Chapter 2

Physical Variational Principle and Governing Equations

Abstract In this chapter, the physical variational principle is used to derive the governing equations of the nonlinear and linear electroelastic analyses in piezoelectric and pyroelectric materials. Some kinds of the physical variational principle are given. Applying the migratory variation of electric potential, the general expression of the static electric force is given. It is shown that for the physical nonlinear problem, such as the electrostrictive materials, in order to get correct governing equations and material constants in experiments, we need to consider the entire system including the dielectric medium, its environment, and their common boundary. When the temperature varies with time, the inertial entropy and generalized inertial entropy theories are used to derive the temperature wave equation and the mass diffusion equation. This theory is consistent with current known thermodynamic theory.

Keywords Physical variational principle • Governing equation • Inertial entropy theory • Temperature wave equation • Mass diffusion equation

2.1 Electric Gibbs Free Energy Variational Principle in Piezoelectric Materials

2.1.1 Electric Gibbs Free Energy and Constitutive Equations

In Sect. 1.6, the physical variational principle (PVP) was proposed as a basic principle in the continuum mechanics for an electrically static state. In this chapter, we shall discuss its applications. At first, the electric Gibbs free energy variational principle in piezoelectric materials under the isothermal case is discussed. According to Eq. (1.69) in Sect. 1.5.4, the electric Gibbs free energy g is

$$g = g\left(\varepsilon_{ij}, E_i\right), \quad \mathrm{d}g = \sigma_{ji}\mathrm{d}u_{i,j} - D_i\mathrm{d}E_i \tag{2.1}$$

Z.-B. Kuang, *Theory of Electroelasticity*, DOI 10.1007/978-3-642-36291-0_2,

Under the small deformation, g can be expanded in the series of $\boldsymbol{\varepsilon}$ and $\boldsymbol{E}$:

$$\begin{aligned} g &= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij} - (1/2)l_{ijkl}E_iE_j\varepsilon_{kl} \\ &\quad - (1/2)\alpha_{km}E_mE_l\varepsilon_{kl} - (1/4)\alpha_{nm}E_mE_n\varepsilon_{kl}\delta_{kl} \\ C_{ijkl} &= C_{jikl} = C_{ijlk} = C_{klij}, \quad l_{ijkl} = l_{jikl} = l_{ijlk} = l_{klij}, \quad e_{kij} = e_{kji} \end{aligned} \tag{2.2}$$

where $\boldsymbol{C}, \boldsymbol{\epsilon}, \boldsymbol{e}, \boldsymbol{l}$ are the elastic coefficient, permittivity, piezoelectric coefficient, and the electrostrictive coefficient, respectively; $\boldsymbol{\alpha}$ is a new asymmetric or symmetric electrostrictive coefficient in order to make $\boldsymbol{l}$ the same symmetries as that in $\boldsymbol{C}$ (Kuang 2007, 2008a). For convenience, the term $\alpha_{nm}E_mE_n\varepsilon_{kl}\delta_{kl}$ is also added.

Because g is a state function, constitutive equations can be derived as

$$\begin{aligned} \sigma_{lk} &= \partial g/\partial\varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j - (1/2)\alpha_{km}E_mE_l - (1/4)\alpha_{nm}E_mE_n\delta_{kl} \\ D_k &= -\partial g/\partial E_k = \left[\epsilon_{kl} + l_{ijkl}\varepsilon_{ij} + (1/2)(\alpha_{ml}\varepsilon_{mk} + \alpha_{mk}\varepsilon_{ml}) + (1/4)(\alpha_{lk} + \alpha_{kl})\varepsilon_{mn}\delta_{mn}\right]E_l \\ &\quad + e_{kij}\varepsilon_{ij} \approx \epsilon_{kl}E_l \end{aligned} \tag{2.3}$$

In general case in Eqs. (2.2) and (2.3), $\varepsilon_{ij} = u_{i,j}, \varepsilon_{ij} \neq \varepsilon_{ji}$, and there are nine components for ε_{ij} and σ_{ij}. For most practical cases, the body couple is neglected; in this case, σ_{ij} and $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$ are symmetric and each of them only has six components. Let $\boldsymbol{\sigma}^{\mathrm{s}}$ and $\boldsymbol{\sigma}^{\mathrm{a}}$ be the symmetric and asymmetric parts of $\boldsymbol{\sigma}$, respectively, we have

$$\begin{aligned} \sigma^{\mathrm{s}}_{lk} &= (1/2)(\sigma_{kl} + \sigma_{lk}) = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j \\ &\quad - (1/4)(\alpha_{km}E_l + \alpha_{lm}E_k)E_m - (1/4)\alpha_{nm}E_mE_n\delta_{kl} \\ \sigma^{\mathrm{a}}_{lk} &= (1/2)(\sigma_{lk} - \sigma_{kl}) = -(1/4)(\alpha_{km}E_l - \alpha_{lm}E_k)E_m \end{aligned} \tag{2.4}$$

where $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$ was used. In Eq. (2.4), terms containing $\boldsymbol{l} : \boldsymbol{\varepsilon}, \boldsymbol{\alpha} : \boldsymbol{\varepsilon}, \boldsymbol{e} : \boldsymbol{\varepsilon}$ had been neglected. In the usual electromagnetic theory, the electromagnetic body couple is $\boldsymbol{P} \times \boldsymbol{E} = \boldsymbol{D} \times \boldsymbol{E}$. In general, $\boldsymbol{\alpha}$ should be determined by experiments. In this book $\boldsymbol{\alpha} = -2\boldsymbol{\epsilon}$ is assumed, so Eqs. (2.3) and (2.4) are reduced to

$$\begin{aligned} \sigma_{lk} &= \partial g/\partial\varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j + \epsilon_{km}E_mE_l + (1/2)\epsilon_{nm}E_mE_n\delta_{kl} \\ \sigma^{\mathrm{s}}_{lk} &= C_{ijkl}\varepsilon_{ji} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j + (1/2)(\epsilon_{km}E_l + \epsilon_{lm}E_k)E_m + (1/2)\epsilon_{nm}E_mE_n\delta_{kl} \\ \sigma^{\mathrm{a}}_{lk} &= (1/2)(\epsilon_{km}E_l - \epsilon_{lm}E_k)E_m \approx (1/2)(D_kE_l - D_lE_k) \end{aligned} \tag{2.5}$$

Equation (2.5) shows that the electric body couple is balanced by the moment produced by the asymmetric stresses (Eringen and Maugin 1989). If the electromagnetic body couple is neglected, all the stresses are symmetric. Using Eq. (2.3), Eq. (2.2) is reduced to

$$g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mathrm{e}}, \quad g^{\mathrm{e}} = -(1/2)\left(D_kE_k + \mathit{\Delta}^{g}_{kl}\varepsilon_{lk}\right); \quad \mathit{\Delta}^{g}_{kl} = e_{mkl}E_m \tag{2.6}$$

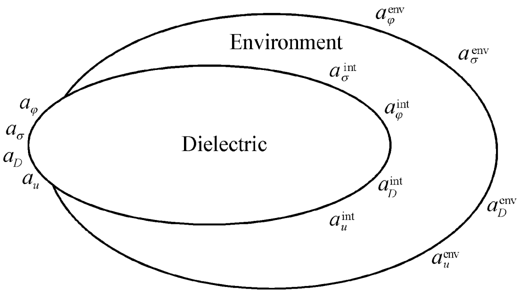


Fig. 2.1 Dielectric and its environment

where g^{e} is the part related to the electric field in g or the energy from the total energy minus the pure deformation energy. The value of the term $\boldsymbol{\Delta}^g : \boldsymbol{\varepsilon}$ is much less than other terms, so it can be neglected.

In the electroelastic analysis, the dielectric medium, its environment, and their common boundary a^{int} consociate a system and should be considered together, because the electric field exists in every material except the ideal conductor. In this book, the variables in the environment will be denoted by a right superscript "env" and the variables on the interface will be denoted by a right superscript "int" (Fig. 2.1). In the environment, Eqs. (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6) are all held.

2.1.2 *Electric Gibbs Free Energy Variational Principle and Governing Equations*

Under the assumption that $\mathbf{u}, \varphi, \mathbf{u}^{\mathrm{env}}, \varphi^{\mathrm{env}}$ satisfy their boundary conditions on their own boundaries $a_u, a_\varphi, a_u^{\mathrm{env}}, a_\varphi^{\mathrm{env}}$ and the continuity conditions on the interface a^{int}. Given the displacement and electric potential virtual increments, the PVP in terms of the electric Gibbs free energy (which is identical to the electric enthalpy in isothermal case) is (Kuang 2007, 2008a, b, 2011a, c)

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0\\
&\delta\Pi_1 = \int_V \delta g \,\mathrm{d}V + \int_V g^{\mathrm{e}} \delta u_{i,i} \,\mathrm{d}V - \delta W\\
&\delta\Pi_2 = \int_{V^{\mathrm{env}}} \delta g^{\mathrm{env}} \,\mathrm{d}V + \int_{V^{\mathrm{env}}} g^{\mathrm{e\ env}} \delta u_{i,i}^{\mathrm{env}} \,\mathrm{d}V - \delta W^{\mathrm{env}}\\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k \,\mathrm{d}V - \int_V \rho_{\mathrm{e}} \delta\varphi \,\mathrm{d}V + \int_{a_\sigma} T_k^* \delta u_k \,\mathrm{d}a - \int_{a_D} \sigma^* \delta\varphi \,\mathrm{d}a\\
&\delta W^{\mathrm{env}} = \int_{V^{\mathrm{env}}} (f_k^{\mathrm{env}} - \rho\ddot{u}_k^{\mathrm{env}})\delta u_k^{\mathrm{env}} \,\mathrm{d}V - \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta\varphi^{\mathrm{env}} \,\mathrm{d}V\\
&\qquad + \int_{a_\sigma^{\mathrm{env}}} T_k^{*\mathrm{env}} \delta u_k^{\mathrm{env}} \,\mathrm{d}a - \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta\varphi^{\mathrm{env}} \,\mathrm{d}a\\
&\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}} \delta u_k \,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta\varphi \,\mathrm{d}a
\end{aligned} \tag{2.7}$$

where $\boldsymbol{f}, \boldsymbol{T}^*, \sigma^*$ are given body force per volume, traction per area, and surface charge density and $\boldsymbol{f}^{\mathrm{env}}, \boldsymbol{T}^{*\mathrm{env}}, \sigma^{*\mathrm{env}}$, and $\boldsymbol{T}^{*\mathrm{int}}, \sigma^{*\mathrm{int}}$ are also given values in the environment and on the interface, respectively. $\boldsymbol{n} = -\boldsymbol{n}^{\mathrm{env}}$ is the outward normal of the interface. It is noted that in Eq. (2.7) the work done by the electric field has the form $q\delta\varphi = (\rho_e dV)\delta\varphi$ with $q = \rho_e dV = const.$, etc. For small deformation, $\delta \int_V g \,\mathrm{d}V = \int_V \delta g \,\mathrm{d}V$ can be used due to small variation of the volume.

The virtual variation of the potential φ is divided into local variation $\delta_\varphi \varphi$ and migratory variation $\delta_u \varphi$, and the similar divisions for $\boldsymbol{E}$, so we have

$$
\begin{aligned}
&\delta\varphi = \delta_\varphi\varphi + \delta_u\varphi, \quad \delta_u\varphi = \varphi_{,p}\delta u_p = -E_p\delta u_p \\
&\partial(\delta\varphi)/\partial x_j = \partial\left(\delta_\varphi\varphi + \varphi_{,p}\delta u_p\right)/\partial x_j = \delta_\varphi\left(\varphi_{,j}\right) + \varphi_{,pj}\delta u_p + \varphi_{,p}\delta u_{p,j} = \delta\left(\varphi_{,j}\right) + \varphi_{,i}\delta u_{i,j} \\
&\delta E_i = -\delta_\varphi\left(\varphi_{,i}\right) - \varphi_{,ip}\delta u_p = \delta_\varphi E_i + \delta_u E_i, \quad \delta_\varphi E_i = -\delta_\varphi\varphi_{,\,i}, \\
&\delta_u E_i = E_{i,p}\delta u_p = E_{p,i}\delta u_p
\end{aligned}
\tag{2.8}
$$

Equation (2.8) shows that $\partial(\delta\varphi)/\partial x_j \neq \delta\left(\partial\varphi/\partial x_j\right)$ when $\delta_u\varphi \neq 0$, and it is discussed also in Eq. (2.130) in Sect. 2.9.1. Using the relation,

$$
\begin{aligned}
&\int_V \delta g \,\mathrm{d}V + \int_V g^{\mathrm{e}} \delta u_{k,k} \,\mathrm{d}V = \int_V \sigma_{ji}\delta u_{i,j} \,\mathrm{d}V - \int_V D_i \delta E_i \,\mathrm{d}V - \int_V (1/2) D_k E_k \delta u_{j,j} \,\mathrm{d}V \\
&\quad = \int_a \sigma_{ji} n_j \delta u_i \,\mathrm{d}a - \int_V \sigma_{ji,j}\delta u_i \,\mathrm{d}V - (1/2)\int_a D_k E_k \delta_{ij} n_j \delta u_i \,\mathrm{d}a \\
&\quad\quad + (1/2)\int_V \left(D_k E_k \delta_{ij}\right)_{,j}\delta u_i \,\mathrm{d}V + \int_a D_i n_i \delta_\varphi\varphi \,\mathrm{d}a \\
&\quad\quad - \int_V D_{i,i}\delta_\varphi\varphi \,\mathrm{d}V - \int_V D_i E_{p,i}\delta u_p \,\mathrm{d}V
\end{aligned}
\tag{2.9}
$$

where $a = a_\sigma + a_u + a^{\mathrm{int}} = a_D + a_\varphi + a^{\mathrm{int}}$, $\sigma_{ji}\delta\varepsilon_{ij} = \sigma_{ji}\delta u_{i,j}$ for asymmetric σ_{ji}. It is noted that $\delta\varphi = 0,\ \delta_\varphi\varphi \neq 0,\ \delta_u\varphi \neq 0$ on a_φ.

Substitution of Eq. (2.9) into $\delta\Pi_1$ in Eq. (2.7) yields

$$
\begin{aligned}
\delta\Pi_1 = {}& \int_a \sigma_{jk} n_j \delta u_k \,\mathrm{d}a - \int_{a_\sigma} T_k^* \delta u_k \,\mathrm{d}a - \int_V \left(\sigma_{jk,j} + f_k - \rho\ddot{u}_k\right)\delta u_k \,\mathrm{d}V \\
& - (1/2)\int_a D_n E_n n_k \delta u_k \,\mathrm{d}a + (1/2)\int_V \left(D_n E_n\right)_{,k}\delta u_k \,\mathrm{d}V - \int_V D_{i,i}\delta_\varphi\varphi \,\mathrm{d}V \\
& + \int_a D_i n_i \delta_\varphi\varphi \,\mathrm{d}a - \int_V \left[\left(D_i E_p\right)_{,i} - D_{i,i}E_p\right]\delta u_p \,\mathrm{d}V + \int_V \rho_{\mathrm{e}}\delta\varphi \,\mathrm{d}V + \int_{a_D} \sigma^*\delta\varphi \,\mathrm{d}a
\end{aligned}
\tag{2.10}
$$

Adding a term $\int_a D_i n_i \left(E_p \delta u_p + \delta_u \varphi\right) \mathrm{d}a = 0$ to Eq. (2.10), we get

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} \left(\sigma_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a - \int_V \left(\sigma_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V - \int_V \left(D_{i,i} - \rho_\mathrm{e}\right) \delta \varphi \, \mathrm{d}V \\
& + \int_{a_D} \left(D_i n_i + \sigma^*\right) \delta \varphi \, \mathrm{d}a + \int_a D_i n_i E_p \delta u_p \, \mathrm{d}a - (1/2) \int_a D_n E_n n_k \delta u_k \, \mathrm{d}a \\
& + (1/2) \int_V \left(D_n E_n\right)_{,k} \delta u_k \, \mathrm{d}V - \int_V \left(D_i E_p\right)_{,i} \delta u_p \, \mathrm{d}V + \int_{a^\mathrm{int}} \sigma_{jk} n_j \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta \varphi \, \mathrm{d}a \\
= & \int_{a_\sigma} \left(S_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a - \int_V \left(S_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V - \int_V \left(D_{i,i} - \rho_\mathrm{e}\right) \delta \varphi \, \mathrm{d}V \\
& + \int_{a_D} \left(D_i n_i + \sigma^*\right) \delta \varphi \, \mathrm{d}a + \int_{a^\mathrm{int}} S_{jk} n_j \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta \varphi \, \mathrm{d}a
\end{aligned} \tag{2.11}$$

In Eq. (2.11), we have

$$\begin{aligned}
& \sigma_{ik}^\mathrm{M} = D_i E_k - (1/2) D_n E_n \delta_{ik}, \\
& S_{kl} = \sigma_{kl} + \sigma_{kl}^\mathrm{M} = C_{ijkl} \varepsilon_{ij} - e_{jkl} E_j - (1/2) l_{ijkl} E_i E_j + \epsilon_{km} E_i E_m + \epsilon_{lm} E_m E_k = S_{kl}
\end{aligned} \tag{2.12}$$

where σ_{ik}^M is the Maxwell stress, $\boldsymbol{S}$ is the pseudo total stress (Jiang and Kuang 2003, 2004) and $\epsilon_{km} E_m = D_k$ has been used. $\boldsymbol{S}$ is a symmetric tensor, but $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\mathrm{M}$ are not. Though the adding term $\int_a D_k n_k E_p \delta u_p \, \mathrm{d}a + \int_a D_k n_k \delta_u \varphi \, \mathrm{d}a$ is zero, the first will be combined with terms of virtual displacements and the second will be combined with terms containing the local variation $\delta_\varphi \varphi$.

Equation (2.10) can also be written as

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} \left(\sigma_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} \sigma_{jk} n_j \delta u_k \, \mathrm{d}a - \int_V \left(\sigma_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V \\
& - (1/2) \int_a D_n E_n n_k \delta u_k \, \mathrm{d}a + (1/2) \int_V \left(D_n E_n\right)_{,k} \delta u_k \, \mathrm{d}V - \int_V \left(D_i E_p\right)_{,i} \delta u_p \, \mathrm{d}V \\
& + \int_{a_D} \left(D_i n_i + \sigma^*\right) \delta \varphi \, \mathrm{d}a + \int_{a_\varphi + a^\mathrm{int}} D_i n_i \delta_\varphi \varphi \, \mathrm{d}a - \int_{a_D} D_i n_i \delta_u \varphi \, \mathrm{d}a - \int_V \left(D_{i,i} - \rho_\mathrm{e}\right) \delta \varphi \, \mathrm{d}V
\end{aligned} \tag{2.13a}$$

Due to the arbitrariness of $\delta \varphi$, it is obtained:

$$D_i n_i + \sigma^* = 0, \quad \text{on} \quad a_D; \quad D_{i,i} - \rho_\mathrm{e} = 0, \quad \text{in} \quad V \tag{2.13b}$$

Substitution of Eq. (2.13b) into Eq. (2.13a) yields

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} \left(\sigma_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} \sigma_{jk} n_j \delta u_k \, \mathrm{d}a - \int_V \left(S_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V \\
& - (1/2) \int_a D_n E_n n_k \delta u_k \, \mathrm{d}a + \int_{a_\varphi + a^\mathrm{int}} D_i n_i \delta_\varphi \varphi \, \mathrm{d}a - \int_{a_D} D_i n_i \delta_u \varphi \, \mathrm{d}a \\
= & \int_{a_\sigma} \left(S_{ij} n_i - T_j^*\right) \delta u_j \, \mathrm{d}a - \int_V \left(S_{ij,i} + f_j - \rho \ddot{u}_j\right) \delta u_j \, \mathrm{d}V + \int_{a^\mathrm{int}} S_{ij} n_i \delta u_j \, \mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta \varphi \, \mathrm{d}a
\end{aligned} \tag{2.13c}$$

In Eq. (2.13c), the following relation was used:

$$\int_{a_\varphi} D_i n_i \delta_\varphi \varphi \,\mathrm{d}a - \int_{a_D} D_i n_i \delta_u \varphi \,\mathrm{d}a = \int_{a_\varphi} D_i n_i \delta\varphi \,\mathrm{d}a - \int_{a} D_i n_i \delta_u \varphi \,\mathrm{d}a = \int_{a} D_i n_i E_p \delta \mathrm{u}_p \,\mathrm{d}a$$

Due to the arbitrariness of $\delta\boldsymbol{u}$ and $\delta\varphi$ from Eq. (2.11) or (2.13), it is obtained:

$$\begin{aligned} & S_{jk,j} + f_k = \rho \ddot{u}_k, \quad D_{i,i} = \rho_\mathrm{e}, \quad \text{in} \quad V \\ & S_{jk} n_j = T_k^*, \quad \text{on} \quad a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D \\ & \delta \Pi_1 = \int_{a^\mathrm{int}} S_{ij} n_i \delta u_j \,\mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta\varphi \,\mathrm{d}a \end{aligned} \tag{2.14a}$$

The momentum equation in Eq. (2.14a) in terms of generalized displacements is

$$\begin{aligned} & C_{ijkl} u_{i,jk} + e_{jkl} \varphi_{,jk} - l_{ijkl} \varphi_{,i} \varphi_{,jk} + \left(\epsilon_{km} \varphi_{,l} \varphi_{,m} + \epsilon_{lm} \varphi_{,m} \varphi_{,k} \right)_{,k} + f_l = \rho \ddot{u}_l, \\ & \left[C_{ijkl} u_{i,j} + e_{jkl} \varphi_{,j} - (1/2) l_{ijkl} \varphi_{,i} \varphi_{,j} + \epsilon_{lm} \varphi_{,k} \varphi_{,m} + \epsilon_{km} \varphi_{,m} \varphi_{,l} \right] n_l = T_k^*, \quad \text{on} \quad a_\sigma; \\ & \epsilon_{kl} \varphi_{,lk} = -\rho_\mathrm{e}; \quad \epsilon_{kl} \varphi_{,l} n_k = -\sigma^*, \quad \text{on} \quad a_D \end{aligned} \tag{2.14b}$$

where the terms containing $\boldsymbol{\varepsilon}$ in φ are neglected. Similarly for the environment, we have

$$\begin{aligned} & S_{ij,i}^\mathrm{env} + f_j^\mathrm{env} = \rho^\mathrm{env} \ddot{u}_j^\mathrm{env}, \qquad D_{i,i}^\mathrm{env} = \rho_\mathrm{e}^\mathrm{env}, \quad \text{in} \quad V^\mathrm{env} \\ & S_{ij}^\mathrm{env} n_i^\mathrm{env} = T_j^{*\mathrm{env}}, \quad \text{on} \quad a_\sigma^\mathrm{env}; \quad D_i^\mathrm{env} n_i^\mathrm{env} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^\mathrm{env} \\ & \delta \Pi_2 = \int_{a^\mathrm{int}} S_{ij}^\mathrm{env} n_i^\mathrm{env} \delta u_j^\mathrm{env} \,\mathrm{d}a + \int_{a^\mathrm{int}} D_i^\mathrm{env} n_i^\mathrm{env} \delta\varphi^\mathrm{env} \,\mathrm{d}a \\ & S_{jk}^\mathrm{env} = \sigma_{jk}^\mathrm{env} + \sigma_{jk}^{\mathrm{M\,env}}; \sigma_{jk}^{\mathrm{M\,env}} = D_j^\mathrm{env} E_k^\mathrm{env} - (1/2) D_n^\mathrm{env} E_n^\mathrm{env} \delta_{jk} \end{aligned} \tag{2.15}$$

Using $n_i = -n_i^\mathrm{env}, u_i = u_i^\mathrm{env}, \varphi = \varphi^\mathrm{env}$ and $\delta\Pi_1 + \delta\Pi_2 = \delta W^{*\mathrm{int}}$, we get

$$(S_{ij} - S_{ij}^\mathrm{env}) n_i = T_j^{*\mathrm{int}}, \quad (D_i - D_i^\mathrm{env}) n_i = -\sigma^{*\mathrm{int}}, \quad \text{on} \quad a^\mathrm{int} \tag{2.16}$$

The above variational principle requests prior that the generalized displacements satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations:

$$\begin{aligned} & u_i = u_i^*, \quad \text{on} \quad a_u; \quad \varphi = \varphi^*, \quad \text{on} \quad a_\varphi \\ & u_i^\mathrm{env} = u_i^{*\mathrm{env}}, \quad \text{on} \quad a_u^\mathrm{env}; \quad \varphi^\mathrm{env} = \varphi^{*\mathrm{env}}; \quad \text{on} \quad a_\varphi^\mathrm{env} \\ & u_i = u_i^\mathrm{env}, \quad \varphi = \varphi^\mathrm{env}; \quad \text{on} \quad a^\mathrm{int} \end{aligned} \tag{2.17}$$

Equations (2.14), (2.15), (2.16), and (2.17) are the governing equations for the electroelastic analysis.

2.1.3 A Note of the Maxwell Stress

In the books of Stratton (1941) and Landau and Lifshitz (1959), the formula of the stress in an isotropic electrostrictive material was

$$\sigma_{ik} = \partial g_0/\partial \varepsilon_{ik} + \sigma^L_{ik}, \quad \sigma^L_{ik} = (1/2)(2\epsilon - a_1)E_k E_i - (1/2)(\epsilon + a_2)E_m E_m \delta_{ik}$$

where $\partial g_0/\partial \varepsilon_{ik}$ is the stress in the media without the electromagnetic field. This formula is just the pseudo total stress $\boldsymbol{S}$ in Eq. (2.12) for the electrostrictive material. For the Maxwell stress and its related problems in literatures, different author had different understanding as shown in Sect. 1.2.7. McMeeking et al. (2005, 2007) considered that in the electroelastic theory the constitutive model can be simplified to one that embraces simultaneously the Cauchy, Maxwell, electrostrictive and electrostatic stresses, which in any case cannot be separately identified from any experiment. In their method authors did not distinguish the local and migratory variations.

From Eq. (2.12), it is known that the Maxwell stress is related to the square of $\boldsymbol{E}$, i.e., $|\boldsymbol{\sigma}| \propto |\boldsymbol{E}|^2$, but the stress introduced by the piezoelectric effect is related to $\boldsymbol{E}$. So for the piezoelectric material when the electric field is not too large and the piezoelectric coefficient is not too small, the Maxwell stress can be neglected. But the isotropic electrostrictive materials do not have the piezoelectric effect, so in this and similar cases, the Maxwell stress should be considered.

Because the strain is accompanied by the change of the distance between the electric particles, the attraction between electric charges or the Maxwell stress and the stress introduced by strains in the material is produced simultaneously. Though they are produced together, their difference is obvious and important. The strength problem in engineering is determined by the Cauchy stress, which is connected with the constitutive equation. However, the Maxwell stress is an external effective electromagnetic force applied to the body and can be obtained by using the migratory variation of φ in the PVP or in the usual energy principle.

