Chapter 5
Some Problems in More Complex Materials with Defects

Abstract In this chapter some electroelastic problems in more complex materials with defects are discussed. It is pointed out that the electroelastic analysis for electrostrictive materials, the entire system including the dielectric medium, its environment, and their common boundary should be considered together. So the Maxwell stress should be considered. The theory illustrated in this chapter is an important complement for the present theory published in literatures. The electroelastic analyses of an infinite isotropic electrostrictive material containing an elliptic hole, containing a crack with and without local saturation electric field near the crack tip, are carried out. The basic theory of the thermo-electro-elastic analysis is given. An elliptic hole in a homogeneous pyroelectric material, interface crack in dissimilar pyroelectric material, point heat source, and its interaction with cracks are discussed. The electroelastic analyses of a functionally graded piezoelectric material are also introduced. These analyses are useful in engineering applications.

Keywords Electroelastic analysis • Electrostrictive material • Maxwell stress • Pyroelectric material • Functionally graded piezoelectric material

5.1 Isotropic Electrostrictive Material

5.1.1 Governing Equations

Some polyurethane elastomers and perovskite-type ceramics can produce large deformation under applied electric field. Their strains are proportional to the square of electric field and larger than $10^{-4}(\mathrm{m/mV})^2E^2$. The electrostrictive effect can occur in all dielectric, such as the electrostrictive ceramic PMN-PT, electrostrictive polymer EPs, and polyurethane PUE. The constitutive equation has been discussed in Sects. 2.2 and 2.6. In this section we only discuss the isotropic electrostrictive

Z.-B. Kuang, *Theory of Electroelasticity*, DOI 10.1007/978-3-642-36291-0_5,

material occupying the region S. The environment occupies S^c. According to Eq. (2.27b) the constitutive equation with independent variables $(\boldsymbol{\varepsilon}, \boldsymbol{E})$ is

$$\begin{aligned}&\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - (1/2)(a_1 E_i E_j + a_2 E_k E_k \delta_{ij})\\&D_i = \tilde{\epsilon}_{ij} E_j, \quad \tilde{\epsilon}_{ij} = \epsilon\delta_{ij} + a_1\varepsilon_{ij} + a_2\varepsilon_{kk}\delta_{ij} \approx \epsilon\delta_{ij}, \quad E_i = -\varphi_{,i}\end{aligned} \tag{5.1}$$

where a_1, a_2 are electrostrictive coefficients. For electrostrictive materials the entire system including the dielectric medium, its environment, and their common boundary should be considered together, as shown in Sect. 2.2. The governing equations are

$$\begin{aligned}&S_{kl,l} + f_k = \rho\ddot{u}_k, \qquad D_{k,k} = \rho_e \qquad \text{in material}\\&S_{ij,j}^{\text{env}} + f_i^{\text{env}} = \rho\ddot{u}_i^{\text{env}}, \quad D_{i,i}^{\text{env}} = \rho_e^{\text{env}}, \quad \text{in environment}\\&S_{kl} = \sigma_{kl} + \sigma_{kl}^{\text{M}} \approx= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(a_2+\epsilon)E_i E_i \delta_{kl} + (1/2)(2\epsilon - a_1)E_k E_l\\&\sigma_{ij}^{M} = E_i D_j - (1/2)E_m D_m \delta_{ij}\end{aligned} \tag{5.2}$$

where $\boldsymbol{S}$ is the pseudo total stress (Jiang and Kuang 2003, 2004). In isotropic case $\boldsymbol{S}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^M$ are all symmetric. The boundary conditions are

$$\begin{aligned}&S_{kl}n_l = T_k^*, \quad \text{on } a_\sigma; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_{\text{D}}; \quad u_i = u_i^*, \quad \text{on} \quad a_u; \quad \varphi = \varphi^*; \quad \text{on} \quad a_\varphi\\&S_{ij}^{\text{env}} n_j^{\text{env}} = T_i^{*\text{env}}, \quad \text{on} \quad a_\sigma^{\text{env}}; \quad D_i^{\text{env}} n_i^{\text{env}} = -\sigma^{*\text{env}}, \quad \text{on} \quad a_{\text{D}}^{\text{env}}\\&u_i^{\text{env}} = u_i^{*\text{env}}, \quad \text{on} \quad a_u^{\text{env}}; \quad \varphi^{\text{env}} = \varphi^{*\text{env}}, \quad \text{on} \quad a_\varphi^{\text{env}}\end{aligned} \tag{5.3}$$

The interface conditions are

$$\left(S_{ij} - S_{ij}^{\text{env}}\right) n_j = T_i^{*\text{int}}, \quad \left(D_i - D_i^{\text{env}}\right) n_i = -\sigma^{*\text{int}}; \quad u_i = u_i^{\text{env}}, \quad \varphi = \varphi^{\text{env}}; \quad \text{on} \quad a^{\text{int}} \tag{5.4}$$

For the ceramic material the difference between $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ is small, but for the electrostrictive polymer ϵ and $a_m\varepsilon_{ij}$ may be in the same order, and the difference between $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ may not be small.

In the case of small strain, it is usually assumed that the electric field is approximately independent to the displacement, i.e., the terms containing strains in $\boldsymbol{D}$ in Eq. (5.1) can be neglected, but the stress field is related to the electric field. So the electric field is decoupled with the elastic field and can be solved independently (Knops 1963; Smith and Warren 1966; McMeeking 1989; Jiang and Kuang 2003, 2004). Assuming the air is charge free, from $\nabla \cdot \boldsymbol{D} = 0$, it is known that φ is a harmonic function, so it can be expressed by the real (or imaginary) part of a complex analytic function $w(z)$, i.e.,

$$w(z) = \varphi(x_1, x_2) + \text{i}A(x_1, x_2), \quad \varphi(x_1, x_2) = \text{Re}\,w(z) = \left[w(z) + \overline{w(z)}\right]\Big/2 \tag{5.5}$$

where A is called the stream function. Comparing Eqs. (3.83) and (5.5), it is found that $w(z) = 2\phi(z)$. These two expressions of the complex electric potential can all be found in literatures. It is noted that in this section $\phi(z)$ denotes the complex stress function. Using Cauchy-Riemann condition $\partial\varphi/\partial x_1 = \partial A/\partial x_2$, $\partial\varphi/\partial x_2 = -\partial A/\partial x_1$ yields

$$\begin{aligned}&\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}\varphi}{\mathrm{d}z} + \mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}z} = \left(\frac{\partial\varphi}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial\varphi}{\partial x_2}\frac{\partial x_2}{\partial z}\right) + \mathrm{i}\left(\frac{\partial A}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial A}{\partial x_2}\frac{\partial x_2}{\partial z}\right) = -\overline{E}\\ &E = E_1 + \mathrm{i}E_2 = -\overline{w'(z)}\end{aligned} \tag{5.6}$$

So the solution of the electric field is reduced to seek a function analytic in the region S.

On a boundary we have

$$\begin{aligned}\int D_n \mathrm{d}s &= \int (D_1 n_1 + D_2 n_2)\mathrm{d}s = \epsilon\int (E_1\mathrm{d}x_2 - E_2\mathrm{d}x_1)\\ &= (\mathrm{i}\epsilon/2)\int\left[w'(z)\mathrm{d}z - \overline{w'(z)\mathrm{d}z}\right] = (\mathrm{i}\epsilon/2)\left[w(z) - \overline{w(z)}\right]\end{aligned} \tag{5.7}$$

where a trivial integral constant is omitted.

For a plane strain problem, we have $\varepsilon_{i3} = 0, \quad (i = 1, 2, 3)$; the constitutive equation expressed by the pseudo total stress $\boldsymbol{S}$ is

$$\begin{aligned}&S_{\alpha\beta} = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2G\varepsilon_{\alpha\beta} + aE_\alpha E_\beta + bE_\gamma E_\gamma\delta_{\alpha\beta}\\ &a = (2\epsilon - a_1)/2, \quad b = -(a_2 + \epsilon)/2, \quad \alpha,\beta = 1,2\end{aligned} \tag{5.8}$$

Using $\varepsilon_{\gamma\gamma} = \left[S_{\gamma\gamma} + (a - 2b)E_\gamma E_\gamma\right]/2(\lambda + G)$, Eq. (5.8) can also be written as

$$\begin{aligned}&2G\varepsilon_{\alpha\beta} = S_{\alpha\beta} - aE_\alpha E_\beta - \left[\lambda S_{\gamma\gamma} + (-\lambda a + 2Gb)E_\gamma E_\gamma\right]\delta_{\alpha\beta}/2(\lambda + G)\\ &\varepsilon_{\alpha\beta} = (1+\nu)\left\{S_{\alpha\beta} - \nu S_{\gamma\gamma}\delta_{\alpha\beta} - aE_\alpha E_\beta + [\nu a - (1-2\nu)b]E_\gamma E_\gamma\delta_{\alpha\beta}\right\}/Y\end{aligned} \tag{5.9}$$

where $Y = 2G(1+\nu)$ is the elastic modulus and ν is the Poisson ratio. Substitution of Eq. (5.9) into the compatible equation $2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}$ finally yields

$$\begin{aligned}2(S_{12} - aE_1E_2)_{,12} = &\left\{S_{11} - aE_1E_1 - \left[\lambda S_{\gamma\gamma} + (-\lambda a + 2Gb)E_\gamma E_\gamma\right]/2(\lambda + G)\right\}_{,22}\\ &+ \left\{S_{22} - aE_2E_2 - \left[\lambda S_{\gamma\gamma} + (-\lambda a + 2Gb)E_\gamma E_\gamma\right]/2(\lambda + G)\right\}_{,11}\end{aligned} \tag{5.10}$$

Let $\tilde{U}$ denote the pseudo total stress function satisfying the equilibrium equation automatically:

$$S_{11} = \tilde{U}_{,22}, \quad S_{22} = \tilde{U}_{,11}, \quad S_{12} = -\tilde{U}_{,12}, \quad \text{or} \quad S_{\alpha\beta} = \nabla^2\tilde{U}\delta_{\alpha\beta} - \tilde{U}_{,\alpha\beta} \tag{5.11}$$

Substituting Eq. (5.11) into Eq. (5.10), after some manipulation, yields

$$\nabla^4\tilde{U}=\kappa\nabla^2\left(E_\gamma E_\gamma\right),\quad \frac{\partial^4\tilde{U}}{\partial z^2\partial\bar{z}^2}=\frac{\kappa}{4}\frac{\partial^4}{\partial z^2\partial\bar{z}^2}[w'(z)\bar{w}'(\bar{z})]=\frac{\kappa}{4}\frac{\partial^2 w'}{\partial z^2}\frac{\partial^2\bar{w}'}{\partial\bar{z}^2}$$
$$\kappa=-G(a_1+2a_2)/2(\lambda+2G)=-(1-2v)(a_1+2a_2)/4(1-v) \tag{5.12}$$

Using the Muskhelishvili's formulas (1975), the general solution of Eq. (5.12) is

$$\tilde{U}(x_1,x_2)=(\kappa/4)w(z)\overline{w(z)}+(1/2)\left[z\overline{\phi(z)}+\bar{z}\phi(z)+\chi(z)+\overline{\chi(z)}\right] \tag{5.13}$$

where $(\kappa/4)w(z)\overline{w(z)}$ is the special solution; $\phi(z),\chi(z)$ are two analytic functions of z. Equation (5.11) yields

$$\begin{aligned}&S_{22}+S_{11}=\kappa w'(z)\overline{w'(z)}+2\left[\phi'(z)+\overline{\phi'(z)}\right]\\&S_{22}-S_{11}+2\mathrm{i}S_{12}=\kappa w''(z)\overline{w(z)}+2[\bar{z}\phi''(z)+\psi'(z)],\quad \psi(z)=\chi'(z)\end{aligned} \tag{5.14}$$

From Eqs. (5.2) and (5.6), it is known that

$$\sigma_{22}^{\mathrm{M}}+\sigma_{11}^{\mathrm{M}}=0,\quad \sigma_{22}^{\mathrm{M}}-\sigma_{11}^{\mathrm{M}}+2\mathrm{i}\sigma_{12}^{\mathrm{M}}=-\epsilon\Omega'(z),\quad \Omega'(z)=[w'(z)]^2 \tag{5.15}$$

The mechanical stresses are

$$\begin{aligned}&\sigma_{22}+\sigma_{11}=\kappa w'(z)\overline{w'(z)}+2\left[\phi'(z)+\overline{\phi'(z)}\right]\\&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12}=\kappa w''(z)\overline{w(z)}+2[\bar{z}\phi''(z)+\psi'(z)]+\epsilon\Omega'(z)\end{aligned} \tag{5.16}$$

and displacements are

$$\begin{aligned}&2G(u_1+iu_2)=K\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}-(\kappa/2)w(z)\overline{w'(z)}+\alpha_1\overline{\Omega(z)}\\&K=(3-4\nu),\quad \alpha_1=(a_1-2\epsilon)/4;\quad \Omega(z)=\int\Omega'(z)\mathrm{d}z\end{aligned} \tag{5.17}$$

The stress boundary condition is

$$\begin{aligned}\mathrm{i}(P_1+iP_2)&=\mathrm{i}\int_A^B\left(\tilde{T}_1+\mathrm{i}\tilde{T}_2\right)\mathrm{d}s=2\left[\partial\tilde{U}/\partial\bar{z}\right]_A^B\\&=\left[z\overline{\phi'(z)}+\phi(z)+\overline{\psi(z)}+(1/2)\kappa w(z)\overline{w'(z)}\right]_A^B\end{aligned} \tag{5.18}$$

where A,B are two points on the boundary; P_1,P_2 are pseudo resultant forces; $\tilde{T}_i=S_{ij}n_j$.

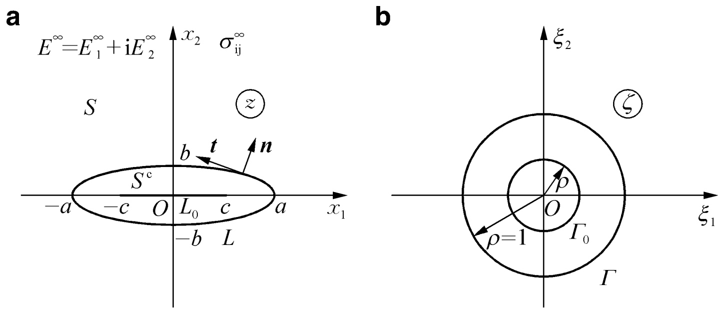

Fig. 5.1 A 2D plane electrostrictive material with an *elliptic hole* or inclusion: (**a**) physical plane z; (**b**) mapping plane ζ

5.1.2 An Impermeable Elliptic Hole in an Isotropic Electrostrictive Material

Let an isotropic electrostrictive material with an elliptic hole of semiaxes a and b directed along the material principle axes x_1 and x_2, respectively, filled by air. The uniform generalized stresses $\boldsymbol{\sigma}^\infty, \boldsymbol{E}^\infty$ are applied at infinity, but the boundary of the hole is free; see Fig. 5.1. A further assumption is that the electric field in the air will be neglected due to the small permittivity comparing with the electrostrictive material. Therefore, in this simple case the Maxwell stress in the hole is neglected, and the electrostrictive material can be studied alone (Jiang and Kuang 2003; Kuang and Jiang 2006). The boundary conditions are

$$\begin{aligned}
&S_{ij} = S_{ij}^\infty, \quad E = E_1 + iE_2 = E^\infty, \text{ at infinity}; \quad S_{ij}n_j = 0, \quad D_n = 0, \text{ on interface} \\
&S_{ij}^\infty = \sigma_{ij}^\infty + \sigma_{ij}^{M\infty}; \quad \sigma_{ij}^{M\infty} = E_i^\infty D_j^\infty - (1/2)E_m^\infty D_m^\infty \delta_{ij} \\
&E^\infty = E_0 e^{i\beta}, \quad E_0 = \sqrt{\left(E_1^\infty\right)^2 + \left(E_2^\infty\right)^2}, \quad \tan\beta = E_2^\infty / E_1^\infty
\end{aligned} \tag{5.19}$$

Electric field The mapping function method is used to solve this problem. The mapping function $z = \omega(\varsigma)$ shown in Eq. (3.82a) is still adopted. In ς plane the general solution of $w(\varsigma)$ can be written as

$$\begin{aligned}
&w(\varsigma) = -\bar{E}^\infty R\left(\varsigma + \alpha\varsigma^{-1}\right) = -R\left(\bar{E}^\infty\varsigma + E^\infty\varsigma^{-1}\right); \quad \alpha = E^\infty/\bar{E}^\infty = e^{2i\beta} \\
&E = E_1 + iE_2 = -\left[\overline{w'(\varsigma)}\Big/\overline{\omega'(\varsigma)}\right] = E^\infty \frac{1 - \bar{\alpha}\bar{\varsigma}^{-2}}{1 - m\bar{\varsigma}^{-2}} = E^\infty \frac{\bar{\varsigma}^2 - \bar{\alpha}}{\bar{\varsigma}^2 - m} = \frac{E^\infty - \bar{E}^\infty\bar{\varsigma}^{-2}}{1 - m\bar{\varsigma}^{-2}} \\
&z = \omega(\varsigma) = R(\varsigma + m/\varsigma); \quad R = (a+b)/2, \quad m = (a-b)/(a+b)
\end{aligned} \tag{5.20}$$

$w(\varsigma)$ expressed by Eq. (5.20) satisfies the boundary at infinity and on the interface. In fact on the interface, we have

$$E_n + \mathrm{i}E_t = \frac{|\mathrm{d}z|}{\mathrm{d}z}(E_1 + \mathrm{i}E_2) = \frac{\bar{\sigma}}{|\sigma|}\frac{\overline{\omega'(\sigma)}}{|\omega'(\sigma)|}\frac{E^\infty - \bar{E}^\infty\sigma^2}{1 - m\sigma^2} = \frac{E^\infty\bar{\sigma} - \bar{E}^\infty\sigma}{|(1 - m\bar{\sigma}^2)|}$$
$$E_n = 0, \quad E_t = 2E_0\sin(\beta - \vartheta)\big/\left|1 - m\mathrm{e}^{-2\mathrm{i}\vartheta}\right|, \quad \Rightarrow \quad D_n = 0$$

Stress field The general solutions of complex potentials $\phi(\varsigma), \psi(\varsigma)$ can be assumed as

$$\phi(\varsigma) = \Gamma_1 R\varsigma + \phi_0(\varsigma), \quad \psi(\varsigma) = \Gamma_2 R\varsigma + \psi_0(\varsigma) \tag{5.21}$$

where $\phi_0(\varsigma),\ \psi_0(\varsigma)$ are undetermined functions analytic in S; Γ_1, Γ_2 are determined by

$$\Gamma_1 = \left(S_{22}^\infty + S_{11}^\infty - \kappa E_k^\infty E_k^\infty\right)/4, \quad \Gamma_2 = \left(S_{22}^\infty - S_{11}^\infty + 2iS_{12}^\infty\right)/2 \tag{5.22}$$

Because the boundary of the hole is free, the boundary condition Eq. (5.18) becomes

$$\omega(\sigma)\overline{\phi'(\sigma)}\Big/\overline{\omega'(\sigma)} + \phi(\sigma) + \overline{\psi(\sigma)} + \kappa w(\sigma)\overline{w'(\sigma)}\Big/2\overline{\omega'(\sigma)} = 0 \tag{5.23}$$

Substitution of Eqs. (5.20) and (5.21) into Eq. (5.23) yields

$$\begin{aligned} &\omega(\sigma)\overline{\phi_0'(\sigma)}\Big/\overline{\omega'(\sigma)} + \phi_0(\sigma) + \overline{\psi_0(\sigma)} + f(\sigma) = 0 \\ &f(\sigma) = R\Gamma_1\frac{\sigma^2 + m}{\sigma(1 - m\sigma^2)} + R\Gamma_1\sigma + \frac{R\bar{\Gamma}_2}{\sigma} + \frac{\kappa RE^\infty\bar{E}^\infty(1 - \bar{\alpha}\sigma^2)(\alpha + \sigma^2)}{2\sigma(1 - m\sigma^2)} \end{aligned} \tag{5.24}$$

Multiplying Eq. (5.24) and its conjugate equation by $\mathrm{d}\sigma/[2\pi\mathrm{i}(\sigma - \varsigma)]$ and using the Cauchy integral formulas we find

$$\begin{aligned} \phi_0(\varsigma) &= \frac{1}{2\pi\mathrm{i}}\int\frac{f(\sigma)\mathrm{d}\sigma}{\sigma - \varsigma} = -\frac{mR\Gamma_1}{\varsigma} - \frac{R\bar{\Gamma}_2}{\varsigma} - \frac{\kappa\alpha RE^\infty\bar{E}^\infty}{2\varsigma} \\ \psi_0(\varsigma) &= \frac{1}{2\pi\mathrm{i}}\int\frac{\overline{f(\sigma)}d\sigma}{\sigma - \varsigma} - \varsigma\frac{1 + m\varsigma^2}{\varsigma^2 - m}\phi_0'(\varsigma) \\ &= -2R\Gamma_1\frac{(1 + m^2)\varsigma}{\varsigma^2 - m} - \frac{R\bar{\Gamma}_2(1 + m\varsigma^2)}{(\varsigma^2 - m)\varsigma} - \frac{\kappa RE^\infty\bar{E}^\infty[1 - \alpha\bar{\alpha} + (\alpha + \bar{\alpha})m]\varsigma}{2(\varsigma^2 - m)} \end{aligned} \tag{5.25}$$

Substitution of Eqs. (5.21), (5.22), and (5.25) into Eq. (5.16) yields the stresses

$$\begin{aligned} \sigma_{22} + \sigma_{11} &= \kappa\frac{w'(\varsigma)}{\omega'(\varsigma)}\frac{\overline{w'(\varsigma)}}{\overline{\omega'(\varsigma)}} + 2\left[\frac{\phi'(\varsigma)}{\omega'(\varsigma)} + \frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}\right] \\ &= \kappa E^\infty\bar{E}^\infty\frac{\varsigma^2 - \alpha}{\varsigma^2 - m}\frac{\bar{\varsigma}^2 - \bar{\alpha}}{\bar{\varsigma}^2 - m} - 2\mathrm{Re}\left[\frac{\kappa\alpha E^\infty\bar{E}^\infty + 2\Gamma_1(\varsigma^2 + m) + 2\bar{\Gamma}_2}{\varsigma^2 - m}\right] \end{aligned} \tag{5.26a}$$

$$
\begin{aligned}
\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} &= \kappa\frac{\mathrm{d}}{\mathrm{d}\varsigma}\left[\frac{w'(\varsigma)}{\omega'(\varsigma)}\right]\overline{w(\varsigma)}+2\left\{\frac{\mathrm{d}}{\mathrm{d}\varsigma}\left[\frac{\phi'(\varsigma)}{\omega'(\varsigma)}\right]\overline{\omega'(\varsigma)}+\frac{\psi'(\varsigma)}{\omega'(\varsigma)}\right\}+\epsilon\left[\frac{w'(\varsigma)}{\omega'(\varsigma)}\right]^2\\
&= \frac{2\kappa E^\infty\bar{E}^\infty(\alpha-m)\varsigma^3(\bar{\varsigma}^2+\bar{\alpha})}{(\varsigma^2-m)^3\bar{\varsigma}}+2\left\{\frac{-(\kappa\alpha E^\infty\bar{E}^\infty+4m\Gamma_1+2\bar{\Gamma}_2)\varsigma^3(\bar{\varsigma}^2+m)}{(\varsigma^2-m)^3\bar{\varsigma}}\right.\\
&\quad+\frac{\Gamma_2\varsigma^2}{\varsigma^2-m}+\frac{\kappa E^\infty\bar{E}^\infty[1-\alpha\bar{\alpha}+(\alpha+\bar{\alpha})m]\varsigma^2(\varsigma^2+m)}{2(\varsigma^2-m)^3}+\frac{2\Gamma_1(1+m^2)\varsigma^2(\varsigma^2+m)}{(\varsigma^2-m)^3}\\
&\quad\left.+\frac{\bar{\Gamma}_2[m\varsigma^4+(m^2+3)\varsigma^2-m]}{(\varsigma^2-m)^3}\right\}+\frac{\epsilon(E^\infty)^2(\varsigma^2+1)^2}{(\varsigma^2-m)^2}
\end{aligned}
\tag{5.26b}
$$

Asymptotic fields near the end of a narrow elliptic hole under the electric load As in Sect. 3.4.6 the asymptotic stress fields near the end of a narrow elliptic hole only under an electric load in the local coordinate system with the origin at the focus of the ellipse are

$$
\begin{aligned}
&\sigma_{22}+\sigma_{11}\approx\kappa E^\infty\bar{E}^\infty(1-\alpha)(1-\bar{\alpha})c/4r\\
&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12}\\
&\quad\approx\kappa E^\infty\bar{E}^\infty\left\{(1-\alpha)[2(\epsilon/\kappa)(1-\alpha)/\alpha-(1-\bar{\alpha})]+2(1-\alpha)(1-\bar{\alpha})\sqrt{\rho_0/r}\right\}c/8r
\end{aligned}
\tag{5.27a}
$$

The electric asymptotic field is

$$
E_1+\mathrm{i}E_2=(1/4)\left\{\sqrt{2}\bar{E}^\infty(1-\bar{\alpha})\sqrt{c/r}+E^\infty\left((3+\bar{\alpha})+4\bar{\alpha}\sqrt{\rho_0/r}\right)\right. \tag{5.27b}
$$

where $\rho_0=b^2/2a$, $c=\sqrt{a^2-b^2}$.

