Zhen-Bang Kuang

Theory of Electroelasticity

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Preface

Since Pierre Curie discovered the piezoelectric effect in 1880, piezoelectric materials have been widely used to make many electromechanical devices, such as transducers for conversion of electrical and mechanical energies, sensors, actuators, filters, resonators, ultrasonic generators, and piezoelectric biosensors. The performance and reality of devices are established on the foundation of electroelastic analyses due to the electromechanical coupling. The foundations of the electroelastic analyses are the Newton's law, Maxwell electrodynamics equations, Lorentz's law, and constitutive equations of materials. In electrically nonlinear case, different authors give different governing equations in electroelastic analyses. In this book, we give a simple theory to discuss simpler electrically nonlinear problem in engineering.

Using the continuum thermodynamics, it is found that the first law of thermodynamics contains a physical variational principle, which can be used as a fundamental natural principle to derive the governing equations in physics and continuum mechanics. This theory will be used to derive the governing equations of the discussed piezoelectric and pyroelectric body and its environment in Chap. 2. The Maxwell stress can be obtained automatically by the migratory variation of the electric potential.

In literatures many works on the static and dynamic generalized stress and displacement analyses in piezoelectric and electrostrictive materials with and without defects have been published. Some important results of piezoelectric materials will be collected, modified, and discussed in a unified version in Chaps. 3 and 4. The results of the electrostrictive, pyroelectric, and functional graded piezoelectric materials will be given in Chap. 5.

The surface wave propagation in or not in a biasing state is discussed in Chap. 6. The reflection and transmission of waves in piezoelectric and pyroelectric materials are disposed by the inhomogeneous wave theory. We also proposed the inertial entropy theory due to the heat inertia. The temperature wave equation with a finite propagation velocity can be derived easily from this theory. In the generalized inertial entropy theory an inertial concentration theory is proposed, which can be extended to more extensive area.

In Chap. 7, the three-dimensional and some practical applied electroelastic problems such as plates and shells in electroelastic theory are discussed.

The failure theories published in literatures are also collected in Chap. 8. In the change of the microstructure and failure process, the energy possesses material structure anisotropic behavior and a modal energy density factor theory is proposed, which can also be used in other area, such as in phase transformation theory.

In order to read easily for readers the fundamental knowledge used in this book is given in Chap. 1. Some basic problems are narrated in detail including the formulation of a problem and the mathematical derivation. But for further problems, the narration is simpler. Because the discussed problems in this book are complicated and the check is difficult, some errors may occur. We wish readers will give comments.

The author hopes that this book is useful for graduate students, scientists, and engineers interesting in this area in the fields of continuum mechanics, material science, solid-state physics, and device engineering.

The literatures are very enormous and cannot be all cited, but readers can get more literatures from our cited papers.

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Shanghai Jiaotong University, Zhen-Bang Kuang
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Chapter 1
Preliminary Knowledge and Continuum Thermodynamics

Abstract In this chapter, some basic knowledge of elastic theory, electrodynamics, and thermodynamics which will be applied in this book are introduced. Some extensions in continuum thermodynamics are proposed. It is shown that together with the first law of thermodynamics, a physical variational principle (PVP) is also held. The physical variational principle gives a true process for all virtual possible process satisfying the geometrical constrained conditions. The physical variational principle is considered to be one of the fundamental physical principles for quasi-static system, which can be used to derive governing equations in continuum mechanics and other fields. When the temperature varies with time, the inertial entropy or inertial heat theory is proposed. This theory is consistent with current classical thermodynamic theory. From this theory, the temperature wave equation with finite phase velocity is derived in a very simple fashion. It is shown that the time arrived to equilibrium of the temperature is about 1 ns $\sim$ 1 ps when an internal heat source with a Heaviside step heat function is applied.

Keywords Basic knowledge • Physical variational principle • Inertial entropy

1.1 Background

Jacques and Pierre Curie brothers discovered the piezoelectric effect in 1880 (Sun and Zhang 1984; Ikeda 1990). They found out that a mechanical stress applied on crystals such as tourmaline, quartz, and Rochelle salt could produce electrical charges, and the voltage was proportional to the stress. Piezoelectric can also work in reverse, generating a strain by the application of an electric field. Centrosymmetric classes of crystals are always not piezoelectric, but a few kinds of crystals are still not piezoelectric though lacking a center of symmetry. The pyroelectric effect was found in eighteenth century (Lang 2005), earlier than piezoelectric effect. Most ferroelectric crystals are strongly piezoelectric and pyroelectric. First applications were piezoelectric ultrasonic submarine detector and quartz clocks

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during the First World War. After the Second World War, many new piezoelectric and pyroelectric materials have been discovered in succession, such as $BaTiO_3$, Pb(Ti.Zr)O_3-PZT, KDP, PMN, $LiNbO_3$, and $LiTaO_3$. In present time, it has been successfully used in various areas, such as in aerospace, transportation, nuclear, and medical.

It is different with the piezoelectric materials, all of the crystals, especially the isotropic electrostrictive materials, have the electrostrictive effect.

The fundamental phenomenological theory of the piezoelectricity was established by Kelvin (1856), Voigt (1910), etc. In the current time, due to the intrinsic mechanical-electric coupling effects, piezoelectric materials have been widely used in engineering structures to detect the responses of the structure by measuring the electric charge (sensing) or to reduce excessive responses by applying additional electric forces or thermal forces (actuating). By integrating the sensing and actuating, it is possible to create the so-called intelligent structures and systems that can adapt to or correct for changing operating condition. Due to its intrinsic electromechanical coupling behavior and its reliability in performance, the electroelastic analysis is necessary and has been paid much attention. A lot of literatures have appeared in journals and books. Here we cannot review all of these literatures, but reader can find more literatures from our cited papers.

The foundations of the electroelastic analyses are the Newton's law, Maxwell electrodynamics equations, Lorentz's law, and constitutive equations of materials. In electrically nonlinear case, different authors give different governing equations in electroelastic analyses. In this book, we give a simple theory to discuss simpler electrically nonlinear problem in engineering.

Using the continuum thermodynamics, it is found that the first law of thermodynamics contains the physical variational principle, which can be used as a fundamental natural principle to derive the governing equations in physics and continuum mechanics. We also proposed the inertial entropy theory due to the heat inertia. The temperature wave equation with a finite propagation velocity can be derived easily from this theory. A failure theory based on the energy principle is proposed in this book, which can also be used in other area, such as in phase transformation theory. Many works on the static and dynamic generalized stress analyses in piezoelectric and electrostrictive materials with defects, the surface wave propagation, and the failure theory are also discussed in this book.

1.2 Foundations of Classical Electrodynamics

1.2.1 Constitutive (or State) Equations

There are many books that discussed the electrodynamics (Landau et al. 1984; Stratton 1941; Cai and Zhu 1985; Moon 1984) and the electric engineering (Kruck 1954). Here, a short discussion is given only.

The constitutive (or state) equations can be written in the following form:

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, T), \quad \boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}, T), \quad \boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E}, T) \tag{1.1a}$$

where $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B},$ and T are the electric field intensity, electric displacement or electric flux density, magnetic field intensity, magnetic induction or the magnetic flux density, and the temperature, respectively; $\boldsymbol{J}$ is the total electric current density. When $\boldsymbol{E}$ and $\boldsymbol{H}$ are small, the linear form is used for isothermal case:

$$\begin{aligned} &\boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E} = \epsilon_0 \cdot \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H} = \mu_0 \cdot (\boldsymbol{H} + \boldsymbol{M}) \\ &\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_e + \boldsymbol{J}_v, \quad \boldsymbol{J}_s = \boldsymbol{\gamma} \cdot \boldsymbol{E}_{\text{ext}}, \quad \boldsymbol{J}_e = \boldsymbol{\gamma} \cdot \boldsymbol{E}, \quad \boldsymbol{J}_v = \boldsymbol{\gamma} \cdot (\boldsymbol{v} \times \boldsymbol{B}) \end{aligned} \tag{1.1b}$$

where $\boldsymbol{J}_s$, $\boldsymbol{J}_e$, and $\boldsymbol{J}_v$ are the given external exciting current density, the induction or eddy current density, and motional electric current density, respectively; $\boldsymbol{\epsilon}$ is the permittivity, $\boldsymbol{\mu}$ the permeability, $\boldsymbol{\gamma}$ the electric conductivity of a material, respectively, ϵ_0 and μ_0 are values of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in the vacuum or air. $\boldsymbol{E}_{\text{ext}}$ is an external field; $\boldsymbol{v}$ is the velocity of a moving body. $\boldsymbol{P}$ and $\boldsymbol{M}$ are the polarization density and magnetization density, respectively.

1.2.2 Conservation Law of Charge

The conservation law of charge is

$$\nabla \cdot \boldsymbol{J} = -\dot{\rho}_{\text{e}} \tag{1.2}$$

where ρ_{e} is the free electric charge density. A superimposed dot indicates partial differentiation with respect to time, i.e., $\dot{()} = \partial()/\partial t$, such as $\dot{\rho}_{\text{e}} = \partial \rho_{\text{e}}/\partial t$.

1.2.3 The Lorentz Force

For a continuous charge distribution in motion, the Lorentz force equation is

$$\boldsymbol{f} = \rho_{\text{e}}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \quad \text{or} \quad \boldsymbol{f} = \rho_{\text{e}}\boldsymbol{E} + \boldsymbol{J}_{\text{e}} \times \boldsymbol{B}; \quad \boldsymbol{J}_{\text{e}} = \rho_{\text{e}}\boldsymbol{v} \tag{1.3a}$$

where $\boldsymbol{f}$ is the force density (force per unit volume) and $\boldsymbol{J}_{\text{e}}$ is the current density. Equation (1.3a) can be extended to the electromagnetic media and approximately expressed as (Pao 1978; Kuang 2011a)

$$\begin{aligned} &\boldsymbol{f} = \rho_{\text{t}}\boldsymbol{E} + \boldsymbol{J}_{\text{t}} \times \boldsymbol{B}; \quad \rho_{\text{t}} = \rho_{\text{e}} + \rho_P, \quad \rho_P = -\nabla \cdot \boldsymbol{P} \\ &\boldsymbol{J}_{\text{t}} = \boldsymbol{J} + \boldsymbol{J}_P + \boldsymbol{J}_M; \quad \boldsymbol{J}_P = \partial \boldsymbol{P}/\partial t = \dot{\boldsymbol{P}}, \quad \boldsymbol{J}_M = \nabla \times \boldsymbol{M} \end{aligned} \tag{1.3b}$$

where ρ_t is the total electric charge density constituted of free and polarized charges and $\boldsymbol{J}_t$ is the total electric current density constituted of conduction, polarized, and magnetization current densities.

1.2.4 Maxwell Equations

The differential and integral Maxwell equations are as follows:

$$\begin{aligned} &\nabla \cdot \boldsymbol{D} = \rho_e && \oint_a D \cdot \mathrm{d}\boldsymbol{a} = \int_V \rho_e \, \mathrm{d}V \\ &\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} && \oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t}\int_a \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \\ &\nabla \cdot \boldsymbol{B} = 0 && \int_a \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = 0\left(\int_V \rho_m \, \mathrm{d}V\right) \\ &\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} && \oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \int_a \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} + \frac{\partial}{\partial t}\int_a \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} \end{aligned} \tag{1.4}$$

where V is the volume occupied by the medium; $\boldsymbol{a}$ is the area vector and a is its absolute value; $\boldsymbol{s}$ is a line element vector of a curve C; $\boldsymbol{\nabla}$ is a differential operator vector.

Taking the divergence of the second and the divergence of the fourth in Eq. (1.4) and using the law of conservation of charge we find respectively,

$$\begin{aligned} &\nabla \cdot \partial \boldsymbol{B}/\partial t = \partial(\nabla \cdot \boldsymbol{B})/\partial t = 0, \\ &\nabla \cdot \boldsymbol{J} + \nabla \cdot \partial \boldsymbol{D}/\partial t = \nabla \partial(\nabla \cdot \boldsymbol{D} - \rho_e)/\partial t = 0 \end{aligned} \tag{1.5}$$

If $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{D} - \rho_e = 0$ at the initial state, they will be held at any time, which are just the third and the first equations in Eq. (1.4). Therefore, the independent equations are the second and the fourth equations in Eq. (1.4) and the charge conservation equation in Eq. (1.2), or other combination.

1.2.5 Electric Scalar Potential and Magnetic Vector Potential

The second and third equations are satisfied automatically if we introduce an electric scalar potential φ and a magnetic vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{E} = -\nabla\varphi - \partial \boldsymbol{A}/\partial t = -\nabla\varphi - \dot{\boldsymbol{A}}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.6}$$

Using the constitutive equation (1.1b) with constant $\boldsymbol{\epsilon} = \epsilon \mathbf{I}$, $\boldsymbol{\mu} = \mu \mathbf{I}$, $\gamma = \gamma \mathbf{I}$ and the relation $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$, the first and forth equations in Maxwell equations are reduced to

$$\begin{aligned}&\nabla^2\varphi + \partial(\nabla\cdot\boldsymbol{A})/\partial t = -\rho_e/\epsilon\\&\nabla^2\boldsymbol{A} - \mu\,\epsilon\partial^2\boldsymbol{A}/\partial t^2 - \nabla(\nabla\cdot\boldsymbol{A} + \mu\epsilon\partial\varphi/\partial t) = -\mu\boldsymbol{J}\end{aligned}\tag{1.7}$$

In order to define $\boldsymbol{A}$ uniquely, a supplement gauge condition must be given. Introducing Lorenz gauge $\nabla\cdot\boldsymbol{A} + \mu\cdot\epsilon\partial\varphi/\partial t = 0$, Maxwell equations can be written compactly in the form:

$$\begin{aligned}&\nabla^2\varphi - \mu\,\epsilon\partial^2\varphi/\partial t^2 = -\rho_e/\epsilon\\&\nabla^2\boldsymbol{A} - \mu\,\epsilon\partial^2\boldsymbol{A}/\partial t^2 = -\mu\boldsymbol{J}\end{aligned}\tag{1.8}$$

1.2.6 Quasi-Stationary Electromagnetic Field

If $\partial\boldsymbol{D}/\partial t$ in Maxwell equations can be neglected, the field is called the quasi-stationary magnetic (MQS) field, and in this case, all radiation effects can be negligible. It is also called the eddy current field problem and is important in the electric machine engineering. If $\partial\boldsymbol{B}/\partial t$ in Maxwell equations can be neglected, the field is called quasi-stationary electric (EQS) field which is less important in engineering. For an eddy current field,

$$\partial\boldsymbol{D}/\partial t = -\epsilon\left(\partial^2\boldsymbol{A}/\partial t^2 + \nabla\partial\varphi/\partial t\right) = \boldsymbol{0},\quad \boldsymbol{J} = \gamma\boldsymbol{E} + \boldsymbol{J}_s + \boldsymbol{J}_v\tag{1.9}$$

so Eq. (1.7) becomes

$$\begin{aligned}&\nabla^2\varphi + \partial(\nabla\cdot\boldsymbol{A})/\partial t = -\rho_e/\epsilon\\&\nabla^2\boldsymbol{A} - \mu\gamma(\partial\boldsymbol{A}/\partial t + \nabla\varphi) - \nabla(\nabla\cdot\boldsymbol{A}) = -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v)\end{aligned}\tag{1.10}$$

Introducing conductivity gauge $\nabla\cdot\boldsymbol{A} + \mu\gamma\varphi = 0$, Eq. (1.10) is reduced to

$$\begin{aligned}&\nabla^2\varphi - \mu\gamma\partial\varphi/\partial t = -\rho_e/\epsilon\\&\nabla^2\boldsymbol{A} - \mu\gamma\partial\boldsymbol{A}/\partial t = -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v)\end{aligned}\tag{1.11}$$

Introducing Coulomb gauge $\nabla\cdot\boldsymbol{A} = 0$, Eq. (1.10) is reduced to

$$\begin{aligned}&\nabla^2\varphi = -\rho_e/\epsilon\\&\nabla^2\boldsymbol{A} - \mu\gamma\partial\boldsymbol{A}/\partial t - \mu\gamma\nabla\varphi = -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v)\end{aligned}\tag{1.12}$$

In current sources and stator laminations, eddy currents are usually neglected. For a constant magnetic field $\boldsymbol{J} = \boldsymbol{J}_s,\ \boldsymbol{J}_e = \boldsymbol{0},\ \boldsymbol{J}_v = \boldsymbol{0}$, Eq. (1.12) becomes $\nabla^2\boldsymbol{A} = -\mu\boldsymbol{J}_s$.

