Chapter 4
Linear Inclusion and Related Problems

Abstract In this chapter, the linear cracks and inclusions are discussed. These problems are mainly reduced to vector Riemann-Hilbert boundary problem with many variables at first, and then the standard method to solve the Riemann-Hilbert boundary problem is used. In general case, the numerical computation is used to get the final results due to its complexity, but for some simpler problems, the analytical solutions can also be obtained. The interface cracks, rigid inclusion, and electrodes in piezoelectric bimaterials are discussed in detail. Some special problems, such as partly insulated and partly conducting crack, the nonideal crack and some other models in a homogeneous piezoelectric material, and contact zone model for interface cracks in a piezoelectric bimaterial, are also discussed shortly. Some interesting problems in engineering, such as interaction of collinear inclusions with singularity loading, interaction of an elliptic hole and a vice-crack, strip electric saturation model of an impermeable crack in a homogeneous material and a strip electric saturation model for mode-III interface crack in a bimaterial, and mode-III problem for a circular inclusion with interface cracks, are also discussed.

Keywords Linear interface crack and inclusion • Singularity • Strip electric saturation model • Circular inclusion

4.1 Vector Riemann-Hilbert Boundary-Value Problem in the z Plane

4.1.1 Fundamental Solution of the Homogeneous Equation

Let n non-intersect line segments $L_k, k = 1 \sim n$, be in the complex z plane, and its assemble is denoted by L. The end points of L_k are a_k, b_k and from a_k to b_k is its positive direction; the left region of $a_k b_k$ is the region S^+, and the right region is S^-.

Z.-B. Kuang, *Theory of Electroelasticity*, DOI 10.1007/978-3-642-36291-0_4,

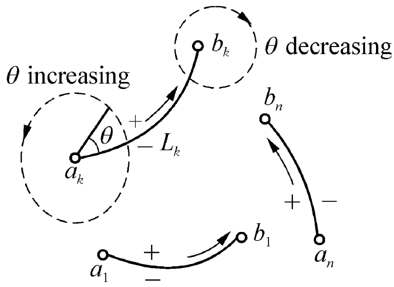


Fig. 4.1 Riemann-Hilbert boundary problem on smooth non-intersecting curves

On L functions $\mathbf{g}(t)$ and $\boldsymbol{\Sigma}_0(t)$ satisfied Hölder condition are given (Fig. 4.1). Now discuss thesolution of the following vector Riemann-Hilbert equation on L (Muskhelishvili 1954, 1975; Hou et al. 1990):

$$\boldsymbol{h}^+(t) - \mathbf{g}\boldsymbol{h}^-(t) = \boldsymbol{\Sigma}_0(t), \quad h_j^+(t) - g_{ij}h_j^-(t) = \Sigma_{0i}(t), \quad i,j = 1 - m; \quad t \in L \tag{4.1}$$

where $\mathbf{g}$ is an $m \times m$ order Hermite matrix and $\det \mathbf{g} \neq 0$ and t is a point on L. The superscripts " + " and " − " indicate the limit values taken from the left and right sides along $a_k b_k$, respectively. The corresponding homogeneous equation is

$$\boldsymbol{h}^+(t) - \mathbf{g}\boldsymbol{h}^-(t) = \mathbf{0}, \quad h_j^+(t) - g_{ij}h_j^-(t) = 0, \quad t \in L \tag{4.2}$$

Let the fundamental solution of the homogeneous equation be

$$\begin{aligned} &\boldsymbol{X}_0(z) = [X_{01}(z), X_{02}(z), \ldots X_{0m}(z))]^{\mathrm{T}} = \boldsymbol{\omega} Y_0(z) \\ &X_{0j}(z) = \omega_j Y_0(z), \quad Y_0(z) = \prod_{k=1}^{n} (z - a_k)^{-\gamma}(z - b_k)^{\gamma - 1} \\ &\boldsymbol{X}_0^-(t) = \mathrm{e}^{-2\pi \mathrm{i}\gamma}\boldsymbol{X}_0^+(t), \quad \text{or} \quad \boldsymbol{X}_0^+(t) = \mathrm{e}^{2\pi \mathrm{i}\gamma}\boldsymbol{X}_0^-(t) \end{aligned} \tag{4.3a}$$

Usually, the single-valued branch of the multi-value function $Y_0(z)$ is selected such that $Y_0(z) \to z^{-n}$ when $z \to \infty$. Substitution of Eq. (4.3a) into Eq. (4.2) yields

$$\mathrm{e}^{2\pi \mathrm{i}\gamma}\boldsymbol{X}_0^-(t) - \mathbf{g}\boldsymbol{X}_0^-(t) = 0 \quad \Rightarrow \quad (\mathrm{e}^{2\pi \mathrm{i}\gamma}\mathbf{I} - \mathbf{g})\boldsymbol{\omega} = 0 \tag{4.4}$$

In order to have nontrivial solution for $\boldsymbol{\omega}$, it must be

$$\left|\mathrm{e}^{2\pi \mathrm{i}\gamma}\mathbf{I} - \mathbf{g}\right| = 0, \quad \mathbf{I} = \mathrm{diag}[1, 1, \ldots, 1]_{m \times m} \tag{4.5}$$

From Eqs. (4.5) and (4.4), we can get m eigenvectors $\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \ldots, \boldsymbol{\omega}^{(m)}$ corresponding to m eigenvalues, $\mathrm{e}^{2\pi \mathrm{i}\gamma_1}, \mathrm{e}^{2\pi \mathrm{i}\gamma_2}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\gamma_m}$, where γ_k is limited within the semi-open interval $[0, 2\pi)$. For a eigenvalue $e^{2\pi i \gamma_k}$, there is only one component of $\boldsymbol{\omega}^{(i)}$ is undetermined. The fundamental solution Eq. (4.3a) becomes

$$\mathbf{X}_0^{(i)}(z) = \boldsymbol{\omega}^{(i)} Y_0^{(i)}(z), \quad X_{0j}^{(i)}(z) = \omega_j^{(i)} Y_0^{(i)}(z)$$
$$Y_0^{(i)}(z) = \prod_{k=1}^{n} (z - a_k)^{-\gamma_i} (z - b_k)^{\gamma_i - 1}, \quad i, j = 1, 2, \ldots, m \tag{4.3b}$$

The complete fundamental solutions form a square matrix $\boldsymbol{P}(z)$:

$$\boldsymbol{P}(z) = \left[\boldsymbol{X}_0^{(1)}(z), \boldsymbol{X}_0^{(2)}(z), \ldots, \boldsymbol{X}_0^{(n)}(z)\right] = \boldsymbol{\Omega Q}(z), \quad P_{ij}(z) = X_{0j}^{(i)}(z)$$
$$\boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z) \right\rangle = \operatorname{diag}\left[Y_0^{(1)}(z), \ldots, Y_0^{(m)}(z)\right], \quad \boldsymbol{\Omega} = \left[\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \ldots, \boldsymbol{\omega}^{(m)}\right] \tag{4.6}$$

4.1.2 First Solving Method

From the behavior of the fundamental solution, it is known that

$$\boldsymbol{P}^+(t) - \mathbf{g}\boldsymbol{P}^-(t) = \mathbf{0}, \quad \mathbf{g} = \boldsymbol{P}^+(t)[\boldsymbol{P}^-(t)]^{-1}, \quad t \in L \tag{4.7}$$

Substitution of Eq. (4.7) into Eq. (4.2) yields

$$[\boldsymbol{P}^+(t)]^{-1}\boldsymbol{h}^+(t) = [\boldsymbol{P}^-(t)]^{-1}\boldsymbol{h}^-(t), \quad t \in L \tag{4.8}$$

The function at the left side in Eq. (4.8) is analytic in S^+, whereas those on the right side are analytic in S^-, and they are continuous on L. So these functions are analytic in whole plane and must be constants. The general solution $\boldsymbol{h}_0(z)$ of Eq. (4.2) is

$$[\boldsymbol{P}(z)]^{-1}\boldsymbol{h}_0(z) = \boldsymbol{C}(z), \quad \boldsymbol{h}_0(z) = \boldsymbol{P}(z)\boldsymbol{C}(z)$$
$$\boldsymbol{C}(z) = \mathbf{c}_n z^n + \mathbf{c}_{n-1} z^{n-1} + \cdots + \boldsymbol{c}_0, \quad \boldsymbol{c}_k = [c_k^{(1)}, c_k^{(2)}, \ldots, c_k^{(m)}]^{\mathrm{T}}, \quad \text{or}$$
$$\boldsymbol{C}(z) = [C^{(1)}(z), C^{(2)}(z), \ldots, C^{(m)}(z)]^{\mathrm{T}}, \quad C^{(k)}(z) = \left[c_n^{(k)} z^n + c_{n-1}^{(k)} z^{n-1} + \cdots + c_0^{(k)}\right] \tag{4.9}$$

If the infinite point is a pole in order p, $\boldsymbol{C}(z)$ is a vector polynomial less than order $n + p$.

Substitution of Eq. (4.7) into the inhomogeneous equation (4.1) yields

$$[\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{h}^{+}(t)-[\boldsymbol{P}^{-}(t)]^{-1}\boldsymbol{h}^{-}(t)=[\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{\Sigma}_0(t),\quad t\in L \tag{4.10}$$

Equation (4.10) is a decoupling Riemann-Hilbert boundary problem of $[\boldsymbol{P}(z)]^{-1}\boldsymbol{h}(z)$. By using the Cauchy formula, the special solution $\boldsymbol{h}_{\mathrm{sp}}$ is

$$[\boldsymbol{P}(z)]^{-1}\boldsymbol{h}_{\mathrm{sp}}(z)=\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\boldsymbol{\Sigma}_0(t)\mathrm{d}t}{\boldsymbol{P}^{+}(t)(t-z)},\quad \boldsymbol{h}_{\mathrm{sp}}(z)=\frac{\boldsymbol{P}(z)}{2\pi\ \mathrm{i}}\int_L\frac{\boldsymbol{\Sigma}_0(t)\mathrm{d}t}{\boldsymbol{P}^{+}(t)(t-z)} \tag{4.11}$$

The general solution of the inhomogeneous equation (4.1) is

$$\begin{aligned}
\boldsymbol{h}(z)&=\boldsymbol{h}_0(z)+\boldsymbol{h}_{\mathrm{sp}}(z)\\
&=\boldsymbol{P}(z)\left[\boldsymbol{C}(z)+\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\tilde{\boldsymbol{\Sigma}}(t)\mathrm{d}t}{(t-z)}\right]=\sum_{k=1}^{m}\mathbf{X}_0^{(k)}(z)\left[\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\tilde{\Sigma}_k(t)}{(t-z)}\mathrm{d}t+C^{(k)}(z)\right]\\
\tilde{\boldsymbol{\Sigma}}(t)&=[\tilde{\Sigma}_1(t),\tilde{\Sigma}_2(t),\ldots,\tilde{\Sigma}_m(t)]^{\mathrm{T}}=[\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{\Sigma}_0(t)
\end{aligned} \tag{4.12}$$

4.1.3 Second Solving Method

Because $\mathbf{g}$ is a Hermite matrix, the eigenvectors corresponding to the different eigenvalues are orthogonal to each other in the complex space. Form a square matrix $\boldsymbol{\Omega}$ consisted of $\boldsymbol{\omega}$ and

$$\begin{aligned}
&\boldsymbol{\Omega}=\left[\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\ldots,\boldsymbol{\omega}^{(m)}\right],\quad \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\boldsymbol{\Lambda},\quad \bar{\boldsymbol{\Omega}}^{-\mathrm{T}}=\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\\
&\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\mathbf{g}\Omega=\boldsymbol{M},\quad \boldsymbol{M}=\mathrm{diag}[\mathrm{e}^{2\pi\ \mathrm{i}\gamma_1}\Lambda_1^2,\ldots,\mathrm{e}^{2\pi\ \mathrm{i}\gamma_m}\Lambda_m^2]\\
&\boldsymbol{\Lambda}=\mathrm{diag}[\Lambda_1^2,\Lambda_2^2,\ldots,\Lambda_m^2],\quad \Lambda_i^2=\bar{\omega}_1^{(i)}\omega_1^{(i)}+\bar{\omega}_2^{(i)}\omega_2^{(i)}+\cdots+\bar{\omega}_m^{(i)}\omega_m^{(i)}
\end{aligned} \tag{4.13}$$

In most cases $\boldsymbol{\Omega}$ is assumed normalized, i.e., $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\boldsymbol{\Lambda}=\mathbf{I}$.

Multiplying on both sides of Eq. (4.1) from left by $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}$ and using Eq. (4.13) we get

$$\begin{aligned}
&\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}^{+}(t)-(\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\mathbf{g}\Omega)\boldsymbol{\Omega}^{-1}\boldsymbol{h}^{-}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}^{+}(t)-\boldsymbol{M}\boldsymbol{\Lambda}^{-1}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}^{-}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(t)\\
&\boldsymbol{M}\boldsymbol{\Lambda}^{-1}=\mathrm{diag}[\mathrm{e}^{2\pi\ \mathrm{i}\gamma_1},\mathrm{e}^{2\pi\ \mathrm{i}\gamma_2},\ldots,\mathrm{e}^{2\pi\ \mathrm{i}\gamma_m}]=\left\langle \mathrm{e}^{2\pi\ \mathrm{i}\gamma_j}\right\rangle
\end{aligned} \tag{4.14}$$

Equation (4.14) can be expressed in the following decoupling form:

$$\begin{aligned}
&\boldsymbol{\Psi}^{+}(t)-\boldsymbol{M}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Psi}^{-}(t)=\boldsymbol{\Sigma}^{*}(t),\quad \Psi_i^{+}(t)-\mathrm{e}^{2\pi\ \mathrm{i}\gamma_i}\Psi_i^{-}(t)=\Sigma_i^{*}(t)\\
&\boldsymbol{\Psi}(z)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z),\quad \boldsymbol{\Sigma}^{*}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(t)
\end{aligned} \tag{4.15}$$

Equation (4.15) is the scalar Riemann-Hilbert boundary-value problem of the component Ψ_i of $\boldsymbol{\Psi}$, soits solution is

$$\begin{aligned}
&\boldsymbol{\Psi}(z)=\boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda C}(z)+\frac{1}{2\pi \mathrm{i}}\int_L\frac{[\boldsymbol{Q}^+(t)]^{-1}\boldsymbol{\Sigma}^*(t)\mathrm{d}t}{(t-z)}\right\};\quad \boldsymbol{h}\ (z)=\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z)\\
&\Psi_i(z)=Y_0^{(i)}(z)\left\{\Lambda_i^2C^{(i)}(z)+\frac{1}{2\pi \mathrm{i}}\int_L\frac{\Sigma_i^*(t)\mathrm{d}t}{Y_0^{(i)+}(t)(t-z)}\right\}\\
&\boldsymbol{Q}(z)=\left\langle Y_0^{(i)}(z)\right\rangle
\end{aligned}\tag{4.16}$$

Solving $\boldsymbol{\Psi}(z)$, $\boldsymbol{h}(z)$ is obtained by $\boldsymbol{h}(z)=\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z)$, where $\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}=[\bar{\boldsymbol{\Omega}}^{\mathrm{T}}]^{-1}=\boldsymbol{\Omega\Lambda}^{-1}$. If we assume $[\boldsymbol{Q}^+(t)]_{ij}^{-1}\Sigma_j^*(t)\to\boldsymbol{\alpha}_qt^q+\cdots+\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_{-1}/t+\cdots$, when $t\to\infty$ and it is single valued, Eq. (4.16) is reduced to

$$\begin{aligned}
&\boldsymbol{\Psi}(z)=\boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda C}(z)+\left\langle 1-\mathrm{e}^{2\pi \mathrm{i}\gamma_i}\right\rangle^{-1}\left[[\boldsymbol{Q}(z)]^{-1}\boldsymbol{\Sigma}^*(z)-\left(\boldsymbol{\alpha}_qz^q+\cdots+\boldsymbol{\alpha}_0\right)\right]\right\}\\
&\Psi_i(z)=Y_0^{(i)}(z)\left\{\Lambda_i^2C^{(i)}(z)+\frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}\gamma_i}}\left[\frac{\boldsymbol{\Sigma}_i^*(z)}{Y_0^{(i)}(z)}-\left(\alpha_q^{(i)}z^q+\cdots+\alpha_0^{(i)}\right)\right]\right\}
\end{aligned}\tag{4.17}$$

where the following integral formula has been used (Shen and Kuang 1998):

$$\begin{aligned}
&\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{G^+(t)\mathrm{d}t}{X^+(t)(t-z)}=\frac{1}{1-\mathrm{g}\mathrm{g}^*}\left[\frac{G(z)}{X(z)}-\alpha_qz^q-\cdots-\alpha_0\right]\\
&X^+(t)-\mathrm{g}X^-(t)=0,\quad G^-(t)=\mathrm{g}^*G^+(t),\quad \mathrm{g}=\mathrm{e}^{2\pi \mathrm{i}\gamma},\quad t\in L
\end{aligned}\tag{4.18}$$

For a single-valued function $G(z)$, $\mathrm{g}^*=1$, Eq. (4.18) is just the formula given by Muskhelishvili (1954).

The two methods are equivalent. In fact by using $\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}=\boldsymbol{\Omega\Lambda}^{-1}$, Eq. (4.15) can be reduced to

$$\begin{aligned}
\boldsymbol{h}(z)=\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z)&=\boldsymbol{\Omega\Lambda}^{-1}\boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda C}(z)+\frac{1}{2\pi \mathrm{i}}\int_L\frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(x_1)\mathrm{d}t}{\boldsymbol{Q}^+(x_1)(x_1-z)}\right\}\\
&=\boldsymbol{\Omega Q}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\boldsymbol{\Sigma}_0(x_1)\mathrm{d}t}{\boldsymbol{\Omega Q}^+(x_1)(x_1-z)}\right\}\\
&=\boldsymbol{P}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\tilde{\boldsymbol{\Sigma}}(x_1)\mathrm{d}x_1}{(x_1-z)}\right\}
\end{aligned}$$

which is identical with Eq. (4.12).

4.2 Interface Cracks in Piezoelectric Bimaterials

4.2.1 General Discussion of an Impermeable Interface Cracks

Discuss a piezoelectric bimaterial with collinear impermeable cracks without generalized loading at infinity (Suo 1990; Suo et al. 1992; Kuang and Ma 2002). Let the material I be located at the upper half plane S^+, $x_2 > 0$; the material II is located at the lower half plane S^-, $x_2 < 0$; $x_2 = 0$ is the interface L, there are collinear cracks, the left end point of the crack L_k is denoted by a_k, and the right end point b_k and its assemble is denoted by L_c. $L - L_c$ is the connected surface (Fig. 4.2). For a single crack with length $2a$, we always let the coordinate origin be selected at the center of the crack. These notations will be used in this whole chapter. Assuming the generalized forces $\boldsymbol{\Sigma}_0(x_1) = [t_1^*, t_2^*, t_3^*, -\sigma^*]^{\mathrm{T}}$ acting on the crack surfaces are self-equilibrium, at infinity, generalized forces are equal to zero, i.e.,

$$\begin{aligned}&\boldsymbol{\Sigma}(x_1) = \boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) = \boldsymbol{\Sigma}_0(x_1), \quad x \in L_c \\ &\boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) = \mathbf{0}, \quad \text{at infinity;} \quad \boldsymbol{\Sigma}_\beta(x_1) = 2\mathrm{Re}[\boldsymbol{B}_\beta \mathbf{F}_\beta(x_1)], \quad \beta = \mathrm{I}, \mathrm{II}\end{aligned} \tag{4.19}$$

On the connected surface, the generalized displacements and traction are continuous:

$$\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = \mathbf{0}, \quad \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1), \quad x \in L - L_c \tag{4.20}$$

where $\hat{\boldsymbol{d}}(x_1)$ is the displacement disconnected value between crack surfaces and the crack opening displacement. Because for any subscript j, $x_{1j} = x_1$ is held on the axis x_1, so

$$\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = 2\mathrm{Re}[\boldsymbol{A}_{\mathrm{I}} \boldsymbol{f}_{\mathrm{I}}(x_1) - \boldsymbol{A}_{\mathrm{II}} \boldsymbol{f}_{\mathrm{II}}(x_1)] \tag{4.21}$$

According to the given conditions, the generalized tractions are continuous on the whole axis x_1, i.e., $\boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1)$, or

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}} \boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}} \overline{\boldsymbol{F}_{\mathrm{I}}(x_1)} = \boldsymbol{B}_{\mathrm{II}} \boldsymbol{F}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{II}} \overline{\boldsymbol{F}_{\mathrm{II}}(x_1)}, \quad -\infty < x_1 < \infty, \quad \text{or} \\ &\boldsymbol{B}_{\mathrm{I}} \boldsymbol{F}_{\mathrm{I}}^+(x_1) - \bar{\boldsymbol{B}}_{\mathrm{II}} \bar{\boldsymbol{F}}_{\mathrm{II}}^+(x_1) = \boldsymbol{B}_{\mathrm{II}} \boldsymbol{F}_{\mathrm{II}}^-(x_1) - \bar{\boldsymbol{B}}_{\mathrm{I}} \bar{\boldsymbol{F}}_{\mathrm{I}}^-(x_1)\end{aligned} \tag{4.22}$$

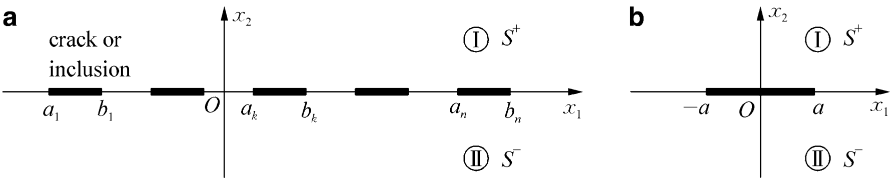


Fig. 4.2 Collinear interface cracks or inclusions: (**a**) general case and (**b**) one crack or inclusion

where the superscripts " + " and " − " indicate the limit values taken from the upper and lower half -planes, respectively. It is known that the functions at the left side in Eq. (4.22) are analytic in the upper half plane $x_2 > 0$, whereas those on the right side are analytic in the lower half plane $x_2 < 0$, and they are continuous on $x_1 = 0$. So, according to Liouville theorem (Lavrenchive and Shabat 1951), these functions are analytic in whole plane and must be constants and equal to zero due to $\boldsymbol{\Sigma}^{\infty} = \mathbf{0}$. So,

$$\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) = \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z), \quad x_2 > 0; \quad \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) = \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z), \quad x_2 < 0 \tag{4.23}$$

From Eqs. (4.21) and (4.23), the dislocation density $\hat{\boldsymbol{d}}'(x_1)$ can be written as

$$\begin{aligned}
\mathrm{i}\hat{\boldsymbol{d}}'(x_1) &= \mathrm{i}\mathrm{d}\hat{\boldsymbol{d}}(x_1)/\mathrm{d}x_1 = [\mathrm{i}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{I}}\overline{\mathbf{F}_{\mathrm{I}}(x_1)}] - [\mathrm{i}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{II}}\overline{\mathbf{F}_{\mathrm{II}}(x_1)}] \\
&= [\mathrm{i}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{B}_{\mathrm{I}}^{-1} - \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}]\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) - [\mathrm{i}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{II}}^{-1} - \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}]\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) \\
&= \boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) - \bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)
\end{aligned} \tag{4.24}$$

where

$$\boldsymbol{H} = \boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}}, \quad \boldsymbol{Y}_{\alpha} = \mathrm{i}\boldsymbol{A}_{\alpha}\boldsymbol{B}_{\alpha}^{-1}, \quad \boldsymbol{Y}_{\alpha} = \begin{pmatrix} \boldsymbol{Y}_{\alpha 11} & \boldsymbol{Y}_{\alpha 14} \\ \boldsymbol{Y}_{\alpha 41} & \boldsymbol{Y}_{\alpha 44} \end{pmatrix}, \quad \alpha = \mathrm{I}, \mathrm{II} \tag{4.25}$$

It is easy shown that $\boldsymbol{Y}_{\alpha}$ and $\boldsymbol{H}$ are all Hermite matrixes. $\boldsymbol{Y}_{\alpha 11}$ is a 3×3 positive definite matrix, $\boldsymbol{Y}_{\alpha 14} = \bar{\boldsymbol{Y}}_{\alpha 41}^{\mathrm{T}}$ is a piezoelectric matrix, and $Y_{\alpha 44}$ is an element of dielectric coefficient. For a stable material, $Y_{\alpha 44} < 0$.

4.2.2 A Simple Method to Get $\boldsymbol{F}_{\beta}(z_j)$

Because on the interface $z_j = x_1$, a simple method to solve the problem can be adopted (Suo 1990; Kuang and Ma 2002). At first we discuss two auxiliary complex functions $\boldsymbol{F}_{\mathrm{I}}(z)$ and $\boldsymbol{F}_{\mathrm{II}}(z)$ in z plane with complex variable z which also satisfy Eqs. (4.19) and (4.24) on the interface and solve the problem in z plane. According to Eq. (4.24), we can construct an auxiliary function $\boldsymbol{h}(z)$ analytic in whole plane except cracks by standard analytic continuation through the connected part on the interface:

$$\boldsymbol{h}(z) = \begin{cases} \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z), & x_2 \geq 0 \\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z), & x_2 \leq 0 \end{cases}, \quad z \notin L_{\mathrm{c}}, \quad z = x_1 + \mathrm{i}x_2 \tag{4.26}$$

The standard analytic continuation will be often used in the following sections. It is obvious that at points $x_1 \notin L_{\mathrm{c}}$ on axis x_1, $\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) = \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)$ is held. Solving $\boldsymbol{h}(z)$, the $\boldsymbol{F}_{\beta}(z)$ can be obtained by the following equations:

$$\begin{aligned}&\boldsymbol{F}_{\mathrm{I}}(z)=\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z),\quad x_2\geq 0;\quad \boldsymbol{F}_{\mathrm{II}}(z)=\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}(z),\quad x_2\leq 0\\&F_{\mathrm{I}j}(z)=B_{\mathrm{I}jl}^{-1}h_l(z),\quad x_2\geq 0;\quad F_{\mathrm{II}j}(z)=B_{\mathrm{II}jm}^{-1}\overline{H}_{\mathrm{m}\,n}^{-1}H_{nl}h_l(z),\quad x_2\leq 0\end{aligned}\tag{4.27}$$

where $B_{\beta jl}^{-1}=\left[\boldsymbol{B}_{\beta}^{-1}\right]_{jl}$; $\beta=\mathrm{I},\mathrm{II}$; $j,l=1-4$.

Substituting Eqs. (4.26) and (4.23) into Eq. (4.19) and noting on x_1 axis all $z_j=x_1$ we find

$$\boldsymbol{h}^{+}(x)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x)=\boldsymbol{\Sigma}_0(x_1),\quad x_1\in L_{\mathrm{c}}\tag{4.28}$$

If let $\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}=-\mathbf{g}$, Eq. (4.28) is identical with Eq. (4.1), which is solved as shown in Sect. 4.2.

A simple method to get $\boldsymbol{F}_{\beta}(z_j)$, for the original piezoelectric problem is replacing z by z_j in $\boldsymbol{F}_{\beta}(z)$. In fact the solution $F_{\beta}(z_j)$ solved by this method are still satisfy Eqs. (4.19) and (4.20) due to on the axis x_1, $\boldsymbol{F}_{\mathrm{I}}(z_j)=\boldsymbol{F}_{\mathrm{I}}(x_1)$, $\boldsymbol{F}_{\mathrm{II}}(z_j)=\boldsymbol{F}_{\mathrm{II}}(x_1)$, and $\boldsymbol{h}(z_j)=\boldsymbol{h}(x_1)$. Outside axis x_1, $\boldsymbol{Af}(z_j)$ and $\boldsymbol{Bf}(z_j)$ satisfy the generalized equilibrium equations due to they are selected as the general solutions given in Eqs. (3.18) and (3.23).

4.2.3 General Solution of the Homogeneous Equation

From Eq. (4.28), the homogeneous vector Riemann-Hilbert equation in the z plane is

$$\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x_1)=\mathbf{0},\quad x_1\in L_{\mathrm{c}}\tag{4.29}$$

If let $\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}=-\mathbf{g}$, Eq. (4.29) is identical with (4.1). So the solution of the homogeneous equation is still expressed by Eqs. (4.3) and (4.6), but Eqs. (4.4) and (4.5) are changed to

$$\left(\mathrm{e}^{2\pi\mathrm{i}\gamma}\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{\omega}=\mathbf{0},\quad \left|\mathrm{e}^{2\pi\mathrm{i}\gamma}\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right|=0\tag{4.30}$$

Let $\gamma=1/2+\mathrm{i}\varepsilon$. Using $\mathrm{e}^{2\pi\mathrm{i}(1/2+\mathrm{i}\varepsilon)}=-\mathrm{e}^{-2\pi\varepsilon}$, Eq. (4.30) and its conjugate equation can be reduced to

$$\begin{aligned}&\left(\mathrm{e}^{-2\pi\varepsilon}\mathbf{I}-\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{\omega}=\mathbf{0},\quad (\mathrm{e}^{-2\pi\varepsilon}\mathbf{I}-\boldsymbol{H}^{-1}\bar{\boldsymbol{H}})\bar{\boldsymbol{\omega}}=\mathbf{0}\\&\left|\bar{\boldsymbol{H}}-\mathrm{e}^{2\pi\varepsilon}\boldsymbol{H}\right|=0,\quad \left|\bar{\boldsymbol{H}}-\mathrm{e}^{-2\pi\varepsilon}\boldsymbol{H}\right|=0\end{aligned}\tag{4.31}$$

It is obvious that ε and $-\varepsilon$ are all the solutions of Eq. (4.31). Because $\boldsymbol{H}$ is a 4×4 order Hermite matrix, it can be decomposed to

$$\boldsymbol{H} = \boldsymbol{\Lambda}_1 + \mathrm{i}\boldsymbol{\Lambda}_2, \quad \bar{\boldsymbol{H}} = \boldsymbol{\Lambda}_1 - \mathrm{i}\boldsymbol{\Lambda}_2 \tag{4.32}$$

where $\boldsymbol{\Lambda}_1$ is a real symmetric matrix and $\boldsymbol{\Lambda}_2$ is an antisymmetric matrix. Let

$$\beta = \tanh(\pi\varepsilon) = \frac{\mathrm{e}^{\pi\varepsilon} - \mathrm{e}^{-\pi\varepsilon}}{\mathrm{e}^{\pi\varepsilon} + \mathrm{e}^{-\pi\varepsilon}} = \frac{\mathrm{e}^{2\pi\varepsilon} - 1}{\mathrm{e}^{2\pi\varepsilon} + 1}, \quad \text{or} \quad \varepsilon = \frac{\mathbf{1}}{2\pi} \ln \frac{1+\beta}{1-\beta} \tag{4.33}$$

Substitution of Eqs. (4.32) and (4.33) into Eq. (4.31) yields

$$\left|\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2 + \mathrm{i}\beta\mathbf{I}\right| = 0, \quad \left|\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2 - \mathrm{i}\beta\mathbf{I}\right| = 0 \tag{4.34}$$

It is known that $\beta, -\beta$ are all roots of the above equation. Expanding above equation, we get

$$\beta^4 + 2b\beta^2 + c = 0, \quad b = (1/4)\mathrm{tr}[(\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2)^2], \quad c = \left|\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2\right| \tag{4.35a}$$

Because $\boldsymbol{\Lambda}_2$ is an even antisymmetric matrix, $|\boldsymbol{\Lambda}_2| \geq 0$, $Y_{\alpha 11}$ positive definite, $Y_{\alpha 44} < 0$, it is derived that $\left|\boldsymbol{\Lambda}_1^{-1}\right| < 0$ and $c < 0$. Therefore

$$\beta_{1,2} = \pm\sqrt{(b^2 - c)^{1/2} - b}, \quad \beta_{3,4} = \pm\mathrm{i}\sqrt{(b^2 - c)^{1/2} + b} \tag{4.35b}$$

Corresponding ε is denoted as

$$\begin{aligned} \varepsilon_1 &= -\varepsilon_2 = \varepsilon_0, \quad \varepsilon_0 = \frac{1}{\pi} \operatorname{ar\,tanh} \sqrt{(b^2 - c)^{1/2} - b} \\ \varepsilon_4 &= -\varepsilon_3 = \mathrm{i}\kappa, \quad \kappa = \frac{1}{\pi} \operatorname{ar\,tan} \sqrt{(b^2 - c)^{1/2} + b} \end{aligned} \tag{4.36}$$

where ε_0, κ are real. From Eqs. (4.31), (4.32), (4.33), (4.34), (4.35a), (4.35b) and (4.36), it is known that $\boldsymbol{\omega}^{(1)}$ and $\bar{\boldsymbol{\omega}}^{(2)}$, $\boldsymbol{\omega}^{(3)}$ and $\bar{\boldsymbol{\omega}}^{(3)}$, and $\boldsymbol{\omega}^{(4)}$ and $\bar{\boldsymbol{\omega}}^{(4)}$ satisfy the same eigen-equation, so we have $\boldsymbol{\omega}^{(1)} = c\bar{\boldsymbol{\omega}}^{(2)}$, where c is a real constant and $\boldsymbol{\omega}^{(3)}$ and $\boldsymbol{\omega}^{(4)}$ are real vectors.

The fundamental solution of Eqs. (4.3), (4.6), and (4.13) can be rewritten in ε as

$$\begin{gathered} \boldsymbol{P}(z) = \left[\boldsymbol{X}_0^{(i)}(z)\right] = \boldsymbol{\Omega}\boldsymbol{Q}(z), \quad \boldsymbol{\Omega} = \left[\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \boldsymbol{\omega}^{(3)}, \boldsymbol{\omega}^{(4)}\right], \quad \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega} = \boldsymbol{\Lambda} = \left\langle \Lambda_j^2 \right\rangle \\ \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{\Omega} = -\boldsymbol{M}, \quad \boldsymbol{M} = \left\langle \mathrm{e}^{2\pi\,\mathrm{i}\gamma_j}\Lambda_j^2 \right\rangle, \quad \boldsymbol{M}\boldsymbol{\Lambda}^{-1} = \left\langle \mathrm{e}^{2\pi\,\mathrm{i}\gamma_j} \right\rangle = \left\langle -\mathrm{e}^{2\pi\,\mathrm{i}\varepsilon_j} \right\rangle \\ \boldsymbol{Q}(z) = \left\langle Y_0^{(i)}(z) \right\rangle, \quad Y_0^{(i)}(z) = \prod_{k=1}^{n} \frac{1}{\sqrt{(z-a_k)(z-b_k)}} \left(\frac{z-b_k}{z-a_k}\right)^{\mathrm{i}\varepsilon_i}, \quad i = 1,2,3,4 \end{gathered} \tag{4.37}$$

In practice $\boldsymbol{\Omega}$ is normalized, i.e., $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega} = \mathbf{I}$.

For a homogeneous material, $\boldsymbol{H}$, $\boldsymbol{\Omega}$ are real, so $\varepsilon_j = \varepsilon = 0, \gamma_j = \gamma = 1/2$, $\boldsymbol{Q}(z) = \langle Y_0(z) \rangle$, $Y_0(z) = Y_0^{(i)}(z) = \prod_{k=1}^{n} [(z - a_k)(z - b_k)]^{-1/2},\ i = 1 - m$.