5.1.3 The Permeable Elliptic Hole

For a permeable elliptic hole, the electric connective conditions in Eq. (5.19) are changed to

$$
\varphi=\varphi^{\mathrm{c}},\quad D_n=D_n^{\mathrm{c}}\quad or\quad \int D_n\mathrm{d}s=\int D_n^{\mathrm{c}}\mathrm{d}s,\quad D_n=D_in_i \tag{5.28}
$$

According to the previous knowledge, it is assumed prior that the electric field in the air is constant (Smith and Warren 1966, 1968; Gao et al. 2010), and the complex electric potential in the media $w(z)$ is in the following form:

$$
\begin{aligned}
&\varphi^{\mathrm{c}}=\mathrm{Re}w^{\mathrm{c}}(z)=-E_1^{\mathrm{c}}x_1-E_2^{\mathrm{c}}x_2\\
&w(z)=\Gamma_3z+w_0(z),\quad \Gamma_3=-\overline{E}^\infty=-\left(E_1^\infty-\mathrm{i}E_2^\infty\right)
\end{aligned}
\tag{5.29}
$$

where $w_0(z)$ is an unknown function analytic in S. Substituting Eqs. (5.5), (5.6), and (5.29) into Eq. (5.28) and using Eq. (5.7) we get

$$\begin{aligned} w_0(\sigma) + \overline{w_0(\sigma)} &= 2\left[\left(E_1^\infty - E_1^{\rm c}\right)x_1 + \left(E_2^\infty - E_2^{\rm c}\right)x_2\right] \\ w_0(\sigma) - \overline{w_0(\sigma)} &= -2{\rm i}\left[\left(D_2^\infty - D_2^{\rm c}\right)x_1 - \left(D_1^\infty - D_1^{\rm c}\right)x_2\right]/\epsilon \end{aligned} \tag{5.30}$$

Substituting $x_1 = a(\sigma + \sigma^{-1})/2,\ x_2 = {\rm i}b(\sigma - \sigma^{-1})/2$ and multiplying ${\rm d}\sigma/[2\pi{\rm i}(\sigma - \varsigma)]$ to two sides of Eq. (5.30) and then integrating the result identity we get

$$\begin{aligned} w_0(\varsigma) &= \left[a\left(E_1^\infty - E_1^{\rm c}\right) + {\rm i}b\left(E_2^\infty - E_2^{\rm c}\right)\right]/\varsigma \\ w_0(\varsigma) &= \left[-{\rm i}a\left(D_2^\infty - D_2^{\rm c}\right) - b\left(D_1^\infty - D_1^{\rm c}\right)\right]/\epsilon\varsigma \end{aligned} \tag{5.31}$$

Equation (5.31) yields

$$a\left(E_1^\infty - E_1^{\rm c}\right) = -b\left(D_1^\infty - D_1^{\rm c}\right)/\epsilon, \quad b\left(E_2^\infty - E_2^{\rm c}\right) = -a\left(D_2^\infty - D_2^{\rm c}\right)/\epsilon \tag{5.32}$$

So, using $D_1^\infty = \epsilon E_1^\infty, D_1^c = \epsilon^c E_1^c$ we obtain

$$\begin{aligned} &D_1^{\rm c} = D_1^\infty \bar{\epsilon}^{\rm c}(1+\bar{b})(1+\bar{\epsilon}^{\rm c}\bar{b})^{-1}, \quad D_2^{\rm c} = D_2^\infty \bar{\epsilon}^{\rm c}(1+\bar{b})(\bar{\epsilon}^{\rm c}+\bar{b})^{-1} \\ &w^{\rm c}(z) = (1+\bar{b})\left[-E_1^\infty(1+\bar{\epsilon}^{\rm c}\bar{b})^{-1} + {\rm i}E_2^\infty(\bar{\epsilon}^{\rm c}+\bar{b})^{-1}\right]z, \quad \sqrt{m} \le |\varsigma| \le 1 \\ &w(\varsigma) = -RE_1^\infty(\varsigma + A/\varsigma) + {\rm i}RE_2^\infty(\varsigma - B/\varsigma), \qquad |\varsigma| \ge 1 \\ &\bar{b} = b/a, \quad \bar{\epsilon}^{\rm c} = \epsilon^{\rm c}/\epsilon, \quad A = (1-\bar{b}\bar{\epsilon}^{\rm c})/(1+\bar{b}\bar{\epsilon}^{\rm c}), \quad B = (\bar{b}-\bar{\epsilon}^{\rm c})/(\bar{b}+\bar{\epsilon}^{\rm c}) \end{aligned} \tag{5.33}$$

Especially for a crack ($b = 0$) we have

$$E_1^\infty = E_1^{\rm c}, \quad D_2^\infty = D_2^{\rm c}, \quad w^{\rm c}(z) = \left(-E_1^\infty + {\rm i}E_2^\infty\epsilon/\epsilon^{\rm c}\right)z, \quad w(z) = \left(-E_1^\infty + {\rm i}E_2^\infty\right)z$$

It means that the electric fields are homogeneous in the crack, but with different constant values. When $E_1^\infty = 0$, the electric asymptotic field near the right crack tip is

$$E_2 = E_2^\infty \frac{\varsigma^2 + (1-\bar{\delta})/(1+\bar{\delta})}{\varsigma^2 - m} \approx E_2^\infty\left[\frac{1}{1+\bar{\delta}}\sqrt{\frac{a}{2r}}e^{-{\rm i}\theta/2} + \frac{1+2\bar{\delta}}{2(1+\bar{\delta})}\right] \tag{5.34}$$

where $\bar{\delta} = \bar{\epsilon}^{\rm c}/\bar{b} = (\epsilon^{\rm c}a)/(\epsilon b)$ is an important parameter; r, θ are polar coordinates in the local coordinate system with the origin at the focus of the ellipse.

The complex stress functions are still expressed by Eq. (5.21). For $E_1^\infty = 0$ finally we find

$$\begin{aligned}
\phi(\varsigma) &= R\left\{\Gamma_1\varsigma - \frac{m\Gamma_1}{\varsigma} - \frac{\bar{\Gamma}_2}{\varsigma} + \frac{\kappa B E_2^{\infty 2}}{2\varsigma}\right\} \\
\psi(\varsigma) &= R\left\{\Gamma_2\varsigma - 2\Gamma_1\frac{1+m^2}{\varsigma^2 - m}\varsigma - \frac{\bar{\Gamma}_2(1+m\varsigma^2)}{(\varsigma^2-m)\varsigma} - \frac{\kappa E_2^{\infty 2}(1-2mB-B^2)\varsigma}{2(\varsigma^2-m)}\right\}
\end{aligned} \tag{5.35}$$

The stress is obtained by Eq. (5.16).

5.1.4 A Rigid Elliptic Conduction Inclusion

In this section we shall discuss a rigid elliptic conducting inclusion with boundary L in an isotropic electrostrictive material (Jiang and Kuang 2004). In this case the problem can be discussed independently in the material region Ω and the boundary conditions are assumed:

$$\begin{aligned}
&\sigma_{ij} = \sigma_{ij}^\infty, \quad E = E_1 + \mathrm{i}E_2 = E^\infty = E_0 \mathrm{e}^{\mathrm{i}\beta}, \quad \text{when} \quad x_1^2 + x_2^2 \to \infty \\
&u_1 = u_1^{\mathrm{c}} = -\omega^{\mathrm{c}} x_2, \quad u_2 = u_2^{\mathrm{c}} = \omega^{\mathrm{c}} x_1; \quad \varphi = 0; \quad \text{on} \quad L \\
&E_0 = \sqrt{\left(E_1^\infty\right)^2 + \left(E_2^\infty\right)^2}, \quad \tan\beta = E_2^\infty / E_1^\infty
\end{aligned} \tag{5.36}$$

where ω^{c} is the rotation angle about axis x_3 of the inclusion. The pseudo total moment $\tilde{M}$, Maxwell stress moment M^e, and mechanical moment M are, respectively,

$$\begin{aligned}
\tilde{M} &= \int_A^B \left(-\tilde{T}_1 x_2 + \tilde{T}_2 x_1\right)\mathrm{d}s = -\left[z\left(\partial\tilde{U}/\partial z\right) + \bar{z}\left(\partial\tilde{U}/\partial\bar{z}\right)\right]_A^B + \tilde{U}_A^B \\
&= \mathrm{Re}\left[\chi(z) - z\psi(z) - z\bar{z}\phi'(z) - (1/2)\kappa w'(z)\overline{w(z)}\right]_A^B + \left[(1/4)\kappa w(z)\overline{w(z)}\right]_A^B \\
M^e &= \int_A^B \left(-\sigma_{1j}^M n_j x_2 + \sigma_{2j}^M n_j x_1\right)\mathrm{d}s = \mathrm{Re}\{(1/2)\epsilon[z\Omega(z) - \Omega_1(z)]\} \\
M &= \tilde{M} - M^e = \mathrm{Re}\Big\{\chi(z) - z\psi(z) - z\bar{z}\phi'(z) - (1/2)\kappa w'(z)\overline{w(z)} \\
&\quad -(1/2)\epsilon[z\Omega(z) - \Omega_1(z)]\Big\}_A^B + \left[(1/4)\kappa w(z)\overline{w(z)}\right]_A^B, \quad \Omega_1(z) = \int \Omega(z)\mathrm{d}z
\end{aligned} \tag{5.37}$$

When there are no body force and free charge, the stress complex potential can be assumed as

$$\begin{aligned}
&w(z) = \Gamma_3 z + w_0(z)\\
&\phi(z) = -[1/8(1-\nu)](\tilde{T}_1 + \mathrm{i}\tilde{T}_2)\ln z + \Gamma_1 z + \phi_0(z)\\
&\psi(z) = [(3-4\nu)/8(1-\nu)]\left(\tilde{T}_1 - \mathrm{i}\tilde{T}_2\right)\ln z + \Gamma_2 z + \psi_0(z)\\
&\Gamma_3 = -\bar{E}^\infty, \quad E^\infty = \left(E_1^\infty + \mathrm{i}E_2^\infty\right) = E_0 \mathrm{e}^{\mathrm{i}\beta}\\
&\Gamma_1 = (1/4)\left(S_{22}^\infty + S_{11}^\infty\right) - (1/4)\kappa E_k^\infty E_k^\infty, \quad \Gamma_2 = (1/2)\left(S_{22}^\infty - S_{11}^\infty + 2\mathrm{i}S_{12}^\infty\right)
\end{aligned} \tag{5.38}$$

where $w_0(z), \phi_0(z), \psi_0(z)$ are complex functions analytic in the region S. $\tilde{T}_i$ is the generalized concentrate force, which is zero in present case, so the terms containing $\ln z$ will be omitted in later.

The conformal mapping method is used to solve the problem. The mapping function is shown in Eq. (3.82). It is easy to prove that the electric field in S can be obtained by changing α to $(-\alpha)$ in Eq. (5.20) discussed in Sect. 5.1.2, i.e.,

$$\begin{aligned}
&w(\varsigma) = -R\bar{E}^\infty\left(\varsigma - \alpha\varsigma^{-1}\right), \quad \overline{w(\varsigma)} = -RE^\infty\left(\bar{\varsigma} - \alpha\bar{\varsigma}^{-1}\right), \quad \alpha = E^\infty/\bar{E}^\infty = \mathrm{e}^{2\mathrm{i}\beta}\\
&E = E_1 + \mathrm{i}E_2 = -\frac{\overline{w'(\varsigma)}}{\overline{\omega'(\varsigma)}} = E^\infty\frac{1+\bar{\alpha}\bar{\varsigma}^{-2}}{1-m\bar{\varsigma}^{-2}} = E^\infty\frac{\bar{\varsigma}^2+\bar{\alpha}}{\bar{\varsigma}^2-m} = \frac{E^\infty + \bar{E}^\infty\bar{\varsigma}^{-2}}{1-m\bar{\varsigma}^{-2}}
\end{aligned} \tag{5.39}$$

Using Eq. (5.17) the displacement boundary condition in Eq. (5.36) can be expressed as

$$2\mathrm{i}G\omega^c z = K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - \frac{\kappa}{2}w(z)\overline{w'(z)} + \alpha_1\overline{\Omega(z)} \tag{5.40}$$

On the mapping plane Eq. (5.40) becomes

$$\Lambda\phi(\varsigma) + \omega(\varsigma)\frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}} + \overline{\psi(\varsigma)} + \frac{\kappa}{2}w(\varsigma)\frac{\overline{w'(\varsigma)}}{\overline{\omega'(\varsigma)}} - \alpha_1\overline{\Omega(\varsigma)} = -2\mathrm{i}G\omega^c\omega(\varsigma) \tag{5.41}$$

where $\Lambda = -K = -3 + 4\nu$ and $\phi(\varsigma)$ and $\psi(\varsigma)$ are given in Eq. (5.38). Noting

$$\begin{aligned}
&\Omega(\varsigma) = \int\frac{[w'(\varsigma)]^2}{\omega'(\varsigma)}\mathrm{d}\varsigma = R(\bar{E}^\infty)^2\left[\frac{\alpha^2}{m\varsigma} + \varsigma - \frac{(m+\alpha)^2\arctan(\varsigma/\sqrt{m})}{m^{3/2}}\right]\\
&\frac{1}{2\pi\mathrm{i}}\int\frac{\Omega(\sigma)\mathrm{d}\sigma}{\sigma-\varsigma} = R\left(\overline{E}^\infty\right)^2\left[-\frac{\alpha^2}{m\varsigma} - \frac{(m+\alpha)^2}{2m^{3/2}}\ln\frac{\varsigma-\sqrt{m}}{\varsigma+\sqrt{m}}\right]\\
&\frac{1}{2\pi\mathrm{i}}\int\frac{\overline{\Omega(\sigma)}\mathrm{d}\sigma}{\sigma-\varsigma} = -R\left(\overline{E}^\infty\right)^2\frac{1}{\varsigma}
\end{aligned} \tag{5.42}$$

The future process to solve the problem is fully similar to that in Sect. 5.1.2. Finally we obtain

$$
\begin{aligned}
\Lambda\phi(\varsigma) &= \Lambda\Gamma_1 R\varsigma - \frac{mR\Gamma_1}{\varsigma} - \frac{R\bar{\Gamma}_2}{\varsigma} + \frac{\kappa\alpha RE^{\infty}\bar{E}^{\infty}}{2\varsigma} + \alpha_1 R(E^{\infty})^2\frac{1}{\varsigma} - 2\mathrm{i}RG\omega^{\mathrm{c}}\frac{m}{\varsigma} \\
\psi(\varsigma) &= \Gamma_2 R\varsigma + \frac{\kappa RE^{\infty}\bar{E}^{\infty}[\alpha\bar{\alpha} - 1 + (\alpha/\Lambda + \bar{\alpha})m]\varsigma^2 + \alpha/\Lambda - \alpha}{2\varsigma(\varsigma^2 - m)} - \frac{R\bar{\Gamma}_2(1 + m\varsigma^2)}{\Lambda(\varsigma^2 - m)\varsigma} \\
&\quad - \frac{R\Gamma_1[(1 + m^2 + \Lambda + m^2/\Lambda)]\varsigma^2 - \Lambda m + m/\Lambda}{\varsigma(\varsigma^2 - m)} + \frac{\alpha_1 R(E^{\infty})^2(1 + m\varsigma^2)}{\Lambda\varsigma(\varsigma^2 - m)} \\
&\quad + \frac{2\mathrm{i}RG\omega^{\mathrm{c}}}{\varsigma} - \frac{2\mathrm{i}RG\omega^{\mathrm{c}}(1 + m\varsigma^2)}{\Lambda\varsigma(\varsigma^2 - m)} + \alpha_1 R(\bar{E}^{\infty})^2\left[\frac{\alpha^2}{m\varsigma} + \frac{(m + \alpha)^2}{2m^{3/2}}\ln\frac{\varsigma - \sqrt{m}}{\varsigma + \sqrt{m}}\right]
\end{aligned}
\tag{5.43}
$$

If ω^{c} is given, the mechanical moment acting on the inclusion can be determined by Eq. (5.37), or

$$
\begin{aligned}
M = \mathrm{Re}\Big\{&\chi(\varsigma) - \omega(\varsigma)\psi(\varsigma) - \omega(\varsigma)\overline{\omega(\varsigma)}\phi'(\varsigma)/\omega'(\varsigma) - (1/2)\kappa w'(\varsigma)/\omega'(\varsigma)\overline{w(\varsigma)} \\
&-(1/2)\epsilon[\omega(\varsigma)\Omega(\varsigma) - \Omega_1(\varsigma)]\Big\}_A^B + \Big[(1/4)\kappa w(\varsigma)\overline{w(\varsigma)}\Big]_A^B
\end{aligned}
\tag{5.44}
$$

In Eq. (5.44) points A and B are the same point, so only multiple value terms containing $\ln\varsigma$ are not zero, i.e., only should keep terms containing $\chi(\varsigma)$ and $\Omega_1(\varsigma)$. From the second equation in Eq. (5.43) we get

$$
\begin{aligned}
\chi(\varsigma) &= \oint \psi(\varsigma)\omega'(\varsigma)\mathrm{d}\varsigma \\
&= \Gamma_2 R^2\left(\frac{1}{2}\varsigma^2 - m\ln\varsigma\right) + \frac{1}{2}\kappa R^2 E^{\infty}\bar{E}^{\infty}\left[\left(\alpha\bar{\alpha} - 1 + \frac{\alpha m}{\Lambda} + \bar{\alpha}m\right)\ln\varsigma + \frac{\alpha\Lambda - \alpha}{2\Lambda\varsigma^2}\right] \\
&\quad - \frac{R^2\bar{\Gamma}_2}{\Lambda}\left(m\ln\varsigma - \frac{1}{2\varsigma^2}\right) - R^2\Gamma_1\left[\left(1 + m^2 + \Lambda + \frac{m^2}{\Lambda}\right)\ln\varsigma - \left(-\Lambda m + \frac{m}{\Lambda}\right)\frac{1}{2\varsigma^2}\right] \\
&\quad + \frac{\alpha_1 R^2(E^{\infty})^2}{\Lambda}\left(m\ln\varsigma - \frac{1}{2\varsigma^2}\right) + \alpha_1 R^2(E^{\infty})^2\left\{\alpha^2\left(\frac{1}{2\varsigma^2} + \frac{\ln\varsigma}{m}\right)\right. \\
&\quad \left. + \frac{(m + \alpha)^2}{2m^{3/2}}\left[2\sqrt{m}\ln\varsigma + \frac{m + \varsigma^2}{\varsigma}\ln\frac{\varsigma - \sqrt{m}}{\varsigma + \sqrt{m}} - 2\sqrt{m}\ln(\varsigma^2 - m)\right]\right\} \\
&\quad + 2\mathrm{i}R^2G\omega^{\mathrm{c}}\left(\ln\varsigma + \frac{m}{2\varsigma^2}\right) - \frac{2\mathrm{i}R^2G\omega^{\mathrm{c}}m}{\Lambda}\left(m\ln\varsigma - \frac{1}{2\varsigma^2}\right)
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
\Omega_1(\varsigma) &= \int \Omega(\varsigma)\omega'(\varsigma)\mathrm{d}\varsigma \\
&= \left(R\overline{E}^{\infty}\right)^2\left[\frac{\alpha^2}{2\varsigma}+\frac{\varsigma^2}{2}-\left(\frac{\alpha^2}{m}-m\right)\ln\varsigma\right] \\
&\quad -\left(R\overline{E}^{\infty}\right)^2\frac{(m+\alpha)^2}{2m^{3/2}}\left\{2\sqrt{m}\left[-\ln\varsigma+\ln\left(\varsigma^2-m\right)+\frac{m+\varsigma^2}{\varsigma}\ln\frac{\varsigma+\sqrt{m}}{-\varsigma+\sqrt{m}}\right]\right\}
\end{aligned} \tag{5.46}
$$

So we have

$$
\begin{aligned}
M =& -2\pi R^2\mathrm{Im}\{-\Gamma_2 m+(1/2)\kappa E^{\infty}\bar{E}^{\infty}(\alpha\bar{\alpha}-1+\alpha m/\Lambda+\bar{\alpha}m) \\
& -\Gamma_1\left(1+m^2+\Lambda+m^2/\Lambda\right)-m\bar{\Gamma}_2/\Lambda+(E^{\infty})^2\alpha_1 m/\Lambda \\
& -[\alpha_1(2\alpha+m)+\epsilon(\alpha+m)](\bar{E}^{\infty})^2+2\mathrm{i}G\omega^{\mathrm{c}}\left(\Lambda-m^2\right)/\Lambda\}
\end{aligned} \tag{5.47}
$$

Noting $\Gamma_1, m, \Lambda, G, \omega^{\mathrm{c}}$ are all real, Eq. (5.47) can be reduced to

$$
\begin{aligned}
M =& -2\pi R^2\mathrm{Im}\{-\Gamma_2 m+(1/2)\kappa E^{\infty}\bar{E}^{\infty}(\alpha\bar{\alpha}-1+\alpha m/\Lambda+\bar{\alpha}m)-m\bar{\Gamma}_2/\Lambda \\
& +\alpha_1 m(E^{\infty})^2/\Lambda-[\alpha_1(2\alpha+m)+\in(\alpha+m)](\bar{E}^{\infty})^2\}-4\pi R^2 G\omega^{\mathrm{c}}\left(\Lambda-m^2\right)/\Lambda
\end{aligned} \tag{5.48}
$$

If there is no moment acting on the inclusion, the ω^{c} is determined by the following equation:

$$
\begin{aligned}
\omega^{\mathrm{c}} =& \left[\Lambda/2G\left(m^2-\Lambda\right)\right]\mathrm{Im}\{-\Gamma_2 m+\frac{1}{2}\kappa E^{\infty}\bar{E}^{\infty}(\alpha\bar{\alpha}-1+\alpha m/\Lambda+\bar{\alpha}m)-m\bar{\Gamma}_2/\Lambda \\
& +\alpha_1 m(E^{\infty})^2/\Lambda-[\alpha_1(2\alpha+m)+\in(\alpha+m)](\bar{E}^{\infty})^2\}
\end{aligned} \tag{5.49}
$$

For a conductor ball $m=0$, from Eq. (5.49), it is seen that $\omega^{\mathrm{c}}=0$, i.e., there is no rotation. It is also noted that for $\beta=n\pi/2, n=1,2,3,4$, $\omega^{\mathrm{c}}=0$ for pure electric loading. ω^{c} is proportional to the square of the electric field and linear of the stress at infinity. Substituting ω^{c} into Eq. (5.43), the stress potentials are obtained and then the stresses are all obtained. The asymptotic field near the right end of a narrow rigid elliptic inclusion under an electric field at infinity is

$$
\begin{aligned}
\sigma_{22}+\sigma_{11} &= (1/8)[\kappa E^{\infty}\bar{E}^{\infty}(1+\alpha)(1+\bar{\alpha})](c/r) \\
\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} &= \Big\{(1/8)(1+\alpha)^2\bar{E}^{\infty}[(2\alpha_1+\epsilon)\bar{E}^{\infty}-\kappa E^{\infty}] \\
&\qquad +(1/4)\kappa E^{\infty}\bar{E}^{\infty}(1+\alpha)(1+\bar{\alpha})\sqrt{\rho_0/r}\Big\}(c/r) \\
E_1+\mathrm{i}E_2 &= \bar{E}^{\infty}\left(1/2m^{3/4}\right)\sqrt{R/r}
\end{aligned} \tag{5.50}
$$

Jiang and Kuang (2005, 2007) discussed a general elliptic inclusion. Liang et al. (1995) discussed piezoelectric materials with a general elliptic inclusion.

5.2 Cracked Infinite Electrostrictive Plate with Local Saturation Electric Field

5.2.1 The Constitutive Equations and Boundary Conditions

For an electrostrictive ceramic with a crack under external high electric field, the mechanical state near the crack tip is elastic, but the electric field may be saturated. Jiang and Kuang (2006) discussed an infinite plate with a central crack of length $2a$, subjected to the electric field $E^{\infty} = E_1^{\infty} + iE_2^{\infty}$ at infinity. It is assumed that the electric field in the region S_0 of the plate is linear, but two zones S_R and S_L near the right and left crack tips are local small-scale saturated (Fig. 5.2). The constitutive equations for an isotropic electrostrictive material are

$$\begin{aligned} &\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2G\varepsilon_{ij} - \left(a_1 D_i D_j + a_2 D_k D_k \delta_{ij}\right)/\left(2\hat{\epsilon}^2\right) \\ &D = \hat{\epsilon}E, \quad \hat{\epsilon} = D(E)/E \end{aligned} \tag{5.51a}$$

where $D(E)$ is the uniaxial dielectric response in the absence of stress. Here it is assumed

$$\begin{aligned} &D_i = \left(\epsilon\delta_{ij} + a_1\varepsilon_{ij} + a_2\varepsilon_{kk}\delta_{ij}\right)E_j, \quad \text{when} \quad |E| = \sqrt{E_k E_k} < E_c \\ &D_i = D_c E_i/|E|, \quad \text{when} \quad |E| = \sqrt{E_k E_k} \geq E_c \end{aligned} \tag{5.51b}$$

where D_c and E_c are the saturation electric displacement and saturation electric field, respectively. For linear case $\hat{\epsilon} = \epsilon$ is constant, but for the nonlinear case $\hat{\epsilon}$ may be dependent to electric field. If the electric field is linear, $\boldsymbol{\sigma}$ in Eq. (5.51a) can also be expressed by

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2G\varepsilon_{ij} - (a_1 E_i E_j + a_2 E_k E_k \delta_{ij})/2 \tag{5.51c}$$

The boundary condition of the problem is

$$\begin{aligned} &\sigma_{ij} = \sigma_{ij}^{\infty}, \quad E = E_1^{\infty} + iE_2^{\infty} = E_0 e^{i\beta}, \quad \text{when} \quad x_k x_k \to \infty \\ &D_2 = 0, \quad \text{on} \quad x_2 = 0, \quad -a < x_1 < a \end{aligned} \tag{5.52}$$

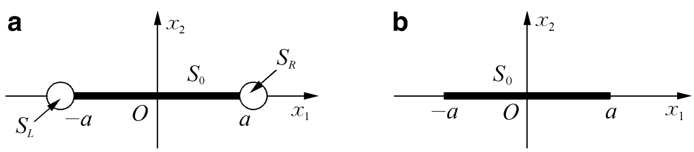

Fig. 5.2 An infinite plane with a *central crack* located at $(-a, a)$: (**a**) local small-scale saturation model at crack tips; (**b**) linear model

5.2.2 *The Electric Field in an Electrostrictive Material with an Impermeable Crack*

1. *The electric field in a linear plate without local saturation region*

According to Eq. (5.6) and approximately taking $\boldsymbol{D} = \epsilon \boldsymbol{E}$ we have

$$D_1 = -(1/2)\ \epsilon\left[\overline{w'(z)} + w'(z)\right], \quad D_2 = (1/2)\mathrm{i}\ \epsilon\left[\overline{w'(z)} - w'(z)\right] \tag{5.53}$$

where $w(z)$ is a complex potential shown in Eq. (5.5). On the crack surface

$$D_2^+ = D_2^- = 0, \quad \text{or} \quad \bar{w}'^+(x_1) - w'^+(x_1) = \bar{w}'^-(x_1) - w'^-(x_1) = 0 \tag{5.54}$$

Equation (5.54) yields

$$\begin{aligned} &[\bar{w}'(x_1) - w'(x_1)]^+ + [\bar{w}'(x_1) - w'(x_1)]^- = 0, \\ &[\bar{w}'(x_1) - w'(x_1)]^+ - [\bar{w}'(x_1) - w'(x_1)]^- = 0 \end{aligned} \tag{5.55}$$

This is a standard Hilbert problem. Noting Eq. (5.52) its solution is

$$\begin{aligned} &w'(z) = \frac{1}{2}(\Gamma_3 - \bar{\Gamma}_3)\frac{z}{\sqrt{z^2 - a^2}} + \frac{1}{2}(\Gamma_3 + \bar{\Gamma}_3) = \mathrm{i}E_2^\infty \frac{z}{\sqrt{z^2 - a^2}} - E_1^\infty \\ &w(z) = \mathrm{i}E_2^\infty\sqrt{z^2 - a^2} - E_1^\infty z; \quad \Gamma_3 = -\bar{E}^\infty \end{aligned} \tag{5.56}$$

The asymptotic field near the crack tip $z = a$ is

$$w'(z) = \frac{\mathrm{i}K_\mathrm{e}}{\sqrt{2\pi(z-a)}}, \quad E_1 - \mathrm{i}E_2 = -\frac{\mathrm{i}K_\mathrm{e}}{\sqrt{2\pi(z-a)}} = -\frac{\mathrm{i}K_\mathrm{e}}{\sqrt{2\pi r}}\mathrm{e}^{-\mathrm{i}\theta/2} \tag{5.57}$$

where $K_\mathrm{e} = E_2^\infty\sqrt{\pi a}$ is the electric field intensity factor, $z - a = re^{i\theta}$.

