Chapter 1
Preliminary Knowledge and Continuum Thermodynamics

Abstract In this chapter, some basic knowledge of elastic theory, electrodynamics, and thermodynamics which will be applied in this book are introduced. Some extensions in continuum thermodynamics are proposed. It is shown that together with the first law of thermodynamics, a physical variational principle (PVP) is also held. The physical variational principle gives a true process for all virtual possible process satisfying the geometrical constrained conditions. The physical variational principle is considered to be one of the fundamental physical principles for quasi-static system, which can be used to derive governing equations in continuum mechanics and other fields. When the temperature varies with time, the inertial entropy or inertial heat theory is proposed. This theory is consistent with current classical thermodynamic theory. From this theory, the temperature wave equation with finite phase velocity is derived in a very simple fashion. It is shown that the time arrived to equilibrium of the temperature is about 1 ns $\sim$ 1 ps when an internal heat source with a Heaviside step heat function is applied.

Keywords Basic knowledge • Physical variational principle • Inertial entropy

1.1 Background

Jacques and Pierre Curie brothers discovered the piezoelectric effect in 1880 (Sun and Zhang 1984; Ikeda 1990). They found out that a mechanical stress applied on crystals such as tourmaline, quartz, and Rochelle salt could produce electrical charges, and the voltage was proportional to the stress. Piezoelectric can also work in reverse, generating a strain by the application of an electric field. Centrosymmetric classes of crystals are always not piezoelectric, but a few kinds of crystals are still not piezoelectric though lacking a center of symmetry. The pyroelectric effect was found in eighteenth century (Lang 2005), earlier than piezoelectric effect. Most ferroelectric crystals are strongly piezoelectric and pyroelectric. First applications were piezoelectric ultrasonic submarine detector and quartz clocks

Z.-B. Kuang, *Theory of Electroelasticity*, DOI 10.1007/978-3-642-36291-0_1,

during the First World War. After the Second World War, many new piezoelectric and pyroelectric materials have been discovered in succession, such as $BaTiO_3$, $Pb(Ti.Zr)O_3$-PZT, KDP, PMN, $LiNbO_3$, and $LiTaO_3$. In present time, it has been successfully used in various areas, such as in aerospace, transportation, nuclear, and medical.

It is different with the piezoelectric materials, all of the crystals, especially the isotropic electrostrictive materials, have the electrostrictive effect.

The fundamental phenomenological theory of the piezoelectricity was established by Kelvin (1856), Voigt (1910), etc. In the current time, due to the intrinsic mechanical-electric coupling effects, piezoelectric materials have been widely used in engineering structures to detect the responses of the structure by measuring the electric charge (sensing) or to reduce excessive responses by applying additional electric forces or thermal forces (actuating). By integrating the sensing and actuating, it is possible to create the so-called intelligent structures and systems that can adapt to or correct for changing operating condition. Due to its intrinsic electromechanical coupling behavior and its reliability in performance, the electroelastic analysis is necessary and has been paid much attention. A lot of literatures have appeared in journals and books. Here we cannot review all of these literatures, but reader can find more literatures from our cited papers.

The foundations of the electroelastic analyses are the Newton's law, Maxwell electrodynamics equations, Lorentz's law, and constitutive equations of materials. In electrically nonlinear case, different authors give different governing equations in electroelastic analyses. In this book, we give a simple theory to discuss simpler electrically nonlinear problem in engineering.

Using the continuum thermodynamics, it is found that the first law of thermodynamics contains the physical variational principle, which can be used as a fundamental natural principle to derive the governing equations in physics and continuum mechanics. We also proposed the inertial entropy theory due to the heat inertia. The temperature wave equation with a finite propagation velocity can be derived easily from this theory. A failure theory based on the energy principle is proposed in this book, which can also be used in other area, such as in phase transformation theory. Many works on the static and dynamic generalized stress analyses in piezoelectric and electrostrictive materials with defects, the surface wave propagation, and the failure theory are also discussed in this book.

1.2 Foundations of Classical Electrodynamics

1.2.1 Constitutive (or State) Equations

There are many books that discussed the electrodynamics (Landau et al. 1984; Stratton 1941; Cai and Zhu 1985; Moon 1984) and the electric engineering (Kruck 1954). Here, a short discussion is given only.

The constitutive (or state) equations can be written in the following form:

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, T), \quad \boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}, T), \quad \boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E}, T) \tag{1.1a}$$

where $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B},$ and T are the electric field intensity, electric displacement or electric flux density, magnetic field intensity, magnetic induction or the magnetic flux density, and the temperature, respectively; $\boldsymbol{J}$ is the total electric current density. When $\boldsymbol{E}$ and $\boldsymbol{H}$ are small, the linear form is used for isothermal case:

$$\begin{aligned} &\boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E} = \epsilon_0 \cdot \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H} = \mu_0 \cdot (\boldsymbol{H} + \boldsymbol{M}) \\ &\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_e + \boldsymbol{J}_v, \quad \boldsymbol{J}_s = \boldsymbol{\gamma} \cdot \boldsymbol{E}_{\text{ext}}, \quad \boldsymbol{J}_e = \boldsymbol{\gamma} \cdot \boldsymbol{E}, \quad \boldsymbol{J}_v = \boldsymbol{\gamma} \cdot (\boldsymbol{v} \times \boldsymbol{B}) \end{aligned} \tag{1.1b}$$

where $\boldsymbol{J}_s$, $\boldsymbol{J}_e$, and $\boldsymbol{J}_v$ are the given external exciting current density, the induction or eddy current density, and motional electric current density, respectively; $\boldsymbol{\epsilon}$ is the permittivity, $\boldsymbol{\mu}$ the permeability, $\boldsymbol{\gamma}$ the electric conductivity of a material, respectively, ϵ_0 and μ_0 are values of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in the vacuum or air. $\boldsymbol{E}_{\text{ext}}$ is an external field; $\boldsymbol{v}$ is the velocity of a moving body. $\boldsymbol{P}$ and $\boldsymbol{M}$ are the polarization density and magnetization density, respectively.

1.2.2 Conservation Law of Charge

The conservation law of charge is

$$\nabla \cdot \boldsymbol{J} = -\dot{\rho}_{\text{e}} \tag{1.2}$$

where ρ_{e} is the free electric charge density. A superimposed dot indicates partial differentiation with respect to time, i.e., $(\dot{)} = \partial()/\partial t$, such as $\dot{\rho}_{\text{e}} = \partial \rho_{\text{e}}/\partial t$.

1.2.3 The Lorentz Force

For a continuous charge distribution in motion, the Lorentz force equation is

$$\boldsymbol{f} = \rho_{\text{e}}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \quad \text{or} \quad \boldsymbol{f} = \rho_{\text{e}}\boldsymbol{E} + \boldsymbol{J}_{\text{e}} \times \boldsymbol{B}; \quad \boldsymbol{J}_{\text{e}} = \rho_{\text{e}}\boldsymbol{v} \tag{1.3a}$$

where $\boldsymbol{f}$ is the force density (force per unit volume) and $\boldsymbol{J}_{\text{e}}$ is the current density. Equation (1.3a) can be extended to the electromagnetic media and approximately expressed as (Pao 1978; Kuang 2011a)

$$\begin{aligned} &\boldsymbol{f} = \rho_{\text{t}}\boldsymbol{E} + \boldsymbol{J}_{\text{t}} \times \boldsymbol{B}; \quad \rho_{\text{t}} = \rho_{\text{e}} + \rho_P, \quad \rho_P = -\nabla \cdot \boldsymbol{P} \\ &\boldsymbol{J}_{\text{t}} = \boldsymbol{J} + \boldsymbol{J}_P + \boldsymbol{J}_M; \quad \boldsymbol{J}_P = \partial \boldsymbol{P}/\partial t = \dot{\boldsymbol{P}}, \quad \boldsymbol{J}_M = \nabla \times \boldsymbol{M} \end{aligned} \tag{1.3b}$$

where ρ_t is the total electric charge density constituted of free and polarized charges and $\boldsymbol{J}_t$ is the total electric current density constituted of conduction, polarized, and magnetization current densities.

1.2.4 Maxwell Equations

The differential and integral Maxwell equations are as follows:

$$\begin{aligned} &\nabla \cdot \boldsymbol{D} = \rho_e & &\oint_a D \cdot \mathrm{d}\boldsymbol{a} = \int_V \rho_e \,\mathrm{d}V \\ &\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} & &\oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t}\int_a \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \\ &\nabla \cdot \boldsymbol{B} = 0 & &\int_a \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = 0\left(\int_V \rho_m \,\mathrm{d}V\right) \\ &\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} & &\oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \int_a \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} + \frac{\partial}{\partial t}\int_a \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} \end{aligned} \tag{1.4}$$

where V is the volume occupied by the medium; $\boldsymbol{a}$ is the area vector and a is its absolute value; $\boldsymbol{s}$ is a line element vector of a curve C; ∇ is a differential operator vector.

Taking the divergence of the second and the divergence of the fourth in Eq. (1.4) and using the law of conservation of charge we find respectively,

$$\begin{aligned} &\nabla \cdot \partial \boldsymbol{B}/\partial t = \partial(\nabla \cdot \boldsymbol{B})/\partial t = 0, \\ &\nabla \cdot \boldsymbol{J} + \nabla \cdot \partial \boldsymbol{D}/\partial t = \nabla \partial(\nabla \cdot \boldsymbol{D} - \rho_e)/\partial t = 0 \end{aligned} \tag{1.5}$$

If $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{D} - \rho_e = 0$ at the initial state, they will be held at any time, which are just the third and the first equations in Eq. (1.4). Therefore, the independent equations are the second and the fourth equations in Eq. (1.4) and the charge conservation equation in Eq. (1.2), or other combination.

1.2.5 Electric Scalar Potential and Magnetic Vector Potential

The second and third equations are satisfied automatically if we introduce an electric scalar potential φ and a magnetic vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{E} = -\nabla\varphi - \partial \boldsymbol{A}/\partial t = -\nabla\varphi - \dot{\boldsymbol{A}}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.6}$$

Using the constitutive equation (1.1b) with constant $\boldsymbol{\epsilon} = \epsilon\mathbf{I}$, $\boldsymbol{\mu} = \mu\mathbf{I}$, $\gamma = \gamma\mathbf{I}$ and the relation $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$, the first and forth equations in Maxwell equations are reduced to

$$\begin{aligned} &\nabla^2\varphi + \partial(\nabla\cdot\boldsymbol{A})/\partial t = -\rho_{\mathrm{e}}/\epsilon \\ &\nabla^2\boldsymbol{A} - \mu\,\epsilon\partial^2\boldsymbol{A}/\partial t^2 - \nabla(\nabla\cdot\boldsymbol{A} + \mu\epsilon\partial\varphi/\partial t) = -\mu\boldsymbol{J} \end{aligned} \tag{1.7}$$

In order to define $\boldsymbol{A}$ uniquely, a supplement gauge condition must be given. Introducing Lorenz gauge $\nabla\cdot\boldsymbol{A} + \mu\cdot\epsilon\partial\varphi/\partial t = 0$, Maxwell equations can be written compactly in the form:

$$\begin{aligned} &\nabla^2\varphi - \mu\,\epsilon\partial^2\varphi/\partial t^2 = -\rho_{\mathrm{e}}/\epsilon \\ &\nabla^2\boldsymbol{A} - \mu\,\epsilon\partial^2\boldsymbol{A}/\partial t^2 = -\mu\boldsymbol{J} \end{aligned} \tag{1.8}$$

1.2.6 Quasi-Stationary Electromagnetic Field

If $\partial\boldsymbol{D}/\partial t$ in Maxwell equations can be neglected, the field is called the quasi-stationary magnetic (MQS) field, and in this case, all radiation effects can be negligible. It is also called the eddy current field problem and is important in the electric machine engineering. If $\partial\boldsymbol{B}/\partial t$ in Maxwell equations can be neglected, the field is called quasi-stationary electric (EQS) field which is less important in engineering. For an eddy current field,

$$\partial\boldsymbol{D}/\partial t = -\epsilon\left(\partial^2\boldsymbol{A}/\partial t^2 + \nabla\partial\varphi/\partial t\right) = \boldsymbol{0}, \quad \boldsymbol{J} = \gamma\boldsymbol{E} + \boldsymbol{J}_s + \boldsymbol{J}_v \tag{1.9}$$

so Eq. (1.7) becomes

$$\begin{aligned} &\nabla^2\varphi + \partial(\nabla\cdot\boldsymbol{A})/\partial t = -\rho_{\mathrm{e}}/\epsilon \\ &\nabla^2\boldsymbol{A} - \mu\gamma(\partial\boldsymbol{A}/\partial t + \nabla\varphi) - \nabla(\nabla\cdot\boldsymbol{A}) = -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v) \end{aligned} \tag{1.10}$$

Introducing conductivity gauge $\nabla\cdot\boldsymbol{A} + \mu\gamma\varphi = 0$, Eq. (1.10) is reduced to

$$\begin{aligned} &\nabla^2\varphi - \mu\gamma\partial\varphi/\partial t = -\rho_{\mathrm{e}}/\epsilon \\ &\nabla^2\boldsymbol{A} - \mu\gamma\partial\boldsymbol{A}/\partial t = -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v) \end{aligned} \tag{1.11}$$

Introducing Coulomb gauge $\nabla\cdot\boldsymbol{A} = 0$, Eq. (1.10) is reduced to

$$\begin{aligned} &\nabla^2\varphi = -\rho_{\mathrm{e}}/\epsilon \\ &\nabla^2\boldsymbol{A} - \mu\gamma\partial\boldsymbol{A}/\partial t - \mu\gamma\nabla\varphi = -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v) \end{aligned} \tag{1.12}$$

In current sources and stator laminations, eddy currents are usually neglected. For a constant magnetic field $\boldsymbol{J} = \boldsymbol{J}_s$, $\boldsymbol{J}_e = \boldsymbol{0}$, $\boldsymbol{J}_v = \boldsymbol{0}$, Eq. (1.12) becomes $\nabla^2\boldsymbol{A} = -\mu\boldsymbol{J}_s$.