4.2.4 General Solution of the Inhomogeneous Equation for Impermeable Cracks

First method. For the inhomogeneous equation (4.1) in z plane, the solution of $\boldsymbol{h}(z)$ is Eq. (4.12), i.e.,

$$\boldsymbol{h}(z) = \boldsymbol{P}(z)\left[\boldsymbol{C}(z) + \frac{1}{2\pi\ \mathrm{i}} \int_L \frac{\tilde{\boldsymbol{\Sigma}}(t)\mathrm{d}t}{(t - z)}\right] = \sum_{k=1}^{4} \mathbf{X}_0^{(k)}(z)\left[\frac{1}{2\pi\ \mathrm{i}} \int_L \frac{\tilde{\Sigma}_k(t)}{(t - z)}\mathrm{d}t + C^{(k)}(z)\right]$$
$$\tilde{\boldsymbol{\Sigma}}(t) = [\tilde{\Sigma}_1(t), \tilde{\Sigma}_2(t), \ldots, \tilde{\Sigma}_4(t)]^{\mathrm{T}} = [\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{\Sigma}_0(t) \tag{4.38}$$

Solving $\boldsymbol{h}(z)$, according to Sect. 4.2.2, $\boldsymbol{F}_\beta(z_j)$ can be solved by the following equations:

$$\boldsymbol{F}_{\mathrm{I}}(z) = \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z), \quad F_{\mathrm{I}j}(z_j) = B_{\mathrm{I}jl}^{-1}h_l(z_j), \quad \boldsymbol{F}_{\mathrm{I}}(z_j) = \left[F_{\mathrm{I}j}(z_j)\right]^{\mathrm{T}}, \quad x_2 \geq 0;$$
$$\boldsymbol{F}_{\mathrm{II}}(z) = \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}(z), \quad F_{\mathrm{II}j}(z_j) = B_{\mathrm{II}jm}^{-1}\overline{H}_{\mathrm{m}n}^{-1}H_{nl}h_l(z_j), \quad \boldsymbol{F}_{\mathrm{II}}(z_j) = \left[F_{\mathrm{II}j}(z_j)\right]^{\mathrm{T}}, \quad x_2 \leq 0 \tag{4.39}$$

The stresses are

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathrm{I}1} &= -2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{I}}\mu_j\boldsymbol{F}_{\mathrm{I}}(z_j)\right], \quad &\boldsymbol{\Sigma}_{\mathrm{I}2} = 2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z_j)\right], \quad & x_2 \geq 0 \\ \boldsymbol{\Sigma}_{\mathrm{II}1} &= -2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{II}}\mu_j\boldsymbol{F}_{\mathrm{II}}(z_j)\right], \quad &\boldsymbol{\Sigma}_{\mathrm{II}2} = 2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z_j)\right], \quad & x_2 \leq 0 \end{aligned} \tag{4.40}$$

Second method. The solution $\boldsymbol{\Psi}(z)$ of Eq. (4.1) is shown in Eqs. (4.16) or (4.17), i.e.,

$$\begin{aligned} \boldsymbol{\Psi}(z) &= \boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda}\boldsymbol{C}(z) + \frac{1}{2\pi\mathrm{i}} \int_L \frac{[\boldsymbol{Q}^{+}(x_1)]^{-1}\boldsymbol{\Sigma}^{*}(t)\mathrm{d}t}{(x_1 - z)}\right\} \\ &= \boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda}\boldsymbol{C}(z) + \left\langle 1 - \mathrm{e}^{2\pi\mathrm{i}\gamma_i}\right\rangle^{-1}\left[[\boldsymbol{Q}(z)]^{-1}\boldsymbol{\Sigma}^{*}(z) - \left(\boldsymbol{\alpha}_q z^q + \cdots + \boldsymbol{\alpha}_0\right)\right]\right\} \\ \boldsymbol{\Psi}(z) &= \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z), \quad \boldsymbol{\Sigma}^{*}(t) = \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(t) \end{aligned} \tag{4.41a}$$

where $[\boldsymbol{Q}^{+}(t)]_{ij}^{-1}\Sigma_j^{*}(t) \to \boldsymbol{\alpha}_q t^q + \cdots + \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_{-1}/t + \cdots$, when $t \to \infty$ is assumed. Combining the similar terms in Eq. (4.41a) yields

$$\boldsymbol{\Psi}(z) = \boldsymbol{Q}(z)\boldsymbol{C}(z) + \left(\mathbf{I} - \boldsymbol{M}\boldsymbol{\Lambda}^{-1}\right)^{-1}\boldsymbol{\Sigma}^{*}(z), \quad \boldsymbol{h}(z) = \bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z) \tag{4.41b}$$

$\boldsymbol{F}_\beta(z_j)$ can be obtained from Eq. (4.39).

The closed solutions of the displacements and stresses are difficult obtained, usually adopted numerical method. But the stress intensity can be expressed analytically.

4.2.5 The Stress Asymptotic Field and the Stress Intensity Factors

Discuss a crack of length $2a$ and its center is selected as the origin (Fig. 4.2b). From Eqs. (4.37) and (4.38), the fundamental solution can be written as

$$\begin{aligned} &Y_0^{(i)}(z) = \frac{1}{\sqrt{z^2 - a^2}}\left(\frac{z-a}{z+a}\right)^{\mathrm{i}\varepsilon_i}, \quad \hat{\boldsymbol{C}}'(z) = \boldsymbol{C}(z) + \frac{1}{2\pi\,\mathrm{i}}\int_L \frac{\tilde{\boldsymbol{\Sigma}}(x_1)\mathrm{d}x_1}{x_1 - z} \\ &\boldsymbol{h}(z) = \boldsymbol{\Omega}\left\langle Y_0^{(i)}(z)\right\rangle\hat{\boldsymbol{C}}'(z), \quad \tilde{\boldsymbol{\Sigma}}(t) = [\boldsymbol{P}^+(t)]^{-1}\boldsymbol{\Sigma}_0(t), \quad i = 1,2,3,4 \end{aligned} \tag{4.42}$$

Near the right crack tip $x_1 = a$, the asymptotic form of $\boldsymbol{h}(z)$ and $\boldsymbol{F}_\beta(z_j)$ is, respectively,

$$\begin{aligned} &\lim_{z\to a}\boldsymbol{h}(z) = \boldsymbol{\Omega}\left\langle (z-a)^{-(1/2)+\mathrm{i}\varepsilon_j}\right\rangle\hat{\boldsymbol{C}}(a), \quad \hat{\boldsymbol{C}}(a) = \left\langle (z+a)^{-(1/2)-\mathrm{i}\varepsilon_j}\right\rangle\hat{\boldsymbol{C}}'(a) \\ &\boldsymbol{F}_{\mathrm{I}}(z) = \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z), \quad F_{\mathrm{I}j}(z_j) = B_{\mathrm{I}jl}^{-1}h_l(z_j), \quad x_2 \ge 0; \\ &\boldsymbol{F}_{\mathrm{II}}(z) = \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}(z), \quad F_{\mathrm{II}j}(z_j) = B_{\mathrm{II}jm}^{-1}\bar{H}_{\mathrm{m}n}^{-1}H_{nl}h_l(z_j), \quad x_2 \le 0 \end{aligned} \tag{4.43}$$

Combining Eqs. (4.40) and (4.43) yields the asymptotic stresses near the right crack tip, but they are complex. However when the stress intensity factors are discussed only, the general expressions of the stress asymptotic field are not needed. Using all $z_j = x_1$ on the axis x_1 yields

$$\begin{aligned} &\boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1) = \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1) \\ &\boldsymbol{h}^+(x_1) = \boldsymbol{h}^-(x_1) = \boldsymbol{h}(x_1), \quad \text{or} \quad \boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) = \bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) \end{aligned}$$

Using Eqs. (4.23), (4.26), and (4.31) yields

$$\begin{aligned} \boldsymbol{\Sigma}_2(x_1) &= \boldsymbol{h}^+(x_1) + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1) = (\mathbf{I} + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H})\boldsymbol{h}(x_1) \\ &= \boldsymbol{\Omega}\left\langle \left(1 + \mathrm{e}^{-2\pi\varepsilon_j}\right)(x_1 - a)^{-(1/2)+\mathrm{i}\varepsilon_j}\right\rangle\hat{\boldsymbol{C}}(a) \end{aligned} \tag{4.44}$$

If $\boldsymbol{H}$ is complex, from Eq. (4.44), it is seen that the stresses are oscillated near the crack tip. The stress intensity factors $\boldsymbol{K}$ of the bimaterial are defined in the way that they can be reduced to the definition in a homogeneous material. According to Eq. (4.44), the $\boldsymbol{K}$ can be defined as

$$\boldsymbol{K} = \sqrt{2\pi}\boldsymbol{\Omega}\left\langle 1 + \mathrm{e}^{-2\pi\varepsilon_j}\right\rangle\hat{\boldsymbol{C}}, \quad \hat{\boldsymbol{C}} = \left(1\big/\sqrt{2\pi}\right)\left\langle 1 + \mathrm{e}^{-2\pi\varepsilon_j}\right\rangle^{-1}\boldsymbol{\Omega}^{-1}\boldsymbol{K} \tag{4.45}$$

The stress asymptotic field can be written as

$$\lim_{x_1\to a}\boldsymbol{\Sigma}_2(x_1) = \frac{1}{\sqrt{2\pi(x_1 - a)}}\boldsymbol{\Omega}\left\langle (x_1 - a)^{\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{K} \tag{4.46}$$

According to Eq. (4.46) $\boldsymbol{K}$ can be expressed by the generalized stresses as

$$\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{x_1\to a}\sqrt{2\pi(x_1 - a)}\boldsymbol{\Omega}\left\langle (x_1 - a)^{-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1) \tag{4.47}$$

where $\boldsymbol{K}$ is real and does not effect by the constant in $\boldsymbol{\Omega}$. For a homogeneous material, $\left\langle (x_1 - a)^{-i\varepsilon_j}\right\rangle = \mathbf{I}$ and $\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{x_1\to a}\sqrt{2\pi(x_1 - a)}\boldsymbol{\Sigma}_2(x_1)$ which is identical with that in Eq. (3.220). In some literatures, the following definition is also used:

$$\begin{aligned} &\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{x_1\to a}\sqrt{2\pi(x_1 - a)}\boldsymbol{\Sigma}_2(x_1)\left\langle (x_1 - a)^{-\mathrm{i}\varepsilon_j}\right\rangle \\ &\lim_{x_1\to a}\boldsymbol{\Sigma}_2(x_1) = \frac{1}{\sqrt{2\pi(x_1 - a)}}\left\langle (x_1 - a)^{\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{K} \end{aligned} \tag{4.48}$$

Beom and Atluri (1996), Shen et al. (1999, 2007), and many other literatures discussed many interesting problems.

4.2.6 *Permeable Crack*

Discuss a permeable crack in an infinite bimaterial. The boundary condition at infinity is

$$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\infty}(x_1), \quad \text{at infinity} \tag{4.49}$$

The mixed boundary conditions on the crack surface and the continuity conditions on the connective interface are

$$\begin{aligned} &\sigma_{\mathrm{I}\,2\,i}(x_1) = \sigma_{\mathrm{II}\,2\,i}(x_1) = 0; \quad E_{\mathrm{I}\,1} = E_{\mathrm{II}\,1}, \quad D_{\mathrm{I}2} = D_{\mathrm{II}2} = D_2; \quad x_1 \in L_{\mathrm{c}}; \quad i = 1, 2 \\ &u_{\mathrm{I}\,2\,i}(x_1) = u_{\mathrm{II}\,2\,i}(x_1), \ \sigma_{\mathrm{I}\,2\,i}(x_1) = \sigma_{\mathrm{II}\,2\,i}(x_1); \quad E_{\mathrm{I}\,1} = E_{\mathrm{II}\,1}, \quad D_{\mathrm{I}2} = D_{\mathrm{II}2} = D_2; \\ &\quad x_1 \in L - L_{\mathrm{c}} \end{aligned} \tag{4.50}$$

The main different of the permeable crack with the impermeable crack is that the electric displacement on an impermeable crack is given and the potential is unknown, but on a permeable crack D_2 is undetermined and the potential is given. Because the generalized stresses are continuous on the whole axis x_1, so

$$\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{x}_1)=\boldsymbol{\Sigma}_2(x_1),\quad -\infty<x_1<\infty \tag{4.51}$$

Noting $\boldsymbol{\Sigma}_2=\boldsymbol{\Sigma}_2^{\infty}(x_1)\neq\mathbf{0}$ at infinity, like Eqs. (4.22) and (4.23), from Eq. (4.51) we get

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{x}_1)=\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{\Delta}^{\infty}\\&\boldsymbol{\Delta}^{\infty}=(1/2)[(\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{\infty}+\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^{\infty})-(\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^{\infty}+\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty})];\quad \boldsymbol{F}_{\alpha}(z),\quad \alpha=\mathrm{I},\mathrm{II}\end{aligned} \tag{4.52}$$

where $\boldsymbol{\Delta}^{\infty}$ is a pure imaginary vector. Analogous to Eq. (4.24),

$$\mathrm{i}\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty} \tag{4.53}$$

Analogous to Eq. (4.26) let,

$$\boldsymbol{h}(z)=\begin{cases}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z), & x_2\geq 0\\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)+\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty}, & x_2\leq 0\end{cases},\quad z\notin L_{\mathrm{c}} \tag{4.54}$$

Using Eq. (4.54), Eq. (4.53) is reduced to

$$\mathrm{i}\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{H}[\boldsymbol{h}^{+}(x_1)-\boldsymbol{h}^{-}(x_1)] \tag{4.55}$$

Substituting Eq. (4.54) into (4.51) and using Eq. (4.52) we get

$$\begin{aligned}\boldsymbol{\Sigma}_2(x_1)&=\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{h}^{+}(x_1)+\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\boldsymbol{\Delta}^{\infty}\\&=\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x_1)-\boldsymbol{\Delta}_1^{\infty},\quad \boldsymbol{\Delta}_1^{\infty}=\bar{\boldsymbol{H}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty}\end{aligned} \tag{4.56}$$

According to Eq. (4.50), on the crack surface $\sigma_{2j}=0$, but D_2 is unknown, so on the crack surface, Eq. (4.56) is reduced to

$$\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x_1)=\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4 D_2(x_1),\quad \mathbf{i}_4=[0,0,0,1]^{\mathrm{T}},\quad z\in L_{\mathrm{c}} \tag{4.57}$$

According to Eq. (4.50), E_1 is continuous on whole axis x_1, so according to Eq. (4.55), we have

$$\boldsymbol{H}_4[\boldsymbol{h}^{+}(x_1)-\boldsymbol{h}^{-}(x_1)]=\mathbf{0},\quad \boldsymbol{H}_4=[H_{41},H_{42},H_{43},H_{44}],\quad |x_1|<\infty \tag{4.58}$$

The solution of Eq. (4.58) in the z plane is

$$\boldsymbol{H}_4\boldsymbol{h}(z)=\boldsymbol{H}_4\boldsymbol{h}(\infty),\quad h_4(z)=-H_{44}^{-1}\sum_{j=1}^{3}H_{4j}h_j(z)+H_{44}^{-1}\boldsymbol{H}_4\boldsymbol{h}(\infty) \tag{4.59}$$

Multiplying both sides of Eq. (4.56) by $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}$, noting on connective surface $\boldsymbol{h}^{+}(x_1)=\boldsymbol{h}^{-}(x_1)$, and when $x_1\to\infty$ we get

$$\boldsymbol{h}(\infty)=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left(\boldsymbol{\Delta}_1^{\infty}+\boldsymbol{\Sigma}_2^{\infty}\right) \tag{4.60}$$

Now, the problem is reduced to solve Eqs. (4.57) and (4.59).

The homogeneous equation corresponding to Eq. (4.57) is identical with Eq. (4.29), so its solution is still expressed by Eq. (4.37). We shall use the second method to solve the inhomogeneous equation (4.57) and adopt the normalized matrix $\boldsymbol{\Omega}$, i.e., $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\mathbf{I}$. Multiplying on both sides of Eq. (4.57) from left by $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}$,

$$\begin{gathered}\boldsymbol{\Psi}^{+}(x_1)-\boldsymbol{M}\boldsymbol{\Psi}^{-}(x_1)=\boldsymbol{\Sigma}^{*}(x_1),\quad \Psi_i^{+}(x_1)-\mathrm{e}^{2\pi\mathrm{i}\gamma_i}\Psi_i^{-}(x_1)=\Sigma_i^{*}(x_1)\\ \boldsymbol{\Psi}(z)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z),\quad \boldsymbol{M}=\left\langle\mathrm{e}^{2\pi\mathrm{i}\gamma_1},\ldots\mathrm{e}^{2\pi\mathrm{i}\gamma_4}\right\rangle,\quad \boldsymbol{\Sigma}^{*}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left[\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(x_1)\right]\end{gathered} \tag{4.61}$$

Analogous to Eqs. (4.14), (4.15), (4.16), and (4.17), the solution of Eq. (4.61) in the z plane is

$$\begin{gathered}\boldsymbol{h}(z)=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left\{\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(z)\right\}+\boldsymbol{\Omega}\left\langle Y_0^{(j)}(z)\right\rangle\boldsymbol{C}(z)\\ \boldsymbol{C}(z)=\boldsymbol{c}_nz^n+\boldsymbol{c}_{n-1}z^{n-1}+\cdots+\boldsymbol{c}_0\end{gathered} \tag{4.62}$$

Using the condition at infinity yields

$$\boldsymbol{h}(\infty)=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left\{\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(\infty)\right\}+\boldsymbol{\Omega}\boldsymbol{C}_n \tag{4.63}$$

Substituting Eq. (4.62) into Eq. (4.59) yields the equation to determine $D_2(z)$:

$$\boldsymbol{H}_4\left\{\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left\{\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(z)\right\}+\boldsymbol{\Omega}\left\langle Y_0^{(j)}(z)\right\rangle\boldsymbol{C}(z)\right\}=\boldsymbol{H}_4\boldsymbol{h}(\infty) \tag{4.64}$$

Comparing Eqs. (4.60) and (4.63) yields

$$\boldsymbol{C}_n=\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_2^{\infty},\quad \tilde{\boldsymbol{\Sigma}}_2^{\infty}=\boldsymbol{\Sigma}_2^{\infty}-\mathbf{i}_4D_2(\infty)=\left[\boldsymbol{\sigma}_2^{\infty},0\right]^{\mathrm{T}} \tag{4.65}$$

Other unknowns in $\boldsymbol{C}(z)$ are determined by the single-valued condition. Using Eq. (4.55) yields

$$\oint_{L_c} h(z)\mathrm{d}z = 0, \quad \text{or} \quad \int_{-a}^{a} (\boldsymbol{U}^+ - \boldsymbol{U}^-)\,\mathrm{d}x_1 = 0 \ (\text{for one crack}) \tag{4.66}$$

Equation (4.54) yields

$$\boldsymbol{F}_{\mathrm{I}}(z) = \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z), \quad x_2 \geq 0; \quad \boldsymbol{F}_{\mathrm{II}}(z) = \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}[\boldsymbol{H}\boldsymbol{h}(z) - (\bar{\boldsymbol{Y}}_{\mathrm{II}} - \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty}], \quad x_2 \leq 0 \tag{4.67}$$

Solving $\boldsymbol{h}(z)$, $\boldsymbol{F}_{\beta}(z_j)$ can be obtained. From Eq. (4.56), the stress on the axis x_1 is

$$\boldsymbol{\Sigma}_2(x_1) = \left(\mathbf{I} + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{h}(x_1) - \boldsymbol{\Delta}_1^{\infty} \tag{4.68}$$

For one crack in a homogeneous material, we have

$$\boldsymbol{A}_{\mathrm{I}} = \boldsymbol{A}_{\mathrm{II}} = \boldsymbol{A}, \boldsymbol{B}_{\mathrm{I}} = \boldsymbol{B}_{\mathrm{II}} = \boldsymbol{B}, \boldsymbol{H}_{\mathrm{I}} = \boldsymbol{H}_{\mathrm{II}} = \boldsymbol{H} = \bar{\boldsymbol{H}}, \boldsymbol{\Delta}_1^{\infty} = \mathbf{0}, \varepsilon_j = 0, \gamma = 1/2$$

and

$$\boldsymbol{h}(z) = (1/2)\mathbf{i}_4 D_2(z) + \boldsymbol{\Omega}\left\langle Y_0^{(j)}(z)\right\rangle \boldsymbol{C}, \quad \boldsymbol{C} = (1/2)\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_2^{\infty}, \quad \tilde{\boldsymbol{\Sigma}}_2^{\infty} = \left[\boldsymbol{\sigma}_2^{\infty}, 0\right]^{\mathrm{T}} \tag{4.69}$$

Gao and Wang (2000, 2001) discussed the collinear permeable cracks and the mutual effect of a crack with a point singularity.

4.3 Other Line Inclusions

4.3.1 Rigid Line Inclusion

Discuss a nonconductive rigid line inclusion in an infinite bimaterial (Zhou et al. 2008). In Fig. 4.2 the crack is replaced by a rigid inclusion. The boundary condition at infinity is

$$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\infty}(x_1), \quad \text{at infinity} \tag{4.70}$$

The mixed boundary conditions on the surface of the rigid line inclusion and the continuity conditions on the connective interface are

$$
\begin{aligned}
&u_{j,1}=u_{\mathrm{I}j,1}=u_{\mathrm{II}j,1}=\omega_0\delta_{j2},\quad x\in L_{\mathrm{c}};\quad E_{\mathrm{I}\,1}=E_{\mathrm{II}\,1},\quad D_{\mathrm{I2}}=D_{\mathrm{II2}},\quad x\in L_{\mathrm{c}}\\
&\boldsymbol{U}_{\mathrm{I}}(x_1)=\boldsymbol{U}_{\mathrm{II}}(x_1),\quad \boldsymbol{\Sigma}_2(x_1)=\boldsymbol{\Sigma}_{\mathrm{I2}}(x_1)=\boldsymbol{\Sigma}_{\mathrm{II2}}(x_1),\quad x\in L-L_{\mathrm{c}}
\end{aligned}
\tag{4.71}
$$

where ω_0 is the rotation angle about axis x_3 of the inclusion. The main difference between the rigid line inclusion and a permeable crack is that in a permeable crack surfaces, the stresses are given, but for a rigid line inclusion, the rotational angles or moments are given.

According to Stroh's formula we have

$$
\begin{aligned}
&\boldsymbol{U}_{\alpha,\,1}=\boldsymbol{A}_\alpha\boldsymbol{F}_\alpha(z)+\overline{\boldsymbol{A}_\alpha\boldsymbol{F}_\alpha(z)};\quad \boldsymbol{\Phi}_{\alpha,\,1}=\boldsymbol{B}_\alpha\boldsymbol{F}_\alpha(z)+\overline{\boldsymbol{B}_\alpha\boldsymbol{F}_\alpha(z)},\\
&\boldsymbol{F}_\alpha(z)=\boldsymbol{f}'_\alpha(z),\quad \alpha=\mathrm{I},\mathrm{II}
\end{aligned}
\tag{4.72}
$$

The generalized displacements are continuous on the whole axis x_1, so analogous to Eq. (4.52) it yields

$$
\begin{aligned}
&\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{x}_1),\quad -\infty<x_1<\infty\\
&\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\boldsymbol{\Delta}^\infty,\quad \boldsymbol{\Delta}^\infty=(1/2)[(\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^\infty+\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^\infty)-(\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^\infty+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^\infty)],\\
&\quad\alpha=\mathrm{I},\mathrm{II}
\end{aligned}
\tag{4.73}
$$

Analogous to previous sections, we have

$$
\begin{aligned}
\boldsymbol{\Delta\Phi}_{,1}(x_1)&=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)-\boldsymbol{\Phi}_{\mathrm{II},1}(x_1)=\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\overline{\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)}\right]-\left[\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)+\overline{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)}\right]\\
&=\mathrm{i}\boldsymbol{R}\left[\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\boldsymbol{R}^{-1}\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^\infty\right]\\
\boldsymbol{Y}_\alpha^{-1}&=-\mathrm{i}\boldsymbol{B}_\alpha\boldsymbol{A}_\alpha^{-1},\quad \boldsymbol{Y}_\alpha=\mathrm{i}\boldsymbol{A}_\alpha\boldsymbol{B}_\alpha^{-1},\quad \boldsymbol{R}=\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}
\end{aligned}
\tag{4.74}
$$

On the connective surface, Eq. (4.74) is zero, so by standard analytic continuation, we can construct a function $\boldsymbol{h}(z)$ analytic in whole plane except the rigid inclusions:

$$
\boldsymbol{h}(z)=\begin{cases}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) & z\in S^+\\ \boldsymbol{R}^{-1}\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)+\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^\infty & z\in S^-\end{cases}
\tag{4.75}
$$

Equations (4.74) and (4.75) yield

$$
\boldsymbol{\Delta\Phi}_{,1}(x_1)=\mathrm{i}\boldsymbol{R}[\boldsymbol{h}^+(x_1)-\boldsymbol{h}^-(x_1)],\quad x_1\in L_{\mathrm{c}};\quad \boldsymbol{\Delta\Phi}_{,1}(x_1)=\boldsymbol{0},\quad x_1\notin L_{\mathrm{c}}
\tag{4.76}
$$

From Eq. (4.71), it is known that $D_2(x_1)$ is continuous on whole axis x_1, so $\Delta\Phi_{4,1}(x_1)=0$, or

$$
\boldsymbol{R}_4[\boldsymbol{h}^+(x_1)-\boldsymbol{h}^-(x_1)]=\boldsymbol{0},\quad -\infty<x_1<\infty,\quad \boldsymbol{R}_4=[R_{41},R_{42},R_{43},R_{44}]
\tag{4.77}
$$

where $\boldsymbol{R}_4$ is the fourth row of $\boldsymbol{R}$ and $\boldsymbol{R}_4^{\mathrm{T}}$ can be seen as a vector. The solution of Eq. (4.77) is

$$\boldsymbol{R}_4\boldsymbol{h}(z) = \boldsymbol{R}_4\boldsymbol{h}^{\infty}, \quad \boldsymbol{h}^{\infty} = \boldsymbol{h}(\infty) \tag{4.78}$$

Using Eq. (4.73) it is easy get

$$\begin{aligned} &\boldsymbol{U}_{\mathrm{I},1}(x_1) = \boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1) = \boldsymbol{h}^{+}(x_1) + \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1) - \boldsymbol{\Delta}_1^{\infty} \\ &\boldsymbol{\Delta}_1^{\infty} = \bar{\boldsymbol{R}}^{-1}\left(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1} - \bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\right)\boldsymbol{\Delta}^{\infty} \end{aligned} \tag{4.79}$$

From Eq. (4.71), it is known that on the inclusion surface, we have

$$\boldsymbol{U}_{\mathrm{I},1}(x_1) = \omega_0\mathbf{i}_2 - E_1(x_1)\mathbf{i}_4, \quad x_1 \in L_{\mathrm{c}}; \quad \mathbf{i}_2 = [0,1,0,0]^{\mathrm{T}}, \quad \mathbf{i}_4 = [0,0,0,1]^{\mathrm{T}} \tag{4.80}$$

where $E_1(x_1)$ is the boundary value of $E_1(z)$ on the inclusion surface and is unknown. So Eq. (4.79) can be reduced to a vector Riemann-Hilbert equation:

$$\boldsymbol{h}^{+}(x_1) + \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1) = \boldsymbol{\Delta}_1^{\infty} + \omega_0\mathbf{i}_2 - E_1(x_1)\mathbf{i}_4, \quad x_1 \in L_{\mathrm{c}} \tag{4.81}$$

Equation (4.81) is identical with (4.28) except using $\boldsymbol{R}$ and $\boldsymbol{\Delta}_1^{\infty} + \omega_0\mathbf{i}_2 - E_1(x_1)\mathbf{i}_4$ instead of $\boldsymbol{H}$ and $\boldsymbol{\Sigma}_0(x_1)$, respectively, but $E_1(x_1)\mathbf{i}_4$ is undetermined, and $\omega_0\mathbf{i}_2$ is given or determined by given moment on the inclusion. The homogeneous equation of Eq. (4.81) is

$$\boldsymbol{h}^{+}(x_1) + \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1) = \mathbf{0}, \quad x_1 \in L_{\mathrm{c}} \tag{4.82}$$

The difference of the homogeneous equation Eqs. (4.82) and (4.29) is only using $\boldsymbol{R}$ instead of $\boldsymbol{H}$. So the fundamental solution of Eq. (4.82) is still expressed by Eq. (4.37), but the eigen-equation is changed to

$$\begin{aligned} &\left(\mathrm{e}^{-2\pi\varepsilon}\mathbf{I} - \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\right)\boldsymbol{\omega} = \mathbf{0}, \quad \left(\mathrm{e}^{2\pi\varepsilon}\mathbf{I} - \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\right))\bar{\boldsymbol{\omega}} = \mathbf{0}, \quad \left|\bar{\boldsymbol{R}} - \mathrm{e}^{2\pi\varepsilon}\boldsymbol{R}\right| = \mathbf{0}, \\ &\left|\bar{\boldsymbol{R}} - \mathrm{e}^{-2\pi\varepsilon}\boldsymbol{R}\right| = \mathbf{0} \end{aligned} \tag{4.83}$$

From Eqs. (4.78) and (4.81), the solution of the inhomogeneous problem is

$$\begin{aligned} \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z) &= \frac{\boldsymbol{Q}(z)}{2\pi\,\mathrm{i}}\int_L \frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\{\boldsymbol{\Delta}_1^{\infty} + \omega_0\mathbf{i}_2 - E_1(x_1)\mathbf{i}_4\}}{\boldsymbol{Q}^{+}(x_1)(x_1 - z)}\mathrm{d}x_1 + \boldsymbol{Q}(z)\boldsymbol{C}(z) \\ &= \left\langle\left(1 + \mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\left\{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left[\boldsymbol{\Delta}_1^{\infty} + \omega_0\mathbf{i}_2 - E_1(z)\mathbf{i}_4\right]\right\} + \boldsymbol{Q}(z)\boldsymbol{C}(z) \\ \boldsymbol{h}(z) &= \boldsymbol{\Omega}\left\langle\left(1 + \mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left[\boldsymbol{\Delta}_1^{\infty} + \omega_0\mathbf{i}_2 - E_1(z)\mathbf{i}_4\right] + \boldsymbol{\Omega}\left\langle Y_0^{(i)}(z)\right\rangle\boldsymbol{C}(z) \\ \boldsymbol{Q}(z) &= \left\langle Y_0^{(j)}(z)\right\rangle, \quad Y_0^{(j)}(z) = \frac{1}{\sqrt{z^2 - a^2}}\left(\frac{z - a}{z + a}\right)^{\mathrm{i}\varepsilon_j}; \quad \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega} = \mathbf{I} \end{aligned} \tag{4.84}$$

where $\boldsymbol{C}(z) = \boldsymbol{C}_1 z + \boldsymbol{C}_0$. $E_1(z)$ can be obtained from Eqs. (4.78) and (4.84):

$$\begin{aligned} &\boldsymbol{R}_4 \boldsymbol{\Omega} \left\langle \left(1 + \mathrm{e}^{2\pi\varepsilon_i}\right)^{-1} \right\rangle \bar{\boldsymbol{\Omega}}^{\mathrm{T}} \mathbf{i}_4 E_1(z) \\ &\quad = \boldsymbol{R}_4 \boldsymbol{\Omega} \left\langle \left(1 + \mathrm{e}^{2\pi\varepsilon_i}\right)^{-1} \right\rangle \bar{\boldsymbol{\Omega}}^{\mathrm{T}} \left(\boldsymbol{\Delta}_1^{\infty} + \omega_0 \mathbf{i}_2\right) + \boldsymbol{\Omega} \left\langle Y_0^{(i)}(z) \right\rangle \boldsymbol{C}(z) - \boldsymbol{R}_4 \boldsymbol{h}^{\infty} \end{aligned} \tag{4.85}$$

The unknown constants are obtained by using the conditions at infinity and the single-valued conditions and the moment condition:

$$\int_L \Delta \boldsymbol{\Phi}_{,1} \mathrm{d}x_1 = \int_{-a}^{a} \Delta \boldsymbol{\Phi}_{,1} \mathrm{d}x_1 = \mathbf{0}, \quad \int_{-a}^{a} \Delta \Phi_{2,\,1} (x_1 - x_0) \mathrm{d}x_1 = M \tag{4.86}$$

The rigid line inclusion is discussed in many literatures (Shi 1997; Deng and Meguid 1998).

4.3.2 A Bimaterial with an Electrode on the Interface

Discuss a thin soft electrode of length $2a$ occupied L_c and let the coordinate origin be located at the center of the electrode (Ru 2000). In Fig. 4.2 the crack is changed to an electrode. The connective surface is denoted by $L - L_\mathrm{c}$. Assume the boundary conditions are

$$\begin{aligned} &\sigma_{\mathrm{I}2i} = \sigma_{\mathrm{II}2i}, \quad u_{\mathrm{I}i} = u_{\mathrm{II}i}, \quad E_{\mathrm{I}1} = E_{\mathrm{II}1}, \quad D_{\mathrm{I}2} = D_{\mathrm{II}2}, \quad x_1 \notin L_\mathrm{c} \\ &\sigma_{\mathrm{I}2i} = \sigma_{\mathrm{II}2i}, \quad u_{\mathrm{I}i} = u_{\mathrm{II}i}, \quad E_{\mathrm{I}1} = E_{\mathrm{II}1} = 0, \quad \int_{L_\mathrm{c}} \delta(x_1) \mathrm{d}x_1 = q, \quad x_1 \in L_\mathrm{c} \\ &\sigma_{ij} \to 0, \quad D_j \to 0, \quad |z| \to \infty \end{aligned} \tag{4.87}$$

where $\delta(x_1) = D_{\mathrm{I}2}(x_1) - D_{\mathrm{II}2}(x_1)$ and q is the total electric charge on the electrode.