2. *The electric field in a plate with local saturation region*

The local saturation model of the electric field at the crack tip is similar to III-type yielding model in an elastoplastic material, so the method used in elastoplastic analysis can also be used here (Cherepanov 1979). The asymptotic solution near a tip of a central crack is the same as that in a semi-infinite crack problem. A local coordinate system Oy_i with the origin located at the crack tip (Fig. 5.3) is also used. A point in it is denoted by $y = y_1 + \mathrm{i}y_2 = z - a$. The boundary value problem is

$$\begin{aligned} &D_{i,i} = 0, \quad \text{when} \quad y \notin (y_2 = 0, -\infty < y_1 \le 0) \\ &D_2^\pm = 0, \quad \text{when} \quad y_2 = 0, \quad -\infty < y_1 \le 0 \\ &\sqrt{D_1^2 + D_2^2} = D_\mathrm{c}, \quad \text{when} \quad y \in S_\mathrm{R}; \quad D_1^2 + D_2^2 = 0, \quad \text{when} \quad y \to \infty \end{aligned} \tag{5.58}$$

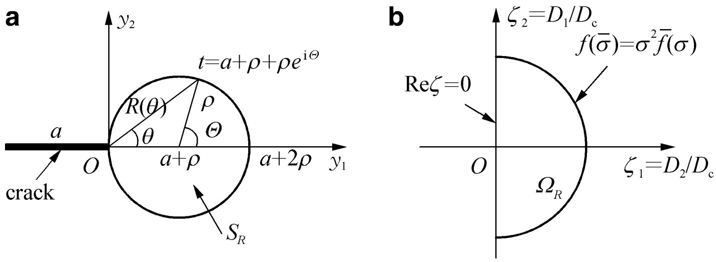

Fig. 5.3 The local saturation zone near crack tip: (**a**) physical plane z; (**b**) mapping plane

where the origin O is not included in S_R. Let

$$\varsigma = \varsigma_1 + i\varsigma_2 = (D_2 + iD_1)/D_c, \quad \text{or} \quad \varsigma = (E_2 + iE_1)/E_c = -iw'(z)/E_c \quad (5.59)$$

According to Eqs. (5.57) and (5.58), it yields (Fig. 5.3)

$$D_1 = -D_c \sin\theta, \quad D_2 = D_c \cos\theta, \quad D_2 + iD_1 = D_c e^{-i\theta}, \quad |\theta| \le \pi/2; \quad \text{in} \quad S_R \quad (5.60)$$

According to Eqs. (5.58) and (5.60), the crack boundary $y_2 = 0,\ y_1 < 0$ in the y plane is transformed to $\theta = \pm\pi/2$ in the ς plane. Let $R(\theta)$ be the boundary of the saturation zone S_R; a point t on the boundary of S_R can be expressed as

$$t = R(\theta)e^{i\theta}; \quad \tan\theta = y_2/y_1, \quad y = y_1 + iy_2 \quad (5.61)$$

According to Eq. (5.60) in the ς plane, the boundary of Ω_R is $e^{-i\theta} = \bar{\sigma}$. In order to simplify the problem, the hodograph transform method is used. The boundary value problem in the ς plane is

$$\begin{array}{llll} y_2 = 0, & -\infty < y_1 \le 0, & \text{when} & \text{Re}\varsigma = 0 \\ y \in R, & & \text{when} & \varsigma = e^{-i\theta} \\ y \to \infty, & & \text{when} & \varsigma = 0 \end{array} \quad (5.62)$$

In the ς plane Eq. (5.62) shows that the zone Ω_R is constituted of a unit semicircle and a line segment $-1 \le \xi_2 \le 1$ on the image axis. The zone inside Ω_R is corresponding to the zone outside S_R. Now we shall solve the problem, Eq. (5.62), in the ς plane. Let

$$R(\theta) = \bar{\sigma} f(\bar{\sigma}) \quad (5.63)$$

where $f(\bar{\sigma})$ is an unknown function. Because $R(\theta)$ is real, so

$$\bar{\sigma} f(\bar{\sigma}) - \sigma \bar{f}(\sigma) = 0, \quad \text{or} \quad f(\bar{\sigma}) = \sigma^2 \bar{f}(\sigma) \quad (5.64)$$

It is considered that the linear asymptotic solution Eq. (5.57) can approximately be used in the present problem, i.e., outside S_R the following relation is held:

$$y = f(\varsigma) = -K_e^2 \big/ 2\pi(E_1 - iE_2)^2 = K_e^2 \big/ 2\pi(E_c\varsigma)^2, \quad (E_c\varsigma)^2 = -(E_1 - iE_2)^2 \tag{5.65}$$

Equation (5.65) also satisfies the condition, $\varsigma = 0$, when $y \to \infty$.

From Eqs. (5.64) and (5.65), it is derived that outside the saturation zone we have

$$y = f(\varsigma) = \frac{K_e^2}{2\pi E_c^2}\left(1 + \varsigma^{-2}\right), \quad w'(z) = \frac{iK_e}{\sqrt{2\pi(z-a) - (K_e/E_c)^2}} \tag{5.66}$$

The boundary of the saturation zone S_R in the ς plane is

$$R(\theta) = \bar{\sigma} f(\bar{\sigma}) = \left(K_e^2/2\pi E_c^2\right)\bar{\sigma}\left(1 + \bar{\sigma}^{-2}\right) = 2\rho\cos\theta; \quad \rho = \left(K_e^2/2\pi E_c^2\right) \tag{5.67}$$

Equation (5.67) shows that the saturation zone S_R in the y plane is a circle with radius ρ. Equation (5.67) can also be obtained if in Eq. (5.57) let $E_1^2 + E_2^2 = E_c^2$.

From Eq. (5.66) it is found that the linear field in S_0 for a material with a saturation zone near the tip is the same as that in a material without a saturation zone, if we use the effective crack length a_{eff} instead of the real crack length a. It is just the method used in the elastoplastic fracture mechanics. The effective crack length is

$$a_{\text{eff}} = a + \rho, \quad \rho = K_e^2/2\pi E_c^2, \; K_e = \left(E_2^\infty\right)\sqrt{\pi(a+\rho)} \tag{5.68}$$

Using the above theory the electric field in S_0 for a central crack problem is

$$\begin{aligned} w(z) &= iE_2^\infty\sqrt{z^2 - (a+\rho)^2} - E_1^\infty z \\ E_2 + iE_1 &= E_2^\infty \frac{z}{\sqrt{z^2 - (a+\rho)^2}} - E_1^\infty \approx \frac{K_e}{\sqrt{\pi(a+\rho)}} \frac{z}{\sqrt{z^2 - (a+\rho)^2}} \end{aligned} \tag{5.69}$$

On the boundary of S_R we have $z = a + \rho + \rho e^{i\Theta}(\Theta = 2\theta)$ (Fig. 5.3a). Substituting it into Eq. (5.69) yields

$$E_2 + iE_1 = \frac{E_2^\infty\left(a + \rho + \rho e^{i\Theta}\right)}{\sqrt{(2a + 2\rho + \rho e^{i\Theta})\rho e^{i\Theta}}} \approx \frac{E_2^\infty\sqrt{a+\rho}}{\sqrt{2}}\sqrt{\frac{1}{\rho}}e^{-i\theta} = E_c e^{-i\theta} + \frac{3\rho}{4(a+\rho)}E_c e^{i\theta}$$

It is seen that on the interface the limit values of the electric field taken from S_0 and S_R are equal in the accuracy of $\rho/(a+\rho)$. Usually $\rho/(a+\rho) \ll 1$, so the above solution is reasonable.

5.2.3 The Stress in an Impermeable Crack with Local Saturation

1. *Stress in linear zone* S_0 This problem in S_0 is similar to that in Sect. 5.1.2. Let $b = 0, m = 1, R = a/2$ and use the effective crack length instead of the real crack length; the solution of the central crack problem can be obtained from the solution of an elliptic hole problem. Equations (5.21), (5.22), (5.23), and (5.24) are still appropriate here, but it should be used the electric field Eq. (5.69) instead of Eq. (5.20). According to above discussions in the ς plane, the stress potentials are determined by the following equations:

$$\begin{aligned}
&\omega(\sigma)\left[\overline{\phi'(\sigma)}\Big/\overline{\omega'(\sigma)}\right] + \phi(\sigma) + \overline{\psi(\sigma)} + (1/2)\kappa w(\sigma)\overline{w'(\sigma)} = 0 \\
&\phi(\varsigma) = \Gamma_1 R\varsigma + \phi_0(\varsigma), \quad \psi(\varsigma) = \Gamma_2 R\varsigma + \psi_0(\varsigma) \\
&w(z) = \mathrm{i}E_2^{\infty}\sqrt{z^2 - (a+\rho)^2} = \mathrm{i}E_2^{\infty}(a/2)\sqrt{[(\varsigma + \varsigma^{-1})]^2 - [2(a+\rho)/a]^2}
\end{aligned} \tag{5.70}$$

where $z = \omega(\varsigma)$ is shown in Eq. (5.20) with $m = 1$. Γ_1, Γ_2 are shown in Eq. (5.22).

Multiplying the first equation in Eq. (5.70) and its conjugate equation by $\mathrm{d}\sigma/[2\pi\mathrm{i}(\sigma - \varsigma)]$ and using the Cauchy integral formulas we find

$$\begin{aligned}
&\phi(\varsigma) = \frac{\Gamma_1 a\varsigma}{2} + \phi_0(\varsigma), \quad \phi_0(\varsigma) = -\frac{\Gamma_1 a}{2\varsigma} - \frac{\bar{\Gamma}_2 a}{2\varsigma} + \frac{\kappa a\left(E_2^{\infty}\right)^2}{2\varsigma} \\
&\psi(\varsigma) = \frac{\Gamma_2 a\varsigma}{2} - \frac{\Gamma_1 a}{2\varsigma} - \frac{\Gamma_1 a}{\varsigma^2 - 1}\varsigma - \frac{\varsigma(1+\varsigma^2)}{(\varsigma^2 - 1)}\phi_0'(\varsigma) + \frac{\kappa a\left(E_2^{\infty}\right)^2}{2\varsigma}
\end{aligned} \tag{5.71}$$

Near the crack tip let $z = a + r\mathrm{e}^{\mathrm{i}\theta}$ (Fig. 5.4); through tedious calculation the pseudo total asymptotic stresses are

$$\begin{aligned}
S_{22} + S_{11} = &\left[\left(2\Gamma_1 + \bar{\Gamma}_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta/2} + \left(2\Gamma_1 + \Gamma_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta/2}\right]\sqrt{a/2r} \\
&+ \kappa E_2^{\infty 2}(a+\rho)/2l \\
S_{22} - S_{11} + 2\mathrm{i}S_{12} = &-\frac{1}{2\sqrt{2}}\mathrm{e}^{-3\mathrm{i}\theta/2}\left\{\left(2\Gamma_1 + \bar{\Gamma}_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta}\right. \\
&\left.-\left(2\Gamma_1 + 2\Gamma_2 - \bar{\Gamma}_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta}\right\}\sqrt{a/r} - \mathrm{e}^{-2\mathrm{i}\Theta}\kappa E_2^{\infty 2}(a+\rho)/2l \\
l = &\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right| \geq \rho, \quad \Theta = \mathrm{Arg}\left(r\mathrm{e}^{\mathrm{i}\theta} - \rho\right)
\end{aligned} \tag{5.72}$$

2. *Stress in saturation zone* S_R In the saturation zone S_R, the electric displacements are finite; the asymptotic stresses near the crack tip will possess singular behavior like $1/\sqrt{r}$ and relate to the size of the saturation zone, so it is assumed

$$S_{ij} = h_{ij}^{(1)}(\theta)/\sqrt{r} + h_{ij}^{(2)}(\theta)/\rho + 0(r) \tag{5.73}$$

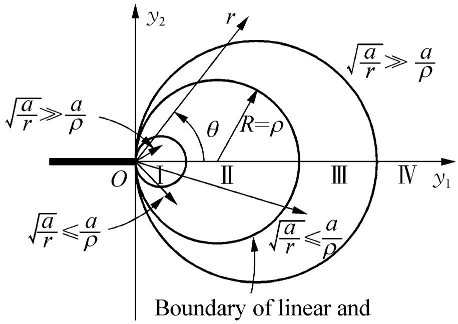

Fig. 5.4 Division regions near the crack tip

Because the electric displacements are continuous on the interface from S_R and S_0, so the Maxwell stress and mechanical and pseudo total stresses are all continuous. So $h_{ij}^{(1)}(\theta), h_{ij}^{(2)}(\theta)$ can be obtained from these continuous conditions:

$$\begin{aligned}
&\frac{h_{11}^{(1)}(\theta)+h_{22}^{(1)}(\theta)}{\sqrt{R(\theta)}}+\frac{h_{11}^{(2)}(\theta)+h_{22}^{(2)}(\theta)}{\rho}=\frac{\kappa E_2^{\infty 2}(a+\rho)}{2l_0}\\
&\quad+\left[\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta/2}+\left(2\Gamma_1+\Gamma_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta/2}\right]\sqrt{\frac{a}{2R(\theta)}}\\
&\frac{h_{22}^{(1)}(\theta)-h_{11}^{(1)}(\theta)+2\mathrm{i}h_{12}^{(1)}(\theta)}{\sqrt{R(\theta)}}+\frac{h_{22}^{(2)}(\theta)-h_{11}^{(2)}(\theta)+2\mathrm{i}h_{12}^{(2)}(\theta)}{\rho}=-\mathrm{e}^{-4\mathrm{i}\theta}\frac{\kappa E_2^{\infty 2}(a+\rho)}{2l_0}\\
&\quad-\frac{1}{2\sqrt{2}}\mathrm{e}^{-3\mathrm{i}\theta/2}\left\{\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta}-\left(2\Gamma_1+2\Gamma_2-\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta}\right\}\sqrt{\frac{a}{R(\theta)}}
\end{aligned} \tag{5.74}$$

where $R(\theta)=2\rho\cos\theta$ and on the interface $\Theta=2\theta$, $l_0=\left|R(\theta)\mathrm{e}^{\mathrm{i}\theta}-\rho\right|=\rho$. If $\rho/a\ll 1$, $(a+\rho)/\left|R(\theta)\mathrm{e}^{\mathrm{i}\theta}-\rho\right|\approx a/\rho$. Comparing the coefficients before $\sqrt{R}$ and $1/\rho$ yields

$$\begin{aligned}
S_{22}+S_{11}&=\left[\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta/2}+\left(2\Gamma_1+\Gamma_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta/2}\right]\sqrt{a/2r}\\
&\quad+\kappa E_2^{\infty 2}a/2\rho\\
S_{22}-S_{11}+2\mathrm{i}S_{12}&=-\left(1\big/2\sqrt{2}\right)\mathrm{e}^{-3\mathrm{i}\theta/2}\left\{\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta}\right.\\
&\quad\left.-\left(2\Gamma_1+2\Gamma_2-\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta}\right\}\sqrt{a/r}-\mathrm{e}^{-4\mathrm{i}\theta}\kappa E_2^{\infty 2}a/2\rho
\end{aligned} \tag{5.75}$$

It is easy to prove that on the interface, the limit values of the stresses taken from S_0 and S_R are equal in the accuracy of $1/\sqrt{r}$ and $1/\rho$ which is consistent of the electric field.

3. *Division region near the crack tip* According to Eqs. (5.72) and (5.73), the stress can be divided into four regions (Fig. 5.4).

Region I: Region I is located in S_R and very near the crack tip, where $\sqrt{a/r} \gg a/\rho$ and the stresses possess the singularity $1/\sqrt{r}$. Under $\sigma_{22}^{\infty}, E_2^{\infty}$ at infinity we have

$$\begin{aligned}
S_{22} &= \frac{1}{2\sqrt{2\pi r}}\left[(K_{\mathrm{I}} - K_{\mathrm{E}})\left(2\cos\frac{\theta}{2} + \sin\theta\sin\frac{3\theta}{2}\right) + K_{\mathrm{I}}\sin\theta\cos\frac{3\theta}{2}\right] \\
S_{11} &= \frac{1}{2\sqrt{2\pi r}}\left[(K_{\mathrm{I}} - K_{\mathrm{E}})\left(2\cos\frac{\theta}{2} - \sin\theta\sin\frac{3\theta}{2}\right) - K_{\mathrm{I}}\left(4\sin\frac{\theta}{2} + \sin\theta\cos\frac{3\theta}{2}\right)\right] \\
S_{12} &= \frac{1}{2\sqrt{2\pi r}}\left[(K_{\mathrm{I}} - K_{\mathrm{E}})\sin\theta\cos\frac{3\theta}{2} + K_{\mathrm{I}}\left(2\cos\frac{\theta}{2} - \sin\theta\sin\frac{3\theta}{2}\right)\right]
\end{aligned} \tag{5.76}$$

Region II: Region II is located in S_R and $\sqrt{a/r} \sim a/\rho$. The stresses should be calculated by Eq. (5.73). The terms containing $\sqrt{a/r}, a/\rho$ all should be considered.

Region III: Region III is in S_0 but neighboring S_R and $\sqrt{a/r} \sim a/\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right|$. The stresses should be calculated by Eq. (5.75).

Region IV: Region IV is in S_0 and $\sqrt{a/r} \gg a/\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right|$. Terms containing $a/\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right|$ can be neglected. If r/a is still small, the stresses can be calculated from Eq. (5.76) also.

5.2.4 Conducting Crack

For the conducting crack or the soft electrode, the boundary conditions are

$$\begin{aligned}
&\sigma_{ij} = 0, \quad E = E_1^{\infty} + \mathrm{i}E_2^{\infty} = E_0\mathrm{e}^{\mathrm{i}\beta}, \quad \text{when} \quad x_k x_k \to \infty \\
&\varphi = 0, \quad \text{or} \quad E_1 = 0, \quad \text{on} \quad x_2 = 0, \quad -a < x_1 < a
\end{aligned} \tag{5.77}$$

1. *The electric field in a linear piezoelectric plate without local saturation zone* According to Eq. (5.6) on the electrode, we have

$$E_1^+ = E_1^- = 0, \quad \text{or} \quad \bar{w}'^+(x_1) + w'^+(x_1) = \bar{w}'^-(x_1) + w'^-(x_1) = 0 \tag{5.78}$$

For the central crack $(-a, a)$ from Eq. (5.78), it can be derived

$$w'(z) = -E_1^{\infty} z\Big/\sqrt{z^2 - a^2} + iE_2^{\infty}, \quad w(z) = -E_1^{\infty}\sqrt{z^2 - a^2} + \mathrm{i}E_2^{\infty} z \tag{5.79}$$

The asymptotic solution near the crack tip $z = a$ is

$$
\begin{aligned}
&w'(z) = -E_1 + \mathrm{i}E_2 \approx -K_\mathrm{e}\Big/\sqrt{2\pi(z-a)} = -\left(K_\mathrm{e}\Big/\sqrt{2\pi r}\right)\mathrm{e}^{-\mathrm{i}\theta} \\
&E_1 = \left(K_\mathrm{e}\Big/\sqrt{2\pi r}\right)\cos\theta, \quad E_2 = \left(K_\mathrm{e}\Big/\sqrt{2\pi r}\right)\sin\theta, \quad K_\mathrm{e} = E_1^\infty\sqrt{\pi a}
\end{aligned}
\tag{5.80}
$$

2. *The electric field in a plate with local saturation zone*
Similar to the impermeable crack the boundary value problem in y plane is

$$
\begin{aligned}
&D_1^\pm = 0, \quad 当 \quad y_2 = 0, \quad -\infty < y_1 \le 0 \\
&\sqrt{D_1^2 + D_2^2} = D_c, \qquad \text{when} \quad y \in R(\theta) \\
&D_1^2 + D_2^2 = 0, \qquad \text{when} \quad y \to \infty
\end{aligned}
\tag{5.81}
$$

where $R = R(\theta)$ is the boundary of the saturation zone in y plane. The hodograph transform method is used. Let

$$
\varsigma = (D_1 - \mathrm{i}D_2)/D_\mathrm{c}, \quad \text{or} \quad \varsigma = (E_1 - \mathrm{i}E_2)/E_\mathrm{c}
\tag{5.82}
$$

According to Eq. (5.80) the electric displacements in the saturation zone is assumed as

$$
D_1 = D_\mathrm{c}\cos\theta, \quad D_2 = D_\mathrm{c}\sin\theta
\tag{5.83}
$$

Obviously Eq. (5.83) satisfies Eq. (5.77). Repeating the discussion in Sect. 5.2.2, the boundary and the radius of the saturation zone are, respectively,

$$
R(\theta) = \left(K_\mathrm{e}^2/\pi E_\mathrm{c}^2\right)\cos\theta, \quad \rho = K_\mathrm{e}^2/\left(2\pi E_\mathrm{c}^2\right)
\tag{5.84}
$$

The remaining discussion is fully similar to Sect. 5.2.3 and omitted here.

5.3 Asymptotic Analysis of a Crack Subjected to Electric Loading

Yang and Suo (1994) and Hao et al. (1996) discussed the ceramic actuators caused by electrostriction; Beom et al. (2006) discussed the asymptotic analysis of an impermeable crack subjected to electric loading. The crack extension criterion in plane strain is mainly determined by the stress field near the crack tip, so they adopted the linear asymptotic solution of a semi-infinite crack as the boundary condition of the asymptotic analysis at infinity (Fig. 5.5). In this analysis the Maxwell stress is not considered. Analogous to Eq. (5.57) we have

$$
D_2 + \mathrm{i}D_1 = K_\mathrm{D}\Big/\sqrt{2\pi y}, \quad y = y_1 + \mathrm{i}y_2 = r\mathrm{e}^{\mathrm{i}\theta}; \quad \text{when} \quad |y| \to \infty
\tag{5.85}
$$

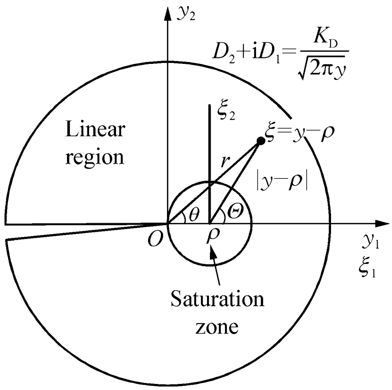

Fig. 5.5 Asymptotic analysis sketch of a crack with local saturation

where K_D is the electric displacement intensity factor. Now we discuss an infinite piezoelectric material with an impermeable crack subjected to electric loading as shown in Eq. (5.85). As shown in Eqs. (5.57) and (5.69), the approximate solutions of the electric displacement can be taken as

$$
\begin{aligned}
&D_2 + iD_1 = K_D \big/ \sqrt{2\pi y}, \quad w'(z) = -D_1 + iD_2 = iK_D \big/ \sqrt{2\pi y}; \quad \text{in} \quad \Omega_0 \\
&D_2 + iD_1 = D_c e^{-i\theta}, \quad \text{in} \quad \Omega_s \\
&\xi = \xi_1 + i\xi_2 = y - \rho = |y-\rho| e^{i\Theta}
\end{aligned} \tag{5.86}
$$

where $\rho = K_D^2 \big/ \left(2\pi D_c^2\right)$ is the radius of the saturation zone, $w(z)$ represents electric displacement complex potential, Ω_0 denotes the linear zone, and Ω_s denotes the saturation zone. Equation (5.86) satisfies the boundary condition on the crack surface and Eq. (5.85) at infinity. On the interface between Ω_0 and Ω_s, $\xi_0 = \rho e^{i\Theta}$. The constitutive equation is shown in Eq. (5.9), but here the slight different form is used:

$$
\varepsilon_{\alpha\beta} = (1+\nu)\left(\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)/Y + Q(1+q)D_\alpha D_\beta - Qq(1+\nu)D_\gamma D_\gamma \delta_{\alpha\beta} \tag{5.87}
$$

where Y is elastic modulus and ν is Poisson ratio, Q and q are the electrostrictive coefficients. Apply the superposition method to solve this problem: Problem (1) is that a plate without crack is subjected to the above electric displacement fields. In this problem on the artificial cut corresponding to the original crack we can get the tractions $\sigma_{22}^c, \sigma_{21}^c$. Problem (2) is that the artificial cut is subjected tractions $-\sigma_{22}^c$, $-\sigma_{21}^c$. The solution of the original problem is the sum of solutions of these two problems.

