad \mathbf{A}^{(k)**} = \int_{-h_k/2}^{h_k/2} \mathbf{C} \mathrm{g}_2 \mathrm{d}x_3, \quad \mathbf{B}^{(k)} = \int_{-h_k/2}^{h_k/2} \mathbf{C} \mathrm{g}_1^2 \mathrm{d}x_3 \\
&\mathbf{B}^{(k)*} = \int_{-h_k/2}^{h_k/2} \mathbf{C} \mathrm{g}_1 \mathrm{g}_2 \mathrm{d}x_3, \quad \mathbf{D} = \int_{-h_k/2}^{h_k/2} \mathbf{C} \mathrm{g}_2^2 \mathrm{d}x_3, \quad F_{44} = \int_{-h_k/2}^{h_k/2} C_{44} \mathrm{g}_{1,3}^2 \mathrm{d}x_3 \\
&F_{55} = \int_{-h_k/2}^{h_k/2} C_{55} \mathrm{g}_{1,3}^2 \mathrm{d}x_3 \beta_1^{(k,j)} = \int_{-h_k/2}^{h_k/2} f_{j,3} \mathrm{d}x_3, \quad \beta_2^{(k,j)} = \int_{-h_k/2}^{h_k/2} \mathrm{g}_1 f_{j,3} \mathrm{d}x_3, \\
&\beta_3^{(k,j)} = \int_{-h_k/2}^{h_k/2} \mathrm{g}_2 f_{j,3} \mathrm{d}x_3, \quad \beta_4^{(k,j)} = \int_{-h_k/2}^{h_k/2} \mathrm{g}_{1,3} f_j \mathrm{d}x_3; \quad \mathbf{C} = \begin{pmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{21} & \bar{C}_{22} & 0 \\ 0 & 0 & \bar{C}_{66} \end{pmatrix} \\
&(\mathbf{A}, \mathbf{A}^*, \mathbf{A}^{**}, \mathbf{B}, \mathbf{B}^*) = \sum_{k=1}^{N} \left(\mathbf{A}^{(k)}, \mathbf{A}^{(k)*}, \mathbf{A}^{(k)**}, \mathbf{B}^{(k)}, \mathbf{B}^{(k)*} \right)
\end{aligned}
\tag{7.174}
$$

Equations (7.170), (7.171), (7.172), (7.173a), (7.173b), (7.173c), (7.173d), and (7.174) are the complete governing equations.

7.5.6 Bending Theory of Timoshenko Beam

A narrow plate can be considered as a beam. The Timoshenko theory (Timoshenko and Woinowsky-Krieger 1959) considering the shear deformation of a beam of a moderate thickness can be obtained from the Mindlin theory. Let all variables be independent to x_2 in Mindlin theory, Eq. (7.149) is reduced to

$$
\begin{aligned}
&\sigma_1 = \sigma_{11} = Y u_{1,1} + e_{31} \varphi_{,3}, \quad \sigma_5 = \sigma_{13} = G(u_{1,3} + u_{3,1}) + e_{15} \varphi_{,2} \\
&D_1 = e_{15}(u_{1,3} + u_{3,1}) - \epsilon_{11} \varphi_{,1}, \quad D_3 = e_{31} u_{1,1} - \epsilon_{33} \varphi_{,3}
\end{aligned}
\tag{7.175}
$$

where Y, G are elastic coefficients. Corresponding to Eq. (7.148), the deformation in Timoshenko beam is assumed as

$$\varepsilon_{11} = u^0_{1,1} + x_3\psi_{,1}, \quad \gamma_{13} = u^0_{3,1} + \psi \tag{7.176}$$

If $\psi = -u^0_{3,1}$, then $\gamma_{13} = 0$,Timoshenko beam is reduced to Bernoulli-Euler beam. Corresponding to Eqs. (7.173) and (7.174), we have

$$\begin{gathered}
\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{Bmatrix} u^0_{1,1} \\ \psi_{,1} \end{Bmatrix} - \begin{Bmatrix} N_e \\ M_e \end{Bmatrix}, \quad q = G\gamma_{13} - q_e \\
(A, B, D) = \sum_{k=1}^{N} \int_{h_k} Y^{(k)}\left(1, x_3.x_3^2\right)\mathrm{d}x_3, \quad G = \sum_{k=1}^{N} \int_{h_k} \Phi^2 G^{(k)} \mathrm{d}x_3 \\
(N_e, M_e) = \sum_{k=1}^{N} \int_{h_k} e^{(k)}_{31} E^{(k)}_3 (1, x_3.)\mathrm{d}x_3, \quad q_e = \sum_{k=1}^{N} \int_{h_k} \Phi^2 e^{(k)}_{15} E^{(k)}_1 \mathrm{d}x_3
\end{gathered} \tag{7.177}$$

Corresponding to Eq. (7.155), we have

$$\rho_0\ddot{u}^0_1 + \rho_1\ddot{\psi}_1 = N_{,1}, \quad \rho_0\ddot{u}^0_3 = q_{,1} + p, \quad \rho_1\ddot{u}^0_1 + \rho_2\ddot{\psi}_1 = M_{,1} - q \tag{7.178}$$

where ρ_0, ρ_1, ρ_2 are shown in Eq. (7.156) and electric displacements and electric fields are shown in Eq. (7.153) or (7.154).

7.5.7 Bending Model of Beams of Crawley

Crawley and De Luis (1987) proposed an extension-bending model to study the simple intelligent beam structure as shown in Fig. 7.8. They assumed that the strain is uniform through the actuator thickness, the beam obeys Bernoulli-Euler rule, and the adhesive layer transfers loads only through shear. The formulas obtained by this model are well for extension, but not bending, especially for a thin plate (Crawley and Aanderson 1989).

7.6 The First-Order Approximate Theory of an Electro-magneto-elastic Thin Plate

7.6.1 Basic Postulations

The nonlinear theory of an electroelastic thin plate is not well established, and different authors proposed different theories. In this section, a first-order approximate theory of the quasi-static electro-magneto-elastic thin plate is recommended for small deformation, when the electromagnetic induction effect can be neglected. Let the origin of the coordinate system axes x_1, x_2 be located on the midplane and x_3 be upward normal to the midplane. The plate is bending upward. The role of x_3 is not the same as that of (x_1, x_2), so it will be discussed alone. For the present plate theory, three following basic postulations are assumed (Kuang 2011):

(1) $\sigma_{33} \ll \sigma_{\alpha 3} \ll \sigma_{\alpha\beta}, (\alpha = 1, 2)$, so σ_{33} is fully neglected and the effect of $\sigma_{\alpha 3}$ is considered partly.

(2) The Kirchhoff assumption is adopted, i.e.,

$$u_k = u_k^0 - x_3 u_{3,k}^0, \quad u_k^0 = u_k^0(x_1, x_2, t), \quad u_{k,3}^0 = 0, \quad (k = 1, 2, 3) \tag{7.179}$$

where $\mathbf{u}^0$ is the displacement on the middle surface and $\mathbf{u}$ is the displacement at a certain point in the plate. According to Eq. (7.179), we have $u_{3,\alpha} + u_{\alpha,3} = 0$, but it is not appropriate for the free boundary and should be modified approximately as shown later.

(3) The electromagnetic field obeys the 3D theory, but in order to consistent with the classical plate theory, the resultant electromagnetic force is reduced to the middle plane S $(\mathrm{d}S = \mathrm{d}x_1 \mathrm{d}x_2)$ or to the contour L of the middle plane. Usually when we solve the electromagnetic field, the electric field due to the direct piezoelectric effect can be approximately neglected compared to the lager applied electric field.