Using $\boldsymbol{D} = D_\mathrm{n}\boldsymbol{n} + D_\mathrm{t}\boldsymbol{t}$ and similar expressions for $\boldsymbol{E}$ and the continuity conditions in Eqs. (2.16) and (2.17) for the $\boldsymbol{D}, \boldsymbol{E}$ on the interface, the mechanical continuous condition on the interface can also be rewritten as

$$\begin{aligned}
&\boldsymbol{n}\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{env}}) = \tilde{\boldsymbol{T}}^{*\mathrm{int}}, \quad \tilde{\boldsymbol{T}}^{*\mathrm{int}} = \boldsymbol{T}^{*\mathrm{int}} + \boldsymbol{n}\cdot\left(\boldsymbol{\sigma}^{\mathrm{M\,env}} - \boldsymbol{\sigma}^{\mathrm{M}}\right)\\
&\boldsymbol{n}\cdot\left(\boldsymbol{\sigma}^{\mathrm{M\,env}} - \boldsymbol{\sigma}^{\mathrm{M}}\right) = [\boldsymbol{n}\cdot(\boldsymbol{D}^{\mathrm{env}}\otimes\boldsymbol{E}^{\mathrm{env}}) - (1/2)(\boldsymbol{D}^{\mathrm{env}}\cdot\boldsymbol{E}^{\mathrm{env}})\boldsymbol{n}]\\
&\quad - [\boldsymbol{n}\cdot(\boldsymbol{D}\otimes\boldsymbol{E}) - (1/2)(\boldsymbol{D}\cdot\boldsymbol{E})\boldsymbol{n}] = (1/2)\left[D_\mathrm{n}\left(E^{\mathrm{env}}_\mathrm{n} - E_\mathrm{n}\right) - \left(D^{\mathrm{env}}_\mathrm{t} - D_\mathrm{t}\right)E_\mathrm{t}\right]\boldsymbol{n}\\
&\quad = [(\epsilon - \epsilon^{env})/2\epsilon\epsilon^{env}]\left(D^2_n + \epsilon\epsilon^{env}E^2_t\right)\boldsymbol{n}
\end{aligned} \tag{2.18}$$

where $\boldsymbol{n}$ is the unit normal, subscripts n and t mean the normal and tangential direction respectively; and there is no sum on n and t. Equation (2.18) shows that in the small strain case, the boundary traction produced by the Maxwell stress is along the normal direction. The Maxwell stress can be naturally obtained by the migratory variation of φ in PVP.

2.2 Alternative Forms of the Physical Variational Principles

2.2.1 *First Alternative Form of the PVP*

From Eqs. (2.14), (2.15), (2.16), and (2.17), it is found that if we use $\boldsymbol{S}$ instead of $\boldsymbol{\sigma}$ and $\boldsymbol{S}^{\text{env}}$ instead of $\boldsymbol{\sigma}^{\text{env}}$ in the governing equations, then the form of governing equations of the physical nonlinear dielectrics is just the same as that in the physical linear electric problem. Therefore, a simpler principle can be obtained: the first alternative form of the variational principle is

$$\begin{aligned}
&\delta\hat{\Pi} = \delta\hat{\Pi}_1 + \delta\hat{\Pi}_2 - \delta W^{\text{int}} = 0 \\
&\delta\hat{\Pi}_1 = \int_V \delta\hat{g}\,\mathrm{d}V - \delta W, \quad \delta\hat{\Pi}_2 = \int_{V^{\text{env}}} \delta\hat{g}^{\text{env}}\,\mathrm{d}V - \delta W^{\text{env}} \\
&\delta\hat{g} = S_{ji}\delta u_{i,j} + D_i\delta\varphi_{,i}, \quad \delta\hat{g}^{\text{env}} = S_{ji}^{\text{env}}\delta u_{i,j}^{\text{env}} + D_i^{\text{env}}\delta\varphi_{,i}^{\text{env}} \\
&S_{kl} = \sigma_{kl} + \sigma_{kl}^{\text{M}}
\end{aligned} \tag{2.19}$$

In Eq. (2.19), the variations of δu and $\delta\varphi$ are all local variations or completely independent, i.e., the migratory variations $\delta_u\varphi$ produced by $\delta\mathbf{u}$ are not needed. $\delta W, \delta W^{\text{env}}$, and δW^{int} are still expressed by Eq. (2.7). An analogous theory was also discussed by Bustamante et al. (2008).

2.2.2 *Second Alternative Form of the Physical Variational Principle*

Introduce the electric body force $f_k^{\text{e}}, f_k^{\text{e env}}$ and traction $T_k^{\text{e}}, T_k^{\text{e env}}$ in media as

$$\begin{gathered}
f_k^{\text{e}} = \sigma_{jk,j}^{\text{M}}, \quad T_k^{\text{e}} = -\sigma_{jk}^{\text{M}} n_j; \quad f_k^{\text{e env}} = \sigma_{jk,j}^{\text{M env}}, \\
T_k^{\text{e env}} = -\sigma_{jk}^{\text{M env}} n_j^{\text{env}} = \sigma_{jk}^{\text{M env}} n_j
\end{gathered} \tag{2.20}$$

The second alternative form of the variational principle is

$$\begin{aligned}
&\delta\Pi' = \delta\Pi_1' + \delta\Pi_2' - \delta W'^{\text{int}} = 0 \\
&\delta\Pi_1' = \int_V \delta g\,\mathrm{d}V - \delta W', \quad \delta\Pi_2' = \int_{V^{\text{env}}} \delta g^{\text{env}}\,\mathrm{d}V - \delta W'^{\text{env}} \\
&\delta W' = \int_V (f_k + f_k^{\text{e}} - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\text{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} (T_k^* + T_k^{\text{e}})\delta u_k\,\mathrm{d}a - \int_{a_D} \sigma^*\delta\varphi\,\mathrm{d}a \\
&\delta W'^{\text{env}} = \int_{V^{\text{env}}} (f_k^{\text{env}} + f_k^{\text{e env}} - \rho^{\text{env}}\ddot{u}_k^{\text{env}})\delta u_k^{\text{env}}\,\mathrm{d}V - \int_{V^{\text{env}}} \rho_{\text{e}}^{\text{env}}\delta\varphi^{\text{env}}\,\mathrm{d}V \\
&\qquad + \int_{a_\sigma^{\text{env}}} \left(T_k^{*\,\text{env}} + T_k^{\text{e env}}\right)\delta u_k^{\text{env}}\,\mathrm{d}a - \int_{a_D^{\text{env}}} \sigma^{*\,\text{env}}\delta\varphi^{\text{env}}\mathrm{d}a \\
&\delta W'^{\text{int}} = \int_{a^{\text{int}}} \left(T_k^{*\,\text{int}} + T_k^{\text{e env}} + T_k^{\text{e}}\right)\delta u_k\,\mathrm{d}a - \int_{a^{\text{int}}} \sigma^{*\,\text{int}}\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{2.21}$$

In Eq. (2.21), the variations of δu and $\delta\varphi$ are also completely independent, i.e., it is also not needed to consider the migratory variation $\delta_u\varphi$. Equation (2.21) is the original form of the PVP Eq. (1.78). The governing equations from Eq. (2.21) are

$$\begin{aligned}
&\sigma_{ij,i} + (f_j + f_j^{\mathrm{e}}) = \rho\ddot{u}_j; \quad D_{i,i} = \rho_{\mathrm{e}}; \quad \text{in} \quad V \\
&\sigma_{ij} n_i = T_j^* + T_j^{\mathrm{e}}, \quad \text{on} \quad a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \\
&\sigma_{ji,j}^{\mathrm{env}} + \left(f_i^{\mathrm{env}} + f_i^{\mathrm{e\,env}}\right) = \rho\ddot{u}_i^{\mathrm{env}}; \quad D_{i,i}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}; \quad \text{in} \quad V^{\mathrm{env}} \\
&\sigma_{ji}^{\mathrm{env}} n_j^{\mathrm{env}} = T_i^{*\mathrm{env}} + T_i^{\mathrm{e\,env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}}; \\
&\left(\sigma_{lk} - \sigma_{lk}^{\mathrm{env}}\right) n_l = T_k^{*\mathrm{int}} + T_k^{\mathrm{e}} + T_k^{\mathrm{e\,env}}, \quad D_k n_k - D_k^{\mathrm{env}} n_k = -\sigma^{*\mathrm{int}}; \quad \text{on} \quad a^{\mathrm{int}}
\end{aligned} \tag{2.22}$$

In many literatures (Pao 1978; Maugin 1988; Moon 1984), the governing equations were expressed in the form of Eq. (2.22) and the electromagnetic force was derived from other methods different with the variational method. In different literatures, $\boldsymbol{f}^{\mathrm{e}}$ and $\boldsymbol{T}^{\mathrm{e}}$ may be different.

2.2.3 *The Medium Fully Surrounded by the Air*

An important engineering problem is that the medium with symmetric material coefficients is fully surrounded by the air. In air, the mechanical stresses and mechanical energy can be neglected and only the electric field and electric energy should be considered. The physical variational formula (2.7) in this case becomes

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0 \\
&\delta\Pi_1 = \delta\int_V g\,\mathrm{d}V - \delta W, \quad \delta\Pi_2 = \delta\int_{V^{\mathrm{env}}} g^{\mathrm{env}}\,\mathrm{d}V - \delta W^{\mathrm{env}} \\
&\delta W^{\mathrm{int}} = \int_{a_\sigma^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a_q^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a, \quad \delta W = -\int_V \rho\ddot{u}_k\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V \\
&\delta W^{\mathrm{env}} = -\int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\mathrm{d}V + \int_{a_q^{\mathrm{env}}} D_i^{*\mathrm{env}} n_i^{env}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a
\end{aligned} \tag{2.23}$$

where the body force is neglected and

$$\begin{aligned}
&g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij} - (1/2)l_{ijkl}E_iE_j\varepsilon_{kl} \\
&g^{\mathrm{env}} = -(1/2)\epsilon_{kl}^{\mathrm{env}}E_k^{\mathrm{env}}E_l^{\mathrm{env}}
\end{aligned} \tag{2.24}$$

2.2.4 Isotropic Materials

For isotropic materials, the constitutive equations are (Kuang 2012)

$$l_{ijkl} = l_1 \delta_{ij}\delta_{kl} + l_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \quad \epsilon_{ij} = \epsilon\delta_{ij}, \quad \mu_{ij} = \mu\delta_{ij}, \quad \alpha_{ij} = -2\epsilon\delta_{ij}, \; e_{kij} = 0 \tag{2.25}$$

In isotropic materials, variables $\boldsymbol{S}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}^{\mathrm{M}}$ are all symmetric. So Eqs. (2.2) and (2.3) are reduced to

$$g = (1/2)\lambda\varepsilon_{ii}\varepsilon_{kk} + G\varepsilon_{ij}\varepsilon_{ij} - (1/2)\epsilon E_k E_k - (1/2)(l_1 - \epsilon)E_i E_i \varepsilon_{kk} - (l_2 - \epsilon)E_i E_j \varepsilon_{ij} \tag{2.26}$$

$$\begin{aligned}
\sigma_{kl} &= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(l_1 - \epsilon)E_i E_i \delta_{kl} - (l_2 - \epsilon)E_k E_l, \\
D_k &= \epsilon E_k + (l_1 - \epsilon)\varepsilon_{ii}E_k + 2(l_2 - \epsilon)\varepsilon_{kl}E_l \approx \epsilon E_k \\
S_{kl} &= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)l_1 E_i E_i \delta_{kl} - (l_2 - 2\epsilon)E_k E_l = \sigma_{kl} + \sigma_{kl}^{\mathrm{M}}
\end{aligned} \tag{2.27a}$$

If we let $l_1 - \epsilon = a_2, l_2 - \epsilon = (1/2)a_1$, then Eq. (2.27a) is reduced to

$$\begin{aligned}
\sigma_{kl} &= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(a_2 E_i E_i \delta_{kl} + a_1 E_k E_l) \\
D_k &= \tilde{\epsilon}_{kl}E_l, \quad \tilde{\epsilon}_{kl} = \epsilon\delta_{kl} + a_1\varepsilon_{kl} + a_2\varepsilon_{ii}\delta_{kl} \approx \epsilon\delta_{kl} \\
S_{kl} &= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(a_2 + \epsilon)E_i E_i \delta_{kl} + (1/2)(2\epsilon - a_1)E_k E_l = \sigma_{kl} + \sigma_{kl}^{\mathrm{M}}
\end{aligned} \tag{2.27b}$$

The first formula in Eq. (2.27b) is just the usual form of the constitutive equation, where a_1 and a_2 are known as electrostrictive coefficients. From Eqs. (2.14), (2.15), (2.16), and (2.17), it is known that solving $\boldsymbol{S}$ is easier than that for $\boldsymbol{\sigma}$, so in experiments, the measured variables usually are $(\boldsymbol{S}, \boldsymbol{\varepsilon}, \boldsymbol{E})$. If the constitutive equation (2.27a) is used, the measured material coefficients are l_1 and $l_2 - 2\epsilon$. If the constitutive equation (2.27b) is used, the measured material coefficients are $2\epsilon - a_1$ and $a_2 + \epsilon$. Therefore, in experiments, the entire system including the dielectric medium, its environment, and their common boundary should be considered together, and appropriate governing equations should be selected.

2.2.5 The Static Electric Force Acting on the Medium by the Electric Field

Comparing Eqs. (2.7) and (2.21), it is found that the difference between them is that in Eq. (2.7), the local variation and the migratory variation are used simultaneously, however in Eq. (2.21), only the local variation is used, but the electric force introduced by electric field is introduced:

$$
\begin{aligned}
\delta W^{\mathrm{e}} = {} & \int_V f_k^{\mathrm{e}} \delta u_k \mathrm{d}V + \int_{a_\sigma} T_k^{\mathrm{e}} \delta u_k \mathrm{d}a + \int_{V^{\mathrm{env}}} f_k^{\mathrm{e\ env}} \delta u_k^{\mathrm{env}} \,\mathrm{d}V + \int_{a_\sigma^{\mathrm{env}}} T_k^{\mathrm{e\ env}} \delta u_k^{\mathrm{env}} \,\mathrm{d}a \\
& + \int_{a^{\mathrm{int}}} \left(T_k^{\mathrm{e}} + T_k^{\mathrm{e\ env}} \right) \delta u_k \,\mathrm{d}a
\end{aligned}
\tag{2.28}
$$

In Eqs. (2.7) and (2.10), the part related to the migratory variations of potential is

$$
\begin{aligned}
\delta_u \Pi = {} & \int_V g_{,E} \cdot \delta_u \boldsymbol{E} \,\mathrm{d}V + \int_V g^{\mathrm{e}} \delta u_{k,k} \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta_u \varphi \,\mathrm{d}V + \int_{a_D} \sigma^* \delta_u \varphi \,\mathrm{d}a + \int_{V^{\mathrm{env}}} g_{,E}^{\mathrm{env}} \cdot \delta_u E^{\mathrm{env}} \,\mathrm{d}V \\
& + \int_{V^{\mathrm{env}}} g^{\mathrm{e\ env}} \delta u_{i,i}^{\mathrm{env}} \,\mathrm{d}V + \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}V + \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta_u \varphi \,\mathrm{d}a \\
= {} & - \int_V D_i E_{p,i} \delta u_p \,\mathrm{d}V - \int_V (1/2) D_k E_k \delta u_{j,j} \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta_u \varphi \,\mathrm{d}V + \int_{a_D} \sigma^* \delta_u \varphi \,\mathrm{d}a \\
& - \int_{V^{\mathrm{env}}} D_i^{\mathrm{env}} \delta_u E_i^{\mathrm{env}} \,\mathrm{d}V - \int_{V^{\mathrm{env}}} (1/2) D_k^{\mathrm{env}} E_k^{\mathrm{env}} \delta u_{j,j}^{\mathrm{env}} \,\mathrm{d}V + \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}V \\
& + \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta_u \varphi \,\mathrm{d}a
\end{aligned}
\tag{2.29a}
$$

Using Eqs. (2.13b) and (2.16) and adding terms $\int_a D_i n_i \left(E_p \delta u_p + \delta_u \varphi \right) \mathrm{d}a = 0$ and $\int_{a^{\mathrm{env}}} D_i^{\mathrm{env}} n_i^{\mathrm{env}} \left(E_p^{\mathrm{env}} \delta u_p^{\mathrm{env}} + \delta_u \varphi^{\mathrm{env}} \right) \mathrm{d}a = 0$ to Eq. (2.29a), then Eq. (2.29a) can be reduced to

$$
\begin{aligned}
\delta_u \Pi = {} & \int_{a_\sigma} \sigma_{ij}^{\mathrm{M}} n_i \delta u_j \,\mathrm{d}a - \int_V \sigma_{ij,i}^{\mathrm{M}} \delta u_j \,\mathrm{d}V + \int_{\mathrm{a}_{\mathrm{\sigma}}^{\mathrm{env}}} \sigma_{ij}^{\mathrm{M\ env}} n_i^{\mathrm{env}} \delta u_j^{\mathrm{env}} \,\mathrm{d}a \\
& - \int_{V^{env}} \sigma_{ij,i}^{\mathrm{M\ env}} \delta u_j^{\mathrm{env}} \,\mathrm{d}V + \int_{a^{\mathrm{int}}} \sigma_{ij}^{\mathrm{M}} n_i \delta u_j \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \sigma_{ij}^{\mathrm{M\ env}} n_i^{\mathrm{env}} \delta u_j^{\mathrm{env}} \mathrm{d}a
\end{aligned}
\tag{2.29b}
$$

Comparing Eqs. (2.20), (2.28), and (2.29), it is found that the static electric force acting on the medium can be obtained from the general energy migratory variational principle (Kuang 2012):

$$
\delta W^{\mathrm{e}} = -\delta_u \Pi \tag{2.30a}
$$

If the environment is neglected, it is obtained:

$$
\begin{aligned}
& \int_V f_k^{\mathrm{e}} \delta u_k \,\mathrm{d}V + \int_{a_\sigma} T_k^{\mathrm{e}} \delta u_k \,\mathrm{d}a \\
& \quad = - \left(\int_V g_{,E} \cdot \delta_u \boldsymbol{E} \,\mathrm{d}V + \int_V g^{\mathrm{e}} \delta u_{k,k} \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta_u \varphi \,\mathrm{d}V + \int_{a_D} \sigma^* \delta_u \varphi \,\mathrm{d}a \right)
\end{aligned}
\tag{2.30b}
$$

2.2.6 Hamilton Principle

In order to use the PVP for moving electroelastic materials, the D'Alembert principle should be used to make the moving media in a state of relative rest. Using D'Alembert principle, the Hamilton principle can easily be obtained from the PVP. Let δu_{k0} and $\delta u_{k\mathrm{f}}$ be displacements at the initial and final times, respectively, in time interval $[t_0, t_\mathrm{f}]$ and assume $\delta u_{k0} = \delta u_{k\mathrm{f}} = 0$, using

$$\begin{aligned}
\delta \int_{t_0}^{t_\mathrm{f}} \int_V K\,\mathrm{d}V\,\mathrm{d}t &= \delta \int_{t_0}^{t_\mathrm{f}} \int_V (1/2)\rho \dot{u}_k \dot{u}_k\,\mathrm{d}V\,\mathrm{d}t = \int_{t_0}^{t_\mathrm{f}} \int_V \rho \dot{u}_k \delta \dot{u}_k\,\mathrm{d}V\,\mathrm{d}t \\
&= \int_V \int_{t_0}^{t_\mathrm{f}} [\rho \mathrm{d}(\dot{u}_k \delta u_k)/\mathrm{d}t - \rho \ddot{u}_k \delta u_k]\mathrm{d}t\,\mathrm{d}V = -\int_{t_0}^{t_\mathrm{f}} \int_V \rho \ddot{u}_k \delta u_k \mathrm{d}V\,\mathrm{d}t
\end{aligned} \tag{2.31}$$

where $K = \rho \dot{u}_k \dot{u}_k/2$ and $K^\mathrm{env} = \rho^\mathrm{env} \dot{u}_k^\mathrm{env} \dot{u}_k^\mathrm{env}/2$ are the kinetic energies in the material and its environment, respectively. Substituting Eq. (2.31) into (2.7) and integrating it from t_0 to t_f then we get the Hamilton principle:

$$\begin{aligned}
&\delta \Pi_H = \delta \Pi_{H1} + \delta \Pi_{H2} - \int_{t_0}^{t_\mathrm{f}} \delta W^\mathrm{int}\,\mathrm{d}t = 0 \\
&\delta \Pi_{H1} = \int_{t_0}^{t_\mathrm{f}} \int_V \delta(g - K)\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_\mathrm{f}} \int_V g^\mathrm{e} \mathrm{d}V\,\mathrm{d}t - \int_{t_0}^{t_\mathrm{f}} \delta W\,\mathrm{d}t \\
&\delta \Pi_{H2} = \int_{t_0}^{t_\mathrm{f}} \int_{V^\mathrm{env}} \delta(g^\mathrm{env} - K^\mathrm{env})\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_\mathrm{f}} \int_V g^\mathrm{e\ env} \mathrm{d}V\,\mathrm{d}t - \int_{t_0}^{t_\mathrm{f}} \delta W^\mathrm{env} \mathrm{d}t \\
&\delta W = \int_V f_k \delta u_k\,\mathrm{d}V - \int_V \rho_\mathrm{e} \delta \varphi\,\mathrm{d}V + \int_{a_\sigma} T_k^* \delta u_k\,\mathrm{d}a - \int_{a_D} \sigma^* \delta \varphi\,\mathrm{d}a \\
&\delta W^\mathrm{env} = \int_{V^\mathrm{env}} f_k^\mathrm{env} \delta u_k^\mathrm{env}\,\mathrm{d}V - \int_{V^\mathrm{env}} \rho_\mathrm{e}^\mathrm{env} \delta \varphi^\mathrm{env}\,\mathrm{d}V + \int_{a_\sigma^\mathrm{env}} T_k^{*\mathrm{env}} \delta u_k^\mathrm{env}\,\mathrm{d}a - \int_{a_D^\mathrm{env}} \sigma^{*\mathrm{env}} \delta \varphi^\mathrm{env}\,\mathrm{d}a \\
&\delta W^\mathrm{int} = \int_{a^\mathrm{int}} T_k^{*\mathrm{int}} \delta u_k\,\mathrm{d}a - \int_{a^\mathrm{int}} \sigma^{*\mathrm{int}} \delta \varphi\,\mathrm{d}a
\end{aligned} \tag{2.32}$$

However the energy conservation law is:

$$g + K + g^{env} + K^{env} = \int_{t_0}^{t_\mathrm{f}} (\mathrm{d}W + dW^{env} + dW^\mathrm{int})\mathrm{d}t \tag{2.33}$$

Equation (2.33) is held for any time interval. It is noted that the energy principle is held in a real process, but the PVP gives a true process for all virtual possible process satisfying the natural constrained conditions, and it is equivalent to the momentum equation. It is also noted that the Hamilton principle is held in four-dimensional space $(\boldsymbol{x}, t)$ and the time boundary conditions should be added. But the PVP is held in three-dimensional (3D) space $(\boldsymbol{x})$ and does not consider the time variation.

It is obvious that Hamilton principle is also a fundamental principle in the physics and continuum mechanics. Using the local and migratory variation theory, the Maxwell stress can also be obtained automatically.

2.2.7 *Physical Variational Principle in Electromagnetic Materials*

In this section, the PVP is extended to electromagnetic materials under static electromagnetic field, without the current and the body electromagnetic couple (Kuang 2011a, b, c).

Let constitutive equations be

$$\begin{aligned}
\sigma_{lk} &= C_{ijkl}\varepsilon_{ij} - e^{\mathrm{e}}_{jkl}E_j - e^{\mathrm{m}}_{jkl}H_j - (1/2)l^{\mathrm{e}}_{ijkl}E_iE_j - (1/2)l^{\mathrm{m}}_{ijkl}H_iH_j \\
&\quad - \epsilon_{km}E_mE_l - \mu_{km}H_mH_l - \beta_{km}H_mE_l - \beta_{km}E_mH_l \\
D_k &= \left[\epsilon_{kl} + l^{\mathrm{e}}_{ijkl}\varepsilon_{ij} + \left(\epsilon_{ml}\varepsilon_{mk} + \epsilon_{mk}\varepsilon_{ml}\right)\right]E_l + \underset{kij}{e^{\mathrm{e}}}\varepsilon_{ij} + \beta_{kl}H_l + (\beta_{lm}H_m + \beta_{km}H_l)\varepsilon_{kl} \\
B_k &= \left[\mu_{kl} + l^{\mathrm{m}}_{ijkl}\varepsilon_{ij} + 2\left(\alpha^{\mathrm{m}}_{ml}\varepsilon_{mk} + \alpha^{\mathrm{m}}_{mk}\varepsilon_{ml}\right)\right]H_l + \underset{kij}{e^{\mathrm{m}}}\varepsilon_{ij} + \beta_{kl}E_l + (\beta_{lm}E_m + \beta_{km}E_l)\varepsilon_{kl}
\end{aligned} \tag{2.34}$$

where e^{e}_{jkl} and e^{m}_{jkl} are piezoelectric and piezomagnetic coefficients, respectively, l^{e}_{ijkl} and l^{m}_{ijkl} are electrostrictive and magnetostrictive coefficients, respectively, and $\beta_{ij} = \beta_{ji}$ is the magnetoelectric coupling coefficient. The electromagnetic body couple is still balanced by the asymmetric stress. In this case, the electromagnetic Gibbs free energy g is

$$\begin{aligned}
g &= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \left(e^{\mathrm{e}}_{kij}E_k + e^{\mathrm{m}}_{kij}H_k\right)\varepsilon_{ij} - (1/2)\left(\epsilon_{ij}E_iE_j + \mu_{ij}H_iH_j\right) - \beta_{kl}E_kH_l \\
&\quad - (1/2)\left(l^{\mathrm{e}}_{ijkl}E_iE_j + l^{\mathrm{m}}_{ijkl}H_iH_j\right)\varepsilon_{kl} - (\epsilon_{km}E_mE_l + \mu_{km}H_mH_l)\varepsilon_{kl} - \beta_{km}(H_mE_l + E_mH_l)\varepsilon_{kl} \\
&= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mathrm{em}} \\
g^{\mathrm{em}} &= -(1/2)(D_kE_k + B_kH_k + \Delta_{kl}\varepsilon_{lk}), \quad \Delta_{kl} = e^{\mathrm{e}}_{mkl}E_m + e^{\mathrm{m}}_{mkl}H_m
\end{aligned} \tag{2.35}$$

For the small deformation $\boldsymbol{\Delta} : \boldsymbol{\varepsilon}$ can still be neglected. The PVP is

$$\begin{aligned}
\delta\Pi &= \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0 \\
\delta\Pi_1 &= \int_V \delta g \,\mathrm{d}V + \int_V g^{\mathrm{em}}\delta u_{i,i}\,\mathrm{d}V - \delta W \\
\delta\Pi_2 &= \int_{V^{\mathrm{env}}} \delta g^{\mathrm{env}}\,\mathrm{d}V + \int_{V^{\mathrm{env}}} g^{\mathrm{em\ env}}\delta u^{\mathrm{env}}_{i,i}\,\mathrm{d}V - \delta W^{\mathrm{env}} \\
\delta W &= \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} T^*_k\delta u_k\,\mathrm{d}a - \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a + \int_{a_\mu} B^*_i n_i\delta\psi\,\mathrm{d}a \\
\delta W^{\mathrm{env}} &= \int_{V^{\mathrm{env}}} (f^{\mathrm{env}}_k - \rho\ddot{u}^{\mathrm{env}}_k)\delta u^{\mathrm{env}}_k\,\mathrm{d}V - \int_{V^{\mathrm{env}}} \rho^{\mathrm{env}}_{\mathrm{e}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}V \\
&\quad + \int_{a^{\mathrm{env}}_\sigma} T^{*\mathrm{env}}_k\delta u^{\mathrm{env}}_k\,\mathrm{d}a - \int_{a^{\mathrm{env}}_D}\sigma^{*\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a + \int_{a_\mu} B^{*\mathrm{env}}_i n^{\mathrm{env}}_i\delta\psi^{\mathrm{env}}\,\mathrm{d}a \\
\delta W^{\mathrm{int}} &= \int_{a^{\mathrm{int}}} T^{*\mathrm{int}}_k\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}}\sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a + \int_{a^{\mathrm{int}}_\mu} B^{*\mathrm{int}}_i n_i\delta\psi^{\mathrm{env}}\,\mathrm{d}a
\end{aligned} \tag{2.36}$$

where $E_i = -\varphi_i, H_i = -\psi_i$. Finishing the variational calculation finally yields

$$\begin{aligned}
&S_{jk,j} + f_k = \rho \ddot{u}_k, \quad D_{i,i} = \rho_{\mathrm{e}}, \quad B_{i,i} = 0, \quad \text{in} \quad V \\
&S_{jk} n_j = T_k^*, \quad \text{on} \quad a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \quad \left(B_i - B_i^*\right) n_i = 0, \quad \text{on} \quad a_\mu \\
&S_{ij,i}^{\mathrm{env}} + f_j^{\mathrm{env}} = \rho^{\mathrm{env}} \ddot{u}_j^{\mathrm{env}}, \quad D_{i,i}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}, \quad B_{i,i} = 0, \quad \text{in} \quad V^{\mathrm{env}} \\
&S_{ij}^{\mathrm{env}} n_i^{\mathrm{env}} = T_j^{*\mathrm{env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}}; \quad B_i^{\mathrm{env}} n_i^{\mathrm{env}} = B_i{}^{*\mathrm{env}} n_i^{\mathrm{env}}, \\
&\quad \text{on} \quad a_\mu^{\mathrm{env}} \\
&(S_{ij} - S_{ij}^{\mathrm{env}}) n_i = T_j^{*\mathrm{int}}, \quad (D_i - D_i^{\mathrm{env}}) n_i = -\sigma^{*\mathrm{int}}, \quad (B_i - B_i^{\mathrm{env}}) n_i = B_i{}^{*\mathrm{int}} n_i, \quad \text{on} \quad a^{\mathrm{int}} \\
&S_{ik} = \sigma_{ik} + \sigma_{ik}^{\mathrm{M}}; \quad \sigma_{ik}^{\mathrm{M}} = D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n)\delta_{ik} \\
&S_{ik}^{\mathrm{env}} = \sigma_{ik}^{\mathrm{env}} + \sigma_{ik}^{\mathrm{M\,env}}; \quad \sigma_{ik}^{\mathrm{M\,env}} = D_i^{\mathrm{env}} E_k^{\mathrm{env}} + B_i^{\mathrm{env}} H_k^{\mathrm{env}} - (1/2)\left(D_n^{\mathrm{env}} E_n^{\mathrm{env}} + B_n^{\mathrm{env}} H_n^{\mathrm{env}}\right)\delta_{ik}
\end{aligned} \tag{2.37}$$