The finite element analysis shows that the results of calculation sometimes are not fully satisfactory when a supplement gauge condition is used.

When $L/c\tau \ll 1$, then $\partial \boldsymbol{D}/\partial t$ and $\partial \boldsymbol{B}/\partial t$ can all be neglected and we call this field as quasi-static electromagnetic field, where $c = 1\big/\sqrt{\mu\epsilon}$ is the optic velocity in media, L is the maximum size of the body, and τ is the concerned time interval. Neglecting $\partial \boldsymbol{D}/\partial t$ and $\partial \boldsymbol{B}/\partial t$, the electric and magnetic fields are independent from each other, so the electric field and magnetic field can be solved independently. When material constants are all constant for static case, the Maxwell equations is reduced to

$$\nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{E}) = \rho_{\mathrm{e}}, \quad \nabla \times (\boldsymbol{\mu}^{-1} \cdot \boldsymbol{B}) = \boldsymbol{J} \tag{1.13}$$

For the static electric field, we can always introduce an electric potential or potential φ. For the case without electric current, i.e., $\boldsymbol{J} = \boldsymbol{0}$, in material, the static magnetic field can also be expressed by a scalar magnetic potential ψ. In this case, we have

$$\begin{aligned} &\boldsymbol{E} = -\nabla\varphi, \quad \nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{E}) = \rho_{\mathrm{e}}, \quad \epsilon_{kl}\varphi_{,lk} = \rho_{\mathrm{e}} \\ &\boldsymbol{H} = -\nabla\psi, \quad \nabla \cdot (\boldsymbol{\mu} \cdot \boldsymbol{H}) = 0, \quad \mu_{kl}\psi_{,lk} = 0 \end{aligned} \tag{1.14}$$

The electromagnetic energy $\mathfrak{A}$ and its Legendre transformation, the electromagnetic Gibbs free energy g, stored in the medium are

$$\mathrm{d}\mathfrak{A} = \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{D} + \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{B}, \quad \mathrm{d}g = \mathrm{d}\mathfrak{A} - \mathrm{d}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) = -\boldsymbol{D} \cdot \mathrm{d}\boldsymbol{E} - \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{H} \tag{1.15}$$

1.2.7 Interface Connective (or Continuity), Boundary, and Initial Conditions

The interface connective conditions of $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}$ of electromagnetic media 1 and 2 are

$$\begin{aligned} &(\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \boldsymbol{n} = \sigma_{\mathrm{s}} \quad \text{or} \quad D_{2n} - D_{1n} = \sigma_{\mathrm{s}} \\ &(\boldsymbol{E}_2 - \boldsymbol{E}_1) \cdot \boldsymbol{n} = (\sigma_{\mathrm{s}} + \sigma_{\mathrm{sp}})/\epsilon_0, \quad \sigma_{\mathrm{sp}} = -(\boldsymbol{P}_2 - \boldsymbol{P}_1) \cdot \boldsymbol{n} \end{aligned} \tag{1.16}$$

$$\begin{aligned} &(\boldsymbol{B}_2 - \boldsymbol{B}_1) \cdot \boldsymbol{n} = 0 \quad \text{or} \quad B_{2n} - B_{1n} = 0 \\ &(\boldsymbol{H}_2 - \boldsymbol{H}_1) \cdot \boldsymbol{n} = -(\boldsymbol{M}_2 - \boldsymbol{M}_1) \cdot \boldsymbol{n}/\mu_0 \end{aligned} \tag{1.17}$$

$$\begin{aligned} &\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_{\mathrm{s}}, \quad \text{or} \\ &\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \mu_0(\boldsymbol{J}_{\mathrm{s}} + \boldsymbol{J}_{\mathrm{sm}}), \quad \boldsymbol{J}_{\mathrm{sm}} = \boldsymbol{n} \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) \end{aligned} \tag{1.18}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = -\nabla(\pi_{\mathrm{s}}/\epsilon) \tag{1.19}$$

In Eqs. (1.16), (1.17), (1.18), and (1.19), $\boldsymbol{n}$ is the normal of the material 1, σ_{sp} is the surface polarization charge density, $\boldsymbol{J}_{\mathrm{sm}}$ is the surface magnetization electric current density, and $\pi_s = \sigma_s d$ is the electric couple density on the interface. There are only two independent interface conditions in Eqs. (1.16), (1.17), (1.18), and (1.19). If material 2 does not exist, let $\boldsymbol{D}_2 = \boldsymbol{E}_2 = \boldsymbol{B}_2 = \boldsymbol{H}_2 = \mathbf{0}$; the boundary conditions can be obtained from Eqs. (1.16), (1.17), (1.18), and (1.19).

On the interface, the conservation condition of the electric current is

$$(\boldsymbol{J}_2 - \boldsymbol{J}_1) \cdot \boldsymbol{n} = -\partial \sigma_s / \partial t = -\dot{\sigma}_s \tag{1.20}$$

The initial conditions are

$$\begin{aligned} &\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(0), \quad \dot{\boldsymbol{E}}(\boldsymbol{x},0) = \dot{\boldsymbol{E}}_0(0), \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(0), \\ &\dot{\boldsymbol{H}}(\boldsymbol{x},0) = \dot{\boldsymbol{H}}_0(0), \quad \boldsymbol{x} \in V \end{aligned} \tag{1.21}$$

In Eq. (1.21), there are only still two independent conditions.

1.2.8 Electromagnetic Force

Multiplying the second equation in Eq. (1.4) by $\boldsymbol{D}$ and the fourth by $\boldsymbol{B}$, then adding the results we obtain

$$\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{D} \times \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) + \frac{\partial \boldsymbol{D}}{\partial t} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{B} = \mathbf{0} \tag{1.22}$$

Using

$$\begin{aligned} &\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) = (\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D} - (\boldsymbol{D} \cdot \nabla)\boldsymbol{E}, \quad \nabla \cdot (\boldsymbol{D} \otimes \boldsymbol{E}) = (\nabla \cdot \boldsymbol{D})\boldsymbol{E} + (\boldsymbol{D} \cdot \nabla)\boldsymbol{E} \\ &(\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D} + (\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} = \nabla \cdot [(\boldsymbol{E} \cdot \boldsymbol{D})\mathbf{I}], \quad \mathbf{I} = \delta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned}$$

and the similar relations for $\boldsymbol{B}, \boldsymbol{H}$, Eq. (1.22) is reduced to

$$\begin{aligned} &\nabla \cdot [(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\mathbf{I} - (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H})] + \frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) \\ &\quad = -\rho_e \boldsymbol{E} - \boldsymbol{J} \times \boldsymbol{B} + (\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} + (\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{H} \end{aligned} \tag{1.23}$$

where the notation $\otimes$ is the tensor product, $\boldsymbol{A} \otimes \boldsymbol{B} = A_i B_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, and $\boldsymbol{e}_i$ is the unit vector on coordinate axis x_i. Using the conservation law of the electric charge, Eq. (1.23) can be written in the form of the electromagnetic momentum equation:

$$\begin{aligned} &\boldsymbol{f}^{\mathrm{M}} = \nabla \cdot \sigma^{\mathrm{M}} - \partial \mathbf{g}^{\mathrm{M}} / \partial t; \qquad \mathbf{g}^{\mathrm{M}} = \boldsymbol{D} \times \boldsymbol{B} \\ &\sigma^{\mathrm{M}} = (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H}) - (1/2)(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\mathbf{I} \\ &\boldsymbol{f}^{\mathrm{M}} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - (1/2)[\,(\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} - (\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D}] \\ &\qquad - (1/2)[(\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{H} - (\nabla \otimes \boldsymbol{H}) \cdot \boldsymbol{B}] \end{aligned} \tag{1.24}$$

where $\sigma^{\mathrm{M}}, \boldsymbol{f}^{\mathrm{M}}$, and $\mathbf{g}^{\mathrm{M}}$ are called the Maxwell stress tensor, electromagnetic body force, and electromagnetic momentum density, respectively.

If $\boldsymbol{D} = \epsilon_0 \cdot \boldsymbol{E} + \boldsymbol{P},\ \boldsymbol{B} = \mu_0 \cdot (\boldsymbol{H} + \boldsymbol{M})$ are used, Maxwell equations become

$$\begin{aligned}
&\epsilon_0 \nabla \cdot \boldsymbol{E} = \rho_{\mathrm{e}} - \nabla \cdot \boldsymbol{P}, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \cdot \boldsymbol{B} = 0, \\
&\mu_0^{-1} \nabla \times \boldsymbol{B} = \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M}
\end{aligned} \tag{1.25}$$

Analogous to the derivation of Eq. (1.24), we get

$$\begin{aligned}
\boldsymbol{f}'^{\mathrm{M}} &= \nabla \cdot \boldsymbol{\sigma}'^{\mathrm{L}} - \partial \mathbf{g}'^{\mathrm{L}} / \partial t, \quad \mathbf{g}'^{\mathrm{L}} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} \\
\boldsymbol{\sigma}'^{\mathrm{L}} &= (\epsilon_0 \cdot \boldsymbol{E} \otimes \boldsymbol{E} + \mu_0^{-1} \boldsymbol{B} \otimes \boldsymbol{B}) - (1/2)(\epsilon_0 E^2 + \mu_0^{-1} B^2)\mathbf{I} \\
\boldsymbol{f}'^{\mathrm{M}} &= \rho_{\mathrm{e}} \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - (\nabla \cdot \boldsymbol{P}) + (\partial \boldsymbol{P} / \partial t + \nabla \otimes \boldsymbol{M}) \times \boldsymbol{B} = \rho_{\mathrm{t}} \boldsymbol{E} + \boldsymbol{J}_{\mathrm{t}} \times \boldsymbol{B}
\end{aligned} \tag{1.26}$$

Using above method it is found that the Maxwell stress and electromagnetic body forces are not unique (Pao 1978). The reason may be that boundary conditions are not considered. The electromagnetic momentum equation can also be discussed by the macroscopic Lorentz force method in Sect. 1.2.3. Let a dielectric medium occupies a volume V enclosed by its surface a. Noting that on the surface there are polarized electric surface density $\boldsymbol{n} \cdot \boldsymbol{P}$ and magnetic current surface density $-\boldsymbol{n} \times \boldsymbol{M}$, so the force acted on the body is

$$\begin{aligned}
\boldsymbol{F}'' = &\int_V [(\rho_{\mathrm{e}} - \nabla \cdot \boldsymbol{P})\boldsymbol{E} + (\boldsymbol{J} + \partial \boldsymbol{P} / \partial t + \nabla \times \boldsymbol{M}) \times \boldsymbol{B}] \mathrm{d}V \\
&+ \int_a [(\boldsymbol{n} \cdot \boldsymbol{P})\boldsymbol{E} - (\boldsymbol{n} \times \boldsymbol{M}) \times \boldsymbol{B}] \mathrm{d}a
\end{aligned} \tag{1.27}$$

After some manipulations, we get

$$\begin{aligned}
\boldsymbol{f}''^{\mathrm{M}} &= \nabla \cdot \boldsymbol{\sigma}''^{\mathrm{M}} - \partial \mathbf{g}''^{\mathrm{M}} / \partial t, \quad \mathbf{g}''^{M} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} \\
\boldsymbol{\sigma}''^{\mathrm{M}} &= (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H}) - (1/2)(\epsilon_0 E^2 + \mu_0^{-1} B^2 - 2\boldsymbol{M} \times \boldsymbol{B})\mathbf{I} \\
\boldsymbol{f}''^{\mathrm{M}} &= \rho_{\mathrm{e}} \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{P} \cdot (\nabla \otimes \boldsymbol{E}) + (\nabla \otimes \boldsymbol{B}) \times \boldsymbol{M} + \partial \boldsymbol{P} / \partial t \times \boldsymbol{B}
\end{aligned} \tag{1.28}$$

Because the macroscopic Lorentz force is related to the polarization and magnetization of the material, so many different formulas can be got. Eq. (1.28) did not strictly get from complete governing equations. Maugin (1988) considered that in order to avoid arbitrary choice of the ponderomotive force and couple in the electromagnetic contributions, he intend to arrive at expressions for these contributions on the basis of a microscopic model, the electron theory of Lorentz (Eringen and Maugin 1989). In electroelastic analyses only the static electromagnetic force will be discussed by the physical variational principle, and it will be discussed in the next chapter. In this book, the theory concerned with the photon motion is not considered.