7.6.2 Governing Equations Derived from the First Method

In engineering the piezoelectric plate is surrounded by air, so the plate has only the interface with the air and does not have its own independent boundary. In air the mechanical stresses can be neglected, so only the electromagnetic field should be considered. It is assumed that there is no body force and body electric charge. According to Eqs. (2.19) and (2.21), there are two methods to establish the thin plate theory. Similar to Eq. (2.36), the first alternative form of the PVP, Eq. (2.19), for the static electromagnetic problem is modified as

$$\begin{aligned}
\delta\hat{\Pi} &= \delta\hat{\Pi}_1 + \delta\hat{\Pi}_2 - \delta W^{\text{int}} \\
&= \int_V S_{kl}\delta u_{k,l}\mathrm{d}V + \int_V \rho\ddot{u}_k\delta u_k\mathrm{d}V - \int_{a_\sigma^{\text{int}}} T_k^{*\,\text{int}}\delta u_k\mathrm{d}a + \int_V D_k\delta\varphi_{,k}\mathrm{d}V + \int_V B_k\delta\psi_{,k}\mathrm{d}V \\
&\quad + \int_{V^{\text{air}}} D_k^{\text{air}}\delta\varphi_{,k}^{\text{air}}\mathrm{d}V + \int_{V^{\text{air}}} \sigma_{kl}^{\text{M air}}\delta u_{k,l}^{\text{air}}\mathrm{d}V + \int_{V^{\text{air}}} B_k^{\text{air}}\delta\psi_{,k}^{\text{air}}\mathrm{d}V - \int_{a_\mu^{\text{air}}} B_i^{*\text{air}} n_i^{\text{air}}\delta\psi^{\text{air}}\mathrm{d}a \\
S_{kl} &= \sigma_{kl} + \sigma_{kl}^{\text{M}}, \quad \sigma_{ik}^{\text{M}} = D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n)\delta_{ik}
\end{aligned} \tag{7.180}$$

S_{kl}^{air} has the similar expression. The mechanical part related to the plate of the PVP is

$$\int_{a_\sigma^{\text{int}}} \left[\left(S_{kl} - \sigma_{kl}^{\text{M air}}\right)n_l - T_k^{*\,\text{int}}\right]\delta u_k \mathrm{d}a - \int_V \left(S_{kl,l} - \rho\ddot{u}_k\right)\delta u_k \mathrm{d}V = 0 \tag{7.181}$$

Therefore, the thin plate theory can apply the usual elastic plate theory, but $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{S}$ and on the interface $\boldsymbol{T}^{*\,\text{int}}$ is replaced by $\boldsymbol{T}^{*\,\text{int}} + \boldsymbol{\sigma}^{\text{M air}} \cdot \boldsymbol{n}$. According to the textbook of elastic plate theory, the bending theory of elastic electromagnetic thin plate can be expressed as

$$\text{Field equation:} \quad \mathrm{M}^{(\mathrm{S})}_{\alpha\beta,\beta\alpha} + \mathrm{q}^* = 0; \quad \text{in S} \tag{7.182}$$

Boundary conditions:

$$\textit{Clamped side} \quad u_3^0 = u_3^{0*}, \quad u_{3,n}^0 = u_{3,n}^{0*} \tag{7.183a}$$

$$\textit{Hinged side} \quad u_3^0 = u_3^{0*}, \quad M_n^{(S)} = M_n^* + M_n^{\text{M air}}, \tag{7.183b}$$

$$\textit{Free side} \quad M_{nt,t}^{(S)} + Q_n^{(S)} = Q_n^* + Q_n^{M\text{ air}}, \quad M_n^{(S)} = M_n^* + M_n^{\text{M air}} \tag{7.183c}$$

The notations in Eqs. (7.182) and (7.183) are

$$\begin{aligned} &M_{\alpha\beta}^{(S)} = \int_{-h}^{h} S_{\alpha\beta}x_3 \mathrm{d}x_3, \quad M_n^{\text{M air}} = M_\alpha^{\text{M air}} n_\alpha, \quad M_\alpha^{\text{M air}} = \int_{-h}^{h} \sigma_{\alpha\beta}^{\text{M air}} x_3 \mathrm{d}x_3 \\ &Q_\alpha^{(S)} = \int_{-h}^{h} S_{\alpha 3}\mathrm{d}x_3, \quad Q_n^{(S)} = Q_\alpha^{(S)} n_\alpha, \quad Q_\alpha^{\text{M air}} = \int_{-h}^{h} \sigma_{\alpha 3}^{\text{M air}} \mathrm{d}x_3 \end{aligned} \tag{7.184}$$

where $2h$ is the thickness of the plate, q^* is the distributed loading on the plate surface, and Q_n^* is the distributed loading on the lateral boundary of the midplane. At first the electromagnetic fields in plate and air are solved under the assumption that the elastic effect can be neglected. After solving the electromagnetic fields, the entire problem is reduced to a linear problem.

7.6.3 Governing Equations Derived from the Second Method

Similar to Eq. (2.36), the second alternative form of the PVP, Eq. (2.21), for the static electromagnetic problem is modified as

$$\begin{aligned} \delta\Pi' &= \delta\Pi_1' + \delta\Pi_2' - \delta W'^{\text{int}} \\ &= \int_V \sigma_{kl}\delta u_{k,l}\mathrm{d}V - \int_V \left(\sigma_{jk,j}^{\text{M}} - \rho\ddot{u}_k\right)\delta u_k \mathrm{d}V - \int_{a_\sigma^{\text{int}}} \left(T_k^{\text{int}\,*} + \sigma_{jk}^{\text{M env}} n_j^{\text{env}} - \sigma_{jk}^{\text{M}} n_j\right)\delta u_k \mathrm{d}a \\ &\quad + \int_V D_k \delta\varphi_{,k}\mathrm{d}V + \int_V B_k \delta\psi_{,k}\mathrm{d}V + \int_{V^{\text{env}}} D_k^{\text{env}}\delta\varphi_{,k}^{\text{env}}\mathrm{d}V + \int_{a_D^{\text{env}}} \sigma^{\text{env}\,*}\delta\varphi^{\text{env}}\mathrm{d}a \\ &\quad + \int_{V^{\text{env}}} B_k^{\text{env}}\delta\psi_{,k}^{\text{env}}\mathrm{d}V - \int_{V^{\text{env}}} \sigma_{jk,j}^{\text{M env}}\delta u_k^{\text{env}}\mathrm{d}V + \int_{a_D} \sigma^*\delta\varphi \mathrm{d}a - \int_{a_\mu^{\text{env}}} B_i^{*\text{env}} n_i^{\text{env}}\delta\psi^{\text{env}}\mathrm{d}a = 0 \\ \sigma_{ik}^{\text{M}} &= D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n)\delta_{ik}, \quad \text{Similar expression for } \sigma_{ik}^{\text{M env}} \end{aligned} \tag{7.185}$$

According to postulation (3), the electromagnetic field obeys the 3D theory, which has been discussed in Chap. 2. Here only the mechanical part related to the plate of the variational principle is discussed in detail, which is

$$\int_V \sigma_{kl}\delta u_{k,l}\mathrm{d}V - \int_V \left(\sigma^{\mathrm{M}}_{jk,j} - \rho\ddot{u}_k\right)\delta u_k \mathrm{d}V - \int_{a^{\mathrm{int}}_\sigma} \left(T^{\mathrm{int}\,*}_k + \sigma^{\mathrm{M\,env}}_{jk} n^{\mathrm{env}}_j - \sigma^{\mathrm{M}}_{jk} n_j\right)\delta u_k \mathrm{d}a \tag{7.186}$$