The finite element analysis shows that the results of calculation sometimes are not fully satisfactory when a supplement gauge condition is used.

When $L/c\tau \ll 1$, then $\partial \boldsymbol{D}/\partial t$ and $\partial \boldsymbol{B}/\partial t$ can all be neglected and we call this field as quasi-static electromagnetic field, where $c = 1\big/\sqrt{\mu\epsilon}$ is the optic velocity in media, L is the maximum size of the body, and τ is the concerned time interval. Neglecting $\partial \boldsymbol{D}/\partial t$ and $\partial \boldsymbol{B}/\partial t$, the electric and magnetic fields are independent from each other, so the electric field and magnetic field can be solved independently. When material constants are all constant for static case, the Maxwell equations is reduced to

$$\nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{E}) = \rho_{\mathrm{e}}, \quad \nabla \times (\boldsymbol{\mu}^{-1} \cdot \boldsymbol{B}) = \boldsymbol{J} \tag{1.13}$$

For the static electric field, we can always introduce an electric potential or potential φ. For the case without electric current, i.e., $\boldsymbol{J} = \boldsymbol{0}$, in material, the static magnetic field can also be expressed by a scalar magnetic potential ψ. In this case, we have

$$\begin{aligned} \boldsymbol{E} &= -\nabla\varphi, \quad \nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{E}) = \rho_{\mathrm{e}}, \quad \epsilon_{kl}\varphi_{,lk} = \rho_{\mathrm{e}} \\ \boldsymbol{H} &= -\nabla\psi, \quad \nabla \cdot (\boldsymbol{\mu} \cdot \boldsymbol{H}) = 0, \quad \mu_{kl}\psi_{,lk} = 0 \end{aligned} \tag{1.14}$$

The electromagnetic energy $\mathfrak{A}$ and its Legendre transformation, the electromagnetic Gibbs free energy g, stored in the medium are

$$\mathrm{d}\mathfrak{A} = \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{D} + \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{B}, \quad \mathrm{d}g = \mathrm{d}\mathfrak{A} - \mathrm{d}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) = -\boldsymbol{D} \cdot \mathrm{d}\boldsymbol{E} - \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{H} \tag{1.15}$$

1.2.7 Interface Connective (or Continuity), Boundary, and Initial Conditions

The interface connective conditions of $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}$ of electromagnetic media 1 and 2 are

$$\begin{aligned} &(\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \boldsymbol{n} = \sigma_{\mathrm{s}} \quad \text{or} \quad D_{2n} - D_{1n} = \sigma_{\mathrm{s}} \\ &(\boldsymbol{E}_2 - \boldsymbol{E}_1) \cdot \boldsymbol{n} = (\sigma_{\mathrm{s}} + \sigma_{\mathrm{sp}})/\epsilon_0, \quad \sigma_{\mathrm{sp}} = -(\boldsymbol{P}_2 - \boldsymbol{P}_1) \cdot \boldsymbol{n} \end{aligned} \tag{1.16}$$

$$\begin{aligned} &(\boldsymbol{B}_2 - \boldsymbol{B}_1) \cdot \boldsymbol{n} = 0 \quad \text{or} \quad B_{2n} - B_{1n} = 0 \\ &(\boldsymbol{H}_2 - \boldsymbol{H}_1) \cdot \boldsymbol{n} = -(\boldsymbol{M}_2 - \boldsymbol{M}_1) \cdot \boldsymbol{n}/\mu_0 \end{aligned} \tag{1.17}$$

$$\begin{aligned} &\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_{\mathrm{s}}, \quad \text{or} \\ &\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \mu_0(\boldsymbol{J}_{\mathrm{s}} + \boldsymbol{J}_{\mathrm{sm}}), \quad \boldsymbol{J}_{\mathrm{sm}} = \boldsymbol{n} \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) \end{aligned} \tag{1.18}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = -\boldsymbol{\nabla}(\pi_{\mathrm{s}}/\epsilon) \tag{1.19}$$

In Eqs. (1.16), (1.17), (1.18), and (1.19), $\boldsymbol{n}$ is the normal of the material 1, σ_{sp} is the surface polarization charge density, $\boldsymbol{J}_{\mathrm{sm}}$ is the surface magnetization electric current density, and $\pi_s = \sigma_s d$ is the electric couple density on the interface. There are only two independent interface conditions in Eqs. (1.16), (1.17), (1.18), and (1.19). If material 2 does not exist, let $\boldsymbol{D}_2 = \boldsymbol{E}_2 = \boldsymbol{B}_2 = \boldsymbol{H}_2 = \boldsymbol{0}$; the boundary conditions can be obtained from Eqs. (1.16), (1.17), (1.18), and (1.19).

On the interface, the conservation condition of the electric current is

$$(\boldsymbol{J}_2 - \boldsymbol{J}_1) \cdot \boldsymbol{n} = -\partial \sigma_s / \partial t = -\dot{\sigma}_s \tag{1.20}$$

The initial conditions are

$$\begin{aligned} &\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(0), \quad \dot{\boldsymbol{E}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{E}}_0(0), \quad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(0), \\ &\dot{\boldsymbol{H}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{H}}_0(0), \quad \boldsymbol{x} \in V \end{aligned} \tag{1.21}$$

In Eq. (1.21), there are only still two independent conditions.

1.2.8 Electromagnetic Force

Multiplying the second equation in Eq. (1.4) by $\boldsymbol{D}$ and the fourth by $\boldsymbol{B}$, then adding the results we obtain

$$\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{D} \times \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) + \frac{\partial \boldsymbol{D}}{\partial t} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{0} \tag{1.22}$$

Using

$$\begin{aligned} &\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) = (\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D} - (\boldsymbol{D} \cdot \nabla)\boldsymbol{E}, \quad \nabla \cdot (\boldsymbol{D} \otimes \boldsymbol{E}) = (\nabla \cdot \boldsymbol{D})\boldsymbol{E} + (\boldsymbol{D} \cdot \nabla)\boldsymbol{E} \\ &(\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D} + (\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} = \nabla \cdot [(\boldsymbol{E} \cdot \boldsymbol{D})\mathbf{I}], \quad \mathbf{I} = \delta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned}$$

and the similar relations for $\boldsymbol{B}, \boldsymbol{H}$, Eq. (1.22) is reduced to

$$\begin{aligned} &\nabla \cdot [(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\mathbf{I} - (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H})] + \frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) \\ &\quad = -\rho_e \boldsymbol{E} - \boldsymbol{J} \times \boldsymbol{B} + (\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} + (\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{H} \end{aligned} \tag{1.23}$$

where the notation $\otimes$ is the tensor product, $\boldsymbol{A} \otimes \boldsymbol{B} = A_i B_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, and $\boldsymbol{e}_i$ is the unit vector on coordinate axis x_i. Using the conservation law of the electric charge, Eq. (1.23) can be written in the form of the electromagnetic momentum equation:

$$\begin{aligned} &\boldsymbol{f}^{\mathrm{M}} = \nabla \cdot \sigma^{\mathrm{M}} - \partial \mathbf{g}^{\mathrm{M}} / \partial t; \qquad \mathbf{g}^{\mathrm{M}} = \boldsymbol{D} \times \boldsymbol{B} \\ &\sigma^{\mathrm{M}} = (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H}) - (1/2)(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\mathbf{I} \\ &\boldsymbol{f}^{\mathrm{M}} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - (1/2)[(\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} - (\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D}] \\ &\qquad - (1/2)[(\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{H} - (\nabla \otimes \boldsymbol{H}) \cdot \boldsymbol{B}] \end{aligned} \tag{1.24}$$

where $\sigma^{\mathrm{M}}, \boldsymbol{f}^{\mathrm{M}}$, and $\mathbf{g}^{\mathrm{M}}$ are called the Maxwell stress tensor, electromagnetic body force, and electromagnetic momentum density, respectively.

If $\boldsymbol{D} = \epsilon_0 \cdot \boldsymbol{E} + \boldsymbol{P},\ \boldsymbol{B} = \mu_0 \cdot (\boldsymbol{H} + \boldsymbol{M})$ are used, Maxwell equations become

$$\begin{aligned}&\epsilon_0\nabla\cdot\boldsymbol{E}=\rho_{\mathrm{e}}-\nabla\cdot\boldsymbol{P},\quad \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\quad \nabla\cdot\boldsymbol{B}=0,\\&\mu_0^{-1}\nabla\times\boldsymbol{B}=\boldsymbol{J}+\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}+\frac{\partial\boldsymbol{P}}{\partial t}+\nabla\times\boldsymbol{M}\end{aligned}\tag{1.25}$$

Analogous to the derivation of Eq. (1.24), we get

$$\begin{aligned}\boldsymbol{f}'^{\mathrm{M}}&=\nabla\cdot\boldsymbol{\sigma}'^{\mathrm{L}}-\partial\mathbf{g}'^{\mathrm{L}}/\partial t,\quad \mathbf{g}'^{\mathrm{L}}=\epsilon_0\boldsymbol{E}\times\boldsymbol{B}\\\boldsymbol{\sigma}'^{\mathrm{L}}&=(\epsilon_0\cdot\boldsymbol{E}\otimes\boldsymbol{E}+\mu_0^{-1}\boldsymbol{B}\otimes\boldsymbol{B})-(1/2)(\epsilon_0E^2+\mu_0^{-1}B^2)\mathbf{I}\\\boldsymbol{f}'^{\mathrm{M}}&=\rho_{\mathrm{e}}\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}-(\nabla\cdot\boldsymbol{P})+(\partial\boldsymbol{P}/\partial t+\nabla\otimes\boldsymbol{M})\times\boldsymbol{B}=\rho_{\mathrm{t}}\boldsymbol{E}+\boldsymbol{J}_{\mathrm{t}}\times\boldsymbol{B}\end{aligned}\tag{1.26}$$

Using above method it is found that the Maxwell stress and electromagnetic body forces are not unique (Pao 1978). The reason may be that boundary conditions are not considered. The electromagnetic momentum equation can also be discussed by the macroscopic Lorentz force method in Sect. 1.2.3. Let a dielectric medium occupies a volume V enclosed by its surface a. Noting that on the surface there are polarized electric surface density $\boldsymbol{n}\cdot\boldsymbol{P}$ and magnetic current surface density $-\boldsymbol{n}\times\boldsymbol{M}$, so the force acted on the body is

$$\begin{aligned}\boldsymbol{F}''=&\int_V[(\rho_{\mathrm{e}}-\nabla\cdot\boldsymbol{P})\boldsymbol{E}+(\boldsymbol{J}+\partial\boldsymbol{P}/\partial t+\nabla\times\boldsymbol{M})\times\boldsymbol{B}]\mathrm{d}V\\&+\int_a[(\boldsymbol{n}\cdot\boldsymbol{P})\boldsymbol{E}-(\boldsymbol{n}\times\boldsymbol{M})\times\boldsymbol{B}]\mathrm{d}a\end{aligned}\tag{1.27}$$

After some manipulations, we get

$$\begin{aligned}\boldsymbol{f}''^{\mathrm{M}}&=\nabla\cdot\boldsymbol{\sigma}''^{\mathrm{M}}-\partial\mathbf{g}''^{\mathrm{M}}/\partial t,\quad \mathbf{g}''^{M}=\epsilon_0\boldsymbol{E}\times\boldsymbol{B}\\\boldsymbol{\sigma}''^{\mathrm{M}}&=(\boldsymbol{D}\otimes\boldsymbol{E}+\boldsymbol{B}\otimes\boldsymbol{H})-(1/2)(\epsilon_0E^2+\mu_0^{-1}B^2-2\boldsymbol{M}\times\boldsymbol{B})\mathbf{I}\\\boldsymbol{f}''^{\mathrm{M}}&=\rho_{\mathrm{e}}\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}-\boldsymbol{P}\cdot(\nabla\otimes\boldsymbol{E})+(\nabla\otimes\boldsymbol{B})\times\boldsymbol{M}+\partial\boldsymbol{P}/\partial t\times\boldsymbol{B}\end{aligned}\tag{1.28}$$

Because the macroscopic Lorentz force is related to the polarization and magnetization of the material, so many different formulas can be got. Eq. (1.28) did not strictly get from complete governing equations. Maugin (1988) considered that in order to avoid arbitrary choice of the ponderomotive force and couple in the electromagnetic contributions, he intend to arrive at expressions for these contributions on the basis of a microscopic model, the electron theory of Lorentz (Eringen and Maugin 1989). In electroelastic analyses only the static electromagnetic force will be discussed by the physical variational principle, and it will be discussed in the next chapter. In this book, the theory concerned with the photon motion is not considered.

1.3 Some Preliminary Knowledge in Electroelasticity

1.3.1 Universal Governing Equations

Universal governing equations must be obeyed by all moving or deforming continuum (Pao 1978; Kuang 2002). In electroelasticity, the universal governing equations are:

1. Mass conservation equation

$$\partial\rho/\partial t+\nabla\cdot(\rho\boldsymbol{v})=\dot{\rho}+\rho\nabla\cdot\boldsymbol{v}=0,\quad \dot{\rho}+\rho v_{k,k}=0,\quad \dot{\rho}=\partial\rho/\partial t+v_k\rho_{,k} \tag{1.29}$$

where ρ is the mass density, $\boldsymbol{v}$ is the velocity.

2. Linear momentum equation

$$\nabla\cdot\boldsymbol{\sigma}+(\mathbf{f}^{\mathrm{m}}+\mathbf{f}^{\mathrm{e}})=\rho\dot{\boldsymbol{v}},\quad \sigma_{ij,i}+\left(f_j^{\mathrm{m}}+f_j^{\mathrm{e}}\right)=\rho\dot{v}_j \tag{1.30}$$

where $\mathbf{f}^{\mathrm{m}}$ and $\mathbf{f}^{\mathrm{e}}$ are the mechanical and electromagnetic forces per volume.