Because the generalized displacements are continuous on whole axis x_1, analogous to Eqs. (4.23) and (4.73) and noting $\sigma_{ij}, D_j \to 0$ at infinity, we have

$$\begin{aligned} &\boldsymbol{A}_\mathrm{I} \boldsymbol{F}_\mathrm{I}(x_1) + \bar{\boldsymbol{A}}_\mathrm{I} \bar{\boldsymbol{F}}_\mathrm{I}(\bar{x}_1) = \boldsymbol{A}_\mathrm{II} \boldsymbol{F}_\mathrm{II}(x_1) + \bar{\boldsymbol{A}}_\mathrm{II} \bar{\boldsymbol{F}}_\mathrm{II}(\bar{x}_1), \quad -\infty < x_1 < \infty \\ &\boldsymbol{A}_\mathrm{I} \boldsymbol{F}_\mathrm{I}(z) = \bar{\boldsymbol{A}}_\mathrm{II} \bar{\boldsymbol{F}}_\mathrm{II}(z), \quad \text{or} \quad \boldsymbol{Y}_\mathrm{I} \boldsymbol{B}_\mathrm{I} \boldsymbol{F}_\mathrm{I}(z) = -\bar{\boldsymbol{Y}}_\mathrm{II} \bar{\boldsymbol{B}}_\mathrm{II} \bar{\boldsymbol{F}}_\mathrm{II}(z); \quad x_2 > 0 \\ &\boldsymbol{A}_\mathrm{II} \boldsymbol{F}_\mathrm{II}(z) = \bar{\boldsymbol{A}}_\mathrm{I} \bar{\boldsymbol{F}}_\mathrm{I}(z), \quad \text{or} \quad \boldsymbol{Y}_\mathrm{II} \boldsymbol{B}_\mathrm{II} \boldsymbol{F}_\mathrm{II}(z) = -\bar{\boldsymbol{Y}}_\mathrm{I} \bar{\boldsymbol{B}}_\mathrm{I} \bar{\boldsymbol{F}}_\mathrm{I}(z); \quad x_2 < 0 \end{aligned} \tag{4.88}$$

According to Eqs. (4.87) and noting $\boldsymbol{\Sigma}_{\mathrm{I}2} - \boldsymbol{\Sigma}_{\mathrm{II}2} = \boldsymbol{\Phi}_{\mathrm{I},1} - \boldsymbol{\Phi}_{\mathrm{II},1}$ yield

$$[\boldsymbol{B}_\mathrm{I} \boldsymbol{F}_\mathrm{I}(x_1) - \bar{\boldsymbol{B}}_\mathrm{II} \bar{\boldsymbol{F}}_\mathrm{II}(x_1)]^+ - [\boldsymbol{B}_\mathrm{II} \boldsymbol{F}_\mathrm{II}(x_1) - \bar{\boldsymbol{B}}_\mathrm{I} \bar{\boldsymbol{F}}_\mathrm{I}(x_1)]^- = \begin{cases} \mathbf{0}, & z \notin L_\mathrm{c} \\ [0, 0, 0, \delta(x_1)]^\mathrm{T}, & z \in L_\mathrm{c} \end{cases} \tag{4.89}$$

From Eq. (4.89), we can construct a function $\boldsymbol{h}(z)$ analytic in whole plane except L_c by the analytic continuation through $L - L_\mathrm{c}$. Using the Sokhotski (Сохоцкий)-Plemelj formula of the Cauchy-type integral, its solution is

$$\boldsymbol{h}(z) = [0, 0, 0, \chi(z)]^\mathrm{T} = \begin{cases} \boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(z) - \bar{\boldsymbol{B}}_\mathrm{II}\bar{\boldsymbol{F}}_\mathrm{II}(z), & z \in S^+ \\ \boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(z) - \bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{F}}_\mathrm{I}(z), & z \in S^- \end{cases}; \quad \chi(z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\delta(x_1)}{x_1 - z} \mathrm{d}x_1 \tag{4.90}$$

Using Eq. (4.88), Eq. (4.90) can be reduced to

$$\begin{aligned} (\boldsymbol{Y}_\mathrm{I} + \bar{\boldsymbol{Y}}_\mathrm{II})\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(z) &= \bar{\boldsymbol{Y}}_\mathrm{II}[0, 0, 0, \chi(z)]^\mathrm{T}, \quad z \in S^+ \\ (\boldsymbol{Y}_\mathrm{II} + \bar{\boldsymbol{Y}}_\mathrm{I})\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(z) &= \bar{\boldsymbol{Y}}_\mathrm{I}[0, 0, 0, \chi(z)]^\mathrm{T}, \quad z \in S^- \end{aligned} \tag{4.91}$$

Using Eq. (4.88) from $E_1^+ = 0$ on L_c, see Eq. (4.87), yields

$$[\boldsymbol{Y}_\mathrm{I}\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(x_1)]^+ + [\boldsymbol{Y}_\mathrm{II}\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(x_1)]^- = [*, *, *, 0]^\mathrm{T}, \quad z \in L_\mathrm{c} \tag{4.92}$$

where "$*$" is not an applied variable and omitted. Substitution of Eq. (4.91) into Eq. (4.92) yields

$$\begin{aligned} &\boldsymbol{Y}_\mathrm{I}(\boldsymbol{Y}_\mathrm{I} + \bar{\boldsymbol{Y}}_\mathrm{II})^{-1}\bar{\boldsymbol{Y}}_\mathrm{II}[0, 0, 0, \chi^+(x_1)]^\mathrm{T} + \boldsymbol{Y}_\mathrm{II}(\boldsymbol{Y}_\mathrm{II} + \bar{\boldsymbol{Y}}_\mathrm{I})^{-1}\bar{\boldsymbol{Y}}_\mathrm{I}[0, 0, 0, \chi^-(x_1)]^\mathrm{T} \\ &\quad = [*, *, *, 0]^\mathrm{T}, \quad z \in L_\mathrm{c} \end{aligned} \tag{4.93}$$

The fourth component of Eq. (4.93) is

$$\chi^+(x_1) - \mathrm{g}\chi^-(x_1) = 0, \quad \mathrm{g} = -\left[\boldsymbol{Y}_\mathrm{II}(\boldsymbol{Y}_\mathrm{II} + \bar{\boldsymbol{Y}}_\mathrm{I})^{-1}\bar{\boldsymbol{Y}}_\mathrm{I}\right]_{44} \Big/ \left[\boldsymbol{Y}_\mathrm{I}(\boldsymbol{Y}_\mathrm{I} + \bar{\boldsymbol{Y}}_\mathrm{II})^{-1}\bar{\boldsymbol{Y}}_\mathrm{II}\right]_{44} \tag{4.94}$$

Equation (4.94) is identical with (4.1) in form, so its solution is

$$\begin{aligned} &\chi(z) = c(z + a)^{-\gamma}(z - a)^{\gamma - 1}, \\ &\quad \gamma = \frac{1}{2\pi\,\mathrm{i}} \ln \mathrm{g} = \frac{1}{2\pi\,\mathrm{i}} \ln \frac{-\left[\boldsymbol{Y}_\mathrm{I}(\boldsymbol{Y}_\mathrm{I} + \bar{\boldsymbol{Y}}_\mathrm{II})^{-1}\bar{\boldsymbol{Y}}_\mathrm{II}\right]_{44}}{\left[\boldsymbol{Y}_\mathrm{II}(\boldsymbol{Y}_\mathrm{II} + \bar{\boldsymbol{Y}}_\mathrm{I})^{-1}\bar{\boldsymbol{Y}}_\mathrm{I}\right]_{44}} \end{aligned} \tag{4.95}$$

For a homogeneous material, we have $\gamma = 1/2$. Comparing Eqs. (4.90) and (4.95) at infinity, it is found that

$$c = -(1/2\pi \mathrm{i}) \int_{L_\mathrm{c}} \delta(x_1)\mathrm{d}x_1 = \mathrm{i}q/2\pi \tag{4.96}$$

Substituting Eq. (4.96) into Eq. (4.91) yields $\boldsymbol{F}_\mathrm{I}(z), \boldsymbol{F}_\mathrm{II}(z)$. Replacing z by z_j in $\boldsymbol{F}_\beta(z)$, the stress potential $\boldsymbol{F}_\beta(z_j)$ is obtained. Ru (2000) discussed the collinear cracks also.

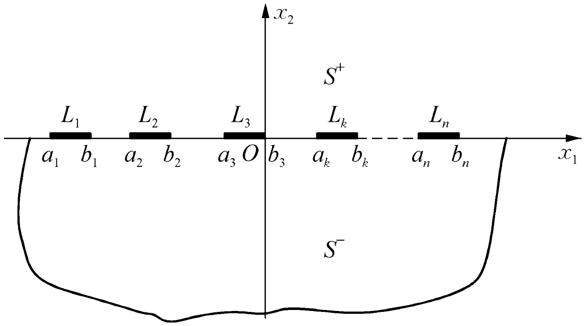


Fig. 4.3 Collinear surface electrodes

4.3.3 Surface Electrodes

In this section, we shall discuss surface electrodes (Fig. 4.3) in details (Zhou et al. 2005a, b; Kuang et al. 2004). In this case, air occupies S^+ and it is assumed that in the air only the electric variables need to be considered; the dielectric occupies S^-. The boundary conditions are

$$\begin{aligned}
&\sigma_{ij} \to 0, \quad D_i \to 0, \quad |z| \to \infty; \quad \sigma_{2i} = 0, \quad D_2 = 0, \quad z \in L - L_c; \quad i,j = 1,2,3 \\
&\sigma_{2i} = 0, \quad E_1 = 0, \quad \text{and} \quad \int_{L_{ck}} D_2^-(x_1)\mathrm{d}x_1 = -q_k, \quad \text{or} \\
&\varphi_k = V_k, \quad \int_{L_c} D_2^-(x_1)\mathrm{d}x_1 = -Q = -\sum_{k=1}^{n} q_k, \quad k = 1,2,\ldots n; \quad z \in L_c
\end{aligned} \tag{4.97}$$

where $D_2^-(x_1)$ is an undetermined function. According to Eq. (4.97), it is known that $\boldsymbol{\Sigma}_2 = \mathbf{0}$ or $\boldsymbol{B}\boldsymbol{F}^-(x_1) + \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^-(\bar{x}_1) = \mathbf{0}$ on $L - L_c$, so we can construct a function $\boldsymbol{h}(z)$ analytic in whole z plane except L_c by the standard analytic continuation method:

$$\boldsymbol{h}(z) = \begin{cases} -\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\bar{\boldsymbol{F}}(z), & z \in S^+ \\ \boldsymbol{F}(z), & z \in S^- \end{cases} \tag{4.98}$$

From Eqs. (4.97) and (4.98) and using $\overline{\boldsymbol{F}^+(x_1)} = \bar{\boldsymbol{F}}^-(x_1), \overline{\boldsymbol{F}^-(x_1)} = \bar{\boldsymbol{F}}^+(x_1)$ we get

$$\begin{aligned}
&\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1) = -\boldsymbol{B}^{-1}\left[\boldsymbol{B}\boldsymbol{F}^-(x_1) + \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^+(x_1)\right] = \mathbf{0}, \quad z \in L - L_c \\
&\boldsymbol{B}\boldsymbol{F}^-(x_1) + \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^+(x_1) = \boldsymbol{\Sigma}_D, \quad \boldsymbol{\Sigma}_D = \left[0,0,0,D_2^-(x_1)\right]^{\mathrm{T}} \\
&\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1) = -\boldsymbol{B}^{-1}\boldsymbol{\Sigma}_D, \quad h_j^+ - h_j^- = -B_{j4}^{-1}D_2^-(x_1), \quad j = 1-4, \quad z \in L_c
\end{aligned} \tag{4.99}$$

Equation (4.99) is a decoupling Riemann-Hilbert boundary problem, and its solution is

$$\boldsymbol{h}(z) = -\boldsymbol{B}^{-1}\frac{1}{2\pi \mathrm{i}}\int_{L''}\frac{\boldsymbol{\Sigma}_D(x_1)}{x_1 - z}\mathrm{d}x_1, \quad \boldsymbol{F}(z) = \boldsymbol{h}(z), \quad z \in S^- \tag{4.100}$$

From the known knowledge, it is assumed

$$\begin{aligned} &D_2(z) = D_2^-(z) = P(z)\Big/\prod_{i=1}^{n}\sqrt{(z-a_i)(z-b_i)}, \quad z \in S^- \\ &P(z) = \mathrm{i}\left(\gamma_{n-1}z^{n-1} + \cdots + \gamma_1 z + \gamma_0\right) \end{aligned} \tag{4.101}$$

where γ_i is a complex constant. Usually, select function $\sqrt{(z-a_i)(z-b_i)} \to z$ when $z \to \infty$ as its single-valued branch. Substitution of Eq. (4.101) into Eq. (4.100) yields

$$\begin{aligned} &F_j(z_j) = (1/2)B_{j4}^{-1}P(z_j)\left(\prod_{i=1}^{n}\sqrt{(z_j-a_i)(z_j-b_i)}\right)^{-1} \\ &f_j(z_j) = \int F_j(z_j)\mathrm{d}z_j + (1/2)\mathrm{i}CB_{j4}^{-1}, \quad z \in S^- \end{aligned} \tag{4.102}$$

where C is a constant. According to Eq. (4.97), it has $E_1 = 0$ on L_c, so

$$\boldsymbol{AF}^-(x_1) + \overline{\boldsymbol{AF}^-(x_1)} = [*,*,*,0], \quad x_1 \in L_\mathrm{c} \tag{4.103}$$

Substituting Eq. (4.102) into Eq. (4.103), on ith electrode, yields

$$\begin{aligned} &A_{4j}B_{j4}^{-1}\frac{\mathrm{i}P(x_1)}{\sqrt{(x_1-a_i)(b_i-x_1)}\prod_{k=1,k\neq i}^{n}\sqrt{(x_1-a_k)(x_1-b_k)}} \\ &\quad -\bar{A}_{4j}\bar{B}_{j4}^{-1}\frac{\mathrm{i}\bar{P}(\bar{x}_1)}{\sqrt{(x_1-a_i)(b_i-x_1)}\prod_{k=1,k\neq i}^{n}\sqrt{(x_1-a_k)(x_1-b_k)}} = 0, \quad x_1 \in L_\mathrm{c} \end{aligned} \tag{4.104}$$

Using $H_{44} = \mathrm{i}A_{4j}B_{j4}^{-1}$ is real, $A_{4j}B_{j4}^{-1}$ is pure imaginary number, and Eq. (4.104) can be reduced to $P(x_1) + \bar{P}(\bar{x}_1) = 0$, it is concluded that all γ_i in $P(z)$ are real.

The generalized stress Σ_{2k} and the generalized displacement U_k are, respectively,

$$\begin{aligned} &\Sigma_{2k} = \mathrm{Re}\sum_{j=1}^{4}{}_{kj}B_{j4}^{-1}P(z_j)\left(\prod_{i=1}^{n}\sqrt{(z_j-a_i)(z_j-b_i)}\right)^{-1} \\ &U_k = 2\mathrm{Re}\left[A_{kj}f_j(z_j)\right] = \mathrm{Re}\left[A_{kj}B_{j4}^{-1}\int\frac{P(z_j)\mathrm{d}z_j}{\prod_{i=1}^{n}\sqrt{(z_j-a_i)(z_j-b_i)}}\right] + H_{44}C \end{aligned} \tag{4.105}$$

If the electric charge on the electrode i is given, we have

$$\begin{aligned}\int_{a_i}^{b_i}\Sigma_{24}(x_1)\mathrm{d}x_1 &= \int_{a_i}^{b_i}D_2^-(x_1)\mathrm{d}x_1 = \int_{a_i}^{b_i}\mathrm{Re}\frac{P(x_1)\mathrm{d}x_1}{\prod_{m=1}^{n}\sqrt{(z_j-a_m)(z_j-b_m)}} \\ &= \int_{a_i}^{b_i}\frac{\mathrm{i}P(x_1)\mathrm{d}x_1}{\sqrt{(x_1-a_i)(b_i-x_1)}\prod_{k=1,k\neq i}^{n}\sqrt{(x_1-a_k)(x_1-b_k)}} = -q_i, \quad i=1-n\end{aligned} \tag{4.106}$$

where n unknowns $\gamma_i(i=0,1,\ldots,n-1)$ are just determined by n equations. Especially when $z\to\infty$, we have

$$\lim_{z\to\infty}F_4(z_4)=\frac{1}{2\pi\,\mathrm{i}z_4}B_{44}^{-1}\int_{L''}D_2(x_1)\mathrm{d}x_1, \quad \lim_{z\to\infty}F_4(z_4)=\frac{\mathrm{i}}{2z_4}B_{44}^{-1}\gamma_{n-1}$$

so

$$\gamma_{n-1}=-\frac{1}{\pi}\int_{L_c}D_2(x_1)\mathrm{d}x_1=-\frac{1}{\pi}(-Q)=\frac{Q}{\pi} \tag{4.107}$$

If the electric potential on the electrode i is given, we have

$$\begin{aligned}(U_4)_i=\varphi_i&=\mathrm{Re}\left[\sum_{j=1}^{4}A_{4j}B_{j4}^{-1}\int_{a_i}^{b_i}\frac{P(x_1)\mathrm{d}x_1}{\prod_{i=1}^{n}\sqrt{(x_1-a_i)(x_1-b_i)}}\right]+H_{44}C \\ &=H_{44}\mathrm{Im}\int_{a_i}^{b_i}\frac{P(x_1)\mathrm{d}x_1}{\prod_{i=1}^{n}\sqrt{(x_1-a_i)(x_1-b_i)}}+H_{44}C=V_i \\ \int_{L_c}D_2^-(x_1)\mathrm{d}x_1&=-Q=\sum_{i=1}^{n}\mathrm{Re}\int_{a_k}^{b_k}\frac{P(x_1)\mathrm{d}x_1}{\prod_{i=1}^{n}\sqrt{(x_1-a_i)(x_1-b_i)}}\end{aligned} \tag{4.108}$$

where $n+1$ unknowns $\gamma_i(i=0,1,\ldots,n-1)$ and C are just determined by $n+1$ equations.

For only one electrode located in $(-a,a)$ case, from Eq. (4.102) by using Eq. (4.107) we get

$$F_j(z_j)=B_{j4}^{-1}\frac{q\mathrm{i}}{2\pi\sqrt{z_j^2-a^2}}, \quad f_j(z_j)=\frac{q\mathrm{i}}{2\pi}B_{j4}^{-1}\left\{\ln\left(z_j+\sqrt{z_j^2-a^2}\right)+\ln\tilde{C}\right\} \tag{4.109}$$

where $\tilde{C}$ is a real constant. Let $\varphi=V_0$ on the electrode, then we have

$$\varphi=2\mathrm{Re}\{A_{4j}f_j(x_1)\}=H_{44}(q/\pi)\mathrm{Re}\left\{\ln\left(x_1-i\sqrt{a^2-x_1^2}\right)+\ln\tilde{C}\right\}=V_0$$

Because $H_{44} = iA_{4j}B_{j4}$ is real, $\mathrm{Re}\ln\left(x_1 - \mathrm{i}\sqrt{a^2 - x_1^2}\right) = \ln a$, from the above equation we get

$$H_{44}(q/\pi)\ln a\tilde{C} = V_0, \quad \text{or} \quad \tilde{C} = (1/a)\exp((\pi V_0/qH_{44})) \tag{4.110}$$

The electric potential and generalized stresses are, respectively,

$$\begin{aligned}
&\varphi = H_{44}(q/\pi)\mathrm{Re}\left[\ln\left(z_j + \sqrt{z_j^2 - a^2}\right)\Big/a\right] + V_0 \\
&\Sigma_{2k} = -(q/\pi)\mathrm{Im}\sum_{j=1}^{4} B_{kj}B_{j4}^{-1}\left(z_j^2 - a^2\right)^{-1/2}, \quad \Sigma_{1k} = \frac{q}{\pi}\mathrm{Im}\sum_{j=1}^{4} B_{kj}B_{j4}^{-1}\mu_j\left(z_j^2 - a^2\right)^{-1/2}
\end{aligned} \tag{4.111}$$

For the dielectric without the piezoelectric effect, we have

$$\begin{aligned}
&Q_{44} = -\epsilon_{11}, \quad R_{44} = -\epsilon_{12}, \quad T_{44} = -\epsilon_{22}; \quad \mu_4 = \left(-\epsilon_{12} + \mathrm{i}\sqrt{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2}\right)\Big/\epsilon_{22} \\
&A_{44} = -\mathrm{i}\sqrt{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2}, \quad B_{44} = -(\epsilon_{12} + \mu_4\epsilon_{22}), \quad H_{44} = -\left(\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2\right)^{-1/2} < 0 \\
&F_4(z_4) = B_{44}^{-1}\frac{q\mathrm{i}}{2\pi\sqrt{z_4^2 - a^2}}, \quad \varphi = V_0 - H_{44}\frac{q}{\pi\sqrt{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2}}\mathrm{Re}\left\{\ln\frac{1}{a}\left(z_4 + \sqrt{z_4^2 - a^2}\right)\right\}
\end{aligned} \tag{4.112}$$

For an isotropic dielectric $\epsilon_{ij} = \epsilon\delta_{ij}$, so it is obtained

$$\varphi = V_0 - (q/\pi\epsilon)\mathrm{Re}\left\{\ln\left[\left(z + \sqrt{z^2 - a^2}\right)\Big/a\right]\right\} \tag{4.113}$$

which is identical with the result in usual textbooks. Kuang et al. (2004) gave numerical examples for the case of two electrodes. Shindo et al. (1998) discussed the surface electrode also.

4.4 Short Discussions on Some Special Problems

4.4.1 Partly Insulated and Partly Conducted Crack in a Homogeneous Material

The impermeable or conducting electric boundary conditions are idealization case. Breakdown of the dielectric inside the crack was observed in experiments, especially near the crack tip region. The local electric discharge may make an impermeable crack conducting electrically and change the failure behavior of piezoelectric materials (Lynch et al. 1995; Zhang et al. 2001). The discharge process at the gap near a crack tip is complex dynamic process. When the electric

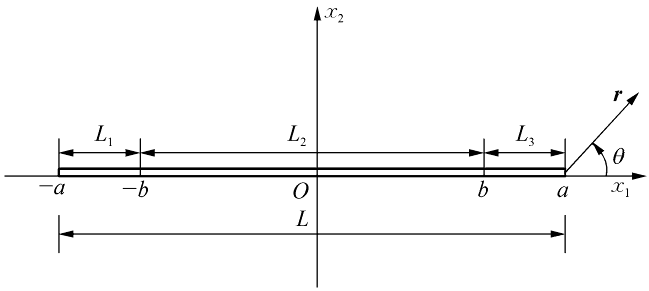


Fig. 4.4 Partly insulated and partly conducted crack

field approaches the critical value, the air breaks down and becomes conducting gas, but after air breakdown, the electric field diminishes quickly and air becomes insulated again. This process will be repeated and form discontinuous electric sparks. For the homogeneous material, Huang and Kuang (2003) proposed an ideal static model: partly insulated and partly conducted crack. Near the crack tip, the conducting boundary condition is adopted, but in the middle part of the crack, it is considered insulated (Fig. 4.4). The boundary conditions are

$$\begin{aligned} &\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_1^{\infty}, \quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\infty}, \quad \text{at infinity} \\ &\sigma_{2j}^{\pm}(x_1, 0) = 0, \quad E_1^{\pm}(x_1, 0) = 0, \quad x_1 \in L_1 \cup L_3 \\ &\sigma_{2j}^{\pm}(x_1, 0) = 0, \quad D_2^{\pm}(x_1, 0) = 0, \quad x_1 \in L_2 \end{aligned} \tag{4.114}$$

where $L_2(-b, b)$ is the insulated region and $L_1(-a, -b)$ and $L_3(b, a)$ are the conducting region. For an electric free crack the single-valued conditions are

$$\begin{aligned} &\int_L \left[u_{j,1}^{+}(x_1, 0) - u_{j,1}^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0, \quad \int_{L_2} \left[\varphi_{,1}^{+}(x_1, 0) - \varphi_{,1}^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0 \\ &\int_{L_1} \left[D_2^{+}(x_1, 0) - D_2^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0, \quad \int_{L3} \left[D_2^{+}(x_1, 0) - D_2^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0 \end{aligned} \tag{4.115}$$

Equation (4.114) can be reduced to the following inhomogeneous Riemann-Hilbert equations:

$$\begin{aligned} &\sum\nolimits_k A_{4k} F_k^{\pm}(x_1) + \sum\nolimits_k \overline{A}_{4k} \overline{F}_k^{\mp}(x_1) = 0, \quad x_1 \in L_1 \cup L_3 \\ &\Sigma_{2j}^{+}(x_1) + \Sigma_{2j}^{-}(x_1) = \sum\nolimits_k \left[B_{jk} F_k^{+} + \overline{B}_{jk} \overline{F}_k^{+} + \overline{B}_{jk} \overline{F}_k^{-} + B_{jk} F_k^{-}\right] = s_1(x_1) \delta_{4j} \\ &\Sigma_{2j}^{+}(x_1) - \Sigma_{2j}^{-}(x_1) = \sum\nolimits_j \left[B_{jk} F_k^{+} - \overline{B}_{jk} \overline{F}_k^{+} + \overline{B}_{jk} \overline{F}_k^{-} - B_{jk} F_k^{-}\right] = s_2(x_1) \delta_{4j} \\ &s_1(x_1) = \begin{cases} 0, & x_1 \in L_2 \\ D_2^{+} + D_2^{-}, & x_1 \in L_1 \cup L_3 \end{cases}; \quad s_2(x_1) = \begin{cases} 0, & x_1 \in L_2 \\ D_2^{+} - D_2^{-}, & x_1 \in L_1 \cup L_3 \end{cases} \end{aligned} \tag{4.116}$$

Because D_2 is unknown on $x_1 \in L_1 \cup L_3$, so $s_1(x_1)$ and $s_2(x_1)$ in Eq. (4.116) are undetermined functions. Eq. (4.116) can be solved as an inhomogeneous Riemann-Hilbert problem by using the analytic continuation method. Finally Huang and Kuang (2003) obtained the solution in z plane

$$
\begin{aligned}
F_j(z) = &\frac{1}{2} B_{j4}^{-1}\left[\gamma_6 z\{X_b(z) - X_a(z)\} + \mathrm{i}\left(\gamma_0 + \gamma_2 z^2\right) X_{ab}(z) - \mathrm{i}\gamma_2\right] \\
&+ \frac{1}{2} B_{jk}^{-1}[\beta_{1k} z X_a(z) + \mathrm{i}\beta_{2k}]; \quad j, k = 1, 2, 3, 4
\end{aligned} \tag{4.117}
$$

It is known that an impermeable crack intensifies an electric field perpendicular to it, but does not perturb an electric field parallel to it. The effect of a conducting crack is just conversely. The singular parts of the generalized stresses are

$$
\begin{aligned}
\sigma_{2j}(x_1) &= \beta_{1j} x_1 X_a(x_1) = \sigma_{2j}^{\infty} x_1 X_a(x_1) \\
D_2(x_1) &= \left(H_{4j}/H_{44}\right) \sigma_{2j}^{\infty} x_1 [X_b(x_1) - X_a(x_1)] + D_2^{\infty} x_1 X_b(x_1) \\
E_1(x_1) &= \left(\beta_{2j} H_{4j}/2\right)\left(I_2/I_1 - x_1^2\right) X_{ab}(x_1) - \sigma_{2j}^{\infty} \mathrm{Im}\left(Y_{4j}\right) x_1 X_a(x_1)
\end{aligned} \tag{4.118}
$$

where

$$
\begin{aligned}
&X_a(z) = 1\Big/\sqrt{z^2 - a^2}, \quad X_b(z) = 1\Big/\sqrt{z^2 - b^2}, \quad X_{ab}(z) = X_a(z) X_b(z) \\
&\Sigma_{2j}^{\infty} = \beta_{1j}, \quad \Sigma_{1j}^{\infty} = -\mathrm{Re}\left[\Sigma_{k=1}^{4} B_{jk} \mu_k B_{km}^{-1} (\beta_{1m} + \mathrm{i}\beta_{2m})\right] \\
&\gamma_2 = H_{4j}\beta_{2j}/H_{44}, \quad \gamma_6 = H_{4j}\beta_{1j}/H_{44}, \quad \gamma_0 = -\gamma_2 I_2/I_1 \\
&I_2 = \int_b^a \left[x_1^2/X_{ab}(x_1)\right] \mathrm{d}x_1, \quad I_1 = \int_b^a [1/X_{ab}(x_1)] \mathrm{d}x_1
\end{aligned} \tag{4.119}
$$

The limit analysis shows that $\gamma_0 = 0$ for $b = 0$ and $\gamma_0 = -a^2\gamma_2$ for $b = a$. These show that the present solution is consistent with solutions of the conventional conducting crack and impermeable crack. For the general situation $0 \ll b < a$ at the tip region, where r and $a - b$ is in the same order, the generalized stresses are related to both r and $a - b$.

In electroelastic fracture mechanics, the energy release rate and $J-$integral (Pak 1990; Suo et al. 1992) is often used. Because there are two singular points, crack tip $x_1 = a, x_2 = 0$ and the tip of the conductive part $x_1 = b, x_2 = 0$, so two $J-$integrals expressed with electric enthalpy are defined as

$$
J_1 = \int_{L_a} \left(g n_1 - n_i \sigma_{ip} u_{p,1} - n_i D_i \varphi_{,1}\right) \mathrm{d}l, \quad J_2 = \int_{L_{a+b}} \left(g n_1 - n_i \sigma_{ip} u_{p,1} - n_i D_i \varphi_{,1}\right) \mathrm{d}l \tag{4.120}
$$

where L_a is the contour only enclosed the crack tip, L_{a+b} is the contour enclosed two singular points, g is the electric enthalpy, and $\boldsymbol{n}$ is the outward normal of the contour.

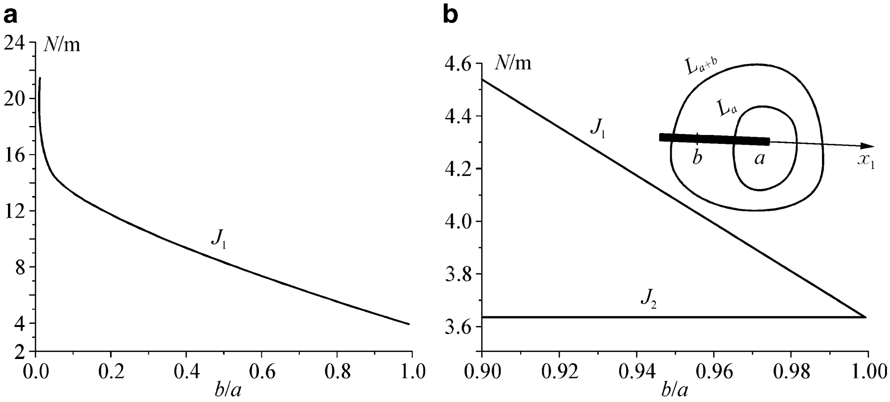


Fig. 4.5 Variation of J-integral value with respect to b/a under loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_1^{\infty} = 0.1\text{MV/m}$: (**a**) J_1 and (**b**) J_1 and J_2

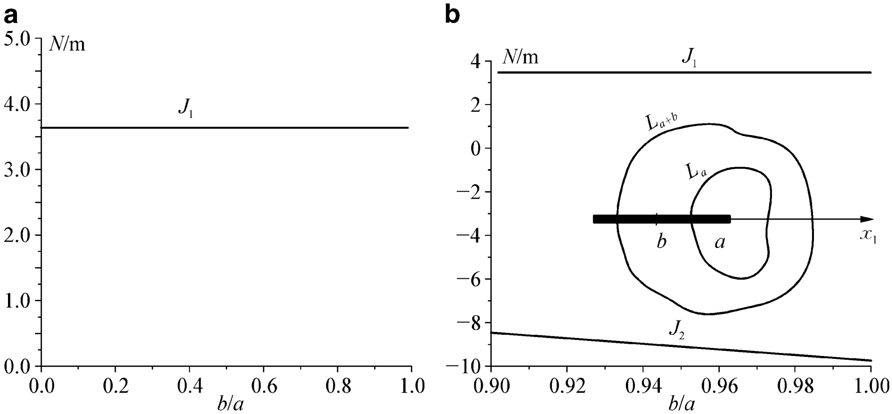


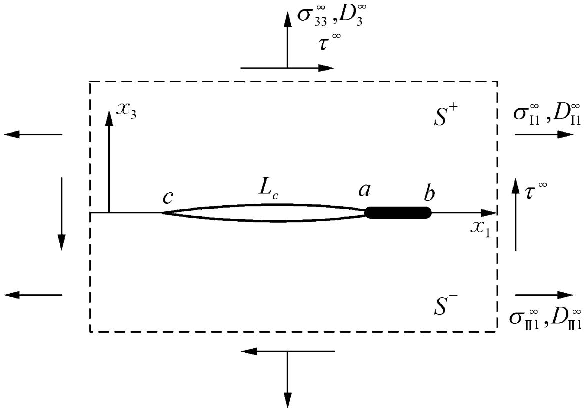
Fig. 4.6 Variation of J-integral value with respect to b/a under loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_2^{\infty} = 0.1\text{MV/m}$: (**a**) J_1 and (**b**) J_1 and J_2

Now give a numerical example. When the poling direction is along axis x_3, the material constants of PZT-4 are

$$C_{11} = 13.9 \times 10^{10}, \quad C_{12} = 7.78 \times 10^{10}, \quad C_{13} = 7.43 \times 10^{10}, \quad C_{33} = 11.3 \times 10^{10},$$
$$C_{44} = 2.56 \times 10^{10}(\text{N/m}^2); \quad e_{31} = -6.98, \quad e_{33} = 13.84, \quad e_{15} = 13.44(\text{C/m}^2)$$
$$\epsilon_{11} = 6.00 \times 10^{-9}, \quad \epsilon_{33} = 5.47 \times 10^{-9}(\text{C/Vm})$$

In the above theoretical analyses, the poling direction is along axis x_2, so the material constants need to be transformed. Figure 4.5 gives the variation of J_1 and J_2 values with respect to b/a under the loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_1^{\infty} = 0.1\text{MV/m}$. Figure 4.6 gives the variation of J_1 and J_2 values with respect to b/a under the

Fig. 4.7 Contact zone model in a bimaterial



loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_2^{\infty} = 0.1\text{MV/m}$. A completely conducting crack can be obtained from J_1 when $b/a \to 0$, while completely impermeable crack can be obtained from J_2 when $b/a \to 1$.