Applying postulations (1) and (2) and noting $n_3 = 1, n_\alpha = 0$ on the midplane, $n_3 = 0$ on the lateral surface, it is obtained:

$$\begin{aligned}
\int_V \sigma_{kl}\delta u_{l,k}\mathrm{d}V &= \int_L N_{\alpha\beta}n_\beta\delta u^0_\alpha \mathrm{d}L - \int_s N_{\alpha\beta,\beta}\delta u^0_\alpha \mathrm{d}s - \int_L M_{\alpha\beta}n_\beta\delta u^0_{3,\alpha}\mathrm{d}L \\
&\quad + \int_L M_{\alpha\beta,\alpha}n_\beta\delta u^0_3 \mathrm{d}L - \int_s M_{\alpha\beta,\beta\alpha}\delta u^0_3 \mathrm{d}s \\
\int_V \rho\ddot{u}_k\delta u_k \mathrm{d}V &= \int_s \left(\rho_0\ddot{u}^0_3 + \rho_1\ddot{u}^0_{\alpha,\alpha} - \rho_2 u^0_{3,\alpha\alpha}\right)\delta u^0_3 \mathrm{d}s \\
&\quad + \int_s \left(\rho_0\ddot{u}^0_\alpha - \rho_1\ddot{u}^0_{3,\alpha}\right)\delta u^0_\alpha \mathrm{d}s - \int_L \left(\rho_1\ddot{u}^0_k - \rho_2 u^0_{3,k}\right)n_k\delta u^0_3 \mathrm{d}L \\
\int_{a^{\mathrm{int}}} \sigma^{\mathrm{M}}_{ip}n_i\delta u_p \mathrm{d}a &= \int_L N^{\mathrm{M}}_{\alpha p}n_\alpha\delta u^0_p \mathrm{d}L - \int_L M^{\mathrm{M}}_{\alpha\beta}n_\alpha\delta u^0_{3,\beta}\mathrm{d}L + \int_S p^{\mathrm{M}}_i\delta u^0_i \mathrm{d}S \\
\int_V \sigma^{\mathrm{M}}_{ip,i}\delta u_p \mathrm{d}V &= \int_s N^{\mathrm{M}}_{i\alpha,i}\delta u^0_\alpha \mathrm{d}S - \int_L M^{\mathrm{M}}_{i\alpha,i}n_\alpha\delta u^0_3 \mathrm{d}L + \int_s M^{\mathrm{M}}_{i\alpha,i\alpha}\delta u^0_3 \mathrm{d}S + \int_s N^{\mathrm{M}}_{i3,i}\delta u^0_3 \mathrm{d}S \\
\int_{a^{\mathrm{int}}_\sigma} T^{*\mathrm{int}}_l\delta u_l \mathrm{d}a &= \int_S p^{*\mathrm{int}}_l\delta u^0_l \mathrm{d}S + \int_{L_\sigma} P^{*\mathrm{int}}_l\delta u^0_l \mathrm{d}L - \int_{L_\sigma} M^{*\mathrm{int}}_l\delta u^0_{3,l}\mathrm{d}L
\end{aligned} \tag{7.187}$$

where $2h$ is the thickness of the plate. The expression of $\sigma^{\mathrm{M\,air}}_{ip}$ is similar to σ^{M}_{ip}. The notations in Eq. (7.187) are

$$\begin{aligned}
&N_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}\mathrm{d}x_3, \quad M_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}x_3\mathrm{d}x_3, \quad N^{\mathrm{M}}_{\alpha p} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{\alpha p}\mathrm{d}x_3, \quad M^{\mathrm{M}}_{\alpha p} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{\alpha p}x_3\mathrm{d}x_3 \\
&p^{\mathrm{M}}_i = N^{\mathrm{M}}_{3i,3} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{3i,3}\mathrm{d}x_3 = \sigma^{\mathrm{M}}_{3i}(h) - \sigma^{\mathrm{M}}_{3i}(-h), \quad M^{\mathrm{M}}_{3i,3} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{3i,3}x_3\mathrm{d}x_3 \\
&\rho_0 = \int_{-h}^{h}\rho\mathrm{d}x_3, \quad \rho_1 = \int_{-h}^{h}\rho x_3\mathrm{d}x_3, \quad \rho_2 = \int_{-h}^{h}\rho x_3^2\mathrm{d}x_3, \\
&p^{*\mathrm{int}}_l = T^{*\mathrm{int}}_l\Big|_{-h}^{h}, \quad \text{on} \quad S; \quad P^{*\mathrm{int}}_l = \int_{-h}^{h}T^{*\mathrm{int}}_l\mathrm{d}x_3, \quad M^{*\mathrm{int}}_l = \int_{-h}^{h}T^{*\mathrm{int}}_l x_3\mathrm{d}x_3, \quad \text{on} \quad L
\end{aligned} \tag{7.188}$$

It is noted that when the Maxwell stresses on the upper and lower surfaces are reduced to the midplane, a distributed couple $\int_S \left(m^{\mathrm{M}}_{3\alpha} - m^{\mathrm{M\,env}}_{3\alpha}\right)\delta u^0_{3,\alpha}\mathrm{d}S,\quad m^{\mathrm{M}}_{3\alpha} = \left[\sigma^{\mathrm{M}}_{3\alpha}(h) + \sigma^{\mathrm{M}}_{3\alpha}(-h)\right](h/2)$ may be produced, but this effect is neglected in

Eq. (7.187) due to small h. It is also noted that Eq. (7.179) is not fully appropriate for the free lateral boundary. In fact from the variational formula, $\sigma_{\alpha 3}$ on the middle plane is approximately considered by $P_3^{\text{int}\,*}$, but $\sigma_{\alpha 3}$ on the free boundary should not be considered. So on the boundary L, a term $N_{3\beta} = \int_{-h}^{h} \sigma_{\beta 3} \mathrm{d}x_3$ should be added to the variational formula.

Substituting Eqs. (7.187) into Eq. (7.186), adding a term $\int_L N_{3\beta} n_\beta \delta u_3^0 \mathrm{d}L$, and finishing the variational calculation, we finally get:

The mechanical governing equations of the plane problem are

$$\begin{aligned} & N_{\alpha\beta,\beta} + N_{\alpha\beta,\beta}^{\mathrm{M}} + p_\alpha^{\mathrm{M\,env}} - p_\alpha^{\mathrm{M}} + p_\alpha^{*\mathrm{int}} = \rho_0 \ddot{u}_\alpha^0 \left(-\rho_1 \ddot{u}_{3,\alpha}^0 \right); \quad \text{in} \quad S \\ & \left(N_{\alpha\beta} + N_{\alpha\beta}^{\mathrm{M}} - N_{\alpha\beta}^{\mathrm{M\,env}} \right) n_\beta = P_\alpha^{*\mathrm{int}}; \quad \text{on} \quad L_\sigma \end{aligned} \tag{7.189}$$

The mechanical governing field equation for the bending problem is

$$\begin{aligned} & M_{\alpha\beta,\beta\alpha} + M_{\alpha i,\alpha i}^{\mathrm{M}} + N_{\alpha 3,\alpha}^{\mathrm{M}} + p_3^{\mathrm{M\,env}} - p_3^{\mathrm{M}} + p_3^{*\mathrm{int}} = \rho_0 \ddot{u}_3^0 + \left(\rho_1 \ddot{u}_{\alpha,\alpha}^0 - \rho_2 u_{3,\alpha\alpha}^0 \right); \quad \text{in} \quad S \\ & \int_L \left[\left(M_{\alpha\beta,\alpha} + M_{\alpha\beta,\alpha}^{\mathrm{M}} \right) n_\beta - P_3^{*\mathrm{int}} + \left(N_{3\beta} + N_{\beta 3}^{\mathrm{M}} - N_{\beta 3}^{\mathrm{M\,env}} \right) n_\beta - \left(\rho_1 \ddot{u}_k^0 - \rho_2 u_{3,k}^0 \right) \right] \delta u_3^0 \mathrm{d}L \\ & - \int_L \left[M_{\alpha\beta} n_\beta + \left(M_{\beta\alpha}^{\mathrm{M}} - M_{\beta\alpha}^{\mathrm{M\,env}} \right) n_\beta - M_\alpha^{*\mathrm{int}} \right] \delta u_{3,\alpha}^0 \mathrm{d}L = 0 \end{aligned} \tag{7.190}$$

In Eq. (7.190) terms $\left(\rho_1 \ddot{u}_{\alpha,\alpha}^0 - \rho_2 u_{3,\alpha\alpha}^0 \right)$ and $\left(\rho_1 \ddot{u}_k^0 - \rho_2 u_{3,k}^0 \right)$ can be neglected. The usual three boundary conditions for the plate bending can easily be derived from Eq. (7.190).