2.3 General Variational Principle

2.3.1 General Variational Principle Not Satisfying Boundary Conditions

This principle does not ask $\boldsymbol{u}, \varphi$ and $\boldsymbol{u}^{\mathrm{env}}, \varphi^{\mathrm{env}}$ to satisfy boundary conditions on their own boundaries a_u, a_φ and $a_u^{\mathrm{env}}, a_\varphi^{\mathrm{env}}$, respectively, and continuity conditions on the interface prior. For small deformation, this principle is

$$\begin{aligned}
\delta\Pi &= \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0 \\
\delta\Pi_1 &= \int_V \delta g \,\mathrm{d}V + \int_V g \delta u_{k,k} \,\mathrm{d}V - \int_V (f_k - \rho \ddot{u}_k) \delta u_k \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta\varphi \,\mathrm{d}V - \int_{a_\sigma} T_k^* \delta u_k \,\mathrm{d}a \\
&\quad + \int_{a_D} \sigma^* \delta\varphi \,\mathrm{d}a - \delta \int_{a_u} T_k^{\mathrm{S}} (u_k - u_k^*) \,\mathrm{d}a + \delta \int_{a_\varphi} \sigma(\varphi - \varphi^*) \,\mathrm{d}a \\
\delta\Pi_2 &= \int_{V^{\mathrm{env}}} \delta g^{\mathrm{env}} \,\mathrm{d}V + \int_{V^{\mathrm{env}}} g^{\mathrm{env}} \delta u_{k,k}^{\mathrm{env}} \,\mathrm{d}V - \int_{V^{\mathrm{env}}} (f_k^{\mathrm{env}} - \rho^{\mathrm{env}} \ddot{u}_k^{\mathrm{env}}) \delta u_k^{\mathrm{env}} \,\mathrm{d}V \\
&\quad + \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta\varphi^{\mathrm{env}} \,\mathrm{d}V - \int_{a_\sigma^{\mathrm{env}}} T_k^{*\mathrm{env}} \delta u_k^{\mathrm{env}} \,\mathrm{d}a + \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta\varphi^{\mathrm{env}} \,\mathrm{d}a \\
&\quad - \delta \int_{a_u^{\mathrm{env}}} T_k^{\mathrm{S\,env}} (u_k^{\mathrm{env}} - u_k^{*\mathrm{env}}) \,\mathrm{d}a + \delta \int_{a_\varphi^{\mathrm{env}}} \sigma^{\mathrm{env}} (\varphi^{\mathrm{env}} - \varphi^{*\mathrm{env}}) \,\mathrm{d}a \\
\delta W^{\mathrm{int}} &= \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}} \delta u_k \,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta\varphi \,\mathrm{d}a \\
&\quad + \delta \int_{a^{\mathrm{int}}} T_k^{\mathrm{S}} (u_k - u_k^{\mathrm{env}}) \,\mathrm{d}a - \delta \int_{a^{\mathrm{int}}} \sigma(\varphi - \varphi^{\mathrm{env}}) \,\mathrm{d}a
\end{aligned} \tag{2.38}$$

In Eq. (2.38), some additional virtual work done on the boundary and interface is added, because the displacements and potentials do not satisfy the boundary conditions and the continuity conditions on the interface. As an example, $\delta \int_{a^{\mathrm{int}}} T_k^{\mathrm{S\ int}}(u_k - u_k^{\mathrm{env}})\,\mathrm{d}a$ is the virtual work introduced by the difference of the virtual displacement $(u_k - u_k^{\mathrm{env}})$ on two sides and the unknown pseudo total traction $\boldsymbol{T}^{\mathrm{S}}$ on the interface. In Eq. (2.38), $\boldsymbol{T}^{\mathrm{S}}$ and σ may also be considered as Lagrange multipliers in the mathematical sense (Kuang 1964, 2002).

Equation (2.38) can be proved as follows. Analogous to the derivation of Eq. (2.11), it is obtained:

$$\begin{aligned}
\delta\Pi_1 = {} & \int_{a_\sigma} \left(S_{jk}n_j - T_k^*\right)\delta u_k\,\mathrm{d}a - \int_V \left(S_{jk,j} + f_k - \rho\ddot{u}_k\right)\delta u_k\,\mathrm{d}V - \int_V \left(D_{i,i} - \rho_{\mathrm{e}}\right)\delta\varphi\,\mathrm{d}V \\
& + \int_{a_D} (D_i n_i + \sigma^*)\delta\varphi\,\mathrm{d}a + \int_{a^{\mathrm{int}}} S_{jk}n_j\delta u_k\,\mathrm{d}a + \int_{a^{\mathrm{int}}} D_i n_i \delta\varphi\,\mathrm{d}a - \int_{a_u} (u_k - u_k^*)\delta T_k^{\mathrm{S}}\,\mathrm{d}a \\
& + \int_{a_u} \left(S_{jk}n_j - T_k^{\mathrm{S}}\right)\delta u_k\,\mathrm{d}a + \int_{a_\varphi} (\varphi - \varphi^*)\delta\sigma\,\mathrm{d}a + \int_{a_\varphi} (D_i n_i + \sigma)\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{2.39}$$

Due to the arbitrariness of $\delta\boldsymbol{u}$ and $\delta\varphi$ from Eq. (2.39), we get

$$\begin{aligned}
& S_{jk,j} + f_k = \rho\ddot{u}_k, \quad D_{i,i} = \rho_{\mathrm{e}}, \quad \text{in} \quad V \\
& S_{jk}n_j = T_k^*, \quad \text{on} \quad a_\sigma; \quad u_k = u_k^*, \quad S_{jk}n_j = T_k^{\mathrm{S}}, \quad \text{on} \quad a_u \\
& D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \quad \varphi = \varphi^*, \quad D_i n_i = -\sigma, \quad \text{on} \quad a_\varphi \\
& \delta\Pi_1 = \int_{a^{\mathrm{int}}} S_{ij}n_i\delta u_j\,\mathrm{d}a + \int_{a^{\mathrm{int}}} D_i n_i\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{2.40}$$

Similarly for the environment we get

$$\begin{aligned}
& S_{ij,i}^{\mathrm{env}} + f_j^{\mathrm{env}} = \rho^{\mathrm{env}}\ddot{u}_j^{\mathrm{env}}, \quad D_{i,i}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}, \quad \text{in} \quad V^{\mathrm{env}} \\
& S_{ij}^{\mathrm{env}} n_i^{\mathrm{env}} = T_j^{*\,\mathrm{env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad u_k^{\mathrm{env}} = u_k^{*\,\mathrm{env}}, \quad S_{jk}^{\mathrm{env}} n_j^{\mathrm{env}} = T_k^{\mathrm{S\ env}}, \quad \text{on} \quad a_u^{\mathrm{env}} \\
& D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}} \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{\mathrm{env}}, \quad \text{on} \quad a_\varphi \\
& \delta\Pi_2 = \int_{a^{\mathrm{int}}} S_{ij}^{\mathrm{env}} n_i^{\mathrm{env}} \delta u_j^{\mathrm{env}}\,\mathrm{d}a + \int_{a^{\mathrm{int}}} D_i^{\mathrm{env}} n_i^{\mathrm{env}} \delta\varphi^{\mathrm{env}}\,\mathrm{d}a
\end{aligned} \tag{2.41}$$

For δW^{int} we have

$$\begin{aligned}
\delta W^{\mathrm{int}} = {} & \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a + \int_{a^{\mathrm{int}}} T_k^{\mathrm{S}}\delta(u_k - u_k^{\mathrm{env}})\,\mathrm{d}a \\
& + \int_{a^{\mathrm{int}}} (u_k - u_k^{\mathrm{env}})\delta T_k^{\mathrm{S}}\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma\delta(\varphi - \varphi^{\mathrm{env}})\,\mathrm{d}a - \int_{a^{\mathrm{int}}} (\varphi - \varphi^{\mathrm{env}})\delta\sigma\,\mathrm{d}a
\end{aligned} \tag{2.42}$$

Due to $\boldsymbol{n} = -\boldsymbol{n}^{\text{env}}$ and arbitrariness of $\delta\boldsymbol{u}, \delta\varphi$, $\delta\boldsymbol{T}^{\text{S}}, \delta\sigma$ from $\delta\Pi = \delta\Pi_1 + \delta\Pi_2 = \delta W^{\text{int}}$ we get

$$(S_{ij} - S_{ij}^{\text{env}})n_i = T_j^{*\text{int}}, \quad (D_i - D_i^{\text{env}})n_i = -\sigma^{*\text{int}}, \quad u_k = u_k^{\text{env}}, \quad \varphi = \varphi^{\text{env}}; \quad \text{on} \quad a^{\text{int}} \tag{2.43}$$

2.3.2 General Variational Principle in Linear Piezoelectric Materials

Kuang (1964, 2002) proposed a Lagrange multiplier method to derive general variational principle (Hu 1981) from the potential energy principle in linear elasticity. This method is easy extended to the linear electroelastic theory where the Maxwell stress is not considered. So the migratory variation of the electric potential is not needed. Using the Lagrange multiplier method, the general variational principle with independent variables $\boldsymbol{u}, \varphi, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{E}$ in the small deformation case is easy obtained. The boundary conditions and continuity conditions on the interface do not satisfied prior. The electric Gibbs free energy for the linear piezoelectric material under the small deformation is

$$\begin{aligned} g &= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij} \\ g^{\text{env}} &= (1/2)C_{ijkl}^{\text{env}}\varepsilon_{ij}^{\text{env}}\varepsilon_{kl}^{\text{env}} - (1/2)\epsilon_{kl}^{\text{env}}E_k^{\text{env}}E_l^{\text{env}} - e_{kij}^{\text{env}}E_k^{\text{env}}\varepsilon_{ij}^{\text{env}} \end{aligned} \tag{2.44}$$

Omitting the derivation process, the PVP is

$$\begin{aligned} &\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0 \\ &\delta\Pi_1 = \delta\int_V \left[g - \sigma_{lk}\varepsilon_{kl} + D_kE_k + (1/2)\sigma_{lk}\left(u_{k,l} + u_{l,k}\right) + D_k\varphi_{,k}\right]\mathrm{d}V - \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V \\ &\qquad + \int_V \rho_{\text{e}}\delta\varphi\,\mathrm{d}V - \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a + \int_{a_D} \sigma^*\delta\varphi\,\mathrm{d}a - \delta\int_{a_u} T_k(u_k - u_k^*)\,\mathrm{d}a + \delta\int_{a_\varphi} \sigma(\varphi - \varphi^*)\,\mathrm{d}a \\ &\delta\Pi_2 = \delta\int_{V^{\text{env}}} \left[g^{\text{env}} - \sigma_{lk}^{\text{env}}\varepsilon_{kl}^{\text{env}} + D_k^{\text{env}}E_k^{\text{env}} + (1/2)\sigma_{lk}^{\text{env}}\left(u_{k,l}^{\text{env}} + u_{l,k}^{\text{env}}\right) + D_k^{\text{env}}\varphi_{,k}^{\text{env}}\right]\mathrm{d}V \\ &\qquad - \int_{V^{\text{env}}} (f_k^{\text{env}} - \rho\ddot{u}_k^{\text{env}})\delta u_k^{\text{env}}\,\mathrm{d}V + \int_{V^{\text{env}}} \rho_e^{\text{env}}\delta\varphi^{\text{env}}\,\mathrm{d}V - \int_{a_\sigma^{\text{env}}} T_k^{*\text{env}}\delta u_k^{\text{env}}\,\mathrm{d}a \\ &\qquad + \int_{a_D^{\text{env}}} \sigma^{*\text{env}}\delta\varphi_k^{\text{env}}\,\mathrm{d}a - \delta\int_{a_u^{\text{env}}} T_k^{\text{env}}(u_k^{\text{env}} - u_k^{*\text{env}})\,\mathrm{d}a + \delta\int_{a_\varphi^{\text{env}}} \sigma^{\text{env}}(\varphi^{\text{env}} - \varphi^{*\text{env}})\,\mathrm{d}a \\ &\delta W^{\text{int}} = \int_{a^{\text{int}}} T_k^{*\text{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\text{int}}} \sigma^{*\text{int}}\delta\varphi\,\mathrm{d}a + \delta\int_{a^{\text{int}}} T_k(u_k - u_k^{\text{env}})\mathrm{d}a - \delta\int_{a^{\text{int}}} \sigma(\varphi - \varphi^{\text{env}})\mathrm{d}a \end{aligned} \tag{2.45}$$

It is easy to show that $\delta\Pi_1$ can be reduced to

$$\begin{aligned}\delta\Pi_1 = \delta\int_V &\Big\{\big(C_{ijkl}\varepsilon_{ij} - e_{kij}E_k - \sigma_{kl}\big)\delta\varepsilon_{kl} + \big(D_k - \epsilon_{kl}E_l - e_{kij}\varepsilon_{ij}\big)\delta E_k + \big(E_k + \varphi_{,k}\big)\delta D_k \\ &+ \big[(1/2)\big(u_{k,l} + u_{l,k}\big) - \varepsilon_{kl}\big]\delta\sigma_{kl} - \big(\sigma_{kl,l} + f_k - \rho\ddot{u}_k\big)\delta u_k - \big(D_{k,k} - \rho_e\big)\delta\varphi\Big\}\mathrm{d}V \\ &+ \int_{a_\sigma}\big(\sigma_{kl}n_l - T_k^*\big)\delta u_k\,\mathrm{d}a + \int_{a_D}\big(D_k n_k + \sigma^*\big)\delta\varphi\,\mathrm{d}a \\ &+ \int_{a_u}\big[\big(\sigma_{kl}n_l - T_k\big)\delta u_k - \big(u_k - u_k^*\big)\delta T_k\big]\mathrm{d}a \\ &+ \int_{a_\varphi}\big[\big(D_k n_k + \sigma\big)\delta\varphi + \big(\varphi - \varphi^*\big)\delta\sigma\big]\mathrm{d}a + \int_{a^{\mathrm{int}}}\sigma_{kl}n_l\delta u_k\,\mathrm{d}a + \int_{a^{\mathrm{int}}}D_k n_k\delta\varphi\,\mathrm{d}a\end{aligned} \tag{2.46}$$

Completing the variational calculation and considering the arbitrariness of $\delta\boldsymbol{u}, \delta\varphi, \delta\boldsymbol{u}^{\mathrm{env}}, \delta\varphi^{\mathrm{env}}$ and $\boldsymbol{T}, \sigma$ finally we get

$$\begin{aligned}&\varepsilon_{kl} = (1/2)\big(u_{k,l} + u_{l,k}\big), \quad E_k = -\varphi_{,k}; \quad D_k = \epsilon_{kl}E_l + e_{kij}\varepsilon_{ij}, \quad \sigma_{kl} = C_{ijkl}\varepsilon_{ij} - e_{ikl}E_i \\ &\sigma_{kl,l} + f_k = \rho\ddot{u}_k; \quad \sigma_{kl}n_l = T_k^*, \quad \text{on} \quad a_\sigma; \quad u_k = u_k^*, \quad \sigma_{kl}n_l = T_k, \quad \text{on} \quad a_u \\ &D_{k,k} = \rho_{\mathrm{e}}; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_D; \quad \varphi = \varphi^*, \quad D_k n_k = -\sigma, \quad \text{on} \quad a_\varphi \\ &\varepsilon_{kl}^{\mathrm{env}} = (1/2)\Big(u_{k,l}^{\mathrm{env}} + u_{l,k}^{\mathrm{env}}\Big), \quad E_k^{\mathrm{env}} = -\varphi_{,k}^{\mathrm{env}}; \quad D_k^{\mathrm{env}} = \epsilon_{kl}E_l^{\mathrm{env}} - e_{kij}\varepsilon_{ij}^{\mathrm{env}}, \\ &\sigma_{kl}^{\mathrm{env}} = C_{ijkl}\varepsilon_{ij}^{\mathrm{env}} - e_{ikl}E_i^{\mathrm{env}} \\ &\sigma_{kl,l}^{\mathrm{env}} + f_k^{\mathrm{env}} = \rho\ddot{u}_k^{\mathrm{env}}; \quad \sigma_{kl}^{\mathrm{env}}n_l^{\mathrm{env}} = T_k^{*\mathrm{env}} \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad u_k^{\mathrm{env}} = u_k^{*\mathrm{env}}, \quad \sigma_{kl}^{\mathrm{env}}n_l^{\mathrm{env}} = T_k^{\mathrm{env}}, \\ &\quad \text{on} \quad a_u^{\mathrm{env}} \\ &D_{k,k}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}; \quad D_k^{\mathrm{env}}n_k^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}}; \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad D_k^{\mathrm{env}}n_k^{\mathrm{env}} = -\sigma^{\mathrm{env}}, \\ &\quad \text{on} \quad a_\varphi^{\mathrm{env}} \\ &\big(\sigma_{kl} - \sigma_{kl}^{\mathrm{env}}\big)n_l = T_k^{*\mathrm{int}}, \quad \big(D_k - D_k^{\mathrm{env}}\big)n_k = -\sigma^{*\mathrm{int}}, \quad u_k = u_k^{\mathrm{env}}, \quad \varphi = \varphi^{\mathrm{env}}; \quad \text{on} \quad a^{\mathrm{int}}\end{aligned} \tag{2.47}$$

Equation (2.47) is the complete governing equation.

2.4 Variational Principle in Piezoelectric Materials Under Finite Deformation

2.4.1 The Electric Gibbs Free Energy in Initial Configuration

Some fundamental formulas and notations for finite deformation shown in Sect. 1.3.4 will be used in this chapter. It is emphasized that the same coordinate system is used in the current and initial configurations. Since the isothermal electric Gibbs free energy $\bar{g}$ in the finite deformation state must be invariant in a rigid body rotation, so the $\bar{g}$ for materials without the electric couple problem should be taken in the following form:

$$\begin{gathered}\bar{g}=(1/2)\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}\bar{\varepsilon}_{KL}-(1/2)\bar{\epsilon}_{kl}\bar{E}_K\bar{E}_L-\bar{e}_{KIJ}\bar{E}_K\bar{\varepsilon}_{IJ}-(1/2)\bar{l}_{IJKL}\bar{E}_I\bar{E}_J\bar{\varepsilon}_{KL}\\ \bar{C}_{IJKL}=\bar{C}_{JIKL}=\bar{C}_{IJLK}=\bar{C}_{KLIJ},\quad \bar{\epsilon}_{KL}=\bar{\epsilon}_{LK},\quad \bar{e}_{KIJ}=\bar{e}_{KJI},\quad \bar{l}_{IJKL}=\bar{l}_{JIKL}=\bar{l}_{IJLK}=\bar{l}_{KLIJ}\end{gathered}\tag{2.48}$$

where $\bar{C}_{IJKL},\bar{\epsilon}_{KL},\bar{e}_{KIJ},\bar{l}_{IJKL}$ are the material coefficients in the initial configuration. It is noted that coefficients in the initial and current configurations are different. From the thermodynamic theory, the constitutive equations are

$$\begin{aligned}\bar{\sigma}_{LK}&=\partial\bar{g}/\partial\bar{\varepsilon}_{KL}=\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}-\bar{e}_{JKL}\bar{E}_J-(1/2)\bar{l}_{IJKL}\bar{E}_I\bar{E}_J\\ \bar{D}_K&=-\partial\bar{g}/\partial\bar{E}_K=(\bar{\epsilon}_{kl}+\bar{l}_{IJKL}\bar{\varepsilon}_{JI})\bar{E}_L+\bar{e}_{KIJ}\bar{\varepsilon}_{IJ}\end{aligned}\tag{2.49}$$

Using Eq. (2.49), Eq. (2.48) can be reduced to

$$\begin{gathered}\bar{g}=(1/2)\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}\bar{\varepsilon}_{KL}+\bar{g}^{\mathrm{e}},\quad \bar{g}^{\mathrm{e}}=-(1/2)\bar{\Gamma}_N\bar{E}_N=-(1/2)\bar{\Gamma}_N\varphi_{,N},\\ \bar{\Gamma}_N=\bar{D}_N+\bar{e}_{NKL}\bar{\varepsilon}_{KL}\end{gathered}\tag{2.50}$$

In $\bar{g}$, the term $(1/2)\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}\bar{\varepsilon}_{KL}$ is the mechanical deformation energy, $(1/2)\bar{D}_K\bar{E}_K$ is the electromagnetic energy, $(1/2)\bar{e}_{NKL}E_N\bar{\varepsilon}_{KL}$ is the mechanical and electromagnetic coupling energy, and $\bar{g}^{\mathrm{e}}$ is the sum of the electromagnetic energy and coupling energy. For the small deformation case, $(1/2)\bar{e}_{NKL}E_N\bar{\varepsilon}_{KL}$ can be neglected, so the coupling energy can also be neglected.

2.4.2 Variational Principle with the Electric Gibbs Free Energy Under Finite Deformation

As in Eq. (2–8), variations of $\varphi,\bar{\boldsymbol{E}}$ are divided into local variation $\delta_\varphi\varphi,\delta_\varphi\bar{\boldsymbol{E}}$ and migratory variation $\delta_u\varphi$, $\delta_u\bar{\boldsymbol{E}}$, i.e.,

$$\begin{aligned}&\delta\varphi=\delta_\varphi\varphi+\delta_u\varphi,\quad \delta_u\varphi=\varphi_{,p}\delta u_p=-E_p\delta u_p=-\bar{E}_LX_{L,p}\delta u_p\\ &\delta\bar{E}_I=\delta_\varphi\bar{E}_I+\delta_u\bar{E}_I,\quad \delta_u\bar{E}_I=\bar{E}_{I,p}\delta u_p=\bar{E}_{I,L}X_{L,p}\delta u_p=\bar{E}_{L,I}X_{L,p}\delta u_p\end{aligned}\tag{2.51}$$

Let the displacement $\mathbf{u}$ and the potential φ satisfy their boundary conditions on their own boundaries $\bar{a}_u,\bar{a}_\varphi,\bar{a}_\psi$ and the continuity conditions on their interface $\bar{a}^{\mathrm{int}}$ (Fig. 2.1). The variational principle with the electric Gibbs free energy under finite deformation for electroelastic media can be expressed in the following form (Kuang 2008b, 2011a):

$$
\begin{aligned}
&\delta\bar{\Pi} = \delta\bar{\Pi}_1 + \delta\bar{\Pi}_2 - \delta\bar{W}^{*\mathrm{int}} = 0 \\
&\delta\bar{\Pi}_1 = \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V} - \delta\bar{W}^{*} \\
&\delta\bar{\Pi}_2 = \int_{\bar{V}^{\mathrm{env}}} \delta\bar{g}^{\mathrm{env}}\,\mathrm{d}\bar{V} + \int_{\bar{V}^{\mathrm{env}}} \bar{g}^{\mathrm{e\ env}}\delta u_{i,i}^{\mathrm{env}}\,\mathrm{d}\bar{V} - \delta\bar{W}^{*\mathrm{env}} \\
&\delta\bar{W}^{*} = \int_{\bar{V}} (\bar{f}_k - \bar{\rho}\ddot{u}_k)\delta u_k\,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{\rho}_{\mathrm{e}}\delta\varphi\,\mathrm{d}\bar{V} + \int_{\bar{a}_\sigma} \bar{T}_k^{*}\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}_D} \bar{\sigma}^{*}\delta\varphi\,\mathrm{d}\bar{a} \\
&\delta\bar{W}^{*\mathrm{env}} = \int_{\bar{V}^{\mathrm{env}}} (\bar{f}_k^{\mathrm{env}} - \bar{\rho}\ddot{u}_k^{\mathrm{env}})\delta u_k^{\mathrm{env}}\,\mathrm{d}\bar{V} - \int_{\bar{V}^{\mathrm{env}}} \bar{\rho}_{\mathrm{e}}^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{V} \\
&\qquad\qquad + \int_{\bar{a}_\sigma^{\mathrm{env}}} \bar{T}_k^{*\mathrm{env}}\delta u_k^{\mathrm{env}}\,\mathrm{d}\bar{a} - \int_{\bar{a}_D^{\mathrm{env}}} \bar{\sigma}^{*\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a} \\
&\delta\bar{W}^{*\mathrm{int}} = \int_{\bar{a}^{\mathrm{int}}} \bar{T}_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}^{\mathrm{int}}} \bar{\sigma}^{*\mathrm{int}}\delta\varphi\,\mathrm{d}\bar{a}
\end{aligned}
\tag{2.52}
$$

where $\bar{T}_k^{*}, \bar{\sigma}^{*}; \bar{T}_k^{*\mathrm{env}}, \bar{\sigma}^{*\mathrm{env}}; \bar{T}_k^{*\mathrm{int}}, \bar{\sigma}^{*\mathrm{int}}$ are the given values on the corresponding surfaces.

