Chapter 3
Generalized Two-Dimensional Electroelastic Problem

Abstract In this chapter the fundamental theory of the generalized two-dimensional (2D) linear electroelastic analyses is discussed. The generalized 2D Stroh method and the extended generalized Lekhnitskii stress function method are studied. The linear electroelastic analyses in an infinite transversely isotropic material with the permeable, impermeable, and conducting elliptic hole; crack; and the rigid elliptic inclusion under plane strain are discussed in detail. Singularities, including generalized dislocation, generalized force, and electric couple, in homogeneous material and bimaterial are researched. Interaction of an elliptic inclusion with a singularity is discussed, and some numerical examples are also given. In this chapter the asymptotic fields near a line inclusion tip in a homogeneous material and Eshelby's eigenstrain problem are also discussed.

Keywords Generalized 2D electroelastic problem • Stroh method • Extended Lekhnitskii method • Transversely isotropic material • Elliptic hole • Crack • Elliptic inclusion • Singularity

3.1 Generalized Two-Dimensional Linear Electroelastic Problem

The generalized two-dimensional (2D) electroelastic problem means that the generalized displacements $(u_i, \varphi;\ i = 1, 2, 3)$ exactly or the generalized stresses $(\sigma_{ij}, D_i;\ i, j = 1, 2, 3)$ approximately depend only on two of the coordinates (x_1, x_2, x_3). It is seen that the generalized 2D problem is a special three-dimensional (3D) problem, which is different with the plane problem (plane strain and generalized plane stress problems). For the linear electroelastic problem with small electric field, the Maxwell stress can be neglected because $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{D})$ depend on $\boldsymbol{E}$ linearly and the Maxwell stress is depended on the square of $\boldsymbol{E}$. The method to solve the electroelastic problem is directly the extension of that in the anisotropic elastic materials, but the problem is more complex.

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In engineering the extensive applied constitutive equations are the second kind and the third kind of the constitutive equations in Eq. (2.83) for the piezoelectric materials. The governing equations are the generalized momentum equations, constitutive equations, and generalized geometric equations. They are, respectively,

$$\sigma_{ij,i} + \left(f_j^{\mathrm{m}} + f_j^{\mathrm{e}}\right) = \rho \ddot{u}_j, \quad D_{i,i} = \rho_{\mathrm{e}} \tag{3.1}$$

$$\begin{aligned} &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad D_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} \quad \text{or} \\ &\varepsilon_{ij} = s_{ijkl}\sigma_{kl} + g_{kij}D_k, \quad E_i = -g_{ijk}\sigma_{jk} + \beta_{ij}D_j \end{aligned} \tag{3.2}$$

$$\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2, \quad E_i = -\varphi_{,i} \tag{3.3}$$

where $\boldsymbol{f}^{\mathrm{m}}$ is the mechanical force per volume and $\boldsymbol{f}^{\mathrm{e}}$ is the static electric force. The boundary conditions and connective conditions on the interface are, respectively,

$$\begin{array}{llll} \sigma_{ij}n_j = T_i^*, & \text{on} \quad a_\sigma; & u_i = u_i^*, & \text{on} \quad a_u \\ D_i n_i = -\sigma^*, & \text{on} \quad a_D; & \varphi = \varphi^*, & \text{on} \quad a_\varphi \end{array} \tag{3.4}$$

$$\sigma_{ij}^+ n_j = \sigma_{ij}^- n_j, \quad u_i^+ = u_i^-, \quad D_i^+ = D_i^-; \quad \varphi^+ = \varphi^-, \quad \text{on} \quad L \tag{3.5}$$

where $\boldsymbol{T}^*, \sigma^*$ are the traction and electric charge per area and the superscripts "+" and " $-$ " denote the values approached from the upper and lower half planes, respectively. For the linear problem, $\boldsymbol{f}^{\mathrm{e}}$ can be neglected. For the static case without the body force and body electric charge, the governing equations in $(\boldsymbol{u}, \varphi)$ are

$$\left(C_{ijlk}u_l + e_{kij}\varphi\right)_{,ik} = 0; \quad \left(-\epsilon_{ik}\varphi + e_{ijk}u_j\right)_{,ik} = 0 \tag{3.6}$$

For the multi-connected domain, the displacement, electric potential must satisfy the uniqueness conditions

$$\oint_L \mathrm{d}U_i = 0 \quad \text{or} \quad \oint_L \mathrm{d}u_i = 0, \quad \oint_L \mathrm{d}\varphi = 0 \tag{3.7}$$

where L is a closed contour and there is no source inside it.

Sometimes the constitutive equations are written in a more compact form:

$$\begin{aligned} &\Sigma_{iJ} = E_{iJKn}Z_{Kn}; \quad \Sigma_{iJ} = \begin{cases} \sigma_{ij}, & J = 1,2,3 \\ D_i, & J = 4 \end{cases}; \quad \Sigma_{iJ,i} = 0 \\ &U_K = \begin{cases} u_k, & K = 1,2,3 \\ \varphi, & K = 4 \end{cases}; \quad Z_{Kn} = \begin{cases} \varepsilon_{kn}, & K = 1,2,3 \\ -E_n, & K = 4 \end{cases}; \\ &E_{iJKn} = \begin{cases} C_{ijkn}, & J,K = 1,2,3 \\ e_{nij}, & J = 1,2,3; \quad K = 4 \\ e_{ikn}, & J = 4; \quad K = 1,2,3 \\ -\epsilon_{in}, & J = K = 4 \end{cases} \end{aligned} \tag{3.8}$$

where a subscript in upper case takes the value 1, 2, 3, or 4 and a subscript in lower case takes the value 1, 2, or 3. U_K, Σ_{iJ}, Z_{Kl}, and E_{iJKl} are the generalized displacement, generalized stress, generalized strain, and generalized stiffness coefficient, respectively. It is noted that the rule of the subscript used here does not hold everywhere and the meaning of the subscript given in corresponding places.

3.2 Generalized Displacement Method in the Piezoelectric Materials

3.2.1 Generalized Displacement Method

For the generalized 2D problem, the Stroh method (Stroh 1958; Suo 1990; Suo et al. 1992; Ting 1996) is often applied. Let

$$\begin{aligned} &\boldsymbol{U} = \boldsymbol{a}f(z), \quad \text{or} \quad U_K = a_K f(z), \quad \text{or} \quad u_i = a_i f(z), \quad \varphi = a_4 f(z) \\ &\boldsymbol{U} = \{U_K\}^{\mathrm{T}} = [u_i, \varphi]^{\mathrm{T}}, \quad \boldsymbol{a} = \{a_K\}^{\mathrm{T}} = [a_i, a_4]^{\mathrm{T}} \\ &U_{K,\alpha} = a_K f'(z)(\delta_{\alpha 1} + \mu\delta_{\alpha 2}), \quad z = x_1 + \mu x_2; \quad z_{,1} = 1, \quad z_{,2} = \mu \end{aligned} \tag{3.9}$$

where the right upper superscript T denotes transpose of a matrix. Substituting Eq. (3.9) into Eq. (3.6) in generalized 2D case yields

$$\begin{aligned} &\left(C_{\alpha jl\beta} a_l + e_{\beta j\alpha} a_4\right) z_{,\alpha} z_{,\beta} = 0, \quad \left(-\epsilon_{\alpha\beta} a_4 + e_{\beta j\alpha} a_j\right) z_{,\alpha} z_{,\beta} = 0; \quad \text{or} \\ &\left(C_{\alpha jl\beta} a_l + e_{\beta\alpha j} a_4\right) z_{,\alpha} z_{,\beta} = 0, \quad \left(-\epsilon_{\alpha\beta} a_4 + e_{\beta\alpha j} a_j\right) z_{,\alpha} z_{,\beta} = 0 \end{aligned} \tag{3.10}$$

where a Greek subscript takes values 1 and 2 and an English subscript takes values 1, 2, and 3. Equation (3.10) can be written in detail as

$$\begin{aligned} &\left[C_{i1k1} + \mu(C_{i1k2} + C_{i2k1}) + \mu^2 C_{i2k2}\right] a_k + \left[e_{1i1} + \mu(e_{2i1} + e_{1i2}) + \mu^2 e_{2i2}\right] a_4 = 0 \\ &\left[e_{1k1} + \mu(e_{2k1} + e_{1k2}) + \mu^2 e_{2k2}\right] a_k - \left[\epsilon_{11} + \mu(\epsilon_{12} + \epsilon_{21}) + \mu^2 \epsilon_{22}\right] a_4 = 0 \end{aligned} \tag{3.11}$$

where the subscripts i and k denote row and column, respectively. In order to obtain nontrivial solutions for (a_k, a_4), the coefficient determinant must be zero, i.e.,

$$|\boldsymbol{D}(\mu)| = \begin{vmatrix} C_{i1k1} + \mu(C_{i1k2} + C_{i2k1}) + \mu^2 C_{i2k2} & e_{1i1} + \mu(e_{2i1} + e_{1i2}) + \mu^2 e_{2i2} \\ e_{1k1} + \mu(e_{2k1} + e_{1k2}) + \mu^2 e_{2k2} & -\epsilon_{11} - \mu(\epsilon_{12} + \epsilon_{21}) - \mu^2 \epsilon_{22} \end{vmatrix} = 0 \tag{3.12}$$

$\boldsymbol{D}(\mu)$ is called the character matrix. If introducing 4×4 matrixes $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}$

$$\boldsymbol{Q} = \begin{bmatrix} C_{i1k1} & e_{1i1} \\ e_{1k1} & -\epsilon_{11} \end{bmatrix}, \ \boldsymbol{R} = \begin{bmatrix} C_{i1k2} & e_{2i1} \\ e_{1k2} & -\epsilon_{12} \end{bmatrix}, \ \boldsymbol{R}^{\mathrm{T}} = \begin{bmatrix} C_{i2k1} & e_{1i2} \\ e_{2k1} & -\epsilon_{12} \end{bmatrix}, \ \boldsymbol{T} = \begin{bmatrix} C_{i2k2} & e_{2i2} \\ e_{2k2} & -\epsilon_{22} \end{bmatrix}$$
$$C_{i2k1} = C_{k1i2}; \quad Q_{JK} = E_{1JK1}, \quad R_{JK} = E_{1JK2}, \quad T_{JK} = E_{2JK2} \tag{3.13}$$

then Eqs. (3.11) and (3.12) can also be written as

$$\begin{aligned} &\boldsymbol{D}(\mu)\boldsymbol{a} = \left[\boldsymbol{Q} + \mu\left(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}\right) + \mu^2\boldsymbol{T}\right]\boldsymbol{a} = \boldsymbol{0}, \quad \text{or} \\ &(\boldsymbol{Q} + \mu\boldsymbol{R})\boldsymbol{a} = -\mu\left(\boldsymbol{R}^{\mathrm{T}} + \mu\boldsymbol{T}\right)\boldsymbol{a}, \quad \left(\boldsymbol{R}^{\mathrm{T}} + \mu\boldsymbol{T}\right)\boldsymbol{a} = -\left(\mu^{-1}\boldsymbol{Q} + \boldsymbol{R}\right)\boldsymbol{a} \\ &|\boldsymbol{D}(\mu)| = \left|\boldsymbol{Q} + \mu\left(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}\right) + \mu^2\right| = 0 \end{aligned} \tag{3.14}$$

$|\boldsymbol{D}(\mu)|$ is a 4×4 determinant, $|\boldsymbol{D}(\mu)| = 0$ is the eighth-order equation of μ, so eigenvalue μ has eight roots. Equation (3.11) or (3.14) is used to determine eigenvector $\boldsymbol{a}$. Because μ is complex (Suo et al 1992; Ting 1996), let

$$\begin{aligned} &\mu_P = \alpha_P + \mathrm{i}\beta_P, \quad \beta_P > 0; \quad \mu_{P+4} = \bar{\mu}_P; \quad (P = 1, 2, 3, 4) \\ &z_P = x_1 + \mu_P x_2; \quad x_1 = (\mu_P \bar{z}_P - \bar{\mu}_P z_P)/(\mu_P - \bar{\mu}_P), \quad x_2 = (z_P - \bar{z}_P)/(\mu_P - \bar{\mu}_P) \end{aligned} \tag{3.15}$$

In fact if we multiply the first equation in (3.10) by a_j and sum over j, multiply the second equation in (3.10) by a_4, the difference of these two results is

$$\left(C_{\alpha jl\beta} a_j a_l + \epsilon_{\alpha\beta} a_4^2\right)(\delta_{\alpha 1} + \mu\delta_{\alpha 2})\left(\delta_{\beta 1} + \mu\delta_{\beta 2}\right) = 0$$

If μ is real, we can choose

$$\begin{aligned} &u_{j,\alpha} = (\delta_{\alpha 1} + \mu\delta_{\alpha 2})a_j, \quad u_{l,\beta} = \left(\delta_{\beta 1} + \mu\delta_{\beta 2}\right)a_l; \quad \varphi_{,\alpha} = (\delta_{\alpha 1} + \mu\delta_{\alpha 2})a_4, \\ &\varphi_{,\beta} = \left(\delta_{\beta 1} + \mu\delta_{\beta 2}\right)a_4 \end{aligned}$$

The expression of the strain energy is

$$C_{\alpha jl\beta} u_{j,\alpha} u_{l,\beta} + \epsilon_{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} = 0$$

However, the strain energy is positive definite and cannot equal zero, so μ must be complex.

3.2.2 Eigenvalues *μ*'s Are All Distinct

When the eigenvalues $\boldsymbol{\mu}$'s in Eq. (3.12) are all distinct, the matrix $\boldsymbol{D}(\mu)$ is called simple. In this case for each μ_P, an independent eigenvector $\boldsymbol{a}_P = [a_{P1}, a_{P2}, a_{P3}, a_{P4}]^{\mathrm{T}}$

can be solved from Eq. (3.11). Corresponding to $\boldsymbol{a}_P$, an arbitrary function $f_P(z_P)$, $z_P = x_1 + \mu_P x_2$, can be assumed. Noting $\boldsymbol{U}$ is real, so the general solution is

$$\begin{aligned} \boldsymbol{U} &= [u_i, \varphi]^{\mathrm{T}} = 2\mathrm{Re}\sum_{P=1}^{4} \boldsymbol{a}_P f_P(z_P) = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z_P)] \\ U_K &= 2\mathrm{Re}\sum_{P=1}^{4} a_{PK} f_P(z_P) = 2\mathrm{Re}\sum_{P=1}^{4} A_{KP} f_P(z_P) \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\boldsymbol{a} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]; \quad \boldsymbol{A} = [A_{KP}], \quad A_{KP} = a_{PK} \\ &\boldsymbol{f}(z_P) = [f_P(z_P)]^{\mathrm{T}} = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^{\mathrm{T}} \end{aligned} \tag{3.17}$$

where symbol Re means the real part of a complex function, $\boldsymbol{A}$ is a 4×4 matrix, and $\boldsymbol{f}(z_P)$ is a vector function and may be called the displacement generation function. It is noted that matrix $\boldsymbol{A}$ and matrix $\boldsymbol{a}$ are identical, but the notations of their components are different. When the number of a summation dummy subscript is larger than 2, we shall directly use the notation Σ as shown in Eq. (3.16). For most engineering problem, $f_P(z_P)$ in Eq. (3.16) can be simplified as $f(z_P)V_P$, where $\boldsymbol{V}$ is a constant vector. So Eq. (3.16) can be reduced to

$$\boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\langle f(z_P)\rangle \boldsymbol{V}], \quad \langle f(z_P)\rangle = \mathrm{diag}[f(z_P)], \quad \boldsymbol{V} = \left[V_j, V_4\right]^{\mathrm{T}} \tag{3.18}$$

Analogous to Eq. (3.10), for any subscript "P," we have

$$\begin{aligned} &\left(C_{i\alpha k\beta}A_{kP} + e_{\beta i\alpha}A_{4P}\right) z_{P,\alpha} z_{P,\beta} = 0, \quad \left(e_{\alpha k\beta}A_{kP} - \epsilon_{\alpha\beta}A_{4P}\right) z_{P,\alpha} z_{P,\beta} = 0; \quad \text{or} \\ &\left(C_{i1k\beta}A_{kP} + e_{\beta i1}A_{4P}\right) z_{P,\beta} = -\mu_P \left(C_{i2k\beta}A_{kP} + e_{\beta i2}A_{4P}\right) z_{P,\beta} \\ &\left(e_{1k\beta}A_{kP} - \epsilon_{\beta 1}A_{4P}\right) z_{P,\beta} = -\mu_P \left(e_{2k\beta}A_{kP} - \epsilon_{\beta 2}A_{4P}\right) z_{P,\beta} \end{aligned} \tag{3.19}$$

Substitution of Eq. (3.16) into Eq. (3.2) yields

$$\begin{aligned} \sigma_{ij} &= 2\mathrm{Re}\sum_{P=1}^{4} \left(C_{ijk\beta}A_{kP} + e_{\beta ij}A_{4P}\right) z_{P,\beta} F_P(z_P) \\ D_i &= 2\mathrm{Re}\sum_{P=1}^{4} \left(e_{ik\beta}A_{kP} - \epsilon_{i\beta}A_{4P}\right) z_{P,\beta} F_P(z_P) \end{aligned} \tag{3.20}$$

where $F_p(z_p) = df_p/dz_p = f_p'(z_p)$ is the derivative of $f_p(z_p)$ with z_p. Substitution of Eq. (3.18) into Eq. (3.2) yields

$$\begin{aligned} \sigma_{ij} &= 2\mathrm{Re}\sum_{P=1}^{4} \left(C_{ijk\beta}A_{kP} + e_{\beta ij}A_{4P}\right) z_{P,\beta} F(z_P) V_P \\ D_i &= 2\mathrm{Re}\sum_{P=1}^{4} \left(e_{ik\beta}A_{kP} - \epsilon_{i\beta}A_{4P}\right) z_{P,\beta} F(z_P) V_P \end{aligned} \tag{3.21}$$

Using Eq. (3.19) from Eq. (3.20), we can get

$$\begin{aligned}
\sigma_{i1} &= 2\mathrm{Re}\sum_{P=1}^{4}\left(C_{i1k\beta}A_{kP}+e_{\beta i1}A_{4P}\right)z_{P,\beta}F_P(z_P) = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P\left(C_{i2k\beta}A_{kP}+e_{\beta i2}A_{4P}\right)z_{P,\beta}F_P(z_P)\\
D_1 &= 2\mathrm{Re}\sum_{P=1}^{4}\left(e_{1k\beta}A_{kP}-\epsilon_{1\beta}A_{4P}\right)z_{P,\beta}F_P(z_P) = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P\left(e_{2k\beta}A_{kP}-\epsilon_{\beta 2}A_{4P}\right)z_{P,\beta}F_P(z_P)\\
\sigma_{i2} &= 2\mathrm{Re}\sum_{P=1}^{4}\left(C_{i2k\beta}A_{kP}+e_{\beta i2}A_{4P}\right)z_{P,\beta}F_P(z_P) = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P^{-1}\left(C_{i1k\beta}A_{kP}+e_{\beta i1}A_{4P}\right)z_{P,\beta}F_P(z_P)\\
D_2 &= 2\mathrm{Re}\sum_{P=1}^{4}\left(e_{2k\beta}A_{kP}-\epsilon_{2\beta}A_{4P}\right)F_P(z_P)z_{P,\beta} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P^{-1}\left(e_{1k\beta}A_{kP}-\epsilon_{\beta 1}A_{4P}\right)z_{P,\beta}F_P(z_P)
\end{aligned} \tag{3.22}$$

Introduce the generalized stress function $\boldsymbol{\Phi}$ satisfying the equilibrium equation automatically:

$$\begin{aligned}
&\boldsymbol{\Phi} = [\Phi_1,\Phi_2,\Phi_3,\Phi_4]^{\mathrm{T}} = [\Phi_i,\Phi_4]^{\mathrm{T}} = 2\mathrm{Re}\sum_{P=1}^{4}\boldsymbol{b}_P f_P(z_P) = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)]\\
&\Sigma_1 = -\boldsymbol{\Phi}_{,2} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P\boldsymbol{b}_P F_P(z_P),\quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Phi}_{,1} = 2\mathrm{Re}\sum_{P=1}^{4}\boldsymbol{b}_P F_P(z_P)\\
&\sigma_{i1} = \Sigma_{i1} = -\Phi_{i,2} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P b_{Pi}F_P(z_P),\quad D_1 = \Sigma_{41} = -\Phi_{4,2} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P b_{P4}F_P(z_P)\\
&\sigma_{i2} = \Sigma_{i2} = \Phi_{i,1} = 2\mathrm{Re}\sum_{P=1}^{4}b_{Pi}F_P(z_P),\quad D_2 = \Sigma_{42} = \Phi_{4,1} = 2\mathrm{Re}\sum_{P=1}^{4}b_{P4}F_P(z_P)
\end{aligned} \tag{3.23}$$

Comparing Eqs. (3.22) and (3.23), it is easily found that

$$\begin{aligned}
b_{Pi} &= B_{iP} = \left(C_{i2k\beta}A_{kP}+e_{\beta i2}A_{4P}\right)z_{P,\beta} = -\mu_P^{-1}\left(C_{i1k\beta}A_{kP}+e_{\beta i1}A_{4P}\right)z_{P,\beta}\\
b_{P4} &= B_{4P} = \left(e_{2k\beta}A_{kP}-\epsilon_{\beta 2}A_{4P}\right)z_{P,\beta} = -\mu_P^{-1}\left(e_{1k\beta}A_{kP}-\epsilon_{\beta 1}A_{4P}\right)z_{P,\beta}\\
\boldsymbol{b} &= [\boldsymbol{b}_P] = [\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4] = [b_{PK}] = [B_{KP}] = B
\end{aligned} \tag{3.24}$$

Combining Eqs. (3.23) and (3.24), we get

$$\begin{aligned}
\boldsymbol{b}_P &= \left(\boldsymbol{R}^{\mathrm{T}}+\mu_P\boldsymbol{T}\right)\boldsymbol{a}_P = -\mu_P^{-1}(\boldsymbol{Q}+\mu_P\boldsymbol{R})\boldsymbol{a}_P\\
\boldsymbol{B} &= \left(\boldsymbol{R}^{\mathrm{T}}+\mu_{P,row}\boldsymbol{T}\right)\boldsymbol{A} = -\mu_{P,row}^{-1}\left(\boldsymbol{Q}+\mu_{P,row}\boldsymbol{R}\right)\boldsymbol{A}\\
\boldsymbol{\Sigma}_1 &= -\boldsymbol{\Phi}_{,2} = -2\mathrm{Re}[\boldsymbol{B}\mu_P\boldsymbol{F}(z_P)],\quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Phi}_{,1} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{F}(z_P)]
\end{aligned} \tag{3.25}$$

where $\mu_{P,row}$ is a special symbol, the subscript P in $\mu_{P,row}$ takes the value of the row number of the matrix $\boldsymbol{A}$ or $\boldsymbol{B}$ under matrix calculation. Similar to $\boldsymbol{a}_P$, components of $\boldsymbol{b}_P$ are $b_{PK}, K = 1,2,3,4$. Because $\sigma_{12} = \sigma_{21}$, we get $\Phi_{1,1} + \Phi_{2,2} = 0$.

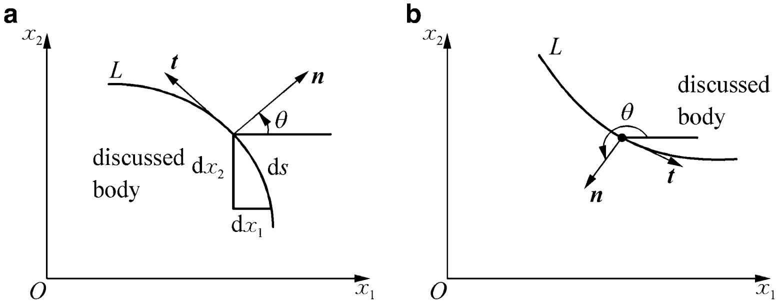


Fig. 3.1 First kind of natural coordinate system on a curve L

Similarly for the general solution Eq. (3.18), we have

$$\begin{aligned}&\boldsymbol{\Phi} = [\Phi_1, \Phi_2, \Phi_3, \Phi_4]^{\mathrm{T}} = [\Phi_i, \Phi_4]^{\mathrm{T}} = 2\mathrm{Re}\sum_{P=1}^{4} \boldsymbol{b}_P f(z_P)\boldsymbol{V}_P = 2\mathrm{Re}[\boldsymbol{B}\langle f(z_P)\rangle \boldsymbol{V}]\\ &\boldsymbol{\Sigma}_1 = -\boldsymbol{\Phi}_{,2} = -2\mathrm{Re}[\boldsymbol{B}\langle \mu_P F(z_P)\rangle \boldsymbol{V}], \quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Phi}_{,1} = 2\mathrm{Re}[\boldsymbol{B}\langle F(z_P)\rangle \boldsymbol{V}]\end{aligned} \tag{3.26}$$

The generalized stress σ_{33} can be obtained by the condition of the generalized plain strain $\varepsilon_{33} = 0$. In the 2D, $D_3 = 0$ is assumed.