According to Eqs. (5.86) and (5.87), the strains in the saturation zone induced by the saturation electric displacements are

$$
\varepsilon_r^s = -QqD_c^2(1+\nu), \quad \varepsilon_\theta^s = QD_c^2(1-q\nu), \quad \varepsilon_{r\theta}^s = 0 \tag{5.88}
$$

The strains in Eq. (5.88) satisfy the compatible equation automatically, so they do not produce stresses. Neglecting the rigid displacements the displacements corresponding to these strains are

$$
\begin{aligned}
&u_r^{\mathrm{s}} = -QqD_{\mathrm{c}}^2(1+\nu)r, \quad u_\theta^s = QD_{\mathrm{c}}^2(1+q)r\theta \\
&u_1^{\mathrm{s}} + \mathrm{i}u_2^{\mathrm{s}} = QD_{\mathrm{c}}^2[-(1+\nu)q + \mathrm{i}(1+q)\theta](\xi+\rho)
\end{aligned}
\tag{5.89}
$$

Analogous to Eqs. (5.6), (5.16), (5.17), and (5.18) in the linear zone we have

$$
\begin{aligned}
&2G(u_1+\mathrm{i}u_2) = K\phi(\xi) - \xi\overline{\phi'(\xi)} - \overline{\psi(\xi)} + hw(\xi)\overline{w'(\xi)} + 4(1-\nu)m^{-1}h\int\left[\overline{w'(\xi)}\right]^2 \mathrm{d}\bar{\xi} \\
&\sigma_{22}+\sigma_{11} = 2\left[\phi'(\xi)+\overline{\phi'(\xi)}\right] - 2hw'(\xi)\overline{w'(\xi)} \\
&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} = 2[\bar{\xi}\phi''(\xi)+\psi'(\xi)] - 2hw''(\xi)\overline{w(\xi)} \\
&\mathrm{i}(P_1+\mathrm{i}P_2) = \left[\xi\overline{\phi'(\xi)}+\phi(\xi)+\overline{\psi(\xi)} - hw(\xi)\overline{w'(\xi)}\right]_A^B
\end{aligned}
\tag{5.90}
$$

where

$$
K = 3-4\nu, \quad h = \frac{1-(1+2\nu)q}{2}\frac{GQ}{1-\nu}, \quad m = 2\frac{1-(1+2\nu)q}{1+q}
\tag{5.91}
$$

In the saturation zone we have

$$
\begin{aligned}
&2G(u_1+\mathrm{i}u_2) = K\phi(\xi) - \xi\overline{\phi'(\xi)} - \overline{\psi(\xi)} + 2G\left(u_1^{\mathrm{s}} + iu_2^{\mathrm{s}}\right) \\
&\sigma_{22}+\sigma_{11} = 2\left[\phi'(\xi)+\overline{\phi'(\xi)}\right] \\
&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} = 2[\bar{\xi}\phi''(\xi)+\psi'(\xi)] \\
&\mathrm{i}(P_1+\mathrm{i}P_2) = \left[\xi\overline{\phi'(\xi)}+\phi(\xi)+\overline{\psi(\xi)}\right]_A^B
\end{aligned}
\tag{5.92}
$$

The two group solutions shown in Eqs. (5.90) and (5.92) should satisfy the continuity conditions of displacements and stresses on the interface between linear and saturation zones. The solution in the linear zone should also satisfy the boundary conditions at infinity.

Solution of problem (1) Assume the solutions are:

$$
\begin{aligned}
&\phi(\xi) = -\sigma_0\rho\ln(\xi/\rho) + \phi_2(\xi), \quad \psi(\xi) = -\sigma_0\rho\ln(\xi/\rho) + \psi_2(\xi); \quad \xi\in\Omega_0 \\
&\phi(\xi) = \phi_1(\xi), \quad \psi(\xi) = \psi_1(\xi), \quad \sigma_0 = \left[(1+q)/8\left(1-\nu^2\right)\right]YQD_{\mathrm{c}}^2; \quad \xi\in\Omega_{\mathrm{s}}
\end{aligned}
\tag{5.93}
$$

Substitution of Eq. (5.93) into Eqs. (5.90) and (5.92) yields

$$
\begin{aligned}
&2G(u_1+\mathrm{i}u_2)+\mathrm{i}(P_1+\mathrm{i}P_2)=4(1-\nu)\left\{[-\sigma_0\rho\ln(\xi/\rho)+\phi_2(\xi)]+m^{-1}h\int\left[\overline{w'(\xi)}\right]^2 d\bar{\xi}\right\};\ \xi\in\Omega_0\\
&2G(u_1+\mathrm{i}u_2)+\mathrm{i}(P_1+\mathrm{i}P_2)=4(1-\nu)\phi_1(\xi)+2G\left(u_1^{\mathrm{s}}+\mathrm{i}u_2^{\mathrm{s}}\right);\quad \xi\in\Omega_{\mathrm{s}}
\end{aligned}
\tag{5.94}
$$

From the continuity conditions of displacements and resultant forces on the interface we have

$$
\phi_2(\xi_0)-\phi_1(\xi_0)=\sigma_0\rho[\ln(\xi_0/\rho)+m/2-1](1+\xi_0/\rho) \tag{5.95}
$$

where $\xi_0=\rho e^{\mathrm{i}\Theta}$ is the value of ξ on the interface. Assuming the displacements vanish at infinity, by the standard analytic continuation theory from Eq. (5.95) we find

$$
\begin{aligned}
\phi_2(\xi)&=\sigma_0\rho\left[\left(1+\frac{\xi}{\rho}\right)\ln\frac{\xi}{\xi+\rho}+1\right],\\
\phi_1(\xi)&=\sigma_0\rho\left[-\left(1+\frac{\xi}{\rho}\right)\left(\ln\frac{\xi+\rho}{\rho}+\frac{1}{2}m-1\right)+1\right]
\end{aligned}
\tag{5.96}
$$

Analogously from the continuity conditions of resultant forces $i(P_1+iP_2)$ on the interface we have

$$
\psi_2(\xi_0)-\psi_1(\xi_0)=-\sigma_0\rho[-(1+m)(\rho/\xi_0)+m/2-1-\ln(\xi_0/\rho)] \tag{5.97}
$$

Assuming the displacements vanish at infinity, by the standard analytic continuation theory from Eq. (5.97)

$$
\begin{aligned}
\psi_2(\xi)&=\sigma_0\rho[(1+m)(\rho/\xi)+\ln[\xi/(\xi+\rho)]-m+4(1-\nu)]\\
\psi_1(\xi)&=\sigma_0\rho[-\ln[(\xi+\rho)/\rho]-m/2+3-4\nu]
\end{aligned}
\tag{5.98}
$$

Finally we have

$$
\begin{aligned}
\phi(\xi)&=\sigma_0\rho\left[-\ln\frac{\xi}{\rho}+\left(1+\frac{\xi}{\rho}\right)\ln\frac{\xi}{\xi+\rho}+1\right]\\
\psi(\xi)&=\sigma_0\rho\left[(1+m)\frac{\rho}{\xi}-\ln\frac{\xi}{\rho}+\ln\frac{\xi}{\xi+\rho}-m+4(1-\nu)\right]\quad\text{in}\quad\Omega_0
\end{aligned}
\tag{5.99}
$$

$$
\begin{aligned}
\varphi(\xi)&=\sigma_0\rho\left[-\left(1+\frac{\xi}{\rho}\right)\left(\ln\frac{\xi+\rho}{\rho}+\frac{1}{2}m-1\right)+1\right]\\
\psi(\xi)&=\sigma_0\rho\left[-\ln\frac{\xi+\rho}{\rho}-\frac{1}{2}m+3-4\nu\right];\quad\text{in}\quad\Omega_{\mathrm{s}}
\end{aligned}
\tag{5.100}
$$

Solution of problem (2) Eqs. (5.90) and (5.99) yield

$$\sigma_{22}^{c} = \sigma_0\left[2\ln\frac{y_1-\rho}{y_1} - (1+m)\left(\frac{\rho}{y_1-\rho}\right)^2\right], \quad \sigma_{21}^{c} = 0 \tag{5.101}$$

When the crack surface is subjected to $-\sigma_{22}^{c}$, the solution is

$$\begin{aligned}\phi'(z) &= \sigma_0\left\{-\ln\frac{y-\rho}{y} + \frac{1}{2}(1+m)\left(\frac{\rho}{y-\rho}\right)^2\right.\\ &\qquad \left. -\frac{1}{4}(1+m)\rho^{3/2}\frac{y+\rho}{\sqrt{y}(y-\rho)^2} - 2\operatorname{arcoth}\sqrt{\frac{y}{\rho}} + 2\sqrt{\frac{\rho}{y}}\right\} \\ \psi'(y) &= -y\phi''(y)\end{aligned} \tag{5.102}$$

Solution of the original problem Superposing solutions of problems (1) and (2), finally we get the following. In the linear zone Ω_0,

$$\begin{aligned}\frac{\sigma_{22}+\sigma_{11}}{2} &= \sigma_0\mathrm{Re}\left\{(1+m)\left(\frac{\rho}{y-\rho}\right)^2 - \frac{1}{2}(1+m)\sqrt{\frac{\rho}{y}}\frac{\rho(y+\rho)}{(y-\rho)^2}\right.\\ &\qquad \left. - 4\operatorname{arcoth}\sqrt{\frac{y}{\rho}} + 4\sqrt{\frac{\rho}{y}}\right\} - \sigma_0 m\frac{\rho}{|y-\rho|} \\ \frac{\sigma_{22}-\sigma_{11}}{2} + \mathrm{i}\sigma_{12} &= \sigma_0\left\{\frac{\bar{y}-\rho}{y-\rho} - \frac{\bar{y}}{y} - (1+m)\left(\frac{\rho}{y-\rho}\right)^2 + m\frac{\rho}{y-\rho}\sqrt{\frac{\bar{y}-\rho}{y-\rho}} + (\bar{y}-y)\right.\\ &\times \left.\left[\left(\sqrt{\frac{\rho}{y}}-1\right)\frac{\rho}{y(y-\rho)} - (1+m)\frac{\rho^2}{(y-\rho)^3} - \frac{1+m}{8}\sqrt{\frac{\rho}{y}}\frac{\rho}{y(y-\rho)^3}\left(\rho^2 - 6\rho y - 3y^2\right)\right]\right\}\end{aligned} \tag{5.103}$$

In the saturation zone Ω_s,

$$\begin{aligned}\frac{\sigma_{22}+\sigma_{11}}{2} &= \sigma_0\mathrm{Re}\left\{-2\ln\frac{y-\rho}{\rho} + (1+m)\left(\frac{\rho}{y-\rho}\right)^2\right.\\ &\qquad \left. -\frac{1}{2}(1+m)\sqrt{\frac{\rho}{y}}\frac{\rho(y+\rho)}{(y-\rho)^2} - 4\operatorname{arcoth}\sqrt{\frac{y}{\rho}} + 4\sqrt{\frac{\rho}{y}}\right\} - \sigma_0 m \\ \frac{\sigma_{22}-\sigma_{11}}{2} + \mathrm{i}\sigma_{12} &= \sigma_0\left\{-\frac{\bar{y}}{y} + (\bar{y}-y)\left[\left(\sqrt{\frac{\rho}{y}}-1\right)\frac{\rho}{y(y-\rho)}\right.\right.\\ &\qquad \left.\left. - (1+m)\frac{\rho^2}{(y-\rho)^3} - \frac{1+m}{8}\sqrt{\frac{\rho}{y}}\frac{\rho}{y(y-\rho)^3}\left(\rho^2 - 6\rho y - 3y^2\right)\right]\right\}\end{aligned} \tag{5.104}$$

It is also found that in the saturation zone the stresses at the real crack tip has the singularity $1/\sqrt{r}$ and at the effective crack tip $(y=\rho)$ has the logarithmic singularity. Because accuracy of the electric field is of the order ρ/a, the accuracy of solutions of the mechanical stresses is still in the same order.

Following the elastoplastic fracture mechanics, Beom et al. (2006) also discussed the modified boundary layer theory, i.e., replaced Eq. (5.85) by

$$D_2 + \mathrm{i}D_1 = K_{\mathrm{D}} \Big/ \sqrt{2\pi z} + \mathrm{i}T; \quad \text{when} \quad |z| \to \infty$$

where T is a finite electric displacement parallel to the crack surface.

Beom (1999)discussed the singular behavior near a crack tip in an electrostrictive material with the elastic behavior shown in Eq. (5.87), and for the electric behavior, he took the Ramberg-Osgood type constitutive equation

$$\begin{aligned} &E_\alpha = -2Q(1+q)\sigma_{\alpha\beta}D_\beta + 2Qq(1+\nu)\sigma_{\beta\beta}D_\alpha + 2YQ^2q^2D_\beta D_\beta D_\alpha + D_\alpha f(D)/D \\ &f(D) = (E_{\mathrm{c}}/D_{\mathrm{c}})D + kE_{\mathrm{c}}(D/D_{\mathrm{c}})^n; \quad n > 3 \end{aligned} \tag{5.105}$$

where k and n are material constants; $E = f(D)$ is the uniaxial dielectric response in the absence of stress. In this case he got $\boldsymbol{\sigma} \propto r^{-1/2}, \boldsymbol{D} \propto r^{-1/(n+1)}$.

5.4 Pyroelectric Material

5.4.1 Generalized Two-Dimensional Linear Thermo-electro-elastic Problem

In engineering the extensive applied governing equation is Eq. (2.89) with independent variables $(\boldsymbol{\varepsilon}, \boldsymbol{E}, \vartheta), \quad \vartheta = T - T_0$ for the pyroelectric materials:

$$\begin{aligned} &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta, \quad D_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta \\ &s = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + C\vartheta/T_0, \quad \vartheta = T - T_0 \end{aligned} \tag{5.106}$$

The thermal conduction and the entropy equations are

$$q_i = -\lambda_{ij}T_{,j}, \quad T_{,j} = \vartheta_{,j} = -\lambda_{ji}^{-1}q_i; \quad -q_{i,i} = T\dot{s} - \dot{r} \tag{5.107}$$

The mechanical, electric, and thermal boundary conditions are

$$\begin{array}{llllll} \sigma_{ij}n_j = T_i^*, & \text{on} & a_\sigma; & \text{or} & u_i = u_i^*, & \text{on} \quad a_u \\ D_i n_i = -\sigma^*, & \text{on} & a_D; & \text{or} & \varphi = \varphi^*, & \text{on} \quad a_\varphi \\ q_i n_i = q_n = q_0^*, & \text{on} & a_q; & \text{or} & T = T^*, & \text{on} \quad a_T \end{array} \tag{5.108}$$

where T^*, σ^*, q_0^* are the traction, electric charge per area, and normal heat flow per area.

The continuity conditions on the interface are

$$\sigma_{ij}^{+}n_j = \sigma_{ij}^{-}n_j, \quad u_i^{+} = u_i^{-}, \quad D_i^{+} = D_i^{-}; \quad \varphi^{+} = \varphi^{-}, T^{+} = T^{-}, \quad q^{+} = q^{-} \quad \text{on} \quad L \tag{5.109}$$

The governing equations in $(\boldsymbol{u}, \varphi, \vartheta)$ are

$$\begin{aligned} &\left(C_{ijlk}u_l + e_{kij}\varphi\right)_{,ik} - \alpha_{ij}\vartheta_{,i} + \left(f_j^{\mathrm{m}} + f_j^{\mathrm{e}}\right) = \rho\ddot{u}_j \\ &\left(-\epsilon_{ik}\varphi + e_{ijk}u_j\right)_{,ik} + \tau_i T_{,i} = \rho_{\mathrm{e}} \\ &\lambda_{ij}T_{,j} = -q_i \end{aligned} \tag{5.110}$$

For a multiply connected domain, the displacement and electric potential must satisfy the uniqueness condition Eq. (3.7).

The thermo-electro-elastic fundamental theory of the pyroelectric material was studied a long time (Tiersten 1971; Mindlin 1974). For a static problem with stationary temperature, from Eq. (5.110) we get

$$\left(C_{ijrs}u_r + e_{sij}\varphi\right)_{,si} = \alpha_{ij}\vartheta_{,i}, \quad \left(-\epsilon_{is}\varphi + e_{irs}u_r\right)_{,si} = -\tau_i\vartheta_{,i}, \quad -q_{i,i} = \left(\lambda_{ij}\vartheta_{,j}\right)_{,i} = 0 \tag{5.111}$$

From Eq. (5.111) it is seen that the generalized displacements are dependent to the temperature, but the temperature is independent to the generalized displacements. So the temperature can be solved independently (Hwu 1992; Shen and Kuang 1998). Because ϑ is real, it is assumed that

$$\vartheta(x_1, x_2) = \mathrm{g}'(z_T) + \overline{\mathrm{g}'(z_T)} = 2\mathrm{Re}\,\mathrm{g}'(z_T), \quad z_T = x_1 + \mu_T x_2 \tag{5.112}$$

Substitution of Eq. (5.112) into the third equation in Eq. (5.111) yields

$$\left(\lambda_{11} + 2\mu_T\lambda_{12} + \mu_T^2\lambda_{22}\right)\mathrm{g}'''(z) = 0 \tag{5.113}$$

As in Sect. 3.2.1, from Eq. (5.113) we get a pair of conjugate complex roots $\mu_T, \bar{\mu}_T$ with $\mathrm{Im}\mu_T > 0$:

$$\begin{aligned} &\lambda_{11} + 2\mu_T\lambda_{12} + \mu_T^2\lambda_{22} = 0, \quad \lambda_{11} + \mu_T\lambda_{12} = -\mu_T(\lambda_{12} + \mu_T\lambda_{22}) \\ &\mu_T = (-\lambda_{12} + \mathrm{i}\alpha)/\lambda_{22}, \quad \alpha = \sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2} = \lambda_{22}(\mu_T - \bar{\mu}_T)/2\mathrm{i} = -\mathrm{i}(\lambda_{21} + \mu_T\lambda_{22}) \end{aligned} \tag{5.114}$$

where α is real. Using Eq. (5.114) the Fourier's law can be written as

$$\begin{aligned} q_i &= -2\mathrm{Re}[(\lambda_{i1} + \mu_T\lambda_{i2})\mathrm{g}''(z_T)], \quad q_n = -2\mathrm{Im}[\alpha(\mu_T n_1 - n_2)\mathrm{g}''(z_T)] \\ q_1 &= -2\mathrm{Re}[(\lambda_{11} + \mu_T\lambda_{12})\mathrm{g}''(z_T)] = 2\mathrm{Re}[\mathrm{i}\alpha\mu_T\mathrm{g}''(z_T)] = -2\mathrm{Im}[\alpha\mu_T\mathrm{g}''(z_T)] \\ q_2 &= -2\mathrm{Re}[(\lambda_{21} + \mu_T\lambda_{22})\mathrm{g}''(z_T)] = -2\mathrm{Re}[\mathrm{i}\alpha\mathrm{g}''(z_T)] = 2\mathrm{Im}[\alpha\mathrm{g}''(z_T)] \end{aligned} \tag{5.115}$$

By using Eq. (3.27) the total heat flow $\hat{q}$ through a line segment from z_0 to z is

$$\hat{q} = \int_{z_0}^{z_T} q_i n_i \mathrm{d}s = 2\mathrm{Re}\int_{z_0}^{z_T} \mathrm{i}\alpha(\mu_T\mathrm{d}x_2 + \mathrm{d}x_1)\mathrm{g}''(z_T) = -2\mathrm{Im}\{\alpha[\mathrm{g}'(z_T) - \mathrm{g}'(z_0)]\} \tag{5.116}$$

When ϑ is solved, the terms in the right side of the first and second equations in Eq. (5.111) become known. The special solution introduced by the temperature ϑ can be assumed as

$$\boldsymbol{U}_T = [U_{TP}]^{\mathrm{T}} = \boldsymbol{c}\mathrm{g}(z_T), \quad U_{Ti} = u_{Ti} = c_i\mathrm{g}(z_T), \quad U_{T4} = \varphi_T = c_4\mathrm{g}(z_T) \tag{5.117}$$

where a subscript in upper case P takes the value 1,2,3, or 4 and a subscript in lower case $i, j, \ldots$ takes the value 1, 2, or 3, as shown in Sect. 3.2.1. Substitution of Eq. (5.117) into the first and second equations in Eq. (5.111) yields the equations to determine $\boldsymbol{c} = [c_1, c_2, c_3, c_4]^{\mathrm{T}}$:

$$\begin{aligned} &\left[C_{j1k1} + \mu_T\left(C_{j1k2} + C_{j2k1}\right) + \mu_T^2 C_{j2k2}\right]c_k + \left[e_{1j1} + \mu_T\left(e_{2j1} + e_{1j2}\right) + \mu_T^2 e_{2j2}\right]c_4 \\ &\quad = \alpha_{1j} + \mu_T\alpha_{2j} \\ &\left[e_{1k1} + \mu_T(e_{2k1} + e_{1k2}) + \mu_T^2 e_{2k2}\right]c_k - \left[\epsilon_{11} + \mu_T(\epsilon_{12} + \epsilon_{21}) + \mu_T^2\epsilon_{22}\right]c_4 = -\tau_1 - \mu_T\tau_2; \quad \text{or} \\ &\left[\boldsymbol{Q} + \mu_T\left(\boldsymbol{R} + \boldsymbol{R}^T\right) + \mu_T^2\boldsymbol{T}\right]\boldsymbol{c} = \boldsymbol{D}(\mu_T)c = \boldsymbol{\chi}_1 + \mu_T\boldsymbol{\chi}_2, \quad \boldsymbol{\chi}_i = [\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, -\tau_i]^{\mathrm{T}} \end{aligned} \tag{5.118}$$

where $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}$ are expressed in Eq. (3.13). The generalized stress introduced by temperature is

$$\begin{aligned} \sigma_{Tij} &= 2\mathrm{Re}[(C_{ijkl}c_k + e_{lij}c_4)z_{T,l} - \alpha_{ij}]\mathrm{g}'(z_T) \\ D_{Ti} &= 2\mathrm{Re}[(e_{ikl}c_k + \epsilon_{il}c_4)z_{T,l} + \tau_i]\mathrm{g}'(z_T) \end{aligned} \tag{5.119}$$

The solution for the thermo-electro-elastic analysis in pyroelectric material is the sum of the special solution and the general solution of the corresponding homogeneous equations. For the stationary temperature the general solution is

$$\boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z_P) + \boldsymbol{c}\mathrm{g}(z_T)], \quad \text{or} \quad \boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\langle f(z_P)\rangle\boldsymbol{V} + \boldsymbol{c}\mathrm{g}(z_T)] \tag{5.120}$$

The stress can be expressed as

$$\begin{aligned} &\boldsymbol{\Sigma}_1 = -2\mathrm{Re}[\boldsymbol{B}\mu_p\boldsymbol{F}(z_P) + \boldsymbol{d}\mu_T g'(z_T)], \quad \boldsymbol{\Sigma}_2 = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{F}(z_P) + \boldsymbol{d}g'(z_T)] \\ &\boldsymbol{d} = \left(\boldsymbol{R}^{\mathrm{T}} + \mu_T\boldsymbol{T}\right)\boldsymbol{c} - \boldsymbol{\chi}_2 = \{-(\boldsymbol{Q} + \mu_T\boldsymbol{R})\boldsymbol{c} + \boldsymbol{\chi}_1\}/\mu_T \\ &d_j = (C_{j2kl}c_k + e_{l2j}c_4)z_{T,l} - \alpha_{2j} = -[(C_{j1kl}c_k + e_{l1j}c_4)z_{T,l} - \alpha_{1j}]/\mu_T \\ &d_4 = (e_{2kl}c_k + \epsilon_{2l}c_4)z_{T,l} + \tau_2 = -[(e_{1kl}c_k + \epsilon_{1l}c_4)z_{T,l} + \tau_2]/\mu_T \end{aligned} \tag{5.121}$$

where $\boldsymbol{F}(z_j) = \boldsymbol{f}'(z_j)$. Introduce the stress function $\boldsymbol{\Phi}$:

$$\begin{aligned} &\boldsymbol{\Phi} = [\Phi_i, \Phi_4]^{\mathrm{T}} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z) + \boldsymbol{d}g(z_T)], \quad \text{or} \quad \boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\langle f(z_P)\rangle\boldsymbol{V} + \boldsymbol{d}g(z_T)] \\ &\Sigma_{2P} = \Phi_{P,1}, \quad \Sigma_{1P} = -\Phi_{P,2}; \quad T_i = -\mathrm{d}\Phi_i/\mathrm{d}s, \quad D_n = -\mathrm{d}\Phi_4/\mathrm{d}s; \quad P = 1,2,4, \quad i = 1,2 \end{aligned} \tag{5.122}$$

Equations (5.112), (5.115), (5.120), and (5.122) are the general solutions of the thermo-electro-elastic analysis in the pyroelectric material. Combining these equations and the appropriate boundary conditions, we can solve all the thermo-electro-elastic problems. For the multi-connected region the generalized displacement and temperature should satisfy the uniqueness condition.