According to the assumption (3), the electromagnetic field is reduced to

$$\begin{aligned} & D_{i,i} = \rho_{\mathrm{e}}, \quad B_{i,i} = 0, \quad \text{in} \quad V; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \quad B_i = B_i^*, \quad \text{on} \quad a_\mu \\ & \left(D_i - D_i^{\mathrm{env}} \right) n_i = -\sigma^{*\mathrm{int}}, \quad \left(B_i - B_i^{\mathrm{env}} \right) = B_i^{*\mathrm{int}}, \quad \text{on} \quad a^{\mathrm{int}} \end{aligned} \tag{7.191}$$

In deriving the above equations, the constitutive equations were not used, so the governing equations can be used for all materials satisfying the basic postulations (1) to (3).

7.6.4 Some Discussions

The soft electromagnetic plate under a uniform transverse magnetic field can be bending or buckling when the magnetic field exceeds a critical value (Moon and Pao 1968; Pao and Yeh 1973). The natural frequency of a soft electromagnetic plate can be changed under a longitudinal magnetic field (Zhou and Miya 1998). Zhou and Zheng (1997) pointed out that there was not a unified theory to discuss the above two problems, and they proposed a variational method attempting to unified deal with these problems. The key problem is to get the electromagnetic force acting on the plate. Though for the electromagnetically static problem and the problem without magnetic field the theory discussed above is appropriate, but for a MQS system ($\partial \boldsymbol{D}/\partial t = 0,\ \partial \boldsymbol{B}/\partial t \neq 0$), such as vibration problem in a magnetic field, it should be modified, the motional electric force should be considered. As an example, the transverse vibration of an elastic electroconductive plate is subjected to the external uniform magnetic field $\boldsymbol{H}_0 = H_{01}\mathbf{i}_1 + H_{02}\mathbf{i}_2$ parallel to the (x_1, x_2) plane only. The induced motional electric field $\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{j}$ in the plate due to the plate motion is (Librescu et al. 2004; Belubekyan et al. 2007)

$$\boldsymbol{e} = -\boldsymbol{v} \times \boldsymbol{B}_0, \quad \boldsymbol{v} = v\,\mathbf{i}_3; \quad \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0); \quad \boldsymbol{j} = \nabla \times \boldsymbol{h} \tag{7.192}$$

The corresponding Maxwell stress is

$$\begin{aligned} &\sigma_{ij}^{\mathrm{M}} = \sigma_{ij}^{\mathrm{M0}} + \sigma_{ij}^{\mathrm{M1}} \\ &\sigma_{ij}^{\mathrm{M0}} = B_{0i}H_{0j} - (1/2)B_{0m}H_{0m}\delta_{ij}, \quad \sigma_{ij}^{\mathrm{M1}} = b_i H_{0j} + B_{0i}h_j - B_{0m}h_m\delta_{ij} \end{aligned} \tag{7.193}$$

Belubekyan et al. (2007) adopted the electromagnetic body force $\boldsymbol{f} = \nabla \cdot \boldsymbol{\sigma}^{\mathrm{M}}$, but Librescu et al. (2004) adopted the Lorentz formula $\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{B}_0$. The MQS problems should be further studied.

7.7 Piezoelectric Composite Shells

7.7.1 First-Order Shear Deformation Theory

Consider a finite, simply supported, N-layered laminated circular cylindrical shell of mean radius R, length L, and thickness H. The shell is constituted of elastic orthotropic or radially polarized piezoelectric materials. The cylindrical coordinates (x, θ, z) are adopted with (x, θ) spanning the mid-surface and z along the normal (radial) direction. Analogous to the Mindlin plate theory, Kapuria et al. (1998) assumed that the displacements can be approximated as

$$u = u^0(x,\theta) + z\psi_1(x,\theta), \quad v = v^0(x,\theta) + z\psi_2(x,\theta), \quad w = w^0(x,\theta) \tag{7.194}$$

where u^0, v^0, w^0 are the displacement components on the mid-surface and ψ_1, ψ_2 are the rotations of its normal. The corresponding strains are

$$\begin{aligned} &\varepsilon_x = u^0_{,x} + z\psi_{1,x}, \quad \varepsilon_\theta = \left(v^0_{,\theta} + +z\psi_{2,\theta} + w^0\right)\Big/(R+z), \quad \varepsilon_z = 0, \quad \gamma_{zx} = \psi_1 + w^0_{,x} \\ &\gamma_{\theta z} = \psi_2 + \left(w^0_{,\theta} - v^0 - z\psi_2\right)\Big/(R+z), \quad \gamma_{\theta x} = v^0_{,x} + \left(u^0_\theta + z\psi_{1,\theta}\right)/(R+z) + z\psi_{2,x} \end{aligned} \tag{7.195}$$

On the mid-surface, the resultant membrane forces $N_x,\ N_\theta,\ N_{x\theta},\ N_{\theta x}$, transverse forces $Q_x,\ Q_\theta$, and resultant moments $M_x,\ M_\theta,\ M_{x\theta},\ M_{\theta x}$ are

$$\begin{bmatrix} N_x,\ N_\theta,\ N_{x\theta},\ N_{\theta x} \\ M_x,\ M_\theta,\ M_{x\theta},\ M_{\theta x} \end{bmatrix} = \int_{-H/2}^{H/2} \begin{bmatrix} 1 \\ z \end{bmatrix} \left[\sigma_x\left(1+\frac{z}{R}\right), \sigma_\theta, \sigma_{x\theta}\left(1+\frac{z}{R}\right), \sigma_{\theta x}\right] \mathrm{d}z$$
$$[Q_x,\ Q_\theta] = \int_{-H/2}^{H/2} [\sigma_{xz}(1+z/R), \sigma_{\theta z}] \mathrm{d}z \tag{7.196}$$