Now the physical meaning of $\boldsymbol{\Phi}$ is discussed. Usually the first natural coordinate system at a point on a curve L is used. Let $\boldsymbol{n}$ be the outward normal to L; when an observer moves along the positive direction of the tangent $\boldsymbol{t}$ around L, the discussed body is located in the left side. θ is directed counterclockwise from the positive x_1-axis to the positive direction of $\boldsymbol{n}$ (Fig. 3.1). Therefore

$$\begin{aligned}&n_1 = t_2 = \cos\theta = \mathrm{d}x_2/\mathrm{d}s, \quad n_2 = -t_1 = \sin\theta = -\mathrm{d}x_1/\mathrm{d}s;\\ &n = n_1 + \mathrm{i}n_2 = -\mathrm{i}\mathrm{d}z/\mathrm{d}s = -\mathrm{i}\mathrm{d}z/|\mathrm{d}z|, \quad t = t_1 + \mathrm{i}t_2 = \mathrm{d}z/\mathrm{d}s = \mathrm{d}z/|\mathrm{d}z| = \mathrm{i}n\end{aligned} \tag{3.27}$$

where ds is the arc length of an infinitesimal element. The traction $\boldsymbol{T}$ on L is

$$\begin{aligned}&T_i = \sigma_{ij}n_j = \sigma_{i1}\mathrm{d}x_2/\mathrm{d}s - \sigma_{i2}\mathrm{d}x_1/\mathrm{d}s = -\Phi_{i,2}\mathrm{d}x_2/\mathrm{d}s - \Phi_{i,1}\mathrm{d}x_1/\mathrm{d}s = -\mathrm{d}\Phi_i/\mathrm{d}s\\ &-\sigma = D_i n_i = D_n = -\Phi_{4,2}\mathrm{d}x_2/\mathrm{d}s - \Phi_{4,1}\mathrm{d}x_1/\mathrm{d}s = -\mathrm{d}\Phi_4/\mathrm{d}s\\ &\boldsymbol{T} = [T_i, -\sigma] = -\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s, \quad \boldsymbol{\Phi}|_0^s = -\int_0^s \boldsymbol{T}\mathrm{d}s, \quad \Phi_i|_0^s = -\int_0^s T_i\mathrm{d}s, \quad \Phi_4|_0^s = -\int_0^s D_n\mathrm{d}s\end{aligned} \tag{3.28}$$

So $-\Delta\boldsymbol{\Phi}$ represents the increased resultant force on Δs of the boundary.

In literatures authors also adopted the second natural coordinate system. In this system authors take the tangent $\boldsymbol{t}'$ and $\boldsymbol{t}' = -\boldsymbol{t}$. This system is often used for a hole or inclusion in a multiply connected region. For this system in literatures, there are

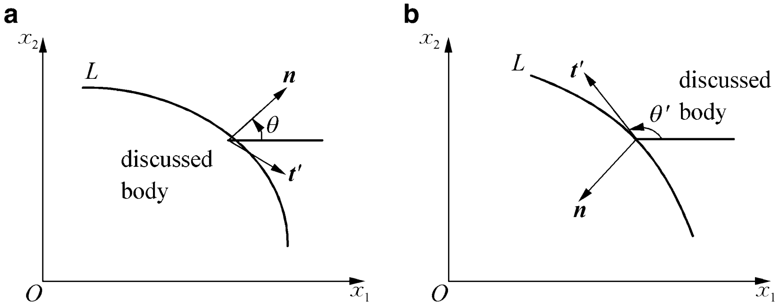


Fig. 3.2 Second kind of natural coordinate system on a curve L

two kinds. The first is that θ is directed counterclockwise from the positive x_1-axis to the direction of $\boldsymbol{n}$ (Fig. 3.2a), so

$$n_1 = -t_2' = \cos\theta = -\mathrm{d}x_2/\mathrm{d}s, \quad n_2 = t_1' = \sin\theta = \mathrm{d}x_1/\mathrm{d}s; \quad n = \mathrm{i}t' = \mathrm{i}\mathrm{d}z/\mathrm{d}s$$
$$\boldsymbol{T} = [t_1, t_2, t_3, -\sigma] = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s, \boldsymbol{\Phi}|_0^s = \int_0^s \boldsymbol{T}\mathrm{d}s, \quad \Phi_i|_0^s = \int_0^s T_i\mathrm{d}s, \quad \Phi_4|_0^s = -\int_0^s \sigma\mathrm{d}s \tag{3.29a}$$

The second is that θ' is directed counterclockwise from the positive x_1-axis to the direction of $\boldsymbol{t}'$ (Fig. 3.2b). In this case we have $\theta = \pi/2 + \theta'$, so we have

$$\boldsymbol{n} = (-\sin\theta', \cos\theta'), \quad \boldsymbol{t}'(= \cos\theta', \sin\theta'); \quad \boldsymbol{T} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s \tag{3.29b}$$

3.2.3 Orthogonality of A and B

From Eq. (3.14) we can get (Ting 1996; Kuang 2011)

$$\begin{bmatrix} -\boldsymbol{Q} & \mathbf{0} \\ -\boldsymbol{R}^{\mathrm{T}} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} = \mu \begin{bmatrix} \boldsymbol{R} & \mathbf{I} \\ \boldsymbol{T} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} \tag{3.30}$$

Multiply on both sides of Eq. (3.30) from left by the following matrix:

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{T}^{-1} \\ \mathbf{I} & -\boldsymbol{R}\boldsymbol{T}^{-1} \end{bmatrix}$$

Equation (3.30) can be reduced to the standard 8×8 eigen-equation

$$\boldsymbol{N}\boldsymbol{\xi} = \mu_P\boldsymbol{\xi}, \quad \boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 & \boldsymbol{N}_1^T \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix}$$
$$\boldsymbol{N}_1 = -\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}}, \quad \boldsymbol{N}_2 = \boldsymbol{T}^{-1}, \quad \boldsymbol{N}_3 = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}} - \boldsymbol{Q} \tag{3.31}$$

where ξ is the right eigenvector. By using Eq. (3.25), from Eq. (3.31), it yields $[\boldsymbol{U}_{,2}\ \boldsymbol{\Phi}_{,2}] = [\boldsymbol{U}_{,1}\ \boldsymbol{\Phi}_{,1}]\boldsymbol{N}^{\mathrm{T}}$. If multiply on both sides of Eq. (3.31) from left by the matrix

$$\boldsymbol{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{J} = \boldsymbol{J}^{\mathrm{T}} = \boldsymbol{J}^{-1}, \quad \boldsymbol{J}\boldsymbol{N} = (\boldsymbol{J}\boldsymbol{N})^{\mathrm{T}} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{J} \tag{3.32}$$

Eq. (3.30) can be reduced to

$$\boldsymbol{J}\boldsymbol{N}\boldsymbol{\xi} = \boldsymbol{N}^{\mathrm{T}}(\boldsymbol{J}\boldsymbol{\xi}) = \mu(\boldsymbol{J}\boldsymbol{\xi}), \quad \text{or} \quad \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\eta} = \mu\boldsymbol{\eta} = \quad \boldsymbol{\eta} = \boldsymbol{J}\boldsymbol{\xi} = [\boldsymbol{b}, \boldsymbol{a}]^{\mathrm{T}} \tag{3.33}$$

where $\boldsymbol{\eta}$ is the left eigenvector. According to the mathematical theory (Ting 1996), the left and right eigenvectors associated with different eigenvalues are orthogonal to each other. So for the normalized ξ and $\boldsymbol{\eta}$, we have

$$\boldsymbol{\eta}_i^{\mathrm{T}}\boldsymbol{\xi}_j = \delta_{ij}, \quad \text{or} \quad \boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{a}_j + \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{b}_j = \delta_{ij}, \quad \text{when} \quad \mu_i \neq \mu_j \tag{3.34a}$$

From Eq. (3.34a) the following identities can be obtained:

$$\begin{gathered} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \bar{\boldsymbol{B}}^{\mathrm{T}}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{B} = \mathbf{I}, \quad \boldsymbol{B}^{\mathrm{T}}\bar{\boldsymbol{A}} + \boldsymbol{A}^{\mathrm{T}}\bar{\boldsymbol{B}} = \bar{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{A} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{B} = \mathbf{0} \\ \boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} + \bar{\boldsymbol{A}}\bar{\boldsymbol{B}}^{\mathrm{T}} = \boldsymbol{B}\boldsymbol{A}^{\mathrm{T}} + \bar{\boldsymbol{B}}\bar{\boldsymbol{A}}^{\mathrm{T}} = \mathbf{I}, \quad \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} + \bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^{\mathrm{T}} = \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}} + \bar{\boldsymbol{B}}\bar{\boldsymbol{B}}^{\mathrm{T}} = \mathbf{0}; \quad \text{or} \\ \begin{pmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} & \bar{\boldsymbol{A}}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}; \quad \begin{pmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} & \bar{\boldsymbol{A}}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \end{gathered} \tag{3.34b}$$

From above equations it is known that $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}$ are pure imaginary. Let

$$\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} = 2\mathrm{i}\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}, \quad \boldsymbol{L} = \boldsymbol{L}^{\mathrm{T}} = -2\mathrm{i}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}, \quad \boldsymbol{S} = \mathrm{i}\left(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} - \mathbf{I}\right) \tag{3.35}$$

where $\boldsymbol{M}$ and $\boldsymbol{L}$ are real positive definite symmetric matrixes and $\boldsymbol{S}$ is a real matrix. It is easy to prove that there are the following relations:

$$\boldsymbol{L}\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{L} = \mathbf{0}, \quad \boldsymbol{M}\boldsymbol{S}^{\mathrm{T}} + \boldsymbol{S}\boldsymbol{M} = \mathbf{0}, \quad \boldsymbol{M}\boldsymbol{L} - \boldsymbol{S}\,\boldsymbol{S} = \mathbf{I} \tag{3.36}$$

From Eqs. (3.35) and (3.36), it is known that $\boldsymbol{S}\boldsymbol{L}^{-1}$ and $\boldsymbol{M}^{-1}\boldsymbol{S}$ are antisymmetric matrix. Using $\boldsymbol{A}\boldsymbol{B}^{-1} = \boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}\left(\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\right)^{-1}, \boldsymbol{B}\boldsymbol{A}^{-1} = \left(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\right)^{-1}$ and the above equations we get

$$\begin{aligned} &\boldsymbol{Y} = \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1} = -\mathrm{i}(\boldsymbol{S} + \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1} = \mathrm{i}\boldsymbol{L}^{-1}\left(\boldsymbol{S}^{\mathrm{T}} - \mathrm{i}\mathbf{I}\right), \quad \bar{\boldsymbol{Y}}^{\mathrm{T}} = \mathrm{i}\left(\boldsymbol{S}\boldsymbol{L}^{-1}\right)^{\mathrm{T}} + \boldsymbol{L}^{-1} \\ &\quad = -\mathrm{i}\boldsymbol{S}\boldsymbol{L}^{-1} + \boldsymbol{L}^{-1} = \boldsymbol{Y} \\ &\boldsymbol{Y}^{-1} = -\mathrm{i}\boldsymbol{B}\boldsymbol{A}^{-1} = -\mathrm{i}\left(\boldsymbol{S}^{\mathrm{T}} + \mathrm{i}\mathbf{I}\right)\boldsymbol{M}^{-1} = \mathrm{i}\boldsymbol{M}^{-1}(\boldsymbol{S} - \mathrm{i}\mathbf{I}) = \boldsymbol{M}^{-1} + \mathrm{i}\boldsymbol{M}^{-1}\boldsymbol{S} \end{aligned} \tag{3.37}$$

It is obvious that $\boldsymbol{Y}$ is a Hermite matrix, i.e., $\boldsymbol{Y} = \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1} = \bar{\boldsymbol{Y}}^{\mathrm{T}}, \boldsymbol{Y}^{-1} = \bar{\boldsymbol{Y}}^{-\mathrm{T}}$.

3.2.4 Semisimple and Degenerate Matrixes

If the eigenvectors in Eq. (3.11) corresponding to each repeated root λ of multiplicity r in Eq. (3.12) have r independent eigenvectors $\lambda_\nu, \nu = 1, \ldots r$, the corresponding matrix $\boldsymbol{D}$ is called semisimple. The eigen-space is complete for the semisimple matrix. For a semisimple matrix, the eigenvectors associated with a repeated eigenvalue are not unique; however, it is possible to establish a set of eigenvectors such that the orthogonality relations hold and normalized. In this case the general solutions Eqs. (3.16) and (3.18) are also held. A real and symmetric matrix or a complex Hermite matrix is always either simple or semisimple, and their eigenvalues are all real. If the number of the independent eigenvectors is less than the multiplicity of a repeat root, the corresponding matrix is called nonsemisimple or degenerate matrix. The eigen-space is not complete for the degenerate matrix. In order to make the eigen-space of the degenerate matrix complete, we can establish the generalized eigenvectors to provide the missing eigenvectors (Ting 1996). Sometimes in the practical calculation, a very small difference between the repeated roots is assumed to approximately satisfy the eigen-equation.

3.2.5 A General Theory of the Generalized Eigenvectors

The general theory of the generalized eigenvectors for the simple, semisimple, and degenerated matrixes is expressed in the following theorem (Dempsey and Sinclair 1979; Yang et al. 1997):

Theorem *Let μ be the eigenvalue of a square matrix $\boldsymbol{D}(\mu)$ of order n (here $n = 4$) and $\boldsymbol{a}$ be the corresponding eigenvector. If rank of matrix $\boldsymbol{D}$ is $m = n - r < n$, where r is the number of the eigenvectors corresponding to a repeated eigenvalue, and if at $\mu = \mu_p$, we have*

$$\boldsymbol{D}\boldsymbol{a} = 0 \tag{3.38}$$

$$\mathrm{d}(\boldsymbol{D}\boldsymbol{a})/\mathrm{d}\mu = (\mathrm{d}\boldsymbol{D}/\mathrm{d}\mu)\boldsymbol{a} + \boldsymbol{D}(\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu) = \mathbf{0} \tag{3.39}$$

$$\mathrm{d}^2(\boldsymbol{D}\boldsymbol{a})/\mathrm{d}\mu^2 = \left(\mathrm{d}^2\boldsymbol{D}/\mathrm{d}\mu^2\right)\boldsymbol{a} + 2(\mathrm{d}\boldsymbol{D}/\mathrm{d}\mu)(\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu) + \mathbf{D}\boldsymbol{D}\left(\mathrm{d}^2\boldsymbol{a}/\mathrm{d}\mu^2\right) = \mathbf{0} \tag{3.40}$$

In order to get nontrivial solution for $\boldsymbol{a}$ in Eq. (3.38), it must be

$$|\boldsymbol{D}| = 0 \tag{3.41}$$

In order to get nontrivial solution for $\boldsymbol{a}$ and $\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu$ in Eq. (3.39), it must be

$$|\boldsymbol{D}| = \mathrm{d}^{n-m}|\boldsymbol{D}|/\mathrm{d}\mu^{n-m} = 0 \tag{3.42}$$

In order to get nontrivial solution for $\boldsymbol{a}$, $\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu$, and $\mathrm{d}^2\boldsymbol{a}/\mathrm{d}\mu^2$ in Eq. (3.40), it must be

$$|\boldsymbol{D}| = \mathrm{d}^{n-m}|\boldsymbol{D}|/\mathrm{d}\mu^{n-m} = \mathrm{d}^{2(n-m)}|\boldsymbol{D}|/\mathrm{d}\mu^{2(n-m)} = 0 \tag{3.43}$$

1. $\mu_1, \mu_2, \mu_3, \mu_4$ are all single roots: $|\boldsymbol{D}|$ is a polynomial containing first power of μ_j, so $\mathrm{d}|\boldsymbol{D}|/\mathrm{d}\mu_j \neq 0$. In this case $r = 1, m = n - 1 = 3$, and Eq. (3.42) is not satisfied. Equation (3.38) has four independent eigenvectors. The general solution of $\boldsymbol{U}$ is expressed by Eq. (3.16).

2. μ_1 is a repeated root with multiplicity 2 and μ_3, μ_4 are single roots: $|\boldsymbol{D}|$ is a polynomial containing second power of μ_j, so $\mathrm{d}^2|\boldsymbol{D}|/\mathrm{d}\mu_1^2 \neq 0$.

(a) There are two independent eigenvectors corresponding to $\mu_1, m = n - 2 = 2$. In this case Eq. (3.42) is not satisfied. The general solution of $\boldsymbol{U}$ is still expressed by Eq. (3.16).

(b) There is only one independent eigenvectors corresponding to μ_1, $m = n - 1 = 3$. In this case Eq. (3.42) can be satisfied; $\boldsymbol{a}_1$ and $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ in Eq. (3.39) all have nontrivial solutions. The general solution of $\boldsymbol{U}$ can be expressed by

$$\begin{aligned} &\boldsymbol{U} = 2\mathrm{Re}\left[\boldsymbol{A}' f(z_*) + x_2 \boldsymbol{a}_1 f_1'(z_1)\right]; \quad \boldsymbol{A}' = [\boldsymbol{a}_1, \mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1, \boldsymbol{a}_3, \boldsymbol{a}_4] \\ &\boldsymbol{f}(z_*) = [f_1(z_1), f_1(z_1), f_3(z_3), f_4(z_4)]^{\mathrm{T}} \end{aligned} \tag{3.44}$$

where $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ is solved from Eq. (3.39).

3. μ_1 is a repeated root with multiplicity 3 and μ_3 is a single root: $|\boldsymbol{D}|$ is a polynomial containing third power of μ_j, so $\mathrm{d}^3|\boldsymbol{D}|/\mathrm{d}\mu_1^3 \neq 0$.

(a) There are three independent eigenvectors corresponding to $\mu_1, m = n - 3 = 1$. In this case it is still that only Eq. (3.41) has nontrivial solution. The general solution of $\boldsymbol{U}$ is still expressed by Eq. (3.16).

(b) There are two independent eigenvectors corresponding to $\mu_1, m = n - 2 = 2$. In this case

Eq. (3.42) can be satisfied; $\boldsymbol{a}_1$ and $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ in Eq. (3.39) have nontrivial solutions. The general solution of $\boldsymbol{U}$ can still be expressed by (3.44).

(c) There is only one independent eigenvector corresponding to $\mu_1, m = n - 1 = 3$. In this case Eq. (3.43) is satisfied. $\boldsymbol{a}_1$, $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$, and $\mathrm{d}^2\boldsymbol{a}_1/\mathrm{d}\mu_1^2$ all have nontrivial solutions. The general solution of $\boldsymbol{U}$ can be expressed by

$$\begin{aligned} &\boldsymbol{U} = 2\mathrm{Re}\left[\boldsymbol{A}'' f(z_*) + x_2 \boldsymbol{a}_1 f_1'(z_1) + 2x_2(\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1) f_1'(z_1) + x_2^2 \boldsymbol{a}_1 f_1''(z_1)\right]; \\ &\boldsymbol{A}'' = \left[\boldsymbol{a}_1, \mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1, \boldsymbol{a}_3, \mathrm{d}^2\boldsymbol{a}_1/\mathrm{d}\mu_1^2\right], \quad \boldsymbol{f}(z_P) = [f_1(z_1), f_1(z_1), f_3(z_3), f_1(z_1)]^{\mathrm{T}} \end{aligned} \tag{3.45}$$

where $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ is solved from Eq. (3.40).

3.2.6 Electric Displacement Tensor Method

The fourth kind of the constitutive equations in Eq. (2.83) is

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - h_{nij}D_n, \quad E_i = -h_{imn}\varepsilon_{mn} + \beta_{in}D_n \tag{3.46}$$

Shen and Kuang (1999a) introduced an antisymmetric tensor $\boldsymbol{G}$ of second order and a vector potential $\boldsymbol{\psi}$ of the electric displacement to satisfy $\nabla \cdot \boldsymbol{D} = 0$ automatically and let

$$D_i = \varpi_{imn}G_{mn}, \quad G_{ij} = \left(\psi_{i,j} - \psi_{j,i}\right)/2, \quad G_{ij} = -G_{ji}; \quad \psi_{i,i} = 0 \tag{3.47}$$

where $\nabla \cdot \boldsymbol{\Psi} = 0$ is the condition to make $\boldsymbol{\Psi}$ unique and $\boldsymbol{\varpi}$ is a permutation tensor:

$$\varpi_{123} = \varpi_{231} = \varpi_{312} = 1, \quad \varpi_{213} = \varpi_{132} = \varpi_{321} = -1, \quad \text{otherwise} \quad \varpi_{ijk} = 0 \tag{3.48}$$

Introduce the electric tensor $\boldsymbol{L}$:

$$E_i = (1/2)\varpi_{imn}L_{mn}, \quad L_{mn} = \varpi_{imn}E_i = -L_{nm} \tag{3.49}$$

Using $E_{i,j} = E_{j,i}$ from Eq. (3.49), we get

$$L_{ij,j} = \varpi_{mij}E_{m,j} = \varpi_{ijm}E_{m,j} = \varpi_{imj}\varphi_{,mj} = 0 \tag{3.50}$$

Using Eqs. (3.47) and (3.49), the constitutive equations Eq. (3.46) can be written as

$$\begin{aligned}
&\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - h_{tij}\varpi_{tmn}G_{mn} = C_{ijmn}\varepsilon_{mn} - \bar{h}_{mnij}G_{mn}\\
&L_{ij} = \varpi_{tij}E_t = \varpi_{tij}\left(-h_{tmn}\epsilon_{mn} + \beta_{tn}\varpi_{npq}\varepsilon_{npq}G_{pq}\right) = -\bar{h}_{ijmn}\varepsilon_{mn} + \bar{\beta}_{ijmn}G_{mn}\\
&\bar{h}_{mnij} = \bar{h}_{mnji} = -\bar{h}_{nmij} = \varpi_{tmn}h_{tij}\\
&\bar{\beta}_{ijmn} = \bar{\beta}_{mnij} = -\bar{\beta}_{jimn} = -\bar{\beta}_{ijnm}\varpi_{tij}\varpi_{npq}\beta_{tn}
\end{aligned} \tag{3.51}$$

Using Eqs. (3.47) and (3.51), the equations $\nabla \cdot \sigma = 0$, $\boldsymbol{E} = -\nabla\varphi$ can be written as

$$\left(C_{ijkl}u_{k,l} - \bar{h}_{klij}\psi_{k,l}\right)_{,j} = 0, \quad \left(\bar{h}_{ijkl}u_{k,l} - \bar{\beta}_{ijkl}\psi_{k,l}\right)_{,j} = 0 \tag{3.52a}$$

When material coefficients are all constants for the general plane problem, Eq. (3.52a) becomes

$$U_{i\alpha\beta,\alpha\beta} = 0, \quad L_{i\alpha\beta,\alpha\beta} = 0; \quad U_{i\alpha\beta} = C_{i\beta k\alpha}u_k - \bar{h}_{k\alpha i\beta}\psi_k, \quad L_{i\alpha\beta} = \bar{h}_{i\beta k\alpha}u_k - \bar{\beta}_{i\beta k\alpha}\psi_k \tag{3.52b}$$

Equation (3.52) is a pretty equation. How to use it in engineering should be studied in the future.