1.3 Some Preliminary Knowledge in Electroelasticity

1.3.1 Universal Governing Equations

Universal governing equations must be obeyed by all moving or deforming continuum (Pao 1978; Kuang 2002). In electroelasticity, the universal governing equations are:

1. Mass conservation equation

$$\partial\rho/\partial t+\nabla\cdot(\rho\boldsymbol{v})=\dot{\rho}+\rho\nabla\cdot\boldsymbol{v}=0,\quad \dot{\rho}+\rho v_{k,k}=0,\quad \dot{\rho}=\partial\rho/\partial t+v_k\rho_{,k} \tag{1.29}$$

where ρ is the mass density, $\boldsymbol{v}$ is the velocity.

2. Linear momentum equation

$$\nabla\cdot\boldsymbol{\sigma}+(\mathbf{f}^{\mathrm{m}}+\mathbf{f}^{\mathrm{e}})=\rho\dot{\boldsymbol{v}},\quad \sigma_{ij,i}+\left(f_j^{\mathrm{m}}+f_j^{\mathrm{e}}\right)=\rho\dot{v}_j \tag{1.30}$$

where $\mathbf{f}^{\mathrm{m}}$ and $\mathbf{f}^{\mathrm{e}}$ are the mechanical and electromagnetic forces per volume.

3. Angular momentum equation

$$\mathbf{e}:\boldsymbol{\sigma}+\mathbf{C}^{\mathrm{e}}=0,\quad e_{kij}\sigma_{ij}+C_k^{\mathrm{e}}=0 \tag{1.31}$$

where $\boldsymbol{C}^{\mathrm{e}}=\boldsymbol{P}\times\boldsymbol{E}+\mu_0\boldsymbol{M}\times\boldsymbol{H}$ is the body couple density per volume. The asymmetric part of the stress is induced by the polarization and magnetization in electromagnetic material. From Eq. (1.31), it is also found that the asymmetric part of the stress is the second-order effect of the electromagnetic field. Let the symmetric part of the stress $\boldsymbol{\sigma}$ be denoted by $\boldsymbol{\sigma}^{\mathrm{s}}$, the asymmetric part by $\boldsymbol{\sigma}^{\mathrm{a}}$, then we get

$$\begin{aligned}&\sigma_{kl}=\sigma_{kl}^{\mathrm{s}}+\sigma_{kl}^{\mathrm{a}},\quad \sigma_{lk}^{\mathrm{s}}=(\sigma_{kl}+\sigma_{lk})/2,\\&\sigma_{kl}^{\mathrm{a}}=(\sigma_{kl}-\sigma_{lk})/2=(E_kP_l-E_lP_k+\mu_0H_kM_l-\mu_0H_lM_k)/2\end{aligned} \tag{1.32}$$

1.3.2 Three-Dimensional Governing Equations in Elasticity with Small Deformation

In this chapter, only the case with symmetric stresses is discussed. Let $\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{f}$ be the displacement, stress, strain, and body force per volume, we have (Ogden 1984; Kuang 2002)

$$\text{Geometric equation}\quad \varepsilon_{ij}=(u_{i,j}+u_{j,i})/2,\quad \boldsymbol{\varepsilon}=(\nabla\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\nabla)/2 \tag{1.33}$$

$$\text{Momentum equation}\quad \sigma_{ij,j}+f_i=\rho\ddot{u}_i,\quad \nabla\boldsymbol{\sigma}+\boldsymbol{f}=\rho\ddot{\boldsymbol{u}} \tag{1.34}$$

$$\text{Constitutive equation}\quad \sigma_{ij}=C_{ijkl}\varepsilon_{kl},\quad \boldsymbol{\sigma}=\boldsymbol{C}:\boldsymbol{\varepsilon};\quad \boldsymbol{\varepsilon}=\boldsymbol{s}:\boldsymbol{\sigma} \tag{1.35}$$

where $\boldsymbol{C}$ is the elastic coefficient tensor, a repeated index implies summation over the range of the index, and a comma followed by index i indicates partial differentiation with respect to x_i. For an isotropic material, the material constants are reduced to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.36}$$

where δ_{ij} is the Kronecker delta, λ and G are Lamé coefficients.

In this book, the Voigt notations, which express the stress and strain tensors as vectors in six spaces, are also used. In Eq. (1.25), if we use the transformation rule of the subscripts that $11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 31 \to 5,\ 12 \to 6$ in the components of stress σ and the material constant matrix $\boldsymbol{C}$, we get the original form of the stress and strain relation in Voigt notation:

$$\begin{gathered}\sigma_i = C_{ij}\varepsilon_j, \quad \varepsilon_i = s_{ij}\sigma_j; \quad \boldsymbol{\sigma} = \boldsymbol{C}\cdot\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{s}\cdot\boldsymbol{\sigma}; \quad \boldsymbol{s} = \boldsymbol{C}^{-1} \\ \boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^{\mathrm{T}}, \quad \boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6]^{\mathrm{T}}\end{gathered} \tag{1.37}$$

where

$$\begin{gathered}\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23},\ \sigma_5 = \sigma_{31},\ \sigma_6 = \sigma_{12} \\ \varepsilon_1 = \varepsilon_{11},\ \varepsilon_2 = \varepsilon_{22},\ \varepsilon_3 = \varepsilon_{33},\ \varepsilon_4 = \gamma_{23} = 2\varepsilon_{23},\ \varepsilon_5 = \gamma_{31} = 2\varepsilon_{31},\ \varepsilon_6 = \gamma_{12} = 2\varepsilon_{12}\end{gathered} \tag{1.38}$$

It is noted that the rule transformed the fourth-order tensor s_{ijkl} to the second-order tensor s_{ij} is slightly different with that from C_{ijkl} to C_{ij} (Kuang and Ma 2002). Let the transform matrix of the coordinate systems ϕ' and ϕ be $\boldsymbol{Q}$, $Q_{kl} = \cos(\boldsymbol{i}'_k, \boldsymbol{i}_l)$. The transform rule for a tensor $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}' = \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^{\mathrm{T}}$, so in the original Voigt vector form, the transform rule is

$$\begin{gathered}\boldsymbol{\sigma}' = \boldsymbol{A} : \sigma, \quad \boldsymbol{\varepsilon}' = \boldsymbol{B} : \boldsymbol{\varepsilon} \\ \boldsymbol{A} = \begin{bmatrix} A_{11} & 2A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} A_{11} & A_{12} \\ 2A_{21} & A_{22} \end{bmatrix}, \quad [A_{11}] = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 \end{bmatrix} \\ [A_{12}] = \begin{bmatrix} Q_{12}Q_{13} & Q_{11}Q_{13} & Q_{12}Q_{11} \\ Q_{22}Q_{23} & Q_{21}Q_{23} & Q_{21}Q_{22} \\ Q_{32}Q_{33} & Q_{31}Q_{33} & Q_{31}Q_{32} \end{bmatrix}, \quad [A_{21}] = \begin{bmatrix} Q_{21}Q_{31} & Q_{22}Q_{32} & Q_{23}Q_{33} \\ Q_{11}Q_{31} & Q_{12}Q_{32} & Q_{13}Q_{33} \\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{13}Q_{23} \end{bmatrix} \\ [A_{22}] = \begin{bmatrix} (Q_{23}Q_{32} + Q_{22}Q_{33}) & (Q_{21}Q_{33} + Q_{23}Q_{31}) & (Q_{22}Q_{31} + Q_{21}Q_{32}) \\ (Q_{12}Q_{33} + Q_{13}Q_{32}) & (Q_{11}Q_{33} + Q_{13}Q_{31}) & (Q_{11}Q_{32} + Q_{12}Q_{31}) \\ (Q_{12}Q_{23} + Q_{13}Q_{22}) & (Q_{13}Q_{21} + Q_{11}Q_{23}) & (Q_{11}Q_{22} + Q_{12}Q_{21}) \end{bmatrix}\end{gathered} \tag{1.39}$$

From Eq. (1.39), it is seen that the stress and strain in forms of Eq. (1.37) do not obey the same transform matrix under the coordinate rotation.

1.3.3 Normalized Stress and Strain Vectors

The following normalized stress and strain vector in 6-D space is more convenient. The normalized stress vector $\bar{\boldsymbol{\sigma}}$ and strain vector $\bar{\boldsymbol{\varepsilon}}$ are defined as (Chen 1984; Arramon et al. 2000; Kuang et al. 2003)

$$\begin{aligned}
&\bar{\boldsymbol{\sigma}} = \left[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{31}, \sqrt{2}\sigma_{12}\right]^{\mathrm{T}} \\
&\bar{\boldsymbol{\varepsilon}} = \left[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}\Big/\sqrt{2}, \gamma_{31}\Big/\sqrt{2}, \gamma_{12}\Big/\sqrt{2}\right]^{\mathrm{T}} \\
&\bar{\boldsymbol{\sigma}} = \boldsymbol{P}\cdot\boldsymbol{\sigma}, \quad \bar{\boldsymbol{\varepsilon}} = \boldsymbol{P}^{-1}\cdot\boldsymbol{\varepsilon}, \quad [\boldsymbol{P}] = \operatorname{diag}\left[1\ 1\ 1\ \sqrt{2}\ \sqrt{2}\ \sqrt{2}\right]
\end{aligned} \tag{1.40}$$

Substitution of Eq. (1.40) into Eq. (1.37) yields the constitutive equation for $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$:

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{C}}\cdot\bar{\boldsymbol{\varepsilon}}, \quad \bar{\boldsymbol{C}} = \boldsymbol{P}\cdot\boldsymbol{C}\cdot\boldsymbol{P}; \quad \bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{s}}\cdot\bar{\boldsymbol{\sigma}}, \quad \bar{\boldsymbol{s}} = \boldsymbol{P}^{-1}\cdot\boldsymbol{s}\cdot\boldsymbol{P}^{-1} \tag{1.41}$$

Under a coordinate system transformation, Eq. (1.41) becomes

$$\begin{aligned}
\bar{\boldsymbol{\sigma}}' &= \boldsymbol{P}\cdot\boldsymbol{\sigma}' = \boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{\sigma} = \boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{P}^{-1}\cdot\bar{\boldsymbol{\sigma}} \\
\bar{\boldsymbol{\varepsilon}}' &= \boldsymbol{P}^{-1}\cdot\boldsymbol{\varepsilon}' = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{\varepsilon} = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{P}\cdot\bar{\boldsymbol{\varepsilon}}
\end{aligned} \tag{1.42}$$

Using Eqs. (1.29), (1.30), (1.31), and (1.32), it is easily shown that

$$\boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{P}^{-1} = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{P} = \boldsymbol{H}; \quad \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} = \boldsymbol{P}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{I} \tag{1.43}$$

From Eqs. (1.42) and (1.43), it is seen that $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are vectors in a same 6-D space.

1.3.4 Some Fundamental Formulas in Finite Deformation

Fundamental formulas for finite deformation can be found in many books (Kuang 2002; Ogden 1984). Here, some fundamental formulas are given which will be used in this book. Notations $\rho, \rho_{\mathrm{e}}, \boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{H}$ and φ, ψ will denote the density, electric charge density, displacement, stress, strain, electric displacement, electric field, magnetic induction, magnetic field, and electric potential and magnetic scalar potential, respectively, in the current configuration. The physical quantities in the initial configuration are expressed by an upper bar '-' on the corresponding physical quantities, such as $\bar{\rho}, \bar{\rho}_{\mathrm{e}}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{E}}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{H}}$ and $\bar{\varphi}, \bar{\psi}$, where $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are the second kind of the Piola-Kirchhoff stress and Green strain, respectively. The coordinate and the subscripts in the current configuration are denoted by small letters, such as x_i, while the initial configuration are denoted by capital letters, such as $X_I, \bar{\sigma}_{IJ}$. Let

$$\begin{gathered}\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X},t)=\boldsymbol{X}+\boldsymbol{u}(\boldsymbol{X},t), \quad \boldsymbol{F}=\partial\boldsymbol{x}/\partial\boldsymbol{X}, \quad \bar{\boldsymbol{\varepsilon}}=(\bar{\boldsymbol{C}}-\mathbf{I})/2, \quad \bar{\boldsymbol{C}}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}, \quad \mathrm{d}\boldsymbol{x}=\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X} \\ F_{kL}=x_{k,L}=\delta_{kL}+u_{k,L}, \quad \bar{C}_{KL}=F_{mK}F_{mL}=x_{m,K}x_{m,L}, \quad \bar{\varepsilon}_{KL}=(x_{m,K}x_{m,L}-\delta_{KL})/2\end{gathered} \tag{1.44}$$

where $\boldsymbol{F}$ and $\bar{\boldsymbol{C}}$ are the deformation gradient and deformation tensor, respectively.