4.4.2 *Contact Zone Model for Interface Cracks in a Piezoelectric Bimaterial*

Figure 4.7 shows a contact zone model in a bimaterial (in x_1–x_3 plane) for an electrically permeable interface crack (Herrmann and Loboda 2000; Loboda 1993). Let material I is located in the upper half space S^+ and material II is located in the lower half space S^-. Let c the left end of the crack, a the right end, and ab the contact zone. The boundary conditions are

$$\begin{aligned}
&\boldsymbol{\Sigma}_{\text{I}} = \boldsymbol{\Sigma}_{\text{II}} = \boldsymbol{\Sigma}^{\infty}, \quad \text{at infinity} \\
&\hat{\boldsymbol{d}}(x_1) = [\![u_3]\!] = u_3^+ - u_3^- = \boldsymbol{0}, \quad [\![\boldsymbol{\Sigma}_3]\!] = \boldsymbol{\Sigma}_{\text{I}3}(x_1) - \boldsymbol{\Sigma}_{\text{II}3}(x_1) = \boldsymbol{0}, \quad x_1 \notin (c, b) \\
&\sigma_{13}^{\pm} = 0, \ \ \sigma_{33}^{\pm} = 0, \quad [\![\varphi]\!] = 0, \quad [\![D_3]\!] = 0, \quad x_1 \in (c, a) \\
&\sigma_{13}^{\pm} = 0, \ \ [\![\sigma_{33}]\!] = 0, \quad [\![u_3]\!] = 0, \quad [\![\varphi]\!] = 0, \quad [\![D_3]\!] = 0, \quad x_1 \in (b, a)
\end{aligned} \tag{4.121}$$

It is assumed that only normal unknown contact stress σ_{33} is acted on the contact zone and no tangential frictional force. Because on whole axis x_1, $\boldsymbol{\Sigma}_{\text{I}3}(x_1) = \boldsymbol{\Sigma}_{\text{II}3}(x_1)$, like Eqs. (4.51), (4.52), (4.53), (4.54), and (4.55), of Sect. 4.2.6 or Eqs. (4.72), (4.73), (4.74), and (4.75) of Sect. 4.3.1, but different notations are adopted, we have

$$\begin{aligned}&\bar{\boldsymbol{F}}_{\mathrm{II}}(z)=\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}(\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)-\boldsymbol{\Delta}^{\infty}),\quad x_3>0;\quad \bar{\boldsymbol{F}}_{\mathrm{I}}(z)=\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}(\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\boldsymbol{\Delta}^{\infty}),\quad x_3<0\\&\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)-\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(x_1)=\boldsymbol{M}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{M}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)+\boldsymbol{\Delta}_1^{\infty}\\&\boldsymbol{M}=\boldsymbol{A}_{\mathrm{I}}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}=(\boldsymbol{A}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1})\boldsymbol{B}_{\mathrm{I}}=-\mathrm{i}\boldsymbol{H}\boldsymbol{B}_{\mathrm{I}},\quad \boldsymbol{\Delta}_1^{\infty}=\left(-\boldsymbol{A}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{II}}^{-1}+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\right)\boldsymbol{\Delta}^{\infty}\end{aligned}\tag{4.122}$$

and

$$\begin{aligned}&\boldsymbol{W}(z)=\begin{cases}\boldsymbol{M}\boldsymbol{F}_{\mathrm{I}}(z), & x_2\ge 0\\ -\bar{\boldsymbol{M}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z)-\boldsymbol{\Delta}_1^{\infty}, & x_2\le 0\end{cases}\\&\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{W}_{\mathrm{I}}(x_1)-\boldsymbol{W}_{\mathrm{II}}(x_1),\quad \boldsymbol{\Sigma}_0(x_1)=\boldsymbol{G}\boldsymbol{W}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{G}}\boldsymbol{W}_{\mathrm{II}}(x_1)-\bar{\boldsymbol{M}}^{-1}\boldsymbol{\Delta}_1^{\infty}\\&\boldsymbol{G}=\boldsymbol{B}_{\mathrm{I}}\boldsymbol{M}^{-1}=\boldsymbol{B}_{\mathrm{I}}\left\{(\boldsymbol{A}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1})\boldsymbol{B}_{\mathrm{I}}\right\}^{-1}=\left(\boldsymbol{A}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\right)^{-1}=\mathrm{i}\boldsymbol{H}^{-1}=-\bar{\boldsymbol{G}}^{\mathrm{T}}\end{aligned}\tag{4.123}$$

where $\boldsymbol{H}$ is shown in Eq. (4.25), $\boldsymbol{\Delta}^{\infty}$ is shown in Eq. (4.52), $\boldsymbol{W}(z)$ is a vector function analytic in whole plane except cracks. For a kind of $6mm$ piezoelectric materials poling along axis x_3, $\boldsymbol{G}$ possesses the following behavior:

$$\begin{aligned}&\boldsymbol{G}=\begin{bmatrix}G_{11} & G_{13} & G_{14}\\ G_{31} & G_{33} & G_{34}\\ G_{41} & G_{43} & G_{44}\end{bmatrix}=\begin{bmatrix}\mathrm{ig}_{11} & \mathrm{g}_{13} & \mathrm{g}_{14}\\ \mathrm{g}_{31} & \mathrm{ig}_{33} & \mathrm{ig}_{34}\\ \mathrm{g}_{41} & \mathrm{ig}_{43} & \mathrm{ig}_{44}\end{bmatrix},\quad \begin{bmatrix}\mathrm{g}_{11} & \mathrm{g}_{13}\\ \mathrm{g}_{31} & \mathrm{g}_{33}\end{bmatrix}\text{ positive definite}\\&\mathrm{g}_{13}=-\mathrm{g}_{31},\ \mathrm{g}_{14}=-\mathrm{g}_{41},\ \mathrm{g}_{34}=\mathrm{g}_{43},\ \mathrm{g}_{44}<0,\quad \text{all } g_{ij} \textit{ is real}\end{aligned}\tag{4.124}$$

and the eigen-equation Eq. (3.12) becomes

$$\begin{bmatrix}C_{11}+C_{44}\mu^2 & (C_{13}+C_{44})\mu & (e_{31}+e_{15})\mu\\ (C_{13}+C_{44})\mu & C_{44}+C_{33}\mu^2 & e_{15}+e_{33}\mu^2\\ (e_{31}+e_{15})\mu & e_{15}+e_{33}\mu^2 & -\epsilon_{11}-\epsilon_{33}\mu^2\end{bmatrix}\begin{Bmatrix}a_1\\ a_3\\ a_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}\tag{4.125}$$

The roots of Eq. (4.125) are $\mu_1=\alpha_1+\mathrm{i}\beta_1,\ \mu_3=-\alpha_1+\mathrm{i}\beta_1,\ \mu_4=\mathrm{i}\beta_4$ where α_1, β_1,β_4 are all real:

$$\begin{aligned}&B_{1j}=C_{44}\left(\mu_jA_{1j}+A_{3j}\right)+e_{15}A_{4j},\quad B_{3j}=C_{13}A_{1j}+C_{33}\mu_jA_{3j}+e_{33}\mu_jA_{4j}\\&B_{4j}=e_{31}A_{1j}+e_{33}\mu_jA_{3j}-\epsilon_{33}\mu_jA_{4j};\quad j=1,3,4,\end{aligned}\tag{4.126}$$

Finally they get

$$\begin{aligned}&K_{\mathrm{I}}=\sqrt{\pi l/2a}\left[\sqrt{1-\lambda}\left(\sigma_{33}^{\infty}\cos\delta+m\sigma_{13}^{\infty}\sin\delta\right)-2\varepsilon\left(\sigma_{33}^{\infty}\sin\delta-m\sigma_{13}^{\infty}\cos\delta\right)\right]\\&K_{\mathrm{II}}=-\sqrt{\pi l/2m^2}\left[\left(\sigma_{33}^{\infty}\sin\delta-m\sigma_{13}^{\infty}\cos\delta\right)+2\varepsilon\sqrt{1-\lambda}\left(\sigma_{33}^{\infty}\cos\delta+m\sigma_{13}^{\infty}\sin\delta\right)\right]\\&K_{\mathrm{D}}=\mathrm{g}_{33}^{-1}\left[\mathrm{g}_{43}-(\mathrm{g}_{31}\mathrm{g}_{43}-\mathrm{g}_{41}\mathrm{g}_{33})\left(\gamma^2-1\right)/2\lambda\rho\right]K_{\mathrm{I}}\end{aligned}\tag{4.127}$$

where

$$\gamma = -(g_{31} + mg_{11})/t, \quad s = (g_{33} + mg_{13})/t, \quad m = \sqrt{-g_{33}/g_{11}}, \quad \rho = t(1+\gamma)$$
$$\delta = \varepsilon \ln\left[\left(1-\sqrt{1-\lambda}\right)\Big/\left(1+\sqrt{1-\lambda}\right)\right], \quad \lambda = (a-b)/l, \quad \varepsilon = (1/2\pi)\ln\gamma, \quad l = a - c \tag{4.128}$$

The contact point b (or the parameter λ) is determined by $K_{\mathrm{I}} = 0$, i.e., under the conditions

$$\sigma_{\mathrm{I}33}(x_1, 0) \le 0, \quad x_1 \in (a, b); \quad [\![u_3(x_1, 0)]\!] \ge 0, \quad x_1 \in (c, a) \tag{4.129}$$

Select the maximum λ_0 from the following equation:

$$\tan\delta = \left[\left(\sqrt{1-\lambda}\sigma_{33}^{\infty} + 2\varepsilon m\sigma_{13}^{\infty}\right)\Big/\left(2\varepsilon\sigma_{33}^{\infty} - \sqrt{1-\lambda}m\sigma_{13}^{\infty}\right)\right] \tag{4.130}$$

For the bimaterial CTS-19(S^+)/PZT-4(S^-) and cadmium sulfide/barium sodium niobate, numerical results show that $\lambda_0 \sim 0.3$, when $\sigma_{13}^{\infty}/\sigma_{33}^{\infty} \to \infty$, and $\lambda_0 \sim 1\big/e^{100}$, $1\big/e^{50}$ when $\sigma_{13}^{\infty}/\sigma_{33}^{\infty} \to 0, 1$, respectively.

Herrmann and Loboda (2000) considered that $\boldsymbol{\Delta}^{\infty}$ can be included in undetermined functions $\boldsymbol{F}_{\mathrm{I}}(z), \boldsymbol{F}_{\mathrm{II}}(z)$, so they let $\boldsymbol{\Delta}^{\infty} = \boldsymbol{0}$. However if let $\boldsymbol{\Delta}^{\infty} = \boldsymbol{0}$, then $\boldsymbol{\Sigma}_{\beta} - 2\mathrm{Re}\left(\boldsymbol{B}_{\beta}\boldsymbol{F}_{\beta}\right) = \boldsymbol{C}_{\beta} \neq \boldsymbol{0}$, where $\boldsymbol{C}_{\beta}$ is a known constant vector. But this does not influence the stress intensity factors and the length of the contact zone.

Herrmann et al. (2001) discussed also the contact zone model of the impermeable crack.

4.4.3 Nonideal Crack in a Homogeneous Piezoelectric Material

In practical structure, the crack cannot be ideal. Now discuss a simple free nonideal crack in a homogeneous piezoelectric material subjected $\boldsymbol{\Sigma}_1^{\infty}, \boldsymbol{\Sigma}_2^{\infty}$ at infinity. Figure 4.8 shows a nonideal symmetric crack expressed by the equation:

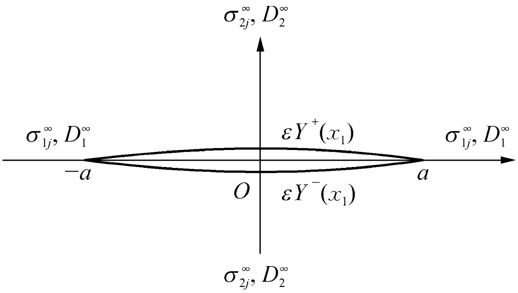
$$\begin{aligned} &x_2 = \varepsilon Y_{\pm}(x_1), \quad Y_+(x_1) - Y_-(x_1) > 0, \quad |x_1| < a \\ &Y'_+(\pm a) - Y'_-(\pm a) = 0 \end{aligned} \tag{4.131a}$$

where ε is a small parameter and $2a$ is the length of the crack. The last equation in Eq. (4.131a) ensures the crack tip idealization.

Huang and Kuang (2001) applied the small parameter method to solve this problem. According to Eq. (4.131a), the points on the crack surfaces in z and z_j planes are denoted respectively by

$$z^0 = x_1 + \mathrm{i}\varepsilon Y_{\pm}(x_1), \quad z_j^0 = x_1 + \varepsilon\mu_j Y_{\pm}(x_1), \quad |x_1| \le a \tag{4.132}$$

Fig. 4.8 Nonideal crack



Expand the complex potential in the piezoelectric material in the series of ε

$$f_j(z_j) = f_j(z_j;\varepsilon) = \sum_{n=0}^{\infty}(\varepsilon^n/n!)f_j^{(n)}(z_j) = f_j^{(0)}(z_j) + \varepsilon f_j^{(1)}(z_j) + \cdots \tag{4.133}$$

On the crack surfaces, we have

$$f_j^{(n)}\left(z_j^0\right) = f_j^{(n)\pm}(x_1) + \varepsilon\mu_j Y_\pm(x_1) f_j'(n)\pm(x_1) + \cdots \tag{4.134}$$

where $f_j^{(n)\pm}(x_1)$ is the value at z_j^0 of $f_j^{(n)}(z_j)$ and $f_j'^{(n)\pm}(z)$ is the derivative of $f_j^{(n)\pm}(z)$ with z. The complex electric potential $\phi(z)$ in the air can be expressed in the same way:

$$\begin{aligned}
&\phi(z) = \phi(z;\varepsilon) = \phi^{(0)}(z) + \varepsilon\phi^{(1)}(z) + \cdots; \quad \varphi^c(z;\varepsilon) = \phi(z;\varepsilon) + \bar{\phi}(\bar{z};\varepsilon)\\
&\phi^{(n)}\left(z^0\right) = \phi^{(n)\pm}(x_1) + \mathrm{i}\varepsilon Y_\pm(x_1)\phi'^{(n)\pm}(x_1) + \cdots\\
&E_1^c = -\varphi_{,1}^c(z) = -2\mathrm{Re}\phi'(z), \quad E_2^c = -\varphi_{,2}^c(z) = 2\mathrm{Im}\phi'(z)
\end{aligned} \tag{4.135}$$

The boundary conditions on a permeable crack surfaces are

$$\begin{aligned}
&2\mathrm{Re}\sum_j B_{kj}f_j\left(z_j^0\right) = 0, \quad k = 1,2,3\\
&2\mathrm{Re}\sum_{j=1}^4 A_{4j}f_j\left(z_j^0\right) = 2\mathrm{Re}\phi\left(z^0\right), \quad 2\mathrm{Re}\sum_{j=1}^4 B_{4j}f_j\left(z_j^0\right) = 2\epsilon_0\mathrm{Im}\phi\left(z^0\right)
\end{aligned} \tag{4.136}$$

The zero-order approximation on the crack surfaces $|x_1| \le a, x_2 = 0$ is

$$\begin{aligned}
&2\mathrm{Re}\sum_{j=1}^4 B_{Pj}f_j^{(0)\pm}(x_1) = T_P^{(0)}(x_1), \quad 2\mathrm{Re}\sum_j A_{4j}f_j^{(0)\pm}(x_1) = 2\mathrm{Re}\phi^{(0)}(x_1)\\
&T_P^{(0)}(x_1) = \left[0,0,0,2\epsilon_0\mathrm{Im}\phi^{(0)}(x_1)\right]^{\mathrm{T}}, \quad P = 1,2,3,4
\end{aligned} \tag{4.137}$$

The first-order approximation on the crack surfaces $|x_1| \le a, x_2 = 0$ is

$$\begin{aligned}
&2\,\mathrm{Re}\sum_{j=1}^{4} B_{Pj} f_j^{(1)\pm}(x_1) = T_P^{(1)\pm}(x_1) \\
&2\,\mathrm{Re}\sum_{j=1}^{4} A_{4j}\left[\mu_j Y_\pm(x_1) f_j'^{(0)\pm}(x_1) + f_j^{(1)\pm}(x_1)\right] = 2\mathrm{Re}\left[\mathrm{i}Y_\pm(x_1)\phi'^{(0)}(x_1) + \phi^{(1)}(x_1)\right] \\
&T_P^{(1)\pm}(x_1) = -2Y_\pm(x_1)\mathrm{Re}\left[\sum_j B_{Pj}\mu_j f_j'^{(0)\pm}(x_1)\right] \\
&\qquad\qquad + 2\delta_{4P}\epsilon_0\,\mathrm{Im}\left[\mathrm{i}Y_\pm(x_1)\phi'^{(0)}(x_1) + \phi^{(1)}(x_1)\right]
\end{aligned} \tag{4.138}$$

The zero-order and first-order approximations at infinity are, respectively,

$$\begin{aligned}
&\lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj}\mu_j f_j'^{(0)}(z_j)\right] = -\Sigma_{1P}^{\infty}, \quad \lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj}\mu_j f_j'^{(1)}(z_j)\right] = 0 \\
&\lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(0)}(z_j)\right] = \Sigma_{2P}^{\infty}, \quad \lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(1)}(z_j)\right] = 0
\end{aligned} \tag{4.139}$$

The single-valued conditions are

$$\begin{aligned}
&\int_{-a}^{a}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(0)+}(x_1) - \sum_{j=1}^{4} B_{Pj} f_j'^{(0)-}(x_1)\right] = 0 \\
&\int_{-a}^{a}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(1)+}(x_1) - \sum_{j=1}^{4} B_{Pj} f_j'^{(1)+}(x_1)\right] = 0
\end{aligned} \tag{4.140}$$

In Eqs. (4.137), (4.138), (4.139), and (4.140), the subscript P takes the values $1, 2, 3, 4$.

From Eqs. (4.137) and (4.138), an inhomogeneous Riemann-Hilbert equations can be obtained.

According to previous sections, it is easy to get their solutions. Finally the stress asymptotic fields near the crack tip are obtained. For a specific symmetric perturbed crack surface configuration,

$$Y_\pm(x_1) = \pm Y(x_1) = \pm\left(a^2 - x_1^2\right)^{3/2}/3a^2 \tag{4.131b}$$

The singular term of the generalized stress fields on the x-axis in piezoelectric material for the zero-order approximation are

$$\Sigma_{2P}^{(0)}(r,\theta) = \sqrt{a/2r}\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj}R_j \bigg/ \sqrt{\Theta_j}\right] + \delta_{4P}C, \quad \Theta_j = \cos\theta + \mu_j \sin\theta$$
$$R_j = \left(B_{jP}^{-1} - B_{j4}^{-1}H_{4P}/H_{44}\right)\Sigma_{2P}^{\infty}, \quad C = (H_{4J}/H_{44})\Sigma_{2J}^{\infty}$$
$$K_{\mathrm{I}}^{(0)} = \sqrt{\pi a}\sigma_{22}^{\infty}, \quad K_{\mathrm{II}}^{(0)} = \sqrt{\pi a}\sigma_{21}^{\infty}, \quad K_{\mathrm{III}}^{(0)} = \sqrt{\pi a}\sigma_{23}^{\infty}, \quad K_{\mathrm{D}}^{(0)} = -\sqrt{\pi a}\sigma_{2j}^{\infty}H_{4j}/H_{44} \tag{4.141}$$

and the electric fields in the air are

$$E_2^{(0)c}(x_1,0) = \frac{D_2^{\infty}}{\epsilon^c} + \frac{H_{4j}\sigma_{2j}^{\infty}}{H_{44}\epsilon^c}, \quad E_1^{(0)c}(x_1,0) = E_1^{\infty} + \mathrm{Re}\sum_{j=1}^{4} A_{4j}R_j \tag{4.142}$$

From Eqs. (4.141) and (4.142), it is seen that the zero-order approximate solution of a permeable crack is consistent with the conducting crack.

The singular term of the generalized stress fields on the x_1-axis in piezoelectric material for the first-order approximation are

$$\Sigma_{2P}^{(1)}(r,0) = \frac{1}{6}\sqrt{\frac{a}{2r}}\left\{\mathrm{Re}\left(\sum_{j=1}^{4} \mathrm{i}B_{Pj}\mu_j R_j\right) - \frac{\delta_{4P}H_{4N}}{H_{44}}\mathrm{Re}\left(\sum_{j=1}^{4} \mathrm{i}B_{Nj}\mu_j R_j\right)\right\}$$
$$K_{\mathrm{I}}^{(1)} = \frac{\sqrt{\pi a}}{6}\mathrm{Re}\left(\sum_{j=1}^{4} \mathrm{i}B_{2j}\mu_j R_j\right), \quad K_{\mathrm{II}}^{(1)} = \frac{\sqrt{\pi a}}{6}\mathrm{Re}\left(\sum_{j=1}^{4} \mathrm{i}B_{1j}\mu_j R_j\right) \tag{4.143}$$
$$K_{\mathrm{III}}^{(1)} = \frac{\sqrt{\pi a}}{6}\mathrm{Re}\left(\sum_{j=1}^{4} \mathrm{i}B_{3j}\mu_j R_j\right), \quad K_{\mathrm{D}}^{(1)} = -\frac{\sqrt{\pi a}H_{4N}}{6H_{44}}\mathrm{Re}\left(\sum_{j} \mathrm{i}B_{Nj}\mu_j R_j\right)$$

and the electric fields in the air are

$$E_1^{(1)c}(x_1,0) = -\frac{1}{2}\left(3\Pi_1\frac{x_1^2}{a^2} + \Pi_2\right), \quad E_2^{(1)c}(x_1,0) = -\frac{1}{\pi \mathrm{i}A_1}\left[(A_2+A_3)\frac{x_1^2}{a^2} - \left(\frac{A_3}{2} + \frac{A_2}{3}\right)\right]$$
$$\varphi[x_1, Y_+(x_1)] - \varphi[x_1, Y_-(x_1)] = -2\varepsilon[Y_+(x_1) - Y_-(x_1)]\left(E_2^{(0)c} + E_2^{(1)c}\right) \tag{4.144}$$

where A, A_2, A_3 and Π_1, Π_2 are known complex constants and functions, respectively. It is found that the generalized stress intensity factors of the zero- and first-order approximations have the same singularity $1/\sqrt{r}$, but the stress angular distributions are different. The future research finds that for an isotropic material, $K_{\mathrm{I}}^{(1)} = K_{\mathrm{D}}^{(1)} = 0$. The electric fields are inhomogeneous in the air gap and the electric potential discontinuity is also inhomogeneous.

In Huang and Kuang's paper (2001), they also discussed the insulated and conducted cracks.

4.4.4 Other Crack Models

Hao and Shen (1994) proposed a model that the electric displacement is dependent on the crack opening displacement. They assumed that the boundary conditions on the crack surfaces are

$$D_2^+ = D_2^-, \quad D_2^+\left(u_2^+ - u_2^-\right) = \epsilon_0(\varphi^- - \varphi^+) \tag{4.145}$$

and discussed a single crack located on the $ox_1(-a, a)$ under the boundary conditions:

$$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^\infty, \quad \text{at infinity;} \quad \boldsymbol{\Sigma}_2 = \mathbf{0}, \quad |x_1| \le a,\ x_2 = 0 \tag{4.146}$$

At first it is assumed $\epsilon_0(\varphi^- - \varphi^+)/\left(u_2^+ - u_2^-\right) = D_2^0$ prior and D_2^0 is a constant determined in the solving process. They applied the stress function method as shown in Sect. 3.3 in the transform planes to solve this problem. The transform function is the same as shown in Eqs. (3.82) and (3.86). Finally they get

$$K_{\text{I}} = \sigma_{22}^\infty\sqrt{\pi a}, \quad K_{\text{II}} = \sigma_{21}^\infty\sqrt{\pi a}, \quad K_{\text{III}} = \sigma_{23}^\infty\sqrt{\pi a}, \quad K_{\text{D}} = \left(D_2^\infty - D_2^0\right)\sqrt{\pi a} \tag{4.147}$$

Their numerical example showed that the smaller external force, the smaller K_{D}. The maximum K_{D} is equal to the electric displacement intensity factor of the insulated crack. It is interest that the boundary conditions Eq. (4.145) can be derived from Eq. (4.144).

Zhang et al. (1998) proposed a self-consistent calculation of a crack profile. They considered that the profile of the opened crack is an elliptic cavity and the ratio of the minor semiaxis to the major semiaxis $\alpha_{\text{s}} = [\![\boldsymbol{A}\boldsymbol{f}(\alpha_{\text{s}}) + \bar{\boldsymbol{A}}\bar{\boldsymbol{f}}(\alpha_{\text{s}})]\!]_2$ (component along x_2) at $x_1 = x_2 = 0$. In the solving process, the current crack profile is used by numerical calculation.

4.5 Interaction of Collinear Inclusions with Singularity

4.5.1 Interaction of an Interface Permeable Crack with a Singularity in a Bimaterial

Let a generalized mechanical singular load with strength $(\boldsymbol{b},\ \boldsymbol{p})$ be located at z_0 in material I occupied the upper half plane $S^+, x_2 > 0$. A permeable crack $(-a, a)$ is located on the interface $x_2 = 0$ (Suo 1990; Gao and Wang 2001; Kuang and Ma 2002). The boundary conditions are

$$\begin{aligned}
&\Sigma_{ij} = \Sigma_{ij}^\infty = 0; \quad |z| \to \infty \\
&\sigma_{2j}^+ = \sigma_{2j}^- = 0; \quad D_2^+ = D_2^- = D_2, \quad E_1^+ = E_1^-; \quad x_1 \in L_{\text{c}} = (-a, a) \\
&\sigma_{2j}^+ = \sigma_{2j}^-, \quad u_j^+ = u_j^-; \quad D_2^+ = D_2^- = D_2, \quad E_1^+ = E_1^-; \quad x_1 \notin L_{\text{c}}
\end{aligned} \tag{4.148}$$

Assume the solution takes the following form:

$$\begin{aligned} &\boldsymbol{F}_{\alpha}(z) = \boldsymbol{F}_{\alpha\,0}(z) + \boldsymbol{G}_{\mathrm{I}}(z)\delta_{\alpha\mathrm{I}}, \quad \alpha = \mathrm{I}, \mathrm{II} \\ &\boldsymbol{G}_{\mathrm{I}}(z) = (1/2\pi\mathrm{i})\boldsymbol{V}\left\langle \left(z_j - z_{0j}\right)^{-1} \right\rangle, \quad \boldsymbol{V} = \left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{p}\right) \end{aligned} \tag{4.149}$$

where $\boldsymbol{G}_{\mathrm{I}}(z)$ is the solution of a singularity in an infinite material I, see Eq. (3.165b). $\boldsymbol{F}_{\alpha\,0}(z)$ is the analytic function in the material α and is zero at infinity, because the generalized stress $\boldsymbol{\Sigma}_2$ is continuous in whole axis x_1. Similar to Eqs. (4.22) and (4.23), it can be obtained:

$$\begin{aligned} &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}\,0}(z) - \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}\,0}(z) + \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z) = \boldsymbol{0}, \quad z \in S^+ \\ &\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}\,0}(z) - \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}\,0}(z) - \boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(z) = \boldsymbol{0}, \quad z \in S^- \end{aligned} \tag{4.150}$$

Equations (4.21), (4.24), and (4.150) yield

$$\begin{aligned} &\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = 2\mathrm{Re}[\boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(x_1) - \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(x_1)] \\ &\mathrm{i}\hat{\boldsymbol{d}}'(x_1) = \boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(x_1) + (\bar{\boldsymbol{Y}}_{\mathrm{II}} - \bar{\boldsymbol{Y}}_{\mathrm{I}})\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1) - \bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(x_1) + (\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1) \end{aligned} \tag{4.151}$$

Because the generalized displacements are continuous on the connective interface, using analytic continuation, a function $\boldsymbol{h}(z)$ analytic in whole z plane except the crack can be constructed:

$$\boldsymbol{h}(z) = \begin{cases} \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(z) + \boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} - \bar{\boldsymbol{Y}}_{\mathrm{I}})\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z), & z \in S^+ \\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(z) - \boldsymbol{H}^{-1}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(z), & z \in S^- \end{cases}; \quad z \notin L_{\mathrm{c}} \tag{4.152}$$

The stress $\boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) = \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)$ on the axis x_1 can be expressed as

$$\boldsymbol{\Sigma}(x_1) = \boldsymbol{h}^+(x_1) + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1) + \bar{\boldsymbol{H}}^{-1}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}} + \boldsymbol{H}^{-1}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}} \tag{4.153}$$

According to Eq. (4.148) on the crack surface, we have $\boldsymbol{\Sigma}(x_1) = D_2(x_1)\mathbf{i}_4,\ \mathbf{i}_4 = [0, 0, 0, 1]^{\mathrm{T}}$, where $D_2(x_1)$ is unknown. So a Riemann-Hilbert equation is obtained:

$$\begin{aligned} &\boldsymbol{h}^+(x_1) + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1) = \tilde{\boldsymbol{\Sigma}}(x_1), \quad x_1 \in L_{\mathrm{c}} \\ &\tilde{\boldsymbol{\Sigma}}(x_1) = D_2(x_1)\mathbf{i}_4 - \bar{\boldsymbol{H}}^{-1}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}} - \boldsymbol{H}^{-1}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}} \end{aligned} \tag{4.154}$$

Equation (4.154) is identical with Eq. (4.28) except using $\tilde{\boldsymbol{\Sigma}}(x_1)$ instead of $\boldsymbol{\Sigma}_0(x_1)$. The form of the solution is still expressed by Eq. (4.41), i.e.,

$$\begin{aligned} &\boldsymbol{\Psi}(z) = \boldsymbol{Q}(z)\left\{\boldsymbol{C}(z) + \frac{1}{2\pi\mathrm{i}}\int_L \frac{[\boldsymbol{Q}^+(x_1)]^{-1}\boldsymbol{\Sigma}^*(x_1)\mathrm{d}x_1}{(x_1 - z)}\right\}, \quad \boldsymbol{C}(z) = \boldsymbol{C}_1 z + \boldsymbol{C}_0 \\ &\boldsymbol{\Psi}(z) = \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z), \quad \boldsymbol{\Sigma}^*(t) = \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}(t), \quad \boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z)\right\rangle, \quad Y_0^{(i)}(z) = \frac{1}{\sqrt{z^2 - a^2}}\left(\frac{z + a}{z - a}\right)^{\mathrm{i}\varepsilon_i} \end{aligned} \tag{4.155}$$

In this problem, it is known that $\boldsymbol{\Psi}(\infty) = \mathbf{0}$ from $\boldsymbol{h}(\infty) = \mathbf{0}$ and $\oint_\Gamma \boldsymbol{\Psi}(z)\mathrm{d}z = \mathbf{0}$ from the single-valued condition. So unknown constant vectors $\boldsymbol{C}_1 = \boldsymbol{C}_0 = \mathbf{0}$.

Equations (4.151) and (4.52) yield

$$\mathrm{i}\hat{\boldsymbol{d}}'(x_1) = \boldsymbol{H}[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] \tag{4.156}$$

Because the electric potential is continuous on whole axis, Eq. (4.156) yields

$$\boldsymbol{H}_4[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] = \mathbf{0}, \quad \boldsymbol{H}_4 = [H_{41}, H_{42}, H_{43}, H_{44}], \quad |x_1| < \infty \tag{4.157}$$

Noting $\boldsymbol{h}(\infty) = \mathbf{0}$ the solution of Eq. (4.157) is

$$\boldsymbol{H}_4\boldsymbol{h}(z) = \boldsymbol{H}_4\boldsymbol{\Omega\Psi} = \mathbf{0}, \quad \boldsymbol{\Omega}\bar{\boldsymbol{\Omega}}^{\mathrm{T}} = \mathbf{I} \tag{4.158}$$

From Eq. (4.158), $D_2(z)$ can be determined and then Eq. (4.155) can be solved. Substituting $\boldsymbol{\Psi}(z)$ into Eq. (4.152) yields $\boldsymbol{F}_{\alpha 0}(z_j)$.

4.5.2 Interaction of an Interface Impermeable Crack with an Interface Singularity

Let a generalized singularity load located at $(x_{01}, 0)$ in front of the right tip of a crack $(-a, a)$ (Wang and Kuang 2002). The superposition method is used to solve this problem, i.e., let

$$\boldsymbol{U}_\alpha = \boldsymbol{U}_{\alpha d} + \boldsymbol{U}_{\alpha c}, \quad \boldsymbol{\Phi}_\alpha = \boldsymbol{\Phi}_{\alpha d} + \boldsymbol{\Phi}_{\alpha c} \tag{4.159}$$

where $\boldsymbol{U}_{\alpha d}, \boldsymbol{\Phi}_{\alpha d}$ are expressed in Eqs. (3.171) and (3.176) representing the solutions of an interface singularity in a bimaterial without crack. This solution introduces the traction $\boldsymbol{\Sigma}_{2d}$. $\boldsymbol{U}_{\alpha c}, \boldsymbol{\Phi}_{\alpha c}$ are the solutions of a crack subjected to $-\boldsymbol{\Sigma}_{2d}$ in a bimaterial.

Using the orthogonal relations of $\boldsymbol{A}$ and $\boldsymbol{B}$ from Eq. (3.171) yields

$$-\boldsymbol{\Sigma}_{2d} = -2\mathrm{Re}\left[\boldsymbol{B}_\alpha\left\langle\frac{1}{x_1 - x_{01}}\right\rangle\boldsymbol{V}_\alpha\right] = -(\boldsymbol{B}_\mathrm{I}\boldsymbol{V}_\mathrm{I} + \bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{V}}_\mathrm{I})\frac{1}{x_1 - x_{01}} = -\frac{1}{x_1 - x_{01}}\frac{\boldsymbol{l}}{\pi} \tag{4.160}$$

where $\boldsymbol{l}$ is expressed in Eq. (3.175). The solution of a crack subjected to $-\boldsymbol{\Sigma}_{2d}$ in a bimaterial can be found in Eq. (4.38). From $\boldsymbol{\Sigma}^\infty = \mathbf{0}$ and the single-valued condition of generalized displacement, it yields $\boldsymbol{C}(z) = \mathbf{0}$ in Eq. (4.38). So the solution is

$$\begin{aligned}\boldsymbol{h}_\mathrm{c}(z) = \boldsymbol{BF}(z) &= \frac{1}{2\pi\mathrm{i}}\boldsymbol{P}(z)\int_L \frac{-\boldsymbol{\Sigma}_{2d}(x_1)\mathrm{d}x_1}{\boldsymbol{P}^+(x_1)(x_1 - z)} = -\frac{1}{2\pi\mathrm{i}}\boldsymbol{P}(z)\int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}\boldsymbol{l}\mathrm{d}x_1}{\pi(x_1 - z)(x_1 - x_{01})} \\ &= -\frac{1}{2\pi\mathrm{i}}\frac{1}{z - x_{01}}\boldsymbol{P}(z)\left\{\int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}\boldsymbol{l}\mathrm{d}x_1}{\pi(x_1 - x_{01})} - \int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}\boldsymbol{l}\mathrm{d}x_1}{\pi(x_1 - z)}\right\}\end{aligned} \tag{4.161}$$

Through some manipulation, we get

$$\boldsymbol{h}_{\mathrm{c}}(z)=\frac{1}{z-x_{01}}\left(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{\Omega}\boldsymbol{Q}(z)\left\langle z-x_{01}-\frac{1}{Y_0(z)}+\frac{1}{Y_0(x_{01})}\right\rangle\boldsymbol{\Omega}^{-1}\frac{\boldsymbol{l}}{\pi} \tag{4.162}$$

From Eqs. (4.44) and (4.162) in front of the crack, the asymptotic stress is

$$\begin{aligned}\boldsymbol{\Sigma}_{2\mathrm{c}}(x_1)&=\boldsymbol{h}_{\mathrm{c}}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}_{\mathrm{c}}^{-}(x_1)=(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H})\boldsymbol{h}_{\mathrm{c}}(x_1)\\&=\boldsymbol{\Omega}\boldsymbol{Q}(z)\left\langle 1-\frac{1}{Y_0(z)(x_1-x_{01})}+\frac{1}{Y_0(x_{01})(x_1-x_{01})}\right\rangle\boldsymbol{\Omega}^{-1}\frac{\boldsymbol{l}}{\pi}\end{aligned} \tag{4.163}$$

According to Eqs. (4.47) and (4.163), the stress intensity factor is

$$\begin{aligned}\boldsymbol{K}&=[K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_{\mathrm{D}}]^{\mathrm{T}}=\lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Omega}\left\langle(x_1-a)^{-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1)\\&=\frac{1}{\sqrt{\pi a}}\boldsymbol{\Omega}\left\langle(2a)^{-\mathrm{i}\varepsilon_*}\left[1+\frac{1}{Y_0(x_{01})(a-x_{01})}\right]\right\rangle\boldsymbol{\Omega}^{-1}(\boldsymbol{\Omega}_1\boldsymbol{b}+\boldsymbol{\Omega}_2\boldsymbol{p})=\boldsymbol{W}_1\boldsymbol{b}+\boldsymbol{W}_2\boldsymbol{p}\end{aligned} \tag{4.164}$$

Sometimes $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are called the weight functions.