The constitutive equation is shown in Eq. (7.149). The equilibrium equations are

$$\begin{aligned} & N_{x,x} + N_{\theta x,\theta}/R + p_x = 0, \quad (Q_\theta + N_{\theta,\theta})/R + N_{x\theta,x} + p_\theta = 0, \\ & Q_{x,x} + (Q_{\theta,\theta} - N_\theta)/R + p_z = 0 \\ & M_{x,x} + M_{\theta x,\theta}/R - Q_x + m_x = 0, \quad M_{\theta,\theta}/R + M_{x\theta,x} - Q_\theta + m_\theta = 0 \\ & (p_x, p_\theta, p_z) = [(1+z/R)(\sigma_{zx}, \sigma_{z\theta}, \sigma_z)]_{-H/2}^{H/2}, \quad (m_x, m_\theta) = [(1+z/R)z(\sigma_{zx}, \sigma_{z\theta})]_{-H/2}^{H/2} \end{aligned} \tag{7.197}$$

where $p_x, p_\theta, p_z, m_x m_\theta$ are the external forces and moments, respectively. The boundary conditions are defined:

$$\begin{aligned} & N_x \quad \text{or} \quad u^0, \quad N_{x\theta} \quad \text{or} \quad v^0, \quad Q_x \quad \text{or} \quad w^0, \quad M_x \quad \text{or} \quad \psi_1, \quad M_{x\theta} \quad \text{or} \quad \psi_2; \\ & \quad \text{at} \quad x = 0 \quad \text{or} \quad L \\ & N_{\theta x} \quad \text{or} \quad u^0, \quad N_\theta \quad \text{or} \quad v^0, \quad Q_\theta \quad \text{or} \quad w^0, \quad M_{\theta x} \quad \text{or} \quad \psi_1, \quad M_\theta \quad \text{or} \quad \psi_2; \\ & \quad \text{at} \quad \theta = 0 \quad \text{or} \quad \theta_0 \end{aligned} \tag{7.198}$$

where θ_0 is the span of the cylindrical panel. Usually σ_z can be neglected. Using Eqs. (7.149) and (7.150), the equilibrium equations in terms of displacements can be obtained. Here it is omitted. The above theory is easily extended to the combined multiply layer shell.

The electric potential φ is assumed to vary linearly across the actuated layer, and the electric field is computed as $E_x = -\varphi_{,x},\ E_\theta = -\varphi_{,\theta}/(R+z),\ E_z = -\varphi_{,z}$.

For the classical shell theory, the transverse shear strains $\gamma_{zx}, \gamma_{\theta z}$ are neglected. Hence,

$$\begin{aligned} & u = u^0 - zw^0_{,x}, \quad v = v^0 - z\left(w^0_{,\theta} - v^0\right)\Big/R, \quad w = w^0(x,\theta); \\ & \varepsilon_x = u^0_{,x} - zw^0_{,xx}, \quad \varepsilon_\theta = v^0_{,\theta}\Big/R + \left(w^0 - zw^0_{,\theta\theta}\right)\Big/(R+z), \quad \varepsilon_z = 0, \\ & \gamma_{\theta x} = v^0_{,x} + \left(u^0_\theta + z\psi_{1,\theta}\right)/(R+z) + z\psi_{2,x}; \quad \psi_1 = -w^0_{,x}, \quad \psi_2 = \left(v^0 - w^0_{,\theta}\right)\Big/R \end{aligned} \tag{7.199}$$

The above thin shell theories yield poor predictions of the transverse stress components $\sigma_{xz}, \sigma_{\theta z}, \sigma_z$, so sometimes a post-processing technique is needed. The transverse stress can approximately be obtained from the 3D equilibrium equations (Kapuria et al. 1998):

$$\begin{aligned}(R+z)^2\sigma_{\theta z} &= -\int_{-h/2}^{z}\left[(R+z)\sigma_{\theta,\theta}+(R+z)^2\sigma_{\theta x,x}\right]\mathrm{d}z + c_1\\ (R+z)^2\sigma_{xz} &= -\int_{-h/2}^{z}\left[\sigma_{\theta x,\theta}+(R+z)\sigma_{x,x}\right]\mathrm{d}z + c_2\\ (R+z)^2\sigma_{z} &= -\int_{-h/2}^{z}\left[\sigma_{\theta}-\sigma_{z\theta,\theta}-(R+z)\sigma_{zx,x}\right]\mathrm{d}z + c_3\end{aligned} \tag{7.200}$$

where c_i is determined by the boundary conditions at the outer shell surface. Kapuria et al. (1998) compared the numerical results of the shell theory with that of the exact 3D theory and gave some comments. Saviz et al. (2007) proposed a layerwise model which is formulated by introducing piecewise continuous approximations through the thickness for each state variables. They showed that the results calculated by this model more consist with that from the 3D theory.

7.7.2 *The Cylindrical Bending of a Laminated Infinitely Long Shell*

The exact analytical solution of a cylindrical shell by 3D theory is difficult, but for some simpler cases, it is possible. Now discuss an infinitely long laminated orthotropic cylindrical shell with simple supported edges under purely cylindrical bending. The top and bottom layers are piezoelectric actuators, and the middle layer is an elastic orthotropic substrate. The cylindrical coordinates r, θ, z are used, where r, θ and z refer to the radial, circumferential, and axial directions, respectively, and u_r, u_θ and u_z are the corresponding displacements (Fig. 7.10).

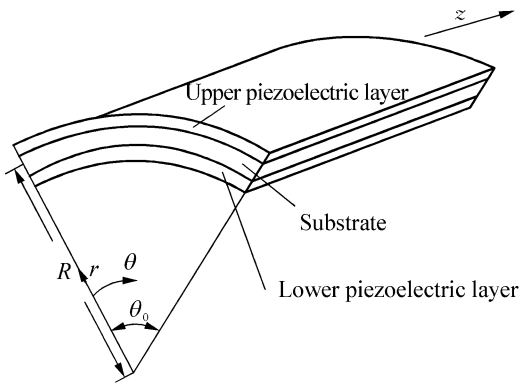
Because the shell is infinitely long, variables can be considered as the functions of (r, θ) only. The equilibrium, geometric, and constitutive equations are, respectively,

$$\begin{aligned}&\sigma_{r,r}+\sigma_{r\theta,\theta}/r+(\sigma_r-\sigma_\theta)/r=0, \quad \sigma_{r\theta,r}+\sigma_{\theta,\theta}/r+2\sigma_{r\theta}/r=0\\ &D_r+rD_{r,r}+D_{\theta,\theta}=0\end{aligned} \tag{7.201}$$

$$\begin{aligned}&\varepsilon_r=u_{r,r}, \quad \varepsilon_\theta=(u_{\theta,\theta}+u_r)/r, \quad \gamma_{r\theta}=(u_{r,\theta}-u_\theta)/r+u_{\theta,r}\\ &E_r=-\varphi_{,r}, \quad E_\theta=-\varphi_{,\theta}/r\end{aligned} \tag{7.202}$$

$$\begin{aligned}&\sigma_r=C_{11}\varepsilon_r+C_{12}\varepsilon_\theta-e_{33}E_r, \quad \sigma_\theta=C_{12}\varepsilon_r+C_{22}\varepsilon_\theta-e_{31}E_r\\ &\sigma_{r\theta}=C_{66}\gamma_{r\theta}-e_{15}E_\theta, \quad D_r=e_{33}\varepsilon_r+e_{31}\varepsilon_\theta+\epsilon_r E_r, \quad D_\theta=e_{15}\gamma_{r\theta}+\epsilon_\theta E_\theta\end{aligned} \tag{7.203}$$

Fig. 7.10 Cylindrical bending of an infinitely long shell



where C_{ij} is the reduced material coefficients. The boundary conditions are

$$\begin{array}{l} 1.\ \text{Simply supported } u_r = \sigma_\theta = \sigma_{\theta z} = 0; \quad \varphi = 0, \quad \text{when} \quad \theta = 0, \quad \theta_0 \\ 2.\ \text{On the interfaces} \quad u_r,\ u_\theta,\ \sigma_r,\ \sigma_{r\theta} \quad \text{continuous and} \quad \varphi = 0 \\ 3.\ \text{Upper surface of the outer actuator } \sigma_r = q_0 \sin p\theta, \quad \sigma_{r\theta} = 0, \quad \varphi = V \sin p\theta \\ 4.\ \text{Lower surface of the inner actuator } \sigma_r = \sigma_{r\theta} = 0; \quad D_r = 0 \end{array} \tag{7.204}$$

where $p = m\pi/\theta_0$,m is an integer, and V and q_0 are given values.