It should be noted that the differentiation with the capital or small letter subscript is different, such as $x_{i,J}=\partial x_i/\partial X_J$ and $u_{i,J}=u_{i,m}x_{m,J}$. Because in this book the same coordinate system in the current and initial configurations is taken, when there is no differential symbol, it also has $X_I=X_i, \bar{\sigma}_{IJ}=\bar{\sigma}_{ij}$, etc. The transform relations of the mechanical variables in the current and initial configurations are

$$\begin{gathered}\bar{\rho}_{\mathrm{e}}=j\rho_{\mathrm{e}}, \quad \bar{\rho}=j\rho, \quad \mathrm{d}V=j\,\mathrm{d}\bar{V}, \quad \bar{f}_K=jf_k, \quad \bar{\mathfrak{A}}=j\mathfrak{A}, \quad \bar{g}=jg, \quad j=\left|x_{i,J}\right| \\ \bar{\sigma}_{IJ}=jX_{I,m}X_{J,n}\sigma_{mn}, \quad \bar{n}_K\mathrm{d}\bar{a}=Jx_{i,K}n_i\mathrm{d}a, \quad \bar{\sigma}^*\mathrm{d}\bar{a}=\sigma^*\mathrm{d}a, \quad J=j^{-1} \\ \bar{T}_{Kl}\bar{n}_K\mathrm{d}\bar{a}=T_{kl}n_k\mathrm{d}a, \quad \bar{T}_k=jX_{J,n}\bar{n}_Jn_nT_k, \quad T_i=Jx_{l,K}n_lN_K\bar{T}_i \\ x_i=X_I+u_i, \quad x_{i,J}X_{J,l}=\delta_{il}, \quad \bar{\varepsilon}_{IJ}=(x_{k,I}x_{k,J}-\delta_{IJ})/2, \quad \delta\bar{\varepsilon}_{IJ}=x_{k,I}\delta u_{k,J}\end{gathered} \tag{1.45}$$

The electric displacement $\bar{D}_I$ and the magnetic induction $\bar{B}_I$ are defined by the principle that the electric charge and "magnetic charge (=0)" do not change with deformation in a closed surface constituted of the same material points, and the electric field $\bar{E}_I$ and $\bar{H}_I$ are defined by the derivative of φ and ψ with x_I, respectively (Eringen and Maugin 1989; Kuang 2002).

So we have

$$\begin{gathered}\oint_{\bar{a}}\bar{D}_{I,I}\mathrm{d}\bar{a}=\oint_a D_{i,i}\mathrm{d}a, \quad \bar{D}_I=jX_{I,m}D_m, \quad D_m=Jx_{m,I}\bar{D}_I \\ \bar{B}_I=jX_{I,m}B_m, \quad B_m=Jx_{m,I}\bar{B}_I, \quad \sigma^*\mathrm{d}a=\bar{\sigma}^*\mathrm{d}\bar{a} \\ \bar{E}_I=-\varphi_{,I}=-\varphi_{,m}x_{m,I}=x_{m,I}E_m, \quad E_m=X_{I,m}\bar{E}_I, \\ \bar{H}_I=-\psi_{,I}=x_{m,I}E_m, \quad H_m=X_{I,m}\bar{H}_I\end{gathered} \tag{1.46}$$

From Eq. (1.46), it is known that $\bar{E}_{I,J}=\bar{E}_{J,I}=-\varphi_{,IJ}$ and $\bar{H}_{I,J}=\bar{H}_{J,I}=-\psi_{,IJ}$.

The generalized momentum equations are

$$\left(\bar{\sigma}_{KL}x_{i,L}\right)_{,K}+\bar{f}_i=\bar{\rho}\ddot{u}_i \tag{1.47}$$

$$\bar{D}_{I,I}=\bar{\rho}_{\mathrm{e}} \tag{1.48}$$

The boundary conditions are

$$x_{i,L}\bar{\sigma}_{KL}\bar{n}_K=\bar{T}_i, \quad \bar{D}_K\bar{n}_K=-\bar{\sigma}, \quad \text{or} \quad \bar{u}_I=\bar{u}_I^*, \quad \bar{\varphi}=\bar{\varphi}^* \tag{1.49}$$

1.4 Classical Thermodynamics

1.4.1 *Basic Concepts*

Classical thermodynamics discusses the thermodynamic system consisted of body, its surroundings, and their common boundary. It is concerned with the state of thermodynamic systems at equilibrium, using macroscopic, empirical properties directly measurable in the laboratory (Wang 1955). Classical thermodynamics study exchanges of energy, work, and heat based on the laws of thermodynamics. The first law of thermodynamics is a principle of conservation of energy and defines a specific internal energy. The second law of thermodynamics is a principle to explain the irreversible phenomenon in nature and defines specific entropy, which will tend to increase over time, approaching a maximum value at equilibrium in an isolated nonequilibrium system. The specific internal energy and specific entropy are two state functions of the system. The thermodynamic state of the system is described by a number of state variables. In classical thermodynamics, state variables usually are pressure p, volume V, temperature T, entropy per volume s, etc. A process means that the state of a system is continuously changed. The succession of states defines the path of the process. The related equations between state parameters are called the constitutive equations or the state equations. An open system is a special class of system with boundaries that matter and energy all can cross. If the matter and energy all cannot cross the boundaries, then the system is an isolated system. If there is no matter exchange between the system and environment, then the system is called the closed system. If there is no heat exchange between the system and environment, then the system is called the insulated system.

1.4.2 *Thermodynamic Laws*

The First Law of Thermodynamics: The first law of thermodynamics states that energy is always conserved; it can be converted from one form into another, but it cannot be created or destroyed. The first law is correct for reversible and irreversible processes. In a quasi-static process for a homogeneous system, the first law can be written in the form

$$\mathrm{d}U = \mathrm{d}W + \mathrm{d}Q \tag{1.50}$$

where W is that the surroundings does work on the system, Q is the heat absorbed by the system from the surroundings, and U is the internal energy, which is a state function.

The Second Law of Thermodynamics: The second law of thermodynamics points out that every natural process has in some senses a preferred direction of action. In a quasi-static reversible process, a mathematical quantity called the entropy of a system can be defined as

$$\mathrm{d}Q = T\,\mathrm{d}S \tag{1.51}$$

where T is the Kelvin temperature, S is the entropy. The second law shows that the entropy of an isolated macroscopic system never decreases, or the total entropy of a system plus its environment cannot decrease.

Combining the first and second law expressions for the gas, it is obtained:

$$\mathrm{d}U = \mathrm{d}W + T\,\mathrm{d}S = -P\,\mathrm{d}V + T\,\mathrm{d}S \tag{1.52}$$

where P is the pressure. Because U is a state function, so

$$P = -\partial U/\partial V, \quad T = \partial U/\partial S, \tag{1.53}$$

The change in entropy for any process between an initial state a and a final state b satisfies

$$\Delta S = S_b - S_a \geq \int_a^b \mathrm{d}Q/T \tag{1.54}$$

where the equality is for a reversible process and the inequality is for an irreversible process.

1.4.3 Thermodynamic Character Functions

From Eq. (1.53), it is known that if $U(V,S)$ is a known function of V and S, then P and T are known, i.e., all state variables are known. The internal energy $U(V,S)$ is called the character function of the thermodynamic system. Using Legendre transformation, we can get the Helmholtz free energy $F(V,T)$, enthalpy $H(P,S)$, and Gibbs free energy (or Gibbs function) $G(P,T)$. These functions are all the character functions:

$$\begin{aligned} &F(V,T) = U - TS, \quad G(P,T) = U + PV - TS, \quad H(P,S) = U + PV \\ &\mathrm{d}F = -P\,\mathrm{d}V - S\,\mathrm{d}T, \quad \mathrm{d}G = V\,\mathrm{d}P - S\,\mathrm{d}T, \quad \mathrm{d}H = V\,\mathrm{d}P + T\,\mathrm{d}S \end{aligned} \tag{1.55}$$

Usually we use the variables per volume or per mass to describe laws of thermodynamics. In this book, $\mathfrak{A}, f, g, h,$ and s denote internal energy, Helmholtz free energy, Gibbs free energy, enthalpy, and entropy per volume, respectively; p is the pressure and v is the initial unit volume. But in finite deformation, these notations represent variables per mass. In this case, Eqs. (1.52) and (1.55) become

$$\begin{gathered}\mathfrak{A}(\mathrm{v},s),\quad f(\mathrm{v},T)=\mathfrak{A}-Ts,\quad g(p,T)=\mathfrak{A}+p\mathrm{v}-Ts,\quad h(p,s)=\mathfrak{A}+p\mathrm{v}\\ \mathrm{d}\mathfrak{A}=-p\,\mathrm{dv}+T\,\mathrm{d}s,\quad \mathrm{d}f=-p\,\mathrm{dv}-s\,\mathrm{d}T,\quad \mathrm{d}g=\mathrm{v}\,\mathrm{d}p-s\,\mathrm{d}T,\quad \mathrm{d}h=\mathrm{v}\,\mathrm{d}p+T\mathrm{d}s\end{gathered} \tag{1.56}$$

1.5 Continuum Thermodynamics and Irreversible Processes

1.5.1 General Concept

In continuum mechanics, the internal behaviors of a body are usually inhomogeneous, the internal processes may be irreversible, and the particle in it may be in motion. Extending the classical thermodynamics to continuum, these distinguished features must be considered. It is assumed that the body can be divided into very fine elements, and in each element, its behaviors are homogeneous and the process deviated from the quasi-static process is small, so the Gibbs equation can be used. It means that the specific entropy is still considered as a state function. The total internal and kinetic energies of a body can be obtained through the integral over the volume. Therefore, equations similar to Eq. (1.56) in continuum mechanics are still applicable.

1.5.2 The First Law in Continuum Thermodynamics

Because a body may be in motion, so it has kinetic energy. In continuum thermodynamics, the first law is in the form

$$\dot{U}+\dot{K}=\dot{W}+\dot{Q} \tag{1.57}$$

where $\dot{U}$ is the rate of the internal energy, $\dot{K}$ is the rate of the kinetic energy, $\dot{W}$ is the rate of the work finished by the generalized external force, and $\dot{Q}$ is the rate of the heat supplied by the surroundings. In the electroelastic material, we have

$$\begin{gathered}U=\int_V \mathfrak{A}\,\mathrm{d}V,\quad K=(1/2)\int_V \rho\dot{u}_i\dot{u}_i\,\mathrm{d}V,\quad \dot{Q}=-\oint_a \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a+\int_V \dot{r}\,\mathrm{d}V\\ \dot{W}=\int_a \boldsymbol{T}^{(n)}\cdot\boldsymbol{v}\,\mathrm{d}a+\int_V \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V+\int_a \varphi\dot{\sigma}\,\mathrm{d}a\end{gathered} \tag{1.58}$$

where $\boldsymbol{f}, \boldsymbol{T}^{(n)}, \boldsymbol{v}, \boldsymbol{q}, \boldsymbol{n}, \varphi, \rho_{\mathrm{e}}, r,$ and σ are the body force per volume, surface force per area, velocity, heat flow vector, normal of the boundary of the surface a, electric potential, volume electric charge density, internal source intensity, and surface electric charge density, respectively. Using the generalized momentum equation, from Eq. (1.58), it yields

$$\begin{aligned}
\dot{W}+\dot{Q} &= \int_a \boldsymbol{T}^{(n)}\cdot\boldsymbol{v}\,\mathrm{da}+\int_V \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V+\int_a \varphi\dot{\sigma}\,\mathrm{d}a-\oint_a \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a+\int_V \dot{r}\,\mathrm{d}V\\
&= \int_a \sigma_{ij}n_j v_i\,\mathrm{da}+\int_V f_i v_i\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V-\int_a \varphi\dot{D}_i n_i\,\mathrm{d}a-\oint_a q_i n_i\,\mathrm{d}a+\int_V \dot{r}\,\mathrm{d}V\\
&= \int_V (\sigma_{ij}v_i)_{,j}\,\mathrm{d}V+\int_V f_i v_i\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V-\int_V (\varphi\dot{D}_i)_{,i}\,\mathrm{d}V-\int_V q_{i,i}\,\mathrm{d}V+\int_V \dot{r}\,\mathrm{d}V\\
&= \int_V \sigma_{ij}v_{i,j}\,\mathrm{d}V+\int_V (\sigma_{ij,j}+f_i)v_i\,\mathrm{d}V+\int_V \varphi(\dot{\rho}_{\mathrm{e}}-\dot{D}_{i,i})\,\mathrm{d}V\\
&\quad -\int_V \varphi_{,i}\dot{D}_i\,\mathrm{d}V+\int_V (\dot{r}-q_{i,i})\,\mathrm{d}V\\
&= \int_V (\sigma_{ij}v_{i,j}-\varphi_{,i}\dot{D}_i-q_{i,i}+\dot{r})\,\mathrm{d}V+\int_V \rho\ddot{u}_i v_i\,\mathrm{d}V=\dot{U}+\dot{K}\\
&= \int_V \dot{\mathfrak{A}}\,\mathrm{d}V+(1/2)\left(\int_V \rho\dot{u}_i\dot{u}_i\,\mathrm{d}V\right)^{\cdot}
\end{aligned}$$

From the above equation, the local form of the energy conservation equation is obtained:

$$\dot{\mathfrak{A}}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}+\boldsymbol{E}\cdot\dot{D}-\nabla\cdot\boldsymbol{q}+\dot{r} \tag{1.59}$$

If there has electric current in the body, a term $\boldsymbol{J}\cdot\boldsymbol{E}$ should be added to the right side of Eq. (1.59). The notation " : " is a double dot production, i.e., $\boldsymbol{A}:\boldsymbol{B}=A_{ij}B_{ij}$.