4.5.3 Interaction of Collinear Rigid Inclusions with a Singularity

Now we discuss the interaction of collinear rigid inclusions with singularity. The singularity is also located at z_0 with strength $(\boldsymbol{b},\ \boldsymbol{p})$ in material I (Zhou et al. 2008). The boundary conditions are assumed:

$$\begin{aligned}&\boldsymbol{\Sigma}_2=\boldsymbol{\Sigma}_2^{\infty}(x_1),\quad |z|\to\infty\\&u_{j,1}=\omega_{\mathrm{r}}\delta_{j2},\quad U_{\mathrm{I}}=U_{\mathrm{II}},\quad E_{\mathrm{I}1}=E_{\mathrm{II}1}=E_{\mathrm{r}1}=-\varphi_{\mathrm{r},1},\quad D_{\mathrm{I}1}=D_{\mathrm{II}1},\quad x_1\in L_{\mathrm{cr}}\\&\boldsymbol{U}_{\mathrm{I}}(x_1)=\boldsymbol{U}_{\mathrm{II}}(x_1),\quad \boldsymbol{\Sigma}_2(x_1)=\boldsymbol{\Sigma}_{\mathrm{I}2}(x_1)=\boldsymbol{\Sigma}_{\mathrm{II}2}(x_1),\quad x\notin L_{\mathrm{c}},\quad L_{\mathrm{c}}=\cup L_{\mathrm{cr}}\end{aligned} \tag{4.165}$$

where ω_{r} is the rotation angle about axis x_3 of the rth inclusion. Comparing with Sect. 4.3.1 (rigid line inclusion) here, only a singularity is added, so the solving process is similar. Assume the solution is in the following form:

$$\begin{aligned}&\boldsymbol{U}_{\alpha,1}=2\mathrm{Re}[\boldsymbol{A}_\alpha\boldsymbol{F}_\alpha(z)+\boldsymbol{A}_\alpha\boldsymbol{G}_{\mathrm{I}}(z)\delta_{\alpha\mathrm{I}}];\quad \boldsymbol{\Phi}_{\alpha,1}=2\mathrm{Re}[\boldsymbol{B}_\alpha\boldsymbol{F}_\alpha(z)+\boldsymbol{B}_\alpha\boldsymbol{G}_{\mathrm{I}}(z)\delta_{\alpha\mathrm{I}}]\\&\boldsymbol{G}_{\mathrm{I}}(z)=(1/2\pi\mathrm{i})\boldsymbol{V}\left\langle(z_j-z_{0j})^{-1}\right\rangle,\quad \boldsymbol{V}=\left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{b}+\boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{p}\right),\quad \alpha=\mathrm{I},\mathrm{II}\end{aligned} \tag{4.166}$$

The generalized displacements are continuous on the whole axis x_1. Like Eqs. (4.73) and (4.150) we have

$$\begin{aligned}&\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(x_1)=\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)=\boldsymbol{\varDelta}^{\infty}\\&\boldsymbol{\varDelta}^{\infty}=(1/2)[(\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{\infty}+\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^{\infty})-(\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^{\infty}+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty})],\quad \alpha=\mathrm{I},\mathrm{II}\end{aligned}\tag{4.167}$$

Like Eq. (4.74), we have

$$\begin{aligned}\Delta\boldsymbol{\Phi}_{,1}(x_1)=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)-\boldsymbol{\Phi}_{\mathrm{II},1}(x_1)=\mathrm{i}\big[&\boldsymbol{R}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\big(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\big)\boldsymbol{\varDelta}^{\infty}\big]\\&+\big(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\big)\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)+\big(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\big)\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)\end{aligned}\tag{4.168}$$

where $\boldsymbol{Y}_{\alpha},\boldsymbol{R}$ are also shown in Eq. (4.74). By the standard analytic continuation through the connective interface $L-L_{\mathrm{c}}$, we can construct a function $\boldsymbol{h}(z)$ analytic in whole plane except the rigid inclusions L_{c} and at infinity $\boldsymbol{h}(\infty)=\boldsymbol{h}^{\infty}$:

$$\boldsymbol{h}(z)=\begin{cases}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)+\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z), & z\in S^{+}\\ \boldsymbol{R}^{-1}\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)+\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\varDelta}^{\infty} & z\in S^{-}\end{cases}\tag{4.169}$$

Equation (4.169) yields

$$\begin{aligned}&\boldsymbol{F}_{\mathrm{I}}(z_j)=\boldsymbol{A}_{\mathrm{I}}^{-1}\big[\boldsymbol{h}(z_j)-\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z_j)\big]\\&\boldsymbol{F}_{\mathrm{II}}(z_j)=\boldsymbol{A}_{\mathrm{II}}^{-1}\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\big[\boldsymbol{h}(z_j)+\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(z_j)-\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\varDelta}^{\infty}\big]\end{aligned}\tag{4.170}$$

Like Eq. (4.79), we have

$$\begin{aligned}\boldsymbol{U}_{\mathrm{I},1}(x_1)&=\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)+\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)\\&=\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)+\bar{\boldsymbol{R}}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)\\&\quad+\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)-\boldsymbol{\varDelta}_{1}^{\infty}\\\boldsymbol{\varDelta}_{1}^{\infty}&=\bar{\boldsymbol{R}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\varDelta}^{\infty}\end{aligned}\tag{4.171}$$

On the surfaces of inclusions, like Eq. (4.80), we have

$$\mathbf{U}_{\mathrm{I},1}=\omega(x_1)\mathbf{i}_2-E_1(x_1)\mathbf{i}_4,\quad \omega(x_1)=\omega_{\mathrm{r}},\quad r=1-n,\quad x_1\in L_{\mathrm{c}}\tag{4.172}$$

From Eqs. (4.171) and (4.172), a Riemann-Hilbert equation is obtained:

$$\begin{aligned}&\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)=\boldsymbol{N}(x_1),\quad x_1\in L_{\mathrm{c}}\\&\boldsymbol{N}(x_1)=\boldsymbol{\varDelta}_{1}^{\infty}+\omega_r(x_1)\mathbf{i}_2-E_1(x_1)\mathbf{i}_4-\bar{\boldsymbol{R}}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)-\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)\end{aligned}\tag{4.173}$$

Equation (4.173) is identical with (4.81) if we use $\boldsymbol{N}$ instead of $\boldsymbol{\Delta}_1^\infty + \omega_0\mathbf{i}_2 - E_1(x_1)\mathbf{i}_4$. Its solution is

$$\begin{aligned}
&\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z) = \boldsymbol{Q}(z)\boldsymbol{C}(z) + \frac{\boldsymbol{Q}(z)}{2\pi\mathrm{i}}\int_L \frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{N}(x_1)}{\boldsymbol{Q}^+(x_1)(x_1 - z)}\mathrm{d}x_1 \\
&\boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z)\right\rangle, \quad Y_0^{(j)}(z) = \prod_{k=1}^{n}\frac{1}{\sqrt{(z-a_k)(z-b_k)}}\left(\frac{z-b_k}{z-a_k}\right)^{\mathrm{i}\varepsilon_j} \\
&C^{(i)}(z) = C_n^{(i)}z^n + C_{n-1}^{(i)}z^{n-1} + \cdots + C_1^{(i)}z + C_0^{(i)}
\end{aligned} \tag{4.174}$$

Equations (4.168), (4.169), and (4.165) yield

$$\begin{aligned}
&\Delta\boldsymbol{\Phi}_{,1}(x_1) = \mathrm{i}\boldsymbol{R}[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)], \quad x_1 \in L_{\mathrm{c}} \\
&\Delta\boldsymbol{\Phi}_{,1}(x_1) = \boldsymbol{0}, \quad x_1 \notin L_{\mathrm{c}}
\end{aligned} \tag{4.175}$$

According to Eq. (4.165), $D_2(x_1)$ is continuous on whole $x_1 = 0$, so $\Delta\Phi_{4,1}(x_1) = 0$, or

$$\boldsymbol{R}_4[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] = \boldsymbol{0}, \quad -\infty < x_1 < \infty \tag{4.176}$$

where $\boldsymbol{R}_4$ is the fourth row of $\boldsymbol{R}$. The solution is

$$\boldsymbol{R}_4\boldsymbol{h}(z) = \boldsymbol{R}_4\boldsymbol{h}^\infty, \quad \boldsymbol{h}^\infty = \boldsymbol{h}(\infty) \tag{4.177}$$

Assume $\boldsymbol{\varepsilon}^\infty = \left[\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + w^\infty, \varepsilon_{13}^\infty + w_3^\infty, -E_1^\infty\right]^{\mathrm{T}}$ at infinity and noting $\boldsymbol{h}^+(x_1) = \boldsymbol{h}^-(x_1)$ on the crack surface and $\boldsymbol{h}^+(\infty) = \boldsymbol{h}^-(\infty)$ we can get $\boldsymbol{h}^\infty$:

$$\begin{aligned}
&\mathbf{U}_{I,1}(\infty) = \left[\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + \omega^\infty, \varepsilon_{13}^\infty + \omega_3^\infty, -E_1^\infty\right] = \boldsymbol{\varepsilon}^\infty = \boldsymbol{h}^+(\infty) + \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^-(\infty) - \boldsymbol{\Delta}_1^\infty \\
&\boldsymbol{h}^\infty = \boldsymbol{\Omega} < \left(1 + \mathrm{e}^{2\pi\varepsilon_i}\right)^{-1} > \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left(\boldsymbol{\varepsilon}^\infty + \boldsymbol{\Delta}_1^\infty\right)
\end{aligned} \tag{4.178}$$

From Eqs. (4.177) and (4.178), $E_1(z)$ can be obtained and then Eq. (4.174) can be solved.

If $\boldsymbol{R} = \bar{\boldsymbol{R}}$ is a real matrix, the solution does not oscillate.

4.5.4 Interaction of a Crack with an Electric Dipole in a Homogeneous Piezoelectric Material

Let an impermeable crack $(-a, a)$ in an infinite piezoelectric material and an electric dipole with strength p_{e} located at z_0 formed an angle θ with positive axis x_1. The distance from z_0 to $(a, 0)$ is $\rho = \left|\overrightarrow{z_0 a}\right|$ and $\overrightarrow{z_0 a}$ form an angle ϕ with the positive direction of x_1 (Fig. 4.9).

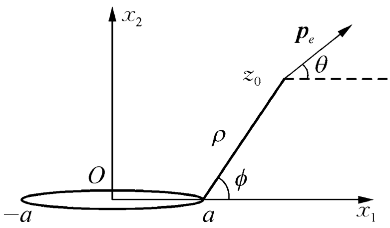


Fig. 4.9 Crack and electric dipole

Wang and Kuang (2000, 2002) discussed the interaction of a crack with an electric dipole in a homogeneous piezoelectric material. Let $\boldsymbol{U}_{\mathrm{p}}, \boldsymbol{\Phi}_{\mathrm{p}}$ as shown in Eq. (3.178) are the solutions of an electric dipole in an infinite piezoelectric material. The generalized traction on the line corresponding to the crack surfaces introduced by this electric dipole is $\boldsymbol{\Sigma}_2$ shown in Eq. (3.179). Assuming $\boldsymbol{h}_{\mathrm{c}}, \boldsymbol{U}_{\mathrm{c}}, \boldsymbol{\Phi}_{\mathrm{c}}$ are the solutions when the crack surfaces are subjected to $-\boldsymbol{\Sigma}_2$, the solutions of a piezoelectric material with a crack and an electric dipole are

$$\boldsymbol{U} = \boldsymbol{U}_{\mathrm{p}} + \boldsymbol{U}_{\mathrm{c}}, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}_{\mathrm{p}} + \boldsymbol{\Phi}_{\mathrm{c}} \tag{4.179}$$

According to Eq. (4.38) and noting $\boldsymbol{\Omega} = \mathbf{I}$ for a homogeneous material, the solution $\boldsymbol{h}_{\mathrm{c}}$ is

$$\boldsymbol{h}_{\mathrm{c}}(z) = \boldsymbol{B}\boldsymbol{F}_{\mathrm{c}}(z) = \boldsymbol{Q}(z)\left\{\boldsymbol{C} + \frac{1}{2\pi\mathrm{i}}\int_L \frac{-\boldsymbol{\Sigma}_2(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1 - z)}\right\}, \quad \boldsymbol{Q}(z) = \langle Y_0(z)\rangle \tag{4.180}$$

where $Y_0(z) = 1\big/\sqrt{z^2 - a^2}$. Using Eq. (4.18) yields

$$\frac{1}{2\pi\mathrm{i}}\int_{-a}^{a}\frac{1}{Y_0^+(x_1 - z)}\mathrm{d}x_1 = \frac{1}{2}\left[\frac{1}{Y_0(z)} - z\right] = \frac{1}{2}\left[\sqrt{z^2 - a^2} - z\right]$$

Substituting Eq. (3.179) and above equation into Eq. (4.180) yields

$$\begin{gathered}
\boldsymbol{h}_{\mathrm{c}}(z) = \frac{1}{2\sqrt{z^2 - a^2}}\mathrm{Re}\left\{\frac{p_{\mathrm{e}}}{\pi\mathrm{i}}\boldsymbol{B}\left\langle\Theta\left[\frac{\sqrt{z_{0j}^2 - a^2}}{(z - z_{0j})^2} - \frac{\sqrt{z^2 - a^2}}{(z - z_{0j})^2} + \frac{z_{0j}}{(z - z_{0j})\sqrt{z_{0j}^2 - a^2}}\right]\right\rangle\boldsymbol{A}^{\mathrm{T}}\mathbf{i}_4\right\} \\
\boldsymbol{\Sigma}_{2\mathrm{c}}(x_1) = 2\mathrm{Re}\boldsymbol{h}_{\mathrm{c}}(x_1), \quad \Theta = \cos\theta + \mu_j\sin\theta
\end{gathered} \tag{4.181}$$

The stress intensity factor is

$$\boldsymbol{K} = \lim_{x_1\to a}\sqrt{2\pi(x_1 - a)}\boldsymbol{\Sigma}_{2\mathrm{c}}(x_1) = p_{\mathrm{e}}\sqrt{\frac{a}{\pi}}\mathrm{Im}\left\{\boldsymbol{B}\left\langle\Theta\left[\frac{1}{(z_{0j} - a)\sqrt{z_{0j}^2 - a^2}}\right]\right\rangle\boldsymbol{A}^{\mathrm{T}}\mathbf{i}_4\right\} \tag{4.182}$$

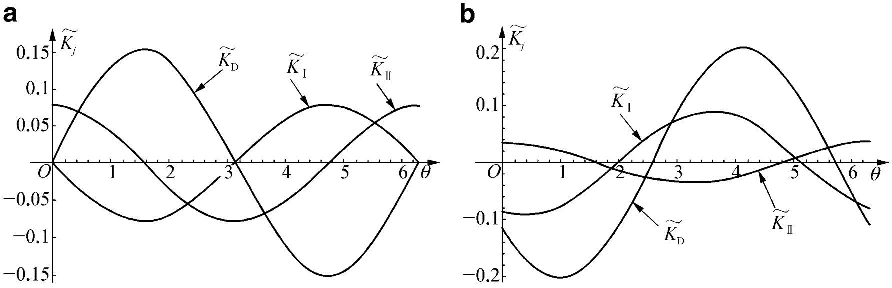


Fig. 4.10 Variation of $\tilde{\boldsymbol{K}}$ with θ: (**a**) dipole located at (2*a*,0) and (**b**) dipole located at (*a*,*a*)

Take a local coordinate system (ρ, ϕ) with the origin at the right crack tip; when $\rho \ll a$, $\boldsymbol{K}$ can be expressed by

$$\boldsymbol{K} = \sqrt{\frac{1}{2\pi}} p_{\mathrm{e}} \frac{1}{\rho\sqrt{\rho}} \mathrm{Im}\left\{ \boldsymbol{B} \left\langle \left[\frac{\Theta}{\left(\cos\phi + \mu_j \sin\phi\right)^{3/2}} \right] \right\rangle \boldsymbol{A}^{\mathrm{T}} \mathbf{i}_4 \right\} \tag{4.183}$$

Figure 4.10 gives the variation of the dimensionless stress intensity factor $\tilde{\boldsymbol{K}} = \boldsymbol{K}/p_{\mathrm{e}} a^{-3/2}$ with the electric dipole direction θ: (a) dipole located at (2*a*,0) and (b) dipole located at (*a*,*a*).

4.5.5 Interaction of a Crack with an Electric Dipole on the Interface in a Bimaterial

Let the electric dipole at $(x_{01}, 0)$ with strength p_{e} on the interface in a bimaterial. The superposition method is used to solve this problem, i.e.,

$$\boldsymbol{U} = \boldsymbol{U}_{\alpha d} + \boldsymbol{U}_{\alpha c}, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}_{\alpha d} + \boldsymbol{\Phi}_{\alpha c} \tag{4.184}$$

where $\boldsymbol{\Phi}_{\alpha d}$ is shown in Eq. (3.180), and $\boldsymbol{\Sigma}_2(x_1)$ on the crack surfaces introduced by $\boldsymbol{\Phi}_{\alpha d}$ is

$$\begin{aligned} \boldsymbol{\Sigma}_2(x_1) &= 2\mathrm{Re}\left[\boldsymbol{B}_\alpha \left\langle \frac{1}{x_1 - x_{01} - d} - \frac{1}{x_1 - x_{01}} \right\rangle \boldsymbol{N}_\alpha q_{\mathrm{e}} \right] \mathbf{i}_4 \\ &= \frac{q_e}{\pi} \left(\frac{1}{x_1 - x_{01} - d} - \frac{1}{x_1 - x_{01}} \right) \boldsymbol{\Omega}_2 \mathbf{i}_4 \end{aligned} \tag{4.185}$$

where $2\mathrm{Re}(\boldsymbol{B}_\alpha \boldsymbol{N}_\alpha) = \boldsymbol{\Omega}_2/\pi$ is used and $\boldsymbol{\Omega}_2$ is shown in Eq. (3.175). Because the generalized stresses are assumed zero at infinity and generalized displacement are single valued, so $\boldsymbol{C}(z)=\mathbf{0}$ in Eq. (4.38). Substituting $\boldsymbol{C}(z)=\mathbf{0}$ and $-\boldsymbol{\Sigma}_2(x_1)$ into Eq. (4.38) yields $\boldsymbol{h}_{\alpha c}$, i.e.,

$$\begin{aligned}\boldsymbol{h}_{\alpha c}(z)&=\boldsymbol{BF}(z)=\frac{1}{2\pi i}\boldsymbol{P}(z)\int_L\frac{-\boldsymbol{\Sigma}_2(x_1)\mathrm{d}x_1}{\boldsymbol{P}^+(x_1)(x_1-z)}\\&=-\frac{1}{2\pi i}\boldsymbol{P}(z)\left\{\int_L\frac{[\boldsymbol{P}^+(x_1)]^{-1}}{(x_1-z)}\left(\frac{1}{x_1-x_{01}-d}-\frac{1}{x_1-x_{01}}\right)\mathrm{d}x_1\right\}\frac{q_e}{\pi}\boldsymbol{\Omega}_2\mathbf{i}_4\\\boldsymbol{P}(z)&=\boldsymbol{\Omega}\boldsymbol{Q}(z),\quad \boldsymbol{Q}(z)=\langle Y_0(z)\rangle,\quad \boldsymbol{\Omega}=\left[\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\boldsymbol{\omega}^{(3)},\boldsymbol{\omega}^{(4)}\right]\end{aligned}\tag{4.186}$$

Using the theory of the singular integral equation, finishing the integral and noting $\lim_{d\to 0}q_e d\to p_e$ we get

$$\begin{aligned}\boldsymbol{h}_{\alpha c}(z)&=\left(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{\Omega}\boldsymbol{Q}(z)\lim_{q_e d\to p_e,d\to 0}\\&\times\left\langle\frac{1}{Y_0(z)}\left(\frac{1}{z-x_{01}}-\frac{1}{z-x_{01}-d}\right)+\frac{1}{Y_0(x_{01}+d)}\frac{1}{z-x_{01}-d}-\frac{1}{Y_0(d)}\frac{1}{z-x_{01}}\right\rangle\boldsymbol{\Omega}^{-1}\frac{q_e}{\pi}\boldsymbol{\Omega}_2\mathbf{i}_4\end{aligned}$$

Taking the approximation in first order, the above equation is reduced to

$$\begin{aligned}\boldsymbol{h}_{\alpha c}(z)&=\left(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{\Omega}\boldsymbol{Q}(z)\\&\times\left\langle-\frac{1}{Y_0(z)}\frac{1}{(z-x_{01})^2}+\frac{1}{Y_0(x_{01})}\frac{1}{z-x_{01}}\left(\frac{1}{z-x_{01}}+\frac{x_{01}-2ia\varepsilon_j}{x_{01}^2-a^2}\right)\right\rangle\boldsymbol{\Omega}^{-1}\frac{q_e}{\pi}\boldsymbol{\Omega}_2\mathbf{i}_4\end{aligned}\tag{4.187}$$

In front of and near the crack tip, the principle singular term is

$$\begin{aligned}\boldsymbol{\Sigma}_{2c}(x_1)&=\boldsymbol{h}^+_{\alpha c}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-_{\alpha c}(x_1)=(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H})\boldsymbol{h}_{\alpha c}(x_1)\\&=\boldsymbol{\Omega}\frac{1}{\sqrt{2a(x_1-a)}}\left(\frac{x_1-a}{2a}\right)^{i\varepsilon_j}\left\langle\frac{1}{Y_0(x_{01})}\frac{1}{a-x_{01}}\left(\frac{1}{a-x_{01}}+\frac{x_{01}-2ia\varepsilon_j}{x_{01}^2-a^2}\right)\right\rangle\boldsymbol{\Omega}^{-1}\frac{p_e}{\pi}\boldsymbol{\Omega}_2\mathbf{i}_4\end{aligned}\tag{4.188}$$

The generalized stress intensity factors at the right tip are

$$\begin{aligned}\boldsymbol{K}&=[K_{II},K_I,K_{III},K_D]^T=\lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Omega}\left\langle(x_1-a)^{-i\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1)\\&=\frac{p_e}{\sqrt{\pi a}}\boldsymbol{\Omega}\left\langle(2a)^{-i\varepsilon_j}\left[\frac{1}{Y_0(x_{01})(a-x_{01})}\right]\left(\frac{1}{a-x_{01}}+\frac{x_{01}-2ia\varepsilon_j}{x_{01}^2-a^2}\right)\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\mathbf{i}_4\\&=p_e\sqrt{\frac{a}{\pi}}\frac{1}{(x_{01}-a)\sqrt{x_{01}^2-a^2}}\boldsymbol{\Omega}\left\langle(2a)^{-i\varepsilon_j}(1+2i\varepsilon_j)\left(\frac{x_{01}+a}{x_{01}-a}\right)^{i\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\mathbf{i}_4\end{aligned}\tag{4.189}$$

When $\rho = x_{01} - a \to 0$, in a region $x_1 - a \ll \rho$, we get

$$\boldsymbol{K} == p_{\mathrm{e}}\left(1\Big/\sqrt{2\pi}\right)\boldsymbol{\Omega}\left\langle\left(1+2\mathrm{i}\varepsilon_j\right)\rho^{-\frac{3}{2}-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\mathbf{i}_4 \tag{4.190}$$

4.6 Interaction of an Elliptic Hole and a Vice-Crack

4.6.1 The Solution Method

Figure 4.11 shows an elliptic hole filled air and a vice-crack in an infinity piezoelectric material subjected $\boldsymbol{\Sigma}^{\infty}$ at infinity. The major and minor axes of the ellipse $(2a, 2b)$ are aligned along x_1 and x_2, respectively. The center of the vice-crack of length $2c_0$ is located at $z^{(0)}\left(x_1^{(0)} + \mathrm{i}x_2^{(0)}\right)$ and forms an angle γ with the positive direction of x_1. The distance from $z^{(0)}$ to $(a, 0)$ is d_0 and $\overrightarrow{z^{(0)}a}$ form an angle α with the positive direction of x_1. Zhou et al. (2005b) used the continuous distribution dislocation method to solve this problem. The main steps of this method are: (1) Problem I. A singularity located in an infinite piezoelectric material with an elliptic hole. The solution of problem I is used as the Green function, which does not produces the traction at infinity and on the boundary of the elliptic hole, but produces tractions on an artificial cut corresponding to the original vice-crack. (2) Problem II. An infinite piezoelectric material with an elliptic cavity filled air subjected to $\boldsymbol{\Sigma}^{\infty}$ at infinity. The solution of problem II produces tractions also on an artificial cut corresponding to the original vice-crack. (3) Problem III. The geometric shape of this problem is identical with the original problem, but the vice-crack is replaced by an artificial generalized continuous distribution dislocation with undetermined density. Add the tractions on the vice-crack surface obtained from problems II and III to satisfy the original boundary conditions, and the unknown dislocation density can be obtained. (4) After solving the unknown

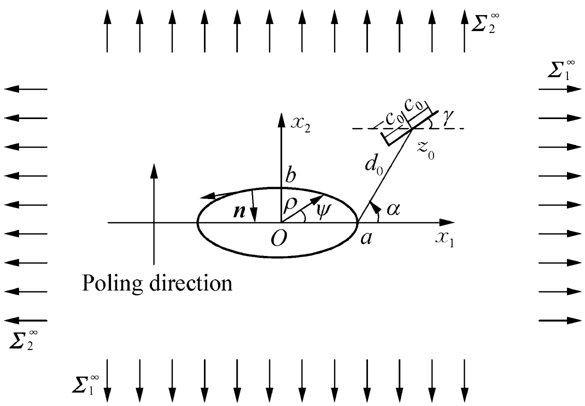


Fig. 4.11 An elliptic hole and a vice-crack

dislocation density, the original problem can be solved. The transform method is used to solve this problem. The transform functions are shown in Eqs. (3.82) and (3.86). The boundary L of the elliptic in the z plane is mapped to the unit circle Γ in the ς plane. In this section, the second natural coordinate system, i.e., use $(\boldsymbol{n}, \boldsymbol{t}')$ in (3.29b) and $\boldsymbol{T} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s$, is used. Some geometric relations can be seen in Eqs. (3.29b) and (3.82b).

4.6.2 Problem I

In this section, a slightly simpler method to solve this problem is used. The problem is decomposed into two subproblems: (1) Problem Ia, a singularity locates at $z_0(x_{01} + \mathrm{i}x_{02})$ in an infinite homogeneous material, and (2) Problem Ib, a distributed loading acts on the boundary of the elliptic hole. (3) Superpose the solutions of problems Ia and Ib, and let the resultant solution satisfy the boundary conditions of the original problem.

(a) According to Eqs. (3.156) and (3.158), the solution of the problem Ia is

$$\begin{aligned}
&\boldsymbol{U}_{\mathrm{I}}^{(a)} = (1/\pi)\mathrm{Im}\left[\boldsymbol{A}\left\langle \ln(z_j - z_{0j})\right\rangle \boldsymbol{V}\right], \quad \boldsymbol{\Phi}_{\mathrm{I}}^{(a)} = (1/\pi)\mathrm{Im}\left[\boldsymbol{B}\left\langle \ln(z_j - z_{0j})\right\rangle \boldsymbol{V}\right] \\
&\ln(z_j - z_{oj}) = \ln(\varsigma_j - \varsigma_{0j}) + \ln\left[c_j\left(1 - d_j/c_j\varsigma_j\varsigma_{0j}\right)\right], \quad \boldsymbol{V} = \boldsymbol{B}^T\boldsymbol{b} + \boldsymbol{A}^T\boldsymbol{p}
\end{aligned} \tag{4.191}$$

On the unit circle Γ in the ς plane, $\varsigma = \varsigma_j = \sigma = \mathrm{e}^{\mathrm{i}\psi}$, so

$$\boldsymbol{\Phi}_{\mathrm{I}}^{(a)}(\sigma) = (1/\pi)\mathrm{Im}\left\{\boldsymbol{B}\left\langle \ln(\sigma - \varsigma_{0j}) + \ln\left[c_j\left(1 - d_j/c_j\sigma\varsigma_{0j}\right)\right]\right\rangle \boldsymbol{V}\right\} \tag{4.192}$$

Using $\mathrm{d}s = \rho(\psi)\mathrm{d}\psi$, $\rho^2 = a^2\sin^2\psi + b^2\cos^2\psi$ given in Eq. (3.82b). Eq. (4.192) can be expanded in the following series

$$\begin{aligned}
\boldsymbol{T}_{\mathrm{I}}^{(a)}(\sigma) = \mathrm{d}\boldsymbol{\Phi}^{(a)}(\sigma)/\mathrm{d}s = (1/\pi\rho(\psi))\mathrm{Im}\Bigg\{\boldsymbol{B}\sum_{k=1}^{\infty}\Big\langle &\left[\left(1/\varsigma_{0j}\right)^k + \left(d_j/c_j\varsigma_{0j}\right)^k\right]\sin k\psi \\
&+\mathrm{i}\left[\left(d_j/c_j\varsigma_{0j}\right)^k - \left(1/\varsigma_{0j}\right)^k\right]\cos k\psi\Big\rangle \boldsymbol{V}\Bigg\}
\end{aligned} \tag{4.193}$$

(b) The solution of the problem Ib can be taken as (Chung and Ting 1996)

$$\begin{aligned}
\boldsymbol{U}_{\mathrm{I}}^{(b)} &= 2\mathrm{Re}\sum_{m=1}^{\infty}\left\{\boldsymbol{A}\left\langle \varsigma_j^{-m}\right\rangle\left(\boldsymbol{A}^{\mathrm{T}}\mathbf{g}_m + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_m\right)\right\} \\
\boldsymbol{\Phi}_{\mathrm{I}}^{(b)} &= 2\mathrm{Re}\sum_{m=1}^{\infty}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-m}\right\rangle\left(\boldsymbol{A}^{\mathrm{T}}\mathbf{g}_m + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_m\right)\right\}
\end{aligned} \tag{4.194}$$

where $\mathbf{g}_m, \boldsymbol{h}_m$ are real vectors determined by the boundary conditions. On Γ we have

$$\begin{aligned}
&\boldsymbol{U}_{\mathrm{I}}^{(b)}(\sigma) = \sum_{m=1}^{\infty} \left[\cos(m\psi)\boldsymbol{h}_m - \sin(m\psi)\hat{\boldsymbol{h}}_m\right] \\
&\boldsymbol{\Phi}_{\mathrm{I}}^{(b)}(\sigma) = \sum_{m=1}^{\infty} [\cos(m\psi)\mathbf{g}_m - \sin(m\psi)\hat{\mathbf{g}}_m] \\
&\boldsymbol{T}_{\mathrm{I}}^{(b)}(\sigma) = \mathrm{d}\boldsymbol{\Phi}^{(b)}(\sigma)/\mathrm{d}s = -[1/\rho(\psi)] \sum_{m=1}^{\infty} m[\sin(m\psi)\mathbf{g}_m + \cos(m\psi)\hat{\mathbf{g}}_m] \\
&\hat{\boldsymbol{h}}_m = \boldsymbol{S}\boldsymbol{h}_m + \boldsymbol{M}\mathbf{g}_m, \quad \hat{\mathbf{g}}_m = \boldsymbol{S}^{\mathrm{T}}\mathbf{g}_m - \boldsymbol{L}\boldsymbol{h}_m
\end{aligned} \tag{4.195}$$

where $\boldsymbol{S}, \boldsymbol{M}, \boldsymbol{L}$ are shown in Eq. (3.35).