5.4.2 A Thermal Impermeable Elliptic Hole in a Pyroelectric Material

As an example in this section, we discuss a generalized 2D problem of a pyroelectric material that occupied the region S with an elliptic hole that occupied the region S^c filled with air under uniform generalized stresses $(\boldsymbol{\sigma}^\infty, \boldsymbol{D}^\infty)$ and heat flow $\boldsymbol{q}^\infty$ (see Fig. 3.3). The interface L between the material and the hole is free of generalized forces and is thermal insulated (Lu et al. 1998; Gao et al. 2002). The boundary conditions are

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{D} = \boldsymbol{D}^\infty, \quad \boldsymbol{q} = \boldsymbol{q}^\infty; \quad \text{at} \quad \text{infinity} \\ &\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0, \quad \varphi = \varphi^c, \quad \boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^c\cdot\boldsymbol{n} = -\epsilon_0(\nabla\cdot\varphi^c)\cdot\boldsymbol{n}; \quad \text{on} \quad L \end{aligned} \tag{5.123}$$

Temperature field in the piezoelectric material with a thermal insulated hole As shown in Sect. 5.4.1, the temperature can be solved independently. As in Sect. 3.4 the transform method is used to solve this problem. The mapping function for z_T plane to ς_T plane is similar to z_j plane to ς_j plane in Eq. (3.86), but μ_j is replaced by μ_T, i.e.,

$$z_T = \omega_T(\varsigma_T) = R_T\left(\varsigma_T + m_T\varsigma_T^{-1}\right); \quad R_T = (a - \mathrm{i}\mu_T b)/2, \quad m_T = (a + \mathrm{i}\mu_T b)/(a - \mathrm{i}\mu_T b) \tag{5.124}$$

The interface L in z plane is mapped to Γ in ς plane. The temperature field can be chosen as

$$g'(z_T) = \beta_T z_T + \hat{g}'_0(z_T) = \beta_T \omega_T(\varsigma_T) + g'_0(\varsigma_T), \quad g'_0(\varsigma_T) = \hat{g}'_0[\omega_T(\varsigma_T)] \tag{5.125}$$

where β_T is a complex constant and $g'_0(\varsigma_T)$ is holomorphic outside the unit circle in ς_T plane. Equations (5.112), (5.115), and (5.125) yield

$$q_1^\infty = 2\mathrm{Re}[\mathrm{i}\alpha\mu_T\beta_T], \quad q_2^\infty = -2\mathrm{Re}[\mathrm{i}\alpha\beta_T]; \quad \beta_T = -\mathrm{i}\left(q_1^\infty + \bar{\mu}_T q_2^\infty\right)/\alpha(\mu_T - \bar{\mu}_T) \tag{5.126}$$

Because the interface is thermal insulated, Eqs. (5.116) and (5.126) yield

$$\begin{aligned} &\mathrm{Re}[\mathrm{i}\alpha g'(\sigma) - \mathrm{i}\alpha g'(\sigma_0)] = 0, \quad \text{or} \\ &\mathrm{i}\left[g'_0(\sigma) - \bar{g}'_0(\bar{\sigma})\right] = \mathrm{i}\left\{\beta_T R_T(\sigma + \bar{m}_T\bar{\sigma}) - \bar{\beta}_T\bar{R}_T(\bar{\sigma} + \bar{m}_T\sigma)\right\} \\ &\quad = (1/2\alpha)\left[a q_2^\infty(\sigma + \bar{\sigma}) + \mathrm{i}b q_1^\infty(\sigma - \bar{\sigma})\right] \end{aligned} \tag{5.127}$$

where σ is the value of ς_T on Γ. Multiplying Eq. (5.127) by $\int_L [\mathrm{d}\sigma/(\sigma - \varsigma)]$ and using the Cauchy integral formula we get

$$g'_0(\varsigma_T) = \delta_T\varsigma_T^{-1}, \quad g'(\varsigma_T) = \beta z_T + g'_0(\varsigma_T); \quad \delta_T = \left[(1/2\mathrm{i}\alpha)\left(a q_2^\infty - \mathrm{i}b q_1^\infty\right)\right] \tag{5.128}$$

From Eqs. (5.125) and (5.128) in z plane, we get

$$\begin{aligned} &g'(z_T) = \beta_T z_T + \delta_T\varsigma_T^{-1}(z_T) \\ &g(z_T) = (1/2)\beta_T z_T^2 + R_T\delta_T \ln\varsigma_T(z_T) + (1/2)R_T m_T\varsigma_T^{-2}(z_T) \end{aligned} \tag{5.129}$$

where $\beta_T z_T$ represents the complex potential of a uniform heat flow $\boldsymbol{q}^\infty$ in an infinite material without hole.

Superposition method By means of superposition, the solution of the original problem can be obtained as the sum of the following three problems:

(1) A pyroelectric material with an elliptic hole under boundary conditions

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{D} = \boldsymbol{D}^\infty, \quad \boldsymbol{q} = \boldsymbol{0}; \quad \text{at} \quad \text{infinity} \\ &\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0, \quad \varphi = \varphi^{\mathrm{c}}, \quad \boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^{\mathrm{c}}\cdot\boldsymbol{n} = -\epsilon_0(\boldsymbol{\nabla}\cdot\varphi^{\mathrm{c}})\cdot\boldsymbol{n}; \quad \text{on} \quad L \end{aligned} \tag{5.130a}$$

Problem (1) can be reduced to the following problem: a piezoelectric material, with an elliptic hole, subjected generalized stresses at infinity under constant temperature, which has been discussed in Sect. 3.4.

(2) A pyroelectric material without elliptic hole under boundary conditions

$$\boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{D} = \mathbf{0}, \quad \boldsymbol{q} = \boldsymbol{q}^{\infty}; \quad \text{at} \quad \text{infinity} \tag{5.130b}$$

The solution is

$$\begin{aligned} &\mathrm{g}'(z_T) = \beta_T z_T, \quad q_1 = q_1^{\infty}, \quad q_2 = q_2^{\infty}; \quad \sigma_{ij} = 0; \quad D_i = 0 \\ &\vartheta(x_1, x_2) = 2\mathrm{Re}(\beta_T z_T) = -\left(\lambda_{11}\lambda_{22} - \lambda_{12}^2\right)^{-1}\left[\left(\lambda_{22}q_1^{\infty} - \lambda_{12}q_2^{\infty}\right)x_1 + \left(\lambda_{11}q_2^{\infty} - \lambda_{12}q_1^{\infty}\right)x_2\right] \end{aligned} \tag{5.131}$$

This temperature field does not affect the generalized stress field, because a linear temperature field always satisfies the strain compatible equation.

(3) A pyroelectric material with an elliptic hole under boundary and single valued conditions

$$\begin{aligned} &\boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{D} = \mathbf{0}, \quad \boldsymbol{q} = \mathbf{0}; \quad \text{at} \quad \text{infinity} \\ &\boldsymbol{\sigma}\cdot\boldsymbol{n} = \mathbf{0}, \quad \boldsymbol{q}\cdot\boldsymbol{n} = -\boldsymbol{q}^{\infty}\cdot\boldsymbol{n}, \quad \varphi = \varphi^{\mathrm{c}}, \quad \boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^{\mathrm{c}}\cdot\boldsymbol{n} = -\epsilon_0(\nabla\cdot\varphi^{\mathrm{c}})\cdot\boldsymbol{n}; \quad \text{on} \quad L \\ &\oint_L \mathrm{d}U = \oint_\Gamma \mathrm{d}U = 0 \end{aligned} \tag{5.130c}$$

Now we discuss the solution of the problem (3) Subtracting the solution of problem (2) from Eq. (5.129), the temperature potential in ς plane of problem (3) can be obtained:

$$\mathrm{g}(\varsigma_T) = R_T\left[\delta_T \ln\varsigma_T + (1/2)m_T\varsigma_T^{-2}\right] \tag{5.132}$$

The electric field inside the hole filled with air is fully the same as that in Sect. 3.4.2 and Eqs. (3.81), (3.82a), (3.82b), (3.83), (3.84), and (3.85) are still held. The complex potential $\phi(\varsigma)$ is still expressed by Eq. (3.85), i.e.,

$$\begin{aligned} &\varphi^{\mathrm{c}}(\rho,\psi) = \phi(\varsigma) + \overline{\phi(\varsigma)} \\ &\phi(\varsigma) = \sum_{k=-\infty}^{\infty} h_k\varsigma^k, \quad h_{-k} = \rho_0^{2k}h_k = m^k h_k \quad (\text{not} \quad \text{summed} \quad \text{on} \quad k), \quad \rho_0 \le |\varsigma| \le 1 \end{aligned} \tag{5.133}$$

From Eq. (5.120) it is seen that $\boldsymbol{f}(z_P)$ and $\mathrm{g}(z_T)$ have the similar role in the generalized displacements, so $\boldsymbol{f}(\varsigma_P)$ in S can be assumed in the following form:

$$\boldsymbol{f}(\varsigma_P) = \langle\ln(\varsigma_P)\rangle\boldsymbol{p} + \boldsymbol{f}_0(\varsigma_P), \quad \boldsymbol{f}_0(\varsigma_P) = \sum_{k=1}^{\infty}\left(\varsigma_P^{-k}\right) \quad \boldsymbol{a}_k; \quad |\varsigma_P| \ge 1 \tag{5.134}$$

Substitution of Eqs. (5.132) and (5.133) into Eq. (5.120) yields

$$\boldsymbol{U} = 2\mathrm{Re}\left\{\boldsymbol{A}[\langle\ln\varsigma_P\rangle\boldsymbol{p} + \boldsymbol{f}_0(\varsigma_P)] + \boldsymbol{c}R_T\delta_T\left[\ln\varsigma_T + (1/2)m_T\varsigma_T^{-2}\right]\right\} \tag{5.135}$$

In Eqs. (5.132), (5.133), (5.134), and (5.135) functions $\mathrm{g}(\varsigma),\boldsymbol{f}(\varsigma),\phi(\varsigma)$ are all the functions of ς, but in Eq. (5.130c) we need their derivatives with s and n on the L in the z plane, so the following relations are needed. Eq. (3.82) yields

$$\begin{aligned}
&x_1 = a\cos\psi, \quad x_2 = b\sin\psi; \quad \mathrm{d}x_1/\mathrm{d}s = -a\sin\psi\mathrm{d}\psi/\mathrm{d}s; \quad \mathrm{d}x_2/\mathrm{d}s = b\cos\psi\mathrm{d}\psi/\mathrm{d}s\\
&\rho\sin\theta = a\sin\psi, \quad \rho\cos\theta = b\cos\psi; \quad \mathrm{d}s = \rho\mathrm{d}\psi, \quad \rho^2 = a^2\sin^2\psi + b^2\cos^2\psi\\
&\partial\varsigma_T/\partial\psi = \partial\varsigma_P/\partial\psi = \mathrm{i}e^{\mathrm{i}\psi} = \mathrm{i}\sigma, \quad \partial z/\partial\psi = -a\sin\psi + \mathrm{i}b\cos\psi = \rho(-\sin\theta + \mathrm{i}\cos\theta)\\
&\partial z_T/\partial\psi_T = \rho(-\sin\theta_T + \mu_T\cos\theta_T), \quad \partial z_j/\partial\psi_j = \rho\left(-\sin\theta_j + \mu_j\cos\theta_j\right); \quad \text{on} \quad \Gamma
\end{aligned} \tag{5.136}$$

Using Eq. (5.136) it is easy to get

$$\begin{aligned}
&\mathrm{g}_{,s} = \frac{\partial \mathrm{g}}{\partial\varsigma_T}\frac{\partial\varsigma_T}{\partial\psi}\frac{\partial\psi_T}{\partial z_T}\left(\frac{\partial z_T}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial z_T}{\partial x_2}\frac{\partial x_2}{\partial s}\right) = \mathrm{i}\frac{\sigma \mathrm{g}'(\sigma)}{\rho} = \mathrm{i}\frac{R_T\delta_T}{\rho}\left(1 - m_T\frac{1}{\sigma^2}\right)\\
&f_{P,s} = \frac{\partial f_P}{\partial\varsigma_P}\frac{\partial\varsigma_P}{\partial\psi}\frac{\partial\psi}{\partial z_P}\left(\frac{\partial z_P}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial z_P}{\partial x_2}\frac{\partial x_2}{\partial s}\right) = \mathrm{i}\frac{\sigma f'_{P,s}(\sigma)}{\rho} = \mathrm{i}\frac{p_P}{\rho} - \frac{\mathrm{i}}{\rho}\sum_{k=1}^{\infty} k a_{Pk}\sigma^{-k}\\
&\phi_{,n} = \frac{\partial\phi}{\partial\varsigma}\frac{\partial\varsigma}{\partial\psi}\frac{\partial\psi}{\partial z}\left(\frac{\partial z}{\partial x_1}n_1 + \frac{\partial z}{\partial x_2}n_2\right) = \frac{\sigma\phi'(\sigma)}{\rho} = \frac{1}{\rho}\sum_{k=1}^{\infty} k\left(h_k\sigma^k - h_{-k}\sigma^{-k}\right)
\end{aligned} \tag{5.137}$$

Substituting Eqs. (5.135), (5.137), and (5.122) into the connective conditions on the interface Γ and the single valued condition in Eq. (5.130c) and then comparing the coefficients of the corresponding terms on both sides in result equations, we get

$$\begin{aligned}
&A_{iP}p_P - \bar{A}_{iP}\bar{p}_P + c_iR_T\delta_T + \bar{c}_i\bar{R}_T\bar{\delta}_T = 0; \quad (\text{single valued condition}), \quad P = 1,2,4\\
&B_{iP}p_P - \bar{B}_{iP}\bar{p}_P + d_iR_T\delta_T - \bar{d}_i\bar{R}_T\bar{\delta}_T = 0; \quad (-\mathrm{d}\Phi_i/\mathrm{d}s = T_i, \quad -\mathrm{d}\Phi_4/\mathrm{d}s = D_n)\\
&kB_{ik}a_{Pk} - \mathrm{i}\epsilon_0 k\left(h_k m^k + \bar{h}_k\right)\delta_{P4} = \begin{cases} -d_iR_T\delta_T m_T, & k = 2\\ 0, & k \neq 2\end{cases}; \quad (\boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^{\mathrm{c}}\cdot\boldsymbol{n})\\
&A_{4k}a_{4k} - \left(h_k m^k + \bar{h}_k\right) = \begin{cases} -(1/2)c_4R_T m_T\delta_T, & k = 2\\ 0 & k \neq 2\end{cases}; \quad (\varphi = \varphi^{\mathrm{c}}); \quad \text{on} \quad \Gamma
\end{aligned} \tag{5.138}$$

where δ_{P4} is Kronecker delta. Solving undetermined coefficients finally yields

$$\begin{aligned}
&\phi(\varsigma) = h_2\left(\varsigma^2 + m^2\varsigma^{-2}\right); \quad \varphi^{\mathrm{c}}(\varsigma) = 2\mathrm{Re}\phi(\varsigma) \quad \rho_0 \leq |\varsigma| \leq 1\\
&\boldsymbol{f}(\varsigma_P) = \langle\ln(\varsigma_P)\rangle\boldsymbol{p} + \boldsymbol{a}_2\left\langle\varsigma_P^{-2}\right\rangle; \quad |\varsigma_j| \geq 1\\
&\mathrm{g}(\varsigma_T) = R_T\delta_T\left[\ln\varsigma_T + (1/2)m_T\varsigma_T^{-2}\right]; \quad |\varsigma_T| \geq 1\\
&\boldsymbol{a}_k = 0, \quad h_k = 0; \quad \text{if} \quad k \neq 2
\end{aligned} \tag{5.139}$$

It is seen from Eq. (5.139) that $\mathrm{g}'(\varsigma_T), f'_P(\varsigma_P) \to 0$ when $|\varsigma_P|,\ |\varsigma_T| \to \infty$. So the boundary conditions at infinity are satisfied also.

In Eq. (5.139) $\varphi^{c}(\varsigma)$ can also be rewritten as

$$\varphi^{c}(x_1, x_2) = -2m(d_2 + \bar{d}_2) + R^{-2}\left(d_2 z^2 + \bar{d}_2 \bar{z}^2\right) \tag{5.140}$$

Therefore, the electric field in the elliptic hole varies linearly with the coordinates.

5.5 Interface Crack in Dissimilar Pyroelectric Material

5.5.1 General Discussion

The fundamental theory of the pyroelectric material has been discussed in Sect. 5.4. Now the interface crack in dissimilar pyroelectric material (see Fig. 4.2) will be discussed (Shen and Kuang 1998; Gao and Wang 2001). The general solutions $\boldsymbol{U}\left(z_j, z_T\right)$, $\boldsymbol{\Phi}\left(z_j, z_T\right)$, and ϑ are shown in Eqs. (5.120), (5.122), and (5.112), respectively. The boundary conditions are assumed:

$$\begin{aligned}
&\boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = \boldsymbol{\Phi}_{\mathrm{II},1}(x_1) = \boldsymbol{\Sigma}_0(x_1), \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1) = q_0(x_1), \quad x \in L_{\mathrm{c}} \\
&\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = \boldsymbol{0}, \quad \boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = \boldsymbol{\Phi}_{\mathrm{II},1}(x_1) \\
&\vartheta_{\mathrm{I}}(x_1) = \vartheta_{\mathrm{II}}(x_1), \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1), \quad x \in L - L_{\mathrm{c}} \\
&\boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) \to \boldsymbol{0}, \quad q_{\mathrm{I}n} = q_{\mathrm{II}n} \to 0; \quad |z| \to \infty
\end{aligned} \tag{5.141}$$

where $\hat{\boldsymbol{d}}$ is the displacement disconnected value between crack surfaces. Equation (5.141) shows that on whole axis x_1 we have

$$\boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = \boldsymbol{\Phi}_{\mathrm{II},1}(x_1), \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1); \quad -\infty < x_1 < \infty, \quad x_2 = 0 \tag{5.142}$$

From Equation (5.115) it is known that $q_2 = -\mathrm{i}\alpha g''(z_T) + \mathrm{i}\alpha \bar{g}''(\bar{z}_T)$, where z_T, α are shown in Eqs. (5.112) and (5.114), respectively. Equation (5.142) yields

$$\begin{aligned}
&-\mathrm{i}\alpha_{\mathrm{I}} g_{\mathrm{I}}''(x_1) + \mathrm{i}\alpha_{\mathrm{I}} \bar{g}_{\mathrm{I}}''(\bar{x}_1) = -\mathrm{i}\alpha_{\mathrm{II}} g_{\mathrm{II}}''(x_1) + \mathrm{i}\alpha_{\mathrm{II}} \bar{g}_{\mathrm{II}}''(\bar{x}_1) \quad \text{or} \\
&\mathrm{i}\alpha_{\mathrm{I}} g_{\mathrm{I}}''^{+}(x_1) + \mathrm{i}\alpha_{\mathrm{II}} \bar{g}_{\mathrm{II}}''^{+}(x_1) = \mathrm{i}\alpha_{\mathrm{II}} g_{\mathrm{II}}''^{-}(x_1) + \mathrm{i}\alpha_{\mathrm{I}} \bar{g}_{\mathrm{I}}''^{-}(x_1)
\end{aligned} \tag{5.143}$$

Analogous to Eq. (4.22) from Eq. (5.143) we have

$$\bar{g}_{\mathrm{II}}''(z_T) = -(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}}) g_{\mathrm{I}}''(z_T), \quad x_2 > 0; \quad \bar{g}_{\mathrm{I}}''(z_T) = -(\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}}) g_{\mathrm{II}}''(z_T), \quad x_2 < 0 \tag{5.144}$$

It is assumed that the temperature satisfies the same equation:

$$\bar{g}_{\mathrm{II}}'(z_T) = -(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}}) g_{\mathrm{I}}'(z_T), \quad x_2 > 0; \quad \bar{g}_{\mathrm{I}}'(z_T) = -(\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}}) g_{\mathrm{II}}'(z_T), \quad x_2 < 0 \tag{5.145}$$

Equations (5.112) and (5.145) yield

$$\begin{aligned}\vartheta_{\mathrm{I}}(x_1) &= \mathrm{g}_{\mathrm{I}}'(x_1) + \bar{\mathrm{g}}_{\mathrm{I}}'(\bar{x}_1) = \mathrm{g}_{\mathrm{I}}'(x_1) - (\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})\mathrm{g}_{\mathrm{II}}'(x_1) \\ \vartheta_{\mathrm{II}}(x_1) &= \mathrm{g}_{\mathrm{II}}'(x_1) + \bar{\mathrm{g}}_{\mathrm{II}}'(\bar{x}_1) = \mathrm{g}_{\mathrm{II}}'(x_1) - (\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\mathrm{g}_{\mathrm{I}}'(x_1)\end{aligned} \tag{5.146}$$

Analogously from Eqs. (5.142), (5.145), and (5.122) we get

$$\begin{aligned}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) + (\boldsymbol{d}_{\mathrm{I}} + \bar{\boldsymbol{d}}_{\mathrm{II}}\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\mathrm{g}_{\mathrm{I}}'(z) &= \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z), \quad x_2 > 0 \\ \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) + (\boldsymbol{d}_{\mathrm{II}} + \bar{\boldsymbol{d}}_{\mathrm{I}}\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})\mathrm{g}_{\mathrm{II}}'(z) &= \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z), \quad x_2 < 0\end{aligned} \tag{5.147}$$

Equations (5.120) and (5.147) yield

$$\begin{aligned}\boldsymbol{U}_{\mathrm{I}}'(x_1) = \boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + c_{\mathrm{I}}\mathrm{g}_{\mathrm{I}}'(x_1) + \bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}[&\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) \\ &+(\boldsymbol{d}_{\mathrm{II}} + \bar{\boldsymbol{d}}_{\mathrm{I}}\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})\mathrm{g}_{\mathrm{II}}'(x_1)] - (\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})\bar{c}_{\mathrm{I}}\mathrm{g}_{\mathrm{II}}'(x_1) \\ \boldsymbol{U}_{\mathrm{II}}'(x_1) = \boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \boldsymbol{c}_{\mathrm{II}}\mathrm{g}_{\mathrm{II}}'(x_1) + \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}[&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) \\ &+(\boldsymbol{d}_{\mathrm{I}} + \bar{\boldsymbol{d}}_{\mathrm{II}}\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\mathrm{g}_{\mathrm{I}}'(x_1)] - (\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\bar{\boldsymbol{c}}_{\mathrm{II}}\mathrm{g}_{\mathrm{I}}'(x_1)\end{aligned} \tag{5.148}$$

5.5.2 The Solution of Temperature

Using Eq. (5.146) and $\vartheta_{\mathrm{I}}(x_1) = \vartheta_{\mathrm{II}}(x_1)$ on the connective surface yields

$$\mathrm{g}_{\mathrm{I}}'(x_1)[1 + (\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})] = \mathrm{g}_{\mathrm{II}}'(x_1)[1 + (\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})], \quad x \notin L_{\mathrm{c}} \tag{5.149}$$

So we can construct a function $\theta(z_T)$ analytic in whole z_T plane except L_{c}:

$$\theta(z_T) = \begin{cases}[1 + (\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})]\mathrm{g}_{\mathrm{I}}(z_T), & x_2 > 0 \\ [1 + (\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})]\mathrm{g}_{\mathrm{II}}(z_T), & x_2 < 0\end{cases}, \quad x \notin L_{\mathrm{c}} \tag{5.150}$$

The heat flow on the crack surface is

$$\begin{aligned}q_{\mathrm{I}2} &= -\lambda_{2j}\vartheta_{,j} = -\mathrm{i}\alpha_{\mathrm{I}}\mathrm{g}_{\mathrm{I}}''(x_1) + \mathrm{i}\alpha_{\mathrm{I}}\bar{\mathrm{g}}_{\mathrm{I}}''(\bar{x}_1) = -\mathrm{i}\alpha_{\mathrm{I}}\mathrm{g}_{\mathrm{I}}''(x_1) - \mathrm{i}\alpha_{\mathrm{II}}\mathrm{g}_{\mathrm{II}}''(x_1) \\ &= -\mathrm{i}[\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}/(\alpha_{\mathrm{I}} + \alpha_{\mathrm{II}})]\left[\theta''^{+}(x_1) + \theta''^{-}(x_1)\right]\end{aligned} \tag{5.151}$$

So the boundary condition of the heat flow on the crack surface is reduced to

$$\theta''^{+}(x_1) + \theta''^{-}(x_1) = \mathrm{i}[(\alpha_{\mathrm{I}} + \alpha_{\mathrm{II}})/\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}]q_0(x_1), \quad x \in L_{\mathrm{c}} \tag{5.152}$$

Its solution is

$$\begin{aligned}\theta''(z_T) &= \frac{\alpha_{\rm I}+\alpha_{\rm II}}{2\pi\alpha_{\rm I}\alpha_{\rm II}} Z_0(z_T)\int_{L_c}\frac{q_0(x_1)\mathrm{d}x_1}{Z_0^+(x_1)(x_1-z_T)} + Z_0(z_T)C(z_T)\\ Z_0(z_T) &= \prod\nolimits_{j=1}^{n}\left(z_T-a_j\right)^{-1/2}\left(z_T-b_j\right)^{-1/2}\end{aligned} \tag{5.153}$$

where $C(z_T)$ is the polynomial degree n of z_T.