1.5.3 The Second Law in Continuum Thermodynamics (Clausius-Duhem Inequality)

In continuum thermodynamics, usually the process is irreversible. As an approximate theory, the entropy is divided into reversible and irreversible parts. The reversible part of the entropy $S^{(\mathrm{r})}$ is produced by the external heat, and the irreversible part $S^{(\mathrm{i})}$ is produced by the heat produced in the internal irreversible process, i.e.,

$$\begin{aligned}
&\boldsymbol{Q}=\int_V T\dot{s}\,\mathrm{d}V=\int_V \dot{r}\,\mathrm{d}V-\int_a \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a=\int_V (\dot{r}-q_{i,i})\,\mathrm{d}V\Rightarrow T\dot{s}=\dot{r}-q_{i,i}\\
&\dot{S}^{(\mathrm{r})}=\int_V \dot{s}^{(\mathrm{r})}\,\mathrm{d}V=\int_V T^{-1}\dot{r}\,\mathrm{d}V-\int_a \dot{\boldsymbol{\eta}}\cdot\boldsymbol{n}\,\mathrm{d}a=\int_V (T^{-1}\dot{r}-\dot{\eta}_{i,i})\,\mathrm{d}V\\
&\dot{s}^{(\mathrm{r})}=T^{-1}\dot{r}-\dot{\eta}_{i,i}=T^{-1}(\dot{r}-q_{i,i}+\dot{\eta}_i T_{,i}),\quad \dot{\boldsymbol{\eta}}=T^{-1}\boldsymbol{q},\quad \boldsymbol{\eta}=\int_0^t T^{-1}\boldsymbol{q}\,\mathrm{d}\tau
\end{aligned} \tag{1.60}$$

where $\boldsymbol{\eta}$ is the entropy displacement vector and $\dot{\boldsymbol{\eta}}$ is the entropy flow vector. $\dot{s}^{(\mathrm{i})}$ is the entropy rate due to the internal dissipative heat per volume and is often denoted by σ. Usually, σ is called the entropy production (rate) and

$$\dot{S}^{(\mathrm{i})} = \int_V \dot{s}^{(\mathrm{i})}\mathrm{d}V = \int_V \sigma\,\mathrm{d}V = \int_V \dot{s}\,\mathrm{d}V - \int_V \dot{s}^{(\mathrm{r})}\mathrm{d}V \tag{1.61}$$

So the local entropy balance equation is

$$\sigma = \dot{s} - \dot{s}^{(\mathrm{r})} = \dot{s} - \dot{r}/T + \boldsymbol{\nabla}\cdot\dot{\eta} \geq 0, \quad \text{or} \quad T\sigma = T\dot{s} - \dot{r} + q_{i,i} - \dot{\eta}_i T_{,i} \geq 0 \tag{1.62a}$$

When the variation of the temperature is not large, Eq. (1.62a) can be reduced to

$$T\sigma \approx T_0\dot{s} - \dot{r} + q_{i,i} - T_0^{-1} q_i \vartheta_{,i} \geq 0; \quad T = T_0 + \vartheta \tag{1.62b}$$

where T_0 is the reference temperature or the temperature in the environment.

Usually it is assumed that the following equations are held:

$$T\dot{s} = \dot{r} - q_{i,i};\ T\sigma = -\dot{\boldsymbol{\eta}}\cdot\nabla T = -\dot{\eta}_i T_{,i} = -T^{-1} q_i \vartheta_{,i} \geq 0 \tag{1.63}$$

Substituting $\dot{r}$ expressed in Eq. (1.59) into Eq. (1.62) yields

$$T\sigma = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{E}\cdot\dot{\boldsymbol{D}} + T\dot{s} - \dot{\mathfrak{A}} - \dot{\boldsymbol{\eta}}\cdot\nabla T \geq 0 \tag{1.64}$$

Using the electric Gibbs function $g = \mathfrak{A} - Ts - ED$ from Eq. (1.64) we get

$$T\sigma = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}\cdot\dot{\boldsymbol{E}} - s\dot{T} - \dot{g} - \dot{\boldsymbol{\eta}}\cdot\nabla T \geq 0 \tag{1.65}$$

Equations (1.63), (1.64), and (1.65) are called Clausius-Duhem inequality. For the reversible process, $\sigma = 0$, but for the irreversible process, $\sigma > 0$.

Because $\mathfrak{A}$ and g are state functions, the approximation in first order we can assume

$$\dot{\mathfrak{A}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{E}\cdot\dot{\boldsymbol{D}} + T\dot{s}, \quad \dot{g} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}\cdot\dot{\boldsymbol{E}} - s\dot{T} \tag{1.66}$$

and

$$\dot{h}_{\mathfrak{A}} = T\sigma = -\dot{\eta}_i T_{,i} \geq 0, \quad \dot{h}_g = -T_{,i}\dot{\eta}_i + (T_{,i}\eta_i)^{\cdot} = \eta_i \dot{T}_{,i}, \quad T\dot{s} = \dot{r} - q_{i,i} \tag{1.67}$$

Equation (1.66) is the Gibbs equation, Eq. (1.67) is the dissipative energy rate, $\dot{h}_{\mathfrak{A}}$, equation and its Legendre transformation or “the complement dissipative energy rate” $\dot{h}_g$. From Eq. (1.66), the constitutive equations can be derived:

$$\begin{aligned} &\sigma_{ij} = \partial\mathfrak{A}/\partial\varepsilon_{ij}, \quad E_i = \partial\mathfrak{A}/\partial D_i, \quad T = \partial\mathfrak{A}/\partial s \\ &\sigma_{ij} = \partial g/\partial\varepsilon_{ij}, \quad D_i = -\partial g/\partial E_i, \quad s = -\partial g/\partial T \end{aligned} \tag{1.68}$$

1.5.4 Thermodynamic Character Functions

In electroelastic mechanics, state variables usually are stress σ, strain ε, electric field strength $\boldsymbol{E}$, electric displacement $\boldsymbol{D}$, temperature $\boldsymbol{T}$, and specific entropy s. Conjugated variable pairs, which form energy rate, are $(\boldsymbol{\sigma},\boldsymbol{\varepsilon}), (\boldsymbol{E},\boldsymbol{D}), (T,S)$. Each one in three pairs can be used as independent variable, so through the Legendre transform, eight character functions in electroelastic materials can be obtained:

$$\begin{aligned} &\text{Internal energy } \mathfrak{u}: \quad \mathrm{d}\mathfrak{A} = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} + E_i\,\mathrm{d}D_i + T\,\mathrm{d}s \\ &\text{Free energy } f: \quad f = \mathfrak{A} - Ts, \quad \mathrm{d}f = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} + E_i\,\mathrm{d}D_i - s\,\mathrm{d}T \\ &\text{Gibbs function}: \quad g^g = f - \sigma_{ij}\varepsilon_{ij} - E_iD_i, \quad \mathrm{d}g^{\mathrm{g}} = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} - D_i\,\mathrm{d}E_i - s\,\mathrm{d}T \\ &\text{Electric Gibbs function}: \quad g = f - E_iD_i, \quad \mathrm{d}g = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} - D_i\,\mathrm{d}E_i - s\,\mathrm{d}T \\ &\text{Elastic Gibbs function}: \quad g^{\mathrm{el}} = f - \sigma_{ij}\varepsilon_{ij}, \quad \mathrm{d}g^{\mathrm{el}} = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} + E_i\,\mathrm{d}D_i - s\,\mathrm{d}T \\ &\text{Enthalpy}: \quad h = \mathfrak{A} - \sigma_{ij}\varepsilon_{ij} - E_iD_i, \quad \mathrm{d}h = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} - D_i\,\mathrm{d}E_i + T\,\mathrm{d}s \\ &\text{Electric enthalpy}: \quad h^{\mathrm{e}} = \mathfrak{A} - E_iD_i, \quad \mathrm{d}h^{\mathrm{e}} = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} - D_i\,\mathrm{d}E_i + T\,\mathrm{d}s \\ &\text{Elastic enthalpy}: \quad h^{\mathrm{el}} = \mathfrak{A} - \sigma_{ij}\varepsilon_{ij}, \quad \mathrm{d}h^{\mathrm{el}} = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} + E_i\,\mathrm{d}D_i + T\,\mathrm{d}s \end{aligned} \tag{1.69}$$

The internal energy $\mathfrak{A}(\varepsilon,\boldsymbol{E},s)$ and the electric Gibbs free energy $g(\boldsymbol{\varepsilon},\boldsymbol{E},T)$ are often used. Especially the electric Gibbs free energy is more convenient in application. It is easy to see that when the temperature effect is not considered, then the internal energy and the free energy are the same, and the electric Gibbs function and the electric enthalpy are the same.

1.5.5 Irreversible Thermodynamics

The first foundation of the irreversible thermodynamics is that the Gibbs equation is applicable, i.e., Eqs. (1.66), (1.67) and (1.68) are applicable for the state slightly deviated from the equilibrium state. The second foundation (De Groet 1952; Gyarmati 1970; Kuang 2002) is that the dissipative rate equation is constituted of the irreversible force and irreversible flow, and the irreversible flow is a function of the irreversible force or vice versa. So in the electroelastic case from Eq. (1.67), we can derive the so-called evolution equation

$$\dot{\eta}_i = \dot{\eta}_i(T_{,j}), \quad \text{or} \quad T_{,i} = T_{,i}(\dot{\eta}_j) \tag{1.70}$$

In the usual thermal conductive theory, the variation of the temperature is not large; Eq. (1.70) can be written in a linear form:

$$T\dot{\eta}_i = q_i = -\lambda_{ij}(\boldsymbol{x}, t)T_{,j}, \quad T_{,j} = -\lambda_{ji}^{-1}T\dot{\eta}_i = -\widehat{\lambda}_{ij}(\boldsymbol{x}, t)T\dot{\eta}_i \tag{1.71}$$

where λ is the thermal conducting coefficient which may be related to $(\boldsymbol{x}, t)$. Equation (1.71) is the Fourier's law. Sometimes Eqs. (1.68) and (1.70) are called the first and second group constitutive equations, respectively (Kuang 2002).