3. Angular momentum equation

$$\mathbf{e}:\boldsymbol{\sigma}+\mathbf{C}^{\mathrm{e}}=0,\quad e_{kij}\sigma_{ij}+C_k^{\mathrm{e}}=0 \tag{1.31}$$

where $\boldsymbol{C}^{\mathrm{e}}=\boldsymbol{P}\times\boldsymbol{E}+\mu_0\boldsymbol{M}\times\boldsymbol{H}$ is the body couple density per volume. The asymmetric part of the stress is induced by the polarization and magnetization in electromagnetic material. From Eq. (1.31), it is also found that the asymmetric part of the stress is the second-order effect of the electromagnetic field. Let the symmetric part of the stress $\boldsymbol{\sigma}$ be denoted by $\boldsymbol{\sigma}^{\mathrm{s}}$, the asymmetric part by $\boldsymbol{\sigma}^{\mathrm{a}}$, then we get

$$\begin{aligned}&\sigma_{kl}=\sigma_{kl}^{\mathrm{s}}+\sigma_{kl}^{\mathrm{a}},\quad \sigma_{lk}^{\mathrm{s}}=(\sigma_{kl}+\sigma_{lk})/2,\\&\sigma_{kl}^{\mathrm{a}}=(\sigma_{kl}-\sigma_{lk})/2=(E_kP_l-E_lP_k+\mu_0H_kM_l-\mu_0H_lM_k)/2\end{aligned} \tag{1.32}$$

1.3.2 Three-Dimensional Governing Equations in Elasticity with Small Deformation

In this chapter, only the case with symmetric stresses is discussed. Let $\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{f}$ be the displacement, stress, strain, and body force per volume, we have (Ogden 1984; Kuang 2002)

Geometric equation $\quad \varepsilon_{ij}=(u_{i,j}+u_{j,i})/2,\quad \boldsymbol{\varepsilon}=(\nabla\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\nabla)/2 \quad$ (1.33)

Momentum equation $\quad \sigma_{ij,j}+f_i=\rho\ddot{u}_i,\quad \nabla\boldsymbol{\sigma}+\boldsymbol{f}=\rho\ddot{\boldsymbol{u}} \quad$ (1.34)

Constitutive equation $\quad \sigma_{ij}=C_{ijkl}\varepsilon_{kl},\quad \boldsymbol{\sigma}=\boldsymbol{C}:\boldsymbol{\varepsilon};\quad \boldsymbol{\varepsilon}=\boldsymbol{s}:\boldsymbol{\sigma} \quad$ (1.35) where $\boldsymbol{C}$ is the elastic coefficient tensor, a repeated index implies summation over the range of the index, and a comma followed by index i indicates partial differentiation with respect to x_i. For an isotropic material, the material constants are reduced to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.36}$$

where δ_{ij} is the Kronecker delta, λ and G are Lamé coefficients.

In this book, the Voigt notations, which express the stress and strain tensors as vectors in six spaces, are also used. In Eq. (1.25), if we use the transformation rule of the subscripts that $11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 31 \to 5,\ 12 \to 6$ in the components of stress σ and the material constant matrix $\boldsymbol{C}$, we get the original form of the stress and strain relation in Voigt notation:

$$\begin{aligned}
&\sigma_i = C_{ij}\varepsilon_j, \quad \varepsilon_i = s_{ij}\sigma_j; \quad \boldsymbol{\sigma} = \boldsymbol{C}\cdot\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{s}\cdot\boldsymbol{\sigma}; \quad \boldsymbol{s} = \boldsymbol{C}^{-1}\\
&\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^{\mathrm{T}}, \quad \boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6]^{\mathrm{T}}
\end{aligned} \tag{1.37}$$

where

$$\begin{aligned}
&\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23},\ \sigma_5 = \sigma_{31},\ \sigma_6 = \sigma_{12}\\
&\varepsilon_1 = \varepsilon_{11},\ \varepsilon_2 = \varepsilon_{22},\ \varepsilon_3 = \varepsilon_{33},\ \varepsilon_4 = \gamma_{23} = 2\varepsilon_{23},\ \varepsilon_5 = \gamma_{31} = 2\varepsilon_{31},\ \varepsilon_6 = \gamma_{12} = 2\varepsilon_{12}
\end{aligned} \tag{1.38}$$

It is noted that the rule transformed the fourth-order tensor s_{ijkl} to the second-order tensor s_{ij} is slightly different with that from C_{ijkl} to C_{ij} (Kuang and Ma 2002). Let the transform matrix of the coordinate systems ϕ' and ϕ be $\boldsymbol{Q}$, $Q_{kl} = \cos(\boldsymbol{i}'_k, \boldsymbol{i}_l)$. The transform rule for a tensor $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}' = \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^{\mathrm{T}}$, so in the original Voigt vector form, the transform rule is

$$\begin{aligned}
&\boldsymbol{\sigma}' = \boldsymbol{A} : \sigma, \quad \boldsymbol{\varepsilon}' = \boldsymbol{B} : \boldsymbol{\varepsilon}\\
&\boldsymbol{A} = \begin{bmatrix} A_{11} & 2A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} A_{11} & A_{12} \\ 2A_{21} & A_{22} \end{bmatrix}, \quad [A_{11}] = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 \end{bmatrix}\\
&[A_{12}] = \begin{bmatrix} Q_{12}Q_{13} & Q_{11}Q_{13} & Q_{12}Q_{11} \\ Q_{22}Q_{23} & Q_{21}Q_{23} & Q_{21}Q_{22} \\ Q_{32}Q_{33} & Q_{31}Q_{33} & Q_{31}Q_{32} \end{bmatrix}, \quad [A_{21}] = \begin{bmatrix} Q_{21}Q_{31} & Q_{22}Q_{32} & Q_{23}Q_{33} \\ Q_{11}Q_{31} & Q_{12}Q_{32} & Q_{13}Q_{33} \\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{13}Q_{23} \end{bmatrix}\\
&[A_{22}] = \begin{bmatrix} (Q_{23}Q_{32} + Q_{22}Q_{33}) & (Q_{21}Q_{33} + Q_{23}Q_{31}) & (Q_{22}Q_{31} + Q_{21}Q_{32}) \\ (Q_{12}Q_{33} + Q_{13}Q_{32}) & (Q_{11}Q_{33} + Q_{13}Q_{31}) & (Q_{11}Q_{32} + Q_{12}Q_{31}) \\ (Q_{12}Q_{23} + Q_{13}Q_{22}) & (Q_{13}Q_{21} + Q_{11}Q_{23}) & (Q_{11}Q_{22} + Q_{12}Q_{21}) \end{bmatrix}
\end{aligned} \tag{1.39}$$

From Eq. (1.39), it is seen that the stress and strain in forms of Eq. (1.37) do not obey the same transform matrix under the coordinate rotation.

1.3.3 Normalized Stress and Strain Vectors

The following normalized stress and strain vector in 6-D space is more convenient. The normalized stress vector $\bar{\boldsymbol{\sigma}}$ and strain vector $\bar{\boldsymbol{\varepsilon}}$ are defined as (Chen 1984; Arramon et al. 2000; Kuang et al. 2003)

$$\begin{aligned}\bar{\boldsymbol{\sigma}} &= \left[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{31}, \sqrt{2}\sigma_{12}\right]^{\mathrm{T}} \\ \bar{\boldsymbol{\varepsilon}} &= \left[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}\Big/\sqrt{2}, \gamma_{31}\Big/\sqrt{2}, \gamma_{12}\Big/\sqrt{2}\right]^{\mathrm{T}} \\ \bar{\boldsymbol{\sigma}} &= \boldsymbol{P}\cdot\boldsymbol{\sigma}, \quad \bar{\boldsymbol{\varepsilon}} = \boldsymbol{P}^{-1}\cdot\boldsymbol{\varepsilon}, \quad [\boldsymbol{P}] = \mathrm{diag}\left[1\ 1\ 1\ \sqrt{2}\ \sqrt{2}\ \sqrt{2}\right]\end{aligned} \tag{1.40}$$

Substitution of Eq. (1.40) into Eq. (1.37) yields the constitutive equation for $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$:

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{C}}\cdot\bar{\boldsymbol{\varepsilon}}, \quad \bar{\boldsymbol{C}} = \boldsymbol{P}\cdot\boldsymbol{C}\cdot\boldsymbol{P}; \quad \bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{s}}\cdot\bar{\boldsymbol{\sigma}}, \quad \bar{\boldsymbol{s}} = \boldsymbol{P}^{-1}\cdot\boldsymbol{s}\cdot\boldsymbol{P}^{-1} \tag{1.41}$$

Under a coordinate system transformation, Eq. (1.41) becomes

$$\begin{aligned}\boldsymbol{\sigma}' &= \boldsymbol{P}\cdot\boldsymbol{\sigma}' = \boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{\sigma} = \boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{P}^{-1}\cdot\bar{\boldsymbol{\sigma}} \\ \boldsymbol{\varepsilon}' &= \boldsymbol{P}^{-1}\cdot\boldsymbol{\varepsilon}' = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{\varepsilon} = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{P}\cdot\bar{\boldsymbol{\varepsilon}}\end{aligned} \tag{1.42}$$

Using Eqs. (1.29), (1.30), (1.31), and (1.32), it is easily shown that

$$\boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{P}^{-1} = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{P} = \boldsymbol{H}; \quad \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} = \boldsymbol{P}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{I} \tag{1.43}$$

From Eqs. (1.42) and (1.43), it is seen that $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are vectors in a same 6-D space.

1.3.4 Some Fundamental Formulas in Finite Deformation

Fundamental formulas for finite deformation can be found in many books (Kuang 2002; Ogden 1984). Here, some fundamental formulas are given which will be used in this book. Notations $\rho, \rho_{\mathrm{e}}, \boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{H}$ and φ, ψ will denote the density, electric charge density, displacement, stress, strain, electric displacement, electric field, magnetic induction, magnetic field, and electric potential and magnetic scalar potential, respectively, in the current configuration. The physical quantities in the initial configuration are expressed by an upper bar '-' on the corresponding physical quantities, such as $\bar{\rho}, \bar{\rho}_{\mathrm{e}}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{E}}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{H}}$ and $\bar{\varphi}, \bar{\psi}$, where $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are the second kind of the Piola-Kirchhoff stress and Green strain, respectively. The coordinate and the subscripts in the current configuration are denoted by small letters, such as x_i, while the initial configuration are denoted by capital letters, such as X_I, $\bar{\sigma}_{IJ}$. Let

$$\begin{aligned}&\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t), \quad \boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}, \quad \bar{\boldsymbol{\varepsilon}} = (\bar{\boldsymbol{C}} - \mathbf{I})/2, \quad \bar{\boldsymbol{C}} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}, \quad \mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}\\ &F_{kL} = x_{k,L} = \delta_{kL} + u_{k,L}, \quad \bar{C}_{KL} = F_{mK}F_{mL} = x_{m,K}x_{m,L}, \quad \bar{\varepsilon}_{KL} = (x_{m,K}x_{m,L} - \delta_{KL})/2\end{aligned} \tag{1.44}$$

where $\boldsymbol{F}$ and $\bar{\boldsymbol{C}}$ are the deformation gradient and deformation tensor, respectively.

It should be noted that the differentiation with the capital or small letter subscript is different, such as $x_{i,J} = \partial x_i/\partial X_J$ and $u_{i,J} = u_{i,m}x_{m,J}$. Because in this book the same coordinate system in the current and initial configurations is taken, when there is no differential symbol, it also has $X_I = X_i$, $\bar{\sigma}_{IJ} = \bar{\sigma}_{ij}$, etc. The transform relations of the mechanical variables in the current and initial configurations are

$$\begin{aligned}&\bar{\rho}_{\mathrm{e}} = j\rho_{\mathrm{e}}, \quad \bar{\rho} = j\rho, \quad \mathrm{d}V = j\,\mathrm{d}\bar{V}, \quad \bar{f}_K = jf_k, \quad \bar{\mathfrak{A}} = j\mathfrak{A}, \quad \bar{g} = jg, \quad j = \left|x_{i,J}\right|\\ &\bar{\sigma}_{IJ} = jX_{I,m}X_{J,n}\sigma_{mn}, \quad \bar{n}_K\mathrm{d}\bar{a} = Jx_{i,K}n_i\mathrm{d}a, \quad \bar{\sigma}^*\mathrm{d}\bar{a} = \sigma^*\mathrm{d}a, \quad J = j^{-1}\\ &\bar{T}_{Kl}\bar{n}_K\mathrm{d}\bar{a} = T_{kl}n_k\mathrm{d}a, \quad \bar{T}_k = jX_{J,n}\bar{n}_J n_n T_k, \quad T_i = Jx_{l,K}n_l N_K\bar{T}_i\\ &x_i = X_I + u_i, \quad x_{i,J}X_{J,l} = \delta_{il}, \quad \bar{\varepsilon}_{IJ} = (x_{k,I}x_{k,J} - \delta_{IJ})/2, \quad \delta\bar{\varepsilon}_{IJ} = x_{k,I}\delta u_{k,J}\end{aligned} \tag{1.45}$$

The electric displacement $\bar{D}_I$ and the magnetic induction $\bar{B}_I$ are defined by the principle that the electric charge and "magnetic charge (=0)" do not change with deformation in a closed surface constituted of the same material points, and the electric field $\bar{E}_I$ and $\bar{H}_I$ are defined by the derivative of φ and ψ with x_I, respectively (Eringen and Maugin 1989; Kuang 2002).

So we have

$$\begin{aligned}&\oint_{\bar{a}} \bar{D}_{I,I}\mathrm{d}\bar{a} = \oint_a D_{i,i}\mathrm{d}a, \quad \bar{D}_I = jX_{I,m}D_m, \quad D_m = Jx_{m,I}\bar{D}_I\\ &\bar{B}_I = jX_{I,m}B_m, \quad B_m = Jx_{m,I}\bar{B}_I, \quad \sigma^*\mathrm{d}a = \bar{\sigma}^*\mathrm{d}\bar{a}\\ &\bar{E}_I = -\varphi_{,I} = -\varphi_{,m}x_{m,I} = x_{m,I}E_m, \quad E_m = X_{I,m}\bar{E}_I,\\ &\bar{H}_I = -\psi_{,I} = x_{m,I}E_m, \quad H_m = X_{I,m}\bar{H}_I\end{aligned} \tag{1.46}$$

From Eq. (1.46), it is known that $\bar{E}_{I,J} = \bar{E}_{J,I} = -\varphi_{,IJ}$ and $\bar{H}_{I,J} = \bar{H}_{J,I} = -\psi_{,IJ}$.

The generalized momentum equations are

$$(\bar{\sigma}_{KL}x_{i,L})_{,K} + \bar{f}_i = \bar{\rho}\ddot{u}_i \tag{1.47}$$

$$\bar{D}_{I,I} = \bar{\rho}_{\mathrm{e}} \tag{1.48}$$

The boundary conditions are

$$x_{i,L}\bar{\sigma}_{KL}\bar{n}_K = \bar{T}_i, \quad \bar{D}_K\bar{n}_K = -\bar{\sigma}, \quad \text{or} \quad \bar{u}_{I} = \bar{u}_I^*, \quad \bar{\varphi} = \bar{\varphi}^* \tag{1.49}$$

1.4 Classical Thermodynamics

1.4.1 Basic Concepts

Classical thermodynamics discusses the thermodynamic system consisted of body, its surroundings, and their common boundary. It is concerned with the state of thermodynamic systems at equilibrium, using macroscopic, empirical properties directly measurable in the laboratory (Wang 1955). Classical thermodynamics study exchanges of energy, work, and heat based on the laws of thermodynamics. The first law of thermodynamics is a principle of conservation of energy and defines a specific internal energy. The second law of thermodynamics is a principle to explain the irreversible phenomenon in nature and defines specific entropy, which will tend to increase over time, approaching a maximum value at equilibrium in an isolated nonequilibrium system. The specific internal energy and specific entropy are two state functions of the system. The thermodynamic state of the system is described by a number of state variables. In classical thermodynamics, state variables usually are pressure p, volume V, temperature T, entropy per volume s, etc. A process means that the state of a system is continuously changed. The succession of states defines the path of the process. The related equations between state parameters are called the constitutive equations or the state equations. An open system is a special class of system with boundaries that matter and energy all can cross. If the matter and energy all cannot cross the boundaries, then the system is an isolated system. If there is no matter exchange between the system and environment, then the system is called the closed system. If there is no heat exchange between the system and environment, then the system is called the insulated system.

1.4.2 Thermodynamic Laws

The First Law of Thermodynamics: The first law of thermodynamics states that energy is always conserved; it can be converted from one form into another, but it cannot be created or destroyed. The first law is correct for reversible and irreversible processes. In a quasi-static process for a homogeneous system, the first law can be written in the form

$$dU = dW + dQ \tag{1.50}$$

where W is that the surroundings does work on the system, Q is the heat absorbed by the system from the surroundings, and U is the internal energy, which is a state function.