In order to satisfy the boundary conditions $u_r = \sigma_\theta = \varphi = 0$ on the edges, Chen et al. (1996) adopted the following generalized displacements for actuators:

$$u_r = u_r^0(r) \sin p\theta, \quad u_\theta = u_\theta^0(r) \cos p\theta, \quad \varphi = \varphi^0(r) \sin p\theta \tag{7.205}$$

Substitution of Eq. (7.205) into Eq. (7.201) for actuators yields

$$\begin{aligned} &C_{11}\left(u''_{r0} + \frac{u'_{r0}}{r}\right) - \left(C_{22} + p^2 C_{66}\right)\frac{u_{r0}}{r^2} - p(C_{66} + C_{12})\frac{u'_{\theta 0}}{r} + p(C_{22} + C_{66})\frac{u_{\theta 0}}{r^2} - e_{31}\frac{\varphi'_0}{r} = 0 \\ &p(C_{12} + C_{66})\frac{u'_{r0}}{r} + p(C_{22} + C_{66})\frac{u_{r0}}{r^2} + C_{66}\left(u''_{\theta 0} + \frac{u'_{\theta 0}}{r}\right) - \left(p^2 C_{22} + C_{66}\right)\frac{u_{\theta 0}}{r^2} + pe_{31}\frac{\varphi'_0}{r} = 0 \\ &e_{31}\frac{u'_{r0}}{r} - pe_{31}\frac{u'_{\theta 0}}{r} - \epsilon_r\left(\varphi''_0 + \frac{\varphi'_0}{r}\right) + p^2 \epsilon_\theta \frac{\varphi_0}{r^2} = 0 \end{aligned} \tag{7.206}$$

where $f' = f_{,r}, \quad f'' = f_{,rr}$ for any f. Let

$$u_{r0}(r) = A_r r^s, \quad u_{\theta 0}(r) = A_\theta r^s, \quad \varphi_0(r) = A_\varphi r^s \tag{7.207}$$

Substitution of Eq. (7.207) into Eq. (7.206) for actuators yields the homogeneous equation of A_r, A_θ, A_φ. In order to have nontrivial solutions for A_r, A_θ, A_φ, the coefficient determinate of them must be zero, so the following character equation is obtained:

$$\begin{aligned}
&As^6 + Bs^4 + Cs^2 + D = 0 \\
&A = -C_{11}C_{66}\,\epsilon_r \\
&B = \left[C_{11}\left(p^2C_{22} + C_{66}\right) + C_{66}\left(C_{22} + p^2C_{66}\right) - p^2(C_{12} + C_{66})^2\right]\epsilon_r \\
&\qquad + p^2C_{11}C_{66}\,\epsilon_\theta + \left(C_{66} + p^2C_{11}\right)e_{31}^2 \\
&C = \left[-\left(C_{22} + p^2C_{66}\right)\left(p^2C_{22} + C_{66}\right) + p^2(C_{22} + C_{66})^2\right]\epsilon_r + \left[p^4(C_{12} + C_{66})^2\right. \\
&\qquad \left.-p^2C_{11}\left(p^2C_{22} + C_{66}\right) - p^2C_{66}\left(C_{22} + p^2C_{66}\right)\right]\epsilon_\theta + \left[2p^2(C_{22} + C_{66})\right. \\
&\qquad \left.-\left(p^2C_{22} + C_{66}\right)e_{31}^2 - p^2\left(C_{22} + p^2C_{66}\right)\right]e_{31}^2 \\
&D = \left[p^2\left(p^2C_{22} + C_{66}\right)\left(C_{22} + p^2C_{66}\right) - p^4(C_{22} + C_{66})^2\right]\epsilon_\theta
\end{aligned} \tag{7.208}$$

From Eq. (7.208) s has 6 real roots $s_j,\ j = 1 - 6$ for piezoelectric material. For each s_j, a group $(A_{rj}, A_{\theta j}, A_{\varphi j})$ with one unknown is obtained, so the general solution of Eq. (7.206) for each actuator is

$$\begin{aligned}
&u_{r0} = \sum_{j=1}^{6} A_j r^{s_j}, \quad u_{\theta 0} = \sum_{j=1}^{6} A_j H_{\theta j} r^{s_j}, \quad \varphi_0 = \sum_{j=1}^{6} A_j H_{\varphi j} r^{s_j} \\
&H_{\theta j} = -\left\{p\left[(C_{12} + C_{66})s_j + (C_{22} + C_{66})\right]\left(-\epsilon_r s_j^2 + p^2\epsilon_\theta\right) - pe_{31}^2 s_j^2\right\}\Big/\Delta \\
&H_{\varphi j} = -\left\{\left(C_{66}s_j^2 - p^2C_{22} - C_{66}\right) + p^2\left[(C_{12} + C_{66})s_j + C_{22} + C_{66}\right]\right\}e_{31}s_j\Big/\Delta \\
&\Delta = \left(C_{66}s_j^2 - p^2C_{22} - C_{66}\right)\left(-\epsilon_r s_j^2 + p^2\epsilon_\theta\right) + pe_{31}^2 s_j^2
\end{aligned} \tag{7.209}$$

The governing equations of the middle orthotropic composite matrix can be obtained if the electric variables in Eq. (7.206) are omitted. Let the generalized displacements in matrix be

$$u_r^{(m)} = a_r r^s \sin p\theta, \quad u_\theta^{(m)} = a_\theta r^s(r)\cos p\theta \tag{7.210}$$

The character equation of a_r, a_θ is

$$\begin{aligned}
&B's^4 + C's^2 + D' = 0 \\
&B' = C_{11}C_{66} \\
&C' = -C_{66}(C_{22} + p^2C_{66}) - C_{11}\left(p^2C_{22} + C_{66}\right) + p^2(C_{12} + C_{66})^2 \\
&D' = \left(C_{22} + p^2C_{66}\right)\left(p^2C_{22} + C_{66}\right) - p^2(C_{12} + C_{66})^2
\end{aligned} \tag{7.211}$$

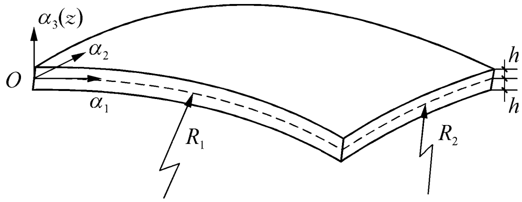


Fig. 7.11 Shallow piezoelectric shell

where the superscript M of C_{ij}^M is omitted. So the solution of the substrate is

$$
\begin{aligned}
&u_{r0} = \sum_{j=1}^{4} a_j H_{rj} r^{s_j}, \quad u_{\theta 0} = \sum_{j=1}^{4} a_j r^{s_j} \\
&H_{rj} = p\left[(C_{12} + C_{66})s_j - (C_{22} + C_{66})\right] \Big/ \left[C_{11}s_j^2 - (C_{22} + p^2 C_{66})\right]
\end{aligned} \tag{7.212}
$$

There are 16 unknowns: 6 A_j of the outer actuator, 6 A_j of the inner actuator, and 4 a_j of the middle substrate. There are also 16 boundary conditions: 4 conditions on each interface, 3 conditions on upper surface, 3 conditions on lower surface, and $\sigma_{\theta z} = 0$ at $\theta = 0, \theta_0$. Therefore, the problem is solved.