1.5.6 The Diffusion Problem

The classical thermal diffusion theory is also assumed that the Gibbs equation is still held:

$$\begin{gathered}\dot{\mathfrak{A}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{E} : \dot{\boldsymbol{D}} + T\dot{s} + \mu\dot{c}, \quad \mathrm{d}\mathfrak{A} = \boldsymbol{\sigma} : \mathrm{d}\varepsilon + \boldsymbol{E} : \mathrm{d}\boldsymbol{D} + T\,\mathrm{d}s + \mu\,\mathrm{d}c \\ T\dot{s} = \dot{r} - q_{i,i} + \mu d_{i,i}, \quad T\dot{s} + \mu\dot{c} = \dot{r} - q_{i,i}, \quad d_{i,i} = -\dot{c}\end{gathered} \tag{1.72}$$

where μ is the chemical potential, $\boldsymbol{d}$ is the flow vector of the diffusing mass, and c is the concentration. Equation (1.72) shows that the total input heat rate $\dot{r} - q_{i,i}$ is balanced by the sum of $T\dot{s}$ and $\mu\dot{c}$. Using relations

$$T^{-1}q_{i,i} = (T^{-1}q_i)_{,i} + T^{-2}q_iT_{,i}, \quad T^{-1}\mu d_{i,i} = (T^{-1}\mu d_i)_{,i} - d_i(T^{-1}\mu)_{,i} \tag{1.73}$$

from Eqs. (1.72) and (1.73), we get (De Groet 1952; Gyarmati 1970; Kuang 2010)

$$\begin{gathered}\dot{s} - T^{-1}\dot{r} + (T^{-1}q_i - T^{-1}\mu d_i)_{,i} = -T^{-2}q_iT_{,i} - d_i(T^{-1}\mu)_{,i} = \dot{s}^{(\mathrm{i})}, \quad \text{or} \\ \dot{s} = \dot{s}^{(\mathrm{r})} + \dot{s}^{(\mathrm{i})}; \quad \dot{s}^{(\mathrm{r})} = (\dot{r}/T) - \nabla \cdot \boldsymbol{J}_s, \quad \boldsymbol{J}_s = (\boldsymbol{q}/T - \mu\boldsymbol{d}/T)\end{gathered} \tag{1.74}$$

From Eq. (1.74), it is obtained:

$$\begin{aligned}&T\dot{s}^{(\mathrm{r})} = \dot{r} - T(\dot{\eta}_i - \mu' d_i)_{,i}, \quad \mu' = \mu/T \\ &T\dot{s}^{(\mathrm{i})} = h_u = -T_{,i}\dot{\eta}_i - \mu'_{,i}\dot{\xi}'_i = \boldsymbol{X}_T \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{X}'_\mu \cdot \dot{\boldsymbol{\xi}}' \geq 0 \\ &\boldsymbol{X}_T = -\nabla T, \quad \boldsymbol{X}'_\mu = -\nabla\mu', \quad \dot{\boldsymbol{\eta}} = \boldsymbol{q}/T, \quad \dot{\boldsymbol{\xi}}' = T\boldsymbol{d} = T\dot{\boldsymbol{\xi}}\end{aligned} \tag{1.75}$$

According to the linear irreversible thermodynamic theory, irreversible flows are proportional to the irreversible forces, so the evolution equations are

$$\begin{aligned} T\dot{\eta}_i &= -\lambda_{ij}T_{,i} - L_{ij}T\mu'_{,i} = -\lambda_{ij}T_{,i} - L_{ij}\mu_{,i} + L_{ij}\mu T_{,i}/T \\ \dot{\xi}_i &= T^{-1}\dot{\xi}'_i = -D_{ij}T\mu'_{,i} - L_{ij}T_{,i} = -D_{ij}\mu_{,i} - L_{ij}T_{,i} + D_{ij}\mu T_{,i}/T \end{aligned} \tag{1.76}$$

where $\boldsymbol{D}$ is the diffusion coefficient and $\boldsymbol{L}$ is the coupling coefficient. When $\mu T_{,i}/T \ll \mu_{,i}$, Eq. (1.76) is reduced to

$$\begin{aligned} &T\dot{s}^{(\mathrm{i})} = h_u \approx \boldsymbol{X}_T \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{X}_\mu \cdot \dot{\boldsymbol{\xi}} = -T_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i \geq 0; \quad \boldsymbol{X}_\mu = -\nabla\mu; \quad \dot{\boldsymbol{\xi}} = \boldsymbol{d} \\ &T\dot{\eta}_i = -\lambda_{ij}(T)T_{,i} - L_{ij}(T)\mu_{,i}, \quad \dot{\xi}_i = -D_{ij}(T)\mu_{,i} - L_{ij}(T)T_{,i}, \quad \text{or} \\ &T_{,i} = -\hat{\lambda}_{ij}(T)T\dot{\eta}_i - \hat{L}_{ij}(T)\dot{\xi}_i, \quad \mu_{,i} = -\hat{D}_{ij}(T)\dot{\xi}_i - \hat{L}_{ij}(T)T\dot{\eta}_i \end{aligned} \tag{1.77}$$

Especially coefficients $\boldsymbol{D}, \boldsymbol{L}, \boldsymbol{\lambda}$ in Eq. (1.77) can be considered as constants. It is noted that the irreversible thermodynamic theory can only give the general form of the evolution equation and the exact form should be given by experiments. For simplicity when the variation of the temperature is not large and considering experimental facts, the extended Fourier's law and Fick's law, Eqs. (1.71) and (1.77) will be used in the future.

In the application of irreversible thermodynamics to more complex materials, such as viscous materials, plastic materials, and materials with phase transformation, the continuum thermodynamics with internal variables is very useful.

1.6 Physical Variational Principle (PVP)

In traditional thermodynamics, the first law is considered as a principle of conservation of energy. But in our papers (Kuang 2007, 2008a, b, 2009a, 2011a, b, c), it was shown that together with the first law of thermodynamics, the known facts show that the physical variational principle (PVP) is also held. The physical variational principle gives a true process for all virtual possible process satisfying the geometrical constrained conditions. If the environment is not considered, the PVP can be expressed as

$$\begin{aligned} &\delta U = \delta W + \delta Q, \quad \text{or} \quad \delta \Pi = \delta U - \delta W - \delta Q = 0 \\ &\delta U = \int_V \delta\mathfrak{A}\,\mathrm{d}V + \int_V \mathfrak{A}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V, \quad \delta W = \int_V \boldsymbol{f}\cdot\delta u\,\mathrm{d}V - \int_V \rho\ddot{\boldsymbol{u}}\cdot\delta u\,\mathrm{d}V + \int_a \boldsymbol{T}\cdot\delta\boldsymbol{u}\,\mathrm{d}a \\ &\delta Q = -\int_a \boldsymbol{q}\delta t\cdot\boldsymbol{n}\,\mathrm{d}a + \int_V \delta r\,\mathrm{d}V \end{aligned} \tag{1.78}$$

where δ is the variation sign and $\boldsymbol{f}, \boldsymbol{T}$, and $\boldsymbol{u}$ are the generalized body force, surface force, and generalized displacement. In complex media, $\mathfrak{A}$ may be the function of strain and other variable, such as electric field $\boldsymbol{E}$. $\mathfrak{A}^{\mathrm{e}}$ is the energy from the total energy minus the pure deformation energy (see later Sect. 2.1.2). In general, W and Q are not the state functions, so they are dependent to path. In general case, the discussed body, environment, and their common interface should be considered together, which will be discussed in Chap. 2 in detail.

Because the thermodynamic laws do not contain the time, in Eq. (1.78), the D'Alembert principle has been used to make the system in an invented quasi-static equilibrium state. So the first law for the electroelastic process can also be written in the form

$$\begin{aligned} &\dot{U} = \dot{W} + \dot{Q} \\ &\dot{W} = \int_a \boldsymbol{T}^{(n)} \cdot \boldsymbol{v}\,\mathrm{da} + \int_V (\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \boldsymbol{v}\,\mathrm{d}V + \int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V + \int_a \varphi\dot{\sigma}\,\mathrm{d}a \end{aligned} \tag{1.79}$$

If the isothermal reversible case with $\delta Q = 0, \delta T = 0$ is discussed, the electroelastic process is reversible. In practice, the specific electric Gibbs free energy $g(\varepsilon, \boldsymbol{E}, T)$ is preferred to use in physical variational principle, i.e., we often use

$$\begin{aligned} &\delta\Pi_g = \int_V \delta g\,\mathrm{d}V + \int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V - \delta W^* - \delta Q = 0 \\ &\delta W^* = \int_a \boldsymbol{T}^{(n)} \cdot \delta\boldsymbol{u}\,\mathrm{da} + \int_V (\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \delta u\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V - \int_{a_{\mathrm{D}}} \sigma^*\delta\varphi\,\mathrm{d}a \end{aligned} \tag{1.80}$$

where W^* is the sum of the work of the external force on the medium and the complementary work of the medium on the electric field and g^{e} is the part containing $\boldsymbol{E}$ or φ in g.

In many problems, the variation of a variable ϕ different with displacement $\boldsymbol{u}$ should be divided into local variation and migratory variation, i.e., the variation $\delta\phi = \delta_\phi\phi + \delta_u\phi$, where the local variation $\delta_\phi\phi$ of ϕ is the variation duo to the change of ϕ itself and the migratory variation $\delta_u\phi$ of ϕ is the variation of change of ϕ due to virtual displacements. It is also noted that the new force produced by the migratory variation $\delta_u\phi$ will enter the virtual work δW or δW^* as the same as the external mechanical force (see later Sect. 2.1.2). It is noted that the local variation does not produce the variation of volume, but the variation of the volume really occurred due to the migratory variation. So in Eq. (1.80), the term $\int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V$ should be considered for the nonlinear electroelastic analysis.

The physical variational principle is considered to be one of the fundamental physical principles for the quasi-equilibrium state case, which can be used to derive governing equations in continuum mechanics and other fields. We can also give it a simple explanation that the true displacement is one kind of the virtual displacement and obviously it satisfies the variational principle. Other virtual displacements cannot satisfy this variational principle; otherwise, the first law is not objective.

The physical variational principle is different to the usual mathematical variational method which is based on the known physical facts. According to this principle, the variational principle can be obtained automatically if the energy expression is given. We consider that the first law of the thermodynamics includes two contents: energy conservation and physical variational principle. The PVP can also be considered as a generalization of the general virtual work principle in some senses.

1.7 Some Extensions in Continuum Thermodynamics

1.7.1 Extension of the First and Second Laws in Continuum Thermodynamics

In continuum mechanics, the state variables in a system are varied not only in space but also with time. Similar to the mechanical kinetic energy in continuum mechanics, an inertial heat can be added to energy equation. For a homogeneous system, the first law in the classical and continuum thermodynamics can be extended to the case where the temperature is varied with time, i.e., the first law; Eqs. (1.50) and (1.57) in continuum mechanics can be extended, respectively, to

$$\mathrm{d}U = \mathrm{d}W + \mathrm{d}Q - \mathrm{d}Q_{\mathrm{T}} \tag{1.81a}$$

$$\dot{U} + \dot{K} = \dot{W} + \dot{Q} - \dot{Q}_{\mathrm{T}}; \quad \dot{Q}_{\mathrm{T}} = \int_V C\rho_{s0}\ddot{T}\,\mathrm{d}V \tag{1.81b}$$

where $\dot{Q}_{\mathrm{T}}$ is the inertial heat rate. $C\rho_{s0}$ can be assumed as a constant, and it can be measured by experiments. The first law in classical thermodynamics is a law in an ideal state, but Eqs. (1.81a) and (1.81b) are approximate formulas in a practical situation. Equations (1.81a) and (1.81b) may be important for the thermodynamic problem with microscopic time and size. It is obvious that the experiments to prove this theory are very important and meaningful. For a homogeneous pure heat problem without heat conduction, using $\dot{U} = C\dot{T}V$, $\dot{Q}_{\mathrm{T}} = C\rho_{s0}\ddot{T}V$, and $\dot{Q} = \dot{r}V$, where C is the specific heat per volume, Eq. (1.81) becomes

$$C(\dot{T} + \rho_{s0}\ddot{T}) = \dot{r} \tag{1.82}$$

The second law, Eq. (1.54), can be extended to

$$\Delta(S + S^{(\mathrm{a})}) = (S + S^{(\mathrm{a})})_b - (S + S^{(\mathrm{a})})_a \geq \int_a^b \mathrm{d}Q/T \tag{1.83}$$

where $s^{(\mathrm{a})}$ is called the inertial entropy, which is reversible, corresponding to the inertial heat.

1.7.2 Inertial Entropy Theory

As an extension of the inertial heat, we modify the thermodynamic entropy equation by adding a term containing inertial entropy in Eqs. (1.60) and (1.63) (Kuang 2009b), i.e.,

$$\begin{aligned}&T\dot{s}+T\dot{s}^{(\mathrm{a})}=\dot{r}-q_{i,i}=\dot{r}-\left(T\dot{\eta}_i\right)_{,i},\quad \dot{s}^{(\mathrm{a})}=\rho_s\ddot{T},\quad \rho_s=\rho_{s0}C/T\\&\dot{s}=\dot{s}^{(\mathrm{r})}+\dot{s}^{(\mathrm{i})};\quad \dot{s}^{(\mathrm{r})}+\dot{s}^{(\mathrm{a})}=\dot{r}/T-\dot{\eta}_{i,i};\quad \dot{\eta}_i=q_i/T\\&T\sigma=T\dot{s}-T\dot{s}^{(\mathrm{r})}=T\dot{s}+T\dot{s}^{(\mathrm{a})}-\dot{r}+T\dot{\eta}_{i,i}=-\dot{\eta}_iT_{,i}\geq 0;\quad \sigma=\dot{s}^{(\mathrm{i})}=-\dot{\eta}_iT_{,i}/T\end{aligned}\tag{1.84}$$

where s is the usual entropy density, $s^{(\mathrm{r})}$ is the reversible part of s produced by the difference of the external heat and the inertial heat, $s^{(\mathrm{i})}$ is the irreversible part of s produced by the internal irreversible process and is often denoted by the entropy production rate $\dot{s}^{(\mathrm{i})}=\sigma$, $s^{(\mathrm{a})}$ is the inertial entropy, and we assume that $\dot{s}^{(\mathrm{a})}$ is proportional to the acceleration of the temperature. ρ_s is called the inertial entropy coefficient, ρ_{s0} is an inertial time constant, and $T\dot{s}^{(\mathrm{a})}=C\rho_{s0}\ddot{T}$ is the inertial heat rate per volume. Comparing Eq. (1.84) with the classical entropy equation, it is found that in Eq. (1.84), $T\dot{s}+T\dot{s}^{(\mathrm{a})}$ is used instead of $T\dot{s}$ in the classical theory. In Eq. (1.84), s is still a state function because $s^{(\mathrm{a})}$ is reversible. If the inertial entropy is omitted, then Eq. (1.84) is reduced to the entropy equation in the classical thermodynamics. The concepts of the inertial entropy and the inertial heat are complementing each other and $\dot{Q}_{\mathrm{T}}=\int_V T\dot{s}^{(\mathrm{a})}\,\mathrm{d}V$. The ideal of the inertial entropy theory is that when the inertial heat is subtracted from the external supplied heat, the system can be studied by the classical thermodynamics. In other words, at any fixed time, the system is located at a local equilibrium state. The supplied heat δQ by the surrounding cannot be absorbed by the system immediately, but only $\delta Q-\delta Q^{(\mathrm{a})}$, where $\delta Q^{(\mathrm{a})}$ is used to overcome the temperature inertia. $\delta Q^{(\mathrm{a})}$ may be positive or negative. So the inertial entropy theory substantially belongs to the framework of local thermodynamic equilibrium theory. When the temperature is inhomogeneous in the space, the space inhomogeneous temperature is dealt with Fourier's law and its variation with time is dealt by the inertial entropy theory. The temperature wave problem is discussed in the time-space 4-D space.