3.3 Stress Function Method

3.3.1 Solution for a General Piezoelectric Material

Using the Voigt notation, the third kind of the constitutive equations in Eq. (2.84) is

$$\varepsilon_i = s_{ij}\sigma_j + g_{\alpha i}D_\alpha, \quad E_\alpha = -g_{\alpha j}\sigma_j + \beta_{\alpha\beta}D_\beta; \quad i,j = 1-6;\ \alpha,\beta = 1-3 \tag{3.53}$$

In this section a subscript in English letter takes the values $1-6$ and a subscript in Greek letter takes the values $1-3$. In the general plane strain problem, $u_{\alpha,3} = 0$ and

$$\varepsilon_3 = u_{3,3} = s_{3j}\sigma_j + g_{\alpha 3}D_\alpha = 0, \quad E_3 = -\varphi_{,3} = -g_{3j}\sigma_j + \beta_{3\alpha}D_\alpha = 0 \tag{3.54}$$

Solving σ_3 and D_3 from Eq. (3.54) yields

$$\begin{aligned}
&\sigma_3 = F_j\sigma_j + G_\alpha D_\alpha, \quad D_3 = H_j\sigma_j + J_\alpha D_\alpha, \quad (j,\alpha \neq 3)\\
&F_j = -\left(g_{33}g_{3j} + s_{3j}\beta_{33}\right)M, \quad G_\alpha = (g_{33}\beta_{3\alpha} - g_{\alpha 3}\beta_{33})M,\\
&H_j = \left(s_{33}g_{3j} - s_{3j}g_{33}\right)M, \quad J_\alpha = -(s_{33}\beta_{3\alpha} + g_{33}g_{\alpha 3})M, \quad M = 1/\left(g_{33}^2 + s_{33}\beta_{33}\right)
\end{aligned} \tag{3.55}$$

Substitution of Eq. (3.55) into Eq. (3.53) yields

$$\begin{aligned}
&u_{1,1} = \kappa_{1j}\sigma_j + \eta_{1\alpha}D_\alpha, \quad u_{2,2} = \kappa_{2j}\sigma_j + \eta_{2\alpha}D_\alpha, \quad u_{3,2} = \kappa_{4j}\sigma_j + \eta_{4\alpha}D_\alpha,\\
&u_{3,1} = \kappa_{5j}\sigma_j + \eta_{5\alpha}D_\alpha, \quad u_{2,1} + u_{1,2} = \kappa_{6j}\sigma_j + \eta_{6\alpha}D_\alpha,\\
&E_1 = -h_{1j}\sigma_j + \xi_{1\alpha}D_\alpha, \quad E_2 = -h_{2j}\sigma_j + \xi_{2\alpha}D_\alpha \quad (j,\alpha \neq 3)
\end{aligned} \tag{3.56}$$

where the reduced constants $\kappa_{ij}, \eta_{i\alpha}, h_{\alpha j}, \xi_{\beta\alpha}$ are

$$\begin{aligned}
&\kappa_{ij} = s_{ij} + s_{i3}F_j + g_{3i}H_j = \kappa_{ji}, \quad \eta_{j\alpha} = g_{\alpha j} + s_{j3}G_\alpha + g_{3j}J_\alpha,\\
&h_{\alpha j} = g_{\alpha j} + g_{\alpha 3}F_j - \beta_{\alpha 3}H_j = \eta_{j\alpha}, \quad \xi_{\beta\alpha} = \beta_{\beta\alpha} - g_{\beta 3}G_\alpha + \beta_{\beta 3}J_\alpha
\end{aligned} \tag{3.57}$$

Applying Lekhnitskii method (1987, 1957), Kosmodamianskii and Lozhkin (1975) discussed the plane stress state of thin piezoelectric plates and gave the expressions with complex potentials. Hao and Shen (1994) and Huang and Kuang (2000a) discussed the general generalized plane problem. They introduced the stress functions Λ, Ψ and the electric potential V to satisfy the generalized equilibrium equations automatically:

$$\sigma_1 = \Lambda_{,22}, \quad \sigma_2 = \Lambda_{,11}, \quad \sigma_6 = -\Lambda_{,12}, \quad \sigma_4 = -\Psi_{,1}, \quad \sigma_5 = \Psi_{,2}; \quad D_1 = V_{,2}, \quad D_2 = -V_{,1} \tag{3.58}$$

Substitution of Eq. (3.58) into Eq. (3.56) finally yields the general compatibility equation

$$L_4\Lambda + L_3\Psi + L_5V = 0, \quad L_3\Lambda + L_2\Psi + L_6V = 0, \quad L_5\Lambda + L_6\Psi - L_9V = 0 \tag{3.59}$$

where

$$\begin{aligned}
L_2 &= \kappa_{55}\frac{\partial^2}{\partial x_2^2} - 2\kappa_{45}\frac{\partial^2}{\partial x_2\partial x_1} + \kappa_{44}\frac{\partial^2}{\partial x_1^2}\\
L_3 &= \kappa_{15}\frac{\partial^3}{\partial x_2^3} - (\kappa_{14}+\kappa_{56})\frac{\partial^3}{\partial x_2^2\partial x_1} + (\kappa_{55}+\kappa_{46})\frac{\partial^3}{\partial x_2\partial x_1^2} - \kappa_{24}\frac{\partial^3}{\partial x_1^3}\\
L_4 &= \kappa_{11}\frac{\partial^4}{\partial x_2^4} - 2\kappa_{16}\frac{\partial^4}{\partial x_2^3\partial x_1} + (2\kappa_{12}+\kappa_{66})\frac{\partial^4}{\partial x_2^2\partial x_1^2} - 2\kappa_{26}\frac{\partial^4}{\partial x_2\partial x_1^3} + \kappa_{22}\frac{\partial^4}{\partial x_1^4}\\
L_5 &= \eta_{11}\frac{\partial^3}{\partial x_2^3} - (\eta_{12}+\eta_{61})\frac{\partial^3}{\partial x_2^2\partial x_1} + (\eta_{21}+\eta_{62})\frac{\partial^3}{\partial x_2\partial x_1^2} + \eta_{22}\frac{\partial^3}{\partial x_1^3}\\
L_6 &= \eta_{51}\frac{\partial^2}{\partial x_2^2} - (\eta_{52}+\eta_{41})\frac{\partial^2}{\partial x_2\partial x_1} + \eta_{42}\frac{\partial^2}{\partial x_1^2}\\
L_7 &= -L_5 = -h_{11}\frac{\partial^3}{\partial x_2^3} + (h_{16}+h_{21})\frac{\partial^3}{\partial x_2^2\partial x_1} - (h_{12}+h_{26})\frac{\partial^3}{\partial x_2\partial x_1^2} - h_{22}\frac{\partial^3}{\partial x_1^3}\\
L_8 &= -L_6 = -h_{15}\frac{\partial^2}{\partial x_2^2} + (h_{14}+h_{25})\frac{\partial^2}{\partial x_2\partial x_1} - h_{24}\frac{\partial^2}{\partial x_1^2}\\
L_9 &= \xi_{11}\frac{\partial^2}{\partial x_2^2} - (\xi_{12}+\xi_{21})\frac{\partial^2}{\partial x_2\partial x_1} + \xi_{22}\frac{\partial^2}{\partial x_1^2}
\end{aligned} \tag{3.60}$$

Eliminating Ψ and V from Eq. (3.59) yields an eighth-order differential equation of Λ:

$$\begin{aligned}
&\left(L_6L_8L_4 - L_9L_4L_2 + L_9L_3^2 - L_5L_3L_8 + L_2L_5L_7 - L_3L_6L_7\right)\Lambda = 0, \quad \text{or}\\
&\left(L_6L_8L_4 - L_9L_4L_2 + L_9L_3^2 + 2L_5L_3L_6 + L_2L_5L_7\right)\Lambda = 0
\end{aligned} \tag{3.61}$$

Its solution is

$$\Lambda = 2\mathrm{Re}\sum_{P=1}^{4}\tilde{f}_P(z_P), \quad z_P = x_1 + \mu_P x_2 \tag{3.62}$$

where $\tilde{f}_P(z_P)$ is an analytic function of z_P and μ_P is the root of the following eigen-equation

$$\begin{aligned}
&l_6l_8l_4 - l_9l_4l_2 + l_9l_3^2 - l_5l_3l_8 + l_2l_5l_7 - l_3l_6l_7 = 0, \quad \text{or}\\
&l_4l_6^2 + l_2l_4l_9 - l_3^2l_9 - 2l_3l_5l_6 + l_2l_5^2 = 0
\end{aligned} \tag{3.63}$$

where $l_i \equiv l_i(\mu)$ can be obtained by using μ instead of the differential operator $\partial/\partial x_2$ in L_i of Eq. (3.60) and the power of μ is the same as the power of $\partial/\partial x_2$, such as

$$l_3(\mu) = \kappa_{15}\mu^3 - (\kappa_{14} + \kappa_{56})\mu^2 + (\kappa_{55} + \kappa_{46})\mu - \kappa_{24}$$

where μ_P is the same as that in Stroh's formula. From Eq. (3.59), it is obtained that

$$\begin{aligned} &\Psi = 2\mathrm{Re}\sum_{p=1}^{4} a_P f_P(z_P), \quad V = 2\mathrm{Re}\sum_{p=1}^{4} b_P f_P(z_P), \quad f_P(z_P) = d\tilde{f}'_p/dz_P \\ &a_P = -\frac{l_5 b_P + l_4}{l_3} = -\frac{l_6 b_P + l_3}{l_2} = -\frac{l_9 b_P + l_7}{l_8}, \quad b_P = \frac{l_3^2 - l_4 l_2}{l_2 l_5 - l_3 l_6} = \frac{l_8 l_3 - l_2 l_7}{l_2 l_9 - l_8 l_6} \end{aligned} \tag{3.64}$$

where $l_i \equiv l_i(\mu_P)$. Substitution of Λ, Ψ and V into Eq. (3.58) yields the generalized stress; then substitution of the result into Eq. (3.56) yields the generalized displacements. Comparing the generalized stress and displacement with that in Stroh's formula, the explicit forms of $\boldsymbol{B}$ and $\boldsymbol{A}$ are obtained:

$$B = \begin{bmatrix} -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 \\ 1 & 1 & 1 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 \\ -b_1 & -b_2 & -b_3 & -b_4 \end{bmatrix}, \quad \mathbf{A} = [A_{ij}] \tag{3.65}$$

$$\begin{aligned} A_{1j} &= \kappa_{11}\mu_j^2 + \kappa_{12} - \kappa_{16}\mu_j + a_j(\kappa_{15}\mu_j - \kappa_{14}) + (\eta_{11}\mu_j - \eta_{12})b_j \\ A_{2j} &= \left[\kappa_{21}\mu_j^2 + \kappa_{22} - \kappa_{26}\mu_j + a_j(\kappa_{25}\mu_j - \kappa_{24}) + (\eta_{21}\mu_j - \eta_{22})b_j\right] \Big/ \mu_j \\ A_{3j} &= \left[\kappa_{41}\mu_j^2 + \kappa_{42} - \kappa_{46}\mu_j + a_j(\kappa_{45}\mu_j - \kappa_{44}) + (\eta_{41}\mu_j - \eta_{42})b_j\right] \Big/ \mu_j \\ A_{4j} &= h_{11}\mu_j^2 + h_{12} - h_{16}\mu_j + a_j(h_{15}\mu_j - h_{14}) + (\xi_{11}\mu_j - \xi_{12})b_j \end{aligned} \tag{3.66}$$

The above results are obtained for the generalized plane strain. For the generalized plane stress, the constants should be simply replaced by

$$\kappa_{ij} = s_{ij} = \kappa_{ji}, \quad \eta_{j\alpha} = g_{\alpha j}, \quad h_{\alpha j} = g_{\alpha j}, \quad \xi_{\beta\alpha} = \beta_{\beta\alpha} \quad (j, \alpha, \beta \neq 3) \tag{3.67}$$

It is also noted that the plane stress deformation can be existed only in the materials with at least one material symmetric plane such as monoclinic material.

From Eq. (3.58), the stress functions can be obtained as

$$\begin{aligned} &\Phi_1 = -\Lambda_{,2} = -2\mathrm{Re}\sum_{P=1}^{4} \mu_P f_P(z_P), \quad \Phi_2 = \Lambda_{,1} = 2\mathrm{Re}\sum_{P=1}^{4} f_P(z_P) \\ &\Phi_3 = -\Psi = -2\mathrm{Re}\sum_{P=1}^{4} a_P f_P(z_P), \quad \Phi_4 = -V = -2\mathrm{Re}\sum_{P=1}^{4} b_P f_P(z_P) \end{aligned} \tag{3.68}$$

Equation (3.68) shows that $\boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)]$ where $\boldsymbol{B}$ is shown in Eq. (3.65). This is consistent with the Stroh's formula Eq. (3.23).

3.3.2 The Transversely Isotropic Material in Plane Strain

Usually in engineering the coordinate system x-y-z is used, and the material constants given in the handbooks are under the condition that the poling direction is along the axis z. For a general piezoelectric material, there are 45 independent material constants: 21 elastic constants, 18 piezoelectric constants, and 6 permittivity constants. For the orthogonal materials in the material principle coordinate system with poling axis z, the plane x-y is an isotropic plane. For an isotropic plane x-y, the in-plane electric field couples only with the out-plane mechanical stress. In the anisotropic plane x-z, y-z, the in-plane electric field couples with the in-plane mechanical stress, and the mechanical behaviors in x-z and y-z planes are the same. In this case when the axis z is taken as the poling axis, there are 17 independent material constants: 9 elastic constants, $s_{11}, s_{12}, s_{13}, s_{22}, s_{23}, s_{33}, s_{44}, s_{55}, s_{66}$ or $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55}, C_{66}$; 5 piezoelectric constants, g_{15}, $g_{24}, g_{31}, g_{32}, g_{33}$ or $e_{15}, e_{24}, e_{31}, e_{32}, e_{33}$; and 3 electric constants, $\beta_{11}, \beta_{22}, \beta_{33}$ or ϵ_{11}, $\epsilon_{22}, \epsilon_{33}$. The second kind of the constitutive equation in Eq. (2.84) is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \\ D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 & 0 & 0 & -e_{32} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & -e_{24} & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 & -e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 & 0 & \epsilon_{22} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 & 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ E_x \\ E_y \\ E_z \end{Bmatrix} \tag{3.69a}$$

The third kind of the constitutive equation in Eq. (2.84) is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ E_x \\ E_y \\ E_z \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 & 0 & 0 & g_{31} \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 & 0 & 0 & g_{32} \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 & 0 & 0 & g_{33} \\ 0 & 0 & 0 & s_{44} & 0 & 0 & 0 & g_{24} & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 & g_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_{15} & 0 & \beta_{11} & 0 & 0 \\ 0 & 0 & 0 & -g_{24} & 0 & 0 & 0 & \beta_{22} & 0 \\ -g_{31} & -g_{32} & -g_{33} & 0 & 0 & 0 & 0 & 0 & \beta_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \\ D_x \\ D_y \\ D_z \end{Bmatrix} \tag{3.69b}$$

It is noted that $[\boldsymbol{s}] = [\boldsymbol{C}]^{-1}$. For the transversely isotropic material, such as piezoelectric ceramic PZT and many other materials, in the material principle coordinate system, there are ten independent material constants because there are relations between material constants:

$$\begin{aligned}
&s_{13} = s_{23}, s_{11} = s_{22}, s_{44} = s_{55}, s_{66} = 2(s_{11} - s_{12}); \quad g_{31} = g_{32}, g_{15} = g_{24}; \quad \beta_{11} = \beta_{22} \\
&C_{13} = C_{23}, C_{11} = C_{22}, C_{44} = C_{55}, C_{66} = (C_{11} - C_{12})/2; \quad e_{31} = e_{32}, e_{15} = e_{24}; \quad \epsilon_{11} = \epsilon_{22}
\end{aligned} \tag{3.70}$$

In this section the plane strain problem is discussed and adopted the third kind of the constitutive equation, Eq. (3.69b). Let

$$\varepsilon_x = \gamma_{zx} = \gamma_{xy} = E_x = 0 \tag{3.71}$$

From Eq. (3.71), it can be obtained that

$$D_x = 0, \quad \tau_{zx} = \tau_{yz} = 0, \quad \sigma_x = -\left(s_{12}\sigma_y + s_{13}\sigma_z + g_{31}D_z\right)/s_{11} \tag{3.72}$$

Analogous to the Voigt expression of the stress and strain in 3D case, we introduce the vector expression of the stress and strain in plane strain case. Let

$$\begin{aligned}
&x_1 = y,\ x_2 = z,\ x_3 = x; \quad \sigma_1 = \sigma_y,\ \sigma_2 = \sigma_z,\ \sigma_3 = \tau_{yz}; \\
&\varepsilon_1 = \varepsilon_y,\ \varepsilon_2 = \varepsilon_z,\ \varepsilon_3 = \gamma_{yz}; \quad D_1 = D_y,\ D_2 = D_z; \quad E_1 = E_y, E_2 = E_z
\end{aligned} \tag{3.73}$$

Substitution of Eqs. (3.71), (3.72), and (3.73) into Eq. (3.69) yields

$$\begin{aligned}
&\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ E_1 \\ E_2 \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & b_{21} \\ a_{12} & a_{22} & 0 & 0 & b_{22} \\ 0 & 0 & a_{33} & b_{13} & 0 \\ 0 & 0 & -b_{13} & k_{11} & 0 \\ -b_{21} & -b_{22} & 0 & 0 & k_{22} \end{pmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ D_1 \\ D_1 \end{Bmatrix} \\
&a_{11} = s_{11} - s_{12}^2/s_{11}, \quad a_{12} = s_{13} - s_{12}s_{13}/s_{11}, \quad a_{22} = s_{33} - s_{13}^2/s_{11}, \quad a_{33} = s_{44} \\
&b_{21} = (1 - s_{12}/s_{11})g_{31}, \quad b_{22} = g_{33} - g_{31}s_{13}/s_{11}, \quad b_{13} = g_{15}, \quad k_{11} = \beta_{11}, \\
&k_{22} = \beta_{33} + g_{31}^2/s_{11}
\end{aligned} \tag{3.74}$$

where a_{ij}, b_{ij}, k_{ij} are reduced material constants and $s_{ij}g_{ij}, \beta_{ij}$ are material constants as shown in Eq. (3.69b). In the plane strain problem, $\sigma_{13} = \sigma_{23} = 0$, so the stress function Ψ in Eq. (3.58) is not needed. The eighth-order differential equation (3.61) is reduced to sixth-order differential equation, and the eighth-order eigen-equation (3.63) is reduced to sixth-order eigen-equation. Repeating the process analogous to Sect. 3.3.1 finally yields (Sosa 1991; Sosa and Khutoryansky 1996; Kuang and Ma 2002)

$$
\begin{aligned}
&\boldsymbol{U} = \boldsymbol{U}(u_1, u_2, \varphi) = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z_P)], \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}(\Phi_1, \Phi_2, \Phi_4) = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)] \\
&\boldsymbol{A} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} -\mu_1 & -\mu_2 & -\mu_3 \\ 1 & 1 & 1 \\ -\eta_1 & -\eta_2 & -\eta_3 \end{pmatrix}
\end{aligned} \tag{3.75}
$$

where μ_P is the root of the following eigen-equation:

$$
\begin{aligned}
&a_{11}k_{11}\mu^6 + \left(a_{11}k_{22} + 2a_{12}k_{11} + a_{33}k_{11} + b_{21}^2 + b_{13}^2 + 2b_{21}b_{13}\right)\mu^4 \\
&\quad + (a_{22}k_{11} + 2a_{12}k_{22} + a_{33}k_{22} + 2b_{21}b_{22} + 2b_{22}b_{13})\mu^2 + a_{22}k_{22} + b_{22}^2 = 0
\end{aligned} \tag{3.76}
$$

and

$$
\begin{aligned}
&p_P = a_{11}\mu_P^2 + a_{12} - b_{21}\eta_P, \quad q_P = \left(a_{12}\mu_P^2 + a_{22} - b_{22}\eta_P\right)/\mu_P \\
&\lambda_P = (b_{13} + k_{11}\eta_P)\mu_P, \quad \lambda_P\mu_P = -\left(b_{21}\mu_P^2 + b_{22} + k_{22}\eta_P\right) \\
&\eta_P = -\left[(b_{21} + b_{13})\mu_P^2 + b_{22}\right]/\left(k_{11}\mu_P^2 + k_{22}\right)
\end{aligned} \tag{3.77}
$$

If the rigid rotation angle ω is considered, we have

$$
\begin{aligned}
&u_1 = 2\mathrm{Re}\sum_{P=1}^{3} p_P f_P(z_P) - \omega x_3, \quad u_2 = 2\mathrm{Re}\sum_{P=1}^{3} q_P f_P(z_P) + \omega x_1, \\
&\varphi = -2\mathrm{Re}\sum_{P=1}^{3} \lambda_P f_P(z_P); \quad \Phi_1 = -2\mathrm{Re}\sum_{P=1}^{3} \mu_P f_P(z_P), \\
&\Phi_2 = 2\mathrm{Re}\sum_{P=1}^{3} f_P(z_P), \quad \Phi_4 = -2\mathrm{Re}\sum_{P=1}^{3} \eta_P f_P(z_P) \\
&\sigma_1 = 2\mathrm{Re}\sum_{j=1}^{3} \mu_j^2 F_j(z_j), \quad \sigma_2 = 2\mathrm{Re}\sum_{j=1}^{3} F_j(z_j), \quad \sigma_3 = -2\mathrm{Re}\sum_{j=1}^{3} \mu_j F_j(z_j) \\
&D_1 = 2\mathrm{Re}\sum_{j=1}^{3} \mu_j \eta_j F_j(z_j), \quad D_2 = -2\mathrm{Re}\sum_{j=1}^{3} \eta_j F_j(z_j)
\end{aligned} \tag{3.78}
$$

3.4 An Elliptic Hole or Inclusion in a Transversely Isotropic Piezoelectric Material

3.4.1 Electrical Permeable Hole

Let a transversely isotropic piezoelectric material with an elliptic hole of semiaxes a and b directed along the material principle axes x_1 and x_2, respectively, be subjected to the uniform generalized stresses at infinity. The hole is filled with air with

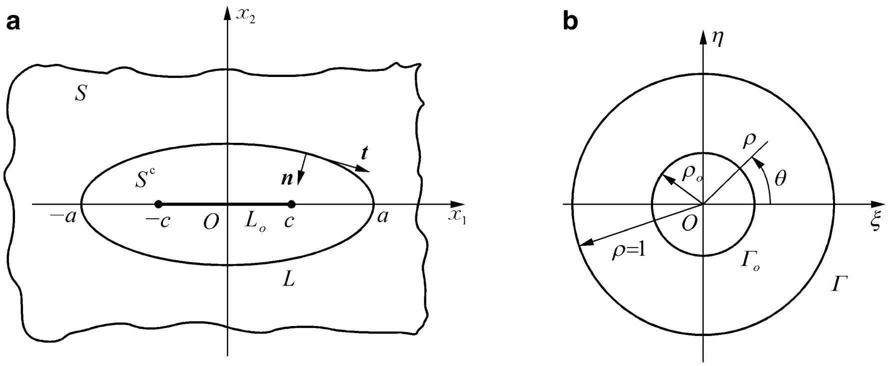


Fig. 3.3 An infinite plane with an elliptic hole: (**a**) physical plane z and (**b**) mapping plane ζ

permittivity ϵ_0 and is mechanically free (Fig. 3.3) (Parton 1976; Sosa and Khutoryansky 1996; Chung and Ting 1996; Zhang et al. 1998; Gao and Fan 1999; Kuang and Ma 2002). The potential electric field and electric displacement in the air region S^c are denoted by $\varphi^c, \boldsymbol{E}^c$ and $\boldsymbol{D}^c$, respectively, and in the piezoelectric material S are denoted by $\varphi, \boldsymbol{E}$ and $\boldsymbol{D}$, respectively. On the interface L, the outward normal is denoted by $\boldsymbol{n}$ directed from the material into the hole, and the first natural coordinate system (see Eqs. (3.27), (3.28)) is adopted. In the air only electric field is researched. Therefore the boundary conditions at infinity are

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{D} = \boldsymbol{D}^\infty \tag{3.79}$$

On the interface the connective conditions are

$$T_1 = T_2 = 0; \quad D_n = D_n^c = -\epsilon_0 \partial\varphi^c/\partial n, \quad \varphi = \varphi^c \quad \text{on} \quad L \tag{3.80}$$

The method solving this problem is the direct extension for the inclusion problem in an elastic anisotropic material (Mura 1987).

3.4.2 *Electric Field Inside the Hole Filled with Air*

It is assumed that there is free of charge in the air; from $\nabla \cdot \boldsymbol{D}^c = 0$, $\boldsymbol{D}^c = \epsilon_0 \boldsymbol{E}^c = -\epsilon_0 \boldsymbol{\nabla}\varphi^c$, the governing equation is

$$\nabla^2 \varphi^c = 0, \quad \text{in} \quad S^c \tag{3.81}$$

The conformal mapping function $\omega(\varsigma)$, transforming an ellipse L in the physical plane $z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta}$ into a unit circle Γ in the mapping plane $\varsigma = \xi_1 + \mathrm{i}\xi_2 = \rho\mathrm{e}^{\mathrm{i}\psi}$, is

$$z=\omega(\varsigma)=R\left(\varsigma+\frac{m}{\varsigma}\right),\quad \varsigma=\frac{z+\sqrt{z^2-4mR^2}}{2R};\quad R=\frac{a+b}{2},\quad m=\frac{a-b}{a+b}\quad \text{or}$$
$$z=R(1+m)\cos\psi+\mathrm{i}R(1-m)\sin\psi=a\cos\psi+\mathrm{i}b\sin\psi;\quad \text{or}\quad x_1=a\cos\psi,$$
$$x_2=b\sin\psi \tag{3.82a}$$

where (r,θ) and (ρ,ψ) are the polar axes and polar angles in the z and ς planes, respectively. Mapping function $\omega(\varsigma)$ transforms L into Γ and one to one for points outside the ellipse L into the outside of Γ, however, only one to one for points inside the ellipse L with a cut L_0 from $-c$ to c on the major axis in z plane into the inside of Γ with a circular cut L_0 of radius ρ_0 in ς plane (Fig. 3.3), where $\rho_0=\sqrt{m}<|z|<1,\ 0\leq\theta<2\pi$ and $c=\sqrt{a^2-b^2}$ is the half of the focal length.

From Eq. (3.82a), it is known that the arc lengths on Γ and on L are respectively

$$\mathrm{d}l^2=\mathrm{d}\varsigma\mathrm{d}\bar{\varsigma}=\mathrm{d}e^{\mathrm{i}\psi}\mathrm{d}e^{-\mathrm{i}\psi}=\mathrm{d}\psi^2;\quad \text{on } \Gamma$$
$$\mathrm{d}s^2=\mathrm{d}z\mathrm{d}\bar{z}=\omega'(\varsigma)\overline{\omega'(\varsigma)}\mathrm{d}\varsigma\mathrm{d}\bar{\varsigma}=\rho^2\mathrm{d}\psi^2,\quad \rho^2=a^2\sin^2\psi+b^2\cos^2\psi;\quad \text{on}\quad L$$
$$\mathrm{d}x_2/\mathrm{d}s=b\cos\psi/\rho,\ \mathrm{d}x_1/\mathrm{d}s=-a\sin\psi/\rho \tag{3.82b}$$

Because φ^c is a harmonic function, it can be expressed by an analytic function $\phi(z)$ as

$$\varphi^c(x_1,x_2)=\phi(z)+\overline{\phi(z)},\quad \text{in } z \text{ plane};\quad \varphi^c(\rho,\psi)=\phi(\varsigma)+\overline{\phi(\varsigma)},\quad \text{in } \zeta \text{ plane}$$
$$E_1^c=-\left[\phi'(z)+\overline{\phi'(z)}\right]=-2\mathrm{Re}\phi'(z),\quad E_2^c=-\mathrm{i}\left[\phi'(z)-\overline{\phi'(z)}\right]=2\mathrm{Im}\phi'(z) \tag{3.83}$$

where $\phi(\varsigma)=\phi[\omega(\varsigma)]$. Because $\phi(z)$ is analytic inside $L-L_0$ and continuous on L_0, so

$$\phi(\rho_0 e^{\mathrm{i}\psi})=\phi(\rho_0 e^{-\mathrm{i}\psi}) \tag{3.84}$$

The solution of $\phi(\varsigma)$ in the annular region $L-L_0$ can be expressed in the Laurent series

$$\phi(\varsigma)=\sum_{k=-\infty}^{\infty}a_k^c\varsigma^k,\quad a_{-k}^c=\rho_0^{2k}a_k^c\ (\text{no sum on } k),\quad \rho_0\leq|\varsigma|\leq 1 \tag{3.85}$$

3.4.3 Generalized Stress Field in the Piezoelectric Material

The general solution in a transversely isotropic material has been given in Sect. 3.3.2. The mapping function $\omega_j(\varsigma), j=1,2,3$ transforming an ellipse L_j in the physical plane $z_j=x_1+\mu_j x_2$ into a unit circle Γ_j in a mapping plane $\varsigma_j=\xi_1+\mu_j\xi_2$ is

$$
\begin{aligned}
& z_j = \omega_j(\varsigma_j) = c_j\varsigma_j + d_j\varsigma_j^{-1} = R_j\left(\varsigma_j + m_j\varsigma_j^{-1}\right) \\
& R_j = c_j = \left(a - \mathrm{i}\mu_j b\right)/2, \quad d_j = \left(a + \mathrm{i}\mu_j b\right)/2, \quad m_j = d_j/c_j \\
& \varsigma_j = \frac{z_j + \sqrt{z_j^2 - \left(a^2 + \mu_j^2 b^2\right)}}{a - \mathrm{i}\mu_j b}, \quad \frac{1}{\varsigma_j} = \frac{z_j - \sqrt{z_j^2 - (a^2 + \mu_j^2 b^2)}}{a + \mathrm{i}\mu_j b}
\end{aligned} \tag{3.86}
$$

When $\mu_j = \mathrm{i}$, Eq. (3.86) is reduced to Eq. (3.82), and the mapping function is conformal. If $\mu_j \neq \mathrm{i}$, the mapping function is not conformal. Because a function $f_j(z_j)$ outside L is analytically transformed into the region outside Γ in the ς plane, so

$$
f_j(z_j) = C_j z_j + f_j^0(\varsigma_j), \quad f_j^0(\varsigma_j) = a_{j0} + \sum_{k=1}^{\infty} a_{jk}\varsigma_j^{-k} \quad (\text{not summed on } k) \tag{3.87}
$$

where $f_j^0(z_j) = f_j^0[\omega(\varsigma_j)]$ is an analytic function in ς_j plane. C_j is determined by the boundary conditions at infinity and a_{jk} is undetermined coefficient. From Eq. (3.78),

$$
\begin{aligned}
& \sigma_1^{\infty} = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2 C_j, \quad \sigma_2^{\infty} = 2\mathrm{Re}\sum_{j=1}^{3} C_j, \quad \sigma_3^{\infty} = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j C_j \\
& D_1^{\infty} = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_j C_j, \quad D_2^{\infty} = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j C_j \\
& \left(E_1 = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j C_j, \quad E_2 = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j\mu_j C_j\right)
\end{aligned} \tag{3.88}
$$

In Eq. (3.88) if one real constant is selected arbitrarily, such as let Im $C_1 = 0$, it does not affect the stresses. So Eq. (3.88) is solvable.