The Second Law of Thermodynamics: The second law of thermodynamics points out that every natural process has in some senses a preferred direction of action. In a quasi-static reversible process, a mathematical quantity called the entropy of a system can be defined as

$$\mathrm{d}Q = T\,\mathrm{d}S \tag{1.51}$$

where T is the Kelvin temperature, S is the entropy. The second law shows that the entropy of an isolated macroscopic system never decreases, or the total entropy of a system plus its environment cannot decrease.

Combining the first and second law expressions for the gas, it is obtained:

$$\mathrm{d}U = \mathrm{d}W + T\,\mathrm{d}S = -P\,\mathrm{d}V + T\,\mathrm{d}S \tag{1.52}$$

where P is the pressure. Because U is a state function, so

$$P = -\partial U/\partial V, \quad T = \partial U/\partial S, \tag{1.53}$$

The change in entropy for any process between an initial state a and a final state b satisfies

$$\Delta S = S_b - S_a \geq \int_a^b \mathrm{d}Q/T \tag{1.54}$$

where the equality is for a reversible process and the inequality is for an irreversible process.

1.4.3 Thermodynamic Character Functions

From Eq. (1.53), it is known that if $U(V,S)$ is a known function of V and S, then P and T are known, i.e., all state variables are known. The internal energy $U(V,S)$ is called the character function of the thermodynamic system. Using Legendre transformation, we can get the Helmholtz free energy $F(V,T)$, enthalpy $H(P,S)$, and Gibbs free energy (or Gibbs function) $G(P,T)$. These functions are all the character functions:

$$\begin{aligned} &F(V,T) = U - TS, \quad G(P,T) = U + PV - TS, \quad H(P,S) = U + PV \\ &\mathrm{d}F = -P\,\mathrm{d}V - S\,\mathrm{d}T, \quad \mathrm{d}G = V\,\mathrm{d}P - S\,\mathrm{d}T, \quad \mathrm{d}H = V\,\mathrm{d}P + T\,\mathrm{d}S \end{aligned} \tag{1.55}$$

Usually we use the variables per volume or per mass to describe laws of thermodynamics. In this book, $\mathfrak{A}, f, g, h,$ and s denote internal energy, Helmholtz free energy, Gibbs free energy, enthalpy, and entropy per volume, respectively; p is the pressure and v is the initial unit volume. But in finite deformation, these notations represent variables per mass. In this case, Eqs. (1.52) and (1.55) become

$$\begin{aligned}&\mathfrak{A}(\mathrm{v},s),\quad f(\mathrm{v},T)=\mathfrak{A}-Ts,\quad g(p,T)=\mathfrak{A}+p\mathrm{v}-Ts,\quad h(p,s)=\mathfrak{A}+p\mathrm{v}\\&\mathrm{d}\mathfrak{A}=-p\,\mathrm{dv}+T\,\mathrm{d}s,\quad \mathrm{d}f=-p\,\mathrm{dv}-s\,\mathrm{d}T,\quad \mathrm{d}g=\mathrm{v}\,\mathrm{d}p-s\,\mathrm{d}T,\quad \mathrm{d}h=\mathrm{v}\,\mathrm{d}p+T\mathrm{d}s\end{aligned} \tag{1.56}$$

1.5 Continuum Thermodynamics and Irreversible Processes

1.5.1 General Concept

In continuum mechanics, the internal behaviors of a body are usually inhomogeneous, the internal processes may be irreversible, and the particle in it may be in motion. Extending the classical thermodynamics to continuum, these distinguished features must be considered. It is assumed that the body can be divided into very fine elements, and in each element, its behaviors are homogeneous and the process deviated from the quasi-static process is small, so the Gibbs equation can be used. It means that the specific entropy is still considered as a state function. The total internal and kinetic energies of a body can be obtained through the integral over the volume. Therefore, equations similar to Eq. (1.56) in continuum mechanics are still applicable.

1.5.2 The First Law in Continuum Thermodynamics

Because a body may be in motion, so it has kinetic energy. In continuum thermodynamics, the first law is in the form

$$\dot{U}+\dot{K}=\dot{W}+\dot{Q} \tag{1.57}$$

where $\dot{U}$ is the rate of the internal energy, $\dot{K}$ is the rate of the kinetic energy, $\dot{W}$ is the rate of the work finished by the generalized external force, and $\dot{Q}$ is the rate of the heat supplied by the surroundings. In the electroelastic material, we have

$$\begin{aligned}&U=\int_V \mathfrak{A}\,\mathrm{d}V,\quad K=(1/2)\int_V \rho\dot{u}_i\dot{u}_i\,\mathrm{d}V,\quad \dot{Q}=-\oint_a \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a+\int_V \dot{r}\,\mathrm{d}V\\&\dot{W}=\int_a \boldsymbol{T}^{(n)}\cdot\boldsymbol{v}\,\mathrm{d}a+\int_V \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V+\int_a \varphi\dot{\sigma}\,\mathrm{d}a\end{aligned} \tag{1.58}$$

where $\boldsymbol{f}, \boldsymbol{T}^{(n)}, \boldsymbol{v}, \boldsymbol{q}, \boldsymbol{n}, \varphi, \rho_{\mathrm{e}}, r,$ and σ are the body force per volume, surface force per area, velocity, heat flow vector, normal of the boundary of the surface a, electric potential, volume electric charge density, internal source intensity, and surface electric charge density, respectively. Using the generalized momentum equation, from Eq. (1.58), it yields

$$\begin{aligned}
\dot{W}+\dot{Q} &= \int_a \boldsymbol{T}^{(n)}\cdot\boldsymbol{v}\,\mathrm{d}a+\int_V \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V+\int_a \varphi\dot{\sigma}\,\mathrm{d}a-\oint_a \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a+\int_V \dot{r}\,\mathrm{d}V\\
&= \int_a \sigma_{ij}n_j v_i\,\mathrm{d}a+\int_V f_i v_i\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V-\int_a \varphi\dot{D}_i n_i\,\mathrm{d}a-\oint_a q_i n_i\,\mathrm{d}a+\int_V \dot{r}\,\mathrm{d}V\\
&= \int_V (\sigma_{ij}v_i)_{,j}\,\mathrm{d}V+\int_V f_i v_i\,\mathrm{d}V+\int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V-\int_V (\varphi\dot{D}_i)_{,i}\,\mathrm{d}V-\int_V q_{i,i}\,\mathrm{d}V+\int_V \dot{r}\,\mathrm{d}V\\
&= \int_V \sigma_{ij}v_{i,j}\,\mathrm{d}V+\int_V (\sigma_{ij,j}+f_i)v_i\,\mathrm{d}V+\int_V \varphi(\dot{\rho}_{\mathrm{e}}-\dot{D}_{i,i})\,\mathrm{d}V\\
&\quad -\int_V \varphi_{,i}\dot{D}_i\,\mathrm{d}V+\int_V (\dot{r}-q_{i,i})\,\mathrm{d}V\\
&= \int_V (\sigma_{ij}v_{i,j}-\varphi_{,i}\dot{D}_i-q_{i,i}+\dot{r})\,\mathrm{d}V+\int_V \rho\ddot{u}_i v_i\,\mathrm{d}V=\dot{U}+\dot{K}\\
&= \int_V \dot{\mathfrak{A}}\,\mathrm{d}V+(1/2)\left(\int_V \rho\dot{u}_i\dot{u}_i\,\mathrm{d}V\right)^{\cdot}
\end{aligned}$$

From the above equation, the local form of the energy conservation equation is obtained:

$$\dot{\mathfrak{A}}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}+\boldsymbol{E}\cdot\dot{D}-\nabla\cdot\boldsymbol{q}+\dot{r} \tag{1.59}$$

If there has electric current in the body, a term $\boldsymbol{J}\cdot\boldsymbol{E}$ should be added to the right side of Eq. (1.59). The notation " : " is a double dot production, i.e., $\boldsymbol{A}:\boldsymbol{B}=A_{ij}B_{ij}$.

1.5.3 The Second Law in Continuum Thermodynamics (Clausius-Duhem Inequality)

In continuum thermodynamics, usually the process is irreversible. As an approximate theory, the entropy is divided into reversible and irreversible parts. The reversible part of the entropy $S^{(\mathrm{r})}$ is produced by the external heat, and the irreversible part $S^{(\mathrm{i})}$ is produced by the heat produced in the internal irreversible process, i.e.,

$$\begin{aligned}
&\boldsymbol{Q}=\int_V T\dot{s}\,\mathrm{d}V=\int_V \dot{r}\,\mathrm{d}V-\int_a \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a=\int_V (\dot{r}-q_{i,i})\,\mathrm{d}V\Rightarrow T\dot{s}=\dot{r}-q_{i,i}\\
&\dot{S}^{(\mathrm{r})}=\int_V \dot{s}^{(\mathrm{r})}\,\mathrm{d}V=\int_V T^{-1}\dot{r}\,\mathrm{d}V-\int_a \dot{\boldsymbol{\eta}}\cdot\boldsymbol{n}\,\mathrm{d}a=\int_V (T^{-1}\dot{r}-\dot{\eta}_{i,i})\,\mathrm{d}V\\
&\dot{s}^{(\mathrm{r})}=T^{-1}\dot{r}-\dot{\eta}_{i,i}=T^{-1}(\dot{r}-q_{i,i}+\dot{\eta}_i T_{,i}),\quad \dot{\boldsymbol{\eta}}=T^{-1}\boldsymbol{q},\quad \boldsymbol{\eta}=\int_0^t T^{-1}\boldsymbol{q}\mathrm{d}\tau
\end{aligned} \tag{1.60}$$

where $\boldsymbol{\eta}$ is the entropy displacement vector and $\dot{\boldsymbol{\eta}}$ is the entropy flow vector. $\dot{s}^{(i)}$ is the entropy rate due to the internal dissipative heat per volume and is often denoted by σ. Usually, σ is called the entropy production (rate) and

$$\dot{S}^{(i)} = \int_V \dot{s}^{(i)} \mathrm{d}V = \int_V \sigma \,\mathrm{d}V = \int_V \dot{s}\,\mathrm{d}V - \int_V \dot{s}^{(r)} \mathrm{d}V \tag{1.61}$$

So the local entropy balance equation is

$$\sigma = \dot{s} - \dot{s}^{(r)} = \dot{s} - \dot{r}/T + \boldsymbol{\nabla} \cdot \dot{\eta} \geq 0, \quad \text{or} \quad T\sigma = T\dot{s} - \dot{r} + q_{i,i} - \dot{\eta}_i T_{,i} \geq 0 \tag{1.62a}$$

When the variation of the temperature is not large, Eq. (1.62a) can be reduced to

$$T\sigma \approx T_0 \dot{s} - \dot{r} + q_{i,i} - T_0^{-1} q_i \vartheta_{,i} \geq 0; \quad T = T_0 + \vartheta \tag{1.62b}$$

where T_0 is the reference temperature or the temperature in the environment.

Usually it is assumed that the following equations are held:

$$T\dot{s} = \dot{r} - q_{i,i};\ T\sigma = -\dot{\boldsymbol{\eta}} \cdot \nabla T = -\dot{\eta}_i T_{,i} = -T^{-1} q_i \vartheta_{,i} \geq 0 \tag{1.63}$$

Substituting $\dot{r}$ expressed in Eq. (1.59) into Eq. (1.62) yields

$$T\sigma = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{E} \cdot \dot{\boldsymbol{D}} + T\dot{s} - \dot{\mathfrak{A}} - \dot{\boldsymbol{\eta}} \cdot \nabla T \geq 0 \tag{1.64}$$

Using the electric Gibbs function $g = \mathfrak{A} - Ts - ED$ from Eq. (1.64) we get

$$T\sigma = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D} \cdot \dot{\boldsymbol{E}} - s\dot{T} - \dot{g} - \dot{\boldsymbol{\eta}} \cdot \nabla T \geq 0 \tag{1.65}$$

Equations (1.63), (1.64), and (1.65) are called Clausius-Duhem inequality. For the reversible process, $\sigma = 0$, but for the irreversible process, $\sigma > 0$.

Because $\mathfrak{A}$ and g are state functions, the approximation in first order we can assume

$$\dot{\mathfrak{A}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{E} \cdot \dot{\boldsymbol{D}} + T\dot{s}, \quad \dot{g} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D} \cdot \dot{\boldsymbol{E}} - s\dot{T} \tag{1.66}$$

and

$$\dot{h}_{\mathfrak{A}} = T\sigma = -\dot{\eta}_i T_{,i} \geq 0, \quad \dot{h}_g = -T_{,i}\dot{\eta}_i + (T_{,i}\eta_i)^{\cdot} = \eta_i \dot{T}_{,i}, \quad T\dot{s} = \dot{r} - q_{i,i} \tag{1.67}$$

Equation (1.66) is the Gibbs equation, Eq. (1.67) is the dissipative energy rate, $\dot{h}_{\mathfrak{A}}$, equation and its Legendre transformation or "the complement dissipative energy rate" $\dot{h}_g$. From Eq. (1.66), the constitutive equations can be derived:

$$\begin{aligned} &\sigma_{ij} = \partial\mathfrak{A}/\partial\varepsilon_{ij}, \quad E_i = \partial\mathfrak{A}/\partial D_i, \quad T = \partial\mathfrak{A}/\partial s \\ &\sigma_{ij} = \partial g/\partial\varepsilon_{ij}, \quad D_i = -\partial g/\partial E_i, \quad s = -\partial g/\partial T \end{aligned} \tag{1.68}$$

1.5.4 Thermodynamic Character Functions

In electroelastic mechanics, state variables usually are stress σ, strain ε, electric field strength $\boldsymbol{E}$, electric displacement $\boldsymbol{D}$, temperature $\boldsymbol{T}$, and specific entropy s. Conjugated variable pairs, which form energy rate, are $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}), (\boldsymbol{E}, \boldsymbol{D}), (T, S)$. Each one in three pairs can be used as independent variable, so through the Legendre transform, eight character functions in electroelastic materials can be obtained:

$$\begin{aligned} &\text{Internal energy } \mathfrak{u}: \quad \mathrm{d}\mathfrak{A} = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} + E_i\,\mathrm{d}D_i + T\,\mathrm{d}s \\ &\text{Free energy } f: \quad f = \mathfrak{A} - Ts, \quad \mathrm{d}f = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} + E_i\,\mathrm{d}D_i - s\,\mathrm{d}T \\ &\text{Gibbs function}: \quad g^g = f - \sigma_{ij}\varepsilon_{ij} - E_iD_i, \quad \mathrm{d}g^g = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} - D_i\,\mathrm{d}E_i - s\,\mathrm{d}T \\ &\text{Electric Gibbs function}: \quad g = f - E_iD_i, \quad \mathrm{d}g = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} - D_i\,\mathrm{d}E_i - s\,\mathrm{d}T \\ &\text{Elastic Gibbs function}: \quad g^{\mathrm{el}} = f - \sigma_{ij}\varepsilon_{ij}, \quad \mathrm{d}g^{\mathrm{el}} = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} + E_i\,\mathrm{d}D_i - s\,\mathrm{d}T \\ &\text{Enthalpy}: \quad h = \mathfrak{A} - \sigma_{ij}\varepsilon_{ij} - E_iD_i, \quad \mathrm{d}h = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} - D_i\,\mathrm{d}E_i + T\,\mathrm{d}s \\ &\text{Electric enthalpy}: \quad h^{\mathrm{e}} = \mathfrak{A} - E_iD_i, \quad \mathrm{d}h^{\mathrm{e}} = \sigma_{ij}\,\mathrm{d}\varepsilon_{ij} - D_i\,\mathrm{d}E_i + T\,\mathrm{d}s \\ &\text{Elastic enthalpy}: \quad h^{\mathrm{el}} = \mathfrak{A} - \sigma_{ij}\varepsilon_{ij}, \quad \mathrm{d}h^{\mathrm{el}} = -\varepsilon_{ij}\,\mathrm{d}\sigma_{ij} + E_i\,\mathrm{d}D_i + T\,\mathrm{d}s \end{aligned} \tag{1.69}$$

The internal energy $\mathfrak{A}(\varepsilon, \boldsymbol{E}, s)$ and the electric Gibbs free energy $g(\boldsymbol{\varepsilon}, \boldsymbol{E}, T)$ are often used. Especially the electric Gibbs free energy is more convenient in application. It is easy to see that when the temperature effect is not considered, then the internal energy and the free energy are the same, and the electric Gibbs function and the electric enthalpy are the same.