Using relations $g^{\mathrm{e}}\,\mathrm{d}V = \bar{g}^{\mathrm{e}}\,\mathrm{d}\bar{V}$, $\int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V = \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V}$ and $\int_V \delta g\,\mathrm{d}V = \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V}$ (Kuang 2008b, 2009a) yields

$$
\begin{aligned}
\int_{\bar{V}} \delta\bar{g}\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V} &= \int_{\bar{V}} (\bar{\sigma}_{JI}\delta\bar{\varepsilon}_{IJ} - \bar{D}_I\delta\bar{E}_I)\,\mathrm{d}\bar{V} + (1/2)\int_{\bar{V}} \bar{\Gamma}_N\varphi_{,N}\delta u_{k,k}\,\mathrm{d}\bar{V} \\
&= \int_{\bar{V}} [\bar{\sigma}_{JI}x_{k,I}\delta u_{k,J} - \bar{D}_I(-\delta_\varphi\varphi_{,I} + \bar{E}_{L,I}X_{L,p}\delta u_p)]\,\mathrm{d}\bar{V} + (1/2)\int_{\bar{V}} \bar{\Gamma}_N\varphi_{,N}X_{J,k}\delta u_{k,J}\,\mathrm{d}\bar{V} \\
&= \int_{\bar{a}} \left[\bar{\sigma}_{JI}x_{k,I} + (1/2)\bar{\Gamma}_N\varphi_{,N}X_{J,k}\right]\bar{n}_J\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{V}} \left[\bar{\sigma}_{JI}x_{k,I} + (1/2)\bar{\Gamma}_N\varphi_{,N}X_{J,k}\right]_{,J}\delta u_k\,\mathrm{d}\bar{V} \\
&\quad + \int_{\bar{a}} \bar{D}_I\bar{n}_I\delta_\varphi\varphi\,\mathrm{d}\bar{a} - \int_{\bar{V}} \bar{D}_{I,I}\delta_\varphi\varphi\,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{D}_I\bar{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\bar{V}
\end{aligned}
\tag{2.53}
$$

where $\delta u_{k,k} = \delta u_{k,J}X_{J,k}$ was used. Substitution of Eq. (2.53) into Eq. (2.52) yields

$$
\begin{aligned}
\delta\bar{\Pi}_1 = &\int_{\bar{a}} [(\bar{\sigma}_{JI}x_{k,I} + (1/2)(\bar{\Gamma}_N\varphi_{,N}X_{J,k})\bar{n}_J]\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}_\sigma} \bar{T}_k^{*}\delta u_k\,\mathrm{d}\bar{a} \\
&- \int_{\bar{V}} [(\bar{\sigma}_{JI}x_{k,I} + (1/2)\bar{\Gamma}_N\varphi_{,N}X_{J,k})_{,J} + \bar{f}_k - \bar{\rho}\ddot{u}_k]\delta u_k\,\mathrm{d}\bar{V} + \int_{\bar{a}_D} (\bar{D}_I\bar{n}_I + \bar{\sigma}^{*})\delta_\varphi\varphi\,\mathrm{d}\bar{a} \\
&+ \int_{\bar{a}^{\mathrm{int}} + a_\varphi} \bar{D}_I\bar{n}_I\delta_\varphi\varphi\,\mathrm{d}\bar{a} - \int_{\bar{V}} (\bar{D}_{I,I} - \bar{\rho}_{\mathrm{e}})\delta_\varphi\varphi\,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{D}_I\bar{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\bar{V} \\
&- \int_{\bar{V}} \bar{\rho}_{\mathrm{e}}E_p\delta u_p\,\mathrm{d}\bar{V} - \int_{\bar{a}_D} \bar{\sigma}^{*}E_p\delta u_p\,\mathrm{d}\bar{a}
\end{aligned}
\tag{2.54a}
$$

The last three terms in (2.54a) can be reduced to

$$
\begin{aligned}
&-\int_{\bar V}\bar D_I\bar E_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\bar V-\int_{\bar V}\bar\rho_{\mathrm{e}}E_p\delta u_p\,\mathrm{d}\bar V-\int_{\bar a_D}\bar\sigma^*E_p\delta u_p\,\mathrm{d}\bar a\\
&\quad=-\int_{\bar a}\bar D_I\bar E_LX_{L,p}\bar n_I\delta u_p\,\mathrm{d}\bar a+\int_{\bar V}\left(\bar D_IX_{L,p}\delta u_p\right)_{,I}\bar E_L\,\mathrm{d}\bar V\\
&\qquad-\int_{\bar V}\bar\rho_{\mathrm{e}}E_p\delta u_p\,\mathrm{d}\bar V-\int_{\bar a_D}\bar\sigma^*E_p\delta u_p\,\mathrm{d}\bar a=-\int_{\bar a_D}(\bar D_I\bar n_I+\bar\sigma^*)E_p\delta u_p\,\mathrm{d}\bar a\\
&\qquad-\int_{\bar a^{\mathrm{int}}+\bar a_\varphi}\bar D_I\bar n_IE_p\delta u_p\,\mathrm{d}\bar a+\int_{\bar V}\bar D_I\bar E_LX_{L,p}\delta u_{p,I}\,\mathrm{d}\bar V+\int_{\bar V}(\bar D_{I,I}-\bar\rho_{\mathrm{e}})E_p\delta u_p\,\mathrm{d}\bar V
\end{aligned}
$$

where $X_{L,p}\delta u_p\bar E_L=E_p\delta u_p$ was used. So Eq. (2.54a) can be reduced to

$$
\begin{aligned}
\delta\bar\Pi_1={}&\int_{\bar a_\sigma}\left[(\bar\sigma_{JI}x_{k,I}+(1/2)\bar\Gamma_N\varphi_{,N}X_{J,k})\bar n_J-\bar T_k^*\right]\delta u_k\,\mathrm{d}\bar a\\
&+\int_{\bar a^{\mathrm{int}}+a_u}(\bar\sigma_{JI}x_{k,I}+\tfrac{1}{2}\bar D_N\varphi_{,N}X_{J,k})\bar n_J\delta u_k\,\mathrm{d}\bar a\\
&-\int_{\bar V}\left[(\bar\sigma_{JI}x_{k,I}+(1/2)\bar\Gamma_N\varphi_{,N}X_{J,k})_{,J}+\bar f_k-\bar\rho\ddot u_k\right]\delta u_k\,\mathrm{d}\bar V+\int_{\bar a_D}(\bar D_I\bar n_I+\bar\sigma^*)\delta_\varphi\varphi\,\mathrm{d}\bar a\\
&-\int_{\bar a_D}(\bar D_I\bar n_I+\bar\sigma^*)E_p\delta u_p\,\mathrm{d}\bar a+\int_{\bar a^{\mathrm{int}}+\bar a_\varphi}\bar D_I\bar n_I\delta_\varphi\varphi\,\mathrm{d}\bar a-\int_{\bar a^{\mathrm{int}}+\bar a_\varphi}\bar D_I\bar n_IE_p\delta u_p\,\mathrm{d}\bar a\\
&-\int_{\bar V}(\bar D_{I,I}-\bar\rho_{\mathrm{e}})\delta_\varphi\varphi\,\mathrm{d}\bar V+\int_{\bar V}\bar D_I\bar E_LX_{L,p}\delta u_{p,I}\,\mathrm{d}\bar V+\int_{\bar V}(\bar D_{I,I}-\bar\rho_{\mathrm{e}})E_p\delta u_p\,\mathrm{d}\bar V\\
={}&\int_{\bar a_\sigma}(\bar S_{IJ}\bar n_I-\bar T_J^*)\delta u_J\,\mathrm{d}\bar a+\int_{\bar a^{\mathrm{int}}}\bar S_{IJ}\bar n_I\delta u_J\,\mathrm{d}\bar a-\int_{\bar V}(\bar S_{IJ,I}+\bar f_J-\bar\rho\ddot u_J)\delta u_J\,\mathrm{d}\bar V\\
&+\int_{\bar a_D}(\bar D_I\bar n_I+\bar\sigma^*)\delta\varphi\,\mathrm{d}\bar a+\int_{\bar a^{\mathrm{int}}}\bar D_I\bar n_I\delta\varphi\,\mathrm{d}\bar a-\int_{\bar V}(\bar D_{I,I}-\bar\rho_{\mathrm{e}})\delta\varphi\,\mathrm{d}\bar V=0
\end{aligned}
\tag{2.54b}
$$

where

$$
\begin{aligned}
&\bar S_{Jk}=\bar\sigma_{JI}x_{k,I}+X_{L,k}\bar\sigma_{JL}^{\mathrm{M}},\\
&\bar\sigma_{JL}^{\mathrm{M}}=\bar D_J\bar E_L-\frac{1}{2}\bar\Gamma_N\bar E_N\delta_{JL}=\bar D_J\bar E_L-\frac{1}{2}(\bar D_N+\bar e_{NML}\bar\varepsilon_{ML})\bar E_N\delta_{JL}
\end{aligned}
\tag{2.55}
$$

$\bar S_{IJ}$ is called the pseudo total stress in the initial configuration, $\bar\sigma_{IJ}^{\mathrm{M}}$ may be called the second kind of the Maxwell stress defined in initial configurations, and $X_{L,k}\bar\sigma_{JL}^{\mathrm{M}}$ may be called the first kind of the Maxwell stress defined in current and initial configurations. From Eq. (2.55), it is known that when the initial configuration is used as the reference configuration, the Maxwell stress is related to strain. But for isotropic materials, the Maxwell stress is still not related to strain due to $\bar e_{NML}=0$.

Due to the arbitrariness of $\delta\bar{u}_i,\ \delta\bar{\varphi}$, from Eq. (2.54b) we get

$$\begin{aligned}
&\bar{S}_{Jk,J}+\bar{f}_k=\bar{\rho}\ddot{u}_k,\quad \bar{D}_{I,I}=\bar{\rho}_{\mathrm{e}}\quad \text{in}\quad \bar{V}\\
&\bar{S}_{Jk}\bar{n}_J=\bar{T}_k^*\quad \text{on}\quad \bar{a}_\sigma,\quad \bar{D}_I\bar{n}_I=-\bar{\sigma}^*\quad \text{on}\quad \bar{a}_D
\end{aligned}\tag{2.56}$$

and

$$\delta\bar{\Pi}_1=\int_{\bar{a}^{\mathrm{itf}}}\bar{S}_{IJ}\bar{n}_I\delta u_J\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{itf}}}\bar{D}_I\bar{n}_I\delta\varphi\,\mathrm{d}\bar{a}\tag{2.57a}$$

Similarly for the environment, we have

$$\begin{aligned}
\delta\bar{\Pi}_2=&\int_{\bar{V}^{\mathrm{env}}}\delta\bar{\mathrm{g}}^{\mathrm{env}}\,\mathrm{d}\bar{V}+\int_{\bar{V}^{env}}\bar{\mathrm{g}}^{\mathrm{env}}\delta u_{k,k}\,\mathrm{d}\bar{V}-\delta\bar{W}_1^{*\mathrm{env}}=\int_{\bar{a}_\sigma^{\mathrm{env}}}(\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}-\bar{T}_J^{*\mathrm{env}})\delta u_i^{\mathrm{env}}\,\mathrm{d}\bar{a}\\
&+\int_{\bar{a}^{\mathrm{int}}}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta u_i^{\mathrm{env}}\,\mathrm{d}\bar{a}-\int_{\bar{V}^{\mathrm{env}}}(\bar{S}_{IJ,I}^{\mathrm{env}}+\bar{f}_J^{\mathrm{env}}-\bar{\rho}\ddot{u}_J^{\mathrm{env}})\delta\ddot{u}_J^{\mathrm{env}}\,\mathrm{d}\bar{V}\\
&+\int_{\bar{a}_D^{\mathrm{env}}}(\overline{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}+\bar{\sigma}^{*\mathrm{env}})\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{itf}}}\overline{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}\\
&-\int_{\bar{V}^{env}}(\overline{D}_{I,I}^{\mathrm{env}}-\bar{\rho}_{\mathrm{e}}^{\mathrm{env}})\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{V}=0
\end{aligned}\tag{2.58}$$

Due to the arbitrariness of $\delta\bar{u}_i^{\mathrm{env}},\delta\bar{\varphi}^{\mathrm{env}}$, from Eq. (2.58), we get

$$\begin{aligned}
&\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}=\bar{T}_J^{*\mathrm{env}}\quad \text{on}\quad \bar{a}_\sigma^{\mathrm{env}},\quad \bar{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}=-\bar{\sigma}^{*\mathrm{env}}\quad \text{on}\quad \bar{a}_{DE}^{\mathrm{env}}\\
&\bar{S}_{IJ,I}^{\mathrm{env}}+\bar{f}_J^{\mathrm{env}}=\bar{\rho}^{\mathrm{env}}\ddot{u}_J^{\mathrm{env}},\quad \bar{D}_{I,I}^{\mathrm{env}}=\bar{\rho}_{\mathrm{e}}^{\mathrm{env}}\quad \text{in}\quad \bar{V}^{\mathrm{env}}
\end{aligned}\tag{2.59}$$

and

$$\delta\bar{\Pi}_2=\int_{\bar{a}^{\mathrm{itf}}}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta u_J^{\mathrm{env}}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{itf}}}\bar{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}\tag{2.57b}$$

Noting $\bar{n}_I=-\bar{n}_I^{\mathrm{env}},\bar{u}_I=\bar{u}_I^{\mathrm{env}},\varphi=\varphi^{\mathrm{env}}$ on the interface, we get

$$\begin{aligned}
\delta\bar{\Pi}=\delta\bar{\Pi}_1+\delta\bar{\Pi}_2-\delta\bar{W}^{*\mathrm{int}}=&\int_{\bar{a}^{\mathrm{int}}}\bar{S}_{IJ}\bar{n}_I\delta u_J\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{int}}}\bar{D}_I\bar{n}_I\delta\varphi\,\mathrm{d}\bar{a}\\
&+\int_{\bar{a}^{\mathrm{int}}}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta u_J^{\mathrm{env}}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{int}}}\bar{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}-\int_{\bar{a}^{\mathrm{int}}}\bar{T}_K^{*\mathrm{int}}\delta u_k\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{int}}}\bar{\sigma}^{*\mathrm{int}}\delta\varphi\,\mathrm{d}\bar{a}=0
\end{aligned}$$

So on the interface, it is obtained:

$$(\bar{S}_{IJ}-\bar{S}_{IJ}^{\mathrm{env}})\bar{n}_I=\bar{T}_J^{*\mathrm{int}},\quad (\bar{D}_I-\bar{D}_I^{\mathrm{env}})\bar{n}_I=-\bar{\sigma}^{*\mathrm{int}};\quad \text{on}\quad \bar{a}^{\mathrm{int}}\tag{2.60}$$

The above variational principle requests prior that the displacements and the potential satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations:

$$\begin{aligned} &u_i = u_i^*, \quad \text{on} \quad a_u; \quad \varphi = \varphi^*, \quad \text{on} \quad a_\varphi \\ &u_i^{\text{env}} = u_i^{*\text{env}}, \quad \text{on} \quad a_u^{\text{env}}; \quad \varphi^{\text{env}} = \varphi^{*\text{env}}; \quad \text{on} \quad a_\varphi^{\text{env}} \\ &u_i = u_i^{\text{env}}, \quad \varphi = \varphi^{\text{env}}; \quad \text{on} \quad a^{\text{int}} \end{aligned} \tag{2.61}$$

Equations (2.55), (2.56), (2.59), (2.57b), (2.60), and (2.61) are the governing equations under the finite deformation. It is noted that for the elastic material, these formulas are reduced to the usual elastic governing equations for elasticity. If in Eq. (2.52) we use $\delta \int_{\bar{V}} \bar{g}\,\mathrm{d}\bar{V}$ instead of $\int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V}$, Eq. (2.52) cannot be reduced to the usual elastic variational formula.

2.5 Internal Energy Variational Principle in Piezoelectric Materials

2.5.1 Internal Energy

It is noted that the constitutive equations of the general electroelastic materials are linear in the elastic part, but are nonlinear in the electric part for small deformation. The internal energy $\mathfrak{A}$ for materials without electric couple is assumed in the following form under small deformation:

$$\begin{aligned} &\mathfrak{A}(\varepsilon_{kl}, D_k) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + (1/2)\beta_{kl}D_kD_l - h_{kij}D_k\varepsilon_{ij} - (1/2)k_{ijkl}D_iD_j\varepsilon_{kl} + \cdots \\ &\beta_{kl} = \beta_{lk}, \quad k_{ijkl} = k_{jikl} = k_{ijlk} = k_{klij}, \quad h_{kij} = h_{kji} \quad C_{ijkl} = C_{jikl} = C_{ijlk} = C \end{aligned} \tag{2.62a}$$

where $h_{kij}, \beta_{kl}, k_{ijkl},$ and C_{ijkl} are material constants. The constitutive equations are

$$\begin{aligned} \sigma_{lk} &= \partial\mathfrak{A}/\partial\varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - h_{kij}D_k - (1/2)k_{ijkl}D_iD_j \\ E_k &= \partial\mathfrak{A}/\partial D_k = (\beta_{kl} - k_{klij}\varepsilon_{ji})D_l - h_{kij}\varepsilon_{ij} \end{aligned} \tag{2.63}$$

Equation (2.62a) can be rewritten as

$$\mathfrak{A}(\varepsilon_{kl}, D_k) = (1/2)C_{ijkl}\varepsilon_{ij} + \mathfrak{A}^e, \quad \mathfrak{A}^e = (1/2)\big(E_kD_k - \Delta_{kl}^{\mathfrak{A}}\varepsilon_{kl}\big); \quad \Delta_{kl}^{\mathfrak{A}} = h_{kij}D_k \tag{2.62b}$$

2.5.2 *Internal Energy Variational Principle under Small Deformation*

Let $\mathbf{u}, \boldsymbol{D}, \mathbf{u}^{\mathrm{env}}, \boldsymbol{D}^{\mathrm{env}}$ satisfy their boundary conditions on their own boundaries a_u, $a_D, a_u^{\mathrm{env}}, a_D^{\mathrm{env}}$ and the continuity conditions on the interface a^{int}, i.e.,

$$\begin{array}{llll} u_i = u_i^*, \quad \text{on} \quad a_u; & D_i n_i = -\sigma^*, & \text{on} & a_D \\ u_i^{\mathrm{env}} = u_i^{*\mathrm{env}}, \quad \text{on} \quad a_u^{\mathrm{env}}; & D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, & \text{on} & a_D^{\mathrm{env}} \\ u_i = u_i^{\mathrm{env}}, \quad (D_i - D_i^{\mathrm{env}}) n_i = -\sigma^{*\mathrm{int}}, \quad \text{on} \quad a^{\mathrm{int}} & & & \end{array} \tag{2.64}$$

where $\boldsymbol{n}$ is the outward normal of the body. Inside the body and environment, it is assumed that

$$\begin{gathered} \rho_{\mathrm{e}} = D_{i,i}, \varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2 \quad \text{in} \quad V; \quad \rho_{\mathrm{e}}^{\mathrm{env}} = D_{i,i}^{\mathrm{env}}, \\ \varepsilon_{ij}^{\mathrm{env}} = \left(u_{i,j}^{\mathrm{env}} + u_{j,i}^{\mathrm{env}}\right)/2 \quad \text{in} \quad V^{\mathrm{env}} \end{gathered} \tag{2.65}$$

Under the above conditions, given the displacement and electric charge virtual increments, the PVP in term of the internal energy is (Kuang 2009a)

$$\begin{aligned} & \delta \Pi = \delta \Pi_1 + \delta \Pi_2 - \delta W^{\mathrm{int}} = 0 \\ & \delta \Pi_1 = \delta \int_V \mathfrak{A}\, \mathrm{d}V - \int_V (f_k - \rho \ddot{u}_k) \delta u_k\, \mathrm{d}V - \int_{a_\sigma} T_k^* \delta u_k\, \mathrm{d}a - \int_V \varphi \delta(\rho_{\mathrm{e}}\, \mathrm{d}V) \\ & \qquad - \int_{a_D} \varphi \delta(\sigma^*\, \mathrm{d}a) - \int_{a_\varphi} \varphi^* \delta(\sigma\, \mathrm{d}a) \\ & \delta \Pi_2 = \delta \int_{V^{\mathrm{env}}} \mathfrak{A}^{\mathrm{env}}\, \mathrm{d}V - \int_{V^{\mathrm{env}}} (f_k^{\mathrm{env}} - \rho^{\mathrm{env}} \ddot{u}_k^{\mathrm{env}}) \delta u_k^{\mathrm{env}}\, \mathrm{d}V - \int_{a_\sigma^{\mathrm{env}}} T_k^{*\mathrm{env}} \delta u_k^{\mathrm{env}}\, \mathrm{d}a \\ & \qquad - \int_{V^{\mathrm{env}}} \varphi^{\mathrm{env}} \delta(\rho_{\mathrm{e}}^{\mathrm{env}}\, \mathrm{d}V) - \int_{a_D^{\mathrm{env}}} \varphi^{\mathrm{env}} \delta(\sigma^{*\mathrm{env}}\, \mathrm{d}a) - \int_{a_\varphi^{\mathrm{env}}} \varphi^{*\mathrm{env}} \delta(\sigma^{\mathrm{env}}\, \mathrm{d}a) \\ & \delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}} \delta u_k\, \mathrm{d}a + \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}} \delta(\sigma\, \mathrm{d}a) \end{aligned} \tag{2.66}$$

where $a = a_\sigma + a_u + a^{\mathrm{int}} = a_D + a_\varphi + a^{\mathrm{int}}, f_k, T_k^*, \varphi$, and φ^* are given body force, traction, potential, and surface potential, respectively. It is noted that the work done by the electric field in Eq. (2.66) has the form $\varphi \delta q$, i.e. the potential is kept constant, but the electric charge $\rho_{\mathrm{e}}\, \mathrm{d}V$, σda etc. have virtual increment. σ^* and $\sigma^{*\mathrm{env}}$ are given constants and do not change when virtual displacements happen, so terms $\sigma^* \mathrm{d}a$ and $\sigma^{*\mathrm{env}}\, \mathrm{d}a^{\mathrm{env}}$ will not be constants. Thus, terms $\int_{a_D} \varphi \delta(\sigma^* \mathrm{d}a)$ and $\int_{a_D^{env}} \varphi^{\mathrm{env}} \delta(\sigma^{*\mathrm{env}} \mathrm{d}a)$ etc. should be added to the variational formula. $T_k^{*\mathrm{int}}$ and $\varphi^{*\mathrm{int}}$ are given surface force and the jump of electric potential on the interface, respectively. Similar to Eq. (2.8),

$$\begin{aligned}
&\delta D_i = \delta_D D_i + \delta_u D_i, \quad \delta_u D_i = D_{i,p}\delta u_p, \quad E_{k,j} = E_{j,k} = -\varphi_{,jk} \\
&\delta\rho_\mathrm{e} = \delta_D\rho_\mathrm{e} + \delta_u\rho_\mathrm{e}, \quad \delta_u\rho_\mathrm{e} = D_{i,ip}\delta u_p = D_{i,pi}\delta u_p
\end{aligned} \tag{2.67}$$

The variation of the differential volume and area, etc. are

$$\begin{aligned}
&\delta(\mathrm{d}V) = \delta u_{k,k}\mathrm{d}V, \quad \delta(n_k\,\mathrm{d}a) = (n_k\delta u_{p,p} - n_p\delta u_{p,k})\,\mathrm{d}a, \\
&\delta(\mathrm{d}a) = (\delta u_{p,p} - \delta u_{p,k}n_p n_k)\,\mathrm{d}a
\end{aligned} \tag{2.68}$$

Neglecting terms containing $(\sigma_{kl}\varepsilon_{kl} + k_{ijkl}D_iD_j\varepsilon_{kl})/2$, it is obtained:

$$\begin{aligned}
\delta\int_V \mathfrak{A}\,\mathrm{d}V &= \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V + \int_V E_j\delta D_j\,\mathrm{d}V + \int_V \mathfrak{A}^e\delta u_{k,k}\,\mathrm{d}V \\
&= \int_a (\sigma_{ji} + E_mD_m\delta_{ij}/2)n_j\delta u_i\,\mathrm{d}a - \int_V (\sigma_{ji} + E_mD_m\delta_{ij}/2)_{,j}\delta u_i\,\mathrm{d}V + \int_V E_j\delta D_j\,\mathrm{d}V
\end{aligned} \tag{2.69a}$$

$$\begin{aligned}
&\int_V \varphi\delta(\rho_\mathrm{e}\,\mathrm{d}V) = \int_V \varphi\delta(D_{i,i}\,\mathrm{d}V) = \int_V \varphi\delta D_{i,i}\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_V \varphi\delta_D D_{i,i}\,\mathrm{d}V \\
&\quad + \int_V \varphi D_{i,ip}\delta u_p\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_a \varphi\delta_D D_i n_i\,\mathrm{d}a - \int_V \varphi_{,i}\delta_D D_i\,\mathrm{d}V \\
&\quad + \int_a \varphi\delta_u D_i n_i\,\mathrm{d}a - \int_V \varphi_{,i}\delta_u D_i\,\mathrm{d}V - \int_V \varphi D_{i,p}\delta u_{p,i}\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_a \varphi\delta D_i n_i\,\mathrm{d}a \\
&\quad - \int_V \varphi_{,i}\delta D_i\,\mathrm{d}V - \int_a \varphi D_{i,p}n_i\delta u_p\,\mathrm{d}a + \int_V (\varphi D_{i,p})_{,i}\delta u_p\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V
\end{aligned} \tag{2.69b}$$

$$\begin{aligned}
\int_{a_D} \varphi\delta(\sigma^*\,\mathrm{d}a) &= -\int_{a_D} \varphi D_i n_i\delta(\mathrm{d}a) = -\int_{a_D} \varphi D_i n_i(\delta u_{p,p} - \delta u_{p,k}n_p n_k)\,\mathrm{d}a \\
&= -\int_{a_D} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a - \int_{a_D} \varphi(D_p\delta u_{i,p} - D_i\delta u_{p,k}n_p n_k)n_i\,\mathrm{d}a \\
&= -\int_a \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a + \int_{a_\varphi + a^{\mathrm{int}}} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a \\
&\quad - \int_{a_D} \varphi(D_p\delta u_{i,p} - D_i\delta u_{p,k}n_p n_k)n_i\,\mathrm{d}a
\end{aligned} \tag{2.69c}$$

$$\begin{aligned}
&\int_{a_\varphi} \varphi^*\delta(\sigma\,\mathrm{d}a) = -\int_{a_\varphi} \varphi^*\delta(D_i n_i\mathrm{d}a) = -\int_{a_\varphi} \varphi^*\delta D_i n_i\,\mathrm{d}a - \int_{a_\varphi} \varphi^* D_i\delta(n_i\,\mathrm{d}a) \\
&\qquad = -\int_{a_\varphi} \varphi^*\delta D_i n_i\,\mathrm{d}a - \int_{a_\varphi} \varphi^*(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a \\
&\int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}\delta(\sigma\,\mathrm{d}a) = -\int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}\delta D_i^{\mathrm{int}} n_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}(D_i^{\mathrm{int}}\delta u_{p,p} - D_p^{\mathrm{int}}\delta u_{i,p})n_i\,\mathrm{d}a
\end{aligned} \tag{2.69d}$$

and

$$
\begin{aligned}
&\int_a \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a - \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_V \varphi_{,i}(D_i\delta u_{p,p} - D_p\delta u_{i,p})\,\mathrm{d}V\\
&\quad + \int_V \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})_{,i}\,\mathrm{d}V - \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_V \varphi_{,i}D_i\delta u_{p,p}\,\mathrm{d}V\\
&\quad - \int_V \varphi_{,i}D_p\delta u_{i,p}\mathrm{d}V - \int_V \varphi D_{p,i}\delta u_{i,p}\,\mathrm{d}V = \int_V \varphi_{,i}D_i\delta u_{p,p}\,\mathrm{d}V - \int_V \left(\varphi D_p\right)_{,i}\delta u_{i,p}\,\mathrm{d}V\\
&\quad = \int_a \left(\varphi_{,i}D_i\delta u_p\right)n_p\,\mathrm{d}a - \int_V \left(\varphi_{,i}D_i\right)_{,p}\delta u_p\,\mathrm{d}V - \int_a \left[\left(\varphi D_p\right)_{,i}\delta u_i\right]n_p\,\mathrm{d}a + \int_V \left(\varphi D_p\right)_{,ip}\delta u_i\,\mathrm{d}V\\
&\quad = -\int_a \left(\varphi D_p\right)_{,i}n_p\delta u_i\,\mathrm{d}a + \int_V \left(\varphi D_p\right)_{,pi}\delta u_i\,\mathrm{d}V + \int_a \varphi_{,i}D_i n_p\delta u_p\,\mathrm{d}a - \int_V \left(\varphi_{,i}D_i\right)_{,p}\delta u_p\,\mathrm{d}V
\end{aligned}
\tag{2.70a}
$$

$$
\begin{aligned}
\int_{a_D} \varphi\delta D_i n_i\,\mathrm{d}a &= \int_{a_D} \varphi\delta(D_i n_i)\,\mathrm{d}a - \int_{a_D} \varphi D_i\delta n_i\,\mathrm{d}a\\
&= -\int_{a_D} \varphi(D_i\delta u_{p,l}n_l n_p n_i - D_i\delta u_{p,i}n_p)\,\mathrm{d}a
\end{aligned}
\tag{2.70b}
$$

In Eq. (2.70b), $D_i n_i = -\sigma^*$ is a given value on a_D, so its variation vanishes on a_D.