3.4.4 The Connective Conditions on the Interface L

Equations (3.75) and (3.28) yield

$$
\begin{aligned}
& \Phi_1 = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j f_j(\sigma) = -\int_0^s T_1^* \mathrm{d}s = 0, \quad \Phi_2 = 2\mathrm{Re}\sum_{j=1}^{3} f_j(\sigma) = -\int_0^s T_2^* \mathrm{d}s = 0 \\
& \Phi_4 = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j(\sigma) = -\int_0^s D_n \mathrm{d}s
\end{aligned} \tag{3.89}
$$

where $\sigma = \mathrm{e}^{\mathrm{i}\psi}$ is a point on Γ. The mechanical connective conditions in Eq. (3.89) can be reduced to

$$\begin{aligned}&\sum_{j=1}^{3}\left\{f_j^0(\sigma)+\overline{f_j^0(\sigma)}\right\}=\bar{l}_1\sigma+l_1\bar{\sigma},\quad \sum_{j=1}^{3}\left\{\mu_j f_j^0(\sigma)+\overline{\mu_j f_j^0(\sigma)}\right\}=\bar{l}_2\sigma+l_2\bar{\sigma}\\&l_1=-(1/2)\left(a\sigma_2^\infty-\mathrm{i}b\sigma_3^\infty\right),\quad l_2=(1/2)\left(a\sigma_3^\infty-\mathrm{i}b\sigma_1^\infty\right)\end{aligned} \tag{3.90}$$

Using Eq. (3.27) and $\mathrm{d}s=|\mathrm{d}z|=|\omega'(\varsigma)||\mathrm{d}\varsigma|, \mathrm{d}\varsigma/|\mathrm{d}\varsigma|=\mathrm{e}^{\mathrm{i}\psi}$ we have

$$\begin{aligned}n&=n_1+\mathrm{i}n_2=\mathrm{d}x_2/\mathrm{d}s-\mathrm{i}\mathrm{d}x_1/\mathrm{d}s=-\mathrm{i}\mathrm{d}z/\mathrm{d}s=\mathrm{e}^{\mathrm{i}\psi}\omega'(\varsigma)/|\omega'(\varsigma)|\\t&=t_1+\mathrm{i}t_2=\mathrm{d}x_1/\mathrm{d}s+\mathrm{i}\mathrm{d}x_2/\mathrm{d}s=\mathrm{d}z/\mathrm{d}s=\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}\omega'(\varsigma)/|\omega'(\varsigma)|\end{aligned} \tag{3.91}$$

where n is the outward normal on the interface of S. Using Eqs. (3.91) and (3.83) yields

$$\begin{aligned}\frac{\partial\varphi^c}{\partial n}&=\frac{\partial\varphi^c}{\partial x_1}n_1+\frac{\partial\varphi^c}{\partial x_2}n_2=\left(\frac{\partial\varphi^c}{\partial z}\frac{\partial z}{\partial x_1}+\frac{\partial\varphi^c}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x_1}\right)n_1+\left(\frac{\partial\varphi^c}{\partial z}\frac{\partial z}{\partial x_2}+\frac{\partial\varphi^c}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x_2}\right)n_2\\&=\left(\frac{\partial\varphi^c}{\partial z}n+\frac{\partial\varphi^c}{\partial\bar{z}}\bar{n}\right)=\phi'(z)n+\overline{\phi'(z)n}\end{aligned} \tag{3.92}$$

If let $\phi(1)=0$ which is not effect on the stress, on L we have

$$\begin{aligned}-\int_0^s\frac{\partial\varphi^c}{\partial n}\mathrm{d}s&=-\int_0^\psi\left[\frac{\phi'(\varsigma)}{\omega'(\varsigma)}\frac{\omega'(\varsigma)}{|\omega'(\varsigma)|}\mathrm{e}^{\mathrm{i}\psi}+\frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}\frac{\overline{\omega'(\varsigma)}}{\overline{|\omega'(\varsigma)|}}\mathrm{e}^{-\mathrm{i}\psi}\right]|\omega'(\varsigma)|\mathrm{d}\psi\\&=-\int_0^\psi\left\{\mathrm{e}^{\mathrm{i}\psi}\phi'\left(\mathrm{e}^{\mathrm{i}\psi}\right)+\mathrm{e}^{-\mathrm{i}\psi}\overline{\phi'(\mathrm{e}^{\mathrm{i}\psi})}\right\}\mathrm{d}\psi=\mathrm{i}\left\{\phi(\sigma)-\overline{\phi(\sigma)}\right\}\end{aligned} \tag{3.93}$$

Substituting Eqs. (3.78), (3.83) and (3.87) and the third equation in Eq. (3.89) and (3.93) into Eq. (3.80), the electric connective conditions are reduced to

$$\begin{aligned}&\sum_{j=1}^{3}\left\{\eta_j f_j^0(\sigma)+\bar{\eta}_j\overline{f_j^0(\sigma)}\right\}=\bar{l}_3'\sigma+l_3'\bar{\sigma}+\mathrm{i}\epsilon^c\left[\phi(\sigma)-\overline{\phi(\sigma)}\right]\\&\sum_{j=1}^{3}\left\{\lambda_j f_j^0(\sigma)+\bar{\lambda}_j\overline{f_j^0(\sigma)}\right\}=\bar{l}_4\sigma+l_4\bar{\sigma}-\left[\phi(\sigma)+\overline{\phi(\sigma)}\right]\\&l_3'=(1/2)\left(aD_2^\infty-\mathrm{i}bD_1^\infty\right), l_4=-(1/2)\left(aE_1^\infty+\mathrm{i}bE_2^\infty\right)\end{aligned} \tag{3.94}$$

3.4.5 Solutions in the Air and Piezoelectric Material

Substituting Eqs. (3.85) and (3.87) into Eqs. (3.90) and (3.94) and neglecting some useless constants yield enough equations to determine the undetermined constants. It is found that only four complex constants $a_{11},\ a_{21},\ a_{31}, a_1^c\ \left(a_{-1}^c = \rho_0^2 a_1^c\right)$ are not zero and they obey the following equations:

$$
\begin{aligned}
&\sum_{j=1}^{3} a_{j1} = l_1, \quad \sum_{j=1}^{3} \mu_j a_{j1} = l_2 \\
&\sum_{j=1}^{3} \eta_j a_{j1} + \mathrm{i}\epsilon^c\left(\bar{a}_1^c - \rho_0^2 a_1^c\right) = l_3', \quad \sum_{j=1}^{3} \lambda_j a_{j1} + \bar{a}_1^c + \rho_0^2 a_1^c = l_4
\end{aligned}
\tag{3.95}
$$

Finally we get

1. *The electric field inside the hole filled with air*

$\varphi^c,\ D_1^c,\ D_2^c$ are constants and obtained from the following equations:

$$
\begin{aligned}
&\varphi^c = -E_1^c x_1 - E_2^c x_2 = 2\left(a_1^c z + \bar{a}_1^c \bar{z}\right)/(a+b); \quad D_1^c = \epsilon_0 E_1^c, \quad D_2^c = \epsilon_0 E_2^c \\
&\left(a - \mathrm{i}b\epsilon_0 \sum_{j=1}^{3} \lambda_j \alpha_{j3}\right) D_1^c + \left(a\epsilon_0 \sum_{j=1}^{3} \lambda_j \alpha_{j3} + \mathrm{i}b\right) D_2^c \\
&\quad = \epsilon_0 \left\{ 2\sum_{j=1}^{3}\sum_{k=1}^{2} \lambda_j \alpha_{jk} l_k + 2\sum_{j=1}^{3} \lambda_j \alpha_{j3} l_3' + aE_1^\infty + \mathrm{i}bE_2^\infty \right\}
\end{aligned}
\tag{3.96}
$$

2. *Solutions in the piezoelectric material*

$$
\begin{aligned}
&f_j(z_j) = C_j z_j + a_{j1}/\varsigma_j = C_j z_j + \sum_{k=1}^{3} \alpha_{jk} l_k/\varsigma_j, \quad a_{j1} = \sum_{k=1}^{3} \alpha_{jk} l_k \\
&F_j(z_j) = C_j + \left(\alpha_{j1} l_1 + \alpha_{j2} l_2 + \alpha_{j3} l_3\right)\left(a + i\mu_j b\right)^{-1} \left\{ 1 - z_j \left[z_j^2 - \left(a^2 + \mu_j^2 b^2\right) \right]^{-1/2} \right\}
\end{aligned}
\tag{3.97}
$$

where

$$
\begin{aligned}
&\boldsymbol{\alpha} = \frac{1}{N} \begin{pmatrix} \mu_2\eta_3 - \mu_3\eta_2 & \eta_2 - \eta_3 & \mu_3 - \mu_2 \\ \mu_3\eta_1 - \mu_1\eta_3 & \eta_3 - \eta_1 & \mu_1 - \mu_3 \\ \mu_1\eta_2 - \mu_2\eta_1 & \eta_1 - \eta_2 & \mu_2 - \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix}^{-1} \\
&N = (\eta_2 - \eta_3)\mu_1 + (\eta_3 - \eta_1)\mu_2 + (\eta_1 - \eta_2)\mu_3
\end{aligned}
\tag{3.98}
$$

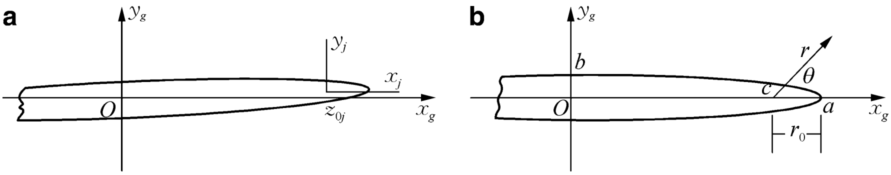


Fig. 3.4 Local coordinate system for a blunt crack

3.4.6 Electroelastic Asymptotic Field Near a Blunt (Slender) Crack Tip

Analogous to the elastic blunt crack (Kuang 1982; Kuang and Ma 2002), Huang and Kuang (2000b) discussed the electroelastic asymptotic field near a blunt (slender) crack tip. Take the global coordinate system $z_g(x_g, y_g)$ and the local coordinate system $z_j(x_j, y_j)$ whose origin is at z_{0j} ($z_0 = x_{01} = c = \sqrt{a^2 - b^2}$; $2c$ is the focal length) (Fig. 3.4). z_{0j} is the point in z_j plane corresponding to the branch point ς_{0j} in ς_j plane with $\omega'(\varsigma_{0j}) = 0$. It has the relation

$$\begin{aligned} z_{0j} &= x_{0j} + \mu_j y_{0j} = \sqrt{a^2 + \mu_j^2 b^2} \approx a + \mu_j^2 r_0, \quad r_0 = b^2/2a \\ x_{0j} &= a^2 - \mu^2 \rho, \quad y_{0j} = 2\alpha_j \rho; \quad \mu_j = \alpha_j + i\beta_j, \quad \mu^2 = \alpha_j^2 + \beta_j^2 \end{aligned} \tag{3.99}$$

where $2r_0$ is the curvature radius at the major end of the slender ellipse. At the local coordinate system, we have

$$z_j = x_j + \mu_j y_j = z_{gj} - z_{0j} = (x_g - x_{0j}) + \mu_j (y_g - y_{0j})$$

From the knowledge of the analytical geometry, it is known that

$$x_g = c + r\cos\theta, \quad y_g = r\sin\theta, \quad a = r_0 + \sqrt{c^2 + r_0^2} \approx c + r_0$$

where r, θ are the polar axis and angle, respectively. Therefore it is easy to derive

$$\begin{aligned} z_k &= r\Theta_k - (1 + \mu_k^2)\rho + 0(\rho^2/c) \approx r\Theta_k\{1 - [(1 + \mu_k^2)\rho/\Theta_k r]\} \\ \Theta_k &= \cos\theta + \mu_k \sin\theta \end{aligned} \tag{3.100}$$

In the local coordinate system, Eq. (3.97) becomes

$$F_j(z_j) = -\sum_{k=1}^{3} \alpha_{jk} l_k \frac{1}{a + \mathrm{i}\mu_j b} \sqrt{\frac{c}{2r\Theta_j}} \left[1 - \frac{1 + \mu_j^2}{\Theta_j} \frac{\rho}{r}\right] + C_j + \frac{1}{a + \mathrm{i}\mu_j b} \sum_{k=1}^{3} \alpha_{jk} l_k \tag{3.101}$$

It is seen that the singularity of the stress near the crack tip not only depends on $1/\sqrt{r}$ but also depends on ρ/r. It can also be seen that the electric field at infinity affects the stress near the blunt crack tip.

3.4.7 Impermeable and Conductive Elliptic Holes

Impermeable elliptic hole. Comparing to a piezoelectric material, in many cases the air is approximately considered as an insulated material, i.e., $\epsilon_0 = 0$ or $D_1^c = D_2^c = 0$, so $D_n^c = 0$ in Eq. (3.80), i.e. the piezoelectric material can be considered alone. On the interface, Eq. (3.94) is reduced to

$$\begin{aligned}
&\sum_{j=1}^{3}\left\{\eta_j f_j^0(\sigma) + \overline{\eta}_j \overline{f_j^0(\sigma)}\right\} = \overline{l'_3}\sigma + l'_3\overline{\sigma} \\
&l'_3 = -\sum_{j=1}^{3}\left\{a\mathrm{Re}(C_j\eta_j) + \mathrm{i}b\mathrm{Re}(C_j\eta_j\mu_j)\right\} = \frac{1}{2}\left(aD_2^\infty - \mathrm{i}bD_1^\infty\right) = l_3
\end{aligned} \tag{3.102}$$

Correspondingly Eq. (3.95) becomes

$$\sum_{j=1}^{3} a_{j1} = l_1, \quad \sum_{j=1}^{3}\mu_j a_{j1} = l_2 \quad \sum_{j=1}^{3}\eta_j a_{j1} = l'_3 \tag{3.103}$$

The solution in the piezoelectric material is still formally expressed by Eqs. (3.97) and (3.98).

Conductive elliptic hole. If the hole is filled with an ideal conductive liquid or on the boundary of the hole deposited a thin flexible layer metal, it can be assumed that the electric potential is equal zero, i.e., $\varphi = 0$ in Eq. (3.80). On interface, Eq. (3.94) is reduced to

$$\begin{aligned}
&\sum_{j=1}^{3}\left\{\lambda_j f_j^0(\sigma) + \overline{\lambda}_j \overline{f_j^0(\sigma)}\right\} = \overline{l}_4\sigma + l_4\overline{\sigma} \\
&l_4 = -\sum_{j=1}^{3}\left\{a\mathrm{Re}\left(C_j\lambda_j\right) + \mathrm{i}b\mathrm{Re}\left(C_j\lambda_j\mu_j\right)\right\} = -\frac{1}{2}\left(aE_1^\infty + \mathrm{i}bE_2^\infty\right)
\end{aligned} \tag{3.104}$$

If we let $1/\epsilon^c = 0$ in Eq. (3.94), Eq. (3.104) can also be obtained. Correspondingly Eq. (3.103) becomes

$$\sum_{j=1}^{3} a_{j1} = l_1, \quad \sum_{j=1}^{3}\mu_j a_{j1} = l_2 \quad \sum_{j=1}^{3}\lambda_j a_{j1} = l_4 \tag{3.105}$$

In this case Eqs. (3.97) and (3.98) become

$$f_j(z_j) = C_j z_j + a_{j1}/\varsigma_j = C_j z_j + \sum_{k=1}^{3} \tilde{\alpha}_{jk} l_k/\varsigma_j, \quad \tilde{a}_{j1} = \sum_{k=1}^{3} \tilde{\alpha}_{jk} l_k \tag{3.106}$$

and

$$\tilde{\boldsymbol{\alpha}} = \frac{1}{\tilde{N}} \begin{pmatrix} \mu_2\lambda_3 - \mu_3\lambda_2 & \lambda_2 - \lambda_3 & \mu_3 - \mu_2 \\ \mu_3\lambda_1 - \mu_1\lambda_3 & \lambda_3 - \lambda_1 & \mu_1 - \mu_3 \\ \mu_1\lambda_2 - \mu_2\lambda_1 & \lambda_1 - \lambda_2 & \mu_2 - \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}^{-1}$$
$$N = (\lambda_2 - \lambda_3)\mu_1 + (\lambda_3 - \lambda_1)\mu_2 + (\lambda_1 - \lambda_2)\mu_3 \tag{3.107}$$

3.4.8 Crack Problem

1. *Permeable crack*. When the length of the minor axis approaches zero, i.e., $b \to 0$, for a permeable crack the solution can be obtained from a permeable elliptic hole. Neglecting terms containing b/a yields

$$l_1 = -\frac{1}{2} a\sigma_2^\infty, \quad l_2 = -\frac{1}{2} a\sigma_3^\infty, \quad l_3' = \frac{1}{2} aD_2^\infty, \quad l_3 = \frac{1}{2} a\left(D_2^\infty - D_2^c\right), \quad l_4 = -\frac{1}{2} aE_1^\infty \tag{3.108}$$

(a) *Electric field in the air*. Equation (3.96) becomes

$$D_1^c + \epsilon_0 \sum_{j=1}^{3} \lambda_j \alpha_{j3} D_2^c = \epsilon_0 \left\{ \sum_{j=1}^{3} \left[-\lambda_j \left(\alpha_{j1}\sigma_2^\infty + \alpha_{j2}\sigma_3^\infty + \alpha_{j3} D_2^\infty \right) \right] + E_1^\infty \right\} \tag{3.109a}$$

Noting $\sum_{j=1}^{3} \lambda_j \alpha_{j2}$ is real (Gao and Fan 1999), separating the real and imaginary parts from Eq. (3.109a) yields

$$D_2^\infty - D_2^c = \mathrm{Im}\left(\sum_{j=1}^{3} \lambda_j \alpha_{j1} \right) \sigma_2^\infty \Big/ \mathrm{Im}\left(\sum_{j=1}^{3} \lambda_j \alpha_{j3} \right)$$
$$E_1^c = E_1^\infty + \mathrm{Re}\left\{ \sum_{j=1}^{3} \lambda_j \left[-a_{j1}\sigma_2^\infty + a_{j2}\sigma_3^\infty + a_{j3}\left(D_2^\infty - D_2^c \right) \right] \right\} \tag{3.109b}$$

(b) *Generalized stress in the piezoelectric material*. Equation (3.97) is reduced to

$$F_j(z_j) = C_j - \frac{1}{2}\left[\alpha_{j1}\sigma_2^\infty - \alpha_{j2}\sigma_3^\infty - \alpha_{j3}\left(D_2^\infty - D_2^c \right) \right] \left[1 - z_j \left(z_j^2 - a^2 \right)^{-1/2} \right] \tag{3.110}$$

It is noted that ϵ_0 is not included in Eqs. (3.109) and (3.110).

The stress intensities at the crack tip $x_1 = a$ are

$$
\begin{aligned}
(K_I, K_{II}, K_e) &= \sqrt{2\pi} \lim_{x_1 \to a} \sqrt{x_1 - a} (\sigma_2, \sigma_3, D_2)_{x_2=0} \\
&= \sqrt{2\pi} \mathrm{Re} \lim_{x_1 \to a} \sqrt{x_1 - a} \sum_{j=1}^{3} f_j'(x_1) \left(1, -\mu_j, -\eta_j\right)_{x_2=0}
\end{aligned}
\tag{3.111a}
$$

or

$$
K_I = \sqrt{\pi a} \sigma_2^\infty, \quad K_{II} = \sqrt{\pi a} \sigma_3^\infty, \quad K_e = \sqrt{\pi a} \left(D_2^\infty - D_2^c\right) \tag{3.111b}
$$

From Eq. (3.111), it is seen that the electric field at infinity does not affect the stress intensity and the mechanical stress at infinity does not affect the electric displacement intensity. This result is obtained from the linear theory.

2. *Impermeable (or insulated) crack*. The correct solution of an impermeable crack can be obtained from the degenerate solution from the insulated elliptic hole, i.e., let $\epsilon_0 = 0$ or $D_2 = 0$ at first and then let $b/a \to 0$. In order to study the electroelastic asymptotic field near a sharp crack tip, the right crack tip should be taken as the origin of the local polar coordinate system, i.e.,

$$
x_1 = a + r\cos\theta, \quad x_2 = a + r\sin\theta \tag{3.112}
$$

When $r \ll 1$, we have

$$
z_j \approx a, \quad \sqrt{z_j^2 - a^2} \approx \sqrt{2ar} \sqrt{\cos\theta + \mu_j \sin\theta} \tag{3.113}
$$

Equation (3.97) is reduced to

$$
F_j(z_j) \approx \left(\alpha_{j1} \sigma_2^\infty - \alpha_{j2} \sigma_3^\infty - \alpha_{j3} D_2^\infty\right) \frac{\sqrt{a}}{2\sqrt{2r}\sqrt{\cos\theta + \mu_j \sin\theta}} \tag{3.114}
$$

Let

$$
C_j = \alpha_{j1} K_I - \alpha_{j2} K_{II} - \alpha_{j3} K_e \tag{3.115}
$$

where K_{I}, K_{II}, K_e is defined by Eq. (3.111). Substituting Eqs. (3.114) and (3.115) into Eq. (3.26) in the Cartesian coordinate system yields (Hoenig 1982)

$$
\begin{aligned}
&\sigma_1 = \left(1 \middle/ \sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \mu_j^2 / \sqrt{\boldsymbol{\Theta}_j}, \quad \sigma_2 = \left(1 \middle/ \sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j / \sqrt{\boldsymbol{\Theta}_j}, \\
&\sigma_3 = -\left(1 \middle/ \sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \mu_j / \sqrt{\boldsymbol{\Theta}_j}, \quad D_1 = \left(1 \middle/ \sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \mu_j \eta_j / \sqrt{\boldsymbol{\Theta}_j}, \\
&D_2 = -\left(1 \middle/ \sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \eta_j / \sqrt{\boldsymbol{\Theta}_j}, \quad \boldsymbol{\Theta}_j = \cos\theta + \mu_j \sin\theta
\end{aligned}
\tag{3.116}
$$

or in the polar coordinate system yields

$$
\begin{aligned}
&\sigma_\theta = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \boldsymbol{\Theta}_j^{3/2}, \quad \sigma_r = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \tilde{\boldsymbol{\Theta}}_j^2 \Big/ \sqrt{\boldsymbol{\Theta}_j},\\
&\sigma_{\theta r} = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \tilde{\boldsymbol{\Theta}}_j \sqrt{\boldsymbol{\Theta}_j}, \quad D_r = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \eta_j \tilde{\boldsymbol{\Theta}}_j \Big/ \sqrt{\boldsymbol{\Theta}_j},\\
&D_\theta = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \eta_j \sqrt{\boldsymbol{\Theta}_j}, \quad \tilde{\Theta}_j = \cos\theta - \mu_j \sin\theta
\end{aligned}
\tag{3.117}
$$

It is seen that the stresses have the singularity $1/\sqrt{r}$ and σ_1^∞, D_1^∞ do not affect the electroelastic asymptotic field. Xu and Rajapakse (1999) discussed an arbitrarily oriented void/crack.

3. *Conducting crack*

The solution of a conducting crack can be obtained from the conducting elliptic hole directly when $b/a \to 0$ or from the general solution when $\epsilon_0 \to \infty$ at first and then when $b/a \to 0$.