(c) The solution of the electric potential inside the cavity hole filled air has been discussed in Sect. 3.4.2. Using $\varphi_{\mathrm{I}}(\sigma) = 2\mathrm{Re}\phi_{\mathrm{I}}(\sigma)$ according to Eq. (3.85) we get

$$\begin{aligned}
&\phi_{\mathrm{I}}(\varsigma) = \sum_{m=1}^{\infty} a_m^c[\varsigma^m + (d/c)^m \varsigma^{-m}], \quad \varsigma = \rho e^{\mathrm{i}\psi} \\
&\varphi_{\mathrm{I}}(\sigma) = 2\mathrm{Re} \sum_{m=1}^{\infty} a_m^c \left[\left(1 + \left(\frac{d}{c}\right)^m\right) \cos m\psi + \mathrm{i}\left(1 - \left(\frac{d}{c}\right)^m\right) \sin m\psi\right] \\
&D_{\mathrm{I}}^c(\sigma) = -2\epsilon^c \mathrm{Im}[\mathrm{d}\phi(\sigma)/\mathrm{d}s] = -(2\epsilon^c/\rho^c)\times \\
&\quad \sum_{m=1}^{\infty} \left\{\left[-m(1 + (d/c)^m)\mathrm{Im}a_m^c\right] \sin m\psi + \left[m(1 - (d/c)^m)\mathrm{Re}a_m^c\right] \cos m\psi\right\}
\end{aligned} \tag{4.196}$$

Comparing $\varphi_{\mathrm{I}}(\sigma)$ in Eq. (4.196) and $\left(\boldsymbol{U}_{\mathrm{I}}^{(b)}\right)_4(\sigma)$ in Eq. (4.195) yields

$$(\boldsymbol{h}_m)_4 = 2[1 + (d/c)^m]\mathrm{Re}a_m^c, \quad (\hat{\boldsymbol{h}}_m)_4 = 2[1 - (d/c)^m]\mathrm{Im}a_m^c, \quad m \geq 1 \tag{4.197}$$

(d) The sum of generalized stresses in problems Ia and Ib on the elliptic boundary must satisfy the original boundary condition:

$$\boldsymbol{T}_{\mathrm{I}}^{(b)} + \boldsymbol{T}_{\mathrm{I}}^{(a)} = D_{\mathrm{I}}^c \mathbf{i}_4, \quad \mathbf{i}_4 = [0, 0, 0, 1]^{\mathrm{T}}; \quad \text{on} \quad \Gamma \tag{4.198}$$

Substitution of Eqs. (4.193), (4.195), and (4.196) into Eq. (4.198) yields

$$\begin{aligned}
&\mathbf{g}_m = \mathbf{g}_{m1} + \mathbf{g}_{m2}, \quad \hat{\mathbf{g}}_m = \hat{\mathbf{g}}_{m1} + \hat{\mathbf{g}}_{m2} \\
&\mathbf{g}_{m1} = (1/m\pi)\mathrm{Im}\left[\boldsymbol{B}\left\langle (1/\varsigma_{0j})^m + (d_j/c_j\varsigma_{0j})^m \right\rangle \boldsymbol{V}\right], \quad \mathbf{g}_{m2} = -2\epsilon^c[1 + (d/c)^m]\mathrm{Im}a_m^c \mathbf{i}_4 \\
&\hat{\mathbf{g}}_{m1} = (1/m\pi)\mathrm{Im}\left[\boldsymbol{B}\left\langle (d_j/c_j\varsigma_{0j})^m - (1/\varsigma_{0j})^m \right\rangle \boldsymbol{V}\right], \quad \hat{\mathbf{g}}_{m2} = 2\epsilon^c[1 - (d/c)^m]\mathrm{Re}a_m^c \mathbf{i}_4
\end{aligned} \tag{4.199}$$

From Eqs. (4.191), (4.194), (4.195), (4.199), and

$$\sum_{m=1}^{\infty} \varsigma_j^{-m}\bar{\varsigma}_{0k}^{-m}/m = -\ln\left(1-\varsigma_j^{-1}\bar{\varsigma}_{0k}^{-1}\right), \quad \text{when} \quad \left|\varsigma_j^{-1}\bar{\varsigma}_{0k}^{-1}\right| < 1$$
$$\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{L}^{-1}\boldsymbol{S}^{\mathrm{T}} = \boldsymbol{B}^{-1}/2, \quad \boldsymbol{B}^{\mathrm{T}}\boldsymbol{L}^{-1} = \mathrm{i}\boldsymbol{B}^{-1}/2$$

the stress functions in the piezoelectric material finally are

$$\begin{aligned}
&\boldsymbol{\Phi}_{\mathrm{I}} = \boldsymbol{\Phi}_{\mathrm{I}}^{(a)} + \boldsymbol{\Phi}_{\mathrm{I}}^{(b)} = \boldsymbol{\Phi}_{\mathrm{I}}^{(1)} + \boldsymbol{\Phi}_{\mathrm{I}}^{(2)} \\
&\boldsymbol{\Phi}_{\mathrm{I}}^{(1)} = (1/\pi)\mathrm{Im}\left\{\boldsymbol{B}\langle\ln(\varsigma_j-\varsigma_{0j})\rangle\boldsymbol{V}\right\} + (1/\pi)\sum_{k=1}^{4}\mathrm{Im}\left\{\boldsymbol{B}\left\langle\ln(\varsigma_j^{-1}-\bar{\varsigma}_{0k})\right\rangle\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{V}}\right\} \\
&\boldsymbol{\Phi}_{\mathrm{I}}^{(2)} = 2\epsilon^c\sum_{m=1}^{\infty}\mathrm{Im}\left\{\boldsymbol{B}\left\langle\varsigma_j^{-m}\right\rangle\boldsymbol{B}^{-1}\left[\bar{a}_m^c - (d/c)^m a_m^c\right]\mathbf{i}_4\right\}
\end{aligned} \tag{4.200}$$

where

$$\begin{aligned}
&a_m^c = \alpha_m/\beta_m, \quad C_m = C_{1m} + \mathrm{i}C_{2m} \\
&\alpha_m = \frac{1}{2}\left\{\overline{C}_m(d/c)^m\left(1-\epsilon^c L_{44}^{-1} + \mathrm{i}\epsilon^c L_{4i}^{-1}S_{4i}\right) - C_m\left(1+\epsilon^c L_{44}^{-1} + \mathrm{i}\epsilon^c L_{4i}^{-1}S_{4i}\right)\right\} \\
&\beta_m = \left[1-(d/c)^{2m}\right]\left[1-\left(\epsilon^c L_{44}^{-1}\right)^2 - \left(\epsilon^c L_{4i}^{-1}S_{4i}\right)^2\right] - 2\epsilon^c\left[1+(d/c)^{2m}\right]L_{44}^{-1}
\end{aligned} \tag{4.201}$$

$$\begin{aligned}
C_{1m} = &\frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\boldsymbol{S}^{\mathrm{T}}\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1+(d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right] - \mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m-1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right]\right\}\boldsymbol{p} \\
&+\frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\boldsymbol{S}^{\mathrm{T}}\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1+(d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right] - \mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m-1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right]\right\}\boldsymbol{b} \\
C_{2m} = &\frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1+(d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right] + \boldsymbol{S}^{\mathrm{T}}\mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m-1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right]\right\}\boldsymbol{p} \\
&+\frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1+(d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right] + \boldsymbol{S}^{\mathrm{T}}\mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m-1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right]\right\}\boldsymbol{b}
\end{aligned} \tag{4.202}$$

where C_{1m}, C_{2m} are real, $\boldsymbol{L}_4^{-1} = \left[L_{41}^{-1}, L_{42}^{-1}, L_{43}^{-1}, L_{44}^{-1}\right]$.

The solution shown in Eq. (4.200) is the solution of the problem I representing a singularity located in an infinite piezoelectric material with an elliptic hole. It is a Green function.

When $b = 0$, the elliptic hole is reduced to a crack and $c = d = c_j = d_j = a/2$. In this case, the Green function is simplified significantly. The stress intensity factor at $x_1 = a$ is

$$\begin{aligned}
\boldsymbol{K}(a) &= \sqrt{2\pi} \lim_{z_j \to a, x_2 = 0} \sqrt{z_j - a}\boldsymbol{\Phi}_{,1} = \sqrt{\pi/a} \lim_{\varsigma_j \to 1} \partial\boldsymbol{\Phi}/\partial\varsigma_j \\
&= \frac{1}{\sqrt{\pi a}} \left\{ \mathrm{Im}\left[\boldsymbol{B} \left\langle 1 - \sqrt{\frac{z_{0j}+a}{z_{0j}-a}} \right\rangle \boldsymbol{B}^{\mathrm{T}} \boldsymbol{b} \right] - \frac{L_{4j}^{-1}}{L_{44}^{-1}} \mathrm{Im}\left[\boldsymbol{B} \left\langle 1 - \sqrt{\frac{z_{0j}+a}{z_{0j}-a}} \right\rangle \boldsymbol{B}^{\mathrm{T}} \boldsymbol{b}\mathbf{i}_4 \right] \right\} \\
K_{\mathrm{D}}(a) &= -\left(L_{4m}^{-1}/L_{44}^{-1}\right) K_m(a), \quad m = 1, 2, 3
\end{aligned} \tag{4.203}$$

4.6.3 Problem II

Problem II can be decomposed into two subproblems. Problem IIa: a homogeneous infinite piezoelectric material subjected $\boldsymbol{\Sigma}^{\infty}$ at infinity. Its solution is

$$\boldsymbol{\Phi}_{\mathrm{II}}^{(a)} = \boldsymbol{\Sigma}_2^{\infty} x_1 - \boldsymbol{\Sigma}_1^{\infty} x_2, \quad \boldsymbol{\Sigma}_{\mathrm{II}\,1}^{(a)} = \boldsymbol{\Sigma}_1^{\infty}, \quad \boldsymbol{\Sigma}_{\mathrm{II}\,2}^{(a)} = \boldsymbol{\Sigma}_2^{\infty} \tag{4.204}$$

Remove a piece of material to form an artificial elliptic hole whose size is identical to the hole in the original problem. Using Eqs. (3.29a) and (3.82b) the generalized traction on this artificial elliptic boundary is $\boldsymbol{\Sigma}_n^{\Gamma}$:

$$\begin{aligned}
\boldsymbol{\Sigma}_n^{\Gamma} &= \left(\boldsymbol{\Sigma}_1^{\infty} n_1 + \boldsymbol{\Sigma}_2^{\infty} n_2 - D_n^c \mathbf{i}_4\right) \\
&= -\frac{b}{\rho(\psi)} \cos\psi \left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c \mathbf{i}_4\right) - \frac{a}{\rho(\psi)} \sin\psi \left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c \mathbf{i}_4\right)
\end{aligned} \tag{4.205}$$

The electric field in the elliptic hole is assumed as unknown constant E_i^c $(i = 1, 2)$:

$$\varphi_{\mathrm{II}}^c = -E_1^c x_1 - E_2^c x_2, \quad D_i^c = \epsilon^c E_i^c, \quad D_n^c = D_i^c n_i \tag{4.206}$$

Problem IIb: $-\boldsymbol{\Sigma}_n^{\Gamma}$ is applied on the artificial elliptic boundary. The general solution of this problem has been shown in Eq. (4.194) and the expression on Γ is given in Eq. (4.195). Comparing φ_{II}^c with $\boldsymbol{U}_{\mathrm{I}}^{(b)}(\sigma)$ and $-\boldsymbol{\Sigma}_n^{\Gamma}$ with $\boldsymbol{T}_{\mathrm{I}}^{(b)}(\sigma)$, it is find that in present problem,

$$\begin{aligned}
&\mathbf{g}_1 = -a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c \mathbf{i}_4\right), \quad \hat{\mathbf{g}}_1 = -b\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c \mathbf{i}_4\right); \quad \mathbf{g}_m = \hat{\mathbf{g}}_m = \mathbf{0}, \text{ for } m \neq 1 \\
&(\boldsymbol{h}_1)_4 = -aE_1^c, \quad \left(\hat{\boldsymbol{h}}_1\right)_4 = bE_2^c
\end{aligned} \tag{4.207}$$

Using the relations between $\boldsymbol{g}_1, \hat{\boldsymbol{g}}_1, \boldsymbol{h}_1, \hat{\boldsymbol{h}}_1$ in Eq. (4.195) the unknown electric displacements D_1^c, D_2^c in the hole are determined by

$$\begin{aligned}
\left(bL_{44}^{-1} - a/\epsilon^c\right) D_1^c - aL_{4i}^{-1} S_{4i} D_2^c &= bL_{4i}^{-1} \sigma_{1i}^{\infty} - aL_{4i}^{-1} S_{ji} \sigma_{2j}^{\infty} \\
bL_{4i}^{-1} S_{4i} D_1^c + \left(aL_{44}^{-1} - b/\epsilon^c\right) D_2^c &= bL_{4i}^{-1} S_{ji} \sigma_{1j}^{\infty} + aL_{4i}^{-1} \sigma_{2i}^{\infty}
\end{aligned} \tag{4.208}$$

Substituting $\boldsymbol{g}_m, \boldsymbol{h}_m$ into Eq. (4.194), $\boldsymbol{\Phi}_{\mathrm{II}}^{(b)}$ can be obtained. The sum of the solutions of the problems IIa and IIb $\boldsymbol{\Phi}_{\mathrm{II}} = \boldsymbol{\Phi}_{\mathrm{II}}^{(a)} + \boldsymbol{\Phi}_{\mathrm{II}}^{(b)}$ is the solution of the problem II. Finally it yields

$$\boldsymbol{\Phi}_{\mathrm{II}} = \boldsymbol{\Sigma}_2^{\infty} x_1 - \boldsymbol{\Sigma}_1^{\infty} x_2 - \mathrm{Re}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-1}\right\rangle \boldsymbol{B}^{-1}\left[a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c \boldsymbol{i}_4\right)\right] - \mathrm{i}b\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c \boldsymbol{i}_4\right)\right\} \tag{4.209}$$

For a crack, $b = 0$, Eqs. (4.208) and (4.209) respectively reduced to

$$\boldsymbol{\Phi}_{\mathrm{II}} = \boldsymbol{\Sigma}_2^{\infty} x_1 - \boldsymbol{\Sigma}_1^{\infty} x_2 - \mathrm{Re}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-1}\right\rangle \boldsymbol{B}^{-1} a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c \boldsymbol{i}_4\right)\right\}, \quad D_2^c = L_{4i}^{-1}\sigma_{2i}^{\infty}/L_{44}^{-1} \tag{4.210}$$

4.6.4 Problem III

For an artificial generalized continuous distribution dislocation instead of the original vice-crack, the solution can be obtained by integrating the Green function Eq. (4.200) with respect to z_{0j} along the vice-crack or the artificial dislocation line, i.e.,

$$\begin{aligned}\boldsymbol{\Phi}_{\mathrm{III}}(\xi) = {} & \frac{1}{\pi}\int_{-c_0}^{c_0}\left\{\mathrm{Im}\left[\boldsymbol{B}\langle \ln(\varsigma_j - \varsigma_{0j})\rangle \boldsymbol{V}\right] + \frac{1}{\pi}\sum\nolimits_{k=1}^{4}\mathrm{Im}\left[\boldsymbol{B}\left\langle \ln(\varsigma_j^{-1} - \bar{\varsigma}_{0k})\right\rangle \boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{V}}\right]\right\}\mathrm{d}\xi_0 \\ & + 2\epsilon^c \int_{-c_0}^{c_0}\sum_{m=1}^{\infty}\mathrm{Im}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-m}\right\rangle \boldsymbol{B}^{-1}\left[\bar{a}_m^c - (d/c)^m a_m^c\right]\mathbf{i}_4\right\}\mathrm{d}\xi_0\end{aligned} \tag{4.211}$$

where $2c_0$ is the length of vice-crack and $\mathrm{d}\xi_0$ is the dislocation differentiate element. Assuming the middle point of the vice-crack is at $z_j^0\left(x_1^0 + \mu_j x_2^0\right)$, the angle of the vice-crack with the positive axis x_1 is γ. A certain point on the vice-crack is at $z_j = z_j^0 + \xi\left(\cos\gamma + \mu_j \sin\gamma\right)$ and the position of a dislocation is at $z_{0j} = z_j^0 + \xi_0(\cos\gamma + \mu_j \sin\gamma)$, where ξ, ξ_0 is the algebraic length calculated from z^0. The traction on the crack surface is $\partial\boldsymbol{\Phi}_{\mathrm{II}}/\partial\xi + \partial\boldsymbol{\Phi}_{\mathrm{III}}/\partial\xi = \mathbf{0}$ due to original vice-crack is free. From this condition, it yields

$$-\frac{1}{\pi}\int_{-c_0}^{c_0}\mathrm{Im}\left[\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}\frac{1}{\xi_0 - \xi}\right]\mathrm{d}\xi_0 + \int_{-c_0}^{c_0}\boldsymbol{K}_1(\xi,\xi_0)\boldsymbol{b}\,\mathrm{d}\xi_0 + \int_{-c_0}^{c_0}\boldsymbol{K}_2(\xi,\xi_0)\mathrm{d}\xi_0 = -\boldsymbol{T}^a(\xi) \tag{4.212}$$

where

$$\begin{aligned}\boldsymbol{T}^a(\xi) &= \boldsymbol{\Sigma}_2^{\infty}\cos\alpha - \boldsymbol{\Sigma}_1^{\infty}\sin\alpha + \mathrm{Re}\left\{\boldsymbol{B}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j^2}\right\rangle\boldsymbol{B}^{-1}a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c\mathbf{i}_4\right) - \mathrm{i}b\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c\mathbf{i}_4\right)\right\} \\ \boldsymbol{K}_1(\xi,\xi_0) &= -\frac{1}{\pi}\mathrm{Im}\left[\boldsymbol{B}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j\left[(c_j/d_j)\varsigma_j\varsigma_{0j} - 1\right]}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right] - \frac{1}{\pi}\sum_{l=1}^{4}\mathrm{Im}\left[\boldsymbol{B}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j(1 - \varsigma_j\bar{\varsigma}_{0l})}\right\rangle\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\mathbf{I}_l\bar{\boldsymbol{B}}^{\mathrm{T}}\right] \\ \boldsymbol{K}_2(\xi,\xi_0) &= -2\epsilon^c\sum_{m=1}^{\infty}\mathrm{Im}\left\{\boldsymbol{B}\left\langle\frac{\delta\partial\varsigma_j/\partial\xi}{\varsigma_j^{m+1}}\right\rangle\boldsymbol{B}^{-1}\left[\bar{a}_m^c - (d/c)^m a_m^c\right]\mathbf{i}_4\right\}\end{aligned} \tag{4.213}$$

For the insulated elliptic hole, $\boldsymbol{K}_2(\xi, \xi_0) = \mathbf{0}$. When the elliptic hole is degenerated into a main crack, the kernel function $\boldsymbol{K}_2(\xi, \xi_0)$ is reduced to

$$\boldsymbol{K}_2 = \frac{1}{\pi}\mathrm{Im}\left\{\boldsymbol{B}\left\langle\frac{L_{4m}^{-1}}{L_{44}^{-1}}\left[B_{ml}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j(\varsigma_j\varsigma_{0l}-1)}\right\rangle B_{pl} - \bar{B}_{ml}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j(\varsigma_j\bar{\varsigma}_{0l}-1)}\right\rangle\bar{B}_{pl}\right]\boldsymbol{b}_p\right\rangle\boldsymbol{B}^{-1}\right\}\mathbf{i}_4 \tag{4.214}$$

Adopt the dimensionless length $l' = \xi_0/c_0, l = \xi/c_0$ and noting the singular behavior of the kernel function, Eq. (4.199) (Muskhelishvili 1975; Erdogan and Gupta 1972) is rewritten as

$$\begin{aligned}&-\frac{1}{\pi}\int_{-1}^{1}\mathrm{Im}\left[\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\right]\frac{\hat{\boldsymbol{b}}(l')}{\sqrt{1-l'^2}}\frac{\mathrm{d}l'}{l'-l} + \int_{-1}^{1}\boldsymbol{K}_1(l', l)\frac{\hat{\mathbf{b}}(l')\mathrm{d}l'}{\sqrt{1-l'^2}} + \int_{-1}^{1}\boldsymbol{K}_2(l', l)\mathrm{d}l' = -\boldsymbol{T}^a(l)\\ &\hat{b} = b\sqrt{1-l'^2}, \quad -1 < l' < 1\end{aligned} \tag{4.215}$$

where $|l| < 1$ and $\hat{\boldsymbol{b}}$ is finite. The generalized displacement single-valued condition is

$$\int_{-1}^{1}\left[\hat{\boldsymbol{b}}(l')\Big/\sqrt{1-l'^2}\right]\mathrm{d}l' = \mathbf{0} \tag{4.216}$$

Equations (4.215) and (4.216) are the singular integral equation system of the original problem and calculated by the numerical method. Here the selected collocation points l_i', l_r in the interval $[-1, 1]$ are

$$l_i' = \cos\frac{(2i-1)\pi}{2n}, \quad l_r = \cos\frac{r\pi}{n}, \quad i = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, n-1 \tag{4.217}$$

and Eq. (4.214) is reduced to a set of algebraic equations:

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{n}\hat{\boldsymbol{b}}(l_i')\left\{\mathrm{Im}\left[\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\right]\frac{1}{l_i'-l_r} - \pi\boldsymbol{K}_1(l_i'-l_r) - \pi\hat{\boldsymbol{K}}_2(l_i'-l_r)\right\} = \boldsymbol{T}^a(l_r)\\ &\sum_{i=1}^{n}\hat{\boldsymbol{b}}(l_i') = 0\\ &\hat{\boldsymbol{K}}_2 = \frac{1}{\pi}\mathrm{Im}\left\{\boldsymbol{B}\left\langle\frac{L_{4m}^{-1}}{L_{44}^{-1}}\left[B_{ml}\left\langle\frac{\partial\varsigma_*/\partial\xi}{\varsigma_*(\varsigma_*\varsigma_{0l}-1)}\right\rangle B_{pl} - \bar{B}_{ml}\left\langle\frac{\partial\varsigma_*/\partial\xi}{\varsigma_*(\varsigma_*\bar{\varsigma}_{0l}-1)}\right\rangle\bar{B}_{pl}\right]\right\rangle\boldsymbol{B}^{-1}\right\}\mathbf{i}_4\end{aligned} \tag{4.218}$$

Equation (4.218) gives $4(n-1)+4=4n$ equations with $4n$ unknowns. Solving $\hat{\boldsymbol{b}}$, the asymptotic field $\boldsymbol{T}(l)$ near the crack tip is

$$\begin{aligned}
&\boldsymbol{T}(l)=\mathrm{i}\boldsymbol{B}\boldsymbol{B}^{T}\hat{\boldsymbol{b}}(l)\Big/\sqrt{l^{2}-1},\quad l=1+\varepsilon,\quad \varepsilon(>0)\to 0\\
&\hat{\boldsymbol{b}}(1)=\frac{1}{n}\sum_{i=1}^{n}\frac{\sin[(2i-1)(2n-1)\pi/4n]}{\sin[(2i-1)\pi/4n]}\hat{\boldsymbol{b}}\left(l_{i}'\right)\\
&\hat{\boldsymbol{b}}(-1)=\frac{1}{n}\sum_{i=1}^{n}\frac{\sin[(2i-1)(2n-1)\pi/4n]}{\sin[(2i-1)\pi/4n]}\hat{\boldsymbol{b}}\left(l_{n+1-i}'\right)
\end{aligned}\tag{4.219}$$

The stress intensity of the right crack tip of the vice-crack is

$$\begin{aligned}
&[K_{\mathrm{I}},K_{\mathrm{II}},K_{\mathrm{III}},K_{\mathrm{D}}]^{\mathrm{T}}=\lim_{l\to\pm 1}\sqrt{2\pi(l-1)}\boldsymbol{Q}\boldsymbol{T}(l)=-\mathrm{i}\sqrt{\pi c}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\hat{\boldsymbol{b}}(1)\\
&\boldsymbol{Q}=\begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{0}\\ \boldsymbol{0} & \mathbf{I}\end{bmatrix},\quad \boldsymbol{Q}_{11}=\begin{bmatrix}-\sin\alpha & \cos\alpha\\ \cos\alpha & \sin\alpha\end{bmatrix},\quad \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}
\end{aligned}\tag{4.220}$$

If the elliptic is degenerated to a main crack, the stress intensity factor of the main crack is

$$\begin{aligned}
&[K_{\mathrm{I}},K_{\mathrm{II}},K_{\mathrm{III}},K_{D}]^{\mathrm{T}}=\boldsymbol{K}^{0}+\hat{\boldsymbol{K}};\quad \boldsymbol{K}^{0}=\sqrt{\pi a}\left(\boldsymbol{\Sigma}_{2}^{\infty}-D_{2}^{c}\mathbf{i}_{4}\right)\\
&\hat{\boldsymbol{K}}=\int_{-c}^{c}\mathrm{d}\boldsymbol{K}=\int_{-1}^{1}\boldsymbol{P}(l')\boldsymbol{b}(l')\mathrm{d}l'=(\pi/n)\sum\nolimits_{i=1}^{n}\boldsymbol{P}(l')\hat{\boldsymbol{b}}(l')
\end{aligned}\tag{4.221}$$

where $\boldsymbol{P}$ is complicated and omitted here.

4.6.5 Example

The matrix piezoelectric material is PZT-4 and the material constants are shown in Sect. 4.4.1. In the following examples, let $\gamma=0,\ d_0/c_0=a/c_0=2$ and $K_{\mathrm{I}}^{(0)}=\sigma_2^{\infty}\sqrt{\pi a}, K_{\mathrm{I}}^{(0m)}=\sigma_2^{\infty}\sqrt{\pi c_0}$. Figure 4.12 shows the distributions of the normalized mechanical stress intensity factors at right tips with α under $\gamma=0$ and different electric loading: (a) $K_{\mathrm{I}}\Big/K_{\mathrm{I}}^{(0)}$ of the main crack ($b=0$) and (b) $K_{\mathrm{I}}^{(m)}\Big/K_{\mathrm{I}}^{(0)}$ of the vice-crack. Figure 4.13 shows (a) the distributions of the normalized stress $\sigma_2\big/\sigma_2^{\infty}$ at right end of the elliptic hole of $b/a=0.1$ with α under $\gamma=0$ and different electric loading and (b) $K_{\mathrm{I}}^{(m)}\Big/K_{\mathrm{I}}^{(0m)}$ of the vice-crack with α under $\gamma=0$ and different electric loading.

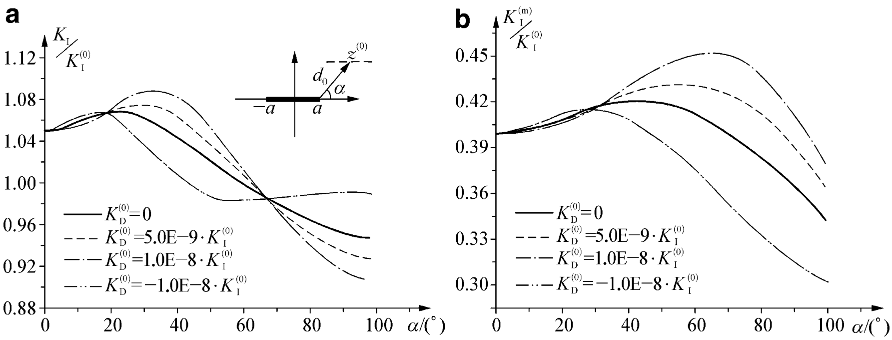


Fig. 4.12 Under $\gamma = 0,\ d_0/c_0 = a/c_0 = 2$: (**a**) variation of $K_{\mathrm{I}}/K_{\mathrm{I}0}$ with α at right tip of main crack and (**b**) variation of $K_{\mathrm{I}}^{m}/K_{\mathrm{I}0}$ with α at right tip of vice-crack

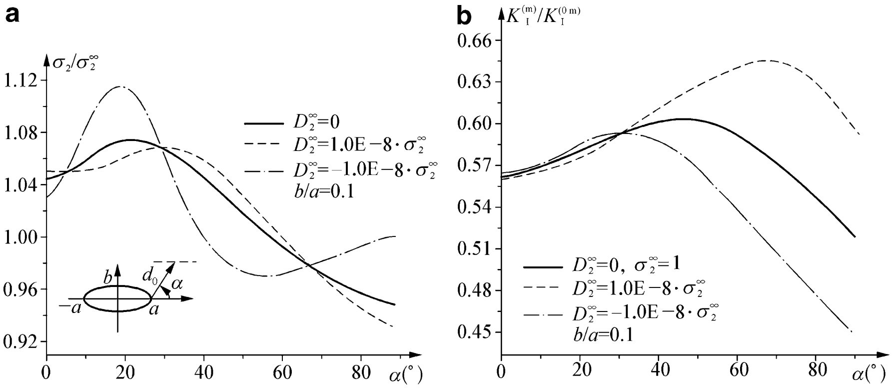


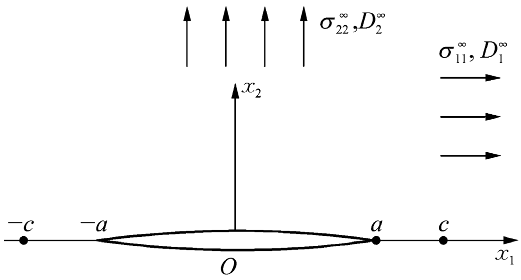
Fig. 4.13 Under $\gamma = 0$: (**a**) variation of $\sigma_2/\sigma_2^{\infty}$ with α at right end of the elliptic hole of $b/a = 0.1$ and (**b**) variation of $K_{\mathrm{I}}^{m}/K_{\mathrm{I}0}$ with α at right tip of vice-crack

4.7 Strip Electric Saturation Model of an Impermeable Crack in a Homogeneous Material

4.7.1 Fundamental Theory

Usually, the mechanical strength of a ceramic is high, and the plastic deformation is very small which can be neglected. Contrarily under high electric field, the crack tip region can be saturated due to the electric field concentration, if breakdown does not happen. Referencing to the Dugdale model in the elastoplastic fracture mechanics, the strip electric saturation model was proposed (Gao et al. 1997; Fulton and Gao 1997; Wang 2000). This model assumes that at crack tip region, the mechanical behavior is elastic, but the electric behavior is saturated. In order to solve this problem, by linear analysis, it is assumed that the electric saturation region is limited on a line segment in front of the tip (Fig. 4.14). The boundary conditions are

Fig. 4.14 Strip electric saturation model



$$
\begin{aligned}
&\boldsymbol{\Sigma}_2^{\infty} = \mathbf{0}, \quad |z| \to \infty \\
&\boldsymbol{\Sigma}_2^{\pm} = -\boldsymbol{T}, \quad \boldsymbol{T} = \left[\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}\right]^T, \quad |x_1| \le a \\
&\boldsymbol{U}^{+} = \boldsymbol{U}^{-}, \quad \boldsymbol{\Sigma}_2^{+} = \boldsymbol{\Sigma}_2^{-} = -\tilde{\boldsymbol{T}}, \quad \tilde{\boldsymbol{T}} = \left[*, *, *, D_2^{\infty} - D_{\mathrm{s}}\right], \quad a \le |x_1| \le c
\end{aligned} \tag{4.222}
$$

where "$*$" denotes variable which does not applied and omitted here, D_{s} is the saturation value, $2a$ is the crack length, and $a \le |x_1| \le c$ is the strip electric saturation region.

Because the generalized stress $\boldsymbol{\Sigma}_2(x_1)$ is continuous on whole axis x_1, similar to Eqs. (4.21), (4.22), (4.23), (4.24), (4.25), and (4.26) in Sect. 4.2.1, we can obtain

$$
\boldsymbol{B}\boldsymbol{F}^{+}(z) = \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^{-}(z), \quad x_2 > 0; \quad \boldsymbol{B}\boldsymbol{F}^{-}(z) = \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^{+}(z), \quad x_2 < 0 \tag{4.223}
$$

the displacement jump $\hat{\boldsymbol{d}}(x_1)$, and the dislocation density $\hat{\boldsymbol{d}}'(x_1)$ are

$$
\begin{aligned}
&\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}^{+}(x_1) - \boldsymbol{U}^{-}(x_1) = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}^{+}(x_1) - \boldsymbol{A}\boldsymbol{f}^{-}(x_1)] \\
&\mathrm{i}\hat{\boldsymbol{d}}'(x_1)(x_1) = \mathrm{i}\mathrm{d}\hat{\boldsymbol{d}}(x_1)/\mathrm{d}x_1 = \mathrm{i}2\mathrm{Re}\{\boldsymbol{A}[\boldsymbol{F}(x_1) - \boldsymbol{F}^{-}(x_1)]\} = \boldsymbol{H}[\boldsymbol{h}^{+}(x_1) + \boldsymbol{h}^{-}(x_1)]
\end{aligned} \tag{4.224}
$$

where the auxiliary function $\boldsymbol{h}(z)$ analytic in whole plane except crack. For a homogeneous material $\boldsymbol{H}$ is real. On the crack surface, we have

$$
\boldsymbol{h}^{+}(x_1) + \boldsymbol{h}^{-}(x_1) = -\boldsymbol{T}, \quad |x_1| < a; \quad \boldsymbol{h}(z) = \boldsymbol{B}\boldsymbol{F}(z) \tag{4.225}
$$

4.7.2 Solution of the Strip Electric Saturation Model for an Impermeable Crack

Introduce a new function $\boldsymbol{\xi}(z)$:

$$
\boldsymbol{\xi}(z) = \boldsymbol{H}\boldsymbol{h}(z), \quad \boldsymbol{h}(z) = \boldsymbol{L}\xi(z), \quad \boldsymbol{L} = \boldsymbol{H}^{-1} \tag{4.226}
$$

Substitution of Eq. (4.226) into Eq. (4.225), in terms of component form, yields

$$\begin{aligned} &L_{ik}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+L_{i4}\left[\xi_4^+(x_1)+\xi_4^-(x_1)\right]=-T_i, \quad i,k=1,2,3 \\ &L_{4k}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+L_{44}\left[\xi_4^+(x_1)+\xi_4^-(x_1)\right]=-T_4, \quad |x_1|<a \end{aligned} \tag{4.227}$$

Eliminating $\xi_4^+(x_1)+\xi_4^-(x_1)$ from Eq. (4.227) yields

$$\begin{aligned} &L_{ik}^*\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]=-T_i^*, \quad i,k=1,2,3, \quad |x_1|<a \\ &L_{ik}^*=L_{ik}-L_{i4}L_{4k}/L_{44}, \quad T_i^*=T_i-T_4L_{i4}/L_{44} \end{aligned} \tag{4.228a}$$

Introducing 3D vectors $\boldsymbol{\xi}^*(z), \boldsymbol{T}^*$, etc., the vector form of Eq. (4.228a) is

$$\begin{aligned} &\boldsymbol{L}^*\left[\boldsymbol{\xi}^{*+}(x_1)+\boldsymbol{\xi}^{*-}(x_1)\right]=-\boldsymbol{T}^*, \quad |x_1|<a \\ &\boldsymbol{\xi}^*(z)=\left[\boldsymbol{\xi}_1(z),\boldsymbol{\xi}_2(z),\boldsymbol{\xi}_3(z)\right]^{\mathrm{T}}, \quad \boldsymbol{T}^*=\left[T_1^*,T_2^*,T_3^*\right]^{\mathrm{T}} \end{aligned} \tag{4.228b}$$

The solution of Eq. (4.228) is

$$\boldsymbol{L}^*\boldsymbol{\xi}^*(z)=\boldsymbol{T}^*F_a(z), \quad F_a(z)=(1/2)\left(z\Big/\sqrt{z^2-a^2}-1\right) \tag{4.229}$$

Equations (4.227), (4.222), and (4.226) yield

$$\begin{aligned} &\xi_4^+(x_1)+\xi_4^-(x_1)=-\left\{L_{4k}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+T_4\right\}/L_{44}, \quad |x_1|<a; \quad k=1,2,3 \\ &\xi_4^+(x_1)+\xi_4^-(x_1)=-\left\{L_{4k}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+T_4-D_{\mathrm{s}}\right\}/L_{44}, \quad a\le|x_1|\le c \end{aligned} \tag{4.230}$$

The solution of Eq. (4.230) is

$$\begin{aligned} &\xi_4(z)=\{-L_{4k}\xi_k(z)+T_4F_c(z)+D_{\mathrm{s}}F_D(z)\}/L_{44}; \quad k=1,2,3 \\ &F_c(z)=\frac{1}{2}\left\{\frac{z}{\sqrt{z^2-c^2}}-1\right\} \\ &F_D(z)=\frac{1}{2}-\frac{1}{2\pi\,\mathrm{i}}\ln\frac{z\sqrt{c^2-a^2}+\mathrm{i}a\sqrt{z^2-c^2}}{z\sqrt{c^2-a^2}-\mathrm{i}a\sqrt{z^2-c^2}}-\frac{1}{\pi}\frac{z}{\sqrt{z^2-c^2}}\arccos\frac{a}{c} \end{aligned} \tag{4.231}$$

where $F_c(z), F_D(z)$ is analytic in z plane except a slit $(-c, c)$ and has the following behavior:

$$F_D^+(x_1)+F_D^-(x_1)=\begin{cases}0, & |x_1|<a \\ 1, & a\le|x_1|\le c\end{cases}, \quad F_D(\infty)=0 \tag{4.232}$$

Equations (4.229) and (4.231) give a complete solution of $\boldsymbol{\xi}(z)$.