1.5.5 Irreversible Thermodynamics

The first foundation of the irreversible thermodynamics is that the Gibbs equation is applicable, i.e., Eqs. (1.66), (1.67) and (1.68) are applicable for the state slightly deviated from the equilibrium state. The second foundation (De Groet 1952; Gyarmati 1970; Kuang 2002) is that the dissipative rate equation is constituted of the irreversible force and irreversible flow, and the irreversible flow is a function of the irreversible force or vice versa. So in the electroelastic case from Eq. (1.67), we can derive the so-called evolution equation

$$\dot{\eta}_i = \dot{\eta}_i(T_{,j}), \quad \text{or} \quad T_{,i} = T_{,i}(\dot{\eta}_j) \tag{1.70}$$

In the usual thermal conductive theory, the variation of the temperature is not large; Eq. (1.70) can be written in a linear form:

$$T\dot{\eta}_i = q_i = -\lambda_{ij}(\boldsymbol{x},t)T_{,j}, \quad T_{,j} = -\lambda_{ji}^{-1}T\dot{\eta}_i = -\widehat{\lambda}_{ij}(\boldsymbol{x},t)T\dot{\eta}_i \tag{1.71}$$

where λ is the thermal conducting coefficient which may be related to $(\boldsymbol{x},t)$. Equation (1.71) is the Fourier's law. Sometimes Eqs. (1.68) and (1.70) are called the first and second group constitutive equations, respectively (Kuang 2002).

1.5.6 The Diffusion Problem

The classical thermal diffusion theory is also assumed that the Gibbs equation is still held:

$$\begin{aligned} &\dot{\mathfrak{A}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{E} : \dot{\boldsymbol{D}} + T\dot{s} + \mu\dot{c}, \quad \mathrm{d}\mathfrak{A} = \boldsymbol{\sigma} : \mathrm{d}\varepsilon + \boldsymbol{E} : \mathrm{d}\boldsymbol{D} + T\,\mathrm{d}s + \mu\,\mathrm{d}c \\ &T\dot{s} = \dot{r} - q_{i,i} + \mu d_{i,i}, \quad T\dot{s} + \mu\dot{c} = \dot{r} - q_{i,i}, \quad d_{i,i} = -\dot{c} \end{aligned} \tag{1.72}$$

where μ is the chemical potential, $\boldsymbol{d}$ is the flow vector of the diffusing mass, and c is the concentration. Equation (1.72) shows that the total input heat rate $\dot{r} - q_{i,i}$ is balanced by the sum of $T\dot{s}$ and $\mu\dot{c}$. Using relations

$$T^{-1}q_{i,i} = (T^{-1}q_i)_{,i} + T^{-2}q_iT_{,i}, \quad T^{-1}\mu d_{i,i} = (T^{-1}\mu d_i)_{,i} - d_i(T^{-1}\mu)_{,i} \tag{1.73}$$

from Eqs. (1.72) and (1.73), we get (De Groet 1952; Gyarmati 1970; Kuang 2010)

$$\begin{aligned} &\dot{s} - T^{-1}\dot{r} + (T^{-1}q_i - T^{-1}\mu d_i)_{,i} = -T^{-2}q_iT_{,i} - d_i(T^{-1}\mu)_{,i} = \dot{s}^{(\mathrm{i})}, \quad \text{or} \\ &\dot{s} = \dot{s}^{(\mathrm{r})} + \dot{s}^{(\mathrm{i})}; \quad \dot{s}^{(\mathrm{r})} = (\dot{r}/T) - \nabla\cdot\boldsymbol{J}_s, \quad \boldsymbol{J}_s = (\boldsymbol{q}/T - \mu\boldsymbol{d}/T) \end{aligned} \tag{1.74}$$

From Eq. (1.74), it is obtained:

$$\begin{aligned} &T\dot{s}^{(\mathrm{r})} = \dot{r} - T(\dot{\eta}_i - \mu' d_i)_{,i}, \quad \mu' = \mu/T \\ &T\dot{s}^{(\mathrm{i})} = h_u = -T_{,i}\dot{\eta}_i - \mu'_{,i}\dot{\xi}'_i = \boldsymbol{X}_T\cdot\dot{\boldsymbol{\eta}} + \boldsymbol{X}'_\mu\cdot\dot{\boldsymbol{\xi}}' \geq 0 \\ &\boldsymbol{X}_T = -\nabla T, \quad \boldsymbol{X}'_\mu = -\nabla\mu', \quad \dot{\boldsymbol{\eta}} = \boldsymbol{q}/T, \quad \dot{\boldsymbol{\xi}}' = T\boldsymbol{d} = T\dot{\boldsymbol{\xi}} \end{aligned} \tag{1.75}$$

According to the linear irreversible thermodynamic theory, irreversible flows are proportional to the irreversible forces, so the evolution equations are

$$\begin{aligned} T\dot{\eta}_i &= -\lambda_{ij}T_{,i} - L_{ij}T\mu'_{,i} = -\lambda_{ij}T_{,i} - L_{ij}\mu_{,i} + L_{ij}\mu T_{,i}/T \\ \dot{\xi}_i &= T^{-1}\dot{\xi}'_i = -D_{ij}T\mu'_{,i} - L_{ij}T_{,i} = -D_{ij}\mu_{,i} - L_{ij}T_{,i} + D_{ij}\mu T_{,i}/T \end{aligned} \tag{1.76}$$

where $\boldsymbol{D}$ is the diffusion coefficient and $\boldsymbol{L}$ is the coupling coefficient. When $\mu T_{,i}/T \ll \mu_{,i}$, Eq. (1.76) is reduced to

$$\begin{aligned} &T\dot{s}^{(\mathrm{i})} = h_u \approx \boldsymbol{X}_T \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{X}_\mu \cdot \dot{\boldsymbol{\xi}} = -T_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i \geq 0; \quad \boldsymbol{X}_\mu = -\nabla\mu; \quad \dot{\boldsymbol{\xi}} = \boldsymbol{d} \\ &T\dot{\eta}_i = -\lambda_{ij}(T)T_{,i} - L_{ij}(T)\mu_{,i}, \quad \dot{\xi}_i = -D_{ij}(T)\mu_{,i} - L_{ij}(T)T_{,i}, \quad \text{or} \\ &T_{,i} = -\hat{\lambda}_{ij}(T)T\dot{\eta}_i - \hat{L}_{ij}(T)\dot{\xi}_i, \quad \mu_{,i} = -\hat{D}_{ij}(T)\dot{\xi}_i - \hat{L}_{ij}(T)T\dot{\eta}_i \end{aligned} \tag{1.77}$$

Especially coefficients $\boldsymbol{D}, \boldsymbol{L}, \boldsymbol{\lambda}$ in Eq. (1.77) can be considered as constants. It is noted that the irreversible thermodynamic theory can only give the general form of the evolution equation and the exact form should be given by experiments. For simplicity when the variation of the temperature is not large and considering experimental facts, the extended Fourier's law and Fick's law, Eqs. (1.71) and (1.77) will be used in the future.

In the application of irreversible thermodynamics to more complex materials, such as viscous materials, plastic materials, and materials with phase transformation, the continuum thermodynamics with internal variables is very useful.

1.6 Physical Variational Principle (PVP)

In traditional thermodynamics, the first law is considered as a principle of conservation of energy. But in our papers (Kuang 2007, 2008a, b, 2009a, 2011a, b, c), it was shown that together with the first law of thermodynamics, the known facts show that the physical variational principle (PVP) is also held. The physical variational principle gives a true process for all virtual possible process satisfying the geometrical constrained conditions. If the environment is not considered, the PVP can be expressed as

$$\begin{aligned} &\delta U = \delta W + \delta Q, \quad \text{or} \quad \delta\Pi = \delta U - \delta W - \delta Q = 0 \\ &\delta U = \int_V \delta\mathfrak{A}\,\mathrm{d}V + \int_V \mathfrak{A}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V, \quad \delta W = \int_V \boldsymbol{f}\cdot\delta u\,\mathrm{d}V - \int_V \rho\ddot{\boldsymbol{u}}\cdot\delta u\,\mathrm{d}V + \int_a \boldsymbol{T}\cdot\delta\boldsymbol{u}\,\mathrm{d}a \\ &\delta Q = -\int_a \boldsymbol{q}\delta t\cdot\boldsymbol{n}\,\mathrm{d}a + \int_V \delta r\,\mathrm{d}V \end{aligned} \tag{1.78}$$

where δ is the variation sign and $\boldsymbol{f}, \boldsymbol{T},$ and $\boldsymbol{u}$ are the generalized body force, surface force, and generalized displacement. In complex media, $\mathfrak{A}$ may be the function of strain and other variable, such as electric field $\boldsymbol{E}$. $\mathfrak{A}^{\mathrm{e}}$ is the energy from the total energy minus the pure deformation energy (see later Sect. 2.1.2). In general, W and Q are not the state functions, so they are dependent to path. In general case, the discussed body, environment, and their common interface should be considered together, which will be discussed in Chap. 2 in detail.

Because the thermodynamic laws do not contain the time, in Eq. (1.78), the D'Alembert principle has been used to make the system in an invented quasi-static equilibrium state. So the first law for the electroelastic process can also be written in the form

$$\begin{aligned}
&\dot{U} = \dot{W} + \dot{Q} \\
&\dot{W} = \int_a \boldsymbol{T}^{(n)} \cdot \boldsymbol{v}\,\mathrm{d}a + \int_V (\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \boldsymbol{v}\,\mathrm{d}V + \int_V \varphi\dot{\rho}_{\mathrm{e}}\,\mathrm{d}V + \int_a \varphi\dot{\sigma}\,\mathrm{d}a
\end{aligned} \tag{1.79}$$

If the isothermal reversible case with $\delta Q = 0, \delta T = 0$ is discussed, the electroelastic process is reversible. In practice, the specific electric Gibbs free energy $g(\varepsilon, \boldsymbol{E}, T)$ is preferred to use in physical variational principle, i.e., we often use

$$\begin{aligned}
&\delta\Pi_g = \int_V \delta g\,\mathrm{d}V + \int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V - \delta W^* - \delta Q = 0 \\
&\delta W^* = \int_a \boldsymbol{T}^{(n)} \cdot \delta\boldsymbol{u}\,\mathrm{d}a + \int_V (\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \delta u\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V - \int_{a_{\mathrm{D}}} \sigma^*\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{1.80}$$

where W^* is the sum of the work of the external force on the medium and the complementary work of the medium on the electric field and g^{e} is the part containing $\boldsymbol{E}$ or φ in g.

In many problems, the variation of a variable ϕ different with displacement $\boldsymbol{u}$ should be divided into local variation and migratory variation, i.e., the variation $\delta\phi = \delta_\phi\phi + \delta_u\phi$, where the local variation $\delta_\phi\phi$ of ϕ is the variation duo to the change of ϕ itself and the migratory variation $\delta_u\phi$ of ϕ is the variation of change of ϕ due to virtual displacements. It is also noted that the new force produced by the migratory variation $\delta_u\phi$ will enter the virtual work δW or δW^* as the same as the external mechanical force (see later Sect. 2.1.2). It is noted that the local variation does not produce the variation of volume, but the variation of the volume really occurred due to the migratory variation. So in Eq. (1.80), the term $\int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V$ should be considered for the nonlinear electroelastic analysis.

The physical variational principle is considered to be one of the fundamental physical principles for the quasi-equilibrium state case, which can be used to derive governing equations in continuum mechanics and other fields. We can also give it a simple explanation that the true displacement is one kind of the virtual displacement and obviously it satisfies the variational principle. Other virtual displacements cannot satisfy this variational principle; otherwise, the first law is not objective.

The physical variational principle is different to the usual mathematical variational method which is based on the known physical facts. According to this principle, the variational principle can be obtained automatically if the energy expression is given. We consider that the first law of the thermodynamics includes two contents: energy conservation and physical variational principle. The PVP can also be considered as a generalization of the general virtual work principle in some senses.

1.7 Some Extensions in Continuum Thermodynamics

1.7.1 Extension of the First and Second Laws in Continuum Thermodynamics

In continuum mechanics, the state variables in a system are varied not only in space but also with time. Similar to the mechanical kinetic energy in continuum mechanics, an inertial heat can be added to energy equation. For a homogeneous system, the first law in the classical and continuum thermodynamics can be extended to the case where the temperature is varied with time, i.e., the first law; Eqs. (1.50) and (1.57) in continuum mechanics can be extended, respectively, to

$$\mathrm{d}U = \mathrm{d}W + \mathrm{d}Q - \mathrm{d}Q_{\mathrm{T}} \tag{1.81a}$$

$$\dot{U} + \dot{K} = \dot{W} + \dot{Q} - \dot{Q}_{\mathrm{T}}; \quad \dot{Q}_{\mathrm{T}} = \int_V C\rho_{s0}\ddot{T}\,\mathrm{d}V \tag{1.81b}$$

where $\dot{Q}_{\mathrm{T}}$ is the inertial heat rate. $C\rho_{s0}$ can be assumed as a constant, and it can be measured by experiments. The first law in classical thermodynamics is a law in an ideal state, but Eqs. (1.81a) and (1.81b) are approximate formulas in a practical situation. Equations (1.81a) and (1.81b) may be important for the thermodynamic problem with microscopic time and size. It is obvious that the experiments to prove this theory are very important and meaningful. For a homogeneous pure heat problem without heat conduction, using $\dot{U} = C\dot{T}V$, $\dot{Q}_{\mathrm{T}} = C\rho_{s0}\ddot{T}V$, and $\dot{Q} = \dot{r}V$, where C is the specific heat per volume, Eq. (1.81) becomes

$$C(\dot{T} + \rho_{s0}\ddot{T}) = \dot{r} \tag{1.82}$$

The second law, Eq. (1.54), can be extended to

$$\Delta(S + S^{(\mathrm{a})}) = (S + S^{(\mathrm{a})})_b - (S + S^{(\mathrm{a})})_a \geq \int_a^b \mathrm{d}Q/T \tag{1.83}$$

where $s^{(\mathrm{a})}$ is called the inertial entropy, which is reversible, corresponding to the inertial heat.