The dissipative energy $h_{\mathfrak{A}}$ and the complement dissipative energy h_g are the same as that in classical irreversible thermodynamic theory, i.e., they are also expressed by Eq. (1.67). Therefore, in the inertial entropy theory, the evolution equation (1.71) is still held. If $\dot{s}=C\dot{\vartheta}/T, \vartheta=T-T_0$, from Eqs. (1.84) and (1.71), a temperature wave equation is obtained:

$$C\left(\dot{\vartheta}+\rho_{s0}\ddot{\vartheta}\right)=\dot{r}+\lambda_{ij}(\boldsymbol{x},t)\vartheta_{,ji}\tag{1.85}$$

For uniform temperature field, Eqs. (1.82) and (1.85) are identical.

1.7.3 Cattaneo-Vernotte Theory of Generalized Thermodynamics

In present temperature wave theory given in literatures, the Cattaneo-Vernotte heat conduction model (Vernotte 1958; Cattaneo 1958; Joseph and Preziosi 1989) was extensively used, which is

$$q_i + \tau_{ij}\dot{q}_j = -\lambda_{ij}T_{,j} \tag{1.86}$$

In a purely thermal conduction case, the energy equation is

$$\mathrm{d}\mathfrak{A}/\mathrm{d}t = \dot{r} - q_{i,i}; \quad \mathrm{d}\mathfrak{A} = C\,\mathrm{d}\vartheta \tag{1.87}$$

Substituting Eq. (1.87) into Eq. (1.86) with $\tau_{ij} = \tau_0\delta_{ij}$ yields

$$C(\dot{\vartheta} + \tau_0\ddot{\vartheta}) = \dot{r} + \tau_0\ddot{r} + \lambda_{ij}\vartheta_{,ji} \tag{1.88}$$

Comparing Eqs. (1.85) and (1.88), especially for the isotropic case, it is seen that these two equations are the same if $r = 0$. But in other cases, they are different. Bertola and Cafaro (2007) considered that discarding Fourier's law is not easy because it is supported by general experimental evidence.

In the phonon theory of the lattice thermal conductivity at low temperatures the similar temperature wave equation can also be seen (Jackson and Walker 1971).

Some biomechanical researchers (Xu et al. 2008) used the non-Fourier heat conduction model to study the bioheat transfer, burn damage, biomechanics, and physiology and used the so-called dual-phase-lag heat conduction model:

$$q_i + \tau_0\dot{q}_i = -\lambda(T_{,i} + \tau_T\dot{T}_{,i}) \tag{1.89}$$

1.7.4 Comparison of Inertial Entropy Theory with the Cattaneo-Vernotte Theory

1. *For a Quasi-Isothermal Case*

For a quasi-isothermal case, we have $\vartheta_{,i} \approx 0, \vartheta = \vartheta(t)$. In this case, Eqs. (1.88) and (1.85) are, respectively, reduced to

$$C(\dot{\vartheta} + \tau_0\ddot{\vartheta}) = \dot{r} + \tau_0\ddot{r}, \quad \text{or} \quad C(\vartheta + \tau_0\dot{\vartheta}) = r + \tau_0\dot{r} \tag{1.90}$$

$$C(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}) = \dot{r}, \quad \text{or} \quad C(\vartheta + \rho_{s0}\dot{\vartheta}) = r \tag{1.91}$$

Let $r = RH(t)$, where $H(t)$ is the Heaviside step function and R is a constant. The solution of Eq. (1.90) is

$$C\vartheta(t) = R\left[1 + \left(\tau_0^{-1} - 1\right)e^{-t/\tau_0}\right]H(t), \quad C\vartheta(0) = R\tau_0^{-1} \tag{1.92}$$

where the Dirac delta function $\delta(t) = \dot{H}(t)$ is used. As a typical value $\tau_0 = 1\,\text{ns} \sim 1$ ps $(1\,\text{ns} = 10^{-9}\,\text{s}, 1\,\text{ps} = 10^{-12}\,\text{s})$, Eq. (1.92) shows that $C\vartheta(0) = 10^9 R \sim 10^{12} R$. It is difficult to explain that this large energy is supported by which body. It means that at least this theory is not appropriate for the quasi-isothermal case.

The solution of Eq. (1.91) is

$$C\vartheta(t) = R\left(1 - e^{-t/\rho_{s0}}\right)H(t), \quad C\vartheta(0) = 0 \tag{1.93}$$

As a typical value $\rho_{s0} = 1\,\text{ns} \sim 1\,\text{ps}$ is taken, the time for $CT = U$ from 0 to R is in the order of $1\,\text{ns} \sim 1\,\text{ps}$. It means that the rise of the temperature in the material introduced by the internal heat source reaches its stable value in a very short time. Equation (1.93) shows that the heat pulse is used to overcome the heat inertia at first time, then the inertial heat converts to the internal energy. In order to explain the energy conversion process more clearly, we discuss the second example. Let us assume that $\dot{r}/C\rho_{s0} = at, a = 10^4\text{K}/(\mu\text{s})^3$ in time interval $[0, 10\,\mu\text{s}]$ and after $t = 10\,\mu\text{s},\ \dot{r} = 0$. Let $\rho_{s0} = 1\text{ns} = 10^{-3}\mu\text{s}, C = 1\text{kJ}/\text{kg} \cdot \text{K}$. From Eq. (1.91) for $t \in [0, 10\,\mu\text{s}]$ we get:

$$\begin{aligned} &\ddot{\vartheta} + \dot{\vartheta}/\rho_{s0} = \dot{r}/C\rho_{s0} = at; \quad t \in [0, 10\,\mu\text{s}] \\ &\vartheta(0) = 0, \quad \dot{\vartheta}(0) = 0 \end{aligned} \tag{1.94}$$

Its solution is

$$\begin{aligned} \vartheta(t) &= a\rho_{s0}\left[(1 - e^{-t/\rho_{s0}})\rho_{s0}^2 - \rho_{s0}t + t^2/2\right] \\ \dot{\vartheta}(t) &= a\rho_{s0}^2\left(e^{-t/\rho_{s0}} - 1\right) + a\rho_{s0}t \end{aligned} \tag{1.95}$$

When $t \in [0, 10\,\mu\text{s}]$, $T\dot{s}^{(a)} = \rho_{s0}C\ddot{\vartheta} = a\rho_{s0}C(1 - e^{-t/\rho_{s0}}) > 0$, i.e. the inertial heat increases with time. For $t > 10\,\mu\text{s}$ we get:

$$\begin{aligned} &\ddot{\vartheta} + \ddot{\vartheta}/\rho_{s0} = 0; \quad t > 10\,\mu\text{s} \\ &\vartheta(10) = 499.9\text{K}, \quad \dot{\vartheta}(10) = 99.99\text{K}/\mu\text{s} \end{aligned} \tag{1.96}$$

where $\vartheta(10)$ and $\ddot{\vartheta}(10)$ are obtained from Eq. (1.95). The solution is

$$\vartheta(t) = 499.9 + 99.99\rho_{s0}(1 - e^{-(t-10)/\rho_{s0}}), \quad \dot{\vartheta}(t) = 99.99e^{-(t-10)/\rho_{s0}} \tag{1.97}$$

So $\theta(\infty) = 500\text{K},\ \dot{\vartheta}(\infty) = 0$. When $t > 10\,\mu\text{s},\ T\dot{s}^{(a)} = -50Ce^{-(t-5)/\rho_{s0}} < 0$, i.e. the inertial heat decreases with time. At $t = \infty$ the internal heat is zero, $Ts^{(a)}(\infty) = 0$. Eqs. (1.95) and (1.97) show that if the internal heat source *r(t)* is supplied in an inteval $[0, t_f]$, then in the interval $[0, t_f]$, is supplied in an interval $[0, t_{\text{f}}]$, then in the interval $[0, t_{\text{f}}]$, the heat supplied by the environment is used to increase the internal energy and overcome the heat inertia; after t_f, the inertial heat converts to the internal energy or increases the temperature.

If the heat inertia is not considered, this absorbed process is instantaneous. The role of the temperature inertia is somewhat different with the mechanical inertia.

2. *For an Anisotropic Material*

In an anisotropic material, τ_{ij} and λ_{ij} in Eq. (1.86) are all tensors. Combining Eqs. (1.86) and (1.87), we get

$$C\dot{\vartheta} - \tau_{ij}\dot{q}_{j,i} = \dot{r} + \lambda_{ij}\vartheta_{,ji} \tag{1.98}$$

Equation (1.98) is difficult to reduce to a simple equation with a single variable ϑ. If we let $\tau_{ij} = \tau_0\delta_{ij}$, Eq. (1.98) is reduced to (1.88). However, if λ_{ij} is a tensor, in general case, it cannot consider τ_{ij} as a scalar. The only possible reason is that τ_{ij} represents an inertial tensor which is a scalar.

3. *About the Entropy Production Rate*

From Eqs. (1.84) and (1.86), it is obtained:

$$\sigma = -\dot{\eta}_i T_{,i}/T = -q_i T_{,i}/T^2 = (1/\lambda T^2)(q_i q_i + \tau_0 q_i \dot{q}_i) \tag{1.99}$$

When $q_i(q_i + \tau_0\dot{q}_i) < 0$, σ may be negative. This violates the Clausius-Duhem inequality $\sigma \geq 0$ in classical thermodynamics based on the local equilibrium. Jou et al. (2001) proposed an extended irreversible thermodynamic theory to improve the Cattaneo-Vernotte model. For a local non-equilibrium state they assumed

$$\begin{aligned} &s = s(\mathfrak{A}, \boldsymbol{q}); \quad \mathrm{d}s = (\partial s/\partial \mathfrak{A})\mathrm{d}\mathfrak{A} + (\partial s/\partial \boldsymbol{q}) \cdot \mathrm{d}\boldsymbol{q} \\ &\partial s/\partial \mathfrak{A} = 1/\Theta, \quad \partial s/\partial \boldsymbol{q} = -(\tau/\lambda T^2)\boldsymbol{q} \end{aligned} \tag{1.100}$$

where Θ is the non-equilibrium temperature. Using Eq. (1.87), for $r = 0$ Eq. (1.100) can be written as

$$\begin{aligned} &\dot{s} = (1/\Theta)\mathfrak{A} - (\tau/\lambda T^2)\boldsymbol{q} \cdot \mathrm{d}\boldsymbol{q} = -\nabla \cdot \boldsymbol{J}_s + \sigma \\ &\boldsymbol{J}_s = \Theta^{-1}\boldsymbol{q}, \quad \sigma = \boldsymbol{q} \cdot \boldsymbol{X} = \boldsymbol{q} \cdot \left[\nabla\Theta^{-1} - (\tau/\lambda T^2)\dot{\boldsymbol{q}}\right] \end{aligned} \tag{1.101}$$

where $\boldsymbol{J}_s$ is the generalized entropy flow. The irreversible thermodynamics gives $\boldsymbol{q} = L\boldsymbol{X}$. Let $L = \lambda T^2, \lambda > 0$ and approximately let $T = \Theta$ we find

$$\boldsymbol{q} = L\boldsymbol{X} = L\left[\nabla\Theta^{-1} - (\tau/\lambda T^2)\dot{\boldsymbol{q}}\right] = -\lambda\nabla\Theta - \tau\dot{\boldsymbol{q}} \tag{1.102}$$

Equation (1.102) is just the Cattaneo-Vernotte heat conduction equation if we use Θ to instead of T in Eq. (1.86). In this extended irreversible thermodynamic theory we always have $\sigma = LX^2 \geq 0$, but some postulated conditions are added.

However to prove the rationality of the definition of the non-equilibrium specific entropy is a difficult problem. Barletta and Zanchini (1997) pointed out that the extended irreversible thermodynamic theories are disagreed each other and no theory has been verified by experiments yet. The above difficult is not occurred in the inertial entropy theory.