Substituting above equations into $\delta\Pi_1$ in Eq. (2.66), it is obtained:

$$
\begin{aligned}
\delta\Pi_1 = &\int_{a_\sigma} \left[\left(\sigma_{ji} + \frac{1}{2}E_m D_m\delta_{ij} + \varphi D_{j,i} - \left(\varphi D_j\right)_{,i} + \varphi_{,p}D_p\delta_{ij}\right)n_j - T_i^*\right]\delta u_i\mathrm{d}a\\
&- \int_V \left[\left(\sigma_{ji} + \frac{1}{2}E_m D_m\delta_{ij} + \varphi D_{j,i} - \left(\varphi D_j\right)_{,i} + \varphi_{,p}D_p\delta_{ij}\right)_{,j} + f_i - \rho\ddot{u}_i\right]\delta u_i\mathrm{d}V\\
&+ \int_V (E_j + \varphi_{,j})\delta D_j\mathrm{d}V + \int_{a_\varphi} (\varphi^* - \varphi)n_i\delta D_i\mathrm{d}a\\
&+ \int_{a_\varphi} (\varphi^* - \varphi)(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\mathrm{d}a\\
&+ \int_{a^{\mathrm{int}}} \left[\left(\sigma_{ji} + \frac{1}{2}E_m D_m\delta_{ij} + \varphi D_{j,i} - \left(\varphi D_j\right)_{,i} + \varphi_{,p}D_p\delta_{ij}\right)n_j\right]\delta u_i\mathrm{d}a\\
&- \int_{a^{\mathrm{int}}} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi\delta D_i n_i\mathrm{d}a
\end{aligned}
\tag{2.71}
$$

From Eqs. (2.66)and (2.71) and the arbitrariness of $\delta u_i,\ \delta D_i$ we get

$$
\begin{aligned}
&S_{ji,j} + f_i = \rho\ddot{u}_i, \quad E_j = -\varphi_{,i}, \quad \text{in} \quad V\\
&S_{ji}n_j = T_i^*, \quad \text{on} \quad a_\sigma; \quad \varphi = \varphi^*, \quad \text{on} \quad a_\varphi; \quad S_{ij} = \sigma_{ij} + \sigma_{ij}^{\mathrm{M}}\\
&\sigma_{ji}^{\mathrm{M}} = \varphi D_{j,i} + \frac{1}{2}E_m D_m\delta_{ij} - \left(\varphi D_j\right)_{,i} + \varphi_{,p}D_p\delta_{ij} = D_j E_i - \frac{1}{2}E_p D_p\delta_{ij}
\end{aligned}
\tag{2.72}
$$

where $\boldsymbol{\sigma}^{\mathrm{M}}$ is the Maxwell stress. Using Eq. (2.72), $\delta\Pi_1$ is reduced to

$$\begin{aligned}\delta\Pi_1 &= \int_{a^{\mathrm{int}}} S_{ji}n_j\delta u_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi\delta D_i n_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a \\ &= \int_{a^{\mathrm{int}}} S_{ji}n_j\delta u_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi\delta D_i n_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi D_i\delta(n_i\,\mathrm{d}a) \end{aligned} \tag{2.73}$$

Similarly for the environment, we have

$$\begin{aligned}&S_{ji,j}^{\mathrm{env}} + f_i^{\mathrm{env}} = \rho\ddot{u}_i^{\mathrm{env}}, \quad E_i^{\mathrm{env}} = -\varphi_{,i}^{\mathrm{env}}, \quad \text{in} \quad V^{\mathrm{env}} \\ &S_{ji}^{\mathrm{env}}n_j^{\mathrm{env}} = T_i^{*\mathrm{env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad \text{on} \quad a_\varphi^{\mathrm{env}}, \quad S_{ij}^{\mathrm{env}} = \sigma_{ij}^{\mathrm{env}} + \sigma_{ij}^{\mathrm{M}} \\ &\delta\Pi_2 = \int_{a^{\mathrm{int}}} S_{ij}^{\mathrm{env}}n_j^{\mathrm{env}}\delta u_i^{\mathrm{env}}\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{\mathrm{env}}\delta D_i^{\mathrm{env}}n_i^{\mathrm{env}}\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{\mathrm{env}}D_i^{\mathrm{env}}\delta\left(n_i^{\mathrm{env}}\mathrm{d}a\right)\end{aligned} \tag{2.74}$$

δW^{int} can be reduced to

$$\begin{aligned}\delta W^{\mathrm{int}} &= \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}n_i\delta D_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a \\ &= \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}n_i\delta D_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{int}D_i\delta(n_i\,\mathrm{d}a)\end{aligned} \tag{2.75}$$

Substituting Eqs. (2.73), (2.74) and (2.75) into Eq. (2.66) and noting $\boldsymbol{n}^{\mathrm{env}} = -\boldsymbol{n}$, $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{env}}$ we have

$$\begin{aligned}\delta\Pi = &\int_{a^{\mathrm{int}}} [(S_{ij} - S_{ij}^{\mathrm{env}})n_j - T_i^{*\mathrm{int}}]\delta u_i\,\mathrm{d}a \\ &- \int_{a^{\mathrm{int}}} (\varphi - \varphi^{\mathrm{env}} - \varphi^{*\mathrm{int}})[\delta(D_in_i) + D_in_i\delta(\mathrm{d}a)]\end{aligned} \tag{2.76}$$

Due to the arbitrariness of δu_i, δD_i, we get

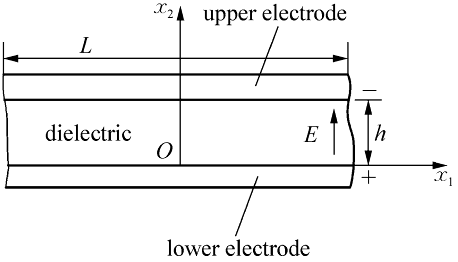
$$(S_{ij} - S_{ij}^{\mathrm{env}})n_j = T_i^{*\mathrm{int}}, \quad \varphi - \varphi^{\mathrm{env}} = \varphi^{*\mathrm{int}}, \quad \text{on} \quad a^{\mathrm{int}} \tag{2.77}$$

Equations (2.72), (2.74), (2.77), (2.64), and (2.65) form the complete governing equations.

2.5.3 The Force Acting on the Dielectric in a Plate Capacitor

As an application of the PVP, we discuss the force acting on the dielectric in a plate capacitor filled the dielectric with permittivity ϵ as shown in Fig. 2.2. Assume both

Fig. 2.2 A plate capacitor



the length and width of the plates are infinitely long, the distance h between two electrodes is small. There is no external mechanical force. The electric field inside the dielectric of the capacitor is homogeneous and $\boldsymbol{E} = E_2\boldsymbol{n}$, where $\boldsymbol{n}$ is along the positive direction of the axis x_2. The electric field inside the electrode is zero. In this simple case, the static electric force can be directly derived from the Maxwell stress and the general equation of the PVP.

1. *The Maxwell Stress Method*

Using the Maxwell stress in Eqs. (2.12) and (2.18), the force acting on the dielectric is

$$\boldsymbol{T} = \boldsymbol{n} \cdot \left(\boldsymbol{\sigma}^{\mathrm{M\,plate}} - \boldsymbol{\sigma}^{\mathrm{M}}\right) = -\boldsymbol{\sigma}^{\mathrm{M}} \cdot \boldsymbol{n} = -(1/2)D_nE_n\boldsymbol{n} = -(1/2)\epsilon E_2^2\boldsymbol{n} \tag{2.78}$$

2. *The Internal Energy Variational Principle*

Let the upper plate electrode possesses negative charge and the lower electrode possesses positive charge. Because on the electrobe the electric charge is given in the internal energy variational principle, the boundary conditions on the whole boundary of the dielectric are known. In Eq. (2.66) we only need to discuss $\delta\Pi_1$. Given the upper electrode a virtual displacement $\delta u_2 = \delta h$, the virtual strain of the dielectric is $\varepsilon_{22} = u_{2,2} = \delta h/h$. Because only δu_2 is considered, the surface integrals in $\delta\Pi_1$ can be neglected due to that the surface area keeps constant in the virtual displacement process. Therefore the variational principle for the volume between unit surfaces of electrodes is

$$\delta\Pi_{\mathfrak{A}} = \delta\Pi_1 = \delta\int_V \mathfrak{A}\,\mathrm{d}V = h(\sigma_{22}\delta u_{2,2} + E_2\delta D_2 + (1/2)E_2D_2\delta u_{2,2}) = 0$$

Because electric charge q on the electrode is constant, so $\delta_D D_2 = 0$. In volume $D_2 = \text{const.}$ due to $D_{2,p} = 0$, so $\delta_u D_2 = 0$. Therefore, it is obtained:

$$\delta\Pi_{\mathfrak{A}} = h(\sigma_{22}\delta u_{2,2} + E_2\delta D_2 + (1/2)E_2D_2\delta u_{2,2}) = h\left[\sigma_{22}\frac{\delta h}{h} + \frac{1}{2}\epsilon\left(\frac{\varphi}{h}\right)^2\frac{\delta h}{h}\right] = 0$$
$$\Rightarrow \quad T_2 = \sigma_{22} = -(\epsilon/2)(\varphi/h)^2$$

The result is identical with that in Eq. (2.78).

3. *The Electric Gibbs Free Energy Variational Principle*

Let the lower electrode possesses positive potential and the upper plate electrode grounded. Analogous to the above problem, but on the electrode the electric potential is given in the electric Gibbs free energy variational principle now. In Eq. (2.7) we only need to discuss $\delta \Pi_1$.

Give a virtual displacement under the constant electric potential on the electrode plate. It is noted that though φ is constant on the plate, but after virtual displacement, φ is changed for the point inside the dielectric. For a fixed $\boldsymbol{x}$, the change of the electric field due to changed φ is

$$\delta_\varphi E_2 = \varphi/(h+\delta h) - \varphi/h = -\varphi \delta h/h^2, \quad -D_2 \delta E_2 = D_2 E_2 \delta h/h.$$

The potential φ on the electrode is constant, $E_{2,p} = 0$, so $\delta E_2 = \delta_\varphi E_2$. Therefore, we have

$$\begin{aligned} \delta \Pi_g &= h\left[\sigma_{22}\delta u_{2,2} - D_2 \delta E_2 - (1/2)E_2 D_2 \delta u_{2,2}\right] \\ &= h D_2\left[\sigma_{22}(\delta h/h) + (1/2)\epsilon E_2^2(\delta h/h)\right] = 0 \quad \Rightarrow \quad T_2 = \sigma_{22} = -D_2^2/2\epsilon \end{aligned}$$

The result is identical with that in Eq. (2.78).

Equation (2.78) shows that the force acting on the dielectric is compressive. It is just the attractive force between two electrodes. This result is identical with that in usual textbooks.

2.6 Constitutive Equations in Electroelasticity

2.6.1 Constitutive Equations

In this section, we only discuss the case with symmetric stresses. When the thermal effect is omitted in Eq. (1.59), there are only four thermodynamic character functions: the internal energy $\mathfrak{A}(\boldsymbol{\varepsilon}, \boldsymbol{D})$ is equivalent to the free energy f, the electric Gibbs function $g(\boldsymbol{\varepsilon}, \boldsymbol{E})$ is equivalent to the electric enthalpy h^{e}, the enthalpy $h(\boldsymbol{\sigma}, \boldsymbol{E})$ is equivalent to the Gibbs function g^g, and the elastic Gibbs function $g^{\mathrm{el}}(\boldsymbol{\sigma}, \boldsymbol{D})$ is equivalent to the elastic enthalpy h^{el}. In general case, there are two groups with four variables: $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$, $(\boldsymbol{E}, \boldsymbol{D})$ in electroelasticity. Because each variable in two groups can be used as the independent variable, there are four group constitutive equations corresponding to four thermodynamic character functions $\mathfrak{A}$ (f), g (h^{e}), h (g^g), and h^{el} (g^{el}). Constitutive equation (2.3) is derived from g; Eq. (2.63) is derived from $\mathfrak{A}$. The enthalpy h and the elastic Gibbs function g^{el} can, respectively, be assumed in the following forms:

$$h = -(1/2)s_{ijkl}\sigma_{ij}\sigma_{kl} - (1/2)\epsilon_{kl}E_k E_l - d_{kij}E_k\sigma_{ij} - (1/2)p_{ijkl}E_i E_j \sigma_{kl} \tag{2.79}$$

$$g^{\mathrm{el}} = -(1/2)s_{ijkl}\sigma_{ij}\sigma_{kl} + (1/2)\beta_{kl}D_k D_l - \mathrm{g}_{kij}D_k\sigma_{ij} - (1/2)q_{ijkl}D_i D_j \sigma_{kl} \tag{2.80}$$

where $\boldsymbol{s}$ is the flexibleness coefficient tensor. From Eqs. (2.79) and (2.80), the following constitutive equations are obtained, respectively:

$$\begin{aligned} \varepsilon_{ij} &= -\partial h/\partial\sigma_{ij} = s_{ijkl}\sigma_{kl} + d_{kij}E_k + (1/2)p_{ijkl}E_kE_l \\ D_i &= -\partial h/\partial E_i = \epsilon_{ij}E_j + d_{ijk}\sigma_{jk} + p_{ijkl}E_j\sigma_{kl} \end{aligned} \tag{2.81}$$

$$\begin{aligned} \varepsilon_{ij} &= -\partial g^{\mathrm{el}}/\partial\sigma_{ij} = s_{ijkl}\sigma_{kl} + \mathrm{g}_{kij}D_k + (1/2)q_{ijkl}D_kD_l \\ E_i &= \partial g^{\mathrm{el}}/\partial D_i = \beta_{ij}D_j - \mathrm{g}_{ijk}\sigma_{jk} - q_{ijkl}D_i\sigma_{kl} \end{aligned} \tag{2.82}$$

Equations (2.3), (2.63), (2.81), and (2.82) are four kinds of constitutive equations for general ferroelectric materials. In these equations, it has been assumed that $\boldsymbol{\sigma} = \boldsymbol{\varepsilon} = \boldsymbol{E} = \boldsymbol{D} = \mathbf{0}$ at the natural state. Constitutive equations of some simpler materials are as follows.

Linear piezoelectric materials. The constitutive equations of the first, second, third, and fourth types of linear piezoelectric materials are

$$\begin{aligned} &\varepsilon_{ij} = s^E_{ijkl}\sigma_{kl} + d^\sigma_{kij}E_k \quad (or\ d^\sigma_{ijk}E_k), \quad D_i = d^E_{ijk}\sigma_{jk} + \epsilon^\sigma_{ij}E_j \\ &\sigma_{ij} = C^E_{ijkl}\varepsilon_{kl} - e^\varepsilon_{kij}E_k, \quad (or\ e^\varepsilon_{ijk}E_k) \quad D_i = e^E_{ikl}\varepsilon_{kl} + \epsilon^\varepsilon_{ij}E_j \\ &\varepsilon_{ij} = s^D_{ijkl}\sigma_{kl} + \mathrm{g}^\sigma_{kij}D_k, \quad (or\ \mathrm{g}^\sigma_{ijk}E_k) \quad E_i = -\mathrm{g}^D_{ijk}\sigma_{jk} + \beta^\sigma_{ij}D_j \\ &\sigma_{ij} = C^D_{ijkl}\varepsilon_{kl} - h^\varepsilon_{kij}D_k, \quad (or\ h^\varepsilon_{ijk}E_k) \quad E_i = -h^D_{ikl}\varepsilon_{kl} + \beta^\varepsilon_{ij}D_j \end{aligned} \tag{2.83}$$

where the superscript letter "ς" of a material constant means that the constant is measured at $\varsigma = \mathrm{const}$. As an example, $\boldsymbol{C}^E_{ijkl}$ means that the constant $\boldsymbol{C}^E_{ijkl}$ is measured at $E_i = \mathrm{const}$. Usually the coefficient measured at constant $\boldsymbol{E}$ is called the closed circuit coefficient, and the coefficient measured at constant $\boldsymbol{D}$ is called the open circuit coefficient. Usually $e \cdot E = e^\varepsilon_{ijk}E_k$ is more convenient than $E \cdot e = e^\varepsilon_{kij}E_k$ in use. If the Voigt notation (see Eq. (1.37) is used, Eq. (2.83) can be rewritten as

$$\begin{aligned} &\boldsymbol{\varepsilon} = \boldsymbol{s} : \boldsymbol{\sigma} + \boldsymbol{d}^{\mathrm{T}} \cdot \boldsymbol{E}, \quad \boldsymbol{D} = \boldsymbol{d} : \boldsymbol{\sigma} + \boldsymbol{\epsilon} \cdot \boldsymbol{E}; \quad \boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} - \boldsymbol{e}^{\mathrm{T}} \cdot \boldsymbol{E}, \quad \boldsymbol{D} = \boldsymbol{e} : \boldsymbol{\varepsilon} + \boldsymbol{\epsilon} \cdot \boldsymbol{E}; \\ &\boldsymbol{\varepsilon} = \boldsymbol{s} : \boldsymbol{\sigma} + \mathbf{g}^{\mathrm{T}} \cdot \boldsymbol{D}, \quad \boldsymbol{E} = -\mathbf{g} : \boldsymbol{\sigma} + \boldsymbol{\beta} \cdot \boldsymbol{D}; \quad \boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} - \boldsymbol{h}^{\mathrm{T}} \cdot \boldsymbol{D}, \quad \boldsymbol{E} = -\boldsymbol{h} : \boldsymbol{\varepsilon} + \boldsymbol{\beta} \cdot \boldsymbol{D} \end{aligned} \tag{2.84}$$

It is noted that though some coefficients have the same notation in different kind of constitutive equations, they should be measured in different conditions.

Electrostrictive materials with symmetric center. For all electrostrictive materials with symmetric center, the material coefficients with odd number subscript are all zero, so the piezoelectric effect disappeared. In Eq. (2.3), if terms containing $\boldsymbol{\alpha}$ are omitted, the constitutive equations have following forms:

$$\begin{aligned} &\varepsilon_{ij} = S^E_{ijkl}\sigma_{kl} + (1/2)p_{ijkl}E_kE_l, \quad D_i = \epsilon^\sigma_{ij}E_j + p_{ijkl}E_j\sigma_{kl} \approx \epsilon^\sigma_{ij}E_j \\ &\sigma_{ij} = C^E_{ijkl}\varepsilon_{kl} - (1/2)l_{ijkl}E_kE_l, \quad D_i = \epsilon^\varepsilon_{ij}E_j + l_{ijkl}E_j\varepsilon_{kl} \approx \epsilon^\varepsilon_{ij}E_j \\ &\varepsilon_{ij} = S^D_{ijkl}\sigma_{kl} + (1/2)q_{ijkl}D_kD_l, \quad E_i = \beta^\sigma_{ij}D_j - q_{ijkl}D_j\varepsilon_{kl} \approx \beta^\sigma_{ij}D_j \\ &\sigma_{ij} = C^D_{ijkl}\varepsilon_{kl} - (1/2)k_{ijkl}D_kD_l, \quad E_i = \beta^\varepsilon_{ij}D_j - k_{ijkl}D_j\varepsilon_{kl} \approx \beta^\varepsilon_{ij}D_j \end{aligned} \tag{2.85}$$

Under the high electric field, usually the electric hysteretic loop of the electrostrictive material, like PMN, is smaller than that of the piezoelectric material, like PZT.

2.6.2 *Relations Between Material Constants of the Linear Piezoelectric Materials*

Equation (2.83) is the four kinds of constitutive equations for the linear piezoelectric materials. Substitution of ***D*** in the second equation into the fourth equation in Eq. (2.83) yields

$$\begin{aligned}
&\sigma_{ij}=C^D_{ijkl}\varepsilon_{kl}-h^E_{kij}\left(e^E_{kmn}\varepsilon_{mn}+\epsilon^\varepsilon_{km}E_m\right)=\left(C^D_{ijkl}-h^E_{pij}e^E_{pkl}\right)\varepsilon_{kl}-h^E_{kij}\epsilon^\varepsilon_{km}E_m=C^E_{ijkl}\varepsilon_{kl}-e^\varepsilon_{mij}E_m\\
&E_i=-h^D_{ikl}\varepsilon_{kl}+\beta^\varepsilon_{ij}\left(e^E_{jmn}\varepsilon_{mn}+\epsilon^\varepsilon_{jm}E_m\right)=\left(-h^D_{ikl}+\beta^\varepsilon_{ij}e^E_{jkl}\right)\varepsilon_{kl}+\beta^\varepsilon_{ij}\epsilon^\varepsilon_{jm}E_m
\end{aligned}$$

and in the similar discussion, we finally get

$$\begin{aligned}
&C^D_{ijkl}s^D_{klmn}=C^E_{ijkl}s^E_{klmn}=\delta_{im}\delta_{jn},\quad \beta^\varepsilon_{ij}\epsilon^\varepsilon_{jm}=\beta^\sigma_{ij}\epsilon^\sigma_{jm}=\delta_{im},\quad \epsilon^\sigma_{ip}-\epsilon^\varepsilon_{ip}=e^E_{ikl}d^\sigma_{pmn},\\
&\beta^\varepsilon_{ip}-\beta^\sigma_{ij}=h^D_{ikl}g^\sigma_{pkl},\quad d^\sigma_{mij}=g^\sigma_{pij}\epsilon^\sigma_{pm},\quad g^D_{ikl}=\beta^\sigma_{ip}d^D_{pkl},\quad e^\varepsilon_{mij}=h^E_{kij}\epsilon^\varepsilon_{km},\quad h^D_{ikl}=e^E_{jkl}\beta^\varepsilon_{ij},\\
&\beta^\varepsilon_{ip}\epsilon^\sigma_{pm}-h^D_{ikl}d^\sigma_{mkl}=\beta^\sigma_{ip}\epsilon^\varepsilon_{pm}+g^D_{ikl}e^\varepsilon_{mkl}=\delta_{im},\quad C^D_{ijkl}-C^E_{ijkl}=h^E_{pij}e^E_{pkl},\quad s^D_{ijkl}-s^E_{ijkl}=-g^\sigma_{pij}d^D_{pkl},\\
&C^D_{ijkl}g^\sigma_{pkl}=h^E_{pij},\quad C^D_{ijkl}d^\sigma_{mkl}=h^E_{pij}\epsilon^\sigma_{pm},\quad C^E_{ijkl}d^\sigma_{pmn}=e^\varepsilon_{pij},\quad g^D_{ikl}C^E_{klmn}=\beta^\sigma_{ip}e^E_{pmn},\\
&h^D_{ikl}s^D_{klmn}=g^D_{imn},\quad e^E_{ikl}s^E_{klmn}=d^D_{imn},\quad h^D_{ikl}s^E_{klmn}=\beta^\varepsilon_{ip}d^D_{pmn},\quad s^D_{ijkl}e^\varepsilon_{mkl}=g^\sigma_{pij}\epsilon^\varepsilon_{pm},\\
&C^D_{ijkl}s^E_{klmn}-h^E_{pij}d^D_{pmn}=s^D_{ijkl}C^E_{klmn}+g^\sigma_{pij}e^E_{pmn}=\delta_{im}\delta_{nj}
\end{aligned}\tag{2.86}$$

For the nonlinear ferroelectric materials, relations between material coefficients of different constitutive equations are difficult expressed in simple unique forms.

2.7 Variational Principle in Pyroelectric Materials and Its Governing Equations

2.7.1 *Internal Energy and Electric Gibbs Function*

According to the continuum thermodynamics in Sect. 1.5, the electric Gibbs function g, the electric complementary dissipative energy rate $\dot{h}_g$, the internal energy $\mathfrak{A}$, and the dissipative energy rate $\dot{h}_\mathfrak{A}$ can be assumed as

$$\begin{aligned}
&g(\varepsilon_{kl},E_k,\vartheta)=(1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}-e_{kij}E_k\varepsilon_{ij}-(1/2)\epsilon_{ij}E_iE_j-\alpha_{ij}\varepsilon_{ij}\vartheta-\tau_iE_i\vartheta-(1/2T_0)C\vartheta^2\\
&\delta h_g=\eta_j\delta\vartheta_{,j}=-\left(\int_0^t\lambda_{ij}T^{-1}\vartheta_{,i}\,\mathrm{d}\tau\right)\delta\vartheta_{,j}\\
&C_{ijkl}=C_{jikl}=C_{ijlk}=C_{klij},\quad e_{kij}=e_{kji},\quad \epsilon_{kl}=\epsilon_{lk},\quad \alpha_{ij}=\alpha_{ji},\quad \lambda_{ij}=\lambda_{ji}
\end{aligned}\tag{2.87}$$

$$\begin{aligned}
&\mathfrak{A}(\varepsilon_{kl}, D_k, s) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - h_{kij}D_k\varepsilon_{ij} + (1/2)\beta_{ij}D_iD_j - \tilde{\alpha}_{ij}\varepsilon_{ji}s - \tilde{\tau}_iD_is + (T_0/2)\tilde{C}s^2\\
&\delta h_{\mathfrak{A}} = \tilde{\lambda}_{ij}T\dot{\eta}_j\delta\eta_i(= T\delta s^{(\mathrm{i})} = -T_{,i}\dot{\eta}_i)\\
&C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad h_{kij} = h_{kji}, \quad \beta_{kl} = \beta_{lk}, \quad \tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}, \quad \tilde{\lambda}_{ij} = \tilde{\lambda}_{ji}
\end{aligned} \tag{2.88}$$

Where $\vartheta = T - T_0$, T_0 is the temperature of the environment. It is noted that in Eq. (2.87), $s = 0$ when $T = T_0$, if $\varepsilon_{ij} = E_i = 0$, but in Eq. (2.88), $s = 0$ when $T = 0$ and $s = s_0$ when $T = T_0$; if $\varepsilon_{ij} = D_i = 0$; $\tilde{\alpha}_{ij}, \tilde{\tau}_i, \tilde{C}, \tilde{\lambda}_{ij}$ are all material constants. In the later sections, this rule will be adopted. Constitutive and evolution equations corresponding to Eq. (2.87) are

$$\begin{aligned}
&\sigma_{ji} = \partial g/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta\\
&D_i = -\partial g/\partial E_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta\\
&s = -\partial g/\partial\vartheta = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0\\
&\eta_i = -\partial h_g/\partial T_{,i} = -\int_0^t T^{-1}\lambda_{ij}\vartheta_{,j}\,\mathrm{d}\tau, \quad T\dot{\eta}_i = q_i = -\lambda_{ij}\vartheta_{,j}
\end{aligned} \tag{2.89}$$

where the evolution equation of temperature has been shown in Eq. (1.71). Corresponding to Eq. (2.88) the constitutive and evolution equations are

$$\begin{aligned}
&\sigma_{ji} = \partial\mathfrak{A}/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - h_{kij}D_k - \tilde{\alpha}_{ij}s\\
&E_i = \partial\mathfrak{A}/\partial D_i = \beta_{ij}D_j - h_{ikl}\varepsilon_{kl} - \tilde{\tau}_is\\
&T = \partial\mathfrak{A}/\partial s = -\tilde{\alpha}_{ij}\varepsilon_{ji} - \tilde{\tau}_iD_i + T_0\tilde{C}s\\
&T_{,i} = -\partial h_{\mathfrak{A}}/\partial\eta_i = -\tilde{\lambda}_{ij}T\dot{\eta}_j = -\tilde{\lambda}_{ij}q_j, \quad \int_0^t T_{,i}\,\mathrm{d}\tau = -T\int_0^t \tilde{\lambda}_{ij}\dot{\eta}_j\,\mathrm{d}\tau
\end{aligned} \tag{2.90}$$

It is obvious that there is $T_{,j} = \vartheta_{,j}, \dot{T} = \dot{\vartheta}$. Using Eqs. (2.89) and (2.90), Eqs. (2.87) and (2.88) can be rewritten, respectively, as

$$\begin{aligned}
&g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{E\,T}, \quad g^{E\,T} = -(1/2)(D_kE_k + s\vartheta + \Delta_{kl}\varepsilon_{kl}), \quad \Delta_{kl} = e_{mkl}E_m + \alpha_{kl}\vartheta\\
&\mathfrak{A} = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \mathfrak{A}^{E\,T}, \quad \mathfrak{A}^{E\,T} = (1/2)(D_kE_k + sT + \Delta'_{kl}\varepsilon_{kl}), \quad \Delta'_{kl} = h_{mkl}D_m + \alpha_{kl}s
\end{aligned} \tag{2.91}$$

In Eq. (2.91), $\Delta_{kl}\varepsilon_{kl}$ and $\Delta'_{kl}\varepsilon_{kl}$ can be neglected for the case of small strain.