1.7.2 Inertial Entropy Theory

As an extension of the inertial heat, we modify the thermodynamic entropy equation by adding a term containing inertial entropy in Eqs. (1.60) and (1.63) (Kuang 2009b), i.e.,

$$\begin{aligned}
&T\dot{s}+T\dot{s}^{(\mathrm{a})}=\dot{r}-q_{i,i}=\dot{r}-(T\dot{\eta}_i)_{,i},\quad \dot{s}^{(\mathrm{a})}=\rho_s\ddot{T},\quad \rho_s=\rho_{s0}C/T\\
&\dot{s}=\dot{s}^{(\mathrm{r})}+\dot{s}^{(\mathrm{i})};\quad \dot{s}^{(\mathrm{r})}+\dot{s}^{(\mathrm{a})}=\dot{r}/T-\dot{\eta}_{i,i};\quad \dot{\eta}_i=q_i/T\\
&T\sigma=T\dot{s}-T\dot{s}^{(\mathrm{r})}=T\dot{s}+T\dot{s}^{(\mathrm{a})}-\dot{r}+T\dot{\eta}_{i,i}=-\dot{\eta}_iT_{,i}\geq 0;\quad \sigma=\dot{s}^{(\mathrm{i})}=-\dot{\eta}_iT_{,i}/T
\end{aligned} \tag{1.84}$$

where s is the usual entropy density, $s^{(\mathrm{r})}$ is the reversible part of s produced by the difference of the external heat and the inertial heat, $s^{(\mathrm{i})}$ is the irreversible part of s produced by the internal irreversible process and is often denoted by the entropy production rate $\dot{s}^{(\mathrm{i})}=\sigma$, $s^{(\mathrm{a})}$ is the inertial entropy, and we assume that $\dot{s}^{(\mathrm{a})}$ is proportional to the acceleration of the temperature. ρ_s is called the inertial entropy coefficient, ρ_{s0} is an inertial time constant, and $T\dot{s}^{(\mathrm{a})}=C\rho_{s0}\ddot{T}$ is the inertial heat rate per volume. Comparing Eq. (1.84) with the classical entropy equation, it is found that in Eq. (1.84), $T\dot{s}+T\dot{s}^{(\mathrm{a})}$ is used instead of $T\dot{s}$ in the classical theory. In Eq. (1.84), s is still a state function because $s^{(\mathrm{a})}$ is reversible. If the inertial entropy is omitted, then Eq. (1.84) is reduced to the entropy equation in the classical thermodynamics. The concepts of the inertial entropy and the inertial heat are complementing each other and $\dot{Q}_{\mathrm{T}}=\int_V T\dot{s}^{(\mathrm{a})}\,\mathrm{d}V$. The ideal of the inertial entropy theory is that when the inertial heat is subtracted from the external supplied heat, the system can be studied by the classical thermodynamics. In other words, at any fixed time, the system is located at a local equilibrium state. The supplied heat δQ by the surrounding cannot be absorbed by the system immediately, but only $\delta Q-\delta Q^{(\mathrm{a})}$, where $\delta Q^{(\mathrm{a})}$ is used to overcome the temperature inertia. $\delta Q^{(\mathrm{a})}$ may be positive or negative. So the inertial entropy theory substantially belongs to the framework of local thermodynamic equilibrium theory. When the temperature is inhomogeneous in the space, the space inhomogeneous temperature is dealt with Fourier's law and its variation with time is dealt by the inertial entropy theory. The temperature wave problem is discussed in the time-space 4-D space.

The dissipative energy $h_{\mathfrak{A}}$ and the complement dissipative energy h_g are the same as that in classical irreversible thermodynamic theory, i.e., they are also expressed by Eq. (1.67). Therefore, in the inertial entropy theory, the evolution equation (1.71) is still held. If $\dot{s}=C\dot{\vartheta}/T, \vartheta=T-T_0$, from Eqs. (1.84) and (1.71), a temperature wave equation is obtained:

$$C(\dot{\vartheta}+\rho_{s0}\ddot{\vartheta})=\dot{r}+\lambda_{ij}(\boldsymbol{x},t)\vartheta_{,ji} \tag{1.85}$$

For uniform temperature field, Eqs. (1.82) and (1.85) are identical.

1.7.3 Cattaneo-Vernotte Theory of Generalized Thermodynamics

In present temperature wave theory given in literatures, the Cattaneo-Vernotte heat conduction model (Vernotte 1958; Cattaneo 1958; Joseph and Preziosi 1989) was extensively used, which is

$$q_i + \tau_{ij}\dot{q}_j = -\lambda_{ij}T_{,j} \tag{1.86}$$

In a purely thermal conduction case, the energy equation is

$$\mathrm{d}\mathfrak{A}/\mathrm{d}t = \dot{r} - q_{i,i}; \quad \mathrm{d}\mathfrak{A} = C\,\mathrm{d}\vartheta \tag{1.87}$$

Substituting Eq. (1.87) into Eq. (1.86) with $\tau_{ij} = \tau_0\delta_{ij}$ yields

$$C(\dot{\vartheta} + \tau_0\ddot{\vartheta}) = \dot{r} + \tau_0\ddot{r} + \lambda_{ij}\vartheta_{,ji} \tag{1.88}$$

Comparing Eqs. (1.85) and (1.88), especially for the isotropic case, it is seen that these two equations are the same if $r = 0$. But in other cases, they are different. Bertola and Cafaro (2007) considered that discarding Fourier's law is not easy because it is supported by general experimental evidence.

In the phonon theory of the lattice thermal conductivity at low temperatures the similar temperature wave equation can also be seen (Jackson and Walker 1971).

Some biomechanical researchers (Xu et al. 2008) used the non-Fourier heat conduction model to study the bioheat transfer, burn damage, biomechanics, and physiology and used the so-called dual-phase-lag heat conduction model:

$$q_i + \tau_0\dot{q}_i = -\lambda(T_{,i} + \tau_T\dot{T}_{,i}) \tag{1.89}$$

1.7.4 Comparison of Inertial Entropy Theory with the Cattaneo-Vernotte Theory

1. *For a Quasi-Isothermal Case*

For a quasi-isothermal case, we have $\vartheta_{,i} \approx 0, \vartheta = \vartheta(t)$. In this case, Eqs. (1.88) and (1.85) are, respectively, reduced to

$$C(\dot{\vartheta} + \tau_0\ddot{\vartheta}) = \dot{r} + \tau_0\ddot{r}, \quad \text{or} \quad C(\vartheta + \tau_0\dot{\vartheta}) = r + \tau_0\dot{r} \tag{1.90}$$

$$C(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}) = \dot{r}, \quad \text{or} \quad C(\vartheta + \rho_{s0}\dot{\vartheta}) = r \tag{1.91}$$

Let $r = RH(t)$, where $H(t)$ is the Heaviside step function and R is a constant. The solution of Eq. (1.90) is

$$C\vartheta(t) = R\left[1 + \left(\tau_0^{-1} - 1\right)e^{-t/\tau_0}\right]H(t), \quad C\vartheta(0) = R\tau_0^{-1} \tag{1.92}$$

where the Dirac delta function $\delta(t) = \dot{H}(t)$ is used. As a typical value $\tau_0 = 1\,\text{ns} \sim 1$ ps $(1\,\text{ns} = 10^{-9}\,\text{s}, 1\,\text{ps} = 10^{-12}\,\text{s})$, Eq. (1.92) shows that $C\vartheta(0) = 10^9 R \sim 10^{12} R$. It is difficult to explain that this large energy is supported by which body. It means that at least this theory is not appropriate for the quasi-isothermal case.

The solution of Eq. (1.91) is

$$C\vartheta(t) = R\left(1 - e^{-t/\rho_{s0}}\right)H(t), \quad C\vartheta(0) = 0 \tag{1.93}$$

As a typical value $\rho_{s0} = 1\,\text{ns} \sim 1\,\text{ps}$ is taken, the time for $CT = U$ from 0 to R is in the order of $1\,\text{ns} \sim 1\,\text{ps}$. It means that the rise of the temperature in the material introduced by the internal heat source reaches its stable value in a very short time. Equation (1.93) shows that the heat pulse is used to overcome the heat inertia at first time, then the inertial heat converts to the internal energy. In order to explain the energy conversion process more clearly, we discuss the second example. Let us assume that $\dot{r}/C\rho_{s0} = at, a = 10^4\text{K}/(\mu\text{s})^3$ in time interval $[0, 10\,\mu\text{s}]$ and after $t = 10\,\mu\text{s}$, $\dot{r} = 0$. Let $\rho_{s0} = 1\text{ns} = 10^{-3}\mu\text{s}, C = 1\text{kJ/kg}\cdot\text{K}$. From Eq. (1.91) for $t \in [0, 10\,\mu\text{s}]$ we get:

$$\begin{aligned} &\ddot{\vartheta} + \dot{\vartheta}/\rho_{s0} = \dot{r}/C\rho_{s0} = at; \quad t \in [0, 10\,\mu\text{s}] \\ &\vartheta(0) = 0, \quad \dot{\vartheta}(0) = 0 \end{aligned} \tag{1.94}$$

Its solution is

$$\begin{aligned} \vartheta(t) &= a\rho_{s0}\left[(1 - e^{-t/\rho_{s0}})\rho_{s0}^2 - \rho_{s0}t + t^2/2\right] \\ \dot{\vartheta}(t) &= a\rho_{s0}^2\left(e^{-t/\rho_{s0}} - 1\right) + a\rho_{s0}t \end{aligned} \tag{1.95}$$

When $t \in [0, 10\,\mu\text{s}]$, $T\dot{s}^{(a)} = \rho_{s0}C\ddot{\vartheta} = a\rho_{s0}C(1 - e^{-t/\rho_{s0}}) > 0$, i.e. the inertial heat increases with time. For $t > 10\,\mu\text{s}$ we get:

$$\begin{aligned} &\ddot{\vartheta} + \ddot{\vartheta}/\rho_{s0} = 0; \quad t > 10\,\mu\text{s} \\ &\vartheta(10) = 499.9\text{K}, \quad \dot{\vartheta}(10) = 99.99\text{K}/\mu\text{s} \end{aligned} \tag{1.96}$$

where $\vartheta(10)$ and $\ddot{\vartheta}(10)$ are obtained from Eq. (1.95). The solution is

$$\vartheta(t) = 499.9 + 99.99\rho_{s0}(1 - e^{-(t-10)/\rho_{s0}}), \quad \dot{\vartheta}(t) = 99.99e^{-(t-10)/\rho_{s0}} \tag{1.97}$$

So $\theta(\infty) = 500\text{K}$, $\dot{\vartheta}(\infty) = 0$. When $t > 10\,\mu\text{s}$, $T\dot{s}^{(a)} = -50Ce^{-(t-5)/\rho_{s0}} < 0$, i.e. the inertial heat decreases with time. At $t = \infty$ the internal heat is zero, $Ts^{(a)}(\infty) = 0$. Eqs. (1.95) and (1.97) show that if the internal heat source *r(t)* is supplied in an inteval $[0, t_f]$, then in the interval $[0, t_f]$, is supplied in an interval $[0, t_\text{f}]$, then in the interval $[0, t_\text{f}]$, the heat supplied by the environment is used to increase the internal energy and overcome the heat inertia; after t_f, the inertial heat converts to the internal energy or increases the temperature.

If the heat inertia is not considered, this absorbed process is instantaneous. The role of the temperature inertia is somewhat different with the mechanical inertia.

2. *For an Anisotropic Material*

In an anisotropic material, τ_{ij} and λ_{ij} in Eq. (1.86) are all tensors. Combining Eqs. (1.86) and (1.87), we get

$$C\dot{\vartheta} - \tau_{ij}\dot{q}_{j,i} = \dot{r} + \lambda_{ij}\vartheta_{,ji} \tag{1.98}$$

Equation (1.98) is difficult to reduce to a simple equation with a single variable ϑ. If we let $\tau_{ij} = \tau_0\delta_{ij}$, Eq. (1.98) is reduced to (1.88). However, if λ_{ij} is a tensor, in general case, it cannot consider τ_{ij} as a scalar. The only possible reason is that τ_{ij} represents an inertial tensor which is a scalar.

3. *About the Entropy Production Rate*

From Eqs. (1.84) and (1.86), it is obtained:

$$\sigma = -\dot{\eta}_i T_{,i}/T = -q_i T_{,i}/T^2 = (1/\lambda T^2)(q_i q_i + \tau_0 q_i \dot{q}_i) \tag{1.99}$$

When $q_i(q_i + \tau_0\dot{q}_i) < 0$, σ may be negative. This violates the Clausius-Duhem inequality $\sigma \geq 0$ in classical thermodynamics based on the local equilibrium. Jou et al. (2001) proposed an extended irreversible thermodynamic theory to improve the Cattaneo-Vernotte model. For a local non-equilibrium state they assumed

$$\begin{gathered} s = s(\mathfrak{A}, \boldsymbol{q}); \quad \mathrm{d}s = (\partial s/\partial\mathfrak{A})\mathrm{d}\mathfrak{A} + (\partial s/\partial \boldsymbol{q})\cdot \mathrm{d}\boldsymbol{q} \\ \partial s/\partial\mathfrak{A} = 1/\Theta, \quad \partial s/\partial\boldsymbol{q} = -(\tau/\lambda T^2)\boldsymbol{q} \end{gathered} \tag{1.100}$$

where Θ is the non-equilibrium temperature. Using Eq. (1.87), for $r = 0$ Eq. (1.100) can be written as

$$\begin{gathered} \dot{s} = (1/\Theta)\mathfrak{A} - (\tau/\lambda T^2)\boldsymbol{q}\cdot \mathrm{d}\boldsymbol{q} = -\nabla\cdot\boldsymbol{J}_s + \sigma \\ \boldsymbol{J}_s = \Theta^{-1}\boldsymbol{q}, \quad \sigma = \boldsymbol{q}\cdot\boldsymbol{X} = \boldsymbol{q}\cdot\left[\nabla\Theta^{-1} - (\tau/\lambda T^2)\dot{\boldsymbol{q}}\right] \end{gathered} \tag{1.101}$$

where $\boldsymbol{J}_s$ is the generalized entropy flow. The irreversible thermodynamics gives $\boldsymbol{q} = L\boldsymbol{X}$. Let $L = \lambda T^2, \lambda > 0$ and approximately let $T = \Theta$ we find

$$\boldsymbol{q} = L\boldsymbol{X} = L\left[\nabla\Theta^{-1} - (\tau/\lambda T^2)\dot{\boldsymbol{q}}\right] = -\lambda\nabla\Theta - \tau\dot{\boldsymbol{q}} \tag{1.102}$$

Equation (1.102) is just the Cattaneo-Vernotte heat conduction equation if we use Θ to instead of T in Eq. (1.86). In this extended irreversible thermodynamic theory we always have $\sigma = LX^2 \geq 0$, but some postulated conditions are added.

However to prove the rationality of the definition of the non-equilibrium specific entropy is a difficult problem. Barletta and Zanchini (1997) pointed out that the extended irreversible thermodynamic theories are disagreed each other and no theory has been verified by experiments yet. The above difficult is not occurred in the inertial entropy theory.

From the above discussions, it is seen that in all published theories, the inertial entropy theory is the simplest one and is the unique consistent with the classical continuum thermodynamics and the constitutive theory. All the above difficulties occurred in theories based on the Cattaneo-Vernotte model do not occurred in the inertial entropy theory. It is also noted that the inertial entropy theory does not reject to use Eq. (1.86) if it is necessary. This theory is also easy to check by experiments in the further.