3.4.9 Eshelby's Elliptic Inclusion Problem in a Piezoelectric Material

Now discuss the Eshelby's elliptic inclusion problem in a piezoelectric material (Ru 1997). In a piezoelectric material, there is a region S^c. In S^c there are the generalized eigenstrains $(\boldsymbol{\varepsilon}^*, E^*)$ and the corresponding additional generalized displacement:

$$
\begin{aligned}
\boldsymbol{U}^* &= (\boldsymbol{u}^*, \varphi^*)\\
&= \left[\varepsilon_{11}^* x_1 + \varepsilon_{12}^* x_2, \varepsilon_{12}^* x_1 + \varepsilon_{22}^* x_2, 2\left(\varepsilon_{13}^* x_1 + \varepsilon_{23}^* x_2\right), -\left(E_1^* x_1 + E_2^* x_2\right)\right]^{\mathrm{T}}
\end{aligned}
\tag{3.118}
$$

The connective conditions on the interface L are

$$
\boldsymbol{u} = \boldsymbol{u}^c + \boldsymbol{u}^*, \quad \varphi = \varphi^c + \varphi^*; \quad z \in L \tag{3.119}
$$

Substituting Eqs. (3.16) and (3.23) into Eq. (3.119), the connective conditions become

$$
\left.
\begin{aligned}
\boldsymbol{Af}(z_P) + \overline{\boldsymbol{Af}(z_P)} &= \boldsymbol{Af}^c(z_P) + \overline{\boldsymbol{Af}^c(z_P)} + \boldsymbol{u}^*\\
\boldsymbol{Bf}(z_P) + \overline{\boldsymbol{Bf}(z_P)} &= \boldsymbol{Bf}^c(z_P) + \overline{\boldsymbol{Bf}^c(z_P)}
\end{aligned}
\right\}, \quad z \in L \tag{3.120}
$$

where $\boldsymbol{f}^c(z_P)$ is the solution in S^c. Multiplying the first equation and second equation by $\boldsymbol{B}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}$, respectively, then adding them, and using Eq. (3.34) yield

$$\boldsymbol{f}(z_P) = \boldsymbol{f}^c(z_P) + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}^*, \quad z \in L \tag{3.121}$$

Using the relations between x_1, x_2 and $z_P, \bar{z}_P$ in Eq. (3.15), for $z \in L$ we have

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}^* = [\xi_P z_P + \eta_P \bar{z}_P]^{\mathrm{T}} = [\xi_1 z_1 + \eta_1 \bar{z}_1, \xi_2 z_2 + \eta_2 \bar{z}_2, \xi_3 z_3 + \eta_3 \bar{z}_3, \xi_4 z_4 + \eta_4 \bar{z}_4]^{\mathrm{T}} \tag{3.122}$$

where ξ_P, η_P are constants determined by $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}^*$. Therefore Eq. (3.121) can be separated into four independent scalar equations:

$$f_P(z_P) = f_P^c(z_P) + \xi_P z_P + \eta_P \bar{z}_P, \quad z_P \in L_P; \quad P = 1 \sim 4 \tag{3.123}$$

Using the mapping function described in Eq. (3.86) yields

$$\begin{aligned} &\bar{z}_P = \bar{\omega}_P(1/\sigma_P) = \bar{c}_P\sigma_P^{-1} + \bar{d}_P\sigma_P, \quad c_P = (a - \mathrm{i}\mu_P b)/2, \\ &d_P = (a + \mathrm{i}\mu_P b)/2; \quad z_P \in L_P \end{aligned}$$

So $\bar{z}_P$ is the boundary value of an analytic function $D_P(\varsigma_P)$ in S_P or in the exterior of S_P^c

$$\begin{aligned} &D_P(\varsigma_P) = \bar{c}_P\varsigma_P^{-1} + \bar{d}_P\varsigma_P \\ &D_P(z_P) = \bar{c}_P\frac{z_P - \sqrt{z_P^2 - (a^2 + \mu_P^2 b^2)}}{a + \mathrm{i}\mu_P b} + \bar{d}_P\frac{z_P + \sqrt{z_P^2 - (a^2 + \mu_P^2 b^2)}}{a - \mathrm{i}\mu_P b}; \quad z_P \in S_P \\ &D_P(z_P) \to h_P z_P, \quad h_P = (a - \mathrm{i}\bar{\mu}_P b)/(a - \mathrm{i}\mu_P b); \quad \text{when} \quad z_P \to \infty \end{aligned} \tag{3.124}$$

Substitution of Eq. (3.124) into Eq. (3.123) yields

$$f_P(z_P) - \xi_P z_P - \eta_P D_P(z_P) = f_P^c(z_P), \quad z_P \in L_P; \quad P = 1 \sim 4 \tag{3.125}$$

Usually the boundary conditions at infinity are $f_P(z_P) \to 0$, when $|z_P| \to \infty$, so the functions in the left- and right-hand side of Eq. (3.125) are all analytic. Therefore we have

$$\left.\begin{aligned} &f_P(z_P) = \eta_P[D_P(z_P) - h_P z_P], \quad && z_P \in S_P \\ &f_P^c(z_P) = -(\xi_P + \eta_P h_P)z_P, && z_P \in S_P^c \end{aligned}\right\}; P = 1 \sim 4, \text{ not summation on } P \tag{3.126}$$

From Eq. (3.126), it is known that the generalized stress field is uniform in the elliptic inclusion.

In Ru's paper (1997), he also discussed the inclusion with arbitrary shape by the mapping function (Muskhelishvili 1954, 1975; Kantorovich and Krylov 1958)

$$z = \omega(\varsigma) = \lambda\varsigma + \sum_{k=0}^{\infty} \lambda_k \varsigma^{-k} \tag{3.127}$$

In many cases the truncation of the infinite series to finite terms $k = N$ offers good approximation (Savin 1961).

Zeng and Rajapakse (2003) discussed the Eshelby's elliptic inclusion problem with specified generalized eigenstrains $(\boldsymbol{\varepsilon}^*, \boldsymbol{D}^*)$.

3.5 Rigid Elliptic Inclusion in Transversely Piezoelectric Material

3.5.1 Basic Theory

Though we can use the theory obtained in Sect. 3.3.2, but in this section we rather use the first kind of the constitutive equation in Eq. (2.83) to discuss the problem, i.e.,

$$\varepsilon_{ij} = s^E_{ijkl}\sigma_{kl} + d^\sigma_{kij}E_k, \quad D_i = d^D_{ijk}\sigma_{jk} + \epsilon^\sigma_{ij}E_j \tag{3.128}$$

Analogous to the derivation in Sect. 3.3.2, for the generalized plane problem in the transversely isotropic material, we have

$$\begin{aligned}
&\varepsilon_x = \gamma_{zx} = \gamma_{xy} = \tau_{zx} = \tau_{yz} = E_x = D_x = 0, \quad \sigma_x = -\left(s_{12}\sigma_y + s_{13}\sigma_z + d_{31}E_z\right)/s_{11} \\
&x_1 = y,\ x_2 = z,\ x_3 = x; \quad \sigma_1 = \sigma_y,\ \sigma_2 = \sigma_z,\ \sigma_3 = \tau_{yz}; \quad \varepsilon_1 = \varepsilon_y,\ \varepsilon_2 = \varepsilon_z,\ \varepsilon_3 = \gamma_{yz} \\
&D_1 = D_y, D_2 = D_z;\ E_1 = E_y, E_2 = E_z \\
&s_{13} = s_{23}, s_{11} = s_{22}, s_{44} = s_{55}, s_{66} = 2(s_{11} - s_{12}); \quad d_{31} = d_{32}, d_{15} = d_{24}; \quad \epsilon_{11} = \epsilon_{22}
\end{aligned} \tag{3.129}$$

Analogous to Eq. (3.74), the constitutive equation in terms of the reduced material constants is

$$\begin{gathered}
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ D_1 \\ D_2 \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & b_{21} \\ a_{12} & a_{22} & 0 & 0 & b_{22} \\ 0 & 0 & a_{33} & b_{13} & 0 \\ 0 & 0 & b_{13} & k_{11} & 0 \\ b_{21} & b_{22} & 0 & 0 & k_{22} \end{pmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ E_1 \\ E_2 \end{Bmatrix} \\
a_{11} = s_{11} - s_{12}^2/s_{11}, \quad a_{12} = s_{13} - s_{12}s_{13}/s_{11}, \quad a_{22} = s_{33} - s_{13}^2/s_{11}, \quad a_{33} = s_{44} \\
b_{21} = (1 - s_{12}/s_{11})d_{31},\ b_{22} = d_{33} - d_{31}s_{13}/s_{11},\ b_{13} = d_{15},\ k_{11} = \epsilon_{11},\ k_{22} = \epsilon_{33} - d_{31}^2/s_{11}
\end{gathered} \tag{3.130}$$

In the present case the generalized equilibrium and compatibility equations are, respectively,

$$\sigma_{1,1} + \sigma_{3,2} = 0, \quad \sigma_{3,1} + \sigma_{2,2} = 0, \quad E_{1,2} - E_{2,1} = 0 \tag{3.131}$$

$$\varepsilon_{1,22} + \varepsilon_{2,11} - \varepsilon_{3,12} = 0, \quad D_{1,1} + D_{2,2} = 0 \tag{3.132}$$

Introduce the stress function Λ and the electric potential φ:

$$\sigma_1 = \Lambda_{,22}, \quad \sigma_2 = \Lambda_{,11}, \quad \sigma_3 = -\Lambda_{,12}, \quad E_1 = -\varphi_{,1}, \quad E_2 = -\varphi_{,2} \tag{3.133}$$

Equation (3.131) is satisfied automatically and Eq. (3.132) becomes

$$\begin{aligned}
&L_4\Lambda - L_3\varphi = 0, \quad L_3\Lambda - L_2\varphi = 0 \\
&L_4 = a_{22}\frac{\partial^4}{\partial x_1^4} + a_{11}\frac{\partial^4}{\partial x_2^4} + (2a_{12} + a_{33})\frac{\partial^4}{\partial x_1^2 \partial x_2^2} \\
&L_3 = b_{21}\frac{\partial^3}{\partial x_2^3} + (b_{22} - b_{13})\frac{\partial^3}{\partial x_2 \partial x_1^2}, \quad L_2 = k_{11}\frac{\partial^2}{\partial x_1^2} + k_{22}\frac{\partial^2}{\partial x_2^2}
\end{aligned} \tag{3.134}$$

Eliminating Λ or φ from Eq. (3.134) yields

$$\left(L_4L_2 - L_3^2\right)\Lambda = 0, \quad \left(L_4L_2 - L_3^2\right)\varphi = 0 \tag{3.135}$$

The solution of Eq. (3.135) is

$$\begin{aligned}
&\Lambda = 2\mathrm{Re}\sum_{j=1}^{3}\tilde{f}_j(z); \quad \varphi = 2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j\left(z_j\right); \quad f_j\left(z_j\right) = \tilde{f}_j'\left(z_j\right) \\
&\eta_j = l_3/l_2 = \mu_j\left[b_{21}\mu_j^2 + (b_{22} - b_{13})\right]\Big/\left(k_{11} + k_{22}\mu_j^2\right); \quad z = x_1 + \mu x_3
\end{aligned} \tag{3.136}$$

where μ_j is the root of the following eigen-equation

$$\begin{aligned}
&l_4(\mu)l_2(\mu) - l_3^2(\mu) = 0; \quad l_4 = a_{22} + a_{11}\mu^4 + (2a_{12} + a_{33})\mu^2 \\
&l_3 = \mu\left[b_{21}\mu^2 + (b_{22} - b_{13})\right], \quad l_2 = k_{11} + k_{22}\mu^2
\end{aligned} \tag{3.137}$$

Equation (3.137) has 6 roots and $\mu_k = \alpha_k + \mathrm{i}\beta_{k,}$ $\alpha_1 = 0,$ $\mu_3 = -\bar{\mu}_2,$ $\mu_{k+3} = \bar{\mu}_k.$ From Eqs. (3.134) and (3.137), we get

$$\varphi = 2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j\left(z_j\right), \quad \eta_j = \frac{l_3}{l_2} = \frac{\mu_j\left[d_{21}\mu^2 + (d_{22} - d_{13})\right]}{\epsilon_{11} + \epsilon_{22}\mu^2} \tag{3.138}$$

where $f_j(z_j) = \tilde{f}_j'(z_j)$. Substituting Eqs. (3.136) and (3.138) into Eqs. (3.133) and (3.130) yields, respectively,

$$\begin{aligned}
&\sigma_1 = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2 F_j(z_j), \quad \sigma_2 = 2\mathrm{Re}\sum_{j=1}^{3} F_j(z_j), \quad \sigma_3 = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j F_j(z_j) \\
&E_1 = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j F_j(z_j), \quad E_2 = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_j F_j(z_j), \quad F_j(z_j) = f_j'(z_j)
\end{aligned} \tag{3.139}$$

$$\begin{aligned}
&u_1 = 2\mathrm{Re}\sum_{j=1}^{3} p_j f_j(z_j) - \omega x_2, \quad p_j = a_{11}\mu_j^2 + a_{12} - b_{21}\mu_j\eta_j \\
&u_2 = 2\mathrm{Re}\sum_{j=1}^{3} q_j f_j(z_j) + \omega x_1, \quad q_j = \left(a_{12}\mu_j^2 + a_{22} - b_{22}\mu_j\eta_j\right)\Big/\mu_j \\
&D_1 = -2\mathrm{Re}\sum_{j=1}^{3}\lambda_j\mu_j F_j(z_j), \quad D_2 = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j F_j(z_j) \\
&\lambda_j\mu_j = b_{13}\mu_j + k_{11}\eta_j, \quad \lambda_j = b_{21}\mu_j^2 + b_{22} - k_{22}\mu_j\eta_j
\end{aligned} \tag{3.140}$$

where ω is the rigid rotation angle.

3.5.2 *Rigid Elliptic Inclusion*

The discussed problem can also be shown in (Fig. 3.3) as that in the Sect. 3.4.1, but here S^c is not a hole, rather a rigid inclusion. The notations are the same as that in Sect. 3.4.1, except the permittivity in the inclusion is denoted by ϵ^c instead of ϵ_0 in the air. The boundary conditions at infinity are assumed $\omega = 0$ and

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{E} = \boldsymbol{E}^\infty \tag{3.141}$$

On the interface the connective equation is

$$\begin{aligned}
&u_1 = u_1^c = -\omega^c x_2, \quad u_2 = u_2^c = \omega^c x_1; \quad \varphi = \varphi^i, \quad \int_0^s D_n \mathrm{d}s = \int_0^s D_n^c \mathrm{d}s; \quad \text{on} \quad L \\
&\int_0^s D_n \mathrm{d}s = \int_0^s (D_1 n_1 + D_2 n_2)\mathrm{d}s = \int_0^s (D_1 \mathrm{d}x_2 - D_2 \mathrm{d}x_1) = 2\mathrm{Re}\sum_{j=1}^{3}\left(\lambda_j f_j\right)\Big|_0^s
\end{aligned} \tag{3.142}$$

Similar to Sect. 3.4, the complex function method is used. The mapping function is shown in Eq. (3.86). Assume that the function $f_j(z_j)$ can be expanded as that in Eq. (3.87) and

$$\begin{aligned} &\sigma_1^\infty = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2 C_j, \quad \sigma_2^\infty = 2\mathrm{Re}\sum_{j=1}^{3} C_j, \quad \sigma_3^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j C_j \\ &E_1^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j C_j, \quad E_2^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_j C_j, \quad \mathrm{Im}C_1 = 0 \end{aligned} \tag{3.143}$$

Substitution of Eqs. (3.140) and (3.87) into first two equations in (3.142) yields

$$\begin{aligned} &2\mathrm{Re}\sum_{j=1}^{3} p_j\left[C_j\left(x_1+\mu_j x_2\right)+a_{j0}+\sum_{k=1}^{\infty}a_{jk}\bar{\sigma}^k\right] = -\omega^c x_2 \\ &2\mathrm{Re}\sum_{j=1}^{3} q_j\left[C_j\left(x_1+\mu_j x_2\right)+a_{j0}+\sum_{k=1}^{\infty}a_{jk}\bar{\sigma}^k\right] = \omega^c x_1, \quad \text{on} \quad \Gamma \end{aligned} \tag{3.144}$$

Noting on Γ, $\varsigma_j = \sigma = \mathrm{e}^{\mathrm{i}\vartheta}(j = 0, 1, 2, 3)$, $x_1 = a(\sigma+\bar{\sigma})/2$, and $x_2 = -\mathrm{i}b(\sigma-\bar{\sigma})/2$. From Eqs. (3.144), we have

$$\begin{aligned} &a_{j0} = 0; \quad a_{jk} = 0, \quad k \geq 2 \\ &\sum_{j=1}^{3} 2p_j a_{j1} + p_j C_j\left(a+\mathrm{i}\mu_j b\right) + \bar{p}_j\bar{C}_j\left(a+\mathrm{i}\bar{\mu}_j b\right) = -\mathrm{i}b\omega^c \\ &\sum_{j=1}^{3} 2q_j a_{j1} + q_j C_j\left(a+\mathrm{i}\mu_j b\right) + \bar{q}_j\bar{C}_j\left(a+\mathrm{i}\bar{\mu}_j b\right) = a\omega^c, \quad \text{on} \quad \Gamma \end{aligned} \tag{3.145}$$

According to the knowledge of the elastic inclusion (Mura 1987) and the solution, Eq. (3.96), it is assumed that the electric field in the inclusion S^c is constant. Let

$$\begin{aligned} &\varphi^c = 2\mathrm{Re}\phi^c(z_0) = 2\mathrm{Re}C_0^c z_0, \\ &z_0 = x_1 + \mu_0 x_3 = (1/2)\left[(a-\mathrm{i}\mu_0 b)\varsigma_0 + (a+\mathrm{i}\mu_0 b)\varsigma_0^{-1}\right] \end{aligned} \tag{3.146}$$

where C_0^c is a constant. From $D_{1,1}^c + D_{2,2}^c = 0$, it can be obtained that

$$\epsilon_{11}^c\varphi_{,11}^c + \epsilon_{22}^c\varphi_{,22}^c = 0, \quad \Rightarrow \quad \epsilon_{11}^c\phi''^c + \epsilon_{22}^c\mu_0^2\phi''^c = 0; \quad \phi''^c = \mathrm{d}^2\phi^c/\mathrm{d}z_0^2 \tag{3.147}$$

From Eqs. (3.146) and (3.147), we find

$$\epsilon_{11}^c + \epsilon_{22}^c\mu_0^2 = 0, \quad \mu_0 = \mathrm{i}\sqrt{\epsilon_{11}^c/\epsilon_{22}^c}, \quad \epsilon_{11}^c = -\mu_0^2\epsilon_{22}^c \tag{3.148}$$

$$\int_0^s D_n^c \mathrm{d}s = \int_0^s \left(D_1^c n_1^c + D_2^c n_2^c\right)\mathrm{d}s = 2\mathrm{Re}\int_0^s \epsilon_{22}^c \mu_0 \phi'^c (\mu_0 \mathrm{d}x_2 + \mathrm{d}x_1) = 2\mathrm{Re}\left[\epsilon_{22}^c \mu_0 \phi^c\right]_0^s \tag{3.149}$$

Substituting Eqs. (3.148) and (3.149) into the last two equations in Eq. (3.140), the electric connective conditions on the interface are reduced to

$$\begin{aligned} 2\mathrm{Re}\sum_{j=1}^{3} \eta_j f_j(z_j) &= 2\mathrm{Re}\sum_{j=1}^{3} \eta_j\left[C_j z_j + a_{j1}\varsigma_j^{-1}\right] = 2\mathrm{Re}\left[C_0^c z_0\right] \\ 2\mathrm{Re}\sum_{j=1}^{3} \lambda_j f_j(z_j) &= 2\mathrm{Re}\sum_{j=1}^{3} \lambda_j\left[C_j z_j + a_{j1}\varsigma_j^{-1}\right] = 2\mathrm{Re}\left[\epsilon_{22}^c \mu_0 \phi^i\right] \end{aligned} \tag{3.150}$$

or

$$\begin{aligned} &\sum_{j=1}^{3}\left\{2\eta_j a_{j1} + \eta_j C_j\left(a + \mathrm{i}b\mu_j\right) + \bar{\eta}_j \bar{C}_j\left(a + \mathrm{i}b\bar{\mu}_j\right)\right\} = C_0^c(a + \mathrm{i}b\mu_0) + \bar{C}_0^c(a + \mathrm{i}\bar{\mu}_0 b) \\ &\sum_{j=1}^{3}\left\{2\lambda_j a_{j1} + \lambda_j C_j\left(a + \mathrm{i}b\mu_j\right) + \bar{\lambda}_j \bar{C}_j\left(a + \mathrm{i}b\bar{\mu}_j\right)\right\} = \epsilon_{22}^c[\mu_0(a + \mathrm{i}b\mu_0) + \bar{\mu}_0(a + \mathrm{i}\bar{\mu}_0 b)] \end{aligned} \tag{3.151}$$

According to Eqs. (3.28) and (3.71), we know that

$$T_1 = -\mathrm{d}\Phi_1/\mathrm{d}s = (\mathrm{d}\Lambda/\mathrm{d}s)_{,2}, \quad T_2 = -\mathrm{d}\Phi_2/\mathrm{d}s = -(\mathrm{d}\Lambda/\mathrm{d}s)_{,1} \tag{3.152}$$

The ω^c is determined by the condition that there is no moment acting on the inclusion, i.e.,

$$\begin{aligned} &M_n = \oint (-T_1 x_2 + T_2 x_1)\,\mathrm{d}s = -\oint \left(\mathrm{d}\left(\Lambda_{,2}\right)x_2 + \mathrm{d}\left(\Lambda_{,1}\right)x_1\right) = 0 \\ &\sigma_1 = \Lambda_{,22}, \quad \sigma_2 = \Lambda_{,11}, \quad \sigma_3 = -\Lambda_{,12}, \quad E_1 = -\varphi_{,1}, \quad E_2 = -\varphi_{,2} \end{aligned} \tag{3.153}$$

Using Eqs. (3.136) and (3.145) and the residual theorem, we finally get

$$\sum_{j=1}^{3}\left[\left(a - \mathrm{i}b\mu_j\right)a_{j1} - \left(a + \mathrm{i}b\bar{\mu}_j\right)\bar{a}_{j1}\right] = 0 \tag{3.154}$$

The undetermined constants $C_j, a_{j1}, C_0^c, \omega^c$ can be determined by Eqs. (3.143), (3.145), (3.151), and (3.154). If ω^c is given, the moment acting on the inclusion is determined by Eq. (3.153).

3.6 Singularity

3.6.1 Singularity in a Homogeneous Material

Let a generalized singularity load be located at a point $z_0(x_{10}, x_{20})$ in an infinite homogeneous material. A generalized singularity load means a generalized dislocation $\boldsymbol{b}(b_1, b_2, b_3, b_4)$ and a generalized force $\boldsymbol{p}(p_1, p_2, p_3, p_4)$, where (b_1, b_2, b_3) are the Burgers vectors representing the displacement increment around the dislocation line and b_4 is the potential increment around the dislocation line. (p_1, p_2, p_3) are the concentrate forces and p_4 is the point electric charge or the electric displacement flux. Let

$$\begin{aligned} &\boldsymbol{g}(z_j) = \langle \ln(z_j - z_{0j}) \rangle c, \quad g_j(z_j) = c_j \ln(z_j - z_{0j}), \quad z_{0j} = x_{01} + \mu_j x_{02} \\ &\boldsymbol{G}(z_j) = \boldsymbol{g}'(z_j) = \left\langle (z_j - z_{0j})^{-1} \right\rangle \boldsymbol{c}, \quad G_j(z_j) = c_j (z_j - z_{0j})^{-1} \end{aligned} \tag{3.155}$$

where $\boldsymbol{c}(c_1, c_2, c_3, c_4)$ is an undetermined constant vector, $\langle \ln(z_j - z_{0j}) \rangle = \mathrm{diag}\left[\ln(z_j - z_{0j}\right]$. Obviously $\ln(z_j - z_{0j})$ is a multivalued function and z_{0j} is a branch point. According to Eqs. (3.16), (3.23), and (3.155), the solutions are assumed as

$$\begin{aligned} &\boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{g}(z_P)], \quad U_i = 2\mathrm{Re}\sum_{j=1}^{4} A_{ij} c_j \ln\left(z_j - z_{0j}\right) \\ &\boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{g}(z_P)], \quad \Phi_i = 2\mathrm{Re}\sum_{j=1}^{4} B_{ij} c_j \ln\left(z_j - z_{0j}\right) \end{aligned} \tag{3.156a}$$

where z_{0j} is the branch point of the ln-function (usually the branch cut is chosen in the negative x_1 direction, from z_{j0} to $-\infty$) and select a single-valued branch that the polar angle is measured from the positive x_1 direction . On the two sides of the cut, it is defined

$$\boldsymbol{b} = \boldsymbol{U}^+ - \boldsymbol{U}^- = 2\pi\mathrm{i}(\boldsymbol{A}\boldsymbol{c} - \bar{\boldsymbol{A}}\bar{\boldsymbol{c}}), \quad \boldsymbol{p} = \boldsymbol{T}^- - \boldsymbol{T}^+ = \boldsymbol{\Phi}^+ - \boldsymbol{\Phi}^- = 2\pi\mathrm{i}(\boldsymbol{B}\boldsymbol{c} - \bar{\boldsymbol{B}}\bar{\boldsymbol{c}}) \tag{3.157}$$

where the superscript "+" and " − " denote the values approached from the upper and lower half planes, respectively. Using the identities (3.34) from Eqs. (3.156a) and (3.157) yields

$$\begin{aligned} &2\pi\mathrm{i}\begin{bmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{c} \\ -\bar{\boldsymbol{c}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{b} \\ \boldsymbol{p} \end{Bmatrix}, \quad \begin{Bmatrix} \boldsymbol{c} \\ -\bar{\boldsymbol{c}} \end{Bmatrix} = \frac{1}{2\pi\mathrm{i}}\begin{bmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} & \bar{\boldsymbol{A}}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{b} \\ \boldsymbol{p} \end{Bmatrix}, \quad \text{or} \\ &\boldsymbol{c} = (1/2\pi)\left\{\boldsymbol{B}^{-1}(\boldsymbol{Y} + \bar{\boldsymbol{Y}})^{-1}\boldsymbol{b} - \boldsymbol{A}^{-1}\left(\boldsymbol{Y}^{-1} + \bar{\boldsymbol{Y}}^{-1}\right)^{-1}\boldsymbol{p}\right\} = (1/2\pi\mathrm{i})\boldsymbol{V}, \\ &\boldsymbol{V} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{p} \end{aligned} \tag{3.158}$$

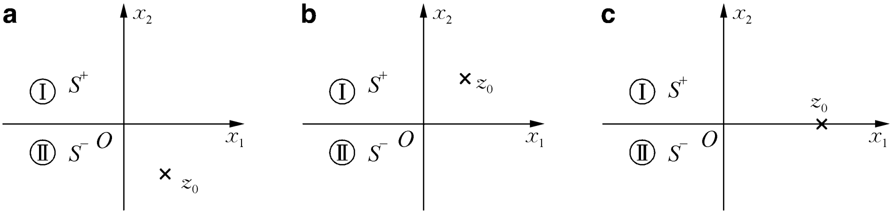


Fig. 3.5 Singularity in a bimaterial, singularity located (**a**) at lower plane, (**b**) at upper plane, and (**c**) on interface

So the solutions in Eq. (3.156a) become

$$\begin{aligned} &\boldsymbol{g}(z_j) = (1/2\pi\mathrm{i})\langle \ln(z_j - z_{0j})\rangle \boldsymbol{V}, \quad \boldsymbol{G}(z_j) = (1/2\pi\mathrm{i})\left\langle (z_j - z_{0j})^{-1}\right\rangle \boldsymbol{V} \\ &\boldsymbol{U} = (1/\pi)\mathrm{Im}\left[\boldsymbol{A}\langle \ln(z_j - z_{0j})\rangle \boldsymbol{V}\right], \quad \boldsymbol{\Phi} = (1/\pi)\mathrm{Im}\left[\boldsymbol{B}\langle \ln(z_j - z_{0j})\rangle \boldsymbol{V}\right] \end{aligned} \tag{3.156b}$$

The solution of the singularity problem can be used as the source function of a general problem.