Using the inertial entropy theory given in Sect. 1.7.2, from Eqs. (2.89) and (1.74), the thermal conductive or energy equation can be obtained:

$$-q_{i,i} = T\dot{s} + T\dot{s}^{(\mathrm{a})} - \dot{r}, \quad \lambda_{ij}T_{,ji} = T(\alpha_{ij}\dot{\varepsilon}_{ij} + \tau_i\dot{E}_i + T_0^{-1}C\dot{\vartheta} + T_0^{-1}C\rho_{s0}\ddot{\vartheta}) - \dot{r} \tag{2.92}$$

If ϑ is much less than T_0, $\vartheta \ll T_0$, then the above equation is reduced to

$$\lambda_{ij}\vartheta_{,ji} = T_0\alpha_{ij}\dot{\varepsilon}_{ij} + T_0\tau_i\dot{E}_i + C(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}) - \dot{r} \tag{2.93}$$

Equations (2.92) and (2.93) are temperature wave equations with finite phase velocity.

2.7.2 Electric Gibbs Function Variational Principle

In this section, we only discuss the PVP of the pyroelectric material with linear elasticity under small deformation. For simplicity, it is assumed that the environment is air. It is also assumed that in the air, the temperature is constant or $\vartheta^{\mathrm{env}} = 0$ and at infinity, $\boldsymbol{E}^{\infty} = \boldsymbol{0},\ \sigma^{*\infty} = 0$. The interface is heat insulated. The heat input and heat output by heat conduction may be occured at some internal boundaries. Analogous to Eq. (2.8), the variation of the temperature ϑ can also be divided into $\delta_{\vartheta}\vartheta$ and $\delta_u\vartheta$, but it is not needed because the final result shows that terms containing $\delta_u\vartheta$ are countervailed each other. So the body and air only have electric connection. However, the contribution of the heat due to the variation of the volume seems to be considered.

Under the assumption that $\boldsymbol{u}, \varphi, \vartheta$ satisfy their own boundary conditions $u_i = u_i^*, \varphi = \varphi^*$ and $\vartheta = \vartheta^*$ on a_u, a_φ and a_T, respectively. $\varphi = \varphi^{\mathrm{env}}, \vartheta = \vartheta^{\mathrm{env}} = 0$ on the interface except at some heat source and sink places. In the medium $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, E_i = -\varphi_{,i}, T\dot{\eta}_j = -\lambda_{ij}\vartheta_{,i}$ and the constitutive equation (2.89) are held. Noting Eq. (1.59), $g = \mathfrak{A} - \boldsymbol{E}\cdot\boldsymbol{D} - Ts$, the PVP in terms of the electric Gibbs function for the pyroelectricity can be written as (Kuang 2009b)

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0\\
&\delta\Pi_1 = \int_V \delta(g + h_g)\,\mathrm{d}V + \int_V g^{E\,T}\delta u_{i,i}\,\mathrm{d}V - \delta Q' - \delta W\\
&\delta Q' = -\int_V\int_0^t (\dot{r}/T)\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V + \int_V s^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V + \int_{a_q}\int_0^t \dot{\eta}^*\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}a - \int_V\int_0^t \dot{s}^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V\\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a\\
&\delta\Pi_2 = \int_{V^{\mathrm{env}}}\delta g^{\mathrm{env}}\,\mathrm{d}V + \int_{V^{\mathrm{env}}} g^{E\,T\,\mathrm{env}}\delta u_{i,i}^{\mathrm{env}}\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi\,\mathrm{d}V\\
&\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a_D^{\mathrm{int}}}\sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{2.94}$$

where $f_k, T_k^*, \rho_{\mathrm{e}}, \sigma^*, \rho_{\mathrm{e}}^{\mathrm{env}}$ and $\dot{\eta}_i^*$ $(\dot{\eta}^* = \dot{\eta}_i^* n_i)$ are the given mechanical body force, traction, body electric charge density, surface electric charge density, body electric charge density in the air, and surface entropy flow, respectively, and a_q is the surface given thermal flow, $g^{\mathrm{env}} = g^{E\,T\,\mathrm{env}} = -(1/2)D_k^{\mathrm{env}}E_k^{\mathrm{env}}, D_k^{\mathrm{env}} = \epsilon_0 E_k^{\mathrm{env}}$. In Eq. (2.94), the term $\int_0^t \dot{s}^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}\tau$ is the electric complement heat rate per unit volume corresponding to the inner electric complement dissipation energy rate δh_g. This is consistent with the laws of the thermodynamics. In order to obtain the heat conduction equation and the boundary condition of the heat flow from the variational principle, the electric complement dissipation energy $\int_V \delta h_g\,\mathrm{d}V$ in $\delta\Pi$ and the inner irreversible electric complement heat $\int_V\int_0^t \dot{s}^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V$ in $\delta Q'$ should be simultaneously included in the variational functional. In Eq. (2.94), the integrands contain the time derivatives of variables and need to integrate with time t, because in the irreversible process the integral is dependent to the integral path. But the time is a parameter and does not join the virtual variation.

It is noted that

$$\begin{aligned}
\int_V \delta g\,\mathrm{d}V &= \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V - \int_V D_k\delta E_k\,\mathrm{d}V - \int_V s\delta\vartheta\,\mathrm{d}V = \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V \\
&\quad + \int_V D_k\delta_\varphi\varphi_{,\,i}\,\mathrm{d}V - \int_V D_iE_{p,i}\delta u_p\,\mathrm{d}V - \int_V s\delta\vartheta\,\mathrm{d}V \\
&= \int_a \sigma_{ji}n_j\delta u_i\,\mathrm{d}a - \int_V \sigma_{ji,j}\delta u_i\,\mathrm{d}V + \int_a D_kn_k\delta_\varphi\varphi\,\mathrm{d}a - \int_V D_{k,k}\delta_\varphi\varphi\,\mathrm{d}V \\
&\quad - \int_V \left(D_iE_p\right)_{,i}\delta u_p\,\mathrm{d}V + \int_V D_{i,i}E_p\delta u_p\,\mathrm{d}V - \int_V s\delta\vartheta\,\mathrm{d}V \\
\int_V g^{E\,T}\delta u_{k,k}\,\mathrm{d}V &= -(1/2)\int_a (D_kE_k + s\vartheta)n_k\delta u_k\,\mathrm{d}V + (1/2)\int_V (D_kE_k + s\vartheta)_{,k}\delta u_k\,\mathrm{d}V \\
\int_V \delta h_g\,\mathrm{d}V &= \int_a \eta_jn_j\delta\vartheta\,\mathrm{d}a - \int_V \eta_{j,j}\delta\vartheta\,\mathrm{d}V, \quad \eta_j = -\int_0^t \lambda_{ij}T^{-1}\vartheta_{,i}\,\mathrm{d}\tau
\end{aligned} \tag{2.95}$$

Substituting Eq. (2.95) into $\delta\Pi_1$ of Eq. (2.94) and adding a term $\int_a D_kn_k \left(E_p\delta u_p + \delta_u\varphi\right)\mathrm{d}a$ to it, similar to the derivation in Sect. 2.1.2, finally we get

$$\begin{aligned}
\delta\Pi_1 = &\int_{a_\sigma} \left(S_{ji}n_j - T_i^*\right)\delta u_i\,\mathrm{d}a - \int_V \left(S_{ji,j} + f_k - \rho\ddot{u}_k\right)\delta u_i\,\mathrm{d}V \\
&+ \int_{a_D} (D_kn_k + \sigma^*)\delta\varphi\,\mathrm{d}a - \int_V \left(D_{k,k} - \rho_\mathrm{e}\right)\delta\varphi\,\mathrm{d}V + \int_{a_q} \left(\eta_jn_j - \eta^*\right)\delta\vartheta\,\mathrm{d}a \\
&+ \int_V \left\{-s - s^{(\mathrm{a})} + \eta_{j,j} + \int_0^t \left(T^{-1}\dot{r} + \dot{s}^{(\mathrm{i})}\right)\mathrm{d}\tau\right\}\delta\vartheta\,\mathrm{d}V \\
&+ \int_{a^\mathrm{int}} S_{ji}n_j\delta u_i\,\mathrm{d}a + \int_{a^\mathrm{int}} D_kn_k\delta\varphi\,\mathrm{d}a + \int_{a^\mathrm{int}} \eta_jn_j\delta\vartheta\,\mathrm{d}a
\end{aligned} \tag{2.96}$$

where

$$\begin{aligned}
&\sigma_{ij}^{\mathrm{M}\,T} = D_iE_j - (1/2)(D_nE_n + s\vartheta)\delta_{ij} \\
&S_{ij} = \sigma_{ij} + \sigma_{ij}^{\mathrm{M}\,T} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta + D_iE_j - (1/2)(D_nE_n + s\vartheta)\delta_{ij}
\end{aligned} \tag{2.97}$$

where $\boldsymbol{\sigma}^{\mathrm{M}\,T}$ is the general Maxwell stress. Whether $\boldsymbol{\sigma}^{\mathrm{M}\,T}$ includes the term $s\vartheta$, it should still be proved by experiments; $\boldsymbol{S}$ is the pseudo total stress (Jiang and Kuang 2003, 2004). In pyroelectric materials, when the electric field and temperature are not too large and the piezoelectric coefficient is not too small, the general Maxwell stress can be neglected.

Due to the arbitrariness of $\delta u_i,\ \delta\varphi$ and $\delta\vartheta$, from Eq. (2.96), it is obtained:

$$\begin{aligned}
&S_{jk,j}+f_k=\rho\ddot{u}_k,\quad D_{k,k}=\rho_{\mathrm{e}};\quad \text{in}\quad V\\
&s+s^{(\mathrm{a})}+\eta_{j,j}=\int_0^t\left(T^{-1}\dot{r}+\dot{s}^{(\mathrm{i})}\right)\mathrm{d}\tau\quad\text{or}\quad T\left(\dot{s}+\rho_{\mathrm{s}}\ddot{\vartheta}\right)=\dot{r}-q_{i,i};\quad\text{in}\quad V\\
&S_{ji}n_j=T_i^*,\quad\text{on}\quad a_\sigma;\quad D_kn_k=-\sigma^*,\quad\text{on}\quad a_D;\quad \eta_jn_j=\eta^*,\quad\text{on}\quad a_q\\
&\delta\Pi_1=\int_{a^{\mathrm{int}}}S_{ji}n_j\delta u_i\,\mathrm{d}a+\int_{a^{\mathrm{int}}}D_kn_k\delta\varphi\,\mathrm{d}a+\int_{a^{\mathrm{int}}}\eta_jn_j\delta\vartheta\,\mathrm{d}a
\end{aligned}\tag{2.98}$$

Analogously in the air,

$$\begin{aligned}
&D_{i,i}^{\mathrm{env}}=\rho_{\mathrm{e}}^{\mathrm{env}},\quad\text{in}\quad\text{air};\quad \delta\Pi_2=\int_{a^{\mathrm{int}}}S_{ji}^{\mathrm{env}}n_j{}^{\mathrm{env}}\delta u_i\,\mathrm{d}a+\int_{a^{\mathrm{int}}}D_i^{\mathrm{env}}n_i{}^{\mathrm{env}}\delta\varphi\,\mathrm{d}a\\
&S_{ij}^{\mathrm{env}}=\sigma_{ij}^{\mathrm{M\ air}}=D_i^{\mathrm{air}}E_j^{\mathrm{air}}-(1/2)D_n^{\mathrm{air}}E_n^{\mathrm{air}}\delta_{ij}
\end{aligned}\tag{2.99}$$

Substituting Eqs. (2.98) and (2.99) into Eq. (2.94) and noting $\boldsymbol{n}^{\mathrm{env}}=-\boldsymbol{n}$ we get

$$\left(S_{ij}-S_{ij}^{\mathrm{env}}\right)n_i=T_j^{*\mathrm{int}},\quad\text{on}\quad a_\sigma^{\mathrm{int}};\quad (D_i-D_i^{\mathrm{env}})n_i-\sigma^{*\mathrm{int}},\quad\text{on}\quad a_D^{\mathrm{int}}\tag{2.100}$$

The governing equations must contain the prior conditions of the variational principle:

$$\begin{aligned}
&\mathbf{u}=\mathbf{u}^*,\quad\text{on}\quad a_u;\quad \varphi=\varphi^*;\quad\text{on}\quad a_\varphi;\quad \vartheta=\vartheta^*,\quad\text{on}\quad a_T\\
&\varphi=\varphi^{\mathrm{env}},\quad\text{on}\quad a_\varphi^{\mathrm{int}};\quad \vartheta=\vartheta^{\mathrm{env}}(=0);\quad\text{on}\quad a_\vartheta^{\mathrm{int}}
\end{aligned}\tag{2.101}$$

If $\vartheta\ll T_0$, the integral can be integrated in Eq. (2.94), and $\delta\Pi_1$ in Eq. (2.94) is reduced to

$$\begin{aligned}
&\delta\Pi_1=\int_V\left(\delta g+\eta_j\delta\vartheta_{,j}\right)\mathrm{d}V+\int_V g^{E\,T}\delta u_{i,i}dV-\delta Q'-\delta W=0\\
&\delta Q'=-T_0^{-1}\int_V r\delta\vartheta\,\mathrm{d}V+\int_{a_q}\eta_0^*\delta\vartheta\,\mathrm{d}a-\int_V S^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}V+\int_V S^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V\\
&\delta W=\int_V(f_k-\rho\ddot{u}_k)\delta u_k\,\mathrm{d}V-\int_V\rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V+\int_{a_\sigma}T_k^*\delta u_k\,\mathrm{d}a-\int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a
\end{aligned}\tag{2.102}$$

where $\eta_0^*=(1/T_0)\int_0^t q^*\,\mathrm{d}t$.

There are eight thermodynamic character functions in pyroelectric materials, so there are eight fundamental variational principles. However, the electric Gibbs function only contains five independent variables $\boldsymbol{u},\varphi,T$ and is convenient in practical application.

2.7.3 An Example for Purely Thermal Conduction

When $\vartheta \ll T_0$ for the purely thermal conduction problem without the internal heat source in an isotropic material, Eq. (2.93) is reduced to

$$\lambda\,\vartheta_{,ii} = C\left(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}\right) \tag{2.103a}$$

Now discuss a simple problem in which the wave propagates along the x_1 direction, i.e.,

$$\begin{aligned} &\lambda\frac{\partial^2\vartheta}{\partial x_1^2} = C\left(\frac{\partial\vartheta}{\partial t} + \rho_{s0}\frac{\partial^2\vartheta}{\partial t^2}\right), \quad \text{or} \quad \frac{\partial^2\vartheta}{\partial x^2} = \frac{\partial\vartheta}{\partial\tau} + \frac{\partial^2\vartheta}{\partial\tau^2} \\ &x = x_1\sqrt{\frac{C}{\lambda\rho_{s0}}} = \frac{x_1}{c\rho_{s0}}, \quad \tau = \frac{t}{\rho_{s0}}; \quad c = \sqrt{\frac{\lambda}{\rho_{s0}C}} \end{aligned} \tag{2.103b}$$

where x is the dimensionless coordinate, τ is the dimensionless time, and c is the phase velocity. Let

$$\begin{aligned} &\text{Boundary conditions:} \quad \vartheta(0,t) = \Theta_0 H(t), \quad \vartheta(\infty,t) = 0; \quad t > 0 \\ &\text{Initial conditions:} \quad \vartheta(x,0) = 0, \quad \dot{\vartheta}(x,0) = 0; \quad x > 0 \end{aligned} \tag{2.104}$$

where $H(t)$ is the Heaviside function and Θ_0 is a constant. The solution of the above problem is

$$\vartheta(x,t) = \Theta_0 H(x-t)\left[e^{-x/2} + x\int_x^\tau e^{-\varsigma/2}\frac{I_1\left(\sqrt{\varsigma^2 - x^2}/2\right)}{2\sqrt{\varsigma^2 - x^2}}\,\mathrm{d}\varsigma\right] \tag{2.105}$$

where $I_1[\cdot]$ is the modified first kind of the Bessel function. Equation (2.105) shows that ϑ is an attenuated advanced wave. At the wave front $x = \tau$ or $x_1 = ct$, ϑ is interrupted with value $e^{-x/2} = e^{-x_1/2c\rho_{s0}}$ which is decreased with time.

For a problem without initial conditions, let

$$\vartheta = \Theta_0\exp(kx - \omega t)$$

where Θ_0 is the amplitude of the wave. Substituting the above equation into Eq. (2.103) yields

$$\begin{aligned} &k^2 = C\lambda^{-1}\omega^2(i\omega^{-1} + \rho_{s0}) \\ &k = \pm(C\lambda^{-1}\rho_{s0})^{\frac{1}{2}}\omega\left[\sqrt{\frac{1}{2}\left(1 + \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}}\right)} + i\sqrt{\frac{1}{2}\left(1 - \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}}\right)}\right] \end{aligned}$$

so

$$\begin{aligned}
&\vartheta = \Theta \mathrm{e}xp[i(kx-\omega t)] = \Theta\exp[ik_1x - k_2x - \omega t)] = \Theta\exp(-k_2x)\exp[ik_1x - \omega t)]\\
&k_1 = \pm(C\lambda^{-1}\rho_{s0})^{\frac{1}{2}}\omega\sqrt{\frac{1}{2}(1+\sqrt{1+\omega^{-2}\rho_{s0}{}^{-2}})},\\
&k_2 = (C\lambda^{-1}\rho_{s0})^{\frac{1}{2}}\omega\sqrt{\frac{1}{2}(1-\sqrt{1+\omega^{-2}\rho_{s0}{}^{-2}})}\\
&c = \frac{\omega}{k_1} = \sqrt{\frac{\lambda}{C\rho_{s0}}}\Big/\sqrt{\frac{1}{2}(1+\sqrt{1+\omega^{-2}\rho_{s0}{}^{-2}})}
\end{aligned} \tag{2.106}$$

Equation (2.106) shows that the temperature wave is an attenuated dispersive wave. When $\rho_{s0} \to 0$, $c \to \sqrt{2\omega\lambda/C}$ which is just the result of the classical heat conduction theory. It shows that when ρ_{s0} is small, the heat inertial effect can be neglected for the problem without initial conditions.

2.8 Variational Principle and Governing Equations in Pyroelectric Materials with Diffusion

2.8.1 Internal Energy, Electrochemical Gibbs Function, and Electric Gibbs Function

In the diffusion theory, mechanical and electrical processes are reversible, but thermal and diffuse processes are irreversible. The internal energy and entropy are all state functions. The Gibbs equation and evolution equation are still expressed by Eqs. (1.72) and (1.77), respectively. According to Eqs. (1.72), (1.77) and (1.69) the internal energy can be given by

$$\begin{aligned}
&\dot{\mathfrak{A}} = \sigma : \dot{\boldsymbol{\varepsilon}} + +\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + T\dot{s} + \mu\dot{c}\\
&\dot{h}_{\mathfrak{A}} = T\sigma \approx \boldsymbol{X}_T\cdot\dot{\boldsymbol{\eta}} + \boldsymbol{X}_\mu\cdot\dot{\boldsymbol{\xi}} = -T_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i \ge 0\\
&g_{\mathrm{c}} = \mathfrak{A} - \boldsymbol{E}\cdot\boldsymbol{D} - Ts - \mu c, \quad \dot{g}_{\mathrm{c}} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}\cdot\dot{\boldsymbol{E}} - s\dot{\vartheta} - c\dot{\mu}\\
&\dot{h}_{gc} \approx -\vartheta_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i + \left(\vartheta_{,i}\eta_i + \mu_{,i}\xi_i\right)^{\cdot} = \eta_i\dot{\vartheta}_{,i} + \xi_i\dot{\mu}_{,i}\\
&g = \mathfrak{A} - Ts - \boldsymbol{E}\cdot\boldsymbol{D}, \quad \dot{g} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}\cdot\dot{\boldsymbol{E}} - s\dot{T} + \mu\dot{c}\\
&\dot{h}_g = \eta_j\dot{\vartheta}_{,j} - \mu_{,k}\dot{\xi}_k
\end{aligned} \tag{2.107}$$

where $\mathfrak{A}, g_{\mathrm{c}}$, and g are the internal energy, electrochemical Gibbs function, and electric Gibbs function. $\dot{h}_{\mathfrak{A}}, \dot{h}_{gc}$ and $\dot{h}_g$ are the corresponding dissipative or complementary dissipative energy rates. In this section, we only discuss electrochemical Gibbs function and electric Gibbs function variational principles. $g_{\mathrm{c}}, \dot{h}_{g\mathrm{c}}$ and $g, \dot{h}_g$ can be assumed as

$$\begin{aligned}
&g_{\mathrm{c}}(\varepsilon_{kl}, E_k, \vartheta, \mu) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}E_k\varepsilon_{ij} - (1/2)\epsilon_{ij}E_iE_j - \alpha_{ij}\varepsilon_{ij}\vartheta - \tau_iE_i\vartheta \\
&\quad - (1/2T_0)C\vartheta^2 - (1/2)b\mu^2 - b_{ij}\varepsilon_{ij}\mu - b_iE_i\mu - a\mu\vartheta \\
&\dot{h}_{\mathrm{c}} = T\dot{s}^{(\mathrm{i})} - \left(Ts^{(\mathrm{i})}\right)^{\cdot} = -s^{(\mathrm{i})}\dot{T} = \boldsymbol{X}_T \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{X}_\mu \cdot \dot{\boldsymbol{\xi}} - \left(\boldsymbol{X}_T \cdot \boldsymbol{\eta} + \boldsymbol{X}_\mu \cdot \boldsymbol{\xi}\right)^{\cdot} = \eta_i\dot{\vartheta}_{,i} + \xi_i\dot{\mu}_{,i}, \\
&e_{kij} = e_{kji}, \quad \epsilon_{kl} = \epsilon_{lk}, \quad \alpha_{ij} = \alpha_{ji}, \quad b_{ij} = b_{ji}, \quad \lambda_{ij} = \lambda_{ji}, \quad D_{ij} = D_{ji}, \quad L_{ij} = L_{ji}
\end{aligned} \tag{2.108}$$

$$\begin{aligned}
&g(\varepsilon_{kl}, E_k, \vartheta, c) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}E_k\varepsilon_{ij} - (1/2)\epsilon_{ij}E_iE_j - \alpha_{ij}\varepsilon_{ij}\vartheta - \tau_iE_i\vartheta \\
&\qquad - (1/2T_0)C\vartheta^2 + (1/2)\hat{b}c^2 - \hat{b}_{ij}\varepsilon_{ij}c - \hat{b}_iE_ic + \hat{a}c\vartheta \\
&h_g = \eta_j\dot{\vartheta}_{,j} - \mu_{,j}\dot{\xi}_j
\end{aligned} \tag{2.109}$$

where $C, C_{ijkl}, e_{kij}, \epsilon_{ij}, \alpha_{ij}, \tau_i, b, b_{ij}, b_i, a, \hat{b}, \hat{b}_{ij}, \hat{b}_i, \hat{a}$ are all material constants.

Constitutive and evolution equations corresponding to g_{c} and g are, respectively,

$$\begin{aligned}
&\sigma_{ji} = \frac{\partial g_{\mathrm{c}}}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta - b_{ij}\mu, \quad D_i = -\frac{\partial g_{\mathrm{c}}}{\partial E_i} = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta + b_i\mu \\
&s = -\frac{\partial g_{\mathrm{c}}}{\partial \vartheta} = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0 + a\mu, \quad c = -\frac{\partial g_{\mathrm{c}}}{\partial \mu} = b\mu + b_{ij}\varepsilon_{ij} + b_iE_i + a\vartheta \\
&\eta_i = \partial h_{\mathrm{c}}/\partial \vartheta_{,i} = -\int_0^t \left(\lambda_{ij}T^{-1}\vartheta_{,j} + L_{ij}T^{-1}\mu_{,j}\right)\mathrm{d}\tau, \quad \xi_i = \partial h_{\mathrm{c}}/\partial \mu_{,i} = -\int_0^t \left(L_{ij}\vartheta_{,j} + D_{ij}\mu_{,j}\right)\mathrm{d}\tau
\end{aligned} \tag{2.110}$$

$$\begin{aligned}
&\sigma_{ji} = \frac{\partial g}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta - \hat{b}_{ij}c, \quad D_i = -\frac{\partial g}{\partial E_i} = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta + \hat{b}_ic \\
&s = -\frac{\partial g}{\partial \vartheta} = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0 - \hat{a}c, \quad \mu = \frac{\partial g}{\partial c} = \hat{b}c - \hat{b}_{ij}\varepsilon_{ij} - \hat{b}_iE_i + \hat{a}\vartheta \\
&\eta_i = \partial h_g/\partial \vartheta_{,i} = -\int_0^t \left(\lambda_{ij}T^{-1}\vartheta_{,j} + L_{ij}T^{-1}\mu_{,j}\right)\mathrm{d}\tau, \quad \mu_{,j} = -\partial h_g/\partial \xi_j = -\hat{L}_{ij}T\dot{\eta}_i - \hat{D}_{ij}\dot{\xi}_i
\end{aligned} \tag{2.111}$$

where the evolution equations of temperature and concentration have been given in Eq. (1.77).

Using Eqs. (2.110) and (2.111), g_{c} and g can be rewritten as

$$\begin{aligned}
&g_{\mathrm{c}} = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mu\,T}, \quad g_{\mathrm{c}}^{\mu\,T} = -(1/2)\left(D_kE_k + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right) \\
&\Delta_{ij}^{\mu}\varepsilon_{ij} = \left(e_{kij}E_k + \alpha_{ij}\vartheta + b_{ij}\mu\right)\varepsilon_{ij} \approx 0
\end{aligned} \tag{2.112}$$

$$\begin{aligned} &g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mathrm{c}\,T}, \quad g^{\mathrm{c}\,T} = -(1/2)\left(D_k E_k + s\vartheta - c\mu + \varDelta_{ij}^{\mathrm{c}}\varepsilon_{ij}\right) \\ &\varDelta_{ij}^{\mathrm{c}}\varepsilon_{ij} = \left(e_{kij}E_k + \alpha_{ij}\vartheta + \hat{b}_{ij}c\right)\varepsilon_{ij} \approx 0 \end{aligned} \tag{2.113}$$

2.8.2 *The Electrochemical Gibbs Function Variational Principle*

In this section and the following Sect. 2.8.3, we only discuss the pyroelectric material with linear elasticity under small deformation and small variation of the temperature; the environment is air. It is assumed that on the interface, there is no diffusion and heat flow, but the electric coupling is allowed, i.e. $\varphi = \varphi^{\mathrm{env}}, q = q^{\mathrm{env}} = 0$ and $d = d^{\mathrm{env}} = 0$. The heat and input and output may be occurred at some internal boundaries. The temperature and concentration problems do not considered in air, but the electric field is discussed and at infinity $\varphi^{*\,\mathrm{env}} = 0$.