1.7.5 Thermoelastic Problem

For a thermoelastic problem, the free energy g and constitutive equations can be assumed as

$$g(\varepsilon_{kl},\vartheta)=\frac{1}{2}C_{ijkl}\varepsilon_{ji}\varepsilon_{lk}-\alpha_{ij}\varepsilon_{ij}\vartheta-\frac{1}{2T_0}C\vartheta^2,\quad C_{ijkl}=C_{jikl}=C_{ijlk}=C_{klij},\quad \alpha_{ij}=\alpha_{ji}$$
$$\sigma_{ij}=\partial g/\partial\varepsilon_{ij}=C_{ijkl}\varepsilon_{kl}-\alpha_{ij}\vartheta,\quad s=-\partial g/\partial\vartheta=\alpha_{ij}\varepsilon_{ij}+C\vartheta/T_0 \tag{1.103}$$

where $\boldsymbol{C}_{ijkl}$ is the elastic coefficient and α_{ij} is the stress-temperature coefficient. Using the inertial entropy theory, from Eqs. (1.84) and (1.103), the temperature wave equation is obtained as

$$\alpha_{ij}\dot{\varepsilon}_{ij}+C\dot{\vartheta}/T_0+\rho_{s0}(C/T)\ddot{\vartheta}=\dot{r}/T+\lambda_{ij}\vartheta_{,ji}/T \tag{1.104}$$

Combined Eq. (1.104) with mechanical dynamic equation $\sigma_{ij,j}=\rho\ddot{u}_i$, the governing equations in generalized displacements for the thermoelastic wave under small variation of temperature are

$$C_{ijkl}u_{k,lj}-\alpha_{ij}\vartheta_{,j}=\rho\ddot{u}_i;\quad T_0\alpha_{ij}\dot{u}_{i,j}+C(\dot{\vartheta}+\rho_{s0}\ddot{\vartheta})=\lambda_{ij}\vartheta_{,ji} \tag{1.105}$$

Equation (1.105) can also be obtained by the physical variational principle (see chapter 2). For one-dimensional problem, Eqs. (1.103) and (1.105) are, respectively, reduced to

$$\sigma=Y\varepsilon-\alpha\vartheta,\quad s=\alpha\varepsilon+C\vartheta/T_0 \tag{1.106}$$

$$C(\rho_{s0}\ddot{\vartheta}+\dot{\vartheta})-\lambda\vartheta''+\alpha T_0\dot{u}'=0;\quad \rho\ddot{u}-Yu''+\alpha\vartheta'=0 \tag{1.107}$$

In Eq. (1.106) and (1.107), Y is the elastic modulus and α is the stress-temperature coefficient, and for any function φ, $\dot{\varphi} = \partial\varphi/\partial t$ and $\varphi' = \partial\varphi/\partial x$ are used. For a plane wave propagating along direction x, we assume

$$\begin{aligned} &\widetilde{u} = U\exp[\mathrm{i}(kx - \omega t)], \quad \widetilde{\vartheta} = \Theta\exp[\mathrm{i}(kx - \omega t)] \\ &u = \mathrm{Re}\,\widetilde{u}, \quad \vartheta = \mathrm{Re}\,\widetilde{\vartheta} \end{aligned} \tag{1.108}$$

where U is the amplitude of the displacement and Θ is the amplitude of the temperature, $k = k_1 + \mathrm{i}k_2$ is the wave number and $\mathrm{i} = \sqrt{-1}$. Substituting Eq. (1.108) into (1.107) yields

$$(Yk^2 - \rho\omega^2)U + \mathrm{i}\alpha k\Theta = 0; \quad \alpha T_0 k\omega U + \left[\lambda k^2 - C(\rho_{s0}\omega^2 + \mathrm{i}\omega)\right]\Theta = 0 \tag{1.109}$$

In order to have nontrivial solutions for U, Θ, the coefficient determinant of Eq. (1.109) must be vanished, i.e.,

$$\begin{vmatrix} Yk^2 - \rho\omega^2 & \mathrm{i}\alpha k \\ \alpha T_0 k\omega & \lambda k^2 - C(\rho_{s0}\omega^2 + \mathrm{i}\omega) \end{vmatrix} = \begin{vmatrix} Yk^2 - \rho\omega^2 & \mathrm{i}\alpha k \\ \alpha T_0 k\omega & \lambda k^2 - Cb \end{vmatrix} = 0 \tag{1.110}$$

where $b = \rho_{s0}\omega^2 + \mathrm{i}\omega$ and $\mathrm{Im}\,b = \omega > 0$. From Eq. (1.110), we get

$$\lambda Y k^4 - (CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)k^2 + \rho\omega^2 Cb = 0 \tag{1.111}$$

Though there are two roots for k^2, but for a wave propagating along positive x-direction only, k is selected and $-k$ is neglected. Let $k = k_T$ be the wave number of the temperature wave and another $k = k_Y$ be the wave number of the elastic wave. Let

$$\begin{aligned} &CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega = r_1 e^{\mathrm{i}\varphi_1} \\ &(CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)^2 - 4C\lambda Y\rho\omega^2 b \\ &\quad = (CYb - \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)^2 + 4\mathrm{i}\lambda\rho\omega^2\alpha^2 T_0\omega = r_2 e^{\mathrm{i}\varphi_2} \end{aligned} \tag{1.112}$$

then we have

$$\begin{aligned} &k_Y = \frac{1}{2\lambda Y}\left[\left(r_1 e^{\mathrm{i}\varphi_1} - \sqrt{r_2}e^{\mathrm{i}\varphi_2/2}\right)\right]^{1/2}; \quad k_T = \frac{1}{2\lambda Y}\left[\left(r_1 e^{\mathrm{i}\varphi_1} + \sqrt{r_2}e^{\mathrm{i}\varphi_2/2}\right)\right]^{1/2} \\ &r_1 e^{\mathrm{i}\varphi_1} = CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega, \quad \mathrm{Im}(r_1 e^{\mathrm{i}\varphi_1}) = (CY + \alpha^2 T_0)\omega > 0 \\ &r_2 e^{\mathrm{i}\varphi_2} = (CYb + \lambda\rho\omega^2 + \mathrm{i}\alpha^2 T_0\omega)^2 - 4C\lambda\rho\omega^2 Yb \\ &\mathrm{Im}\left(r_2 e^{\mathrm{i}\varphi_2}\right) = 2\left[(CY\rho_{s0} - \lambda\rho)CY + (CY\rho_{s0} + \lambda\rho)\alpha^2 T_0\right]\omega^3 \end{aligned} \tag{1.113}$$

where the notation "Im" denotes the image part of a complex function. From Eq. (1.109), we can get ϑ and u.

From Eq. (1.113), it is known that $\mathrm{Im}\,k_T$ is always positive, so the temperature wave is always attenuated. However, if $r_1 \sin\varphi_1 < \sqrt{r_2}\sin(\varphi_2/2)$, $\mathrm{Im}\,k_Y < 0$, and in this case, the elastic wave is enlarged. This phenomenon may be applied in the acoustic wave technique.

From this phenomenon, three possible cases can be assumed: (1) The inertial time constant ρ_{s0} is determined by the equation

$$r_1 \sin\varphi_1 = \sqrt{r_2}\sin(\varphi_2/2) \tag{1.114}$$

to make $\operatorname{Im} k_Y = 0$. (2) The second possibility is that the elastic viscosity need be considered. If the elastic viscosity is considered, Eq. (1.114) need be modified. (3) The third possibility is that this phenomenon may also be the inertial effect of the temperature and does not violate the energy conservation law. This phenomenon is also found in the numerical calculation for the elastic wave in pyroelectric material (Yuan and Kuang 2008).

1.7.6 Generalized Inertial Entropy Theory

Analogous to the inertial entropy concept in Eq. (1.84), an inertial concentration concept is introduced in the thermodiffusion problem (Kuang 2010). In the generalized inertial entropy theory, the Gibbs equation, Eq. (1.72), of the classical thermal diffusion theory is changed to

$$\begin{aligned} &T(\dot{s}+\dot{s}^{(\mathrm{a})})+\mu(\dot{c}+\dot{c}^{(\mathrm{a})})=\dot{r}-\nabla\cdot\boldsymbol{q} \\ &\dot{s}^{(\mathrm{a})}=\rho_s\ddot{T}=\rho_{s0}(C/T)\ddot{T},\quad \dot{c}^{(\mathrm{a})}=\rho_\mu\ddot{\mu};\quad \dot{c}=-d_{i,i} \end{aligned} \tag{1.115}$$

where μ is the chemical potential, $\boldsymbol{d}$ is the flow vector of the diffusing mass and c the concentration, and ρ_s and ρ_μ are inertial entropy and inertial concentration coefficients, respectively. Similar to the derivation in Sect. 1.5.6, it can be obtained:

$$\begin{aligned} &(\dot{s}+\dot{s}^{(\mathrm{a})})+\frac{\mu\dot{c}^{(\mathrm{a})}}{T}=\frac{\dot{r}}{T}-\frac{q_{i,i}}{T}+\frac{\mu d_{i,i}}{T}=\frac{\dot{r}}{T}-\left(\frac{q_i}{T}\right)_{,i}-\frac{q_iT_{,i}}{T^2}+\left(\frac{\mu d_i}{T}\right)_{,i}-\left(\frac{\mu}{T}\right)_{,i}d_i \\ &T\dot{s}=T\dot{s}^{(\mathrm{r})}+T\dot{s}^{(\mathrm{i})};\quad T(\dot{s}^{(\mathrm{r})}+\dot{s}^{(\mathrm{a})})+\mu\dot{c}^{(\mathrm{a})}=\dot{r}+T(\dot{\eta}_i-\mu'\dot{\xi}_i)_{,i} \\ &T\sigma=T\dot{s}^{(\mathrm{i})}=-T_{,i}\dot{\eta}_i-\mu'_{,i}\dot{\xi}'_i\approx-T_{,i}\dot{\eta}_i-\mu_{,i}\dot{\xi}_i\geq 0 \end{aligned} \tag{1.116}$$

This theory shows that the heat rate $\int_V \dot{r}\,\mathrm{d}V-\int_a T\dot{\boldsymbol{\eta}}\cdot\boldsymbol{n}\,\mathrm{d}a$ supplied by the environment is transformed to the heat rate stored in the medium, dissipation energy rate, and the inertial heat rate, which is introduced by the inertial entropy and the inertial concentration.

The evolution equation, Eq. (1.77), is still held in the generalized inertial entropy theory.

1.8 The SI System (International System of Units)

1.8.1 SI Base Units

Length (meter, m), Mass (kilogram, kg), Time (second, s), Electrical current (ampere, A), Thermodynamic temperature (Kelvin, K), Amount of substance (mole, mol), Luminous (candela, cd)

1.8.2 Some SI-Derived Units

1. *Mechanics*: Force (Newton, N, kg ms^{-2}), pressure (pascal, Pa, kg/(m s^2) = (N/m^2)), stress (MPa), velocity (or speed) (m s^{-1}), acceleration (m s^{-2}), work, energy, heat (joule, J, m^2 kg s^{-2}, Nm), power or radiant flux (watt, W, kg m^2 s^{-3}, J/s), frequency (hertz, Hz, s^{-1}), modulus of elasticity (GPa), wave number (m^{-1})

2. *Electromagnetism*: Electrical charge (coulomb, C, A s), electrical potential (Volt, V, $\text{N} \cdot \text{m}/\text{A} \cdot \text{s}$, W/A), electric field strength ($\text{V}/\text{m} = \text{N}/\text{C}$), electric displacement (electric flux density) (C/m^2), electrical capacitance (farad, F, $\text{m}^{-2}\,\text{kg}^{-1}\,\text{s}^4\,\text{A}^2$, C/V), electrical inductance (Henry, H, m^2 kg s^{-2} A^{-2}), electrical resistance (ohm, w, m^2 kg s^{-3} A^{-2}, V/A), magnetic flux (weber, Wb, m^2 kg s^{-2} A^{-1}, V s), magnetic flux density (tesla, T, $\text{N}/\text{A} \cdot \text{m}$, Wb/$\text{m}^2$), magnetic field strength (A m^{-1}), permeability (H/m), permittivity (F/m)

3. *Thermodynamics*: Concentration (of amount of substance) (mol m^{-3}), heat capacity (J/K), thermal conductivity (W/(m K)), coefficient of heat transfer (W/(m^2 K)), specific energy (J/kg, m^2 s^{-2}), specific heat capacity (or specific entropy) (J/(kg K), m^2 s^{-2} K^{-1}), heat flow rate (W, J/s), heat flux density (W/m^2)

References

Arramon YP, Mchrabadi MM, Martin DW, Cowin SC (2000) A multi-dimensional anisotropic strength criterion based on Kelvin modes. Int J Solids Struct 37:2915–2935

Barletta A, Zanchini E (1997) Hyperbolic heat conduction and local equilibrium: a second law analysis. Int J Heat Mass Transf 40:1007–1016

Bertola V, Cafaro E (2007) On the speed of heat. Phys Lett A 372:1–4

Cai S-S, Zhu Y (1985) Classical electrodynamics. Shanghai Fudan University Publishing House, Shanghai (in Chinese)

Cattaneo C (1958) Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanee. Comptes Rendus de l'Académie des Sciences Paris 247:431–433

Chen ST (1984) New concepts of elasticity theory and an application. Acta Mech Sin 16:259–274 (in Chinese)

De Groet SR (1952) Thermodynamics of irreversible processes. North-Holland Publishing Company, Amsterdam

Eringen AC, Maugin GA (1989) Electrodynamics of continua. Springer, New York

Gyarmati I (1970) Non-equilibrium thermodynamics, field theory and variational principles. Springer, Berlin/Heidelberg/New York
Ikeda T (1990) Fundamentals of piezoelectricity. Oxford University Press, Oxford
Jackson HE, Walker CT (1971) Thermal conductivity, second sound and phonon-phonon interactions in NaF. Phys Rev B 3:1428–1439
Joseph DD, Preziosi L (1989) Heat waves. Rev Mod Phys 61:41–73
Joseph DD, Preziosi L (1990) Addendum to the paper: heat waves. Rev Mod Phys 62:375–391
Jou D, Casas-Vázquez J, Lebon G (2001) Extended irreversible thermodynamics (third edition), Springer, Berlin, Heidelberg
Kelvin L (1856) Elements of a mathematical theory of elasticity. Philos Trans R Soc 166:481–498
Kruck A (1954) Foundations of electrotechniques. National Electrical Press, Moscow; Кругк А (1954) Оснвыэлектротехники. Государственное энергетическое издадельство, Москва
Kuang Z-B (2002) Nonlinear continuum mechanics. Shanghai Jiaotong University Publishing House, Shanghai (in Chinese)
Kuang Z-B (2007) Some problems in electrostrictive and magnetostrictive materials. Acta Mech Solida Sin 20:219–227
Kuang Z-B (2008a) Some variational principles in elastic dielectric and elastic magnetic materials. Eur J Mech A/Solids 27:504–514
Kuang Z-B (2008b) Some variational principles in electroelastic media under finite deformation. Sci China, Ser G 51:1390–1402
Kuang Z-B (2009a) Internal energy variational principles and governing equations in electroelastic analysis. Int J Solids Struct 46:902–911
Kuang Z-B (2009b) Variational principles for generalized dynamical theory of thermopiezoelectricity. Acta Mech 203:1–11
Kuang Z-B (2010) Variational principles for generalized thermodiffusion theory in pyroelectricity. Acta Mech 214:275–289
Kuang Z-B (2011a) Theory of electroelasticity. Shanghai Jiaotong University Publishing House, Shanghai (in Chinese)
Kuang Z-B (2011b) Physical variational principle and thin plate theory in electro-magneto-elastic analysis. Int J Solids Struct 48:317–325
Kuang Z-B (2011c) Some thermodynamic problems in continuum mechanics. In: Piraján JCM (ed) Thermodynamics – kinetics of dynamic systems. Intech Open Access Publisher, Croatia
Kuang Z-B (2013) Two theoretical problems in electro-magneto-elastic analysis. Acta Mech 224:1213–1224
Kuang Z-B, Ma F-S (2002) Crack tip field. Xian Jiaotong University Publishing House, Xian (in Chinese)
Kuang Z-B, Zhou Z-D, Chen Y, Zhao S-X (2003) Eigen-material constants, mode and failure criterion for piezoelectric media. Int J Appl Electromagn Mech 18:235–250
Landau LD, Lifshitz EM, Pitaevskii LP (1984) Electrodynamics of continuum media, 2nd edn. Pergamon Press, Oxford; Ландау Л Д, Лифшиц ЕМ (1959) Электродинамика сплошных сред. Москва: ФИЗМАТГИЗ
Lang SB (2005) Pyroelectricity: from ancient curiosity to modern imaging tool. Phys Today 58:31–36
Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland Amsterdam, New York, Oxford, Tokyo
Moon EC (1984) Magneto-solid mechanics. A Wiley-Interscience Publication, Wiley, New York/Chichester/Brisbane/Toronto/Singapore
Ogden RW (1984) Nonlinear elastic deformations. Ellis Horwood/Halsted Press, Chichester/New York
Pao YH (1978) Electromagnetic forces in deformable continua. // Nemat-nasser. Mech Today. Bath: printed in Great Britain by Pitman
Stratton JA (1941) Electromagnetic theory. McGraw-Hill, New York

Sun K, Zhang FX (1984) Piezoelectricity. National Defence Industry Publishing House, Beijing (in Chinese)
Vernotte P (1958) Les paradoxes de la théorie continue de l'equation de la chaleur. Comptes Rendus de l'Académie des Sciences Paris 246:3154–3155
Voigt W (1910) Lehrbuch der kristall physic. Teubner, Leipzig
Wang ZX (1955) Thermodynamics. Higher Education Publishing House, Beijing (in Chinese)
Xu F, Seffen KA, Lu TJ (2008) Non-Fourier analysis of skin biothermomechanics. Int J Heat Mass Transf 51:2237–2259
Yuan X-G, Kuang Z-B (2008) Waves in pyroelectrics. J Therm Stress 31:1190–1211

Chapter 2
Physical Variational Principle and Governing Equations

Abstract In this chapter, the physical variational principle is used to derive the governing equations of the nonlinear and linear electroelastic analyses in piezoelectric and pyroelectric materials. Some kinds of the physical variational principle are given. Applying the migratory variation of electric potential, the general expression of the static electric force is given. It is shown that for the physical nonlinear problem, such as the electrostrictive materials, in order to get correct governing equations and material constants in experiments, we need to consider the entire system including the dielectric medium, its environment, and their common boundary. When the temperature varies with time, the inertial entropy and generalized inertial entropy theories are used to derive the temperature wave equation and the mass diffusion equation. This theory is consistent with current known thermodynamic theory.