3.6.2 *Singularity in a Bimaterial*

Let the material I be located at the upper half-plane $S^+, x_2 > 0$, and the material II be located at the lower half-plane $S^-, x_2 < 0; x_1 = 0$ is the interface L; a singularity load is located at $z_0(x_{10}, x_{20})$ in the material II (Fig. 3.5a). At first the problem is discussed in the z plane. Let (Tucker 1969; Barnett and Lothe 1975)

$$\boldsymbol{f}(z, z_0) = \begin{cases} \boldsymbol{f}_{\mathrm{I}}(z, z_0) & z \in S^+ \\ \boldsymbol{f}_{\mathrm{II}}(z, z_0) + \boldsymbol{g}_{\mathrm{II}}(z, z_0) & z \in S^- \end{cases} \tag{3.159}$$

where $\boldsymbol{g}_{\mathrm{II}}(z, z_0) = \boldsymbol{g}_{\mathrm{II}}(z)$ is the solution in a homogeneous material, i.e., the solution when the material II is extended to whole infinite plane, so $\boldsymbol{g}_{\mathrm{II}}(z, z_0)$ is analytic in the material I.

$$\boldsymbol{g}_{\mathrm{II}}(z_j) = \boldsymbol{c}_{\mathrm{II}}\langle \ln(z - z_0)\rangle; \quad \boldsymbol{c}_{\mathrm{II}} = (1/2\pi\mathrm{i})\boldsymbol{V}_{\mathrm{II}}, \quad \boldsymbol{V}_{\mathrm{II}} = \left(\boldsymbol{B}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{p}\right) \tag{3.160}$$

On the interface $x_2 = 0$, we have $\boldsymbol{U}_{\mathrm{I}} = \boldsymbol{U}_{\mathrm{II}}, \boldsymbol{\Phi}_{\mathrm{I}} = \boldsymbol{\Phi}_{\mathrm{II}}$ or

$$\begin{aligned} \boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{I}}\overline{\boldsymbol{f}_{\mathrm{I}}(x_1)} &= \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{II}}\overline{\boldsymbol{f}_{\mathrm{II}}(x_1)} + \boldsymbol{A}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01}) + \bar{\boldsymbol{A}}_{\mathrm{II}}\overline{\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01})} \\ \boldsymbol{B}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}}\overline{\boldsymbol{f}_{\mathrm{I}}(x_1)} &= \boldsymbol{B}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{II}}\overline{\boldsymbol{f}_{\mathrm{II}}(x_1)} + \boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01}) + \bar{\boldsymbol{B}}_{\mathrm{II}}\overline{\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01})} \end{aligned} \tag{3.161}$$

It is noted (Muskhelishvili 1954, 1975) that

$$\overline{\boldsymbol{f}^{+}(x_1)} = \bar{\boldsymbol{f}}^{-}(x_1),\ \overline{\boldsymbol{f}^{-}(x_1)} = \bar{\boldsymbol{f}}^{+}(x_1);\quad \boldsymbol{f}_{\mathrm{I}}(x_1) = \boldsymbol{f}^{+}(x_1),\ \boldsymbol{f}_{\mathrm{II}}(x_1) = \boldsymbol{f}^{-}(x_1) \tag{3.162}$$

where the superscripts " + " and " − " indicate the limit values taken from the upper and lower half planes, respectively. By using Eq. (3.162), Eq. (3.161) can be reduced to

$$\begin{aligned}
\boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}^{+}(x_1) - \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}^{+}(x_1) - \boldsymbol{A}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01}) = \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}^{-}(x_1) - \bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}^{-}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(x_1, x_{01}) \\
\boldsymbol{B}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}^{+}(x_1) - \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}^{+}(x_1) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01}) = \boldsymbol{B}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}^{-}(x_1) - \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}^{-}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(x_1, x_{01})
\end{aligned} \tag{3.163}$$

It is known that the functions at the left side in Eq. (3.163) are analytic in the upper half-plane $x_2 > 0$, whereas those on the right side are analytic in the lower half-plane $x_2 < 0$, and they are continuous on $x_1 = 0$. So according to Liouville theorem, these functions are analytic in whole plane and must be constants. If there are no generalized external forces and displacements acting at infinite, these constants must be zero. So we have

$$\begin{aligned}
&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(z) - \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}(z) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(z) = \boldsymbol{0}, \quad \boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(z) - \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}(z) - \boldsymbol{A}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(z) = \boldsymbol{0}; \quad z \in S^{+} \\
&\boldsymbol{B}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(z) - \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}(z) + \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z)] = \boldsymbol{0}, \quad \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(z) - \bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}(z) + \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z)] = \boldsymbol{0}; \quad z \in S^{-}
\end{aligned} \tag{3.164}$$

From Eq. (3.164) we get

$$\begin{aligned}
\boldsymbol{f}_{\mathrm{I}}(z) &= \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} + \boldsymbol{Y}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(z), \quad z \in S^{+} \\
\boldsymbol{f}_{\mathrm{II}}(z) &= \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} - \bar{\boldsymbol{Y}}_{\mathrm{I}})\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z), \quad z \in S^{-}
\end{aligned} \tag{3.165}$$

where $\boldsymbol{H} = \boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}}, \quad \boldsymbol{Y} = \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1}$. It is noted that the above theory will be often used in the following sections and we only give a simple illustration there.

Finally the solution of the problem is

$$\begin{aligned}
\boldsymbol{U}_{\mathrm{I}} &= 2\mathrm{Re}[\boldsymbol{A}_{\mathrm{I}}\langle \boldsymbol{f}_{\mathrm{I}}(z_P)\rangle], \quad z \in S^{+}; \quad \boldsymbol{U}_{\mathrm{II}} = 2\mathrm{Re}[\boldsymbol{A}_{\mathrm{II}}\langle \boldsymbol{f}_{\mathrm{II}}(z_P) + \boldsymbol{g}_{\mathrm{II}}(z_P)\rangle], \quad z \in S^{-} \\
\boldsymbol{\Phi}_{\mathrm{I}} &= 2\mathrm{Re}[\boldsymbol{B}_{\mathrm{I}}\langle \boldsymbol{f}_{\mathrm{I}}(z_P)\rangle], \quad z \in S^{+}; \quad \boldsymbol{\Phi}_{\mathrm{II}} = 2\mathrm{Re}[\boldsymbol{B}_{\mathrm{II}}\langle \boldsymbol{f}_{\mathrm{II}}(z_P) + \boldsymbol{g}_{\mathrm{II}}(z_P)\rangle], \quad z \in S^{-}
\end{aligned} \tag{3.166}$$

Some special cases are discussed as follows:

1. *Semi-infinite material*. If the material I is not existed, i.e., $x_2 = 0$ is a free plane, i.e., $\boldsymbol{f}_{\mathrm{I}}(z_j) = \boldsymbol{0}, \boldsymbol{\Phi}_{\mathrm{II}}(x_1, 0) = \boldsymbol{0}$. Let $\boldsymbol{A}_{\mathrm{II}} = \boldsymbol{A},\ \boldsymbol{B}_{\mathrm{II}} = \boldsymbol{B},$ then

$$\boldsymbol{f}(z_j) = \boldsymbol{g}_{\mathrm{II}}(z_j) - \boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z_j) \tag{3.167}$$

2. *Material I is rigid*. $x_2 = 0$ is a fixed plane, i.e., $\boldsymbol{f}_{\mathrm{I}}(z_j) = \mathbf{0}, \boldsymbol{U}_{\mathrm{II}}(x_1, 0) = \boldsymbol{0}$. Let $\boldsymbol{A}_{\mathrm{II}} = \boldsymbol{A}, \quad \boldsymbol{B}_{\mathrm{II}} = \boldsymbol{B},$ then

$$\boldsymbol{f}(z_j) = \boldsymbol{g}_{\mathrm{II}}(z_j) - \boldsymbol{A}^{-1}\bar{\boldsymbol{A}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z_j) \tag{3.168}$$

3. *Singularity at the upper semi-infinite plane*. If a singularity $z_0(x_{10}, x_{20})$ is located in the material I (Fig. 3.5b), then

$$\begin{aligned} &\boldsymbol{F}(z_j) = \boldsymbol{f}'(z_j) = \begin{cases} \boldsymbol{F}_{\mathrm{I}}(z_j) + \boldsymbol{G}_{\mathrm{I}}(z_j) & z \in S^+ \\ \boldsymbol{F}_{\mathrm{II}}(z_j) & z \in S^- \end{cases} \\ &\boldsymbol{G}_{\mathrm{I}}(z_j) = \boldsymbol{c}_{\mathrm{I}}\left\langle (z_j - z_{0j})^{-1} \right\rangle, \quad \boldsymbol{c}_{\mathrm{I}} = (1/2\pi\mathrm{i})\boldsymbol{V}_{\mathrm{I}}, \quad \boldsymbol{V}_{\mathrm{I}} = \left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{p}\right) \end{aligned} \tag{3.169}$$

$$\begin{aligned} \boldsymbol{F}_{\mathrm{I}}(z) &= \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{I}} - \bar{\boldsymbol{Y}}_{\mathrm{II}})\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z) = \left(\bar{\boldsymbol{A}}_{\mathrm{II}}^{-1}\boldsymbol{A}_{\mathrm{I}} - \bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\right)^{-1}\left(\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\bar{\boldsymbol{B}}_{\mathrm{I}} - \bar{\boldsymbol{A}}_{\mathrm{II}}^{-1}\bar{\boldsymbol{A}}_{\mathrm{I}}\right)\bar{\boldsymbol{G}}_{\mathrm{I}}(z) \\ \boldsymbol{F}_{\mathrm{II}}(z) &= \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{I}} + \boldsymbol{Y}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(z) = \left(\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}} - \bar{\boldsymbol{A}}_{\mathrm{I}}^{-1}\boldsymbol{A}_{\mathrm{II}}\right)^{-1}\left(\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{I}} - \bar{\boldsymbol{A}}_{\mathrm{I}}^{-1}\boldsymbol{A}_{\mathrm{I}}\right)\boldsymbol{G}_{\mathrm{I}}(z) \end{aligned} \tag{3.170}$$

3.6.3 Singularity on the Interface in a Bimaterial

Let a singularity $z_0(x_{01}, x_{02} = 0)$ be located on the interface in a biomaterial (Fig. 3.5c). Wang and Kuang (2000, 2002) took the following solution:

$$\begin{aligned} \boldsymbol{U}_{\alpha d} &= 2\mathrm{Re}\left[\boldsymbol{A}_\alpha \left\langle \ln(z_{\alpha j} - x_{01j}) \right\rangle \boldsymbol{V}_\alpha\right], \quad \boldsymbol{V}_\alpha = (1/\pi)\left(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{l}_\alpha + \boldsymbol{B}_\alpha^{\mathrm{T}}\mathbf{g}_\alpha\right) \\ \boldsymbol{\Phi}_{\alpha d} &= 2\mathrm{Re}\left[\boldsymbol{B}_\alpha \left\langle \ln(z_{\alpha j} - x_{01j}) \right\rangle \boldsymbol{V}_\alpha\right], \quad \alpha = \mathrm{I}, \mathrm{II} \end{aligned} \tag{3.171}$$

where $\boldsymbol{l}_\alpha, \mathbf{g}_\alpha$ are undetermined vectors. Draw a cut from x_{01} to $-\infty$; the jump value on the cut (between crack surfaces) of the generalized displacement and traction are

$$\boldsymbol{U}_{\mathrm{I}}(x_1, 0^+) - \boldsymbol{U}_{\mathrm{II}}(x_1, 0^-) = \boldsymbol{b}, \quad x_1 < 0; \quad \boldsymbol{\Phi}_{\mathrm{I}}(x_1, 0^+) - \boldsymbol{\Phi}_{\mathrm{II}}(x_1, 0^-) = \boldsymbol{p}\delta(x_{01}) \tag{3.172}$$

where $\delta(x_1)$ is the Dirac function. Using the following result (Qu and Li 1991),

$$\begin{aligned} &\lim_{x_2 \to \pm 0} \ln(x_1 + \mu x_2) = \ln|x_1| \pm \mathrm{i}\pi H(x_1), \\ &\lim_{x_2 \to \pm 0} \frac{1}{x_1 + \mu x_2} = \frac{1}{x_1} \mp \mathrm{i}\pi\delta(x_1), \quad \text{if } Im\,\mu > 0 \end{aligned} \tag{3.173}$$

where $H(x_1)$ is the Heaviside unit step function. Substituting Eqs. (3.171) and (3.173) into Eq. (3.172) and using Eq. (3.34) we get

$$\begin{aligned}\boldsymbol{b} &= (2/\pi)\mathrm{Re}\left\{\left(\left[\boldsymbol{A}_{\mathrm{I}}\langle \ln|x_1 - x_{01}| + \mathrm{i}\pi H(x_1)\rangle\left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\mathbf{g}_1 + \boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{l}_1\right)\right]\right.\right.\\ &\quad \left.-\left[\boldsymbol{A}_{\mathrm{II}}\langle \ln|x_1 - x_{01}| - \mathrm{i}\pi H(x_1)\rangle\left(\boldsymbol{B}_{\mathrm{II}}^{\mathrm{T}}\mathbf{g}_2 + \boldsymbol{A}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{l}_2\right)\right]\right\}\\ &= (1/\pi)\ln|x_1 - x_{01}|(\mathbf{g}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}}) + \boldsymbol{S}_{\mathrm{I}}\mathbf{g}_{\mathrm{I}} + \boldsymbol{S}_{\mathrm{II}}\mathbf{g}_{\mathrm{II}} + \boldsymbol{M}_{\mathrm{I}}\boldsymbol{l}_{\mathrm{I}} + \boldsymbol{M}_{\mathrm{II}}\boldsymbol{l}_{\mathrm{II}}\\ \boldsymbol{p} &= (1/\pi x_1)(\boldsymbol{l}_{\mathrm{I}} - \boldsymbol{l}_{\mathrm{II}}) + \left(\boldsymbol{S}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{l}_{\mathrm{I}} + \boldsymbol{S}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{l}_{\mathrm{II}} - \boldsymbol{L}_{\mathrm{I}}\mathbf{g}_{\mathrm{I}} - \boldsymbol{L}_{\mathrm{II}}\mathbf{g}_{\mathrm{II}}\right)\end{aligned} \tag{3.174}$$

where $\boldsymbol{S}, \boldsymbol{M}$ and $\boldsymbol{L}$ are shown in Eq. (3.35) and all real matrixes. From Eq. (3.174) we get

$$\begin{aligned}&\mathbf{g}_I = \mathbf{g}_{\mathrm{II}} = \mathbf{g}, \quad \boldsymbol{l}_{\mathrm{I}} = \boldsymbol{l}_{\mathrm{II}} = \boldsymbol{l}, \quad \begin{Bmatrix}\boldsymbol{l}\\ \mathbf{g}\end{Bmatrix} = \begin{bmatrix}\boldsymbol{\Omega}_1 & \boldsymbol{\Omega}_2\\ \boldsymbol{\Omega}_3 & \boldsymbol{\Omega}_4\end{bmatrix}\begin{Bmatrix}\boldsymbol{b}\\ \boldsymbol{p}\end{Bmatrix}\\ &\boldsymbol{\Omega}_1 = \left\{(\boldsymbol{M}_1 + \boldsymbol{M}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)(\boldsymbol{L}_1 + \boldsymbol{L}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}\right\}^{-1}\\ &\boldsymbol{\Omega}_2 = \left\{(\boldsymbol{M}_1 + \boldsymbol{M}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)(\boldsymbol{L}_1 + \boldsymbol{L}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}\right\}(\boldsymbol{S}_1 + \boldsymbol{S}_2)(\boldsymbol{L}_1 + \boldsymbol{L}_2)^{-1}\\ &\boldsymbol{\Omega}_3 = \left\{(\boldsymbol{L}_1 + \boldsymbol{L}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}(\boldsymbol{M}_1 + \boldsymbol{M}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)\right\}^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}(\boldsymbol{M}_1 + \boldsymbol{M}_2)^{-1}\\ &\boldsymbol{\Omega}_4 = -\left\{(\boldsymbol{L}_1 + \boldsymbol{L}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}(\boldsymbol{M}_1 + \boldsymbol{M}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)\right\}^{-1}\end{aligned} \tag{3.175}$$

where $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4$ are all real matrix. Substitution of Eq. (3.175) into Eq. (3.171) yields

$$\boldsymbol{V}_\alpha = \boldsymbol{M}_\alpha\boldsymbol{b} + \boldsymbol{N}_\alpha\boldsymbol{p}, \quad \boldsymbol{M}_\alpha = (1/\pi)(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{\Omega}_1 + \boldsymbol{B}_\alpha^{\mathrm{T}}\boldsymbol{\Omega}_3), \quad \boldsymbol{N}_\alpha = (1/\pi)(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{\Omega}_2 + \boldsymbol{B}_\alpha^{\mathrm{T}}\boldsymbol{\Omega}_4) \tag{3.176}$$

Zhou et al. (2007) discussed a generalized screw dislocation in a piezoelectric tri-material body.

3.6.4 Electric Dipole

Wang and Kuang (2000, 2002) discussed the electric dipole in a piezoelectric material. The electric dipole may be useful in the discussion on the electric switching wake. Let a generalized concentrate load $\boldsymbol{p} = -q_{\mathrm{e}}\mathbf{I}_4, \mathbf{I}_4 = [0, 0, 0, 1]^{\mathrm{T}}$ be acted at z_0 and $\boldsymbol{p} = q_{\mathrm{e}}\mathbf{I}_4$ acted at z_1. Solutions of these problems are $\boldsymbol{U}_0, \boldsymbol{\Phi}_0$ and $\boldsymbol{U}_1, \boldsymbol{\Phi}_1$, respectively:

$$\begin{aligned}&\boldsymbol{U}_0 = \mathrm{Re}\left[-(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{A}\left\langle \ln(z_j - z_{0j})\right\rangle\boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right], \quad \boldsymbol{\Phi}_0 = \mathrm{Re}\left[-(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{B}\left\langle \ln(z_j - z_{0j})\right\rangle\boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right]\\ &\boldsymbol{U}_1 = \mathrm{Re}\left[(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{A}\left\langle \ln(z_j - z_{1j})\right\rangle\boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right], \quad \boldsymbol{\Phi}_1 = \mathrm{Re}\left[(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{B}\left\langle \ln(z_j - z_{1j})\right\rangle\boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right]\end{aligned}$$

Using the relation,

$$\begin{aligned}
&z_1 - z_0 = d(\cos\theta + \mathrm{i}\sin\theta) \to 0, \quad z_{1j} - z_{0j} = d\left(\cos\theta + \mu_j \sin\theta\right) \to 0 \\
&\lim_{d\to 0, q_\mathrm{e} d \to p} \left\{q_\mathrm{e} \ln(z_j - z_{1j}) - q_\mathrm{e} \ln(z_j - z_{0j})\right\} = \lim_{d\to 0, q_\mathrm{e} d\to p} q_\mathrm{e} \ln\left[\left(z_j - z_{1j}\right)/\left(z_j - z_{0j}\right)\right] \\
&= -p_\mathrm{e}\left[\Theta_j/\left(z_j - z_{0j}\right)\right]; \qquad \lim_{q_\mathrm{e}\to\infty, d\to 0} q_\mathrm{e} d = p_\mathrm{e}, \quad \Theta_j = \cos\theta + \mu_j \sin\theta
\end{aligned} \tag{3.177}$$

where p_e is the electric pole couple and d is the distance from the negative charge to the positive charge. Thus the solution of an electric dipole in a homogeneous material is

$$\begin{aligned}
\boldsymbol{U}_p &= \boldsymbol{U}_1 - \boldsymbol{U}_0 = \operatorname{Re}\left[\mathrm{i}(p_\mathrm{e}/\pi)\boldsymbol{A}\left\langle \Theta_j\left(z_j - z_0\right)^{-1}\right\rangle \boldsymbol{A}^\mathrm{T}\mathbf{I}_4\right] \\
\boldsymbol{\Phi}_p &= \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_0 = \operatorname{Re}\left[\mathrm{i}(p_\mathrm{e}/\pi)\boldsymbol{B}\left\langle \Theta_j\left(z_j - z_0\right)^{-1}\right\rangle \boldsymbol{A}^\mathrm{T}\mathbf{I}_4\right]
\end{aligned} \tag{3.178}$$

$$\begin{aligned}
\boldsymbol{\Sigma}_2 &= \boldsymbol{\Phi}_{,1} = \operatorname{Re}\left[(p_\mathrm{e}/\pi\mathrm{i})\boldsymbol{B}\left\langle \Theta_j\left(z_j - z_0\right)^{-2}\right\rangle \boldsymbol{A}^\mathrm{T}\mathbf{I}_4\right] \\
\boldsymbol{\Sigma}_1 &= -\boldsymbol{\Phi}_{,2} = -\operatorname{Re}\left[(p_\mathrm{e}/\pi\mathrm{i})\boldsymbol{B}\left\langle \mu_j\Theta_j\left(z_j - z_0\right)^{-2}\right\rangle \boldsymbol{A}^\mathrm{T}\mathbf{I}_4\right]
\end{aligned} \tag{3.179}$$

For an electric dipole on the interface in a bimaterial consisted of materials I and II, the solution can be obtained from Eqs. (3.171) and (3.176):

$$\begin{aligned}
\boldsymbol{U}_{\alpha d} &= 2\operatorname{Re}\left[\boldsymbol{A}_\alpha\left\langle \ln\left(z_{\alpha j} - x_{01} - d\right) - \ln\left(z_{\alpha j} - x_{01}\right)\right\rangle_\alpha \boldsymbol{N}_\alpha q_\mathrm{e}\right]\mathbf{I}_4 \\
&= -2p_\mathrm{e}\operatorname{Re}\left[\boldsymbol{A}_\alpha\left\langle \left(z_{\alpha j} - x_{01}\right)^{-1}\right\rangle \boldsymbol{N}_\alpha\right]\mathbf{I}_4 \\
\boldsymbol{\Phi}_{\alpha d} &= 2\operatorname{Re}\left[\boldsymbol{B}_\alpha\left\langle \ln\left(z_{\alpha_j} - x_{01} - d\right) - \ln\left(z_{\alpha j} - x_{01}\right)\right\rangle_\alpha \boldsymbol{N}_\alpha q_\mathrm{e}\right]\mathbf{I}_4 \\
&= -2p_\mathrm{e}\operatorname{Re}\left[\boldsymbol{B}_\alpha\left\langle \left(z_{\alpha j} - x_{01}\right)^{-1}\right\rangle \boldsymbol{N}_\alpha\right]\mathbf{I}_4; \ \alpha = \mathrm{I.II}
\end{aligned} \tag{3.180}$$

3.6.5 General Case

Now we discuss a multiply connected plate. Let the kth singularity be located at $z_k^{(0)}$ and its total number be N, the kth inclusion occupy the region S_k and its total number be M, the region occupied by the piezoelectric material be denoted by S with the outer profile L_0, and the interface between S_k and S be L_k (Fig. 3.6). The complex stress function $\boldsymbol{\Phi}(z_P)$ can beassumed in the following form and complete determined by the boundary conditions:

$$\begin{aligned}
&f_P(z_P) = C_P z_P + \sum_{k=1}^{N} \alpha_{Pk} \ln\left(z_P - z_{Pk}^{(0)}\right) + \sum_{k=1}^{M} \beta_{Pk} \ln(z_P - z_{Pk}) + f_{0P}(z_P) \\
&\boldsymbol{\alpha}_k = (1/2\pi\mathrm{i})\boldsymbol{V}_k, \quad \boldsymbol{\Phi} = 2\operatorname{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)], \quad \boldsymbol{\Phi}_k - \boldsymbol{\Phi}_0 = -\oint_{L_k} \boldsymbol{T}\mathrm{d}s, \quad \oint_{L_k} \mathrm{d}\boldsymbol{U} = \boldsymbol{b}_k - \boldsymbol{b}_0
\end{aligned} \tag{3.181}$$

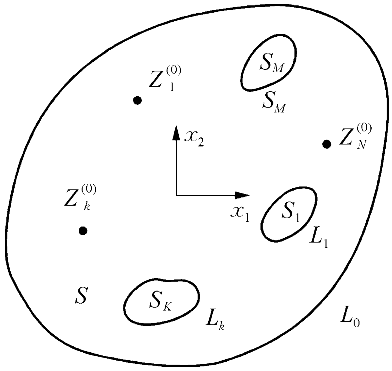


Fig. 3.6 General multiply connected plane zone

where z_{Pk} is a point inside the contour L_k and can be selected arbitrarily, $f_{0P}(z_P)$ is a single-valued function analytic in S, and $\boldsymbol{\Phi}_0, \boldsymbol{b}_0$ are constant vectors. If the singularity is considered as an infinitesimal inclusion, the terms containing singularity can be omitted. C_P can be determined by the stress condition at infinity and for a finite body $C_P = \mathbf{0}$; β_{Pk} can be expressed by the external generalized resultant force and the generalized dislocation acted on S_k. When we use the stress function method given in Sect. 3.3, the generalized stress function and displacement are expressed by $\boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{Bf}(z_P)]$ and $\boldsymbol{U} = (1/\pi)\mathrm{Im}[\boldsymbol{Af}(z_P)]$, respectively, where $\boldsymbol{B}$ and $\boldsymbol{A}$ are expressed by Eqs. (3.65) and (3.66).