Under assumptions that $\boldsymbol{u}$, φ, ϑ, and μ satisfy their own boundary conditions $\boldsymbol{u} = \boldsymbol{u}^*$, $\varphi = \varphi^*$, $\vartheta = \vartheta^*$, and $\mu = \mu^*$ on a_u, a_φ, a_T, and a_μ, respectively and on the interface $\varphi = \varphi^{\mathrm{env}}$ are satisfied prior. Analogous to Sect. 2.7, the PVP in terms of the electro-chemical Gibbs function is (Kuang 2010, 2011c)

$$\begin{aligned} &\delta\varPi = \delta\varPi_1 + \delta\varPi_2 - \delta W^{\mathrm{int}} = 0 \\ &\delta\varPi_1 = \int_V \delta(g_{\mathrm{c}} + h_{\mathrm{c}})\,\mathrm{d}V + \int_V g_{\mathrm{c}}^{\mu\,T}\delta u_{k,k}\,\mathrm{d}V - \delta Q' - \delta\varPhi - \delta W = 0 \\ &\delta Q' = -\int_V \left(\int_0^t T^{-1}\dot{r}\,\mathrm{d}\tau\right)\delta\vartheta\,\mathrm{d}V + \int_V s^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V + \int_{a_q} \eta^*\delta\vartheta\,\mathrm{d}a \\ &\qquad + \int_V \int_0^t T^{-1}\left(T_{,i}\dot{\eta}_i + \mu_{,i}\dot{\xi}_i\right)\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V - \int_a \int_0^t T^{-1}\mu\dot{\xi}_i n_i\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}a \\ &\delta\varPhi = \int_V c^{(\mathrm{a})}\delta\mu\,\mathrm{d}V + \int_{a_d} \xi^*\delta\mu\,\mathrm{d}a \\ &\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V - \int_{a_D} \sigma^*\delta\varphi\,\mathrm{d}a \\ &\delta\varPi_2 = \int_{V^{\mathrm{env}}} \delta g_{\mathrm{c}}^{\mathrm{env}}\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi\,\mathrm{d}V \\ &\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a_D^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a \end{aligned} \tag{2.114}$$

In Eq. (2.114),$f_k, T_k^*, T_k^*, \sigma^*, \rho_{\mathrm{e}}^{\mathrm{env}}, \sigma^{*\mathrm{env}}, T_k^{*\mathrm{int}}, \sigma^{*\mathrm{int}}, \dot{\eta}^* = \dot{\eta}_i^* n_i$, and $\dot{\xi}^* = \dot{\xi}_i^* n_i$ are given values; $\delta Q'$ is related to heat (including the irreversible heat produced by the irreversible process in the material and the inertial heat); $\delta\varPhi$ is related to the diffusion energy. Equation (2.108) shows that there is no term in $\int_V \delta h_{\mathrm{c}}\,\mathrm{d}V$

corresponding to the term $-\int_a \int_0^t T^{-1}\mu\dot{\xi}_i n_i \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}a$, so it should not be included in $\delta Q'$, as shown in Eq. (2.114). It is also noted that

$$\int_V \int_0^t T^{-1}\mu_{,i}\dot{\xi}_i \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}V - \int_a \int_0^t T^{-1}\mu\dot{\xi}_i n_i \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}a = -\int_V \int_0^t T^{-1}\mu\dot{\xi}_{i,i} \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}V$$

$$\begin{aligned}
\delta\int_V \mathrm{g}_\mathrm{c} \,\mathrm{d}V &= \int_V \sigma_{ji}\delta u_{i,j} \,\mathrm{d}V - \int_V D_k \delta E_k \,\mathrm{d}V - \int_V s\delta\vartheta \,\mathrm{d}V - \int_V c\delta\mu \,\mathrm{d}V \\
&= \int_a \sigma_{ji} n_j \delta u_i \,\mathrm{d}a - \int_V \sigma_{ji,j}\delta u_i \,\mathrm{d}V + \int_a D_k n_k \delta_\varphi \varphi \,\mathrm{d}a - \int_V D_{k,k}\delta_\varphi \varphi \,\mathrm{d}V \\
&\quad - \int_V \left(D_i E_p\right)_{,i} \delta u_p \,\mathrm{d}V + \int_V D_{i,i} E_p \delta u_p \,\mathrm{d}V - \int_V s\delta\vartheta \,\mathrm{d}V - \int_V c\delta\mu \,\mathrm{d}V \\
\int_V g_\mathrm{c}^{\mu\,T} \delta u_{k,k} \,\mathrm{d}V &= -(1/2)\int_a \left(D_k E_k + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right) n_k \delta u_k \,\mathrm{d}V \\
&\quad + (1/2)\int_V \left(D_k E_k + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right)_{,k} \delta u_k \,\mathrm{d}V \\
\delta\int_V h_\mathrm{c} \,\mathrm{d}V &= \int_V \left(\eta_j \delta\vartheta_{,j} + \xi_j \delta\mu_{,j}\right) \mathrm{d}V = \int_a \left(\eta_j n_j \delta\vartheta + \xi_j n_j \delta\mu\right) \mathrm{d}a - \int_V \left(\eta_{j,j}\delta\vartheta + \xi_{j,j}\delta\mu\right) \mathrm{d}V \\
\eta_j &= -\int_0^t \left(\lambda_{ij} T^{-1}\vartheta_{,i} + L_{ij} T^{-1}\mu_{,i}\right) \mathrm{d}\tau, \quad \xi_j = -\int_0^t \left(L_{ij}\vartheta_{,i} + D_{ij}\mu_{,i}\right) \mathrm{d}\tau
\end{aligned} \tag{2.115}$$

Finishing the variational calculation yields

$$\begin{aligned}
&S_{ik,i} + f_k = \rho\ddot{u}_k, \quad D_{k,k} = \rho_\mathrm{e}; \quad \text{in} \quad V \\
&\int_0^t \left(\dot{s} + \rho_\mathrm{s}\ddot{\vartheta}\right) \mathrm{d}\tau = \int_0^t \left(T^{-1}\dot{r} - T^{-1}q_{j,j} + T^{-1}\mu\dot{\xi}_{i,i}\right) \mathrm{d}\tau, \quad \text{or} \quad T\left(\dot{s} + \rho_\mathrm{s}\ddot{\vartheta}\right) = \dot{r} - q_{i,i} + \mu\dot{\xi}_{i,i} \\
&\int_0^t \left(\dot{c} + \rho_\mu\ddot{\mu}\right) \mathrm{d}\tau = \int_0^t \dot{\xi}_{j,j} \,\mathrm{d}\tau, \quad \text{or} \quad \dot{c} + \rho_\mu\ddot{\mu} = -\dot{\xi}_{j,j}; \quad \text{in} \quad V \\
&S_{ji} n_j = T_i^*, \quad \text{on} \quad a_\sigma; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_D; \\
&\eta_j n_j = \eta^*, \quad \text{or} \quad q_i = q_i^* \quad \text{on} \quad a_q; \quad \dot{\xi}_j n_j = \dot{\xi}^*, \quad \text{or} \quad d_i = d_i^* \quad \text{on} \quad a_d
\end{aligned} \tag{2.116}$$

where

$$\begin{aligned}
&\sigma_{ij}^{\mathrm{M}\,\mu T} = D_i E_j - (1/2)\left(D_n E_n + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right)\delta_{ij} \\
&S_{ij} = \sigma_{ij} + \sigma_{ij}^{\mu\,T} \approx C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta + D_i E_j - (1/2)(D_n E_n + s\vartheta + c\mu)\delta_{ij}
\end{aligned} \tag{2.117}$$

From the second and third equations of Eq. (2.116) we find $T\left(\dot{s} + \rho_\mathrm{s}\ddot{\vartheta}\right) + \mu\left(\dot{c} + \rho_\mu\ddot{\mu}\right) = \dot{r} - q_{i,i}$, which is identical with Eq. (1.62).

In the air and on the interface, there are

$$
\begin{aligned}
&D^{\mathrm{env}}_{i,i} = \rho_{\mathrm{e}}^{\mathrm{env}}, \quad \text{in} \quad V^{\mathrm{env}} \\
&\left(S_{ij} - S^{\mathrm{env}}_{ij}\right) n_i = T_j^{*\mathrm{int}}, \quad \text{on} \quad a_\sigma^{\mathrm{int}}; \quad \left(D_i - D_i^{\mathrm{env}}\right) n_i = -\sigma^{*\mathrm{int}}, \quad \text{on} \quad a_D^{\mathrm{int}} \\
&S_{ij}^{\mathrm{env}} = \sigma_{ij}^{\mathrm{M\ air}} = D_i^{\mathrm{air}} E_j^{\mathrm{air}} - (1/2) D_n^{\mathrm{air}} E_n^{\mathrm{air}} \delta_{ij}
\end{aligned}
\tag{2.118}
$$

The above variational principle requests prior that the $\boldsymbol{u}, \varphi, \vartheta$ and μ satisfy their own boundary conditions, so in governing equations, the following equations should also be added:

$$
\begin{aligned}
&\boldsymbol{u} = \boldsymbol{u}^*, \quad \text{on} \quad a_u; \ \varphi = \varphi^*, \quad \text{on} \quad a_\varphi; \ \vartheta = \vartheta^*, \quad \text{on} \quad a_T; \\
&\mu = \mu^*, \quad \text{on} \quad a_\mu; \ \varphi = \varphi^{\mathrm{env}}, \quad \text{on} \quad a^{\mathrm{int}}
\end{aligned}
\tag{2.119}
$$

Equations (2.116), (2.117), (2.118), and (2.119) are the governing equations of the generalized thermodiffusion theory.

If we neglect the term $\mu\left(\dot{c} + \dot{c}^{(\mathrm{a})}\right)$ in Eq. (1.77), or let $T\left(\dot{s} + \dot{s}^{(\mathrm{a})}\right) = \dot{r} - q_{i,i}$, then we get

$$
\begin{aligned}
&T\left(\dot{s} + \rho_{\mathrm{s}} \ddot{\vartheta}\right) = \dot{r} - q_{j,j}, \quad \dot{c} + \rho_{\mathrm{c}} \ddot{\mu} = -\dot{\xi}_{j,j}; \quad \text{In medium} \\
&\dot{\eta}_j n_j = \dot{\eta}^*, \quad \text{or} \quad q_n = q_n^* \quad \text{on} \quad a_q \\
&\dot{\xi}_j n_j = \dot{\xi}^*, \quad \text{or} \quad d_n = d_n^* \quad \text{on} \quad a_d \quad \text{and} \quad a_q
\end{aligned}
\tag{2.120}
$$

If we also assume that $T_{,i}$ and $\mu_{,j}$ are not dependent with each other, for $\dot{r} = 0$, Eq. (2.120) becomes

$$
\begin{aligned}
&T\left(\alpha_{ij} \dot{u}_{i,j} + C\dot{\vartheta}/T_0 + a\dot{\mu} + \rho_{\mathrm{s}} \ddot{\vartheta}\right) = \lambda_{ij} \vartheta_{,j} \\
&b\dot{\mu} + b_{ij} \dot{u}_{i,j} + a\dot{\vartheta} + \rho_{\mathrm{c}} \ddot{\mu} = D_{ij} \mu_{,j}; \quad \text{In medium}
\end{aligned}
\tag{2.121}
$$

The formulas in literatures analogous to Eq. (2.121) can be found, such as in the paper of Sherief et al. (2004), where they used the Maxwell-Cattaneo formula. Genin and Xu (1999) discussed the thermoelastic plastic metals with mass diffusion.

2.8.3 The Electric Gibbs Function Variational Principle

Under assumptions that $\boldsymbol{u}$, φ, ϑ, and c satisfy their own boundary conditions $\boldsymbol{u} = \boldsymbol{u}^*$, $\varphi = \varphi^*$, $\vartheta = \vartheta^*$, and $c = c^*$ on a_u, a_φ, a_T, and a_{c}, respectively. The PVP in terms of the electric Gibbs function for the thermo-electro-elasto-diffusive problem is (Kuang 2010)

$$
\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0\\
&\delta\Pi_1 = \int_V \delta(g + h_g)\,\mathrm{d}V + \int_V g^{\mathrm{c}\,T}\delta u_{k,k}\mathrm{d}V - \delta Q' + \delta\Phi - \delta W = 0\\
&\delta Q' = -\int_V \left(\int_0^t T^{-1}\dot{r}\,\mathrm{d}\tau\right)\delta\vartheta\mathrm{d}V + \int_V s^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V + \int_{a_q}\eta^*\delta\vartheta\,\mathrm{d}a\\
&\qquad + \int_V\int_0^t T^{-1}\left(\vartheta_{,i}\dot{\eta}_i + \mu_{,i}\dot{\xi}_i\right)\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V - \int_a\int_0^t T^{-1}\mu\dot{\xi}_i n_i\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}a\\
&\delta\Phi = \int_V \mu_{,j}^{(\mathrm{a})}\delta\xi_j\,\mathrm{d}V + \int_{a_d}\mu^*\delta\xi\,\mathrm{d}a\\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V - \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a\\
&\delta\Pi_2 = \int_{V^{\mathrm{env}}}\delta g^{\mathrm{env}}\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi\,\mathrm{d}V\\
&\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a_D^{\mathrm{int}}}\sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a
\end{aligned}
\tag{2.122}
$$

where the symbols are the same as that in Sect. 2.8.2, but the gradient of the inertial chemical potential $\mu_{,i}^{(\mathrm{a})} = \rho_{\mathrm{c}}\ddot{\xi}_i$ is introduced, and μ^* is given value.

Finishing the variational calculation finally yields

$$
\begin{aligned}
&S_{ik,i} + f_k = \rho\ddot{u}_k, \quad D_{k,k} = \rho_{\mathrm{e}}; \quad \text{in} \quad V\\
&\int_0^t \left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right)\mathrm{d}\tau = \int_0^t \left(T^{-1}\dot{r} - T^{-1}q_{j,j} + T^{-1}\mu\dot{\xi}_{i,i}\right)\mathrm{d}\tau, \quad \text{or} \quad T\left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right) = \dot{r} - q_{i,i} + \mu\dot{\xi}_{i,i}\\
&\mu_{,j} + \rho_\mu\ddot{\xi}_j = -\hat{D}_{ij}\dot{\xi}_i - \hat{L}_{ij}T\dot{\eta}_i; \quad \text{in} \quad V\\
&S_{ji}n_j = T_i^*, \quad \text{on} \quad a_\sigma; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_D;\\
&\eta_j n_j = \eta^*, \quad \text{or} \quad q_i = q_i^* \quad \text{on} \quad a_q;
\end{aligned}
\tag{2.123}
$$

If differentiating the equation of the chemical potential with $\boldsymbol{x}$ in Eq. (2.123), it is obtained:

$$
\begin{aligned}
&\mu_{,jj} + \rho_\mu\ddot{\xi}_{j,j} + \hat{D}_{ij}\dot{\xi}_{i,j} + \hat{L}_{ij}(T\dot{\eta}_i)_{,j} = 0;\\
&\left(\hat{b}c - \hat{b}_{ij}\varepsilon_{ij} - \hat{b}_i E_i + \hat{a}\vartheta\right)_{,jj} + \rho_\mu\ddot{\xi}_{j,j} + \hat{D}_{ij}\dot{\xi}_{i,j} + \hat{L}_{ij}(T\dot{\eta}_i)_{,j} = 0; \quad \text{in} \quad V
\end{aligned}
\tag{2.124}
$$

If $\hat{D}_{ij} = \hat{D}\delta_{ij}, \lambda_{ij} = \lambda\delta_{ij}, \hat{L}_{ij} = 0$ from Eq. (2.124), a simpler diffusion equation can be obtained:

$$
\rho_{\mathrm{c}}\ddot{c} + \hat{D}\dot{c} = \left(bc - b_{ik}\varepsilon_{ik} + b_i\varphi_{,i} + a\vartheta\right)_{,jj}; \quad \text{in} \quad V \tag{2.125}
$$

Governing equations in the air are the same as that in Eq. (2.118).

2.8.4 Constitutive Equations

In general case, there are three groups with six variables: $(\boldsymbol{\sigma},\boldsymbol{\varepsilon}),(\boldsymbol{E},\boldsymbol{D}),(\vartheta,s)$ for pyroelectric materials. Because each variable in three groups can be used as the independent variable, there are eight group constitutive equations which just correspond to eight thermodynamic character functions in Eq. (1.59). Equations (2.89) and (2.90) are the constitutive equations corresponding to electric Gibbs function g and internal energy $\mathfrak{A}$. However for pyroelectric materials with diffusion there are four groups with eight variables: $(\boldsymbol{\sigma},\boldsymbol{\varepsilon}),(\boldsymbol{E},\boldsymbol{D}),(\vartheta,s),(\mu,c)$. So there are sixteen group constitutive equations. Equations (2.110) and (2.111) are the constitutive equations corresponding to electrochemical Gibbs function g_{c} and electric Gibbs function g.

2.9 Conservation Integrals in Piezoelectric Materials

2.9.1 Noether Theory

In previous sections of this chapter, it is found that the electroelastic governing equations can be obtained from the extreme value of a variational functional. The governing equation is just the Euler-Lagrange equation of that functional. Based on the theory of Noether's invariant variational problem (1918), conservation laws (integrals) can be easily obtained (Fletcher 1976; Honein and Herrmann 1997). These conservation integrals are very useful in fracture mechanics due to their path independence property. Here, some conservation laws for inhomogeneous materials (Shi and Kuang 2003) will be obtained by using the Noether's invariant variational principle.

Let the variational functional in the continuum mechanics be

$$\mathfrak{I}=\int_{\mathrm{V}}\mathrm{L}(\mathrm{x_i},\psi_{\alpha,\mathrm{j}})\,\mathrm{dV} \tag{2.126}$$

where $L(x_i,\psi_{\alpha,j})$ is the Lagrange density function and $\boldsymbol{x},\boldsymbol{\psi}$ are the independent and dependent variables, respectively. The Euler- Lagrange equation of $\mathfrak{I}$ is

$$\frac{\partial}{\partial x_j}\frac{\partial L(x_i,\psi_{\alpha,j})}{\partial\psi_{\alpha,j}}=0 \tag{2.127}$$

Give an infinitesimal transform as

$$\begin{aligned}&x_i\to x'_i=x_i+\delta x_i(x_j,\psi_\alpha),\quad \psi_\alpha(x_i)\to\psi'_\alpha(x'_i)=\psi_\alpha(x_i)+\delta\psi_\alpha(x_i,\psi_\beta)\\&\delta\psi_\alpha=\psi'_\alpha(x'_i)-\psi_\alpha(x_i)=[\psi_\alpha(x_i+\delta u_i)+\delta_\psi\psi_\alpha(x_i)]-\psi_\alpha(x_i)\\&\qquad=\delta_\psi\psi_\alpha+\delta_u\psi_\alpha=\delta_\psi\psi_\alpha+\psi_{\alpha,i}\delta u_i\end{aligned} \tag{2.128}$$

Using

$$\frac{\partial x'_i}{\partial x_j} \approx \delta_{ij} + \frac{\partial \delta x_i}{\partial x_j}, \quad \frac{\partial x_i}{\partial x'_j} \approx \delta_{ij} - \frac{\partial \delta x_i}{\partial x_j}, \quad j = \left| \frac{\partial x'_i}{\partial x_j} \right| \approx 1 + \frac{\partial \delta x_i}{\partial x_i} \tag{2.129}$$

from Eq. (2.128) yields

$$\begin{aligned} \delta(\psi_{\alpha,j}) &= \frac{\partial \psi'_\alpha(x'_i)}{\partial x'_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_j} = \frac{\partial[\psi_\alpha(x_i) + \delta\psi_\alpha(x_i, \psi_\beta)]}{\partial x_k} \frac{\partial x_k}{\partial x'_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_j} \\ &= \frac{\partial \delta\psi_\alpha(x_i, \psi_\beta)}{\partial x_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_k} \frac{\partial \delta x_k}{\partial x_j} \end{aligned} \tag{2.130a}$$

Equation (2.130a) can also be reduced to

$$\begin{aligned} \delta(\psi_{\alpha,j}) &= \frac{\partial\left[\delta_\psi \psi_\alpha + \psi_{\alpha,i}\delta x_i\right]}{\partial x_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_k} \frac{\partial \delta x_k}{\partial x_j} \\ &= \frac{\partial(\delta_\psi \psi_\alpha)}{\partial x_j} + \frac{\partial(\psi_{\alpha,i}\delta x_i)}{\partial x_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_k} \frac{\partial \delta x_k}{\partial x_j} = \frac{\partial(\delta_\psi \psi_\alpha)}{\partial x_j} + \frac{\partial(\psi_{\alpha,i})}{\partial x_j} \delta x_i \\ \frac{\partial}{\partial x_i}(\delta\psi_\alpha) &= \frac{\partial}{\partial x_i}(\delta_\psi \psi_\alpha + \psi_{\alpha,i}\delta u_i) = \frac{\partial \delta_\psi \psi_\alpha}{\partial x_i} + \psi_{\alpha,ij}\delta u_i + \psi_{\alpha,i}\delta u_{i,j} = \delta(\psi_{\alpha,j}) + \psi_{\alpha,i}\delta u_{i,j} \end{aligned} \tag{2.130b}$$

Equation (2.130b) is identical with that in Eq. (2.8) in Sect. 2.1.2.

Because $\delta\psi_\alpha(x_i, \psi_\beta)$ is the function of x_i, ψ_β, so

$$\frac{\partial \delta\psi_\alpha(x_i, \psi_\beta)}{\partial x_j} = \frac{\bar{\partial} \delta\psi_\alpha(x_i, \psi_\beta)}{\partial x_j} + \frac{\partial \delta\psi_\alpha(x_i, \psi_\beta)}{\partial \psi_\beta} \frac{\partial \psi_\beta}{\partial x_j} \tag{2.131}$$

where the notation $\bar{\partial}/\partial x_i$ is the partial derivative with respect to explicit x_i in ψ_α. If under the transform, Eq. (2.128), on the accuracy of the first order of $\delta x_i, \delta\psi_\alpha$, $\delta\psi_{\alpha,j}$, the following equality holds:

$$\int_{V'} L'(x'_i, \psi'_{\alpha,j})\,\mathrm{d}V' = \int_V L'(x'_i, \psi'_{\alpha,j}) j\,\mathrm{d}V = \int_V L(x_i, \psi_{\alpha,j})\,\mathrm{d}V \tag{2.132}$$

the group of transform Eq. (2.128) is called the symmetric group of a system. From Eq. (2.132), some conservation laws can be found.

Appling Eqs. (2.128) and (2.130), the following relation can be obtained:

$$\begin{aligned} &\left(\frac{\partial L}{\partial \psi_{\alpha,j}} \delta\psi_\alpha - \frac{\partial L}{\partial \psi_{\alpha,j}} \psi_{\alpha,i}\delta x_i\right)_{,j} + \left(\frac{\partial L}{\partial \psi_{\alpha,j}} \psi_{\alpha,i}\right)_{,j} \delta x_i = \frac{\partial L}{\partial \psi_{\alpha,j}} \left(\frac{\partial \delta\psi_\alpha}{\partial x_j} - \psi_{\alpha,i} \frac{\partial \delta x_i}{\partial x_j}\right) \\ &= \frac{\partial L}{\partial \psi_{\alpha,j}} \delta(\psi_{\alpha,j}) \end{aligned}$$

So that

$$\begin{aligned} L'(x'_i, \psi'_{\alpha,j}) &= L\left[x_i + \delta x_i, \psi_{\alpha,j} + \delta(\psi_{\alpha,j})\right] = L(x_i, \psi_{\alpha,j}) + \frac{\bar{\partial} L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \psi_{\alpha,j}}\delta(\psi_{\alpha,j}) \\ &= L(x_i, \psi_{\alpha,j}) + \frac{\bar{\partial} L}{\partial x_i}\delta x_i + \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha - \frac{\partial L}{\partial \psi_{\alpha,j}}\psi_{\alpha,i}\delta x_i\right)_{,j} + \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\psi_{\alpha,i}\right)_{,j}\delta x_i \end{aligned} \tag{2.133}$$

Substituting the identity

$$\left(\frac{\partial L}{\partial \psi_{\alpha,j}}\psi_{\alpha,i}\right)_{,j}\delta x_i = (L\delta x_i)_{,i} - L(\delta x_i)_{,i} - \frac{\bar{\partial} L}{\partial x_i}\delta x_i$$

into Eq. (2.133) yields

$$\begin{aligned} &L'(x'_i, \psi'_{\alpha,j}) = L(x_i, \psi_{\alpha,j}) + \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right)_{,j} - L(\delta x_i)_{,i} \\ &P_{ij} = L\delta_{ij} - \left(\partial L/\partial \psi_{\alpha,j}\right)\psi_{\alpha,i} \end{aligned} \tag{2.134}$$

P_{ij} is called the energy-momentum tensor of matter. Substitution of Eq. (2.134) into Eq. (2.132) yields

$$\int_V \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right)_{,j} \mathrm{d}V = 0, \quad \text{or} \quad \int_a \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right) n_j \, \mathrm{d}a = 0 \tag{2.135}$$

Equations (2.134) and (2.135) are the invariant conditions under the infinitesimal transform. The second equation in Eq. (2.135) is a path independence integral. Due to the arbitrariness of the volume from Eq. (2.135), the invariant condition in the differential form is

$$\left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right)_{,j} = 0 \tag{2.136}$$

In the above discussion, it is assumed that there is no body force, body electric charge, etc.

For the electroelastic problem without body couple let $\boldsymbol{\psi} = [u_i, \varphi]^T, L \equiv g$, we have

$$\begin{aligned}
&L \equiv g = (1/2)C_{ijkl}u_{i,j}u_{k,l} - (1/2)\epsilon_{kl}\varphi_{,k}\varphi_{,l} + e_{kij}u_{i,j}\varphi_{,k} = (1/2)\,\Sigma_{aj}\,\psi_{a,j} \\
&\Sigma_{\alpha j} = \partial g/\partial\psi_{\alpha,j} = \begin{cases} \sigma_{ij} = \partial g/\partial u_{i,j} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j, & i,j,k,l = 1,2,3 \\ \sigma_{4j} = D_j = -\partial g/\partial\psi_{4,j} = \partial g/\partial\varphi_{,j} = \epsilon_{jl}E_l + e_{jkl}\varepsilon_{kl}, & \alpha = 1,2,3,4 \end{cases} \\
&P_{ij} = g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i} = g\delta_{ij} - \sigma_{mj}u_{m,i} - D_j\varphi_{,i} \\
&\left(\Sigma_{\alpha j}\delta\psi_\alpha + P_{ij}\delta u_i\right)_{,j} = \left(\sigma_{ij}\delta u_i + D_j\delta\varphi + P_{ij}\delta u_i\right)_{,j} = 0
\end{aligned} \tag{2.137}$$

where $\Sigma_{\alpha j}$ is the generalized stress and $\psi_{\alpha,j}$ is the generalized strain and the Greek indices take 1–4.

2.9.2 *Conservation Integral in a Homogeneous Material*

For a homogeneous material, L is independent to $\boldsymbol{x}$, so $L = L(\psi_{\alpha,j}),\ \bar{\partial}/\partial x_i = 0$.

1. *Infinitesimal translation of general displacement*. Let

$$\delta x_i = 0, \quad \delta\psi_\alpha = \varepsilon c_\alpha \tag{2.138}$$

where c_α is a constant and ε is an infinitesimal parameter. Substitution of Eq. (2.138) into Eq. (2.136) yields the invariant condition:

$$\left(\Sigma_{\alpha j}\delta\psi_\alpha\right)_{,j} = \Sigma_{\alpha j,j}\delta\psi_\alpha = 0 \quad \Rightarrow \quad \Sigma_{\alpha j,j} = 0 \tag{2.139}$$

Equation (2.139) is just the generalized momentum equation.