From the above discussions, it is seen that in all published theories, the inertial entropy theory is the simplest one and is the unique consistent with the classical continuum thermodynamics and the constitutive theory. All the above difficulties occurred in theories based on the Cattaneo-Vernotte model do not occurred in the inertial entropy theory. It is also noted that the inertial entropy theory does not reject to use Eq. (1.86) if it is necessary. This theory is also easy to check by experiments in the further.

1.7.5 Thermoelastic Problem

For a thermoelastic problem, the free energy g and constitutive equations can be assumed as

$$\begin{aligned} & g(\varepsilon_{kl}, \vartheta) = \frac{1}{2} C_{ijkl} \varepsilon_{ji} \varepsilon_{lk} - \alpha_{ij} \varepsilon_{ij} \vartheta - \frac{1}{2T_0} C \vartheta^2, \quad C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad \alpha_{ij} = \alpha_{ji} \\ & \sigma_{ij} = \partial g / \partial \varepsilon_{ij} = C_{ijkl} \varepsilon_{kl} - \alpha_{ij} \vartheta, \quad s = -\partial g / \partial \vartheta = \alpha_{ij} \varepsilon_{ij} + C \vartheta / T_0 \end{aligned} \tag{1.103}$$

where $\boldsymbol{C}_{ijkl}$ is the elastic coefficient and α_{ij} is the stress-temperature coefficient. Using the inertial entropy theory, from Eqs. (1.84) and (1.103), the temperature wave equation is obtained as

$$\alpha_{ij} \dot{\varepsilon}_{ij} + C \dot{\vartheta} / T_0 + \rho_{s0} (C/T) \ddot{\vartheta} = \dot{r} / T + \lambda_{ij} \vartheta_{,ji} / T \tag{1.104}$$

Combined Eq. (1.104) with mechanical dynamic equation $\sigma_{ij,j} = \rho \ddot{u}_i$, the governing equations in generalized displacements for the thermoelastic wave under small variation of temperature are

$$C_{ijkl} u_{k,lj} - \alpha_{ij} \vartheta_{,j} = \rho \ddot{u}_i; \quad T_0 \alpha_{ij} \dot{u}_{i,j} + C(\dot{\vartheta} + \rho_{s0} \ddot{\vartheta}) = \lambda_{ij} \vartheta_{,ji} \tag{1.105}$$

Equation (1.105) can also be obtained by the physical variational principle (see chapter 2). For one-dimensional problem, Eqs. (1.103) and (1.105) are, respectively, reduced to

$$\sigma = Y\varepsilon - \alpha\vartheta, \quad s = \alpha\varepsilon + C\vartheta / T_0 \tag{1.106}$$

$$C(\rho_{s0} \ddot{\vartheta} + \dot{\vartheta}) - \lambda \vartheta'' + \alpha T_0 \dot{u}' = 0; \quad \rho \ddot{u} - Y u'' + \alpha \vartheta' = 0 \tag{1.107}$$

In Eq. (1.106) and (1.107), Y is the elastic modulus and α is the stress-temperature coefficient, and for any function φ, $\dot{\varphi} = \partial\varphi/\partial t$ and $\varphi' = \partial\varphi/\partial x$ are used. For a plane wave propagating along direction x, we assume

$$\begin{aligned} &\widetilde{u} = U\exp[\mathrm{i}(kx - \omega t)], \quad \widetilde{\vartheta} = \Theta\exp[\mathrm{i}(kx - \omega t)] \\ &u = \operatorname{Re}\widetilde{u}, \quad \vartheta = \operatorname{Re}\widetilde{\vartheta} \end{aligned} \tag{1.108}$$

where U is the amplitude of the displacement and Θ is the amplitude of the temperature, $k = k_1 + \mathrm{i}k_2$ is the wave number and $\mathrm{i} = \sqrt{-1}$. Substituting Eq. (1.108) into (1.107) yields

$$(Yk^2 - \rho\omega^2)U + \mathrm{i}\alpha k\Theta = 0; \quad \alpha T_0 k\omega U + \left[\lambda k^2 - C(\rho_{s0}\omega^2 + \mathrm{i}\omega)\right]\Theta = 0 \tag{1.109}$$

In order to have nontrivial solutions for U, Θ, the coefficient determinant of Eq. (1.109) must be vanished, i.e.,

$$\begin{vmatrix} Yk^2 - \rho\omega^2 & \mathrm{i}\alpha k \\ \alpha T_0 k\omega & \lambda k^2 - C(\rho_{s0}\omega^2 + \mathrm{i}\omega) \end{vmatrix} = \begin{vmatrix} Yk^2 - \rho\omega^2 & \mathrm{i}\alpha k \\ \alpha T_0 k\omega & \lambda k^2 - Cb \end{vmatrix} = 0 \tag{1.110}$$

where $b = \rho_{s0}\omega^2 + \mathrm{i}\omega$ and $\operatorname{Im} b = \omega > 0$. From Eq. (1.110), we get

$$\lambda Yk^4 - (CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)k^2 + \rho\omega^2 Cb = 0 \tag{1.111}$$

Though there are two roots for k^2, but for a wave propagating along positive x-direction only, k is selected and $-k$ is neglected. Let $k = k_T$ be the wave number of the temperature wave and another $k = k_Y$ be the wave number of the elastic wave. Let

$$\begin{aligned} &CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega = r_1 e^{\mathrm{i}\varphi_1} \\ &(CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)^2 - 4C\lambda Y\rho\omega^2 b \\ &\quad = (CYb - \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)^2 + 4\mathrm{i}\lambda\rho\omega^2\alpha^2 T_0\omega = r_2 e^{\mathrm{i}\varphi_2} \end{aligned} \tag{1.112}$$

then we have

$$\begin{aligned} &k_Y = \frac{1}{2\lambda Y}\left[\left(r_1 e^{\mathrm{i}\varphi_1} - \sqrt{r_2}e^{\mathrm{i}\varphi_2/2}\right)\right]^{1/2}; \quad k_T = \frac{1}{2\lambda Y}\left[\left(r_1 e^{\mathrm{i}\varphi_1} + \sqrt{r_2}e^{\mathrm{i}\varphi_2/2}\right)\right]^{1/2} \\ &r_1 e^{\mathrm{i}\varphi_1} = CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega, \quad \operatorname{Im}(r_1 e^{\mathrm{i}\varphi_1}) = (CY + \alpha^2 T_0)\omega > 0 \\ &r_2 e^{\mathrm{i}\varphi_2} = (CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)^2 - 4C\lambda\rho\omega^2 Yb \\ &\operatorname{Im}\left(r_2 e^{\mathrm{i}\varphi_2}\right) = 2\left[(CY\rho_{s0} - \lambda\rho)CY + (CY\rho_{s0} + \lambda\rho)\alpha^2 T_0\right]\omega^3 \end{aligned} \tag{1.113}$$

where the notation "Im" denotes the image part of a complex function. From Eq. (1.109), we can get ϑ and u.

From Eq. (1.113), it is known that $\operatorname{Im} k_T$ is always positive, so the temperature wave is always attenuated. However, if $r_1 \sin\varphi_1 < \sqrt{r_2}\sin(\varphi_2/2)$, $\operatorname{Im} k_Y < 0$, and in this case, the elastic wave is enlarged. This phenomenon may be applied in the acoustic wave technique.

From this phenomenon, three possible cases can be assumed: (1) The inertial time constant ρ_{s0} is determined by the equation

$$r_1 \sin\varphi_1 = \sqrt{r_2}\sin(\varphi_2/2) \tag{1.114}$$

to make $\operatorname{Im} k_Y = 0$. (2) The second possibility is that the elastic viscosity need be considered. If the elastic viscosity is considered, Eq. (1.114) need be modified. (3) The third possibility is that this phenomenon may also be the inertial effect of the temperature and does not violate the energy conservation law. This phenomenon is also found in the numerical calculation for the elastic wave in pyroelectric material (Yuan and Kuang 2008).

1.7.6 Generalized Inertial Entropy Theory

Analogous to the inertial entropy concept in Eq. (1.84), an inertial concentration concept is introduced in the thermodiffusion problem (Kuang 2010). In the generalized inertial entropy theory, the Gibbs equation, Eq. (1.72), of the classical thermal diffusion theory is changed to

$$\begin{aligned}
&T(\dot{s}+\dot{s}^{(\mathrm{a})})+\mu(\dot{c}+\dot{c}^{(\mathrm{a})})=\dot{r}-\nabla\cdot\boldsymbol{q}\\
&\dot{s}^{(\mathrm{a})}=\rho_s\ddot{T}=\rho_{s0}(C/T)\ddot{T},\quad \dot{c}^{(\mathrm{a})}=\rho_\mu\ddot{\mu};\quad \dot{c}=-d_{i,i}
\end{aligned} \tag{1.115}$$

where μ is the chemical potential, $\boldsymbol{d}$ is the flow vector of the diffusing mass and c the concentration, and ρ_s and ρ_μ are inertial entropy and inertial concentration coefficients, respectively. Similar to the derivation in Sect. 1.5.6, it can be obtained:

$$\begin{aligned}
&(\dot{s}+\dot{s}^{(\mathrm{a})})+\frac{\mu\dot{c}^{(\mathrm{a})}}{T}=\frac{\dot{r}}{T}-\frac{q_{i,i}}{T}+\frac{\mu d_{i,i}}{T}=\frac{\dot{r}}{T}-\left(\frac{q_i}{T}\right)_{,i}-\frac{q_iT_{,i}}{T^2}+\left(\frac{\mu d_i}{T}\right)_{,i}-\left(\frac{\mu}{T}\right)_{,i}d_i\\
&T\dot{s}=T\dot{s}^{(\mathrm{r})}+T\dot{s}^{(\mathrm{i})};\quad T(\dot{s}^{(\mathrm{r})}+\dot{s}^{(\mathrm{a})})+\mu\dot{c}^{(\mathrm{a})}=\dot{r}+T(\dot{\eta}_i-\mu'\dot{\xi}_i)_{,i}\\
&T\sigma=T\dot{s}^{(\mathrm{i})}=-T_{,i}\dot{\eta}_i-\mu'_{,i}\dot{\xi}'_i\approx-T_{,i}\dot{\eta}_i-\mu_{,i}\dot{\xi}_i\geq 0
\end{aligned} \tag{1.116}$$

This theory shows that the heat rate $\int_V \dot{r}\,\mathrm{d}V-\int_a T\dot{\boldsymbol{\eta}}\cdot\boldsymbol{n}\,\mathrm{d}a$ supplied by the environment is transformed to the heat rate stored in the medium, dissipation energy rate, and the inertial heat rate, which is introduced by the inertial entropy and the inertial concentration.

The evolution equation, Eq. (1.77), is still held in the generalized inertial entropy theory.

1.8 The SI System (International System of Units)

1.8.1 SI Base Units

Length (meter, m), Mass (kilogram, kg), Time (second, s), Electrical current (ampere, A), Thermodynamic temperature (Kelvin, K), Amount of substance (mole, mol), Luminous (candela, cd)

1.8.2 Some SI-Derived Units

1. *Mechanics*: Force (Newton, N, kg ms^{-2}), pressure (pascal, Pa, kg/(m s^2) = (N/m^2)), stress (MPa), velocity (or speed) (m s^{-1}), acceleration (m s^{-2}), work, energy, heat (joule, J, m^2 kg s^{-2}, Nm), power or radiant flux (watt, W, kg m^2 s^{-3}, J/s), frequency (hertz, Hz, s^{-1}), modulus of elasticity (GPa), wave number (m^{-1})

2. *Electromagnetism*: Electrical charge (coulomb, C, A s), electrical potential (Volt, V, $\mathrm{N} \cdot \mathrm{m}/\mathrm{A} \cdot \mathrm{s}$, W/A), electric field strength ($\mathrm{V}/\mathrm{m} = \mathrm{N}/\mathrm{C}$), electric displacement (electric flux density) (C/m^2), electrical capacitance (farad, F, $\mathrm{m}^{-2}\,\mathrm{kg}^{-1}\,\mathrm{s}^4\,\mathrm{A}^2$, C/V), electrical inductance (Henry, H, m^2 kg s^{-2} A^{-2}), electrical resistance (ohm, w, m^2 kg s^{-3} A^{-2}, V/A), magnetic flux (weber, Wb, m^2 kg s^{-2} A^{-1}, V s), magnetic flux density (tesla, T, $\mathrm{N}/\mathrm{A} \cdot \mathrm{m}$, Wb/$\mathrm{m}^2$), magnetic field strength (A m^{-1}), permeability (H/m), permittivity (F/m)

3. *Thermodynamics*: Concentration (of amount of substance) (mol m^{-3}), heat capacity (J/K), thermal conductivity (W/(m K)), coefficient of heat transfer (W/(m^2 K)), specific energy (J/kg, m^2 s^{-2}), specific heat capacity (or specific entropy) (J/(kg K), m^2 s^{-2} K^{-1}), heat flow rate (W, J/s), heat flux density (W/m^2)

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