7.7.3 Approximate Theory of a Functionally Graded Shallow Piezoelectric Shell

Figure 7.11 shows a functionally graded shallow piezoelectric shell of thickness $2h$; (α_1, α_2) are the orthogonal curvilinear coordinates on the mid-surface; and its corresponding Lamé parameters are H_1, H_2 and radii of curvatures are R_1, R_2. α_3 is a linear coordinate and normal to the mid-surface. For a thin shell, $h \ll R_i$, $(i = 1, 2)$, R_i is approximately independent of α_3. Let (u_1, u_2, u_3, φ) be the generalized displacements in the orthogonal curvilinear coordinates, then

$$
\begin{aligned}
&\varepsilon_{11} = \frac{\partial u_1}{H_1 \partial \alpha_1} + \frac{u_2 \partial H_1}{H_1 H_2 \partial \alpha_2} + \frac{u_3}{R_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{H_2 \partial \alpha_2} + \frac{u_1 \partial H_2}{H_1 H_2 \partial \alpha_1} + \frac{u_3}{R_2} \\
&\varepsilon_{33} = \frac{\partial u_3}{\partial \alpha_3}, \quad \gamma_{23} = \frac{\partial u_2}{\partial \alpha_3} + \frac{\partial u_3}{H_2 \partial \alpha_2} - \frac{u_2}{R_2}, \quad \gamma_{13} = \frac{\partial u_1}{\partial \alpha_3} + \frac{\partial u_3}{H_1 \partial \alpha_1} - \frac{u_1}{R_1} \\
&\gamma_{12} = \frac{\partial u_1}{H_2 \partial \alpha_2} + \frac{\partial u_2}{H_1 \partial \alpha_1} - \frac{u_2 \partial H_2}{H_1 H_2 \partial \alpha_1} - \frac{u_1 \partial H_1}{H_1 H_2 \partial \alpha_2} \\
&E_1 = -\frac{\partial \varphi}{H_1 \partial \alpha_1}, \quad E_2 = -\frac{\partial \varphi}{H_2 \partial \alpha_2}, \quad E_3 = -\frac{\partial \varphi}{\partial \alpha_3}
\end{aligned} \tag{7.213}
$$

Let $C_{ijkl}(z), e_{kij}(z), \epsilon_{ij}(z)$, $z \equiv \alpha_3$. Wu et al. (2002) assumed

$$\begin{aligned} u_1 &= u_1^0(\alpha_1,\alpha_2) + zu_1^1(\alpha_1,\alpha_2), \quad u_2 = u_2^0(\alpha_1,\alpha_2) + zu_2^1(\alpha_1,\alpha_2) \\ u_3 &= u_3^0(\alpha_1,\alpha_2) + zu_3^1(\alpha_1,\alpha_2), \quad \varphi = \varphi^0(\alpha_1,\alpha_2) + z\varphi^1(\alpha_1,\alpha_2) + z^2\varphi^{(2)}(\alpha_1,\alpha_2) \end{aligned} \tag{7.214}$$

Substitution of Eq. (7.214) into Eq. (7.213) yields the generalized strains, then substitutes the strains into the variational formula Eq. (7.160). Approximately take $1 + z/R_1 \approx 1,\ 1 + z/R_2 \approx 1$. Noting $\mathrm{d}V = H_1H_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}z$ and finishing the variational calculation, the approximate equations of the thin shell can be obtained. The generalized momentum equation is

$$\begin{aligned}
\delta u_1^0 &: \frac{\partial(H_2N_{11})}{\partial\alpha_1} + \frac{\partial(H_1N_{12})}{\partial\alpha_2} + N_{12}\frac{\partial H_1}{\partial\alpha_2} - N_{22}\frac{\partial H_2}{\partial\alpha_1} + N_{31}\frac{H_1H_2}{R_1} \\
&\quad + H_1H_2\left(T_1^{*+} + T_1^{*-}\right) = H_1H_2\left(\rho_0\ddot{u}_1^0 + \rho_1\ddot{u}_1^1\right) \\
\delta u_2^0 &: (H_2N_{12})_{,1} + (H_1N_{22})_{,2} + N_{12}H_{2,1} - N_{11}H_{1,2} + N_{32}H_1H_2/R_2 \\
&\quad + H_1H_2\left(T_2^{*+} + T_2^{*-}\right) = H_1H_2\left(\rho_0\ddot{u}_2^0 + \rho_1\ddot{u}_2^1\right) \\
\delta u_3^0 &: (H_2N_{13})_{,1} + (H_1N_{32})_{,2} - N_{11}H_1H_2/R_1 - N_{22}H_1H_2/R_2 \\
&\quad + H_1H_2\left(T_3^{*+} + T_3^{*-}\right) = H_1H_2\left(\rho_0\ddot{u}_3^0 + \rho_1\ddot{u}_3^1\right)_3 \\
\delta u_1^1 &: (H_2M_{11})_{,1} + (H_1M_{12})_{,2} + M_{12}H_{1,2} - M_{22}H_{2,1} + M_{31}H_1H_2/R_1 \\
&\quad - N_{13}H_1H_2 + H_1H_2\left(hT_1^{*+} - hT_1^{*-}\right) = H_1H_2\left(\rho_1\ddot{u}_1^0 + \rho_2\ddot{u}_1^1\right) \\
\delta u_2^1 &: (H_2M_{12})_{,1} + (H_1M_{22})_{,2} + M_{12}H_{2,1} - M_{11}H_{1,2} + M_{32}H_1H_2/R_2 \\
&\quad - N_{23}H_1H_2 + H_1H_2\left(hT_2^{*+} - hT_2^{*-}\right) = H_1H_2\left(\rho_1\ddot{u}_2^0 + \rho_2\ddot{u}_2^1\right) \\
\delta u_3^1 &: (H_2M_{13})_{,1} + (H_1M_{23})_{,2} - M_{11}H_1H_2/R_1 - M_{22}H_1H_2/R_2 \\
&\quad - N_{33}H_1H_2 + H_1H_2\left(hT_3^{*+} - hT_3^{*-}\right) = H_1H_2\left(\rho_1\ddot{u}_3^0 + \rho_2\ddot{u}_3^1\right) \\
\delta\varphi^0 &: \left(H_2D_1^0\right)_{,1} + \left(H_1D_2^0\right)_{,2} + H_1H_2\left(\sigma_1^{*+} + \sigma_1^{*-}\right) = 0 \\
\delta\varphi^1 &: \left(H_2D_1^1\right)_{,1} + \left(H_1D_2^1\right)_{,2} - H_1H_2D_3^0 + H_1H_2\left(h\sigma_1^{*+} - h\sigma_1^{*-}\right) = 0 \\
\delta\varphi^{(2)} &: \left(H_2D_1^{(2)}\right)_{,1} + \left(H_1D_2^{(2)}\right)_{,2} - H_1H_2D_3^1 + H_1H_2\left(h^2\sigma_1^{*+} - h^2\sigma_1^{*-}\right) = 0
\end{aligned} \tag{7.215}$$

The natural boundary conditions are

$$\begin{aligned}
&\delta u_1^0 : N_{11}n_1 + N_{12}n_2 = \bar{T}_1, && \delta u_2^0 : N_{12}n_1 + N_{22}n_2 = \bar{T}_2 \\
&\delta u_3^0 : N_{13}n_1 + N_{32}n_2 = \bar{T}_3, && \delta u_1^1 : M_{11}n_1 + M_{12}n_2 = \bar{M}_1 \\
&\delta u_2^1 : M_{12}n_1 + M_{22}n_2 = \bar{M}_2, && \delta u_3^1 : M_{13}n_1 + M_{23}n_2 = \bar{M}_3 \\
&\delta\varphi^0 : D_1^0n_1 + D_2^0n_2 = -\sigma^{0*}, && \delta\varphi^1 : D_1^1n_1 + D_2^1n_2 = -\sigma^{1*} \\
&\delta\varphi^{(2)} : D_1^{(2)}n_1 + D_2^{(2)}n_2 = -\sigma^{2*}
\end{aligned} \tag{7.216}$$