2. *Infinitesimal translation of coordinate*. $\boldsymbol{x}$ Let

$$\delta x_i = \varepsilon c_i, \quad \delta\psi_\alpha = 0 \tag{2.140}$$

Substitution of Eq. (2.140) into Eq. (2.136) yields

$$\left(P_{ij}\delta x_i\right)_{,j} = P_{ij,j}\delta x_i = g_{,j}\delta_{ij}\delta x_i = g_{,i}\delta x_i = \delta g = 0 \quad \Rightarrow \quad g = \text{const.} \tag{2.141}$$

Equation (2.141) is just the energy conservative equation.

3. *Infinitesimal translation of coordinate and generalized displacement*. Let

$$\delta x_i = \varepsilon c_i, \quad \delta\psi_\alpha = \varepsilon\Omega_\alpha \tag{2.142}$$

where c_i, Ω_α are constants and ε is an infinitesimal parameter. Substituting Eq. (2.142) into Eq. (2.136) and noting $\Sigma_{\alpha j,j} = 0$ we find

$$\varepsilon P_{ij,j} c_i = 0, \quad \text{or} \quad \int_V P_{ij,j}\,\mathrm{d}V = \int_a \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right) n_j\,\mathrm{d}a = 0 \tag{2.143}$$

Equation (2.143) shows that the integral value is zero along a closed surface for the integrated function $\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right) n_j$. For two open surfaces initiated from a same closed curve, Eq. (2.143) shows that the integral values for two open surfaces are the same. In the two-dimensional (2D) problem, it represents the path independence J integral, J_i. Equation (2.143) can also be obtained by taking the divergence of g. In fact using $\nabla g = (\partial g / \partial x_i)\boldsymbol{e}_i$ and the equilibrium equation, we have

$$\frac{\partial g}{\partial x_i} = \frac{\bar{\partial} g}{\partial x_i} + \frac{\partial g}{\partial \psi_{\alpha,j}} \psi_{\alpha,ji} = \Sigma_{\alpha j}\psi_{\alpha,ji}, \quad \text{or} \quad g_{,i} - \Sigma_{\alpha j}\psi_{\alpha,ji} = \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)_{,j} = 0 \tag{2.144}$$

This method was adopted by many authors (Delph 1982; Pak 1992; Wang and Shen 1996).

4. *Infinitesimal expansion of coordinate and generalized displacement*. Let

$$\delta x_i = \varepsilon x_i, \;\; \delta\psi_\alpha = -(1/2)\varepsilon\psi_\alpha \tag{2.145}$$

where ε is an infinitesimal parameter. Substitution of Eq. (2.145) into (2.136) yields

$$\begin{gathered} \varepsilon\left[-(1/2)\Sigma_{\alpha j}\psi_\alpha + P_{ij} x_i\right]_{,j} = 0, \quad \text{or} \\ \int_a \left[\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right) x_i - (1/2)\Sigma_{\alpha j}\psi_\alpha\right] n_j\,\mathrm{d}a = 0 \end{gathered} \tag{2.146}$$

In the two-dimensional problem Eq. (2.146) represents the path independence M integral

$$\begin{aligned} M &= \int_{\Gamma_1} \left[\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right) x_i - (1/2)\Sigma_{\alpha j}\psi_\alpha\right] n_j\,\mathrm{d}l \\ &= \int_{\Gamma_1} \left[\left(g\delta_{ij} - \sigma_{ij}u_{i,j} - D_j\varphi_{,i}\right) x_i - (1/2)\left(\sigma_{ij}u_i + D_j\varphi\right)\right] n_j\,\mathrm{d}l \end{aligned} \tag{2.147}$$

5. *Infinitesimal rotation about the axis* x_3. Let

$$\delta x_1 = \varepsilon x_2, \quad \delta x_2 = -\varepsilon x_1, \quad \delta\psi_1 = \varepsilon\psi_2, \quad \delta\psi_2 = -\varepsilon\psi_1, \quad \delta x_3 = \delta\psi_3 = \delta\psi_4 = 0 \tag{2.148}$$

Substitution of Eq. (2.148) into Eq. (2.136) yields

$$\begin{gathered} \left(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1\right)_{,k} = 0, \quad \text{or} \\ \int_a \left(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1\right) n_k\,\mathrm{d}a = 0 \end{gathered} \tag{2.149}$$

6. *The conservative integral in pyroelectric material.* Wang and Kuang (2001) discussed the conservative integral in pyroelectric material by Noether theory and got

$$J_i = \int \left(g\delta_{ij} - \sigma_{ij}u_{i,j} - D_j\varphi_{,i} - s_j\vartheta_{,i}\right)n_j\,\mathrm{d}l$$
$$M = \int \left[\left(g\delta_{ij} - \sigma_{ij}u_{i,j} - D_j\varphi_{,i} - \dot{\eta}_j\vartheta_{,i}\right)x_i + \frac{1}{2}\left(\sigma_{ij}u_i + D_j\varphi + s_j\vartheta\right)\right]n_j\,\mathrm{d}l \qquad (2.150)$$

where $\dot{\eta}_j = q_j/T_0, \vartheta = T - T_0$.

2.9.3 The Force Acting on a Defect in an Inhomogeneous Material

For an inhomogeneous material, L is dependent to $\boldsymbol{x}$, so $L = L(x_i, \psi_{\alpha,j}), \bar{\partial}/\partial x_i \neq 0$.

1. *Infinitesimal translation of* $\boldsymbol{x}$ *and generalized displacement.* $\delta x_i, \delta u_\alpha$ are also given in Eq. (2.142). The invariant condition under infinitesimal transformation is still $\varepsilon P_{ij,j}c_i = 0$, but

$$P_{ij,j} = \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)_{,j} = \frac{\bar{\partial} g}{\partial x_i} + \frac{\partial g}{\partial \psi_{\alpha,j}}\psi_{\alpha,ji} - \Sigma_{\alpha j}\psi_{\alpha,ij} = \frac{\bar{\partial} g}{\partial x_i}$$

So the integral in Eq. (2.143) in an inhomogeneous material becomes

$$\int_V P_{ij,j}\mathrm{d}V = \int_a \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)n_j\,\mathrm{d}a = \int_V \left(\bar{\partial} g/\partial x_i\right)\mathrm{d}V \qquad (2.151)$$

Though Eq. (2.151) is not a conservative integral, it still has important meaning. Eshelby (1956, 1975) pointed out that $P_{ij,j} - \bar{\partial} g/\partial x_i = 0$, so the negative derivative of the electric Gibbs function with $\boldsymbol{x}$, $-\bar{\partial} g/\partial x_i$, is the so-called material inhomogeneity force with the dimension of force.

2. *Infinitesimal expansion of coordinate and general displacement.* $\delta x_i, \delta u_\alpha$ are also given in Eq. (2.145). The invariant condition under infinitesimal transformation is still $\varepsilon\left[-(1/2)\Sigma_{\alpha j}\psi_\alpha + P_{ij}x_i\right]_{,j} = 0$, but

$$\left[-(1/2)\Sigma_{\alpha j}\psi_\alpha + P_{ij}x_i\right]_{,j} = P_{ij,j}x_i + P_{ij}x_{i,j} - (1/2)\Sigma_{\alpha j}\psi_{\alpha,j} = x_i\bar{\partial} g/\partial x_i$$

So the integral in Eq. (2.146) in an inhomogeneous material becomes

$$\int_a \left[\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)x_i - (1/2)\Sigma_{\alpha j}\psi_\alpha\right]n_j\,\mathrm{d}a = \int_V x_i\bar{\partial} g/\partial x_i\,\mathrm{d}V \qquad (2.152)$$

where $-x_i\bar{\partial} g/\partial x_i$ is the so-called material inhomogeneity moment.

3. *Infinitesimal rotation about the axis.* x_3 $\delta x_i, \delta u_\alpha$ are also given in Eq. (2.148). The invariant condition under the infinitesimal transformation is still expressed by $(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1)_{,k} = 0$, but

$$(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1)_{,k} = x_1\bar{\partial}g/\partial x_2 - x_2\bar{\partial}g/\partial x_1 \tag{2.153}$$

2.9.4 Conservation Integral in an Inhomogeneous Material

1. *Infinitesimal translation of coordinate and general displacement is related to an undetermined function.* Let

$$\delta x_i = \varepsilon c_i, \quad \delta\psi_\alpha = \varepsilon\Omega_\alpha \tag{2.154}$$

where c_i is a constant, Ω_α is an undetermined function, and ε is an infinitesimal parameter. Substituting Eq. (2.154) into Eq. (2.136) and noting $P_{ij,j} = (g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i})_{,j} = (\bar{\partial}g/\partial x_i)$, $\Sigma_{\alpha j,j} = 0$, we find

$$\Sigma_{\alpha j}\Omega_{\alpha,i} + c_i(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i})_{,j} = \Sigma_{\alpha j}\Omega_{\alpha,i} + c_i\bar{\partial}g/\partial x_i = 0 \tag{2.155}$$

2. *Infinitesimal translation of coordinate, infinitesimal translation, and expansion of general displacement.* Let

$$\delta x_i = \varepsilon x_i, \quad \delta\psi_\alpha = \varepsilon[-(1/2)\psi_\alpha + \Omega_\alpha] \tag{2.156}$$

where Ω_α is an undetermined function and ε is an infinitesimal parameter. Substituting Eq. (2.156) into (2.136) yields

$$\Sigma_{\alpha j}\Omega_{\alpha,j} + \left[P_{ij}x_i - (1/2)\Sigma_{\alpha j}\Omega_\alpha\right]_{,j} = 0 \tag{2.157}$$

From Eqs. (2.156) and (2.157), it is known that when material constants obey definite distribution and select appropriate c_i, Ω_α, the conservative integrals can be obtained; otherwise, the conservative integrals do not exist. In the following sections, some examples will be given to illustrate the above theory.

2.9.5 One-Directional Gradient Material

In engineering, the combined material is often used to improve the material behavior, such as on the substrate covering a surface heat-resisting layer to defense the high temperature environment. In order to reduce the stress on the interface between the substrate and heat-resisting layer, a transient layer constituted of the gradient material is added. One-directional gradient material is often used.

1. *Material constants varied as exponential function*. Assume material constants varied as $C^0_{ijkl}e^{\lambda x_1}, e^0_{kij}e^{\lambda x_1}, \epsilon^0_{ij}e^{\lambda x_1}$, where $C^0_{ijkl}, e^0_{kij}, \epsilon^0_{ij}, \lambda$ are all constants. Let

$$\delta x_1 = \varepsilon, \quad \delta x_2 = \delta x_3 = 0, \quad \delta\psi_\alpha = \varepsilon b\psi_\alpha, \quad \alpha = 1,2,3,4 \tag{2.158}$$

Substituting Eq. (2.158) into (2.136) yields

$$\left(\Sigma_{\alpha j}b\psi_\alpha + P_{1j}\right)_{,j} = b\Sigma_{\alpha j}\psi_{\alpha,j} + P_{1j,j} = b\Sigma_{\alpha j}\psi_{\alpha,j} + \bar{\partial}g/\partial x_1 = 0 \tag{2.159}$$

Because $\Sigma_{\alpha j}\psi_{\alpha,j} = 2g$, $\bar{\partial}g/\partial x_1 = \lambda g$, from Eq. (2.159), we get $2b + \lambda = 0$ or $b = -\lambda/2$. Substituting these results into Eq. (2.158) and then into Eq. (2.159) we get

$$\left(-\lambda\Sigma_{\alpha j}\psi_\alpha/2 + P_{1j}\right)_{,j} = 0 \tag{2.160}$$

Using the relation

$$\left(P_{1j} - \lambda\Sigma_{\alpha j}\psi_\alpha/2\right)_{,j} = \left(g\delta_{1j} - \Sigma_{\alpha j}\psi_{\alpha,1} - \lambda\Sigma_{\alpha j}\psi_\alpha/2\right)_{,j} = g_{,1} - \Sigma_{\alpha j}\psi_{\alpha,1j} - \lambda\Sigma_{\alpha j}\psi_{\alpha,j}/2$$

It is easy to get the path independence integral

$$\int_a \left(g\delta_{j1} - \Sigma_{\alpha j}\psi_{\alpha,1} - \lambda\Sigma_{\alpha j}\psi_\alpha/2\right)n_j \mathrm{d}a = 0 \tag{2.161}$$

2. *Material constants varied as power function*. Assume material constants varied as $C_{ijkl} = C^0_{ijkl}(1 + px_1)^q$, $e_{kij} = e^0_{kij}(1 + px_1)^q$, and $\epsilon_{ij} = \epsilon^0_{ij}(1 + px_1)^q$, where $C^0_{ijkl}, e^0_{kij}, \epsilon^0_{ij}, p, q$ are constants. Let

$$\delta x_i = \varepsilon(1 + px_i), \quad \delta\psi_\alpha = \varepsilon[(1 + p)\Omega_\alpha + p\psi_\alpha/2] \tag{2.162}$$

Substitution of Eq. (2.162) into Eq. (2.136) yields

$$\left\{\Sigma_{\alpha j}[(1 + p)\Omega_\alpha + p\psi_\alpha/2] + P_{ij}(1 + px_i)\right\}_{,j} = 0 \tag{2.163}$$

The relations between material constants are

$$\begin{aligned} &\frac{\partial C_{ijkl}}{\partial x_1} = \frac{pq}{1 + px_1}C_{ijkl}, \quad \frac{\partial e_{kij}}{\partial x_1} = \frac{pq}{1 + px_1}e_{kij}, \quad \frac{\partial \epsilon_{ij}}{\partial x_1} = \frac{pq}{1 + px_1}\epsilon_{ij} \\ &\frac{\bar{\partial}g}{\partial x_1} = \frac{pq}{1 + px_1}g, \quad \frac{\bar{\partial}g}{\partial x_2} = \frac{\bar{\partial}g}{\partial x_3} = 0 \end{aligned} \tag{2.164}$$

Substitution of Eq. (2.164) into Eq. (2.163) and using (2.164), we find

$$(1+p)\Sigma_{\alpha j}\Omega_{\alpha,j}+\frac{p}{2}\Sigma_{\alpha j}\psi_{\alpha,j}+(1+px_1)\frac{\bar{\partial}g}{\partial x_1}+pP_{ij}\frac{\partial x_i}{\partial x_j} \tag{2.165}$$

Using the relation $pP_{ij}\partial x_i/\partial x_j=-pg$ and Eq. (2.161), Eq. (2.165) is reduced to

$$(1+p)\Sigma_{\alpha j}\Omega_{\alpha,j}+pqg=0 \quad \Rightarrow \quad \Omega_\alpha=-\frac{pq}{2(1+p)}\psi_\alpha \tag{2.166}$$

Finally, the path independence integral is obtained:

$$\begin{gathered}\left[P_{ij}+px_iP_{ij}+(1/2)p(1-q)\Sigma_{\alpha j}\psi_\alpha\right]_{,j}=0\\ \int_a\left[(1+px_i)P_{ij}+(1/2)p(1-q)\Sigma_{\alpha j}\psi_\alpha\right]n_j\,\mathrm{d}a=0\end{gathered} \tag{2.167}$$

2.9.6 Transversely Isotropic Materials

For transversely isotropic materials, the infinitesimal transformation is taken as

$$\delta x_i=\varepsilon\omega_i(x_j), \quad \delta\psi_\alpha=\varepsilon W_\alpha(\psi_\beta)=\varepsilon A_{\alpha\beta}\psi_\beta \tag{2.168}$$

where ω_i is undetermined function, $A_{\alpha\beta}$ is an undetermined constant. Substitution of Eq. (2.168) into (2.136) yields

$$\omega_j\frac{\bar{\partial}g}{\partial x_j}+g\frac{\bar{\partial}\omega_j}{\partial x_j}+\Sigma_{\alpha j}\left(A_{\alpha\beta}\psi_{\beta,j}-\psi_{\alpha,i}\frac{\bar{\partial}\omega_i}{\partial x_j}\right)=0 \tag{2.169}$$

From Eq. (2.169), we obtain

$$K_{ijkl}u_{i,j}u_{k,l}+M_{kij}\varphi_{,k}u_{i,j}+N_{ij}\varphi_{,i}\varphi_{,j}=0 \tag{2.170}$$

where

$$\begin{aligned}&K_{ijkl}=(1/2)\left[\partial(\omega_nC_{ijkl})/\partial x_n\right]+A_{mi}C_{mjkl}-\omega_{j,m}C_{imkl}+A_{4i}e_{jkl}\\ &M_{kij}=\partial(\omega_ne_{kij})/\partial x_n+A_{44}e_{kij}+A_{mi}e_{kmj}-\omega_{j,m}e_{kim}-\omega_{k,m}e_{mij}+A_{m4}C_{mkij}-A_{4i}\epsilon_{jk}\\ &N_{ij}=-(1/2)\partial(\omega_n\epsilon_{ij})/\partial x_n-A_{44}\epsilon_{ij}+\omega_{i,m}\epsilon_{mj}+A_{m4}e_{imj}\end{aligned} \tag{2.171}$$

From Eq. (2.170) we have

$$K_{ijkl}+K_{klij}=0, \quad M_{kij}=0, \quad N_{ij}+N_{ji}=0 \tag{2.172}$$

As an example, we discuss the transversely isotropic piezoelectric ceramic (such as PZT). Assume x_3 is the poling direction, so x_1x_2 is the isotropic plane. Now the plane x_1x_3 is discussed. Appling Voigt notation, the constitutive equation is

$$\begin{gathered}
\{\sigma\} = [C]\{\varepsilon\} - [e]^{\mathrm{T}}\{E\}, \quad \{D\} = [\epsilon]\{E\} + [e]\{\varepsilon\} \\
[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2C_{66} \end{bmatrix}, \\
[e] = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad [\epsilon] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}
\end{gathered} \tag{2.173}$$

where $C_{11} = C_{1111}, C_{12} = C_{1122}, C_{13} = C_{1133}, C_{33} = C_{3333}, C_{44} = C_{1313}; e_{15} = e_{113}$, $e_{31} = e_{311}, e_{33} = e_{333}$.

Though the number of the undetermined constants in Eq. (2.168) is less than the number of the equation in Eq. (2.170), the undetermined constants can still be determined by special selection of constants. Finally, we get

$$\begin{aligned}
&\omega_1 = (b - A_{11})x_1 + A_{12}x_2 + C_1, \quad \omega_2 = -A_{12}x_1 + (b - A_{11})x_2 + C_2 \\
&\omega_3 = (b - A_{33})x_3 + C_3, \quad W_1 = A_{11}u_1 + A_{12}u_2, \quad W_2 = -A_{12}u_1 + A_{11}u_2, \\
&W_3 = A_{33}u_3 + A_{34}\varphi, \quad W_4 = A_{44}\varphi
\end{aligned} \tag{2.174}$$

where C_i is a new arbitrary constant. When coefficients in Eq. (2.168) take values given in Eq. (2.174), we can get a group of linear partial differential equation to determine the unknown coefficients by using the invariant conditions Eq. (2.172). This group linear partial differential equation is

$$\begin{aligned}
&\omega_n \frac{\partial C_{11}}{\partial x_n} + (2A_{11} - A_{33} + b)C_{11} = 0, \quad \omega_n \frac{\partial C_{33}}{\partial x_n} + (-2A_{11} + 3A_{33} + b)C_{33} = 0, \\
&\omega_n \frac{\partial C_{12}}{\partial x_n} + (2A_{11} - A_{33} + b)C_{12} = 0, \quad \omega_n \frac{\partial C_{13}}{\partial x_n} + (A_{33} + b)C_{13} = 0, \\
&\omega_n \frac{\partial C_{44}}{\partial x_n} + (A_{33} + b)C_{44} = 0, \quad \omega_n \frac{\partial \epsilon_{11}}{\partial x_n} + (2A_{44} - A_{33} + b)\epsilon_{11} - 2A_{34}e_{15} = 0, \\
&\omega_n \frac{\partial \epsilon_{33}}{\partial x_n} + (2A_{44} + A_{33} - 2A_{11} + b)\epsilon_{33} - 2A_{34}e_{33} = 0, \\
&\omega_n \frac{\partial e_{15}}{\partial x_n} + (A_{44} + b)e_{15} + A_{34}C_{44} = 0, \quad \omega_n \frac{\partial e_{31}}{\partial x_n} + (A_{44} + b)e_{31} + A_{34}C_{13} = 0, \\
&\omega_n \frac{\partial e_{33}}{\partial x_n} + (A_{44} + 2A_{33} - 2A_{11} + b)e_{33} + A_{34}C_{33} = 0
\end{aligned} \tag{2.175}$$

If Eq. (2.175) has solution, the infinitesimal transform given by Eq. (2.168) can be obtained. Substitution of Eq. (2.168) into Eq. (2.136) yields the conservative integral:

$$(P_{ij}\omega_i + \sigma_{ij}W_i + D_jW_4)_{,j} = 0, \qquad \int_a (P_{ij}\omega_i + \sigma_{ij}W_i + D_jW_4)n_j\,\mathrm{d}a = 0 \tag{2.176}$$

For a homogeneous material, Eq. (2.175) is reduced to linear equations and its solutions are

$$A_{34} = 0, \quad A_{11} = A_{33} = A_{44} \tag{2.177}$$

where $C_1, C_2, C_3, A_{12},$ and A_{11} are arbitrary constants. In this case, Eq. (2.176) is reduced to

$$\begin{aligned}
&\{C_1P_{ij} + C_2P_{2j} + C_3P_{3j} - 2A_{11}[P_{ij}x_1 - (1/2)(\sigma_{ij}u_i + D_j\varphi)] \\
&\quad + A_{12}(P_{1j}x_2 - P_{2j}x_1 + \sigma_{1j}u_2 - \sigma_{2j}u_1)\}_{,j} = 0, \\
&\int_a \{C_1P_{ij} + C_2P_{2j} + C_3P_{3j} + 2A_{11}\left[P_{ij}x_1 - \frac{1}{2}(\sigma_{ij}u_i + D_j\varphi)\right] \\
&\quad + A_{12}(P_{1j}x_2 - P_{2j}x_1 + \sigma_{1j}u_2 - \sigma_{2j}u_1)\}n_j\,\mathrm{d}a = 0
\end{aligned} \tag{2.178}$$

Equation (2.178) can be divided into five group independent conservative integrals due to the arbitrariness of constants. The independent conservative integrals corresponding to C_1, A_{11}, A_{12} are identical with Eqs. (2.143), (2.147), and (2.149). There are no new conservative integrals corresponding C_2, C_3 because $P_{2j,j} = P_{3j,j} = 0$ are the special cases of $P_{ij,j} = 0$.

References

Bustamante R, Dorfmann A, Ogden RW (2008) On variational formulations in nonlinear magnetoelastostatics. Math Mech Solids 13:725–745

Delph TJ (1982) Conservation laws in linear elasticity based upon divergence transformations. J Elast 12:385–393

Eringen AC, Maugin GA (1989) Electrodynamics of continuua. Springer-Verlag, New York

Eshelby JD (1956) The continuum theory of lattice defects. Solid State Phys 3:79–144// Edited by Seitz and TuruBull

Eshelby JD (1975) The elastic energy-momentum tensor. J Elast 5:321–335

Fletcher DC (1976) Conservation laws in linear elastodynamics. J Ration Mech Anal 60:330–353

Genin J, Xu W (1999) Thermoelastic plastic metals with mass diffusion. Z Angew Math Phys 50:511–528

Guillot FM, Jarzynski J, Balizer JE (2001) Measurement of electrostrictive coefficients of polymer films. J Acoust Soc Am 110:2980–2990

Honein T, Herrmann G (1997) Conservation laws in nonhomogeneous plane elastostatics. J Mech Phys Solids 45:789–805

Hu HC (1981) Variational principle and its applications in elasticity. Science Press, Beijing

Jiang Q, Kuang Z-B (2003) Stress analysis of electrostrictive material with an elliptic defect. Sci China Ser G 46:492–500

Jiang Q, Kuang Z-B (2004) Stress analysis in two dimensional electrostrictive material with an elliptic rigid conductor. Eur J Mech A/Solids 23:945–956

Kuang Z-B (1964) General variational principle and the theory of thin plate. Scientific Report of Xian Jiaotong University, Xian (in Chinese)

Kuang Z-B (2002) Nonlinear continuum mechanics. Shanghai Jiaotong University Press, Shanghai (in Chinese)

Kuang Z-B (2007) Some problems in electrostrictive and magnetostrictive materials. Acta Mech Solida Sin 20:219–227

Kuang Z-B (2008a) Some variational principles in elastic dielectric and elastic magnetic materials. Eur J Mech A/Solids 27:504–514

Kuang Z-B (2008b) Some variational principles in electroelastic media under finite deformation. Sci China Ser G 51:1390–1402

Kuang Z-B (2009a) Internal energy variational principles and governing equations in electroelastic analysis. Int J Solids Struct 46:902–911

Kuang Z-B (2009b) Variational principles for generalized dynamical theory of thermopiezoelectricity. Acta Mech 203:1–11

Kuang Z-B (2010) Variational principles for generalized thermodiffusion theory in pyroelectricity. Acta Mech 214:275–289

Kuang Z-B (2011a) Theory of electroelasticity. Shanghai Jiaotong University Press, Shanghai (in Chinese)

Kuang Z-B (2011b) Physical variational principle and thin plate theory in electro-magneto-elastic analysis. Int J Solids Struct 48:317–325

Kuang Z-B (2011c) Some thermodynamic problems in continuum mechanics. In: Piraján JCM (ed) Thermodynamics – kinetics of dynamic systems. Intech Open Access Publisher, Croatia

Kuang Z-B (2012) Static electric force and measurement principle of material constants in electrostrictive material. Smart Mater Res. Article ID 712103, 8 p, doi:10.1155/2012/712103

Kuang Z-B (2013) Two theoretical problems in electro-magneto-elastic analysis. Acta Mech 224:1213–1224

Landau LD, Lifshitz EM, Pitaevskii LP (1984) Electrodynamics of continuum media, 2nd edn. Pergamon Press, Oxford; Ландау Л Д, Лифшиц Е М. Электродинамика сплошных сред. Москва: ФИЗМАТГИЗ, 1959

Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam

Mcmeeking RM, Landis CM (2005) Electrostatic forces and stored energy for deformable dielectric materials. J Appl Mech 72:581–590

Mcmeeking RM, Landis CM, Jimenez A (2007) Principle of virtual work for combined electrostatic and mechanical loading of materials. Int J Nonlinear Mech 42:831–838

Moon EC (1984) Magneto-solid mechanics. A Wiley-Interscience Publication, John Wiley & Sons. New York, Chichester, Brisbane, Toronto, Singapore

Nother E (1918) Invariant variations problem. Göttinger Nachrichten Mathematisch-physicalische Klasse 2:235–253

Pak YE (1992) Linear electro-elastic fracture mechanics of piezoelectric materials. Int J Fract 54:79–100

Pao YH (1978) Electromagnetic forces in deformable continua // Nemat-nasser. Mechanics today. Bath: printed in Great Britain by Pitman

Sherief HH, Hamza FA, Saleh HA (2004) The theory of generalized thermoelastic diffusion. Int J Eng Sci 42:591–608

Shi W-C, Kuang Z-B (2003) Conservation laws in non-homogeneous electro-magneto-elastic materials. Eur J Mech A/Solids 22:217–230

Shkel YM, Klingenberg DJ (1996) Material parameters for electrostriction. J Appl Phys 80:4566–4571

Stratton JA (1941) Electromagnetic theory. McGraw-Hill, New York

Wang J-W, Kuang Z-B (2001) The conservative law of the non-conservative dynamic system in the pyroelectric material. Q Mech 22:154–161 (in Chinese)

Wang X-M, Shen Y-P (1996) The conservation laws and path-independent integrals with an application for linear electro-magneto-elastic media. Int J Solids Struct 33:865–878

Zhang QM, Su J, Kim CH (1997) An experimental investigation of electromechanical responses in a polyurethane elastomer. J Appl Phys 81:2770–2776