4.7.3 The Size of the Strip Region and the Stress Intensity Factor

According to $\boldsymbol{\Sigma}_2(x_1) = \boldsymbol{h}^+(x_1) + \boldsymbol{h}^-(x_1)$, the electric displacement in front of the crack is

$$D_2 = L_{4k}\left[\xi_k^+(x_1) + \xi_k^-(x_1)\right] + L_{44}\left[\xi_4^+(x_1) + \xi_4^-(x_1)\right]; \quad |x_1| \geq c \quad k = 1, 2, 3$$

Substitution of Eqs. (4.229) and (4.231) into the above equation yields

$$\begin{aligned} D_2 = 2T_4 f_c'(x_1) + D_\mathrm{s}\left[F_D^+(x_1) + F_D^-(x_1)\right] &= \left(D_2^\infty - \frac{2}{\pi} D_\mathrm{s} \cos^{-1}\frac{a}{c}\right)\frac{x_1}{\sqrt{x_1^2 - c^2}} \\ &- D_2^\infty + D_\mathrm{s}\left(1 - \frac{1}{\pi\,\mathrm{i}}\ln\frac{x_1\sqrt{c^2 - a^2} + \mathrm{i}a\sqrt{x_1^2 - c^2}}{x_1\sqrt{c^2 - a^2} - \mathrm{i}a\sqrt{x_1^2 - c^2}}\right), \quad |x_1| \geq c \end{aligned} \tag{4.233}$$

In order to make D_2 finite, it is necessary that

$$D_2^\infty - (2/\pi) D_\mathrm{s} \arccos(a/c) = 0, \quad \text{or} \quad a/c = \cos\left(\pi D_2^\infty / 2D_\mathrm{s}\right) \tag{4.234}$$

The size of the strip region is $c - a$.

According to $\boldsymbol{\Sigma}_2(x_1) = \boldsymbol{h}^+(x_1) + \boldsymbol{h}^-(x_1)$, the stress in front of the crack on the axis x_1 is

$$\begin{aligned} \sigma_{2i} &= L_{ik}\left[\xi_k^+(x_1) + \xi_k^-(x_1)\right] + L_{i4}\left[\xi_4^+(x_1) + \xi_4^-(x_1)\right] \\ &= L_{ik}^*\left(\xi_k^+ + \xi_k^-\right) + (L_{i4}/L_{44})\left\{D_2^\infty\left(F_c^+ + F_c^-\right) + D_\mathrm{s}\left(F_D^+ + F_D^-\right)\right\} \\ &= T_i^*\left(x_1 \Big/ \sqrt{x_1^2 - c^2} - 1\right) + (L_{i4}/L_{44})\left(D_\mathrm{s} - D_2^\infty\right) \end{aligned} \tag{4.235}$$

It is noted that adding $\boldsymbol{\Sigma}_2^\infty$ to the solution Eq. (4.231), the solution of a free crack under $\boldsymbol{\Sigma}_2^\infty$ at infinity is obtained. In this case, the stress and stress intensity factors are

$$\begin{aligned} &\sigma_{2i} = T_i^* x_1 \Big/ \sqrt{x_1^2 - c^2} + (L_{i4}/L_{44}) D_\mathrm{s} \\ &K_\mathrm{I} = \sqrt{\pi a}\left(\sigma_{22}^\infty - \frac{L_{24}}{L_{44}} D_2^\infty\right), \quad K_\mathrm{II} = \sqrt{\pi a}\left(\sigma_{21}^\infty - \frac{L_{14}}{L_{44}} D_2^\infty\right), \\ &K_\mathrm{III} = \sqrt{\pi a}\left(\sigma_{23}^\infty - \frac{L_{34}}{L_{44}} D_2^\infty\right) \end{aligned} \tag{4.236}$$

and the electric displacement is finite due to electric saturation.

Ru and Mao (1999) discussed the strip electric saturation model for a conducting crack. Their results showed that when the electric loading is parallel to the poling axis, then (1) for a conducting crack perpendicular to the poling axis, in front of the crack tip, a saturation strip is existed and the stresses and electric displacements are all finite. (2) For a conducting crack parallel to the poling axis, behind the crack tip, a saturation strip is existed and the stress intensity factors are identical to those predicted by the linear piezoelectric model and the electric loading does not induce any nonzero stress intensity factor.

4.8 Strip Electric Saturation Model of a Mode-III Interface Crack in a Bimaterial

4.8.1 Fundamental Theory

For a transversely isotropic piezoelectric material with poling direction along axis x_3, plane (x_1, x_2) is isotropic. The mode-III (antiplane shear) problem in a piezoelectric material means that the mechanical loading is applied out of plane (x_1, x_2), but the electric loading is in-plane (x_1, x_2), i.e.,

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2); \quad E_1 = E_1(x_1, x_2), \quad E_2 = E_2(x_1, x_2), \quad E_3 = 0 \tag{4.237}$$

Shen et al. (2000) discussed the strip electric saturation model for a mechanical III-type interface crack. From Eqs. (3.1), (3.2), and (3.3), the governing equations for III-type problem are

$$\begin{aligned} &\sigma_{31,1} + \sigma_{32,2} = 0, \quad D_{1,1} + D_{2,2} = 0 \\ &\sigma_{31} = C_{44}u_{3,1} - e_{15}E_1, \quad \sigma_{32} = C_{44}u_{3,2} - e_{15}E_2, \\ &D_1 = e_{15}u_{3,1} + \epsilon_{11}E_1, \quad D_2 = e_{15}u_{3,2} + \epsilon_{11}E_2 \end{aligned} \tag{4.238}$$

Using $\boldsymbol{E} = -\nabla\varphi$ the equilibrium equation in terms of the generalized displacements is

$$C_{44}\nabla^2 u_3 + e_{15}\nabla^2\varphi = 0, \quad e_{15}\nabla^2 u_3 - \epsilon_{11}\nabla^2\varphi = 0; \quad \text{or} \quad \nabla^2 u_3 = 0, \quad \nabla^2\varphi = 0 \tag{4.239}$$

Figure 4.15 shows a III-type strip electric saturation model for an interface crack of length $2a$ in a bimaterial. The material I and II are located at the upper and lower half planes respectively. Let the boundary conditions are

$$\begin{aligned} &\boldsymbol{\Sigma}_2^\infty = \boldsymbol{0}, \quad |z| \to \infty \\ &\sigma_{23} = -\tau^\infty, \quad D_2 = -D^\infty, \quad |x_1| \le a, \quad x_2 = 0 \\ &\boldsymbol{U}_{\mathrm{I}}(x_1) = \boldsymbol{U}_{\mathrm{II}}(x_1), \quad \boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1) = \boldsymbol{\Sigma}_2(x_1); \quad |x_1| > a, \quad x_2 = 0 \end{aligned} \tag{4.240}$$

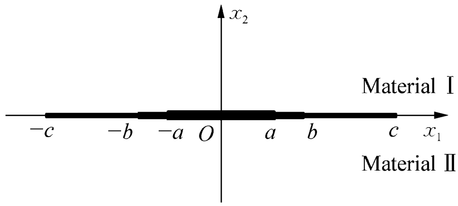


Fig. 4.15 Strip electric saturation model of a mode-III interface crack

where $\boldsymbol{U}_\beta = [U_{\beta 3}, \varphi_\beta]^T$. The single-valued condition is

$$\begin{aligned} &\int_{-a}^{a} \boldsymbol{\Psi}(x_1)\mathrm{d}x_1 = 0, \quad \boldsymbol{\Psi}(x_1) = [\psi_1(x_1), \psi_2(x_1)] \\ &\hat{\boldsymbol{d}}(x_1) = \Delta\boldsymbol{U}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1, 0) - \boldsymbol{U}_{\mathrm{II}}(x_1, 0), \quad \boldsymbol{\Psi}(x_1) = \Delta\boldsymbol{U}'(x_1) \end{aligned} \tag{4.241}$$

where $\boldsymbol{\Psi}(x_1)$ is called the dislocation density. On the connective surface, $\Delta\boldsymbol{U}(x_1) = \mathbf{0}$.

The Fourier transform method is used to solve this problem. For a function $f(x_1, x_2)$, the Fourier transform and the corresponding inverse transform are, respectively,

$$\tilde{f}(s, x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\mathrm{e}^{-\mathrm{i}\,s\,x_1}\mathrm{d}x_1, \quad f(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(s, x_2)\mathrm{e}^{\mathrm{i}\,x_1 s}\mathrm{d}s \tag{4.242}$$

where $f(t)$ is called the original function, $\tilde{f}(s)$ is the image function, and s is a real number. We have

$$\begin{aligned} &\int_{-\infty}^{\infty} f^{(n)}(x_1, x_2)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1 = (\mathrm{i}s)^n\tilde{f}(s, x_2); \quad \text{if} \quad f^{(n)}(x_1, x_2) \to 0, \quad \text{when} \quad \sqrt{x_1^2 + x_2^2} \to \infty \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}^{(n)}(s, x_2)\mathrm{e}^{\mathrm{i}x_1 s}\mathrm{d}s = (-\mathrm{i}x_1)^n f(x_1, x_2); \quad \text{if} \quad \int_{-\infty}^{\infty} \left|x_1^n f(x_1, x_2)\right|\mathrm{d}x_1 < \infty \end{aligned} \tag{4.243}$$

where $f^{(n)} = \partial^n f / \partial x_1^n$. Using Eqs. (4.242) and (4.243), Eq. (3.239) is transformed to

$$\int_{-\infty}^{\infty}\left(\frac{\partial^2\boldsymbol{U}_\beta}{\partial x_1^2} + \frac{\partial^2\boldsymbol{U}_\beta}{\partial x_2^2}\right)\mathrm{e}^{-\mathrm{i}sx_2}\mathrm{d}x_2 = -s^2\tilde{\boldsymbol{U}}_\beta(s, x_2) + \frac{\partial^2\tilde{\boldsymbol{U}}_\beta(s, x_2)}{\partial x_2^2} = 0, \ \beta = \mathrm{I}, \mathrm{II} \tag{4.244}$$

The Fourier transform of the constitutive equation in Eq. (4.238) is

$$\tilde{\boldsymbol{\Sigma}}_{\beta 2} = \begin{Bmatrix} \tilde{\sigma}_{\beta 23} \\ \tilde{D}_{\beta 2} \end{Bmatrix} = \boldsymbol{B}_\beta \frac{\partial\tilde{\boldsymbol{U}}_\beta(s, x_2)}{\partial x_2}, \quad \tilde{\boldsymbol{U}}_\beta = \begin{Bmatrix} \tilde{u}_{\beta 3} \\ \tilde{\varphi}_\beta \end{Bmatrix}, \quad \boldsymbol{B}_\beta = \begin{bmatrix} C_{\beta 44} & e_{\beta 15} \\ e_{\beta 15} & -\epsilon_{\beta 11} \end{bmatrix}, \quad \beta = \mathrm{I}, \mathrm{II} \tag{4.245}$$

Because $\tilde{\boldsymbol{U}}_\beta$ is finite at infinity, the solution of Eq. (4.244) takes the following form:

$$\begin{aligned} &\tilde{\boldsymbol{U}}_{\mathrm{I}}(s,x_2)=\mathrm{e}^{sx_2}\boldsymbol{G}_{\mathrm{I}}(s),\quad \text{if}\quad s<0;\qquad \tilde{\boldsymbol{U}}_{\mathrm{I}}(s,x_2)=\mathrm{e}^{-sx_2}\boldsymbol{F}_{\mathrm{I}}(s),\quad \text{if}\quad s>0\\ &\tilde{\boldsymbol{U}}_{\mathrm{II}}(s,x_2)=\mathrm{e}^{-sx_2}\boldsymbol{F}_{\mathrm{II}}(s),\quad \text{if}\quad s<0;\quad \tilde{\boldsymbol{U}}_{\mathrm{II}}(s,x_2)=\mathrm{e}^{sx_2}\boldsymbol{G}_{\mathrm{II}}(s),\quad \text{if}\quad s>0 \end{aligned} \tag{4.246}$$

where $G_{\mathrm{I}}\,(s)$, $F_{\mathrm{I}}\,(s)$, $G_{\mathrm{II}}\,(s)$, $F_{\mathrm{II}}\,(s)$ are undetermined functions. The generalized stress can be expressed by

$$\begin{aligned} &\tilde{\boldsymbol{\Sigma}}_{\mathrm{I}2}(s,x_2)=\boldsymbol{R}_{\mathrm{I}}\tilde{\boldsymbol{U}}_{\mathrm{I}}(s,x_2);\quad \boldsymbol{R}_{\mathrm{I}}=s\boldsymbol{B}_{\mathrm{I}},\quad \text{if}\quad s<0;\quad \boldsymbol{R}_{\mathrm{I}}=-s\boldsymbol{B}_{\mathrm{I}},\quad \text{if}\quad s>0\\ &\tilde{\boldsymbol{\Sigma}}_{\mathrm{II}2}(s,x_2)=\boldsymbol{R}_{\mathrm{II}}\tilde{\boldsymbol{U}}_{\mathrm{II}}(s,x_2);\quad \boldsymbol{R}_{\mathrm{II}}=-sB_{\mathrm{II}},\quad \text{if}\quad s<0;\quad \boldsymbol{R}_{\mathrm{II}}=s\boldsymbol{B}_{\mathrm{II}},\quad \text{if}\quad s>0 \end{aligned} \tag{4.247}$$

It is known from Eq. (4.240) that on whole axis x_1, the generalized stress is continuous, so

$$\boldsymbol{R}_{\mathrm{I}}(s)\tilde{\boldsymbol{U}}_{\mathrm{I}}(s,0)=\boldsymbol{R}_{\mathrm{II}}(s)\tilde{\boldsymbol{U}}_{\mathrm{II}}(s,0);\quad |x_1|<\infty,\quad x_2=0 \tag{4.248}$$

The Fourier transform of Eq. (4.241) is

$$\begin{aligned} &\int_{-\infty}^{\infty}\boldsymbol{\Psi}(x_1)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1=\mathrm{i}s\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s),\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s)=\tilde{\boldsymbol{U}}_{\mathrm{I}}(s)-\tilde{\boldsymbol{U}}_{\mathrm{II}}(s)\\ &\quad=-(\mathrm{i}/s)\int_{-a}^{a}\boldsymbol{\Psi}(x_1)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1 \end{aligned} \tag{4.249}$$

Combining Eqs. (4.248) and (4.249) yields

$$\begin{aligned} &\tilde{\boldsymbol{U}}_{\mathrm{I}}=\boldsymbol{P}_{\mathrm{I}}\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s),\quad \tilde{\boldsymbol{U}}_{\mathrm{II}}=\boldsymbol{P}_{\mathrm{II}}\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s);\quad \boldsymbol{P}_{\mathrm{I}}=\frac{\boldsymbol{R}_{\mathrm{II}}}{\boldsymbol{R}_{\mathrm{II}}-\boldsymbol{R}_{\mathrm{I}}}=\frac{\boldsymbol{B}_{\mathrm{II}}}{\boldsymbol{B}_{\mathrm{II}}+\boldsymbol{B}_{\mathrm{I}}},\\ &\boldsymbol{P}_{\mathrm{II}}=\frac{\boldsymbol{R}_{\mathrm{I}}}{\boldsymbol{R}_{\mathrm{II}}-\boldsymbol{R}_{\mathrm{I}}}=-\frac{\boldsymbol{B}_{\mathrm{I}}}{\boldsymbol{B}_{\mathrm{II}}+\boldsymbol{B}_{\mathrm{I}}} \end{aligned} \tag{4.250}$$

Combining Eqs. (4.247) and (4.250) and inversely transforming the obtained results yield

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1,0)&=(1/2\pi)\int_{-\infty}^{\infty}\tilde{\boldsymbol{\Sigma}}_{\mathrm{II}2}(s,x_2)\mathrm{e}^{\mathrm{i}x_1s}\mathrm{d}s=(1/2\pi)\int_{-\infty}^{\infty}\boldsymbol{R}_{\mathrm{II}}\boldsymbol{P}_{\mathrm{II}}\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s)\mathrm{e}^{\mathrm{i}x_1s}\mathrm{d}s\\ &=-(1/2\pi)\int_{-\infty}^{\infty}\boldsymbol{R}_{\mathrm{II}}\boldsymbol{P}_{\mathrm{II}}\left[(\mathrm{i}/s)\int_{-a}^{a}\boldsymbol{\Psi}(t)\mathrm{e}^{-\mathrm{i}s\xi}\mathrm{d}t\right]\mathrm{e}^{\mathrm{i}x_1s}\mathrm{d}s\\ &=-(\mathrm{i}/2\pi)\int_{-a}^{a}\left[\int_{-\infty}^{\infty}(1/s)\boldsymbol{R}_{\mathrm{II}}(s)\boldsymbol{P}_{\mathrm{II}}(s)\mathrm{e}^{-\mathrm{i}s(\xi-x_1)}\mathrm{d}s\right]\boldsymbol{\Psi}(\xi)\mathrm{d}\xi \end{aligned} \tag{4.251}$$

And for a certain s, the following relations hold:

$$\frac{1}{s}\boldsymbol{R}_{\mathrm{II}}(s)\boldsymbol{P}_{\mathrm{II}}(s)=\frac{1}{s}\frac{\boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}}{\boldsymbol{R}_{\mathrm{II}}-\boldsymbol{R}_{\mathrm{I}}}=-\frac{s}{|s|}\boldsymbol{V},\quad \boldsymbol{V}=\frac{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{I}}}{\boldsymbol{B}_{\mathrm{I}}+\boldsymbol{B}_{\mathrm{II}}} \tag{4.252}$$

$\boldsymbol{V}$ is a real symmetric matrix. Using the following formula:

$$\int_{-\infty}^{\infty}\frac{s}{|s|}\mathrm{e}^{-\mathrm{i}s(t-x_1)}\mathrm{d}s=-\frac{2\mathrm{i}}{t-x_1},\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}s(t-x_1)}\mathrm{d}s=\delta(t-x_1) \tag{4.253}$$

we can get the solution of Eq. (4.251) as

$$(1/\pi)\boldsymbol{V}\int_{-a}^{a}[\boldsymbol{\Psi}(\xi)/(\xi-x_1)]\mathrm{d}\xi=\boldsymbol{\Sigma}_{\mathrm{II}2}(x_1,0)=[t_{01}(x_1),t_{02}(x_1)]^{\mathrm{T}},\quad |x_1|<\infty,\quad x_2=0 \tag{4.254}$$

4.8.2 Solution for Longer Electric Saturation Size

The strip electric saturation model of a mode-III interface crack in a bimaterial is that: Let c and b are the right ends of the electric saturation and mechanical yielding regions respectively and $c>b$, the following boundary conditions are assumed (Fig. 4.15):

$$t_{01}(x_1)=\begin{cases}-\tau^{\infty}, & \text{if}\quad |x_1|<a\\ -\tau^{\infty}+\tau_{\mathrm{s}} & \text{if}\quad a<|x_1|<b\end{cases};\quad t_{02}(x_1)=\begin{cases}-D^{\infty}, & \text{if}\quad |x_1|<a\\ -D^{\infty}+D_{\mathrm{s}} & \text{if}\quad a<|x_1|<c\end{cases}$$
$$\psi_1(x_1)=0,\quad |x_1|>b;\quad \psi_2(x_1)=0,\quad |x_1|>c;\quad c>b \tag{4.255}$$

where τ_{s} is the yielding stress, D_{s} is the saturation electric displacement, and they take the smaller values of materials I and II. Equation (4.254) yields

$$\begin{aligned}&(1/\pi)\int_{-b}^{b}[\psi_1(t)/(t-x_1)]\mathrm{d}t=G_{1j}t_{0j}(x_1),\quad |x_1|<b\\ &(1/\pi)\int_{-l}^{l}\left[V_{2j}\psi_j(t)/(t-x_1)\right]\mathrm{d}t=t_{02}(x_1),\quad |x_1|<c,\quad j=1,2\end{aligned} \tag{4.256}$$

where $\boldsymbol{G}=\boldsymbol{V}^{-1}$. Because the stress is not singular at $x_1=\pm b, \psi_1(t)$ must be finite at $x_1=\pm b$. Analogously, the electric displacement is not singular at $x_1=\pm c; H_{2j}\psi_j(t)$ must be finite at $x_1=\pm c$. Since the first and second equations in Eq. (4.256) are solvable, the following conditions should be satisfied, respectively (Muskhelishvili 1975; Hou et al. 1990; Barnett and Asaro 1972):

$$\int_{-b}^{b}\left[G_{1j}t_{0j}(\xi)\Big/\sqrt{b^2-\xi^2}\right]\mathrm{d}\xi=0,\quad \int_{-c}^{c}\left[t_{02}(\xi)\Big/\sqrt{c^2-\xi^2}\right]\mathrm{d}\xi=0 \tag{4.257}$$

Using

$$\begin{aligned}\int_{-b}^{b}\frac{G_{1j}t_{0j}(\xi)}{\sqrt{b^2-\xi^2}}\mathrm{d}\xi&=[G_{11}(\tau_{\mathrm{s}}-\tau^{\infty})+G_{12}(D_{\mathrm{s}}-D^{\infty})]\left(\int_{-b}^{-a}\frac{\mathrm{d}\xi}{\sqrt{b^2-\xi^2}}+\int_{a}^{b}\frac{\mathrm{d}\xi}{\sqrt{b^2-\xi^2}}\right)\\&-[G_{11}\tau^{\infty}+G_{12}D^{\infty}]\int_{-a}^{a}\frac{\mathrm{d}\xi}{\sqrt{b^2-\xi^2}}\\&=[G_{11}(\tau_{\mathrm{s}}-\tau^{\infty})+G_{12}(D_{\mathrm{s}}-D^{\infty})][\pi-2\arcsin(a/b)]-[G_{11}\tau^{\infty}+G_{12}D^{\infty}]2\arcsin(a/b)\end{aligned}$$

and $\arcsin(a/b)=\pi/2-\arccos(a/b)$, from the first equation of Eq. (4.257),we get the size of the plastic region:

$$b/a=\sec[\pi(G_{11}\tau^{\infty}+G_{12}D^{\infty})/2(G_{11}\tau_{\mathrm{s}}+G_{12}D_{\mathrm{s}})] \tag{4.258}$$

Analogously, the size of the electric saturation region is

$$c/a=\sec(\pi D^{\infty}/2D_{\mathrm{s}}) \tag{4.259}$$

From Eqs. (4.258) and (4.259), it is known that $c/a>b/a,$ if $D^{\infty}/D_{\mathrm{s}}>\tau^{\infty}/\tau_{\mathrm{s}}$.

Under condition Eq. (4.257), the solution of the first equation in Eq. (4.256) is

$$\begin{aligned}\psi_1(x_1)&=\frac{1}{\pi}\sqrt{b^2-x_1^2}\int_{-b}^{b}\frac{G_{1j}t_{0j}(\xi)}{\sqrt{b^2-\xi^2}(\xi-x_1)}\mathrm{d}\xi\\&=(1/\pi)(G_{11}\tau_{\mathrm{s}}+G_{12}D_{\mathrm{s}})[\omega(x_1,a,b)-\omega(-x_1,a,b)],\quad |x_1|<b\\\omega(x_1,a,b)&=\operatorname{ar\,cosh}\left|\frac{b^2-a^2}{b(a-x_1)}+\frac{a}{b}\right|\end{aligned} \tag{4.260}$$

and the solution of the second equation in Eq. (4.256) is

$$V_{2j}\psi_j(x_1)=\frac{1}{\pi}\sqrt{c^2-x_1^2}\int_{-c}^{c}\frac{t_{02}(\xi)}{\sqrt{c^2-\xi^2}(\xi-x_1)}\mathrm{d}\xi=\frac{D_{\mathrm{s}}}{\pi}[\omega(x_1,a,c)-\omega(-x_1,a,c)] \tag{4.261}$$

Equation (4.261) yields

$$\psi_2(x_1)=\frac{D_{\mathrm{s}}}{\pi V_{22}}[\omega(x_1,a,c)-\omega(-x_1,a,c)]-\frac{V_{21}}{V_{22}}\psi_1(x_1),\quad x_1<c \tag{4.262}$$

The generalized crack opening displacements are

$$\begin{aligned}
&\Delta u_3(x_1) = -\int_b^{x_1} \psi_1(\xi)\mathrm{d}\xi, \quad \Delta\varphi(x_1) = -\int_b^{x_1} \psi_2(\xi)\mathrm{d}\xi \\
&\Delta u_3(x_1) = \frac{G_{11}\tau_s + G_{12}D_\mathrm{s}}{\pi}[(a-x_1)\omega(x_1,a,b) + (a+x_1)\omega(-x_1,a,b)], \quad |x_1| < b \\
&\Delta\varphi(x_1) = \frac{D_\mathrm{s}}{\pi V_{22}}[(a-x_1)\omega(x_1,a,c) + (a+x_1)\omega(-x_1,a,c)] - \frac{V_{21}}{V_{22}}\Delta u_3(x_1), \quad |x_1| < c
\end{aligned} \tag{4.263}$$

The generalized crack tip opening displacements are

$$\begin{aligned}
\Delta u_3(a) &= \frac{2a}{\pi}(G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s}) \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{11}\tau^\infty + G_{12}D^\infty}{G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s}}\right)\right] \\
\Delta\varphi(a) &= \frac{2aD_\mathrm{s}}{\pi V_{22}} \ln\left[\sec\left(\frac{\pi D^\infty}{2D_\mathrm{s}}\right)\right] - \frac{V_{21}}{V_{22}}\Delta u_3(a)
\end{aligned} \tag{4.264}$$

The energy release rate is

$$\begin{aligned}
J = \tau_\mathrm{s}\Delta u_3(a) + D_\mathrm{s}\Delta\varphi(a) = \frac{2a}{\pi}\left\{\left(\tau_\mathrm{s} - \frac{V_{21}}{V_{22}}D_\mathrm{s}\right)(G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s})\right. \\
\left.\times \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{11}\tau^\infty + G_{12}D^\infty}{G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s}}\right)\right] + \frac{D_\mathrm{s}^2}{V_{22}} \ln\left[\sec\left(\frac{\pi D^\infty}{2D_\mathrm{s}}\right)\right]\right\}
\end{aligned} \tag{4.265}$$

For the small-scale saturation and yielding, we have $c/a \sim b/a \sim 1$, so

$$J = \frac{\pi a}{4}[\tau^\infty \;\; D^\infty][\boldsymbol{G}]\begin{Bmatrix} \tau^\infty \\ D^\infty \end{Bmatrix} \tag{4.266}$$

It is also noted that all the singular integrals are in the sense of the Cauchy principle value.

4.8.3 Solution for Longer Mechanical Yielding Size

In this case, the size of the mechanical yielding region is c and the size of the electric saturation region is b and $c > b$. Equation (4.254) yields

$$\begin{aligned}
&(1/\pi)\int_{-b}^{b} [\psi_2(t)/(t-x_1)]\mathrm{d}t = G_{2j}t_{0j}(x_1), \quad |x_1| < b \\
&(1/\pi)\int_{-l}^{l} \left[V_{1j}\psi_j(t)/(t-x_1)\right]\mathrm{d}t = t_{01}(x_1), \quad |x_1| < c, \quad j = 1,2
\end{aligned} \tag{4.267}$$

The sizes of the yielding and saturating regions are, respectively,

$$\frac{b}{a} = \sec\left(\frac{\pi}{2}\frac{G_{21}\tau^{\infty} + G_{22}D^{\infty}}{G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}}\right), \quad \frac{c}{a} = \sec\left(\frac{\pi\tau^{\infty}}{2\tau_{\mathrm{s}}}\right) \tag{4.268}$$

The generalized crack tip opening displacements are

$$\begin{aligned} \Delta\varphi(a) &= \frac{2a}{\pi}(G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}) \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{21}\tau^{\infty} + G_{22}D^{\infty}}{G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}}\right)\right] \\ \Delta u_3(a) &= \frac{2a\tau_{\mathrm{s}}}{\pi V_{11}} \ln\left[\sec\left(\frac{\pi\tau^{\infty}}{2\tau_{\mathrm{s}}}\right)\right] - \frac{V_{12}}{V_{11}}\Delta\varphi(a) \end{aligned} \tag{4.269}$$

The energy release rate is

$$\begin{aligned} J = \frac{2a}{\pi}\Bigg\{ &\left(D_{\mathrm{s}} - \frac{V_{12}}{V_{11}}\tau_{\mathrm{s}}\right)(G_{11}\tau_{\mathrm{s}} + G_{12}D_{\mathrm{s}}) \\ &\times \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{21}\tau^{\infty} + G_{22}D^{\infty}}{G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}}\right)\right] + \frac{\tau_{\mathrm{s}}^2}{V_{11}} \ln\left[\sec\left(\frac{\pi\tau^{\infty}}{2\tau_{\mathrm{s}}}\right)\right]\Bigg\} \end{aligned} \tag{4.270}$$

4.9 Mode-III Problem for a Circular Inclusion with Interface Cracks

4.9.1 Fundamental Equations

The generalized equilibrium and constitutive equations of a mode-III problem (antiplane shear) are shown in Eq. (4.238), and the equilibrium equations in terms of generalized displacements are shown in Eq. (4.239), i.e.,

$$\nabla^2 u_3 = 0, \quad \nabla^2\varphi = 0 \tag{4.271}$$

where ∇^2 is the 2D Laplace operator. Introduce two analytical functions $\phi_1(z)$ and $\phi_2(z)$. Let

$$\begin{aligned} &u_3(x_1, x_2) = \left[\phi_1(z) + \overline{\phi_1(z)}\right], \quad \varphi = \left[\phi_2(z) + \overline{\phi_2(z)}\right]; \\ &z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta}, \quad z_{,\theta} = \mathrm{i}z \end{aligned} \tag{4.272}$$

Note

$$u_{3,\theta} = u_{3,z}z_{,\theta} + u_{3,\bar{z}}\bar{z}_{,\theta} = \mathrm{i}\left[z\phi_1'(z) - \overline{z\phi_1'(z)}\right], \quad \varphi_{,\theta} = \mathrm{i}\left[z\phi_2'(z) - \overline{z\phi_2'(z)}\right] \tag{4.273}$$

where $\phi_i'(z) = \mathrm{d}\phi_i(z)/\mathrm{d}z$. Equations (4.272) and (4.273) yield

$$\begin{aligned}
&\sigma_{31} - \mathrm{i}\sigma_{32} = 2\left[G\phi_1'(z) + e_{15}\phi_2'(z)\right], \quad D_1 - \mathrm{i}D_2 = 2\left[e_{15}\phi_1'(z) - \epsilon_{11}\phi_2'(z)\right] \\
&\sigma_{3r} - \mathrm{i}\sigma_{3\theta} = 2\mathrm{e}^{\mathrm{i}\theta}\left[G\phi_1'(z) + e_{15}\phi_2'(z)\right], \quad D_r - \mathrm{i}D_\theta = 2\mathrm{e}^{\mathrm{i}\theta}\left[e_{15}\phi_1'(z) - \epsilon_{11}\phi_2'(z)\right] \\
&E_1 - \mathrm{i}E_2 = -2\phi_2'(z), \quad E_r - iE_\theta = -2\mathrm{e}^{\mathrm{i}\theta}\phi_2'(z)
\end{aligned} \tag{4.274}$$

Let

$$\begin{aligned}
&\boldsymbol{f}(z) = \begin{Bmatrix} \phi_1(z) \\ \phi_2(z) \end{Bmatrix}, \quad \boldsymbol{F}(z) = \begin{Bmatrix} \phi_1'(z) \\ \phi_2'(z) \end{Bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} G & e_{15} \\ e_{15} & -\epsilon_{11} \end{bmatrix}, \\
&\boldsymbol{\Sigma}_r = \begin{Bmatrix} \sigma_{3r} \\ D_r \end{Bmatrix}, \quad \boldsymbol{U}_{,\theta} = \begin{Bmatrix} u_{3,\theta}/r \\ -E_\theta \end{Bmatrix}
\end{aligned} \tag{4.275}$$

Notations used in this section may be different with other sections. In Eq. (4.275), $\boldsymbol{B}$ is real, so $\boldsymbol{B} = \bar{\boldsymbol{B}}$. Adopting notations in Eq. (4.275) yields

$$\boldsymbol{\Sigma}_r = \left\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{B}\boldsymbol{F}(z) + \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{B}\boldsymbol{F}(z)}\right\}, \quad \boldsymbol{U}_{,\theta} = \mathrm{i}\left\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{F}(z) - \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{F}(z)}\right\} \tag{4.276}$$

On the interface, Eq. (4.276) is reduced to

$$\boldsymbol{\Sigma}_r = (z/a)\left\{\boldsymbol{B}\boldsymbol{F}(z) + (a/z)^2\overline{\boldsymbol{B}\boldsymbol{F}(z)}\right\}, \quad \boldsymbol{U}_{,\theta} = \mathrm{i}(z/a)\left\{\boldsymbol{F}(z) - (a/z)^2\overline{\boldsymbol{F}(z)}\right\}; \quad z \in L \tag{4.277}$$

4.9.2 Permeable Crack

Figure 4.16a shows an infinite matrix II occupied region S^- including a circular inclusion I of radius a occupied region S^+. Materials I and II are all transversely isotropic. The entire interface is denoted by L and there are n circular arc cracks on it. The ends of cracks are successively counterclockwise denoted by a_k, b_k and its whole is denoted by L_c. The origin of the coordinate system (x_1, x_2) or (r, θ) is selected at the center of the inclusion. The boundary conditions are

$$\begin{aligned}
&\sigma_{31} = \sigma_{31}^\infty, \quad \sigma_{32} = \sigma_{32}^\infty, \quad D_2 = D_2^\infty, \quad D_1 = D_1^\infty; \quad |z| \to \infty \\
&\sigma_{\mathrm{I}r3} = \sigma_{\mathrm{II}r3} = 0, \quad D_{\mathrm{I}r} = D_{\mathrm{II}r}, \quad \varphi_\mathrm{I} = \varphi_\mathrm{II}\ (E_{\mathrm{I}\theta} = E_{\mathrm{II}\theta}); \quad z \in L_\mathrm{c} \\
&\sigma_{\mathrm{I}r3} = \sigma_{\mathrm{II}r3}, \quad D_{\mathrm{I}r} = D_{\mathrm{II}r}, \quad u_{\mathrm{I}3} = u_{\mathrm{II}3}\ \left(u_{\mathrm{I}3,\theta} = u_{\mathrm{II}3,\theta}\right), \quad \varphi_\mathrm{I} = \varphi_\mathrm{II}(E_{\mathrm{I}\theta} = E_{\mathrm{II}\theta}); \\
&\quad z \in L - L_\mathrm{c}
\end{aligned} \tag{4.278}$$

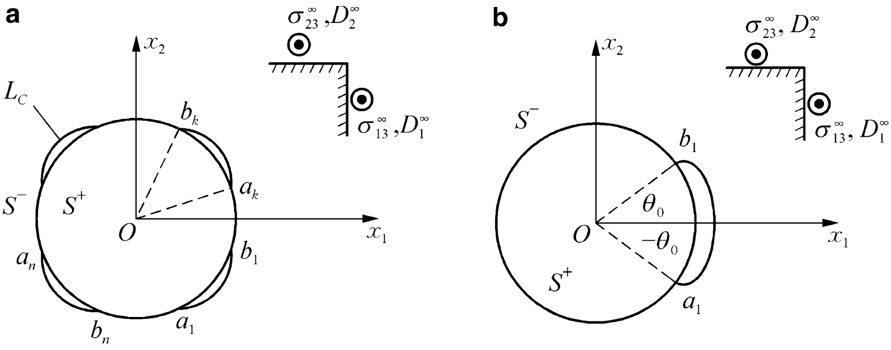


Fig. 4.16 A circular interface inclusions with interface cracks: (**a**) general case and (**b**) one crack