3.7 Interaction of an Elliptic Inclusion with a Singularity

3.7.1 Green Function for a Singularity Outside the Elliptic Inclusion

Let an elliptic inclusion I with major axis $2a$– and minor axis $2b$–occupied S^+ be imbedded in an infinite piezoelectric material matrix II–occupied S^-. L is their interface. A singularity with strength $(\boldsymbol{b}, \boldsymbol{p})$ is acted at $z_0 = x_{01} + ix_{02}$ located in the matrix (Fig. 3.7). Huang and Kuang (2001b) discussed this problem under the conditions

$$\begin{aligned} &\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{\infty} = \mathbf{0}, \quad |z| \to \infty \\ &\boldsymbol{U}^{\mathrm{I}} = \boldsymbol{U}^{\mathrm{II}}, \quad \boldsymbol{\Phi}^{\mathrm{I}} = \boldsymbol{\Phi}^{\mathrm{II}}, \quad z \in L \end{aligned} \tag{3.182}$$

In this problem the second natural coordinate system is used, i.e., use $(\boldsymbol{n}, \boldsymbol{t}')$ in (3.29b) and $\boldsymbol{T} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s$. The transform method is used and the transform functions $\omega(\varsigma)$ and $\omega_j(\varsigma_j)$ are shown in Eqs. (3.82) and (3.86) respectively.

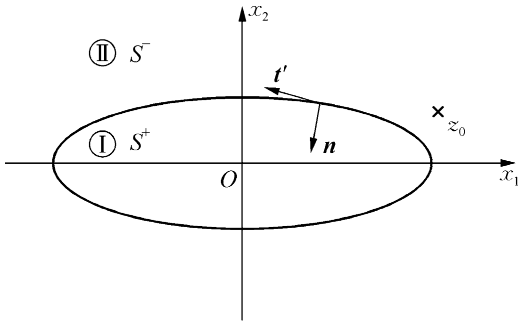


Fig. 3.7 An elliptic inclusion

In this section for clarity, the notations I, II will be written as superscripts. According to Ting (1996) and Huang and Kuang (2001b), the solution for a singularity outside the elliptic inclusion is assumed in the following form:

$$\begin{aligned}\boldsymbol{U}^{\mathrm{II}} &= (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \varsigma_{0j}^{\mathrm{II}}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] + (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left[\left(1\Big/\varsigma_j^{\mathrm{II}}\right) - \bar{\varsigma}_{0\beta}^{\mathrm{II}}\right]\right\rangle \boldsymbol{V}_\beta''\right] \\ &\quad + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \left(1\Big/k\varsigma_j^{\mathrm{II}\,k}\right)\right\rangle \mathbf{g}_k\right]\end{aligned} \tag{3.183}$$

$$\boldsymbol{U}^{\mathrm{I}} - \boldsymbol{U}_0^{\mathrm{I}} = (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle \ln\left(z_j^{\mathrm{I}} - y_{j\beta}^{\mathrm{I}}\right)\right\rangle \boldsymbol{V}_\beta'\right] + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{A}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle \boldsymbol{h}_k\right] \tag{3.184}$$

where $\boldsymbol{u}_0^{\mathrm{I}}, \boldsymbol{V}_\beta', \boldsymbol{V}_\beta'', \mathbf{g}_\delta, \boldsymbol{h}_\delta$ are undetermined vectors and

$$\begin{aligned}&\boldsymbol{V}^{\mathrm{II}} = \boldsymbol{B}^{\mathrm{IIT}}\boldsymbol{b} + \mathbf{A}^{\mathrm{IIT}}\boldsymbol{p}, \quad f_{jk}^{\mathrm{I}} = \left(\varsigma_j^{\mathrm{I}}\right)^k + \left(m_j^{\mathrm{I}}\right)^k\left(\varsigma_j^{\mathrm{I}}\right)^{-k} \\ &z_{0j}^{\mathrm{II}} = x_{01} + \mu_j^{\mathrm{II}}x_{02} = c_j^{\mathrm{II}}\varsigma_{0j}^{\mathrm{II}} + d_j^{\mathrm{II}}\left(\varsigma_{0j}^{\mathrm{II}}\right)^{-1}, \quad c_j^{\mathrm{II}} = R_j^{\mathrm{II}}, \quad d_j^{\mathrm{II}} = R_j^{\mathrm{II}}m_j^{\mathrm{II}} \\ &y_{j\beta}^{\mathrm{I}} = y_{j\beta1} + \mu_j^{\mathrm{I}}y_{j\beta2} = c_j^{\mathrm{I}}\varsigma_{0\beta}^{\mathrm{II}} + d_j^{\mathrm{I}}\left(\varsigma_{0\beta}^{\mathrm{II}}\right)^{-1}, \quad c_j^{\mathrm{I}} = R_j^{\mathrm{I}}, \quad d_j^{\mathrm{I}} = R_j^{\mathrm{I}}m_j^{\mathrm{I}}\end{aligned} \tag{3.185}$$

where f_{jk}^{I} is analytic in an annular region $\sqrt{m} < |\varsigma_i| < 1,\ 0 \le \theta < 2\pi$ (see Sect. 3.4.2).

Now Eqs. (3.183) and (3.184) will be explained in detail. In Eq. (3.183) the first term is the solution of a singularity when matrix II extended to whole space. It is noted that $\varsigma_{0\beta}^{\mathrm{II}}$ and $1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}$ are mirror images of each other with respect to the unit circle γ in ς plane, so $|\varsigma_{0\beta}^{\mathrm{II}}||1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}| = 1$. From $(1/\varsigma_j^{\mathrm{II}}) - \bar{\varsigma}_{0\beta}^{\mathrm{II}} = 0$, it is known that this singularity is located at $(\varsigma_j^{\mathrm{II}}) = (1/\bar{\varsigma}_{0\beta}^{\mathrm{II}})$, so the second term represents solutions of 4 image singularities located at $1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}$, $|1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}| < 1$ inside the inclusion I, in ς plane, or total 16 image singularities located at $z_{0j}^{\mathrm{I}} = [c_j(1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}) + d_j\bar{\varsigma}_{0\beta}^{\mathrm{II}}]$ in four z_j plane. Similarly the first term in Eq. (3.184) represents the solution for material I, its 16 image singularities located at $(y_{j\beta1}, y_{j\beta2})$ in the matrix II. For an impermeable hole and conductive rigid inclusion, $\boldsymbol{\Phi}^{\mathrm{I}}$ is not needed, so the third term in Eq. (3.183) can be omitted. This source function method is often used in the static electromagnetic field and stationary ideal fluid mechanics but here more complex. Using the relation

$$
\begin{aligned}
&z_j^{\mathrm{I}} - y_{j\beta}^{\mathrm{I}} = c_j^{\mathrm{I}}\left(\varsigma_j^{\mathrm{I}} - \varsigma_{0\beta}^{\mathrm{II}}\right) + d_j^{\mathrm{I}}\left[\left(\varsigma_j^{\mathrm{I}}\right)^{-1} - \left(\varsigma_{0\beta}^{\mathrm{II}}\right)^{-1}\right] = c_j^{\mathrm{I}}\left(\varsigma_j^{\mathrm{I}} - \varsigma_{0\beta}^{\mathrm{II}}\right)\left[1 - \tau_{j\beta}^{\mathrm{I}}\left(\varsigma_j^{\mathrm{I}}\right)^{-1}\right]\\
&\tau_{j\beta}^{\mathrm{I}} = m_j^{\mathrm{I}}\left(\varsigma_{0\beta}^{\mathrm{II}}\right)^{-1}
\end{aligned}
\tag{3.186}
$$

and $\mathrm{Im}(F) = -\mathrm{Im}(\bar{F})$, Eqs. (3.183) and (3.184), respectively, take

$$
\begin{aligned}
\boldsymbol{U}^{\mathrm{II}} = \frac{1}{\pi}\mathrm{Im}\Bigg[&-\bar{\boldsymbol{A}}^{\mathrm{II}}\left\langle \ln\left(\mathrm{e}^{-\mathrm{i}\psi} - \bar{\varsigma}_{0j}^{\mathrm{II}}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}} + \sum_{\beta=1}^{4} \ln\left(\mathrm{e}^{-\mathrm{i}\psi} - \bar{\varsigma}_{0\beta}^{\mathrm{II}}\right)\boldsymbol{A}^{\mathrm{II}}\boldsymbol{V}_{\beta}''\\
&+ \sum_{k=1}^{\infty}(1/k)\mathrm{e}^{-\mathrm{i}k\psi}\boldsymbol{A}^{\mathrm{II}}\mathbf{g}_k\Bigg]
\end{aligned}
\tag{3.187}
$$

$$
\begin{aligned}
\boldsymbol{U}^{\mathrm{I}} - \boldsymbol{U}_0^{\mathrm{I}} = (1/\pi)\mathrm{Im}\Bigg\{&\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}_{\beta}'\right) - \ln\left(\mathrm{e}^{-\mathrm{i}\psi} - \bar{\varsigma}_{0\beta}^{\mathrm{II}}\right)\bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{V}}_{\beta}'\\
&+ \boldsymbol{A}^{\mathrm{I}}\left\langle \ln\left(1 - \tau_{j\beta}^{\mathrm{I}}\mathrm{e}^{-\mathrm{i}\psi}\right)\boldsymbol{V}_{\beta}'\right\rangle\Bigg] + \sum_{k=1}^{\infty}(1/k)\mathrm{e}^{-\mathrm{i}k\psi}\left[-\bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k + \boldsymbol{A}^{\mathrm{I}}\left\langle\left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k\right]\Bigg\}
\end{aligned}
\tag{3.188}
$$

In Eqs. (3.183) and (3.187) replacing $\boldsymbol{A}^{\mathrm{II}}$ by $\boldsymbol{B}^{\mathrm{II}}$, we get $\boldsymbol{\Phi}^{\mathrm{II}}$, and in Eqs. (3.184) and (3.188) replacing $\boldsymbol{A}^{\mathrm{I}}$ by $\boldsymbol{B}^{\mathrm{I}}$ and $\boldsymbol{u}_0^{\mathrm{I}}$ by $\boldsymbol{\Phi}_0^{\mathrm{I}}$, we get $\boldsymbol{\Phi}^{\mathrm{I}}$:

$$
\begin{aligned}
\boldsymbol{\Phi}^{\mathrm{II}} &= (1/\pi)\mathrm{Im}\left[\boldsymbol{B}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \varsigma_{0j}^{\mathrm{II}}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] + (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{B}^{\mathrm{II}}\left\langle \ln\left(1\big/\varsigma_j^{\mathrm{II}} - \bar{\varsigma}_{0\beta}^{\mathrm{II}}\right)\right\rangle \boldsymbol{V}_{\beta}''\right]\\
&\quad + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left(\boldsymbol{B}^{\mathrm{II}}\left\langle 1\big/k\varsigma_j^{\mathrm{II}\,k}\right\rangle \mathbf{g}_k\right)\\
\boldsymbol{\Phi}^{\mathrm{I}} - \boldsymbol{\Phi}_0^{\mathrm{I}} &= (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{B}^{\mathrm{I}}\left\langle \ln\left(z_j^{\mathrm{I}} - y_{j\beta}^{\mathrm{I}}\right)\right\rangle \boldsymbol{V}_{\beta}'\right] + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{B}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle \boldsymbol{h}_k\right]
\end{aligned}
\tag{3.189}
$$

Substituting these equations into the continuity conditions (3.182) on the interface and noting $\ln(1-x) = -\sum_{k=1}^{\infty} x^k/k$, the following equations to determine unknown functions are obtained:

$$\boldsymbol{U}_0^{\mathrm{I}} = -(1/\pi)\mathrm{Im}\left(\boldsymbol{A}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}'\right), \quad \boldsymbol{\Phi}_0^{\mathrm{I}} = -(1/\pi)\mathrm{Im}\left(\boldsymbol{B}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}'\right), \quad \boldsymbol{V}' = \sum_{\beta=1}^{4} \boldsymbol{V}'_\beta \tag{3.190a}$$

$$\begin{aligned}&\boldsymbol{A}^{\mathrm{II}}\boldsymbol{V}''_\beta + \bar{\boldsymbol{A}}^{\mathrm{I}}\boldsymbol{V}'_\beta = \bar{\boldsymbol{A}}^{\mathrm{II}}\mathbf{I}_\beta \bar{\boldsymbol{V}}^{\mathrm{II}}, \quad \boldsymbol{B}^{\mathrm{II}}\boldsymbol{V}''_\beta + \bar{\boldsymbol{B}}^{\mathrm{I}}\boldsymbol{V}'_\beta = \bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_\beta \bar{\boldsymbol{V}}^{\mathrm{II}}\\ &\mathbf{I}_1 = \langle 1,0,0,0\rangle,\ \mathbf{I}_2 = \langle 0,1,0,0\rangle,\ \mathbf{I}_3 = \langle 0,0,1,0\rangle,\ \mathbf{I}_4 = \langle 0,0,0,1\rangle\end{aligned} \tag{3.190b}$$

$$\begin{aligned}\boldsymbol{A}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k &= \boldsymbol{A}^{\mathrm{I}}\left[\left\langle \left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4}\left\langle\left(\tau_{j\beta}^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{V}'_\beta\right]\\ \boldsymbol{B}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{B}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k &= \boldsymbol{B}^{\mathrm{I}}\left[\left\langle \left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4}\left\langle\left(\tau_{j\beta}^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{V}'_\beta\right]\end{aligned} \tag{3.190c}$$

From Eq. (3.190b) we can get

$$\begin{aligned}&\boldsymbol{B}^{\mathrm{II}}\boldsymbol{V}'' = \bar{\boldsymbol{H}}^{-1}\left(\bar{\boldsymbol{Y}}^{\mathrm{I}} - \bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_\beta\bar{\boldsymbol{V}}^{\mathrm{II}}, \quad \bar{\boldsymbol{B}}^{\mathrm{I}}\boldsymbol{V}'_\beta = \bar{\boldsymbol{H}}^{-1}\left(\boldsymbol{Y}^{\mathrm{II}} + \bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_\beta\bar{\boldsymbol{V}}^{\mathrm{II}}\\ &\boldsymbol{Y}^\alpha = \mathrm{i}\boldsymbol{A}^\alpha(\boldsymbol{B}^\alpha)^{-1}, \quad \boldsymbol{H} = \boldsymbol{Y}^{\mathrm{I}} + \bar{\boldsymbol{Y}}^{\mathrm{II}}\end{aligned} \tag{3.191}$$

3.7.2 *Green Function for a Singularity Inside the Elliptic Inclusion*

When a singularity is located inside the elliptic inclusion, the solution can be assumed:

$$\begin{aligned}\boldsymbol{U}^{\mathrm{II}} = {}&(1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \varsigma_{0j}^{\mathrm{I}}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right]\\ &+ (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left[\left(1\big/\varsigma_j^{\mathrm{II}}\right) - \left(1\big/\varsigma_{0\beta}^{\mathrm{I}}\right)\right]\right\rangle \boldsymbol{V}''_\beta\right]\\ &+ (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle\left(1\big/k\varsigma_j^{\mathrm{II}\,k}\right)\right\rangle \mathbf{g}_k\right]\end{aligned} \tag{3.192}$$

$$\begin{aligned}\boldsymbol{U}^{\mathrm{I}} - \boldsymbol{U}_0^{\mathrm{I}} = {}&(1/\pi)\mathrm{Im}\left\{\boldsymbol{A}^{\mathrm{I}}\ln\left(z_j^{\mathrm{I}} - z_{0j}^{\mathrm{I}}\right)\boldsymbol{V}^{\mathrm{I}}\right\}\\ &+ (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle \ln\left(z_j^{\mathrm{I}} - \hat{y}_{j\beta}^{\mathrm{I}}\right)\right\rangle\left(\boldsymbol{V}'_\beta - \mathbf{I}_\beta\boldsymbol{V}^{\mathrm{I}}\right)\right]\\ &+ (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{A}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle \boldsymbol{h}_k\right]\end{aligned} \tag{3.193}$$

where

$$
\begin{aligned}
&\boldsymbol{V}^{\alpha} = \boldsymbol{B}^{\alpha\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}^{\alpha\mathrm{T}}\boldsymbol{p}, \quad \alpha = \mathrm{I}, \mathrm{II} \\
&z_{0j}^{\mathrm{I}} = c_j^{\mathrm{I}}\varsigma_{0j}^{\mathrm{I}} + d_j^{\mathrm{I}}\Big/\varsigma_{0j}^{\mathrm{I}}, \quad \hat{y}_{j\beta}^{\mathrm{I}} = \hat{y}_{j\beta1} + \mu_j^{\mathrm{I}}\hat{y}_{j\beta2} = c_j^{\mathrm{I}}\Big/\bar{\varsigma}_{0\beta}^{\mathrm{I}} + d_j^{\mathrm{I}}\bar{\varsigma}_{0\beta}^{\mathrm{I}}
\end{aligned}
\tag{3.194}
$$

When z_0 is located in the inclusion, the single-valued cut from z_0 to $-\infty$ goes through the material II. So the first terms in Eqs. (3.192) and (3.193) are all discontinuous through this branch cut. The second term in Eq. (3.192) represents solutions for material II of 16 image singularities located at $\varsigma_{0\beta}^{\mathrm{I}}$ with $|\varsigma_{0\beta}^{\mathrm{I}} < 1|$ in the z_j plane. Similarly the second term in Eq. (3.193) represents solutions for material I of 16 image singularities located at $(\hat{y}_{j1}^{\mathrm{I}}, \hat{y}_{j2}^{\mathrm{I}})$ outside the elliptic inclusion.

In Eq. (3.192) replacing $\boldsymbol{A}^{\mathrm{II}}$ by $\boldsymbol{B}^{\mathrm{II}}$, we get $\boldsymbol{\Phi}^{\mathrm{II}}$, and in Eq. (3.193) replacing $\boldsymbol{A}^{\mathrm{I}}$ by $\boldsymbol{B}^{\mathrm{I}}$ and $\boldsymbol{u}_0^{\mathrm{I}}$ by $\boldsymbol{\Phi}_0^{\mathrm{I}}$, we get $\boldsymbol{\Phi}^{\mathrm{I}}$.

Substituting the solutions into the continuity conditions in Eq. (3.182) on the interface yields

$$
\begin{aligned}
\boldsymbol{U}_0^{\mathrm{I}} &= -(1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left\{\boldsymbol{A}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}' + \ln\left(-\varsigma_{0\beta}^{\mathrm{I}}\right)\left(\boldsymbol{A}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{A}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right)\right\} \\
\boldsymbol{\Phi}_0^{\mathrm{I}} &= -(1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left\{\boldsymbol{B}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}' + \ln\left(-\varsigma_{0\beta}^{\mathrm{I}}\right)\left(\boldsymbol{B}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{B}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right)\right\}
\end{aligned}
\tag{3.195a}
$$

$$
\begin{aligned}
\boldsymbol{A}^{\mathrm{II}}\boldsymbol{V}_{\beta}'' + \bar{\boldsymbol{A}}^{\mathrm{I}}\boldsymbol{V}_{\beta}' &= \bar{\boldsymbol{A}}^{\mathrm{II}}\mathbf{I}_{\beta}\bar{\boldsymbol{V}}^{\mathrm{II}} + 2\mathrm{Re}\left(\boldsymbol{A}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{A}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right) \\
\boldsymbol{B}^{\mathrm{II}}\boldsymbol{V}_{\beta}'' + \bar{\boldsymbol{B}}^{\mathrm{I}}\boldsymbol{V}_{\beta}' &= \bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_{\beta}\bar{\boldsymbol{V}}^{\mathrm{II}} + 2\mathrm{Re}\left(\boldsymbol{B}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{B}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right)
\end{aligned}
\tag{3.195b}
$$

$$
\begin{aligned}
&\boldsymbol{A}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k = \boldsymbol{A}^{\mathrm{I}}\left[\left\langle\left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k - \left\langle\left(\tau_{0j}^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{V}^{\mathrm{I}} - \sum_{\beta=1}^{4}\left\langle\left(\hat{\tau}_{j\beta}^{\mathrm{I}}\right)^k\right\rangle\left(\boldsymbol{V}_{\beta}' - \mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}}\right)\right] \\
&\boldsymbol{B}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{B}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k = \boldsymbol{B}^{\mathrm{I}}\left[\left\langle\left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k - \left\langle\left(\tau_{0j}^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{V}^{\mathrm{I}} - \sum_{\beta=1}^{4}\left\langle\left(\hat{\tau}_{j\beta}^{\mathrm{I}}\right)^k\right\rangle\left(\boldsymbol{V}_{\beta}' - \mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}}\right)\right] \\
&\tau_{0j}^{\mathrm{I}} = m_j^{\mathrm{I}}\left(1\Big/\varsigma_{0j}^{\mathrm{I}}\right), \quad \hat{\tau}_{j\beta}^{\mathrm{I}} = m_j^{\mathrm{I}}\bar{\varsigma}_{0\beta}^{\mathrm{II}}
\end{aligned}
\tag{3.195c}
$$

3.7.3 Green Function for a Singularity on the Interface

When a singularity is located at $z_0 = a\cos\psi_0 + \mathrm{i}b\sin\psi_0$ on the elliptic boundary, Eqs. (3.183) and (3.184) become

$$
\begin{aligned}
\boldsymbol{U}^{\mathrm{II}} &= (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \mathrm{e}^{\mathrm{i}\psi_0}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] + (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left[\left(1\Big/\varsigma_j^{\mathrm{II}}\right) - \mathrm{e}^{-\mathrm{i}\psi_0}\right]\right\rangle \boldsymbol{V}''\right] \\
&\quad + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left(\boldsymbol{A}^{\mathrm{II}}\left\langle 1\Big/k\varsigma_j^{\mathrm{II}\,k}\right\rangle \mathbf{g}_k\right)
\end{aligned}
\tag{3.196}
$$

$$\boldsymbol{U}^{\mathrm{I}}-\boldsymbol{U}_{0}^{\mathrm{I}}=(1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle\ln\left(z_{j}^{\mathrm{I}}-z_{0j}^{\mathrm{I}}\right)\right\rangle\boldsymbol{V}'\right]+(1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{A}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle\mathbf{h}_{k}\right] \tag{3.197}$$

where

$$\begin{aligned}\boldsymbol{V}''&=\sum_{\beta=1}^{4}\boldsymbol{V}''_{\beta}=\left(\boldsymbol{B}^{\mathrm{II}}\right)^{-1}\left(\boldsymbol{Y}^{\mathrm{II}}+\bar{\boldsymbol{Y}}^{\mathrm{I}}\right)^{-1}\left(\bar{\boldsymbol{Y}}^{\mathrm{I}}-\bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\bar{\boldsymbol{V}}^{\mathrm{II}}\\ \boldsymbol{V}'&=\sum_{\beta=1}^{4}\boldsymbol{V}'_{\beta}=\left(\boldsymbol{B}^{\mathrm{I}}\right)^{-1}\left(\boldsymbol{Y}^{\mathrm{I}}+\bar{\boldsymbol{Y}}_{\mathrm{II}}\right)^{-1}\left(\boldsymbol{Y}^{\mathrm{II}}+\bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\bar{\boldsymbol{V}}^{\mathrm{II}}\end{aligned} \tag{3.198}$$

In Eq. (3.196) replacing $\boldsymbol{A}^{\mathrm{II}}$ by $\boldsymbol{B}^{\mathrm{II}}$, we get $\boldsymbol{\Phi}^{\mathrm{II}}$, and in Eq. (3.197) replacing $\boldsymbol{A}^{\mathrm{I}}$ by $\boldsymbol{B}^{\mathrm{I}}$ and $\boldsymbol{u}_0^{\mathrm{I}}$ by $\boldsymbol{\Phi}_0^{\mathrm{I}}$, we get $\boldsymbol{\Phi}^{\mathrm{I}}$.

From Eqs. (3.192) and (3.193), we still get the same result.