5.5.3 The Solution of Generalized Stress

Because on $L-L_c$ $i\boldsymbol{d}'(x_1)=0$, so

$$\begin{aligned}&\boldsymbol{H}\boldsymbol{B}_{\rm I}\boldsymbol{F}_{\rm I}(x_1)+\left\{\mathrm{i}[\boldsymbol{c}_{\rm I}+(\alpha_{\rm I}/\alpha_{\rm II})\bar{\boldsymbol{c}}_{\rm II}]+\bar{\boldsymbol{Y}}_{\rm II}[\boldsymbol{d}_{\rm I}+(\alpha_{\rm I}/\alpha_{\rm II})\bar{\boldsymbol{d}}_{II}]\right\}g_{\rm I}'(x_1)\\ &\quad=\bar{\boldsymbol{H}}\boldsymbol{B}_{\rm II}\boldsymbol{F}_{\rm II}(x_1)+\left\{\mathrm{i}[\boldsymbol{c}_{\rm II}+(\alpha_{\rm II}/\alpha_{\rm I})\bar{\boldsymbol{c}}_{\rm I}]+\bar{\boldsymbol{Y}}_{\rm I}[\boldsymbol{d}_{\rm II}+(\alpha_{\rm II}/\alpha_{\rm I})\bar{\boldsymbol{d}}_{\rm I}]\right\}g_{\rm II}'(x_1),\quad x\notin L_c\end{aligned} \tag{5.154}$$

where $\boldsymbol{H}=\boldsymbol{Y}_{\rm I}+\bar{\boldsymbol{Y}}_{\rm II},\ \boldsymbol{Y}_\alpha=\mathrm{i}\boldsymbol{A}_\alpha\boldsymbol{B}_\alpha^{-1}$ $(\alpha={\rm I},{\rm II})$. So we can construct a function $\boldsymbol{h}(z)$ analytic in whole z plane except L_c:

$$\boldsymbol{h}(z)=\begin{cases}\boldsymbol{B}_{\rm I}\boldsymbol{F}_{\rm I}(z)+(\alpha_{\rm I}+\alpha_{\rm II})^{-1}\boldsymbol{H}^{-1}[i(\alpha_{\rm II}\boldsymbol{c}_{\rm I}+\alpha_{\rm I}\bar{\boldsymbol{c}}_{\rm II})\\ \quad+\bar{\boldsymbol{Y}}_{\rm II}(\alpha_{\rm II}\boldsymbol{d}_{\rm I}+\alpha_{\rm I}\bar{\boldsymbol{d}}_{\rm II})]\theta'(z),\quad x_2>0\\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\Big\{\boldsymbol{B}_{\rm II}\boldsymbol{F}_{\rm II}(z)+(\alpha_{\rm I}+\alpha_{\rm II})^{-1}\bar{\boldsymbol{H}}^{-1}[i(\alpha_{\rm I}\boldsymbol{c}_{\rm II}+\alpha_{\rm II}\bar{\boldsymbol{c}}_{\rm I})\\ \quad+\bar{\boldsymbol{Y}}_{\rm I}(\alpha_{\rm I}\boldsymbol{d}_{\rm II}+\alpha_{\rm II}\bar{\boldsymbol{d}}_{\rm I})]\theta'(z)\Big\},\quad x_2<0\end{cases} \tag{5.155}$$

Using Eqs. (5.145), (5.147), and (5.155), Eq. (5.122) can be reduced to

$$\begin{aligned}\boldsymbol{\Phi}_{{\rm I},1}(x_1)&=\boldsymbol{h}^+(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1)-\boldsymbol{\eta}_1\theta'^+(x_1)-\boldsymbol{\eta}_2\theta'^-(x_1)\\ \boldsymbol{\eta}_1&=-\bar{\boldsymbol{\eta}}_2=(\alpha_{\rm I}+\alpha_{\rm II})^{-1}\boldsymbol{H}^{-1}\left\{[\mathrm{i}(\alpha_{\rm II}\boldsymbol{c}_{\rm I}+\alpha_{\rm I}\bar{\boldsymbol{c}}_{\rm II})+\bar{\boldsymbol{Y}}_{\rm II}(\alpha_{\rm II}\boldsymbol{d}_{\rm I}+\alpha_{\rm I}\bar{\boldsymbol{d}}_{\rm II})]-\alpha_{\rm II}\boldsymbol{d}_{\rm I}\right\}\\ \boldsymbol{\eta}_2&=-\bar{\boldsymbol{\eta}}_1=(\alpha_{\rm I}+\alpha_{\rm II})^{-1}\bar{\boldsymbol{H}}^{-1}\left\{[\mathrm{i}(\alpha_{\rm I}\boldsymbol{c}_{\rm II}+\alpha_{\rm II}\bar{\boldsymbol{c}}_{\rm I})+\bar{\boldsymbol{Y}}_{\rm I}(\alpha_{\rm I}\boldsymbol{d}_{\rm II}+\alpha_{\rm II}\bar{\boldsymbol{d}}_{\rm I})]-\alpha_{\rm I}\boldsymbol{d}_{\rm II}\right\}\end{aligned} \tag{5.156}$$

Substituting Eq. (5.156) into the generalized stress boundary condition in (5.141) yields

$$\boldsymbol{h}^+(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1)=\tilde{\boldsymbol{\Sigma}}_0(x_1),\quad \tilde{\boldsymbol{\Sigma}}_0(x_1)=\boldsymbol{\Sigma}_0(x_1)+\boldsymbol{\eta}_1\theta'^+(x_1)+\boldsymbol{\eta}_2\theta'^-(x_1) \tag{5.157}$$

Equation (5.157) is identical with (4.28) except using $\tilde{\boldsymbol{\Sigma}}_0(x_1)$ instead of $\boldsymbol{\Sigma}_0(x_1)$, so its solution is still expressed by Eqs. (4.41a) and (4.9):

$$\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z)=\boldsymbol{Q}(z)\left[\boldsymbol{C}(z)+\frac{1}{2\pi\mathrm{i}}\int_L\frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_0(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1-z)}\right],\quad \boldsymbol{Q}(z)=\left\langle Y_0^{(j)}(z)\right\rangle \tag{5.158}$$

From Eq. (5.157) it is seen that its homogeneous equation is fully identical with (4.29) and does not relate to the temperature, so the eigenvalues and eigenvectors of both equations are also the same. Therefore, $\boldsymbol{Q}(z)$ and $\boldsymbol{\Omega}$ in Eq. (5.158) are still expressed by Eq. (4.37).

On the connective surface $\boldsymbol{h}^+(x_1)=\boldsymbol{h}^-(x_1)=\boldsymbol{h}(x_1)$, $\theta'^+(x_1)=\theta'^-(x_1)=\theta'(x_1)$, so we have

$$\boldsymbol{\Sigma}_2(x_1)=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)=\left(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{h}(x_1)-(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\theta'(x_1),\quad x\in L-L_{\mathrm{c}} \tag{5.159}$$

The open displacement disconnected value $\hat{\boldsymbol{d}}$ behind the crack tip is

$$\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{U}'_{\mathrm{I}}(x_1)-\boldsymbol{U}'_{\mathrm{II}}(x_1)=-\mathrm{i}\boldsymbol{H}[\boldsymbol{h}^+(x_1)-\boldsymbol{h}^-(x_1)],\quad x\notin L_{\mathrm{c}} \tag{5.160}$$

5.5.4 A Single Interface Crack

In the case of a crack of length $2a$, we have $Z_0(z_T)=\sqrt{z_T^2-a^2}$. If only the normal heat flow q_0 on the crack surface, Eq. (5.153) yields

$$\begin{aligned}\theta''(z_T)&=\mathrm{i}q_0^*\left[1-\left(z_T\Big/\sqrt{z_T^2-a^2}\right)\right]+C_1z_T+C_0\\ \theta'(z_T)&=\mathrm{i}q_0^*\left(z_T-\sqrt{z_T^2-a^2}\right),\quad q_0^*=q_0[(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})/2\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}]\end{aligned} \tag{5.161}$$

where $C_1=0$ due to $\boldsymbol{q}\cdot\boldsymbol{n}=\mathbf{0}$ at infinity, and $C_0=0$ due to the temperature single value condition $\int_{-a}^{a}\left[\theta''^+(x_1)-\theta''^-(x_1)\right]\mathrm{d}x_1=0$. Equation (5.150) yields

$$\mathrm{g}_I''(z_T)=[\alpha_{\mathrm{II}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})]\theta''(z_T),\quad \mathrm{g}_{\mathrm{II}}''(z_T)=[\alpha_{\mathrm{I}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})]\theta''(z_T) \tag{5.162}$$

Because Eq. (5.158) is decoupling, on the crack surface, for normalized Ω we have

$$\boldsymbol{h}(z)=\boldsymbol{\Omega}\boldsymbol{Q}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{\tilde{\boldsymbol{\Sigma}}_0(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1-z)}\right\},\quad Y_0^{(i)}(z)=\frac{1}{\sqrt{z^2-a^2}}\left(\frac{z-a}{z+a}\right)^{\mathrm{i}\varepsilon_i} \tag{5.163}$$

In Eq. (5.163) the integrated function containing $\sqrt{z^2-a^2}$, so when use Eq. (4.18), $g^*=-1$ should be used due to $\lim_{z\to x^-}\sqrt{z^2-a^2}=-\lim_{z\to x^+}\sqrt{z^2-a^2}$. These integrals are

$$\begin{aligned}
\frac{1}{2\pi\,\mathrm{i}}\int_{-a}^{a}\frac{\mathrm{d}x_1}{Y_0^{(j)+}(x_1)(x_1-z)} &= \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\left\{\frac{1}{Y_0^{(j)}(z)}-\left(z+2\mathrm{i}\varepsilon_j a\right)\right\}\\
\frac{1}{2\pi\,\mathrm{i}}\int_{-a}^{a}\frac{x_1\mathrm{d}x_1}{Y_0^{(j)+}(x_1)(x_1-z)} &= \frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\left\{\frac{z}{Y_0^{(j)}(z)}-\left[z^2+2\mathrm{i}\varepsilon_j az-\frac{a^2}{2}\left(1+4\varepsilon_j^2\right)\right]\right\}\\
\frac{1}{2\pi\,\mathrm{i}}\int_{-a}^{a}\frac{\mathrm{i}\sqrt{a^2-x_1^2}\mathrm{d}x_1}{Y_0^{(j)+}(x_1)(x_1-z)} &= \frac{1}{1-\mathrm{e}^{2\pi\varepsilon_j}}\left\{\frac{\sqrt{z^2-a^2}}{Y_0^{(j)}(z)}-\left[z^2+2\mathrm{i}\varepsilon_j az-a^2\left(1+2\varepsilon_j^2\right)\right]\right\}
\end{aligned}\tag{5.164}$$

Using Eq. (5.164), Eq. (5.163) is reduced to

$$\begin{aligned}
\boldsymbol{h}(z) &= \boldsymbol{\Omega Q}(z)(\boldsymbol{C}_1 z+\boldsymbol{C}_0)+\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle 1-(z+2\mathrm{i}\varepsilon_* a)Y_0^{(j)}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0\\
&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle z-\left[z^2+2\mathrm{i}\varepsilon_j az-\frac{a^2}{2}\left(1+4\varepsilon_j^2\right)\right]Y_0^{(j)}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\\
&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1-\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle\sqrt{z^2-a^2}-\left[z^2+2\mathrm{i}\varepsilon_j az-a^2(1+2\varepsilon j)\right]Y_0^{(j)}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)
\end{aligned}\tag{5.165}$$

At infinity, $\boldsymbol{Q}(z)\to\boldsymbol{I}/z$, $\theta'(z_T)\to 0$, $\boldsymbol{\Sigma}_2(x_1)=\boldsymbol{0}$, from Eqs. (5.159) and (5.165) we get

$$\begin{aligned}
\boldsymbol{C}_1 &= \mathrm{i}q_0^*\left\langle\frac{2\mathrm{i}\varepsilon_j a}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)+\mathrm{i}q_0^*\left\langle\frac{2\mathrm{i}\varepsilon_j a}{1-\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)\langle 2\mathrm{i}\varepsilon_j a\rangle\\
C_1^{(j)} &= \mathrm{i}q_0^*\left\{\frac{2\mathrm{i}\varepsilon_j a}{1+\mathrm{e}^{2\pi\varepsilon_j}}\bar{\Omega}_{jk}^{\mathrm{T}}(\eta_{1k}+\eta_{2k})+\frac{2\mathrm{i}\varepsilon_j a}{1-\mathrm{e}^{2\pi\varepsilon_j}}\bar{\Omega}_{jk}^{\mathrm{T}}(\eta_{2k}-\eta_{1k})\right\},\quad \bar{\Omega}_{jk}^{\mathrm{T}}=\Omega_{kj}
\end{aligned}\tag{5.166}$$

Substitution of Eq. (5.166) into Eq. (5.165) yields

$$\begin{aligned}
\boldsymbol{h}(z) &= \boldsymbol{\Omega Q}(z)\boldsymbol{C}_0+\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle 1-\left(z+2\mathrm{i}\varepsilon_j a\right)Y_0^{(j)}(z)\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0\\
&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle z-\left[z^2-\frac{a^2}{2}\left(1+4\varepsilon_j^2\right)\right]Y_0^{(j)}(z)\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\\
&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1-\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle\sqrt{z^2-a^2}-\left[z^2-a^2\left(1+2\varepsilon_j^2\right)\right]Y_0^{(j)}(z)\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)
\end{aligned}\tag{5.167}$$

$\boldsymbol{C}_0$ is determined by the single value condition, and according to Eq. (5.160) it is equivalent to

$$\boldsymbol{H}\int_{-a}^{a}[\boldsymbol{h}^{+}(x_1)-\boldsymbol{h}^{-}(x_1)]\mathrm{d}x_1=0,\quad H_{ij}\int_{-a}^{a}\left[h_j^{+}(x_1)-h_j^{-}(x_1)\right]\mathrm{d}x_1=0 \tag{5.168}$$

On the crack surface there has $\left\langle Y_0^{(j)-}(x_1)\right\rangle=-\left\langle \mathrm{e}^{2\pi\varepsilon_j}Y_0^{(j)+}(x_1)\right\rangle$ or $\boldsymbol{Q}^{+}-\boldsymbol{Q}^{-}=\left\langle 1+\mathrm{e}^{2\pi\varepsilon_j}\right\rangle\boldsymbol{Q}^{+}$. Using the following equation (Shen and Kuang 1998)

$$\int_{-a}^{a}\frac{x^n}{\sqrt{a^2-x^2}}\left(\frac{a-x}{a+x}\right)^{\mathrm{i}\varepsilon}\mathrm{d}x=\begin{cases}\pi/\cosh\pi\varepsilon & \text{when } n=0\\ -2\mathrm{i}\pi a\varepsilon/\cosh\pi\varepsilon & \text{when } n=1\\ (1-4\varepsilon^2)\pi a^2/2\cosh\pi\varepsilon & \text{when } n=2\end{cases} \tag{5.169}$$

and noting $\int_{-a}^{a}\sqrt{x_1^2-a^2}\mathrm{d}x_1=\pm\,\mathrm{i}\pi a^2/2$, from the single valued condition we get

$$-\boldsymbol{C}_0=\mathrm{i}q_0^{*}a^2\left\langle\frac{4\varepsilon_j^2}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)+\mathrm{i}q_0^{*}a^2\left\langle\frac{1+8\varepsilon_j^2+2\mathrm{i}\cosh\pi\varepsilon_j}{2(1-\mathrm{e}^{2\pi\varepsilon_j})}\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1) \tag{5.170}$$

The stress intensity is

$$\boldsymbol{K}=[K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_D]^{\mathrm{T}}=\lim_{x_1\to b_j}\sqrt{2\pi(x_1-b_j)}\boldsymbol{\Omega}\left\langle\left(x_1-b_j\right)^{-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1) \tag{5.171}$$

where $\boldsymbol{\Sigma}_2(x_1)$ is determined by Eq. (5.159).

For a homogeneous material $\boldsymbol{A}_{\mathrm{I}}=\boldsymbol{A}_{\mathrm{II}}=\boldsymbol{A}$, and $\boldsymbol{H}=\bar{\boldsymbol{H}}$, $\boldsymbol{C}_0=\boldsymbol{0}$, $Y_0^{(j)}=1\Big/\sqrt{z_j^2-a^2}$. So the solution is

$$\begin{aligned}
&\theta''(z_T)=\mathrm{i}q_0^{*}\left\{1-\frac{z_T}{\sqrt{z_T^2-a^2}}\right\},\quad \theta'(z_T)=\mathrm{i}q_0^{*}\left(z_T-\sqrt{z_T^2-a^2}\right)\\
&\mathrm{g}_{\mathrm{I}}''(z_T)=\mathrm{g}_{\mathrm{II}}''(z_T)=\frac{1}{2}\theta''(z_T)=\frac{\mathrm{i}q_0^{*}}{2}\left\{1-\frac{z_T}{\sqrt{z_T^2-a^2}}\right\}\\
&\boldsymbol{h}(z)=\frac{1}{2}\boldsymbol{\Omega}\langle 1-z\boldsymbol{Q}(z)\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0+\mathrm{i}q_0^{*}\boldsymbol{\Omega}\left\langle z-\frac{2z^2-a^2}{2}\boldsymbol{Q}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\eta}
\end{aligned} \tag{5.172}$$

And the asymptotic stress field near the crack tip $x_1=a$ is

$$\begin{aligned}
\boldsymbol{\Sigma}_2(x_1)&=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)=2\boldsymbol{h}(x_1)-2\boldsymbol{\eta}\theta'(x_1)\\
&=-\boldsymbol{\Omega}\left\langle\sqrt{\frac{a}{2}}\frac{1}{\sqrt{x_1-a}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0-\mathrm{i}q_0^{*}a\boldsymbol{\Omega}\left\langle\sqrt{\frac{a}{2}}\frac{1}{\sqrt{x_1-a}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\eta}
\end{aligned} \tag{5.173}$$

The stress intensity factor at $x_1 = a$ is

$$\begin{aligned} \boldsymbol{K} &= [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_D]^{\mathrm{T}} = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)}\boldsymbol{\Sigma}_2(x_1) \\ &= -\sqrt{\pi a}\left(\boldsymbol{\Omega}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0 + \mathrm{i}q_0^* a\boldsymbol{\Omega}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\eta}\right) = -\sqrt{\pi a}\left(\boldsymbol{\Sigma}_0 + \mathrm{i}q_0^* a\boldsymbol{\eta}\right) \end{aligned} \tag{5.174}$$

5.6 Point Heat Source and Interaction with Cracks

5.6.1 Point Heat Source in Piezoelectric and Bi-piezoelectric Material

Hwu (1990) discussed the thermal stress in an anisotropic elastic material. Shen et al. (1995) and Shen and Kuang (1998) discussed the thermal stress in a pyroelectric material, the point heat source, and their interactions.

1. *Heat source in a homogeneous material* For a point heat source, the temperature $\vartheta = T - T_0$ can be expressed as

$$\begin{aligned} &\vartheta(x_1, x_2) = 2\mathrm{Re}\,\mathrm{g}'(z_T), \quad z_T = x_1 + \mu_T x_2, \quad \mu_T = (-\lambda_{12} + i\alpha)/\lambda_{22} \\ &\mathrm{g}'(z_T) = \mathrm{g}_0'(z_T) = c\ln(z_T - z_{T0}), \quad \mathrm{g}_0''(z_T) = c/(z_T - z_{T0}) \end{aligned} \tag{5.175}$$

According to Eq. (5.116) for a point heat source with strength M located at $z_0(x_{10}, x_{20})$ in an infinite homogeneous pyroelectric material, c is determined by the following equation:

$$\begin{aligned} &M = \oint q_n \mathrm{d}s = -2\mathrm{Im}\{\alpha[\mathrm{g}'(z_T) - \mathrm{g}'(z_0)]\}_0^{2\pi} = -4\pi\alpha c; \quad c = -M/4\pi\alpha \\ &q_1 = 2\mathrm{Re}[\mathrm{i}\alpha\mu_T \mathrm{g}''(z_T)], \quad q_2 = -2\mathrm{Re}[\mathrm{i}\alpha\mathrm{g}''(z_T)]; \quad \alpha = \lambda_{22}(\mu_T - \bar{\mu}_T)/2\mathrm{i} \end{aligned} \tag{5.176}$$

So finally the solution of the temperature in an infinite homogeneous pyroelectric material is

$$\vartheta = 2\mathrm{Re}\,\mathrm{g}_0'(z_T) = -(M/2\pi\alpha)\mathrm{Re}\ln(z_T - z_{T0}), \quad z_{T0} = x_{10} + \mu_T x_{20} \tag{5.177}$$

2. *Heat source in a bimaterial* The solving method of a heat source in a bimaterial is analogous to that in Paragraph 3.6.2. Let the point heat source with strength M be located at $z_0(x_{10}, x_{20})$ in material II that occupied $S^-, x_2 < 0$. The solution can be assumed as

$$\begin{aligned} &\mathrm{g}'(z_T) = \begin{cases} \mathrm{g}_{\mathrm{I}}'(z_T), & z_T \in S^+ \\ \mathrm{g}_{\mathrm{II}}'(z_T) + \mathrm{g}_0'(z_T), & z_T \in S^- \end{cases} \\ &\mathrm{g}_0'(z_T) = c_{\mathrm{II}}\ln(z_T - z_{T0}), \quad \mathrm{g}_0''(z_T) = c_{\mathrm{II}}/(z_T - z_{T0}), \quad c_{\mathrm{II}} = -M/4\pi\alpha_{\mathrm{II}} \end{aligned} \tag{5.178}$$

Because heat flow and temperature are continuous in whole axis x_1, so according to Eqs. (5.115) and (5.112) it yields

$$\begin{aligned}&\alpha_{\mathrm{I}}g_{\mathrm{I}}''(x_1)-\alpha_{\mathrm{I}}\overline{g_{\mathrm{I}}''(x_1)}=\alpha_{\mathrm{II}}g_{\mathrm{II}}''(x_1)-\alpha_{\mathrm{II}}\overline{g_{\mathrm{II}}''(x_1)}+\alpha_{\mathrm{II}}g_0''(x_1)-\alpha_{\mathrm{II}}\overline{g_0''(x_1)}\\&g_{\mathrm{I}}'(x_1)+\overline{g_{\mathrm{I}}'(x_1)}=g_{\mathrm{II}}'(x_1)+g_0'(x_1)+\overline{g_{\mathrm{II}}'(x_1)}+\overline{g_0'(x_1)}\end{aligned}\tag{5.179}$$

If $\mathbf{q}\to\mathbf{0},T\to 0$ when $|z|\to\infty$, like Eqs. (3.161), (3.162), (3.163), (3.164), (3.165) or (4.22), (4.23) we have

$$\begin{aligned}&\alpha_{\mathrm{I}}g_{\mathrm{I}}''(z_T)+\alpha_{\mathrm{II}}\overline{g_{\mathrm{II}}''}(z_T)-\alpha_{\mathrm{II}}g_0''(z_T)=0,\quad \alpha_{\mathrm{II}}g_{\mathrm{II}}''(z_T)+\alpha_{\mathrm{I}}\overline{g_{\mathrm{I}}''}(z_T)-\alpha_{\mathrm{II}}\overline{g_0''}(z_T)=0\\&g_{\mathrm{I}}'(z_T)-\overline{g_{\mathrm{II}}'}(z_T)-g_0'(z_T)=0,\quad g_{\mathrm{II}}'(z_T)-\overline{g_{\mathrm{I}}'}(z_T)+\overline{g_0'}(z_T)=0\end{aligned}\tag{5.180}$$

Equations (5.178), (5.179), and (5.180) yield

$$\begin{aligned}&g'(z_T)=\begin{cases}g_{\mathrm{I}}'(z_T)=2\alpha_2 g_0'(z_T), & z\in S^+\\ g_{\mathrm{II}}'(z_T)+g_0'(z_T)=(\alpha_2-\alpha_1)\bar{g}_0'(z_T)+g_0'(z_T), & z\in S^-\end{cases}\\&\alpha_1=\alpha_{\mathrm{I}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}}),\quad \alpha_2=\alpha_{\mathrm{II}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})\end{aligned}\tag{5.181}$$

On the interface $x_2=0$ we have

$$q_2=-\mathrm{i}\alpha_{\mathrm{I}}g_{\mathrm{I}}''(z_T)+\mathrm{i}\alpha_{\mathrm{I}}\overline{g_{\mathrm{I}}''(z_T)}=-2\mathrm{i}\alpha_{\mathrm{I}}\alpha_2\left[g_0''(x_1)-\bar{g}_0''(x_1)\right]\tag{5.182}$$

Because the generalized stress and displacement are continuous on whole axis x_1, according to Eqs. (3.161), (3.162), (3.163), (3.164), and (3.165), we can derive

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)-\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z)-[(\alpha_2-\alpha_1)\bar{\boldsymbol{d}}_{\mathrm{II}}-2\alpha_2\boldsymbol{d}_{\mathrm{I}}+\boldsymbol{d}_{\mathrm{II}}]g_0'(z)=0\\&\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z)+[(\alpha_2-\alpha_1)\boldsymbol{d}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{d}}_{\mathrm{I}}+\bar{\boldsymbol{d}}_{\mathrm{II}}]\bar{g}_0'(z)=0\end{aligned}\tag{5.183}$$

and

$$\begin{aligned}&\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z)-[(\alpha_2-\alpha_1)\bar{\boldsymbol{c}}_{\mathrm{II}}-2\alpha_2\boldsymbol{c}_{\mathrm{I}}+\boldsymbol{c}_{\mathrm{II}}]g_0'(z)=0\\&\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z)+[(\alpha_2-\alpha_1)\boldsymbol{c}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{c}}_{\mathrm{I}}+\bar{\boldsymbol{c}}_{\mathrm{II}}]\bar{g}_0'(z)=0\end{aligned}\tag{5.184}$$

From Eqs. (5.183) and (5.184), the stress functions are

$$\begin{aligned}\boldsymbol{F}_{\mathrm{I}}(z_j)&=\mathrm{i}\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}[(\alpha_2-\alpha_1)\bar{\boldsymbol{c}}_{\mathrm{II}}-2\alpha_2\boldsymbol{c}_{\mathrm{I}}+\boldsymbol{c}_{\mathrm{II}}]g_0'(z_j)\\&\quad+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}\boldsymbol{Y}_{\mathrm{II}}[(\alpha_2-\alpha_1)\bar{\boldsymbol{d}}_{\mathrm{II}}-2\alpha_2\boldsymbol{d}_{\mathrm{I}}+\boldsymbol{d}_{\mathrm{II}}]g_0'(z_j)\\\boldsymbol{F}_{\mathrm{II}}(z_j)&=-\mathrm{i}\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}[(\alpha_2-\alpha_1)\boldsymbol{c}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{c}}_{\mathrm{I}}+\bar{\boldsymbol{c}}_{\mathrm{II}}]\bar{g}_0'(z_j)\\&\quad-\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\bar{\boldsymbol{Y}}_{\mathrm{I}}[(\alpha_2-\alpha_1)\boldsymbol{d}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{d}}_{\mathrm{I}}+\bar{\boldsymbol{d}}_{\mathrm{II}}]\bar{g}_0'(z_j)\end{aligned}\tag{5.185}$$

According to Eq. (5.121) on $x_2=0$ we have

$$\boldsymbol{\Sigma}_2(x_1)=2\mathrm{Re}\left\{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)+\boldsymbol{d}_{\mathrm{II}}g_{\mathrm{II}}'(x_1)\right\}=2\mathrm{Re}\left\{\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\boldsymbol{d}_{\mathrm{I}}g_{\mathrm{I}}'(x_1)\right\}\tag{5.186}$$

5.6.2 The Point Heat Source Located at the External of an Elliptic Inclusion

Let an infinite piezoelectric material II occupied region Ω^- with an elliptic inclusion I, occupied region Ω^+ of major semiaxis a and minor axis b directed along the material principle axes x_1 and x_2, respectively. The interface of Ω^- and Ω^+ is denoted by L, its normal is denoted by $\boldsymbol{n}$ directed the inside of the inclusion or the outside of the piezoelectric material. At infinity $\boldsymbol{T}_{\mathrm{II}} = \mathbf{0},\ q_n = 0$ and the connective conditions on the L are

$$\boldsymbol{T}_{\mathrm{I}} = \boldsymbol{T}_{\mathrm{II}}, \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1), \quad x \in L \tag{5.187}$$

Let a point heat source at z_0 with strength M be located in the piezoelectric material. We shall use the transform method to solve this problem (Qin 1998, 1999). The transform function from z plane to ς plane is shown in Eqs. (3.82) and (3.86). The point ς_0 in ς plane is corresponding to point z_0 in z plane. L is transformed to Γ. Let $\mathrm{g}_0(\varsigma_T)$ be the fundamental solution in the ς plane when the piezoelectric material occupies the whole space and as in Sect. 5.6.1 we take

$$\mathrm{g}_0'(\varsigma_T) = c\ln(\varsigma_T - \varsigma_{0T}), \quad c = -M/4\pi\alpha_{\mathrm{II}}; \quad \vartheta_0(x_1, x_2) = 2\mathrm{Re}\,\mathrm{g}_0'(\varsigma_T) \tag{5.188}$$

Obviously $\mathrm{g}_0(\varsigma_T)$ is analytic in the inclusion Ω^+. Assume the solution of the problem is

$$\mathrm{g}'(\varsigma_T) = \begin{cases} \mathrm{g}_{\mathrm{I}}'(\varsigma_T), & \varsigma_T \in \Omega^+ - \Omega_0 \\ \mathrm{g}_{\mathrm{II}}'(\varsigma_T) + \mathrm{g}_0'(\varsigma_T), & \varsigma_T \in \Omega^- \end{cases} \tag{5.189}$$

where $\mathrm{g}_{\mathrm{I}}'(\varsigma_T)$ and $\mathrm{g}_{\mathrm{II}}'(\varsigma_T)$ are analytic functions in $\Omega^+ - \Omega_0$ and Ω^-, and Ω_0 is the region $\rho \le \rho_0 = \sqrt{m}, 0 \le \theta < 2\pi$ and on $\Omega_0\, \phi(\rho_0 \mathrm{e}^{\mathrm{i}\psi}) = \phi(\rho_0 \mathrm{e}^{-\mathrm{i}\psi})$ (see Sect. 3.4.2).