Let T_i^{*+} and T_i^{*-} denote the traction on the upper and lower surfaces, respectively, and

$$(\bar{T}_i, \bar{M}_i) = \int_{-h}^{h} T_i^*(1, z)\mathrm{d}z, \quad \left(D_i^0, D_i^1, D_i^{(2)}\right) = \int_{-h}^{h} D_i\left(1, z, z^2\right)\mathrm{d}z,$$
$$\left(\sigma^{0*}, \sigma^{1*}, \sigma^{(2)*}\right) = \int_{-h}^{h} \sigma^*\left(1, z, z^2\right)\mathrm{d}z \tag{7.217}$$
$$(\rho_0, \rho_1, \rho_2) = \sum_{k=1}^{N} \int_{h_k} \rho^{(k)}\left(1, x_3, x_3^2\right)\mathrm{d}x_3, \quad \left(N_{ij}, M_{ij}\right) = \int_{-h}^{h} (1, z)\sigma_{ij}\mathrm{d}z$$

It is noted that the generalized displacements must satisfy the boundary conditions when the variational formula Eq. (7.160) is used. Using the constitutive equation, the governing equations in terms of the generalized displacements are easily obtained.

7.7.4 Free Vibration of a Functionally Graded Piezoelectric Hollow Cylinder Filled with Compressible Fluid

Consider an orthotropic piezoelectric hollow cylinder of inner radius R, thickness h, and length L. Chen et al. (2004) adopted the cylindrical coordinates r, θ, z and adopted the state space method to analyze the free vibration of a functionally graded piezoelectric hollow cylinder filled with compressible fluid. Assumed all material constants and mass are the functions of r. From the constitutive equations, geometric equations, and motion equations, the state equation can be obtained as

$$\boldsymbol{Y}_{,r} = \boldsymbol{MY}, \quad \boldsymbol{Y} = [u_z, u_\theta, \sigma_r, D_r, \sigma_{rz}, \sigma_{r\theta}, u_r, \varphi]^{\mathrm{T}} \tag{7.218}$$

where $\boldsymbol{Y}$ is the state vector and $\boldsymbol{M}$ is 8×8 matrix. Chen et al. (2004) discussed the simply supported case with the boundary conditions:

$$u_r = u_\theta = \sigma_z = D_z = 0, \quad \text{at} \quad z = 0, L \tag{7.219}$$

In order to satisfy Eq. (7.219), they assumed that the solution of state vector can be expanded in double trigonometric series:

$$\begin{Bmatrix} u_z \\ u_\theta \\ \sigma_r \\ D_r \\ \sigma_{rz} \\ \sigma_{r\theta} \\ u_r \\ \varphi \end{Bmatrix} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \begin{Bmatrix} R_0\bar{u}_z(\eta)\cos m\pi\varsigma\cos n\theta \\ R_0\bar{u}_\theta(\eta)\sin m\pi\varsigma\sin n\theta \\ C_{44}^{\mathrm{out}}\bar{\sigma}_r(\eta)\sin m\pi\varsigma\cos n\theta \\ \sqrt{C_{44}^{\mathrm{out}}\epsilon_{33}^{\mathrm{out}}}\bar{D}_r(\eta)\cos m\pi\varsigma\cos n\theta \\ C_{44}^{\mathrm{out}}\bar{\sigma}_{rz}(\eta)\cos m\pi\varsigma\cos n\theta \\ C_{44}^{\mathrm{out}}\bar{\sigma}_{r\theta}(\eta)\sin m\pi\varsigma\sin n\theta \\ R_0\bar{u}_r(\eta)\sin m\pi\varsigma\cos n\theta \\ R_0\sqrt{C_{44}^{\mathrm{out}}/\epsilon_{33}^{\mathrm{out}}}\bar{\varphi}(\eta)\cos m\pi\varsigma\cos n\theta \end{Bmatrix} \mathrm{e}^{\mathrm{i}\omega t} \tag{7.220}$$

where $R_0 = R + h/2$, $\eta = r/R_0$, and $\varsigma = z/L$; m and n are integers; and ω is the angular frequency. Variables at the outer cylindrical surface $r = R + h$ are denoted by right superscript "out," and at the inner surface $r = R$ will be denoted by right superscript "inn." Substitution of Eq. (7.220) into Eq. (7.218) yields

$$\bar{\boldsymbol{Y}}_{,\eta} = \boldsymbol{N}\bar{\boldsymbol{Y}}, \quad \bar{\boldsymbol{Y}} = [\bar{u}_z, \bar{u}_\theta, \bar{\sigma}_r, \bar{D}_r, \bar{\sigma}_{rz}, \bar{\sigma}_{r\theta}, \bar{u}_r, \bar{\varphi}]^{\mathrm{T}} \tag{7.221}$$

where $\bar{\boldsymbol{Y}}$ is a constant vector and $\boldsymbol{N}$ is a 8×8 matrix and is not constant, so the solution cannot be obtained directly from Eq. (7.221). The approximate laminated model, for which the cylinder is divided into N thin layers, is adopted. For ith layer, $\boldsymbol{N}_i$ can be assumed constant and takes its value at midplane. Using the transfer matrix method as shown in Section 6.4.1, $\boldsymbol{Y}^{\mathrm{out}} = \boldsymbol{T}\boldsymbol{Y}^{\mathrm{inn}}$ can be obtained, through the transfer matrix $\boldsymbol{T}$, and the variables on the outer surface are expressed by the inner variables. The boundary condition on the inner cylindrical surface is solved by the fluid-solid coupling theory. For a nonviscous fluid, the connected conditions on the inner surface are

$$v_r = v_{\mathrm{f}r}, \quad p_{\mathrm{f}} + \sigma_r = 0, \quad \sigma_{rz} = \sigma_{r\theta} = 0, \quad \text{at} \quad r = R \tag{7.222}$$

where $v_{\mathrm{f}r}, v_r$ are the radial components of the velocity of the fluid and solid, respectively, and p_{f} is the fluid pressure. Finally, the boundary conditions on the inner and outer surfaces are

$$\begin{aligned} &\sigma_r^{\mathrm{inn}} = -\Omega^2 Q(\beta) u_r \rho_{\mathrm{f}}/\rho^{\mathrm{out}}, \quad \sigma_{rz}^{\mathrm{inn}} = \sigma_{r\theta}^{\mathrm{inn}} = 0, \quad \sigma_r^{\mathrm{out}} = \sigma_{rz}^{\mathrm{out}} = \sigma_{r\theta}^{\mathrm{out}} = 0 \\ &\Omega^2 = R^2\omega^2\rho^{\mathrm{out}}/C_{44}^{\mathrm{out}}, \quad \beta = \omega R/c_{\mathrm{f}}^2 - m\pi R/L \end{aligned} \tag{7.223}$$

where, $\rho_{\mathrm{f}}, \rho^{\mathrm{out}}, c_{\mathrm{f}}$ are the fluid density, solid density at $r = R + h$, and the sound velocity in fluid, respectively.

The electrically boundary conditions on $r = R,\ R + h$ are as follows:

Electrically open, $D_r = 0$, or electrically shorted, $\varphi = 0$, at

$$r = R,\ R + h \tag{7.224}$$

Substituting the boundary conditions at the inner and outer surfaces into Eq. (7.221), the frequency equation can be obtained. The frequency equations for the electrically shorted case and the electrically open case are different.

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