For convenience the following mapping function is used:

$$z = \omega(\varsigma) = a\varsigma, \quad z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta}, \quad \varsigma = \xi + \mathrm{i}\eta = R\mathrm{e}^{\mathrm{i}\theta}; \quad r = aR \tag{4.279}$$

Under this transformation, the circle with radius a in z plane is transformed to a unit circle in ς plane and L, L_c is transformed to Γ, Γ_c, respectively. In ς plane, the matrix is located in the region S^-, $|\varsigma| > 1$. The inclusion is located in the region S^+, $|\varsigma| < 1$. The ends of cracks are all on the unit circle and denoted by $\sigma_k^{(1)}, \sigma_k^{(2)}$ in the ς plane. It is noted that

$$\boldsymbol{f}(z) = \boldsymbol{f}[\omega(\varsigma)] = \boldsymbol{f}(\varsigma), \quad \boldsymbol{F}(z) = \boldsymbol{f}'(z) = \boldsymbol{f}'(\varsigma)/\omega'(\varsigma) = \boldsymbol{F}(\varsigma)/a \tag{4.280}$$

In ς plane, Eq. (4.276) is reduced to

$$\boldsymbol{\Sigma}_r = (1/a)\left[\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{BF}(\varsigma) + \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{BF}(\varsigma)}\right], \quad \boldsymbol{U}_{,\theta} = (\mathrm{i}/a)\left\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{F}(\varsigma) - \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{F}(\varsigma)}\right\} \tag{4.281}$$

On the interface Γ, $\sigma = \mathrm{e}^{\mathrm{i}\theta}$. Equation (4.277) is reduced to

$$\begin{aligned} &\boldsymbol{\Sigma}_r = (1/a)\left[\sigma\boldsymbol{BF}(\sigma) + \bar{\sigma}\overline{\boldsymbol{BF}(\sigma)}\right], \quad \boldsymbol{U}_{,\theta} = (\mathrm{i}/a)\left[\sigma\boldsymbol{F}(\sigma) - \bar{\sigma}\overline{\boldsymbol{F}(\sigma)}\right] \\ &E_r - iE_\theta = -(2/a)\sigma\phi_2'(\sigma), \quad E_r = -(2/a)\mathrm{Re}\left[\sigma\phi_2'(\sigma)\right], \quad \sigma \in \Gamma \end{aligned} \tag{4.282}$$

4.9.3 Reduced to Riemann-Hilbert Equation

According to Eq. (4.278) on whole interface, $\boldsymbol{\Sigma}_{\mathrm{I}r} = \boldsymbol{\Sigma}_{\mathrm{II}r}$, so Eq. (4.282) yields

$$\sigma\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(\sigma) + \bar{\sigma}\bar{\boldsymbol{B}}_{\mathrm{I}}\overline{\mathbf{F}_{\mathrm{I}}(\sigma)} = \sigma\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(\sigma) + \bar{\sigma}\bar{\boldsymbol{B}}_{\mathrm{II}}\overline{\mathbf{F}_{\mathrm{II}}(\sigma)}; \quad \varsigma \in \Gamma \tag{4.283}$$

For a unit circular region, if $\mathrm{g}(\varsigma)$ is analytic in $S^+(S^-)$, $\mathrm{g}_*(\varsigma) = \bar{\mathrm{g}}(1/\varsigma)$ is analytic in $S^-(S^+)$ (Muskhelishvili 1954) and

$$\mathrm{g}_*^-(\sigma) = \bar{\mathrm{g}}^+(\bar{\sigma}), \quad \mathrm{g}_*^+(\sigma) = \bar{\mathrm{g}}^-(\bar{\sigma}), \quad \mathrm{g}_*(\varsigma) = \bar{\mathrm{g}}(1/\varsigma) \tag{4.284}$$

Rewrite Eq. (4.283) as

$$\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^+(\sigma) - \bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}*}^+(\sigma) = \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^-(\sigma) - \bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}^-; \quad \sigma \in \Gamma \tag{4.285}$$

Now research the behavior of $\boldsymbol{F}_\alpha(\varsigma)(\alpha = \mathrm{I}, \mathrm{II})$. Denote $\boldsymbol{F}_{\mathrm{I}}(\varsigma)$ is analytic in S^+ and rewritten as $\boldsymbol{F}_{\mathrm{I}0}(\varsigma)$; $\boldsymbol{F}_{\mathrm{II}}(\varsigma)$ is analytic in S^- except at infinite and can be expressed as

$$\boldsymbol{F}_{\mathrm{II}}(\varsigma) = \boldsymbol{F}_{\mathrm{II}}^\infty + \boldsymbol{F}_{\mathrm{II}0}(\varsigma), \quad \boldsymbol{F}_{\mathrm{II}}^\infty = \boldsymbol{F}_{\mathrm{II}}(\infty), \quad \varsigma \in S^- \tag{4.286}$$

Because there is no generalized force and dislocation in a finite region, from Eqs. (4.274), (4.275), and (4.278), it is easy to obtain

$$\boldsymbol{F}_{\mathrm{II}}(\infty) = \frac{a}{2}\boldsymbol{B}_{\mathrm{II}}^{-1}\begin{Bmatrix} \sigma_{31}^\infty - \mathrm{i}\sigma_{32}^\infty \\ D_1^\infty - \mathrm{i}D_2^\infty \end{Bmatrix} \tag{4.287}$$

In Eq. (4.285), $\bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}*}^+(\sigma)$ is the boundary value on Γ of the function $(1/\varsigma^2)\bar{\boldsymbol{B}}_{\mathrm{II}}$ $\boldsymbol{F}_{\mathrm{II}*}(\varsigma) = (1/\varsigma^2)\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(1/\varsigma)$ which is analytic in S^+ except the pole point $\varsigma = 0$. $\bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}^-$ is the boundary value on Γ of the function $(1/\varsigma^2)\overline{\boldsymbol{B}}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}(\varsigma) = (1/\varsigma^2)\overline{\boldsymbol{B}}_{\mathrm{I}}\overline{\boldsymbol{F}}_{\mathrm{I}}$ $(1/\varsigma)$ which is analytic in S^-. These two functions can be analytic continuation through the connective parts on Γ. The function after analytic continuation and the original function must possess the same pole points and values at infinity. Let

$$\begin{aligned} &\boldsymbol{G}_{\mathrm{II}}(\varsigma) = \boldsymbol{F}_{\mathrm{II}*}(\varsigma)/\varsigma^2 = \bar{\boldsymbol{F}}_{\mathrm{II}}^\infty/\varsigma^2 + \boldsymbol{G}_{\mathrm{II}0}(\varsigma), \quad \boldsymbol{G}_{\mathrm{II}0}(\varsigma) = \bar{\boldsymbol{F}}_{\mathrm{II}0}(1/\varsigma)/\varsigma^2, \quad \varsigma \in S^+ \\ &\boldsymbol{G}_{\mathrm{I}0}(\varsigma) = \boldsymbol{F}_{\mathrm{I}*}(\varsigma)/\varsigma^2 = \bar{\boldsymbol{F}}_{\mathrm{I}0}(1/\varsigma)/\varsigma^2, \quad \varsigma \in S^- \end{aligned} \tag{4.288}$$

Using $\boldsymbol{B}_{\mathrm{I}} = \bar{\boldsymbol{B}}_{\mathrm{I}},\ \boldsymbol{B}_{\mathrm{II}} = \bar{\boldsymbol{B}}_{\mathrm{II}}$, it can be assumed

$$\begin{aligned} &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(\varsigma) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{G}_{\mathrm{II}}(\varsigma) = \mathbf{g}(\varsigma), \quad \varsigma \in S^+ \\ &\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(\varsigma) - \boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}0}(\varsigma) = \mathbf{g}(\varsigma), \quad \varsigma \in S^- \\ &\mathbf{g}(\varsigma) = -\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^\infty/\varsigma^2 + \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^\infty \end{aligned} \tag{4.289}$$

Substituting Eqs. (4.286) and (4.288) into Eq. (4.289) yield

$$\begin{aligned} &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(\varsigma) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{G}_{\mathrm{II}0}(\varsigma) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^\infty = 0, \quad \varsigma \in S^+ \\ &\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(\varsigma) - \boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}0}(\varsigma) + (1/\varsigma^2)\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^\infty = 0, \quad \varsigma \in S^- \end{aligned} \tag{4.290}$$

On the interface, we have

$$\boldsymbol{G}_{\mathrm{II}0}(\sigma)=-\boldsymbol{F}_{\mathrm{II}}^{\infty}+\boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(\sigma),\quad \boldsymbol{G}_{\mathrm{I}0}(\sigma)=\bar{\sigma}^2\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(\sigma) \tag{4.291}$$

According to Eq. (4.282), the jump $\hat{\boldsymbol{d}}'$ of the direction derivative $U_{,\theta}$ is

$$\begin{aligned}
\hat{\boldsymbol{d}}' &= \left(\boldsymbol{U}_{\mathrm{I},\theta}-\boldsymbol{U}_{\mathrm{II},\theta}\right)=\mathrm{i}(\sigma/a)\left\{\left[\boldsymbol{F}_{\mathrm{I}}(\sigma)-\bar{\sigma}^2\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{\sigma})\right]-\left[\boldsymbol{F}_{\mathrm{II}}(\sigma)-\bar{\sigma}^2\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{\sigma})\right]\right\},\quad \text{or}\\
(a/\sigma)\hat{\boldsymbol{d}}' &= \mathrm{i}\left[\boldsymbol{F}_{\mathrm{I}}(\sigma)+\bar{\sigma}^2\boldsymbol{F}_{\mathrm{II}*}^{+}(\sigma)\right]-\left[\boldsymbol{F}_{\mathrm{II}}(\sigma)+\bar{\sigma}^2\boldsymbol{F}_{\mathrm{I}*}^{-}(\sigma)\right]\\
&= \mathrm{i}\left[\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma)-2\boldsymbol{F}_{\mathrm{II}}^{\infty}+\boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma)\right]\\
&\quad -\mathrm{i}\left[\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma)-\bar{\sigma}^2\left(\mathbf{I}-\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma)\right]
\end{aligned} \tag{4.292}$$

Construct a function $\boldsymbol{h}(\varsigma)$ analytic in whole plane except cracks and $\varsigma=0$ by the analytic continuation method through $\Gamma-\Gamma_{\mathrm{c}}$:

$$\boldsymbol{h}(\varsigma)=\begin{cases}\boldsymbol{F}_{\mathrm{I}0}(\varsigma)+\boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(\varsigma)-2\,\boldsymbol{F}_{\mathrm{II}}^{\infty}\\ \boldsymbol{F}_{\mathrm{II}0}(\varsigma)+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(\varsigma)-\left(1/\varsigma^2\right)\left(\mathbf{I}-\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\,\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\end{cases} \tag{4.293}$$

According to Eqs. (4.282) and (4.290), on the crack surface, we have

$$\begin{aligned}
(a/\sigma)\boldsymbol{\Sigma}_{\mathrm{I}\,r} &= \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(\sigma)+\bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{\sigma})=\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{+}(\sigma)+\bar{\sigma}^2\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}^{-}(\sigma)\\
&= \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{+}(\sigma)+\bar{\sigma}^2\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}+\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma),\quad \varsigma\in L_{\mathrm{c}}
\end{aligned} \tag{4.294}$$

Equation (4.293) yields

$$\begin{aligned}
\boldsymbol{h}^{+}(\sigma)+\boldsymbol{h}^{-}(\sigma) &= \left(\mathbf{I}+\boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\right)\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma)-2\boldsymbol{F}_{\mathrm{II}}^{\infty}+\left(\mathbf{I}+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\boldsymbol{F}_{\mathrm{II}0}^{-}-\bar{\sigma}^2\left(\mathbf{I}-\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\\
&= \boldsymbol{H}\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma)+\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma)+\bar{\sigma}^2\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\right]-2\boldsymbol{F}_{\mathrm{II}}^{\infty}-2\bar{\sigma}^2\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}
\end{aligned} \tag{4.295}$$

where $\boldsymbol{H}=\boldsymbol{B}_{\mathrm{I}}^{-1}+\boldsymbol{B}_{\mathrm{II}}^{-1}$. Comparing Eqs. (4.294) and (4.295), the Riemann-Hilbert equation on the crack surface is obtained:

$$(a/\sigma)\boldsymbol{\Sigma}_{\mathrm{I}\,r}=\boldsymbol{H}^{-1}[\boldsymbol{h}^{+}(\sigma)+\boldsymbol{h}^{-}(\sigma)]+\boldsymbol{p}^{\infty}+\bar{\boldsymbol{p}}^{\infty}\bar{\sigma}^2,\quad \boldsymbol{p}^{\infty}=2\boldsymbol{H}^{-1}\boldsymbol{F}_{\mathrm{II}}^{\infty} \tag{4.296}$$

After $\boldsymbol{h}(\varsigma)$ is solved, from Eq. (4.293), $\boldsymbol{F}_{\mathrm{I}0}\left(\varsigma_j\right),\boldsymbol{F}_{\mathrm{II}0}\left(\varsigma_j\right)$ can be obtained:

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{I}0}\left(\varsigma_j\right) &= \left(\mathbf{I}+\boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\right)^{-1}\left[\boldsymbol{h}\left(\varsigma_j\right)+2\boldsymbol{F}_{\mathrm{II}}^{\infty}\right]\\
\boldsymbol{F}_{\mathrm{II}0}\left(\varsigma_j\right) &= \left(\mathbf{I}+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)^{-1}\left[\boldsymbol{h}\left(\varsigma_j\right)+\left(1\big/\varsigma_j^2\right)\left(\mathbf{I}-\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\right]
\end{aligned} \tag{4.297}$$

4.9.4 Solution for Permeable Crack

From Eqs. (4.292) and (4.293), it is found that

$$-\mathrm{i}(a/\sigma)\hat{\boldsymbol{d}}' = \boldsymbol{h}^{+}(\sigma) - \boldsymbol{h}^{-}(\sigma) \tag{4.298}$$

From Eq. (4.278), it is known that on the crack surface, $E_{\mathrm{I}\theta} = E_{\mathrm{II}\theta}$, so on the whole interface L, $\hat{d}_2' = 0$, or $h_2^{+}(\sigma) - h_2^{-}(\sigma) = 0,\ \sigma \in L$. So $h_2(\varsigma)$ is analytic in whole plane. Because $h_2(\infty) = 0$, so $h_2(\varsigma) = 0$.

Using the boundary conditions Eq. (4.278) and $h_2(\varsigma) = 0$, Eq. (4.296) yields

$$\begin{aligned} (a/\sigma)\sigma_{\mathrm{I}r} &= H_{11}^{-1}\left[h_1^{+}(\sigma) + h_1^{-}(\sigma)\right] + p_1^{\infty} + \bar{p}_1^{\infty}\bar{\sigma}^2 = 0 \\ (a/\sigma)D_{\mathrm{I}r} &= H_{21}^{-1}\left[h_1^{+}(\sigma) + h_1^{-}(\sigma)\right] + p_2^{\infty} + \bar{p}_2^{\infty}\bar{\sigma}^2 \end{aligned} \tag{4.299}$$

Equation (4.299) yields

$$D_{\mathrm{I}r} = (1/a)\mathrm{Re}\left[\sigma p_2^{\infty} - \left(H_{21}^{-1}/H_{11}^{-1}\right)\sigma p_1^{\infty}\right] \tag{4.300}$$

Because the traction on the crack surface is zero and $D_{\mathrm{I}r}$ is shown in Eq. (4.300), Eq. (4.296) can be reduced to

$$\left[\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right]^{+} + \left[\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right]^{-} = \boldsymbol{P}^{\infty} + \bar{\boldsymbol{P}}^{\infty}\bar{\sigma}^2; \quad \boldsymbol{P}^{\infty} = -p_1^{\infty}\left(\mathbf{i}_1 + \left(H_{21}^{-1}/H_{11}^{-1}\right)\mathbf{i}_2\right) \tag{4.301}$$

The general solution is

$$\begin{aligned} &\boldsymbol{H}^{-1}\boldsymbol{h}(\varsigma) = (1/2)\left(\boldsymbol{P}^{\infty} + \bar{\boldsymbol{P}}^{\infty}/\varsigma^2\right) + (1/2)X(\varsigma)\left[\boldsymbol{C}(\varsigma) + \boldsymbol{C}_{-1}/\varsigma + \boldsymbol{C}_{-2}/\varsigma^2\right] \\ &X(\varsigma) = \prod_{k=1}^{n}\left(\varsigma - \sigma_k^{(1)}\right)^{-1/2}\left(\varsigma - \sigma_k^{(2)}\right)^{-1/2}, \quad \boldsymbol{C}(\varsigma) = \boldsymbol{C}_n\varsigma^n + \cdots + \boldsymbol{C}_0 \end{aligned} \tag{4.302}$$

where n is the number of cracks. It is noted that

$$\lim_{\varsigma\to 0} X(\varsigma) \approx \left[\prod_{k=1}^{n}(-1)^n\left(1\Big/\sqrt{\sigma_k^{(1)}\sigma_k^{(2)}}\right)\right]\left[1 + (\varsigma/2)\sum_{k=1}^{n}\left(1\Big/\sigma_k^{(1)} + 1\Big/\sigma_k^{(2)}\right)\right] \tag{4.303}$$

Substituting Eq. (4.303) into Eq. (4.302) and comparing the order of ς yield

$$\begin{aligned} \varsigma^{-1}: &\quad \boldsymbol{C}_{-1} = -(1/2)\sum_{k=1}^{n}\left(1\Big/\sigma_k^{(1)} + 1\Big/\sigma_k^{(2)}\right)\boldsymbol{C}_{-2} \\ \varsigma^{-2}: &\quad \boldsymbol{C}_{-2} = (-1)^{n+1}\prod_{k=1}^{n}\sqrt{\sigma_k^{(1)}\sigma_k^{(2)}}\,\bar{\boldsymbol{P}}^{\infty} \end{aligned} \tag{4.304}$$

When $\varsigma \to \infty$, we get

$$\lim_{\varsigma\to\infty} X(\varsigma) \approx 1/\varsigma^{n} + (1/2\varsigma)\sum_{k=1}^{n}\left(\sigma_k^{(1)} + \sigma_k^{(2)}\right)\left(1/2\varsigma^{n+1}\right)$$
$$\varsigma^0: \ \boldsymbol{C}_n = -\boldsymbol{P}^{\infty}, \quad \varsigma^{-1}: \ \boldsymbol{C}_{n-1} = -(1/2)\sum_{k=1}^{n}\left(\sigma_k^{(1)} + \sigma_k^{(2)}\right)\boldsymbol{C}_n \tag{4.305}$$

Other coefficients are determined by single-valued conditions of the generalized displacement:

$$\int_{L_c} \hat{\boldsymbol{d}}' \mathrm{d}\sigma = 0, \quad \text{or} \quad \int_{L_c} [\boldsymbol{h}^{+}(\sigma) - \boldsymbol{h}^{-}(\sigma)]\mathrm{d}\sigma = 0 \tag{4.306}$$

4.9.5 Single Crack

Figure 4.16b shows a single crack with $a_1 = a\mathrm{e}^{-\mathrm{i}\theta_0}$, $b_1 = a\mathrm{e}^{\mathrm{i}\theta_0}$, where $2\theta_0$ is the center angle spanning by the crack. In this case we have

$$\sigma_k^{(1)} = \mathrm{e}^{-\mathrm{i}\theta_0}, \quad \sigma_k^{(2)} = \mathrm{e}^{\mathrm{i}\theta_0}, \quad X(\varsigma) = \left(\varsigma - \mathrm{e}^{-\mathrm{i}\theta_0}\right)^{-1/2}\left(\varsigma - \mathrm{e}^{\mathrm{i}\theta_0}\right)^{-1/2}$$
$$\boldsymbol{C}(\varsigma) = \boldsymbol{C}_1\varsigma + \boldsymbol{C}_0, \quad \boldsymbol{C}_1 = -\boldsymbol{P}^{\infty}, \quad \boldsymbol{C}_0 = \cos\theta_0\boldsymbol{P}^{\infty}, \quad \boldsymbol{C}_{-2} = \bar{\boldsymbol{P}}^{\infty}, \quad \boldsymbol{C}_{-1} = -\cos\theta_0\bar{\boldsymbol{P}}^{\infty} \tag{4.307}$$

The solution is

$$\boldsymbol{H}^{-1}\boldsymbol{h}(\varsigma) = (1/2)\left(\boldsymbol{P}^{\infty} + \bar{\boldsymbol{P}}^{\infty}/\varsigma^2\right)$$
$$+ (1/2)\left(\varsigma^2 - 2\cos\theta_0 + 1\right)^{-1/2}\left[-\boldsymbol{P}^{\infty}\varsigma + \cos\theta_0\boldsymbol{P}^{\infty} - \cos\theta_0\bar{\boldsymbol{P}}^{\infty}/\varsigma + \bar{\boldsymbol{P}}^{\infty}/\varsigma^2\right] \tag{4.308}$$

$\boldsymbol{F}_{\mathrm{I}0}\left(\varsigma_j\right), \boldsymbol{F}_{\mathrm{II}0}\left(\varsigma_j\right)$ can be obtained from Eq. (4.297). So the generalized stress and displacement in any point can also be obtained. It is noted that

$$X(\sigma) = \left(\sigma^2 - 2\cos\theta_0 + 1\right)^{-1/2} = \mathrm{e}^{-\mathrm{i}\theta/2}[2\sin\theta_0(\theta_0 - \theta)]^{-1/2}$$
$$\sigma\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma) = (1/2)\sigma X(\varsigma)\left[-\boldsymbol{P}^{\infty}\sigma + \cos\theta_0\boldsymbol{P}^{\infty} - \cos\theta_0\bar{\boldsymbol{P}}^{\infty}\bar{\sigma} + \bar{\sigma}^2\bar{\boldsymbol{P}}^{\infty}\right]$$
$$= \frac{\mathrm{e}^{\mathrm{i}\theta/2}}{2\sqrt{2\sin\theta_0(\theta_0 - \theta)}}\left\{-\mathrm{i}\sin\theta\boldsymbol{P}^{\infty} - \mathrm{i}\bar{\sigma}\sin\theta\bar{\boldsymbol{P}}^{\infty}\right\} = -\frac{\sqrt{\sin\theta_0}}{2\sqrt{2(\theta - \theta_0)}}\left\{\sqrt{\sigma}\boldsymbol{P}^{\infty} + \sqrt{\bar{\sigma}}\bar{\boldsymbol{P}}^{\infty}\right\}$$
$$\boldsymbol{p}^{\infty} = 2\boldsymbol{H}^{-1}\boldsymbol{F}_{\mathrm{II}}^{\infty} = a\boldsymbol{H}^{-1}\boldsymbol{B}_{\mathrm{II}}^{-1}\begin{Bmatrix} \sigma_{31}^{\infty} - \mathrm{i}\sigma_{32}^{\infty} \\ D_1^{\infty} - \mathrm{i}D_2^{\infty} \end{Bmatrix} = a\boldsymbol{M}\begin{Bmatrix} \sigma_{31}^{\infty} - \mathrm{i}\sigma_{32}^{\infty} \\ D_1^{\infty} - \mathrm{i}D_2^{\infty} \end{Bmatrix}, \quad \boldsymbol{M} = (\boldsymbol{B}_{\mathrm{II}}\boldsymbol{H})^{-1}$$
$$\boldsymbol{P}^{\infty} = -p_1^{\infty}\left(\mathbf{i}_1 + \mathbf{i}_2H_{21}^{-1}/H_{11}^{-1}\right), \quad p_1^{\infty} = a\left\{M_{11}\left(\sigma_{31}^{\infty} - \mathrm{i}\sigma_{32}^{\infty}\right) + M_{12}\left(D_1^{\infty} - \mathrm{i}D_2^{\infty}\right)\right\} \tag{4.309}$$

The stress intensity factors can be directly obtained from $\boldsymbol{h}(\sigma)$ and is only related to the singular parts of the generalized stress. Using Eqs. (4.282), (4.296), and (4.309), the stress intensity factor at $\varsigma = \mathrm{e}^{\mathrm{i}\theta_0}\left(\text{or } z = a\mathrm{e}^{\mathrm{i}\theta_0}\right)$ is

$$\begin{aligned}
\boldsymbol{K} &= [K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{\theta\to\theta_0}\sqrt{2\pi a(\theta-\theta_0)}\boldsymbol{\Sigma}_r = (2/a)\sqrt{2\pi a(\theta-\theta_0)}\mathrm{Re}\left[\sigma\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right] \\
&= 2\sqrt{\pi a\sin\theta_0}\left[\cos(\theta_0/2)\left(M_{11}\sigma_{31}^{\infty} + M_{12}D_1^{\infty}\right) + \sin(\theta_0/2)\left(M_{11}\sigma_{32}^{\infty} + M_{12}D_2^{\infty}\right)\right] \\
&\quad \times\left[\mathbf{i}_1 + \left(H_{21}^{-1}/H_{11}^{-1}\right)\mathbf{i}_2\right] \\
K_{\alpha E} &= \lim_{\theta\to\theta_0}\sqrt{2\pi a(\theta-\theta_0)}E_{\alpha r} = -(2/a)\lim_{\theta\to\theta_0}\sqrt{2\pi a(\theta-\theta_0)}\boldsymbol{B}_{\alpha}^{-1}\mathrm{Re}\left[\sigma\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right]_2
\end{aligned} \tag{4.310}$$

For a homogeneous material, $\boldsymbol{M} = \mathbf{I}/2$, so

$$\begin{aligned}
\boldsymbol{K} &= \sqrt{\pi a\sin\theta_0}\left[\sigma_{31}^{\infty}\cos(\theta_0/2) + \sigma_{32}^{\infty}\sin(\theta_0/2)\right]\left(\mathbf{i}_1 + \mathbf{i}_2H_{21}^{-1}/H_{11}^{-1}\right) \\
K_{\alpha E} &= -\sqrt{\pi a\sin\theta_0}\left[\sigma_{31}^{\infty}\cos(\theta_0/2) + \sigma_{32}^{\infty}\sin(\theta_0/2)\right]\left(B_{\alpha 21}^{-1} + B_{\alpha 22}^{-1}H_{21}^{-1}/H_{11}^{-1}\right)
\end{aligned} \tag{4.311}$$

From Eq. (4.311), it is known that for a permeable crack, the stress intensity factors do not depend to the external electric field.

4.9.6 Impermeable Crack

For an impermeable crack, $D_{\mathrm{I}r} = D_{\mathrm{II}r} = 0$ on the crack surface are known and $\boldsymbol{P}^{\infty} = -\boldsymbol{p}^{\infty}$. $\boldsymbol{H}^{-1}\boldsymbol{h}(\varsigma)$ is still expressed by Eqs. (4.302), (4.303), (4.304), (4.305), and (4.306). The stress intensity factor is

$$\begin{aligned}
\boldsymbol{K} &= \begin{Bmatrix} K_{\mathrm{III}} \\ K_{\mathrm{D}} \end{Bmatrix} = \frac{2}{a}\sqrt{\pi a\sin\theta_0}\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta_0/2}p^{\infty}\right) = 2\sqrt{\pi a\sin\theta_0}\boldsymbol{M}\mathrm{Re}\left[\left\{\mathrm{e}^{\mathrm{i}\theta_0/2}\begin{Bmatrix} \sigma_{31}^{\infty} - \mathrm{i}\sigma_{32}^{\infty} \\ D_1^{\infty} - \mathrm{i}D_2^{\infty} \end{Bmatrix}\right\}\right] \\
&= 2\sqrt{\pi a\sin\theta_0}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{Bmatrix} \sigma_{32}^{\infty}\sin(\theta_0/2) + \sigma_{31}^{\infty}\cos(\theta_0/2) \\ D_2^{\infty}\sin(\theta_0/2) + D_1^{\infty}\cos(\theta_0/2) \end{Bmatrix}
\end{aligned} \tag{4.312}$$

From Eq. (4.312), it is known that for an impermeable crack, the stress intensity factors are dependent to the external electric field.

Zhong and Meguid (1997), Gao and Balke (2003), and Liu and Fang (2004) et al. discussed the similar problem.

References

Barnett DM, Asaro RJ (1972) The fracture mechanics of slit-like cracks in anisotropic elastic media. J Mech Phys Solids 20:353–366

Beom HG, Atluri SN (1996) Near-tip fields and intensity factors for interfacial cracks in dissimilar anisotropic piezoelectric media. Int J Fract 75:163–183

Chung MY, Ting TCT (1996) Piezoelectric solid with an elliptic inclusion or hole. Int J Solids Struct 33:3343–3361

Deng W, Meguid SA (1998) Analysis of conducting rigid inclusion at the interface of two dissimilar piezoelectric materials. J Appl Mech 65:76–94

Erdogan F, Gupta GD (1972) On the numerical solution of singular integral equations. Q Appl Math 32:525–553

Fulton CC, Gao H (1997) Electrical nonlinearity in fracture of piezoelectric ceramics. Appl Mech Rev 50:556–563

Gao C-F, Balke H (2003) Fracture analysis of circular-arc interface cracks in piezoelectric materials. Int J Solids Struct 40:3507–3522

Gao C-F, Wang M-Z (2000) Collinear permeable cracks between dissimilar piezoelectric materials. Int J Solids Struct 37:4969–4986

Gao C-F, Wang M-Z (2001) Green's functions of an interfacial crack between two dissimilar piezoelectric media. Int J Solids Struct 38:5323–5334

Gao HJ, Zhang TY, Tong P (1997) Local and global energy release rates for an electrically yielded crack in a piezoelectric ceramic. J Mech Phys Solids 45:491–510

Hao TH, Shen ZY (1994) A new electric boundary condition of electric fracture mechanics and its applications. Eng Fract Mech 47:793–802

Herrmann KP, Loboda VV (2000) Fracture-mechanical assessment of electrically permeable interface cracks in piezoelectric bimaterials by consideration of various contact zone models. Arch Appl Mech 70:127–143

Herrmann KP, Loboda VV, Govorukha VB (2001) On contact zone models for an electrically impermeable interface crack in a piezoelectric bimaterial. Int J Fract 111:203–227

Hou Z-Y, Li M-Z, Zhang W-G (1990) Singular integral equations and its applications. Shanghai Science and Technology Publishing House, Shanghai (in Chinese)

Huang Z-Y, Kuang Z-B (2001) A first order perturbation analysis of a non-ideal crack in a piezoelectric material. Int J Solids Struct 38:7261–7281

Huang Z-Y, Kuang Z-B (2003) A mixed electric boundary value problem for a two-dimensional piezoelectric crack. Int J Solids Struct 40:1433–1453

Kuang Z-B, Ma F-S (2002) Crack tip field. Xian Jiaotong University Press, Xian (in Chinese)

Kuang Z-B, Zhou Z-D, Zhou K-L (2004) Electroelastic analysis of a piezoelectric half-plane with finite surface electrodes. Int J Eng Sci 42:1603–1619

Lavrenchive MA, Shabat VA (1951) Method and theory of complex functions. National Technical and Theoretical Literature Press, Moscow; Лаврентьев М А, Шабат В А. Методтеориифукцийкоплекснгопеременого. Государственноеиздадельство. технико-теоретическийлитературы. Москва (1951)

Liu Y-W, Fang Q-H (2004) Electroelastic interaction between a piezoelectric screw dislocation and a circular inhomogeneity with interfacial cracks. Appl Math Mech 25:1428–1436

Loboda VV (1993) The quasi-invariant in the theory of interface cracks. Eng Fract Mech 44:573–580

Lynch CS, Yang W, Collier L, Suo Z, McCMeeking RM (1995) Electric field induced cracking in ferroelectric ceramics. Ferroelectrics 166:11–30

Muskhelishvili NI (1975) Some basic problems of mathematical theory of elasticity. Noordhoof, Leyden; Мусхелишвили Н Е. Некоторые осноные задачи математической теории упругости, Масква: Издательство академии наук СССР (1954)

Muskhelishvili NI (1975) Singular integral equations. Noordhoff, Gronigen

Pak YE (1990) Force on a piezoelectric screw dislocation. J Appl Mech 57:863–869

Ru CQ (2000) Exact solution for finite electrode layers embedded at the interface of two piezoelectric half-planes. J Mech Phys Solids 48:693–708
Ru CQ, Mao X (1999) Conducting cracks in a piezoelectric ceramic of limited electrical polarization. J Mech Phys Solids 47:2125–2146
Shen S-P, Kuang Z-B (1998) Interface crack in bi-piezothermoelastic media and the interaction with a point heat source. Int J Solids Struct 35:3899–3915
Shen S-P, Kuang Z-B, Hu SL (1999) Interface crack problem of a laminated piezoelectric plate. Eur J Mech A/Solids 18:219–238
Shen S-P, Nishioka T, Kuang Z-B, Liu Z-X (2000) Nonlinear electromechanical interface fracture for piezoelectric materials. Mech Mater 32:57–64
Shen MH, Chen FM, Chen SN (2007) Piezoelectric study on singularities interacting with interfaces in an anisotropic medium. Int J Solids Struct 44:5598–5610
Shi W-C (1997) Rigid line inclusions under anti-plane deformation and in-plane electric field in piezoelectric materials. Eng Fract Mech 56:265–274
Shindo Y, Narita F, Sosa H (1998) Electroelastic analysis of piezoelectric ceramics with surface electrodes. Int J Eng Sci 36:1001–1009
Suo Z (1990) Singularities interfaces and cracks in dissimilar anisotropic media. Proc R Soc Lond A 427:331–358
Suo Z, Kuo CM, Barnett DM, Willis JR (1992) Fracture mechanics for piezoelectric ceramics. J Mech Phys Solids 40:739–765
Wang T-C (2000) Analysis of strip electric saturation model of crack problem in piezoelectric materials. Int J Solids Struct 37:6031–6049
Wang J-W, Kuang Z-B (2000) The interaction between crack and electric dipole of piezoelectricity. Acta Mech Solida Sin 13:283–289
Wang J-W, Kuang Z-B (2002) The electric dipole and crack on the interface in a bi-piezoelectric material. Acta Mech Sin 34:192–199 (in Chinese)
Zhang T-Y, Qian C-F, Tong P (1998) Linear electro-elastic analysis of a cavity or a crack in a piezoelectric material. Int J Solids Struct 35:2121–2149
Zhang TY, Zhao M, Tong P (2001) Fracture of piezoelectric ceramics. Adv Appl Mech 38:147–289
Zhong Z, Meguid SA (1997) Interfacial debonding of a circular inhomogeneity in piezoelectric materials. Int J Eng Sci 34:1965–1984
Zhou K-L, Zhou Z-D, Kuang Z-B (2005a) Surface electrode problems in piezoelectric materials. Adv Mater Res 9:191–198
Zhou Z-D, Zhao S-X, Kuang Z-B (2005b) Stress and electric displacement analysis in piezoelectric media with an elliptic hole and a small crack. Int J Solids Struct 42:2803–2822
Zhou Z-D, Zhao S-X, Kuang Z-B (2008) Electroelastic analysis of bi-piezoelectrics embedded interfacial rigid lines with generalized piezoelectric dislocation. Acta Mech Solida Sin 29:1–8