According to Eqs. (5.175) and (5.176), the continuity conditions of temperature ϑ and heat $q_n\mathrm{d}s$ through a differential arc on Γ can be reduced to

$$\begin{aligned} &\mathrm{g}_{\mathrm{I}}'(\sigma) + \overline{\mathrm{g}_{\mathrm{I}}'(\sigma)} = \mathrm{g}_{\mathrm{II}}'(\sigma) + \overline{\mathrm{g}_{\mathrm{II}}'(\sigma)} + \mathrm{g}_0'(\sigma) + \overline{\mathrm{g}_0'(\sigma)}, \\ &\alpha_{\mathrm{I}}\mathrm{g}_{\mathrm{I}}'(\sigma) - \alpha_{\mathrm{I}}\overline{\mathrm{g}_{\mathrm{I}}'(\sigma)} = \alpha_{\mathrm{II}}\mathrm{g}_{\mathrm{II}}'(\sigma) - \alpha_{\mathrm{II}}\overline{\mathrm{g}_{\mathrm{II}}'(\sigma)} + \alpha_{\mathrm{II}}\mathrm{g}_0'(\sigma) - \alpha_{\mathrm{II}}\overline{\mathrm{g}_0'(\sigma)} \end{aligned} \tag{5.190}$$

It is noted that $\mathrm{g}_{\mathrm{I}}'(\sigma)$ is analytic only in an annular region $\Omega^+ - \Omega_0$. Similar to Eqs. (3.84) and (3.85), it yields

$$\begin{aligned} &\mathrm{g}_{\mathrm{I}}'(\varsigma_T) = \sum_{k=1}^{\infty}\left(d_k\varsigma_T^k + d_{-k}\varsigma_T^{-k}\right) = \sum_{k=1}^{\infty} d_k\left(\varsigma_T^k + \upsilon_k\varsigma_T^{-k}\right) \\ &\upsilon_k = \rho_0^{2k} = m_{T\,\mathrm{I}}^k = [(a + \mathrm{i}\mu_{T\,\mathrm{I}}b)/(a - \mathrm{i}\mu_{T\,\mathrm{I}}b)]^k, \quad \rho_0 \le |\varsigma| \le 1 \end{aligned} \tag{5.191}$$

So Eq. (5.190) can be reduced to

$$\sum_{k=1}^{\infty}(d_k+\bar{\upsilon}_k\bar{d}_k)\sigma^k-\bar{\mathrm{g}}'_{\mathrm{II}}(1/\sigma)-\mathrm{g}'_0(\sigma)=\mathrm{g}'_{\mathrm{II}}(\sigma)+\bar{\mathrm{g}}'_0(1/\sigma)-\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\sigma^{-k}$$
$$(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\sum_{k=o}^{\infty}(d_k-\bar{d}_k\bar{\varpi}_k)\sigma^k+\bar{\mathrm{g}}'_{\mathrm{II}}(1/\sigma)-\mathrm{g}'_0(\sigma)$$
$$=\mathrm{g}'_{\mathrm{II}}(\sigma)-\bar{\mathrm{g}}'_0(1/\sigma)+(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\sum_{k=o}^{\infty}(\bar{d}_k-\upsilon_k d_k)\sigma^{-k} \tag{5.192}$$

From Eq. (5.192) it is known that the functions at the left side in Eq. (5.192) are analytic in the region Ω^+, whereas those on the right side are analytic in the region Ω^-, and they are continuous on Γ. So these functions are analytic in whole plane and must be constants. So we have

$$\theta_1(\varsigma)=\begin{cases}\sum_{k=1}^{\infty}(d_k+\bar{\upsilon}_k\bar{d}_k)\sigma^k-\overline{\mathrm{g}'_{\mathrm{II}}}(1/\varsigma)-\mathrm{g}'_0(\varsigma), & \varsigma\in\Omega^+\\ \mathrm{g}'_{\mathrm{II}}(\varsigma)-\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\sigma^{-k}+\overline{\mathrm{g}'_0}(1/\varsigma), & \varsigma\in\Omega^-\end{cases}$$
$$\theta_2(\varsigma)=\begin{cases}\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(d_k-\bar{\upsilon}_k\bar{d}_k)\sigma^k+\alpha_{\mathrm{II}}\overline{\mathrm{g}'_{\mathrm{II}}}(1/\varsigma)-\alpha_{\mathrm{II}}\mathrm{g}'_0(\varsigma), & \varsigma\in\Omega^+\\ \alpha_{\mathrm{II}}\mathrm{g}'_{\mathrm{II}}(\varsigma)+\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(\bar{d}_k-\upsilon_k d_k)\sigma^{-k}-\alpha_{\mathrm{II}}\overline{\mathrm{g}'_0}(1/\varsigma), & \varsigma\in\Omega^-\end{cases} \tag{5.193}$$

If there are no generalized external forces acting at infinite, these constants must be zero, i.e., $\theta_1(\infty)=\theta_2(\infty)=0$, so $\theta_1(\varsigma)=\theta_2(\varsigma)=0$ and from Eq. (5.193) we get

$$\begin{aligned}&\sum_{k=1}^{\infty}(d_k+\bar{\upsilon}_k\bar{d}_k)\sigma^k=\bar{\mathrm{g}}'_{\mathrm{II}}(1/\varsigma)+\mathrm{g}'_0(\varsigma),\\&\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(d_k-\bar{\upsilon}_k\bar{d}_k)\sigma^k=-\alpha_{\mathrm{II}}\bar{\mathrm{g}}'_{\mathrm{II}}(1/\varsigma)+\alpha_{\mathrm{II}}\mathrm{g}'_0(\varsigma), \qquad \varsigma\in\Omega^+\\&\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\sigma^{-k}=\mathrm{g}'_{\mathrm{II}}(\varsigma)+\bar{\mathrm{g}}'_0(1/\varsigma),\\&\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(\bar{d}_k-\upsilon_k d_k)\sigma^{-k}=-\alpha_{\mathrm{II}}\mathrm{g}'_{\mathrm{II}}(\varsigma)+\alpha_{\mathrm{II}}\bar{\mathrm{g}}'_0(1/\varsigma), \quad \varsigma\in\Omega^-\end{aligned} \tag{5.194}$$

Solving Eq. (5.194) yields

$$\begin{aligned}&\sum_{k=1}^{\infty}[(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})d_k+(\alpha_{\mathrm{II}}-\alpha_{\mathrm{I}})\bar{\upsilon}_k\bar{d}_k]\varsigma^k=2\alpha_{\mathrm{II}}\mathrm{g}'_0(\varsigma)\\&\mathrm{g}'_{\mathrm{II}}(\varsigma)=-\overline{\mathrm{g}'_0}(1/\varsigma)+\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\varsigma^{-k}\end{aligned} \tag{5.195}$$

Solving $d_k, \mathrm{g}'_{\mathrm{II}}(\varsigma)$ and using ς_T instead of ς, from Eq. (5.189), $\mathrm{g}'(\varsigma_T)$ is obtained:

$$\mathrm{g}'(\varsigma_T) = \begin{cases} \mathrm{g}'_{\mathrm{I}}(\varsigma_T) = \sum\limits_{k=1}^{\infty} d_k \left[\varsigma_T^k + \upsilon_k \varsigma_T^{-k} \right], & \varsigma_T \in \Omega^+ - \Omega_0 \\ \tilde{\mathrm{g}}'_{\mathrm{II}}(\varsigma_T) = \mathrm{g}'_0(\varsigma_T) - \bar{\mathrm{g}}'_0(1/\varsigma) + \sum\limits_{k=1}^{\infty} (\bar{d}_k + \upsilon_k d_k) \varsigma_T^{-k}, & \varsigma_T \in \Omega^- \end{cases}$$

$$\vartheta(x_1, x_2) = 2\mathrm{Re}\, \mathrm{g}'(\varsigma_T), \quad \mathrm{g}'_0(z_T) = -(M/4\pi\alpha_{II}) \ln(\varsigma_T - \varsigma_{0T}) \tag{5.196}$$

5.6.3 Interaction of an Impermeable Crack with a Singularity in a Piezoelectric Bimaterial

Let a mechanical singular generalized load with strength $(\boldsymbol{b}, \boldsymbol{p})$ be located at z_0 in material II that occupied the lower half-plane $\Omega^-, x_2 < 0$. An insulated crack $(-a, a)$ is located on the interface $x_2 = 0$. According to Eqs. (3.165), (3.166), and (3.160), the generalized stress on the interface introduced by the singularity in a piezoelectric material is

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathrm{I}2} = \boldsymbol{\Sigma}_{\mathrm{II}2} = \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = 2\mathrm{Re}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) = 2\mathrm{Re}\left[\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} + \boldsymbol{Y}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{II}}\mathbf{g}_{\mathrm{II}}(z)\right] \\ = 2\mathrm{Re}\left[\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} + \boldsymbol{Y}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{II}}\langle 2\pi\mathrm{i}(x_1 - z_0)\rangle^{-1}\left(\bar{\boldsymbol{B}}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{V}\right)\right] \end{aligned} \tag{5.197}$$

The original problem can be simply solved by the superposition method: The singularity in a piezoelectric material without crack and an external force $-\boldsymbol{\Sigma}_2(x_1)$ applies on the crack surface. The last problem has been solved in Sect. 4.2.4.

From Eq. (5.197) it is seen that the effect of a mechanical singularity is equivalent to adding an external force $-\boldsymbol{\Sigma}_2(x_1)$ on the crack surface, and it does not affect the heat flow.

A point heat source with strength M located at z_0 in material II, $\Omega^-, x_2 < 0$ will produce the heat flow, as shown in Eq. (5.82), and generalized surface traction, as shown in Eq. (5.186), i.e., the point heat source affects both the stress and temperature fields. A point heat source in a bimaterial is equivalent to a point heat source in an infinite homogeneous piezoelectric material II and on the crack surface superposed the following loads:

$$q_T = -q_2 = -\mathrm{i}\alpha_1\alpha_2\left[\mathrm{g}''_0(x_1) - \bar{\mathrm{g}}''_0(x_1)\right], \quad \boldsymbol{t}_2 = -\boldsymbol{\Sigma}_2(x_1) = -2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \boldsymbol{d}_{\mathrm{I}}\mathrm{g}'_{\mathrm{I}}(x_1)\right] \tag{5.198}$$

where $\mathrm{g}'_0(x_1), \mathrm{g}'_{\mathrm{I}}(x_1), \boldsymbol{F}_{\mathrm{I}}(x_1)$ are calculated from Eqs. (5.175), (5.178), and (5.185), respectively.

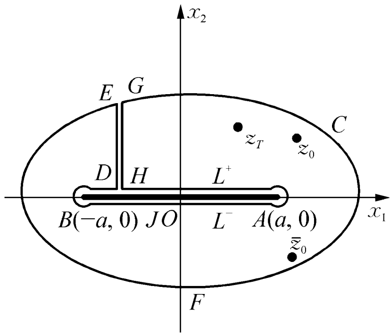

Fig. 5.6 Integral path Λ for the integral Φ

As an example we discuss the interaction of the above point heat source with a single crack located at $(-a, a)$ (Shen and Kuang 1998). It is assumed that the boundary conditions are

$$\begin{aligned} &\boldsymbol{\Phi}_{\mathrm{I},i}(x_1) = \boldsymbol{\Phi}_{\mathrm{II,i}}(x_1) = \mathbf{0}, \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1) = 0, \quad x \in L_{\mathrm{c}} \\ &\boldsymbol{\Phi}_{\mathrm{I},i}(x_1) = \boldsymbol{\Phi}_{\mathrm{II,i}}(x_1) = \mathbf{0}, \quad q_i = T = 0, \quad |z| \to \infty; \quad i = 1, 2 \end{aligned} \tag{5.199}$$

Substitution of Eqs. (5.198) and (5.178) into Eq. (5.153) yields

$$\begin{aligned} \theta''(z_T) &= \frac{\alpha_{\mathrm{I}} + \alpha_{\mathrm{II}}}{2\pi\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}} Z_0(z_T) \int_{L_{\mathrm{c}}} \frac{q_T(x_1)\mathrm{d}x_1}{Z_0^+(x_1)(x_1 - z_T)} + Z_0(z_T)C(z_T) \\ &= -\frac{M\mathrm{i}}{4\pi^2\alpha_{\mathrm{II}}} Z_0(z_T) \int_{L_{\mathrm{c}}} \left(\frac{1}{x_1 - z_0} - \frac{1}{x_1 - \bar{z}_0} \right) \frac{1}{Z_0^+(x_1)(x_1 - z_T)} \mathrm{d}x_1 + Z_0(z_T)C(z_T) \end{aligned} \tag{5.200}$$

where $Z_0(z_T) = \left(z_T^2 - a^2\right)^{-1/2}$. The integral in Eq. (5.200) can be integrated. At first we discuss the contour integral

$$\Phi = \int_{\Lambda} \frac{1}{Z_0^+(x_1)(x_1 - z_T)(x_1 - z_0)(x_1 - \bar{z}_0)} \mathrm{d}x_1$$

where Λ is shown in Fig. 5.6. Inside the contour there are three poles: $z_T, z_0, \bar{z}_0$. Using the residual theorem the Φ is reduced to

$$\Phi = 2\pi\mathrm{i} \left\{ \frac{\sqrt{z_T^2 - a^2}}{(z - z_0)(z - \bar{z}_0)} + \frac{\sqrt{z_0^2 - a^2}}{(z_0 - z)(z_0 - \bar{z}_0)} + \frac{\sqrt{\bar{z}_0^2 - a^2}}{(\bar{z}_0 - z)(\bar{z}_0 - z_0)} \right\}$$

On the other hand it is easy to prove that the integral Φ on the path *DEFGH* vanishes whereas on the path *HJD* equals

$$2 \int_{L_{\mathrm{c}}} \frac{z_0 - \bar{z}_0}{Z_0^+(x_1)(x_1 - z_T)(x_1 - z_0)(x_1 - \bar{z}_0)} \mathrm{d}x_1 = \int_{L_{\mathrm{c}}} \left(\frac{1}{x_1 - z_0} - \frac{1}{x_1 - \bar{z}_0} \right) \frac{1}{Z_0^+(x_1)(x_1 - z_T)} \mathrm{d}x_1$$

From the condition at infinity in Eq. (5.199) and the single valued condition of temperature we find $C(z_T) = 0$. So Eq. (5.200) is reduced to

$$\theta''(z_T) = \frac{M}{2\pi\alpha_{\mathrm{II}}} \frac{1}{\sqrt{z_T^2 - a^2}} \left\{ \frac{\sqrt{\bar{z}_0^2 - a^2}}{z_T - \bar{z}_0} - \frac{\sqrt{z_0^2 - a^2}}{z_T - z_0} \right\} + \frac{M}{2\pi\lambda_0} \left\{ \frac{1}{z_T - z_0} - \frac{1}{z_T - \bar{z}_0} \right\} \tag{5.201}$$

Using $T = 0$ at infinity finally we get

$$\theta'(z_T) = \frac{M}{4\pi\alpha_{\mathrm{II}}} \ln \left\{ \frac{\sqrt{\bar{z}_0^2 - a^2} + \bar{z}_0}{\sqrt{z_0^2 - a^2} + z_0} \frac{\sqrt{z_T^2 - a^2}\sqrt{z_0^2 - a^2} + z_T z_0 - a^2}{\sqrt{z_T^2 - a^2}\sqrt{\bar{z}_0^2 - a^2} + z_T \bar{z}_0 - a^2} \right\} \tag{5.202}$$

Substituting Eq. (5.202) into (5.158) $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z)$ can be obtained:

$$\begin{gathered}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z) = \boldsymbol{Q}(z)\left[\boldsymbol{C}(z) + \frac{1}{2\pi \mathrm{i}} \int_L \frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}} \tilde{\boldsymbol{\Sigma}}_0(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1 - z)}\right], \quad \boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z) \right\rangle \\ \tilde{\boldsymbol{\Sigma}}_0(x_1) = \boldsymbol{\eta}_1\theta'^{+}(x_1) + \boldsymbol{\eta}_2\theta'^{-}(x_1)\end{gathered} \tag{5.203}$$

From Eq. (5.155) $\boldsymbol{F}_{\mathrm{I}}(z_j)$ and $\boldsymbol{F}_{\mathrm{II}}(z_j)$ can be obtained.

Gao and Wang (2001) discussed the permeable crack problem, Herrmann and Loboda (2003) discussed the contact zone model in pyroelectric material, and Norris (1994) discussed the dynamic Green function in piezoelectric material.

5.7 Functionally Graded Piezoelectric Material

5.7.1 Fundamental Equations in Antiplane Shear Problem

Functionally graded piezoelectric material (FGPM) is a kind of material with continuously varying properties (Wu et al. 1996) which is very useful as a transit layer instead of the bonding agent in order to avoid the very large stresses near the interface. Li and Weng (2002) discussed the antiplane crack problem (Fig. 5.7) with varied material constants for a transversely material:

$$\begin{gathered}C_{44}(x_2) = C_{44}^0(1 + \alpha|x_2|)^k, \quad e_{15} = e_{15}^0(1 + \alpha|x_2|)^k, \quad \epsilon_{11} = \epsilon_{11}^0(1 + \alpha|x_2|)^k \\ \alpha = \left(\sqrt[k]{C_{44}^h/C_{44}^0} - 1\right)\Big/ h = \left(\sqrt[k]{e_{15}^h/e_{15}^0} - 1\right)\Big/ h = \left(\sqrt[k]{\epsilon_{11}^h/\epsilon_{11}^0} - 1\right)\Big/ h\end{gathered} \tag{5.204}$$

where $C_{44}^0, e_{15}^0, \epsilon_{11}^0$ are the values at $x_2 = 0$ and $C_{44}^h, e_{15}^h, \epsilon_{11}^h$ are the values at $x_2 = \pm h$; k and α are material constants. It is assumed that the geometry, material behavior,

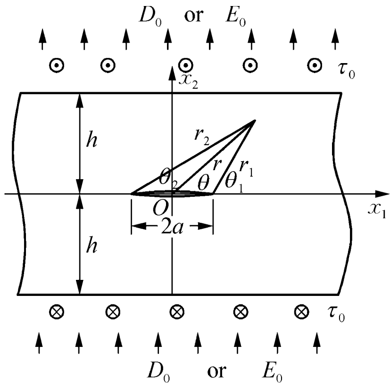

Fig. 5.7 An antiplane crack in FGPM

and applied loading are symmetric about the x_2-axis, so we only need to study the part of $x_1 \geq 0, x_2 \geq 0$ and $|x_2| = x_2$. The fundamental equations (4.238) and (4.239) of antiplane shear problem discussed in Sect. 4.8.1 are still held in a FGPM, but the material constants are functions of coordinates.

Substitution of Eq. (5.204) into Eq. (4.239) yields

$$\begin{aligned} &C_{44}^0\left[\nabla^2 u_3 + (k\alpha/\xi)u_{3,2}\right] + e_{15}^0\left[\nabla^2\varphi + (k\alpha/\xi)\varphi_{,2}\right] = 0 \\ &e_{15}^0\left[\nabla^2 u_3 + (k\alpha/\xi)u_{3,2}\right] - \epsilon_{11}^0\left[\nabla^2\varphi + (k\alpha/\xi)\varphi_{,2}\right] = 0, \quad \xi = 1 + \alpha x_2 \end{aligned} \tag{5.205}$$

where $\nabla^2 = \partial/\partial x_1^2 + \partial/\partial x_2^2$. In general case $\left(e_{15}^0\right)^2 + C_{44}^0\epsilon_{11}^0 \neq 0$, so we also have

$$\nabla^2 u_3 + (k\alpha/\xi)u_{3,2} = 0, \quad \nabla^2\varphi + (k\alpha/\xi)\varphi_{,2} = 0 \tag{5.206}$$

The boundary and connective conditions on $x_2 = 0$ are

$$\begin{aligned} &\sigma_{32}(x_1,0) = 0, \quad E_1(x_1,0^+) = E_1^c(x_1,0^-), \quad D_2(x_1,0^+) = D_2^c(x_1,0^-), \quad 0 \leq x_1 < a \\ &u_3(x_1,0) = 0, \quad \varphi(x_1,0) = 0, \quad \sigma_{32}(x_1,0^+) = \sigma_{32}(x_1,0^-), \quad a \leq x_1 < \infty \end{aligned} \tag{5.207a}$$

where the right superscript "c" means that the related variable is in the air. The boundary conditions on $x_2 = h$ are divided into two forms dependent to giving D_2 or E_2:

$$\begin{aligned} &\text{Case 1:} \quad D_2(x_1,h) = D_0, \quad \sigma_{32}(x_1,h) = \tau_h = \left(C_{44}^{h*}/C_{44}^h\right)\tau_0 - \left(e_{15}^h/\epsilon_{11}^h\right)D_0 \\ &\text{Case 2:} \quad E_2(x_1,h) = E_0, \sigma_{32}(x_1,h) = \tau_h = \tau_0 - e_{15}^h E_0 \ ; \quad 0 \leq x_1 < \infty \end{aligned} \tag{5.207b}$$

where D_0 and E_0 are the external electric displacement and field, respectively; τ_0 is the stress at zero electric loading, $C_{44}^{h*} = C_{44}^h + \left(e_{15}^h\right)^2/\epsilon_{11}^h$.