3.7.4 Material Force Between the Singularity and the Elliptic Inclusion

Eshelby (1956) defined the material force as the negative gradient of the total mechanical and electrical energy with respect to the position variation of the defect. For a linear electroelasticity, we can also use the total electric enthalpy (Eq. (1.55)) instead of total energy. The general method calculating the material force is given in many literatures, such as Lardner (1974), Pak (1990), Wen and Hwu (1994), and Kuang et al. (1998). The total electric enthalpy of the system for a dislocation at (x_{01}, x_{02}) can be defined as the work required to introduce the dislocation in the material, i.e.,

$$H=(1/2)\int_{x_{01}+\delta}^{\Lambda}(\sigma_{2i}b_i+D_2b_4)\mathrm{d}x_1,\quad \delta\to 0,\quad \Lambda\to\infty \tag{3.199}$$

1. *Dislocation is inside the matrix*. Equation (3.199) becomes

$$\begin{aligned}&H=(1/2)\boldsymbol{b}^{\mathrm{T}}\cdot\boldsymbol{\Phi}^{\mathrm{II}}\Big|_{z_{0j}^{\mathrm{II}}+\delta_j}^{z_{0j}^{\mathrm{II}}=\Lambda_j}\\ &\boldsymbol{\Phi}^{\mathrm{II}}\Big|_{z_{0j}^{\mathrm{II}}+\delta_j}^{z_{0j}^{\mathrm{II}}=\Lambda_j}=(1/\pi)\mathrm{Im}\Big[\boldsymbol{B}^{\mathrm{II}}\langle\ln(\Lambda_j/\delta_j)\rangle\boldsymbol{V}^{\mathrm{II}}+\boldsymbol{B}^{\mathrm{II}}\left\langle\ln\left(1-m_j^{\mathrm{II}}\Big/\varsigma_{0j}^{\mathrm{II}}\varsigma_{0j}^{\mathrm{II}}\right)\right\rangle\boldsymbol{V}^{\mathrm{II}}\\ &\qquad-\sum_{\beta=1}^{4}\boldsymbol{B}^{\mathrm{II}}\left\langle\ln\left(1-1\Big/\varsigma_{0j}^{\mathrm{II}}\bar{\varsigma}_{0\beta}^{\mathrm{II}}\right)\right\rangle\boldsymbol{V}''_{\beta}-\sum_{k=1}^{\infty}\boldsymbol{B}^{\mathrm{II}}\left\langle 1\Big/k\varsigma_{0j}^{\mathrm{II}\,k}\right\rangle\mathbf{g}_k\Big]\end{aligned} \tag{3.200}$$

By excluding singular part of the dislocation enthalpy itself, the interaction enthalpy part of the media with the dislocation is obtained as

$$\begin{aligned} H_{\text{int}}^{\text{II}} = (1/2\pi)\boldsymbol{b}^{\text{T}} \cdot \operatorname{Im}\Big[&\boldsymbol{B}^{\text{II}}\Big\langle \ln\Big(1 - m_j^{\text{II}} \Big/ \varsigma_{0j}^{\text{II}}\varsigma_{0j}^{\text{II}}\Big)\Big\rangle \boldsymbol{V}^{\text{II}} \\ &- \sum_{\beta=1}^{4} \boldsymbol{B}^{\text{II}}\Big\langle \ln\Big(1 - 1\Big/\varsigma_{0j}^{\text{II}}\bar{\varsigma}_{0\beta}^{\text{II}}\Big)\Big\rangle \boldsymbol{V}'' - \sum_{k=1}^{\infty} \boldsymbol{B}^{\text{II}}\Big\langle 1\Big/k\varsigma_{0j}^{\text{II}\,k}\Big\rangle \mathbf{g}_k\Big] \end{aligned} \tag{3.201}$$

2. *Dislocation is inside the inclusion*. Equation (3.199) becomes

$$\begin{aligned} H = (1/2)\boldsymbol{b}^{\text{T}} \cdot \left(\boldsymbol{\Phi}^{\text{II}}\Big|_{z_{0j}^{\text{II}*}}^{z_{0j}^{\text{II}}=\Lambda_j} + \boldsymbol{\Phi}^{\text{I}}\Big|_{z_{0j}^{\text{I}}+\delta_j}^{z_{0j}^{\text{I}*}}\right) = \frac{1}{2\pi}\boldsymbol{b}^{\text{T}} \cdot \operatorname{Im}\Big[&\boldsymbol{B}^{\text{II}}\langle \ln\Lambda_j\rangle \boldsymbol{V}^{\text{II}} - \boldsymbol{B}^{\text{I}}\langle \ln\delta_j\rangle \boldsymbol{V}^{\text{I}} \\ - \sum_{k=1}^{\infty}(1/k)\boldsymbol{B}^{\text{I}}\Big\langle f_{jk}^{\text{I}}\Big\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4}\boldsymbol{B}^{\text{I}}\Big\langle \ln\Big(z_{0j}^{\text{I}} - \hat{y}_{j\beta}^{\text{I}}\Big)\Big\rangle \Big(\boldsymbol{V}'_\beta - \text{I}_\beta \boldsymbol{V}^{\text{I}}\Big) \\ - \sum_{\beta=1}^{4}\ln\Big(-\varsigma_{0\beta}^{\text{I}}\Big)\Big(-\boldsymbol{B}^{\text{II}}\boldsymbol{V}''_\beta + \boldsymbol{B}^{\text{I}}\text{I}_\beta \boldsymbol{V}^{\text{I}} - \boldsymbol{B}^{\text{II}}\text{I}_\beta \boldsymbol{V}^{\text{II}}\Big)\Big] \end{aligned} \tag{3.202}$$

where $z_{0j}^{\text{II}*} = z_{0j}^{\text{I}*}$ is the same point on the interface. By excluding singular part of the dislocation enthalpy itself, the interaction enthalpy part of the media with the dislocation is obtained as

$$\begin{aligned} H_{\text{int}}^{\text{I}} = \frac{1}{2\pi}\mathbf{b}^{\text{T}} \cdot \operatorname{Im}\Big[&-\sum_{k=1}^{\infty}(1/k)\boldsymbol{B}^{\text{I}}\Big\langle f_{jk}^{\text{I}}\Big\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4}\boldsymbol{B}^{\text{I}}\Big\langle \ln\Big(z_{0j}^{\text{I}} - \hat{y}_{j\beta}^{\text{I}}\Big)\Big\rangle \Big(\boldsymbol{V}'_\beta - \text{I}_\beta \boldsymbol{V}^{\text{I}}\Big) \\ &- \sum_{\beta=1}^{4}\ln\Big(-\varsigma_{0\beta}^{\text{I}}\Big)\Big(-\boldsymbol{B}^{\text{II}}\boldsymbol{V}''_\beta + \boldsymbol{B}^{\text{I}}\text{I}_\beta \boldsymbol{V}^{\text{I}} - \boldsymbol{B}^{\text{II}}\text{I}_\beta \boldsymbol{V}^{\text{II}}\Big)\Big] \end{aligned} \tag{3.203}$$

The generalized interaction force per unit length $\boldsymbol{F}$ along the direction $\boldsymbol{s}$ on the dislocation is

$$\boldsymbol{F} = -\partial H_{\text{int}}/\partial \boldsymbol{s} \tag{3.204}$$

which is usually obtained by numerical calculation.

3.7.5 Numerical Example

Let the matrix be PZT-5H and the inclusions be epoxy, insulated void, and rigid conductor, respectively. Usually material constants are given in the material principle coordinate system (X_1, X_2, X_3) with poling axis X_3. For PZT-5H matrix,

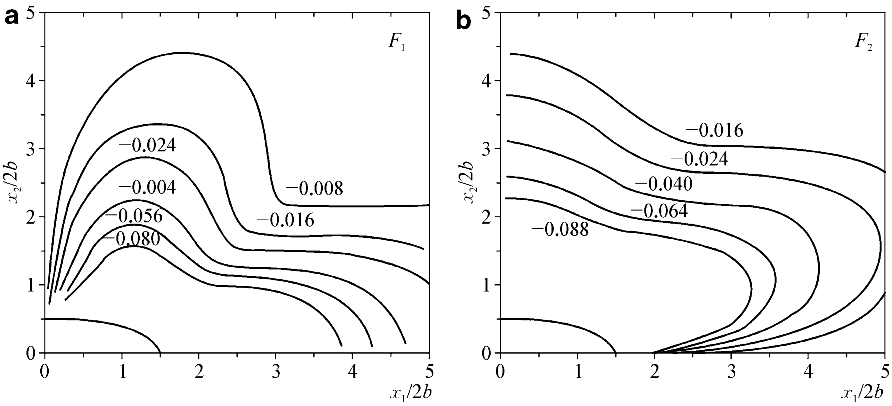


Fig. 3.8 PZT-5H/epoxy under loading $\mathbf{b}/2b = (1,0,0,0)$: (**a**) contour plots of the dimensionless glide force F_1 and (**b**) contour plots of the dimensionless climb force F_2

$$
\begin{aligned}
&C_{11}^{\mathrm{II}} = 126, \quad C_{33}^{\mathrm{II}} = 117, \quad C_{44}^{\mathrm{II}} = 35.3, \quad C_{12}^{\mathrm{II}} = 55, \quad C_{13}^{\mathrm{II}} = 53(\mathrm{GPa}) \\
&e_{31}^{\mathrm{II}} = -6.5, \quad e_{33}^{\mathrm{II}} = 23.3, \quad e_{15}^{\mathrm{II}} = 17.0(\mathrm{C/m^2}) \\
&\epsilon_{11}^{\mathrm{II}} = 15.1 \times 10^{-9}, \quad \epsilon_{33}^{\mathrm{II}} = 13.0 \times 10^{-9}(\mathrm{C^2/Nm^2})
\end{aligned}
$$

For inclusion epoxy,

$$
\begin{aligned}
&C_{11}^{\mathrm{I}} = 6.43, \quad C_{33}^{\mathrm{I}} = 6.429, \quad C_{44}^{\mathrm{I}} = 1.07, \quad C_{12}^{\mathrm{I}} = 4.29, \quad C_{13}^{\mathrm{I}} = 4.289(\mathrm{GPa}) \\
&e_{31}^{\mathrm{I}} = e_{33}^{\mathrm{I}} = e_{15}^{\mathrm{I}} = 0(\mathrm{C/m^2}), \quad \epsilon_{11}^{\mathrm{I}} = 5.0 \times 10^{-9}, \quad \epsilon_{33}^{\mathrm{I}} = 5.001 \times 10^{-9}(\mathrm{C^2/Nm^2})
\end{aligned}
$$

Material constants for epoxy were be modified slightly to avoid repeated eigenvalues. It is noted that in above analyses of this section, the coordinates (x_1, x_2, x_3) with polarized x_2-axis are used, so in numerical calculation, materials should be converted. The corresponding relation between (X_1, X_2, X_3) and (x_1, x_2, x_3) is $X_3 \to x_2, X_1 \to x_1, X_2 \to x_3$.

The dimensionless glide force F_1 and climb force F_2 of the interaction between the inclusion and dislocation are defined as

$$
F_1 = -\left(\partial H_{\mathrm{int}}^e/\partial x_1\right)/\left(L_{11} \times 10^{-9}/4\pi\right), \quad F_2 = -\left(\partial H_{\mathrm{int}}^e/\partial x_2\right)/\left(L_{11} \times 10^{-9}/4\pi\right) \tag{3.205}
$$

where L_{11} is shown in Eq. (3.35). F_1 and F_2 will be numerically studied. The positive glide and climb forces show that the dislocation is repelled from x_2- and x_1-axes, respectively. Figures 3.8 and 3.9 show the contour plots of F_1 and F_2 under two cases: (1) $\mathbf{b}/2b = (1,0,0,0)$, only mechanical dislocation b_1, and (2) $\mathbf{b}/2b = (0,0,0,10^9\ \mathrm{V/m})$, only electric dislocation b_4.

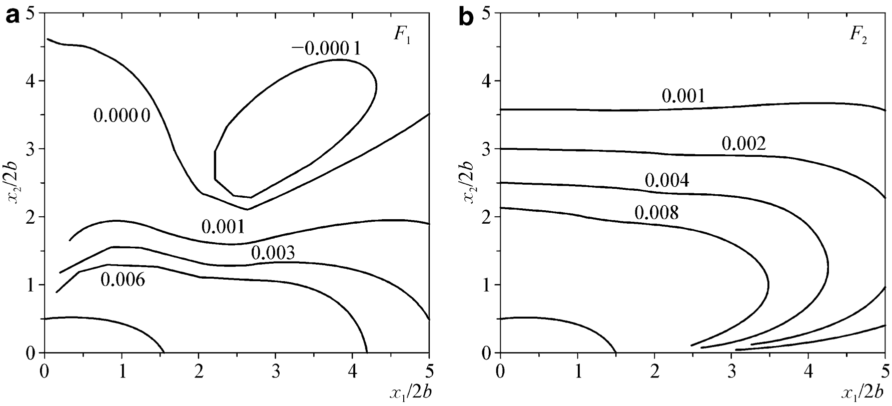


Fig. 3.9 PZT-5H/epoxy under loading $\mathbf{b}/2b = (0, 0, 0, 10^9\ \text{V/m})$: (**a**) contour plots of the dimensionless glide force F_1 and (**b**) contour plots of the dimensionless climb force F_2

Interaction of an elliptic inclusion with a singularity was discussed in many literatures, such as Meguid and Deng (1998), Deng and Meguid (1999), Liu et al. (1999), and Fan et al. (2005).

3.8 Asymptotic Fields near a Line Inclusion Tip in a Homogeneous Material

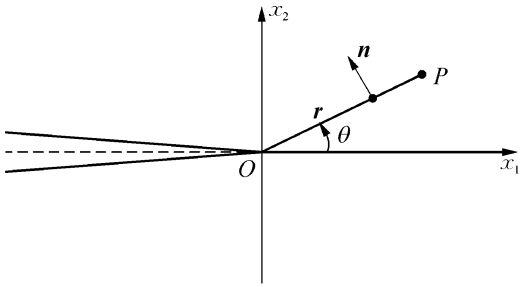
3.8.1 A General Form of the Asymptotic Fields near a Line Inclusion Tip

Discuss a homogeneous material with a line inclusion. It is assumed that the size of the line inclusion is much smaller than that of the material. The region near the tip to be suitable for an asymptotic analysis is much smaller than that of the line inclusion, so the asymptotic fields near a line inclusion tip in a practical structure are almost the same as that in a semi-infinite line inclusion. Let a semi-infinite line inclusion be along the axis x_1 from the origin to negative infinite, i.e., the region Ω of the material is $0 \leq r \leq \infty, -\pi < \theta \leq \pi$, where θ is the polar angle (Fig. 3.10). The asymptotic fields near the right tip can be assumed in the following form (Ting 1996; Kuang and Ma 2002):

$$\begin{aligned} f_j(z_j) &= V_j z_j^{\lambda+1}/(\lambda+1), \quad z_j = x_1 + \mu_j x_2 = r(\cos\theta + \mu_j \sin\theta) \\ F_j(z_j) &= f_j'(z_j) = V_j z_j^{\lambda}, \quad \mu_j = \alpha_j + i\beta_j \end{aligned} \tag{3.206}$$

where $\boldsymbol{V}$ is an undetermined complex constant, λ is an undetermined singular index, and $\lambda > -1/2$ to keep a finite strain energy density at the tip region. In plane problem we often use $\boldsymbol{f}(z_j), \boldsymbol{F}(z_j)$ instead of $\boldsymbol{f}(z_P), \boldsymbol{F}(z_P)$, where $j = 1, 2, 3$ and $j = 3$

Fig. 3.10 Local coordinate system at a crack tip



represent the electric variables, as shown in Eq. (3.206). From Eqs. (3.26) and (3.206), it is known that

$$\begin{aligned}
\boldsymbol{U} &= (\lambda+1)^{-1}\sum_{j=1}^{4}\left(\boldsymbol{a}_j z_j^{\lambda+1}V_j + \bar{\boldsymbol{a}}_j \bar{z}_j^{\lambda+1}\bar{V}_j\right) = (\lambda+1)^{-1}\left[\boldsymbol{A}\left\langle z_j^{\lambda+1}\right\rangle \boldsymbol{V} + \bar{\boldsymbol{A}}\left\langle \bar{z}_j^{\lambda+1}\right\rangle \bar{\boldsymbol{V}}\right]\\
\Phi &= (\lambda+1)^{-1}\sum_{j=1}^{4}\left(\boldsymbol{b}_j z_j^{\lambda+1}V_j + \bar{\boldsymbol{b}}_j \bar{z}_j^{\lambda+1}\bar{V}_j\right) = (\lambda+1)^{-1}\left[\boldsymbol{B}\left\langle z_j^{\lambda+1}\right\rangle \boldsymbol{V} + \bar{\boldsymbol{B}}\left\langle \bar{z}_j^{\lambda+1}\right\rangle \bar{\boldsymbol{V}}\right]\\
\boldsymbol{\Sigma}_1 &= -\boldsymbol{\Phi}_{,2} = -\sum_{j=1}^{4}\left(\mu_j \boldsymbol{b}_j z_j^{\lambda}V_j + \bar{\mu}_j \bar{\boldsymbol{b}}_j \bar{z}_j^{\lambda}\bar{V}_j\right) = \boldsymbol{B}\left\langle \mu_j z_j^{\lambda}\right\rangle \boldsymbol{V} + \bar{\boldsymbol{B}}\left\langle \bar{\mu}_j \bar{z}_j^{\lambda}\right\rangle \bar{\boldsymbol{V}}\\
\boldsymbol{\Sigma}_2 &= \boldsymbol{\Phi}_{,1} = \sum_{j=1}^{4}\left(\boldsymbol{b}_j z_j^{\lambda}V_j + \bar{\boldsymbol{b}}_j \bar{z}_j^{\lambda}\bar{V}_j\right) = \boldsymbol{B}\left\langle z_j^{\lambda}\right\rangle \boldsymbol{V} + \bar{\boldsymbol{B}}\left\langle \bar{z}_j^{\lambda}\right\rangle \bar{\boldsymbol{V}}
\end{aligned} \tag{3.207}$$

In the polar coordinate system, the normal of a radial plane is $\boldsymbol{n}\ (-\sin\theta, \cos\theta)$ which is identical with the tangent $\boldsymbol{t}$ of a circle with the center at the coordinate origin (Fig. 3.10). The traction $\boldsymbol{T}$ on the radial plane is

$$\begin{aligned}
T_i &= \sigma_{i1}n_1 + \sigma_{i2}n_2 = -\sigma_{i1}\sin\theta + \sigma_{i2}\cos\theta = \Phi_{i,2}\sin\theta + \Phi_{i,1}\cos\theta\\
&= \Phi_i'(z_j)\left(z_j/r\right) = 2\mathrm{Re}\sum_{j=1}^{4}\left(B_{ij}z_j^{\lambda+1}V_j r^{-1}\right), \quad \boldsymbol{T} = 2\mathrm{Re}\left\{r^{-1}\boldsymbol{B}\left\langle z_j^{\lambda+1}\right\rangle \boldsymbol{V}\right\}
\end{aligned} \tag{3.208}$$

where $\Phi_i'(z_j) = \mathrm{d}\Phi_i(z_j)\big/\mathrm{d}z_j$. Equation (3.208) can be used to discuss the asymptotic field near a wedge, but in this book we only discuss the line inclusion.

3.8.2 *The Stress Singularity*

The stress singularity near a tip is related to the boundary conditions of the inclusion.

1. *Two sides of the line crack are free*. The boundary conditions are

$$\boldsymbol{T}(r, \pm\pi) = \boldsymbol{0}, \quad \text{or} \quad \boldsymbol{\Sigma}_2(r, \pm\pi) = \boldsymbol{0} \tag{3.209}$$

Substituting Eq. (3.207) or (3.208) into Eq. (3.209) and noting $x_{1j} = x_1; z_j(r,0) = r, z_j(r,\pm\pi) = re^{\pm i\pi}$ on axis x_1 yield

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{BV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\mathbf{B}\boldsymbol{V} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \text{or} \\ &\sum_{j=1}^{4}\left(\mathrm{e}^{\mathrm{i}\lambda\pi}V_j\boldsymbol{b}_j + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{b}}_j\right) = \boldsymbol{0}, \quad \sum\nolimits_{j=1}^{4}\left(\mathrm{e}^{-\mathrm{i}\lambda\pi}V_j\boldsymbol{b}_j + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{b}}_j\right) = \boldsymbol{0} \end{aligned} \tag{3.210}$$

Equation (3.210) yields

$$\left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right)^4\boldsymbol{BV} = \boldsymbol{0}, \quad \text{or} \quad \left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right)\boldsymbol{b}_jV_j = \boldsymbol{0}; \quad j = 1-4 \tag{3.211}$$

Because B is not singular, so we have the eigenvalue equation

$$\left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right)^4 = 0, \quad \text{or} \quad \left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right) = 0 \tag{3.212}$$

From Eq. (3.212) it is known that $\lambda = -1/2, 0, m/2$, where m is an integer. When $\lambda = -1/2$, the generalized stresses are singular with the singular index $-1/2$.

2. *Two sides of the line crack are fixed (rigid inclusion with zero electric potential)*. The boundary conditions are

$$\boldsymbol{U}(\pm\pi) = \boldsymbol{0} \tag{3.213}$$

We have

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \text{or} \\ &\sum_{j=1}^{4}\left(\mathrm{e}^{\mathrm{i}\lambda\pi}V_j\boldsymbol{a}_j + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{a}}_j\right) = \boldsymbol{0}, \quad \sum_{j=1}^{4}\left(\mathrm{e}^{-\mathrm{i}\lambda\pi}V_j\boldsymbol{a}_j + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{a}}_j\right) = \boldsymbol{0} \end{aligned} \tag{3.214}$$

It is also found that the eigenvalue equation is Eq. (3.212), so we also have $\lambda = -1/2, 0, m/2$; m is an integer.

3. *One side free and one side fixed*. The boundary conditions are

$$\boldsymbol{T}(r,\pi) = \boldsymbol{0}, \quad \boldsymbol{U}(-\pi) = \boldsymbol{0} \tag{3.215}$$

We have

$$\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{BV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0} \tag{3.216}$$

The eigenvalue equation is

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{BV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0}; \quad \text{or} \\ &\left(\mathrm{e}^{-2\mathrm{i}\lambda\pi}\boldsymbol{Y} + \mathrm{e}^{2\mathrm{i}\lambda\pi}\bar{\boldsymbol{Y}}\right)\boldsymbol{BV} = \boldsymbol{0}; \quad \boldsymbol{Y} = \mathrm{i}\boldsymbol{AB}^{-1}, \quad \bar{\boldsymbol{Y}} = -\mathrm{i}\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}^{-1} \end{aligned} \tag{3.217}$$

Substituting Eq. (3.37), $\boldsymbol{Y} = -\mathrm{i}(\boldsymbol{S} + \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1}, \boldsymbol{S} = \mathrm{i}\left(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} - \mathbf{I}\right)$, into Eq. (3.217), the eigen-equation in a more convenient form is obtained:

$$\left[-\mathrm{i}e^{-2\mathrm{i}\lambda\pi}(\boldsymbol{S} + \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1} + \mathrm{i}e^{2\mathrm{i}\lambda\pi}(\boldsymbol{S} - \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1}\right]\boldsymbol{B}\boldsymbol{V} = \mathbf{0} \quad \Rightarrow \quad (\boldsymbol{S} + \cot 2\lambda\mathbf{I})\boldsymbol{L}^{-1}\boldsymbol{B}\boldsymbol{V} = \mathbf{0} \tag{3.218}$$

From the above analyses, it is known that the singular index λ is independent to the selected z_j plane.

3.8.3 The Stress Asymptotic Field near a Crack Tip

From Eqs. (3.212) and(3.214), it is known that when $\lambda = -1/2$, the stresses are singular. Substituting it into Eqs. (3.206) and (3.207) yields

$$\begin{aligned}
&F_j(z_j) = V_j\Big/\sqrt{z_j} = V_j\Big/\sqrt{r\Theta_j}, \quad \Theta_j = \cos\theta + \mu_j\sin\theta \\
&\Sigma_{1i} = -2\mathrm{Re}\sum_{j=1}^{4}\left(\mu_j b_{ji}V_j\Big/\sqrt{z_j}\right) = -2\mathrm{Re}\sum_{j=1}^{4}B_{ij}\mu_j V_j\Big/\sqrt{r\Theta_j} \\
&\Sigma_{2i} = 2\mathrm{Re}\sum_{j=1}^{4}\left(b_{ji}V_j\Big/\sqrt{z_j}\right) = 2\mathrm{Re}\sum_{j=1}^{4}B_{ij}V_j\Big/\sqrt{r\Theta_j} \\
&\boldsymbol{\Sigma}_1 = -2\mathrm{Re}\boldsymbol{B}\left\langle\mu_j\Big/\sqrt{z_j}\right\rangle\boldsymbol{V}, \quad \boldsymbol{\Sigma}_2 = 2\mathrm{Re}\boldsymbol{B}\left\langle 1\Big/\sqrt{z_j}\right\rangle\boldsymbol{V}
\end{aligned} \tag{3.219}$$

Define the stress intensities as

$$\boldsymbol{K} = (K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_D)^T = \lim_{r\to 0}\sqrt{2\pi r}(\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2)^{\mathrm{T}}\big|_{\theta=0} = \lim_{r\to 0}\sqrt{2\pi r}\,\boldsymbol{\Sigma}_2\big|_{\theta=0} \tag{3.220}$$

Let

$$\boldsymbol{V} = 1\Big/\left(2\sqrt{2\pi}\right)\boldsymbol{B}^{-1}\boldsymbol{K}, \quad V_i = 1\Big/\left(2\sqrt{2\pi}\right)B_{ij}^{-1}K_j, \quad B_{ij}^{-1} = \left[\boldsymbol{B}^{-1}\right]_{ij} \tag{3.221}$$

Substitution of Eqs. (3.220) and (3.221) into Eq. (3.219) yields

$$\begin{aligned}
&\Sigma_{1i} = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{4}B_{ij}\mu_j B_{jl}^{-1}K_l\Big/\sqrt{\Theta_j}, \quad \Sigma_{2i} = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{4}B_{ij}B_{jl}^{-1}K_l\Big/\sqrt{\Theta_j} \\
&\boldsymbol{\Sigma}_1 = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\boldsymbol{B}\left\langle\mu_j\Big/\sqrt{\Theta_j}\right\rangle\boldsymbol{B}^{-1}\boldsymbol{K}, \quad \boldsymbol{\Sigma}_2 = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\boldsymbol{B}\left\langle 1\Big/\sqrt{\Theta_j}\right\rangle\boldsymbol{B}^{-1}\boldsymbol{K}
\end{aligned} \tag{3.222}$$

It is noted that in general situation $B_{ij}B_{jl}^{-1}K_l\Big/\sqrt{\Theta_j} \neq K_l\Big/\sqrt{\Theta_j}$. But when $\theta = 0$ and $\Theta_j = 1$, $B_{ij}B_{jl}^{-1}K_l\Big/\sqrt{\Theta_j} = B_{ij}B_{jl}^{-1}K_l = K_l$ and $\boldsymbol{\Sigma}_2 = \boldsymbol{K}\Big/\sqrt{2\pi r}$.

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