5.7.2 Solution of the Antiplane Shear Problem

Considering the symmetry about x_2-axis, Li and Weng (2002) used the Fourier cosine transforms to solve this problem. Let

$$\begin{aligned}u_3(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty \xi^{-\beta}\{A_1(s)\mathrm{I}_\beta(\xi s/\alpha)+A_2(s)\mathrm{K}_\beta(\xi s/\alpha)\}\cos(sx_1)\mathrm{d}s + a_1x_2\\ \varphi(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty \xi^{-\beta}\{B_1(s)\mathrm{I}_\beta(\xi s/\alpha)+B_2(s)\mathrm{K}_\beta(\xi s/\alpha)\}\cos(sx_1)\mathrm{d}s - b_1x_2\end{aligned} \tag{5.208}$$

where $\beta=(k-1)/2$; I_β and K_β are the first and second kind of modified Bessel's functions, respectively; $A_i(s)$ and $B_i(s)$ are undetermined functions; a_1, b_1 are real constants. Equation (5.208) yields

$$\begin{aligned}\sigma_{31}(x_1,x_2) =& -\frac{2}{\pi}\int_0^\infty s\xi^{-\beta}\big[(C_{44}A_1+e_{15}B_1)\mathrm{I}_\beta(\xi s/\alpha)\\ &+(C_{44}A_2+e_{15}B_2)K_\beta(\xi s/\alpha)\big]\sin(sx_1)\mathrm{d}s\\ \sigma_{32}(x_1,x_2) =& -\frac{2}{\pi}\int_0^\infty \Big\{(C_{44}A_1+e_{15}B_1)\Big[\beta\alpha\xi^{-\beta-1}\mathrm{I}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{I}'_\beta(\xi s/\alpha)\Big]\\ &+(C_{44}A_2+e_{15}B_2)\Big[\{\beta\alpha\xi^{-\beta-1}\mathrm{K}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{K}'_\beta(\xi s/\alpha)\Big]\Big\}\cos(sx_1)\mathrm{d}s\\ &+C_{44}a_1-e_{15}b_1\end{aligned} \tag{5.209}$$

$$\begin{aligned}D_1(x_1,x_2) =& -\frac{2}{\pi}\int_0^\infty s\xi^{-\beta}\big[(e_{15}A_1-\epsilon_{11}B_1)\mathrm{I}_\beta(\xi s/\alpha)\\ &+(e_{15}A_2-\epsilon_{11}B_2)\mathrm{K}_\beta(\xi s/\alpha)\big]\sin(sx_1)\mathrm{d}s\\ D_2(x_1,x_2) =& -\frac{2}{\pi}\int_0^\infty \Big\{\beta\alpha\xi^{-\beta-1}\mathrm{I}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{I}'_\beta(\xi s/\alpha)\Big\}\\ &\times(e_{15}A_1-\epsilon_{11}B_1)+\Big[\beta\alpha\xi^{-\beta-1}\mathrm{K}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{K}'_\beta(\xi s/\alpha)\Big]\\ &\times(e_{15}A_2-\epsilon_{11}B_2)\}\cos(sx_1)\mathrm{d}s+e_{15}a_1+\epsilon_{11}b_1\end{aligned} \tag{5.210}$$

$$\begin{aligned}E_1(x_1,x_2) =& \frac{2}{\pi}\int_0^\infty s\xi^{-\beta}\big[B_1\mathrm{I}_\beta(\xi s/\alpha)+B_2\mathrm{K}_\beta(\xi s/\alpha)\big]\sin(sx_1)\mathrm{d}s\\ E_2(x_1,x_2) =& \frac{2}{\pi}\int_0^\infty \Big\{B_1\Big[\beta\alpha\xi^{-\beta-1}\mathrm{I}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{I}'_\beta(\xi s/\alpha)\Big]\\ &+B_2\Big[\beta\alpha\xi^{-\beta-1}\mathrm{K}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{K}'_\beta(\xi s/\alpha)\Big]\Big\}\cos(sx_1)\mathrm{d}s+b_1\end{aligned} \tag{5.211}$$

where $\mathrm{I}'_\beta, \mathrm{K}'_\beta$ are the derivatives of $\mathrm{I}_\beta, \mathrm{K}_\beta$.

In the air between the crack surfaces, we have

$$D_1^{\mathrm{c}} = \epsilon^{\mathrm{c}} E_1^{\mathrm{c}}, \quad D_2^{\mathrm{c}} = \epsilon^{\mathrm{c}} E_2^{\mathrm{c}}, \quad \nabla^2 \varphi^{\mathrm{c}} = 0 \tag{5.212}$$

Its solution can be assumed as

$$\begin{aligned} &\varphi^{\mathrm{c}}(x_1, x_2) = \frac{2}{\pi} \int_0^{\infty} C(s) \sinh(s x_2) \cos(s x_1) \mathrm{d}s, \quad 0 \le x_1 < a \\ &D_1^{\mathrm{c}}(x_1, 0) = 0, \quad D_2^{\mathrm{c}}(x_1, 0) = -\frac{2}{\pi} \int_0^{\infty} \epsilon^c s C(s) \cos(s x_1) \mathrm{d}s \\ &E_1^{\mathrm{c}}(x_1, 0) = 0, \quad E_2^{\mathrm{c}}(x_1, 0) = -\frac{2}{\pi} \int_0^{\infty} s C(s) \cos(s x_1) \mathrm{d}s \end{aligned} \tag{5.213}$$

where $C(s)$ is an unknown function. Using the boundary conditions on $x_2 = h$ yields

$$\begin{aligned} &A_2(s) = R A_1(s), \quad B_2(s) = R B_1(s) \\ &R = -\frac{\beta\alpha(1+\alpha h)^{-1} \mathrm{I}_\beta[(1+\alpha h)s/\alpha] - s\mathrm{I}'_\beta(1+\alpha h)s/\alpha}{\beta\alpha(1+\alpha h)^{-1} \mathrm{K}_\beta[(1+\alpha h)s/\alpha] - s\mathrm{K}'_\beta(1+\alpha h)s/\alpha} \\ &a_1 = \left(e_{15}^h D_0 + \epsilon_{11}^h \tau_0\right) / C_{44}^{h*} \epsilon_{11}^h, \quad b_1 = \left(C_{44}^h D_0 - e_{15}^h \tau_0\right) / C_{44}^{h*} \epsilon_{11}^h \quad \text{(case 1)} \\ &a_1 = \left(e_{15}^h E_0 + \tau_0\right) / C_{44}^h, \quad b_1 = E^{\infty} \quad \text{(case 2)} \end{aligned} \tag{5.214}$$

Substituting $E_1(x_1, 0), \varphi(x_1, 0), E_1^{\mathrm{c}}(x_1, 0)$ into the corresponding boundary conditions in Eq. (5.207) yields the following dual integral equation:

$$\begin{aligned} &\int_0^{\infty} s B_1(s) \left\{ I_\beta(s/\alpha) + R K_\beta(s/\alpha) \right\} \sin(s x_1) \mathrm{d}s = 0, \quad 0 \le x_1 < a \\ &\int_0^{\infty} B_1(s) \left\{ I_\beta(s/\alpha) + R K_\beta(s/\alpha) \right\} \cos(s x_1) \mathrm{d}s = 0, \quad a \le x_1 < \infty \end{aligned} \tag{5.215}$$

If let

$$B_1(s) \left\{ I_\beta(s/\alpha) + R K_\beta(s/\alpha) \right\} = \left(\pi a^2/2\right) \int_0^1 \sqrt{\eta} \Phi(\eta) J_0(s a \eta) \mathrm{d}\eta \tag{5.216}$$

where J_0 is the zero-order Bessel function of the first kind, then the second equation in Eq. (5.215) is satisfied automatically and the first equation in Eq. (5.215) requires $\Phi(\eta) = 0$. So it is easy to obtain $B_1(s) = 0$ and then straightly $B_2(s) = 0$.

Substituting $\sigma_{32}(x_1, 0), u_3(x_1, 0)$ into the corresponding boundary conditions in Eq. (5.207) and noting $B_1(s) = B_2(s) = 0$, the following dual integral equation is obtained:

$$\begin{aligned} &\int_0^{\infty} s F(s) A(s) \cos(s x_1) \mathrm{d}s = (\pi/2) \left(C_{44}^0 a_1 - e_{15}^0 b_1\right) / C_{44}^0, \quad 0 \le x_1 < a \\ &\int_0^{\infty} A(s) \cos(s x_1) \mathrm{d}s = 0, \quad a \le x_1 < \infty \end{aligned} \tag{5.217}$$

where

$$\begin{aligned} A(s) &= A_1(s)\left[I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right] \\ F(s) &= \frac{\left[\beta\alpha I_\beta(s/\alpha) - sI'_\beta(s/\alpha)\right] + R\left[\beta\alpha K_\beta(s/\alpha) - sK'_\beta(s/\alpha)\right]}{s\left[I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right]} \end{aligned} \tag{5.218}$$

The solution of Eq. (5.217) can be written as

$$A(s) = \frac{\pi a^2}{2}\frac{\hat{C}_{44}^0}{C_{44}^0}\int_0^1 \sqrt{\eta}\Psi(\eta)J_0(sa\eta)\mathrm{d}\eta, \quad \hat{C}_{44}^0 = C_{44}^0 a_1 - e_{15}^0 b_1 \tag{5.219}$$

Equation (5.219) satisfies the second equation in Eq. (5.217) automatically. In order to satisfy the first equation in Eq. (5.217), $\Psi(\eta)$ should be satisfied by the following Fredholm integral equation of the second kind:

$$\begin{aligned} &\Psi(\eta) + \int_0^1 \psi(\eta)G(\eta,\eta')\mathrm{d}\eta' = \sqrt{\eta} \\ &G(\eta,\eta') = \sqrt{\eta\eta'}\int_0^\infty s[F(s/a) - 1]J_0(s\eta)J_0(s\eta')\mathrm{d}s \end{aligned} \tag{5.220}$$

5.7.3 *The Generalized Stress Asymptotic Fields Near the Crack Tip*

The singular generalized stress fields are determined by the behavior of the solution when $s \to \infty$. Using integration by parts to decompose Eq. (5.219) into singular and regular parts,

$$A(s) = \frac{\pi a}{2}\frac{\hat{C}_{44}^0}{C_{44}^0}\frac{1}{s}\left\{\Psi(1)J_1(sa) - \int_0^1 \eta J_1(sa\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{\Psi(\eta)}{\sqrt{\eta}}\right]\mathrm{d}\eta\right\} \tag{5.221}$$

where J_1 is the first-order Bessel function of the first kind. The integral in Eq. (5.221) is finite at the crack tip $x_1 = \pm a$, and the singular behavior is determined by the term containing $\Psi(1)$. It is noted that the modified Bessel functions have the following behaviors:

$$\begin{aligned} &\lim_{s\to\infty} \mathrm{I}_\beta(s) = \left(1\Big/\sqrt{2\pi s}\right)\mathrm{e}^s, \quad \lim_{s\to\infty} \mathrm{I}'_\beta(s) = \left(1\Big/\sqrt{2\pi s}\right)\mathrm{e}^s \\ &\lim_{s\to\infty} \mathrm{K}_\beta(s) = \left(\sqrt{\pi/2s}\right)\mathrm{e}^{-s}, \quad \lim_{s\to\infty} \mathrm{K}'_\beta(s) = -\left(\sqrt{\pi/2s}\right)\mathrm{e}^{-s} \end{aligned} \tag{5.222}$$

After complex derivation we obtain

$$\begin{aligned}
&\sigma_{31} = -\hat{C}^0_{44} a\Psi(1)\xi^{k/2} f_1(s), \quad \sigma_{32} = -\hat{C}^0_{44} a\Psi(1)\xi^{k/2} f_2(s) \\
&D_1 = -\frac{e^0_{15}\hat{C}^0_{44}}{C^0_{44}} a\Psi(1)\xi^{k/2} f_1(s), \quad D_2 = -\frac{e^0_{15}\hat{C}^0_{44}}{C^0_{44}} a\Psi(1)\xi^{k/2} f_2(s) \\
&E_1 = 0, \quad E_2 = E_0
\end{aligned} \tag{5.223}$$

where

$$\begin{aligned}
f_1(s) &= \int_0^\infty J_1(sa)\mathrm{e}^{-sx_2}\sin(sx_1)\mathrm{d}s = -\frac{r}{a\sqrt{r_1 r_2}}\sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \\
f_2(s) &= \int_0^\infty J_1(sa)\mathrm{e}^{-sx_2}\cos(sx_1)\mathrm{d}s = \frac{1}{a} - \frac{r}{a\sqrt{r_1 r_2}}\cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right)
\end{aligned} \tag{5.224}$$

where the meanings of $r, r_1, r_2, \theta, \theta_1, \theta_2$ can be seen in Fig. 5.7. Let $\theta \to 0, \theta_2 \to 0$, $r_2 \to 2a, r \to a$ from Eq. (5.223) we get

$$\begin{aligned}
&\sigma_{31} = -\left(K_{\mathrm{III}}\Big/\sqrt{2\pi r_1}\right)\sin(\theta_1/2), \quad \sigma_{32} = \left(K_{\mathrm{III}}\Big/\sqrt{2\pi r_1}\right)\cos(\theta_1/2), \\
&D_1 = -\left(K^D_{\mathrm{III}}\Big/\sqrt{2\pi r_1}\right)\sin(\theta_1/2), \quad D_2 = \left(K^D_{\mathrm{III}}\Big/\sqrt{2\pi r_1}\right)\cos(\theta_1/2), \quad E_1 = E_2 = 0 \\
&K_{\mathrm{III}} = \hat{C}^0_{44}\sqrt{\pi a}\Psi(1), \quad K^D_{\mathrm{III}} = \left(e^0_{15}/C^0_{44}\right)K_{\mathrm{III}} = \left(\hat{C}^0_{44}e^0_{15}/C^0_{44}\right)\sqrt{\pi a}\Psi(1)
\end{aligned} \tag{5.225}$$

It is found that for the functional gradient piezoelectric material, the asymptotic fields of the generalized stress still have the singularity $1/\sqrt{r}$. Because a_1, b_1 is enclosed in $\hat{C}^0_{44}$ (see Eq. (5.219)), the generalized stress intensity factors are different for two different electric boundary conditions. It is also found that the electric field does not have singularity at the crack tip.

Yang et al. (2004) also discussed the electric field gradient effects in antiplane problems of polarized ceramics.

5.7.4 Plane Strain Problem

The constitutive equations of the in-plane problem are

$$\begin{aligned}
&\sigma_{11} = C_{11}u_{1,1} + C_{13}u_{3,3} - e_{31}E_3, \quad \sigma_{33} = C_{13}u_{1,1} + C_{33}u_{3,3} - e_{33}E_3 \\
&\sigma_{13} = C_{44}\left(u_{1,3} + u_{3,1}\right) - e_{15}E_1, \quad D_1 = e_{15}\left(u_{1,3} + u_{3,1}\right) + \epsilon_{11}E_1 \\
&D_3 = e_{31}u_{1,1} + e_{33}u_{3,3} + \epsilon_{33}E_3
\end{aligned} \tag{5.226}$$

It is assumed that the material properties are one dimensional dependent to x_3 as

$$(C_{11}, C_{13}, C_{33}, C_{44}, e_{31}, e_{33}, e_{15}, \epsilon_{11}, \epsilon_{33}) = \left(C_{11}^0, C_{13}^0, C_{33}^0, C_{44}^0, e_{31}^0, e_{33}^0, e_{15}^0, \epsilon_{11}^0, \epsilon_{33}^0\right) e^{\beta|x_3|} \tag{5.227}$$

where β is a material constant. The equilibrium equations in terms of generalized displacements are

$$\begin{aligned}
&C_{11}^0 u_{1,11} + C_{44}^0 u_{1,33} + \left(C_{13}^0 + C_{44}^0\right) u_{3,13} + \left(e_{31}^0 + e_{15}^0\right)\varphi_{,13} + \beta\left[C_{44}^0\left(u_{1,3} + u_{3,1}\right) + e_{15}^0 \varphi_{,1}\right] = 0 \\
&C_{44}^0 u_{3,11} + C_{33}^0 u_{3,33} + \left(C_{13}^0 + C_{44}^0\right) u_{1,13} + e_{15}^0 \varphi_{,11} + e_{33}^0 \varphi_{,33} + \beta\left(C_{13}^0 u_{1,1} + C_{33}^0 u_{3,3} + \epsilon_{33}^0 \varphi_{,3}\right) = 0 \\
&e_{15}^0 u_{3,11} + e_{33}^0 u_{3,33} + \left(e_{31}^0 + e_{15}^0\right) u_{1,13} - \epsilon_{11}^0 \varphi_{,11} - \epsilon_{33}^0 \varphi_{,33} + \beta\left(e_{31}^0 u_{1,1} + e_{33}^0 u_{3,3} - \epsilon_{33}^0 \varphi_{,3}\right) = 0
\end{aligned} \tag{5.228}$$

In the air between the crack surfaces, the governing equations are still shown in Eq. (5.212).

As in Sect. 5.7.1 it is assumed that the geometry, material behavior, and applied loading are all symmetric about the x_3-axis, so we only need to study the part of $x_1 \geq 0, x_3 \geq 0$ and $|x_2| = x_2$. The boundary conditions on the crack and connective surfaces are

$$\begin{aligned}
&\sigma_{33}(x_1, 0) = 0, \quad E_1(x_1, 0^+) = E_1^c(x_1, 0^-), \quad D_3(x_1, 0^+) = D_3^c(x_1, 0^-), \quad 0 \leq |x_1| < a \\
&u_3(x_1, 0) = 0, \quad \varphi(x_1, 0) = 0, \quad a \leq |x_1| < \infty; \quad \sigma_{31}(x_1, 0^+) = 0, \quad 0 \leq |x_1| < \infty
\end{aligned} \tag{5.229a}$$

The boundary conditions on the edge $x_3 = h$ are divided into two forms:

$$\begin{aligned}
&\text{case 1:} \quad \sigma_{33}(x_1, h) = \sigma_h = \frac{C_{33}^{0*}}{C_{33}^0}\sigma_0 - \frac{e_{33}^0}{\epsilon_{33}^0} D_0, \quad \sigma_{13}(x_1, h) = 0, \quad D_3(x_1, h) = D_0 \\
&\text{case 2:} \quad \sigma_{33}(x_1, h) = \sigma_h = \sigma_0 - e_{33}^0 E_0 e^{\beta h}, \quad \sigma_{13}(x_1, h) = 0, \; E_3(x_1, h) = E_0
\end{aligned} \tag{5.229b}$$

where D_0 and E_0 are the external electric displacement and field, respectively, and σ_0 is the stress at zero electric loading, $C_{33}^{0*} = C_{33}^0 + \left(e_{33}^0\right)^2 / \epsilon_{33}^0$.

The single valued condition of the generalized displacements is

$$\int_{-a}^{a} \psi(x_1)\mathrm{d}x_1 = 0, \quad \psi(x_1) = \mathrm{d}[U_3(x_1, 0^+) - U_3(x_1, 0^-)]/\mathrm{d}x_1, \quad 0 \leq |x_1| < a \tag{5.230}$$

where $\psi(x_1)$ is the generalized dislocation density and on the connective surface $\psi = 0$.

Ueda (2005) adopted the Fourier integral transform techniques to solve this problem. Let

$$
\begin{aligned}
u_1(x_1,x_3) &= \frac{2}{\pi}\sum_{j=1}^{6}\int_0^{\infty} a_j A_j(s)\mathrm{e}^{s\gamma_j x_3}\sin(sx_1)\mathrm{d}s \\
u_3(x_1,x_3) &= \frac{2}{\pi}\sum_{j=1}^{6}\int_0^{\infty} A_j(s)\mathrm{e}^{s\gamma_j x_3}\cos(sx_1)\mathrm{d}s + a_0\left(1-\mathrm{e}^{-\beta x_3}\right) \\
\varphi(x_1,x_3) &= -\frac{2}{\pi}\sum_{j=1}^{6}\int_0^{\infty} b_j A_j(s)\mathrm{e}^{s\gamma_j x_3}\cos(sx_1)\mathrm{d}s - b_0\left(1-\mathrm{e}^{-\beta x_3}\right)
\end{aligned}
\tag{5.231}
$$

where $A_j(s)$ is undetermined function and a_0, b_0 are unknown constants; $\gamma_j(s), a_j(s)$, $b_j(s)$ are known functions. $\gamma_j(s)$ is the root of the following equation:

$$
\begin{aligned}
&(f_3q_3+g_2p_4)\gamma^6+(f_3q_2+g_2p_3+f_2q_3+g_1p_4)\gamma^5+(f_3q_1+g_2p_2+f_2q_2+g_1p_3 \\
&\quad +f_1q_3+g_0p_4)\gamma^4+(f_3q_0+g_2p_1+f_2q_1+g_1p_2+f_1q_2+g_0p_3+f_0q_3)\gamma^3 \\
&\quad +(f_2q_0+g_2p_0+f_1q_1+g_1p_1+f_0q_2+g_0p_2)\gamma^2+(f_1q_0+g_1p_0+f_0q_1+g_0p_1)\gamma \\
&\quad +(f_0q_0+g_0p_0)=0
\end{aligned}
\tag{5.232}
$$

For convenience let $\mathrm{Re}\gamma_j(s) < \mathrm{Re}\gamma_{j+1}(s), j = 1-5$. $a_j(s), b_j(s)$ are determined by

$$
\begin{aligned}
a_j(s) &= \frac{q_3\gamma_j^3+q_2\gamma_j^2+q_1\gamma_j+q_0}{g_2\gamma_j^2+g_1\gamma_j+g_0} \\
b_j(s) &= \frac{\left[\left(C_{13}^0+C_{44}^0\right)s\gamma_j+C_{13}^0\beta\right]a_j+C_{33}^0 s\gamma_j^2+C_{33}^0\beta\gamma_j-C_{44}^0 s}{e_{33}^0 s\gamma_j^2+e_{33}^0\beta\gamma_j-e_{15}^0 s}
\end{aligned}
\tag{5.233}
$$

where

$$
\begin{aligned}
f_0 &= -\left(e_{31}^0e_{15}^0+C_{13}^0\epsilon_{11}^0\right)\beta s \\
f_1 &= \left(e_{31}^0e_{33}^0+C_{13}^0\epsilon_{33}^0\right)\beta^2-\left[e_{15}^0\left(e_{15}^0+e_{31}^0\right)+\epsilon_{11}^0\left(C_{13}^0+C_{44}^0\right)\right]s^2 \\
f_2 &= \left[\epsilon_{33}^0\left(2C_{13}^0+C_{44}^0\right)+e_{33}^0\left(2e_{31}^0+e_{15}^0\right)\right]\beta s \\
f_3 &= \left[\epsilon_{33}^0\left(C_{13}^0+C_{44}^0\right)+e_{33}^0\left(e_{31}^0+e_{15}^0\right)\right]s^2
\end{aligned}
\tag{5.234a}
$$

$$
\begin{aligned}
&p_0 = \left(C_{44}^0\epsilon_{11}^0+e_{15}^{02}\right)s^2, \quad p_1 = -\left(\epsilon_{33}^0C_{44}^0+2e_{33}^0e_{15}^0+\epsilon_{11}^0C_{33}^0\right)\beta s \\
&p_2 = \left(e_{33}^{02}+C_{33}^0\epsilon_{33}^0\right)\beta^2-\left(\epsilon_{33}^0C_{44}^0+2e_{33}^0e_{15}^0+\epsilon_{11}^0C_{33}^0\right)s^2 \\
&p_3 = 2\left(e_{33}^{02}+C_{33}^0\epsilon_{33}^0\right)\beta s, \quad p_4 = \left(e_{33}^{02}+C_{33}^0\epsilon_{33}^0\right)s^2
\end{aligned}
\tag{5.234b}
$$

$$
\begin{aligned}
g_0 &= e_{15}^0\left(e_{31}^0 e_{33}^0 + C_{13}^0 \epsilon_{33}^0\right)\beta^2 + C_{11}^0\left(e_{15}^0 \epsilon_{33}^0 - e_{33}^0 \epsilon_{11}^0\right)s^2 \\
g_1 &= \left\{e_{15}^0\left[\epsilon_{33}^0\left(C_{13}^0 + C_{44}^0\right) + e_{33}^0\left(e_{31}^0 + e_{15}^0\right)\right] + \left(e_{31}^0 + e_{15}^0\right)\left(e_{31}^0 e_{33}^0 + C_{13}^0 \epsilon_{33}^0\right)\right. \\
&\quad \left. -C_{44}^0\left(e_{15}^0 \epsilon_{33}^0 - e_{33}^0 \epsilon_{11}^0\right)\right\}\beta s \\
g_2 &= \left\{\left(e_{15}^0 + e_{31}^0\right)\left[\epsilon_{33}^0\left(C_{13}^0 + C_{44}^0\right) + e_{33}^0\left(e_{31}^0 + e_{15}^0\right)\right] - C_{44}^0\left(e_{15}^0 \epsilon_{33}^0 - e_{33}^0 \epsilon_{11}^0\right)\right\}s^2
\end{aligned}
\tag{5.234c}
$$

$$
\begin{aligned}
q_0 &= e_{33}^0\left(C_{44}^0 \epsilon_{11}^0 + e_{15}^{02}\right)\beta s \\
q_1 &= -e_{15}^0\left(e_{33}^{02} + C_{33}^0 \epsilon_{33}^0\right)\beta^2 + \left[\left(\epsilon_{33}^0 C_{44}^0 + e_{33}^0 e_{15}^0\right) - \left(e_{15}^0 \epsilon_{33}^0 - e_{31}^0 \epsilon_{11}^0\right)\left(C_{13}^0 + C_{44}^0\right)\right]s^2 \\
q_2 &= -\left(e_{31}^0 + 2e_{15}^0\right)\left(e_{33}^{02} + C_{33}^0 \epsilon_{33}^0\right)\beta s, \quad q_3 = -\left(e_{31}^0 + e_{15}^0\right)\left(e_{33}^{02} + C_{33}^0 \epsilon_{33}^0\right)s^2
\end{aligned}
\tag{5.234d}
$$

Let

$$
\varphi^c(x_1, x_3) = \frac{2}{\pi}\int_0^\infty B(s)\sinh(sx_3)\cos(sx_1)\mathrm{d}s, \quad -a < x_1 < a \tag{5.235}
$$

where $B(s)$ is undetermined function.

Using the dislocation density $\psi(x_1)$, $\sigma_{33}(x_1, 0) = 0,\ 0 \le x_1 < a$, other boundary conditions, and Eq. (5.235), finally we can get the following singular integral equation:

$$
\frac{1}{\pi}\int_{-a}^{a}\psi(t)\left[\frac{1}{t - x_1} + M_1(t, x_1) + M_2(t, x_1)\right]\mathrm{d}t = \frac{\sigma_h}{Z_0^\infty} \tag{5.236}
$$

The expressions of $M_1(t, x_1)$ and $M_2(t, x_1)$ are omitted here.

Equations (5.236) and (5.231) form a singular integral equation system. Let

$$
\psi(t) = \left(\sigma_h / Z_0^\infty\right)\Phi(u) \Big/ \sqrt{1 - u^2}, \quad u = t/a \tag{5.237}
$$

Substitute Eq. (5.237) into (5.236) and then use the Gauss-Jacobi numerical integral technique to solve the integral equation. The generalized stress intensity factors are

$$
K_{\mathrm{I}} = \lim_{x \to a^+}\sqrt{2\pi(x_1 - a)}\sigma_{33}(x_1, 0) = \sigma_0\sqrt{\pi a}\Phi(1), \quad K_{\mathrm{D}} = \left(Z_3^\infty / Z_0^\infty\right)K_{\mathrm{I}} \tag{5.238}
$$

A lot of literatures studied the functional graded piezoelectric materials, such as Zhou and Chen (2008), Chen et al. (2003), and Wang and Zhang (2004).

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