Keywords Physical variational principle • Governing equation • Inertial entropy theory • Temperature wave equation • Mass diffusion equation

2.1 Electric Gibbs Free Energy Variational Principle in Piezoelectric Materials

2.1.1 Electric Gibbs Free Energy and Constitutive Equations

In Sect. 1.6, the physical variational principle (PVP) was proposed as a basic principle in the continuum mechanics for an electrically static state. In this chapter, we shall discuss its applications. At first, the electric Gibbs free energy variational principle in piezoelectric materials under the isothermal case is discussed. According to Eq. (1.69) in Sect. 1.5.4, the electric Gibbs free energy g is

$$g = g(\varepsilon_{ij}, E_i), \quad \mathrm{d}g = \sigma_{ji}\mathrm{d}u_{i,j} - D_i\mathrm{d}E_i \tag{2.1}$$

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Under the small deformation, g can be expanded in the series of $\boldsymbol{\varepsilon}$ and $\boldsymbol{E}$:

$$\begin{aligned} g &= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij} - (1/2)l_{ijkl}E_iE_j\varepsilon_{kl} \\ &\quad - (1/2)\alpha_{km}E_mE_l\varepsilon_{kl} - (1/4)\alpha_{nm}E_mE_n\varepsilon_{kl}\delta_{kl} \\ C_{ijkl} &= C_{jikl} = C_{ijlk} = C_{klij}, \quad l_{ijkl} = l_{jikl} = l_{ijlk} = l_{klij}, \quad e_{kij} = e_{kji} \end{aligned} \tag{2.2}$$

where $\boldsymbol{C}, \boldsymbol{\epsilon}, \boldsymbol{e}, \boldsymbol{l}$ are the elastic coefficient, permittivity, piezoelectric coefficient, and the electrostrictive coefficient, respectively; $\boldsymbol{\alpha}$ is a new asymmetric or symmetric electrostrictive coefficient in order to make $\boldsymbol{l}$ the same symmetries as that in $\boldsymbol{C}$ (Kuang 2007, 2008a). For convenience, the term $\alpha_{nm}E_mE_n\varepsilon_{kl}\delta_{kl}$ is also added.

Because g is a state function, constitutive equations can be derived as

$$\begin{aligned} \sigma_{lk} &= \partial g/\partial\varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j - (1/2)\alpha_{km}E_mE_l - (1/4)\alpha_{nm}E_mE_n\delta_{kl} \\ D_k &= -\partial g/\partial E_k = \left[\epsilon_{kl} + l_{ijkl}\varepsilon_{ij} + (1/2)(\alpha_{ml}\varepsilon_{mk} + \alpha_{mk}\varepsilon_{ml}) + (1/4)(\alpha_{lk} + \alpha_{kl})\varepsilon_{mn}\delta_{mn}\right]E_l \\ &\quad + e_{kij}\varepsilon_{ij} \approx \epsilon_{kl}E_l \end{aligned} \tag{2.3}$$

In general case in Eqs. (2.2) and (2.3), $\varepsilon_{ij} = u_{i,j}, \varepsilon_{ij} \neq \varepsilon_{ji}$, and there are nine components for ε_{ij} and σ_{ij}. For most practical cases, the body couple is neglected; in this case, σ_{ij} and $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$ are symmetric and each of them only has six components. Let $\boldsymbol{\sigma}^{\mathrm{s}}$ and $\boldsymbol{\sigma}^{\mathrm{a}}$ be the symmetric and asymmetric parts of $\boldsymbol{\sigma}$, respectively, we have

$$\begin{aligned} \sigma_{lk}^{\mathrm{s}} &= (1/2)(\sigma_{kl} + \sigma_{lk}) = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j \\ &\quad - (1/4)(\alpha_{km}E_l + \alpha_{lm}E_k)E_m - (1/4)\alpha_{nm}E_mE_n\delta_{kl} \\ \sigma_{lk}^{\mathrm{a}} &= (1/2)(\sigma_{lk} - \sigma_{kl}) = -(1/4)(\alpha_{km}E_l - \alpha_{lm}E_k)E_m \end{aligned} \tag{2.4}$$

where $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$ was used. In Eq. (2.4), terms containing $\boldsymbol{l}:\boldsymbol{\varepsilon}, \boldsymbol{\alpha}:\boldsymbol{\varepsilon}, \boldsymbol{e}:\boldsymbol{\varepsilon}$ had been neglected. In the usual electromagnetic theory, the electromagnetic body couple is $\boldsymbol{P}\times\boldsymbol{E} = \boldsymbol{D}\times\boldsymbol{E}$. In general, $\boldsymbol{\alpha}$ should be determined by experiments. In this book $\boldsymbol{\alpha} = -2\boldsymbol{\epsilon}$ is assumed, so Eqs. (2.3) and (2.4) are reduced to

$$\begin{aligned} \sigma_{lk} &= \partial g/\partial\varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j + \epsilon_{km}E_mE_l + (1/2)\epsilon_{nm}E_mE_n\delta_{kl} \\ \sigma_{lk}^{\mathrm{s}} &= C_{ijkl}\varepsilon_{ji} - e_{jkl}E_j - (1/2)l_{ijkl}E_iE_j + (1/2)(\epsilon_{km}E_l + \epsilon_{lm}E_k)E_m + (1/2)\epsilon_{nm}E_mE_n\delta_{kl} \\ \sigma_{lk}^{\mathrm{a}} &= (1/2)(\epsilon_{km}E_l - \epsilon_{lm}E_k)E_m \approx (1/2)(D_kE_l - D_lE_k) \end{aligned} \tag{2.5}$$

Equation (2.5) shows that the electric body couple is balanced by the moment produced by the asymmetric stresses (Eringen and Maugin 1989). If the electromagnetic body couple is neglected, all the stresses are symmetric. Using Eq. (2.3), Eq. (2.2) is reduced to

$$g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mathrm{e}}, \quad g^{\mathrm{e}} = -(1/2)\left(D_kE_k + \Delta_{kl}^{g}\varepsilon_{lk}\right); \quad \Delta_{kl}^{g} = e_{mkl}E_m \tag{2.6}$$

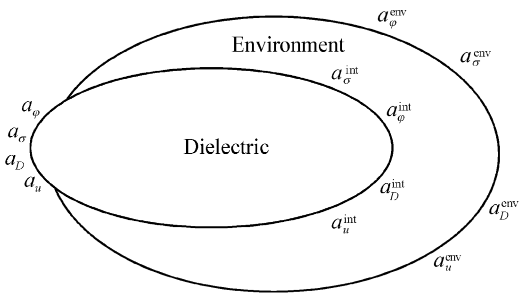


Fig. 2.1 Dielectric and its environment

where g^{e} is the part related to the electric field in g or the energy from the total energy minus the pure deformation energy. The value of the term $\boldsymbol{\Delta}^g : \boldsymbol{\varepsilon}$ is much less than other terms, so it can be neglected.

In the electroelastic analysis, the dielectric medium, its environment, and their common boundary a^{int} consociate a system and should be considered together, because the electric field exists in every material except the ideal conductor. In this book, the variables in the environment will be denoted by a right superscript "env" and the variables on the interface will be denoted by a right superscript "int" (Fig. 2.1). In the environment, Eqs. (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6) are all held.

2.1.2 *Electric Gibbs Free Energy Variational Principle and Governing Equations*

Under the assumption that $\mathbf{u}, \varphi, \mathbf{u}^{\mathrm{env}}, \varphi^{\mathrm{env}}$ satisfy their boundary conditions on their own boundaries $a_u, a_\varphi, a_u^{\mathrm{env}}, a_\varphi^{\mathrm{env}}$ and the continuity conditions on the interface a^{int}. Given the displacement and electric potential virtual increments, the PVP in terms of the electric Gibbs free energy (which is identical to the electric enthalpy in isothermal case) is (Kuang 2007, 2008a, b, 2011a, c)

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0\\
&\delta\Pi_1 = \int_V \delta g\,\mathrm{d}V + \int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V - \delta W\\
&\delta\Pi_2 = \int_{V^{\mathrm{env}}} \delta g^{\mathrm{env}}\,\mathrm{d}V + \int_{V^{\mathrm{env}}} g^{\mathrm{e\ env}}\delta u_{i,i}^{\mathrm{env}}\,\mathrm{d}V - \delta W^{\mathrm{env}}\\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_{a_D} \sigma^*\delta\varphi\,\mathrm{d}a\\
&\delta W^{\mathrm{env}} = \int_{V^{\mathrm{env}}} (f_k^{\mathrm{env}} - \rho\ddot{u}_k^{\mathrm{env}})\delta u_k^{\mathrm{env}}\,\mathrm{d}V - \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}V\\
&\qquad + \int_{a_\sigma^{\mathrm{env}}} T_k^{*\mathrm{env}}\delta u_k^{\mathrm{env}}\,\mathrm{d}a - \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a\\
&\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{2.7}$$

where $\boldsymbol{f}, \boldsymbol{T}^{*}, \sigma^{*}$ are given body force per volume, traction per area, and surface charge density and $\boldsymbol{f}^{\text{env}}, \boldsymbol{T}^{*\text{env}}, \sigma^{*\text{env}}$, and $\boldsymbol{T}^{*\text{int}}, \sigma^{*\text{int}}$ are also given values in the environment and on the interface, respectively. $\boldsymbol{n} = -\boldsymbol{n}^{\text{env}}$ is the outward normal of the interface. It is noted that in Eq. (2.7) the work done by the electric field has the form $q\delta\varphi = (\rho_e dV)\delta\varphi$ with $q = \rho_e dV = const.$, etc. For small deformation, $\delta \int_V g\,\mathrm{d}V = \int_V \delta g\,\mathrm{d}V$ can be used due to small variation of the volume.

The virtual variation of the potential φ is divided into local variation $\delta_\varphi \varphi$ and migratory variation $\delta_u \varphi$, and the similar divisions for $\boldsymbol{E}$, so we have

$$
\begin{aligned}
&\delta\varphi = \delta_\varphi \varphi + \delta_u \varphi, \quad \delta_u \varphi = \varphi_{,p}\delta u_p = -E_p \delta u_p \\
&\partial(\delta\varphi)/\partial x_j = \partial\left(\delta_\varphi \varphi + \varphi_{,p}\delta u_p\right)/\partial x_j = \delta_\varphi\left(\varphi_{,j}\right) + \varphi_{,pj}\delta u_p + \varphi_{,p}\delta u_{p,j} = \delta\left(\varphi_{,j}\right) + \varphi_{,i}\delta u_{i,j} \\
&\delta E_i = -\delta_\varphi\left(\varphi_{,i}\right) - \varphi_{,ip}\delta u_p = \delta_\varphi E_i + \delta_u E_i, \quad \delta_\varphi E_i = -\delta_\varphi \varphi_{,\,i}, \\
&\delta_u E_i = E_{i,p}\delta u_p = E_{p,i}\delta u_p
\end{aligned}
\tag{2.8}
$$

Equation (2.8) shows that $\partial(\delta\varphi)/\partial x_j \neq \delta\left(\partial\varphi/\partial x_j\right)$ when $\delta_u\varphi \neq 0$, and it is discussed also in Eq. (2.130) in Sect. 2.9.1. Using the relation,

$$
\begin{aligned}
&\int_V \delta g\,\mathrm{d}V + \int_V g^{\mathrm{e}}\delta u_{k,k}\,\mathrm{d}V = \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V - \int_V D_i \delta E_i\,\mathrm{d}V - \int_V (1/2) D_k E_k \delta u_{j,j}\,\mathrm{d}V \\
&\quad = \int_a \sigma_{ji} n_j \delta u_i\,\mathrm{d}a - \int_V \sigma_{ji,j}\delta u_i\,\mathrm{d}V - (1/2)\int_a D_k E_k \delta_{ij} n_j \delta u_i\,\mathrm{d}a \\
&\qquad + (1/2)\int_V \left(D_k E_k \delta_{ij}\right)_{,j}\delta u_i\,\mathrm{d}V + \int_a D_i n_i \delta_\varphi \varphi\,\mathrm{d}a \\
&\qquad - \int_V D_{i,i}\delta_\varphi\varphi\,\mathrm{d}V - \int_V D_i E_{p,i}\delta u_p\,\mathrm{d}V
\end{aligned}
\tag{2.9}
$$

where $a = a_\sigma + a_u + a^{\text{int}} = a_D + a_\varphi + a^{\text{int}}$, $\sigma_{ji}\delta\varepsilon_{ij} = \sigma_{ji}\delta u_{i,j}$ for asymmetric σ_{ji}. It is noted that $\delta\varphi = 0,\ \delta_\varphi\varphi \neq 0,\ \delta_u\varphi \neq 0$ on a_φ.

Substitution of Eq. (2.9) into $\delta\Pi_1$ in Eq. (2.7) yields

$$
\begin{aligned}
\delta\Pi_1 = &\int_a \sigma_{jk} n_j \delta u_k\,\mathrm{d}a - \int_{a_\sigma} T_k^{*}\delta u_k\,\mathrm{d}a - \int_V \left(\sigma_{jk,j} + f_k - \rho\ddot{u}_k\right)\delta u_k\,\mathrm{d}V \\
&- (1/2)\int_a D_n E_n n_k \delta u_k\,\mathrm{d}a + (1/2)\int_V (D_n E_n)_{,k}\delta u_k\,\mathrm{d}V - \int_V D_{i,i}\delta_\varphi\varphi\,\mathrm{d}V \\
&+ \int_a D_i n_i \delta_\varphi\varphi\,\mathrm{d}a - \int_V \left[\left(D_i E_p\right)_{,i} - D_{i,i}E_p\right]\delta u_p\,\mathrm{d}V + \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V + \int_{a_D} \sigma^{*}\delta\varphi\,\mathrm{d}a
\end{aligned}
\tag{2.10}
$$

Adding a term $\int_a D_i n_i \left(E_p \delta u_p + \delta_u \varphi\right) \mathrm{d}a = 0$ to Eq. (2.10), we get

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} \left(\sigma_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a - \int_V \left(\sigma_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V - \int_V \left(D_{i,i} - \rho_\mathrm{e}\right) \delta \varphi \, \mathrm{d}V \\
& + \int_{a_D} (D_i n_i + \sigma^*) \delta \varphi \, \mathrm{d}a + \int_a D_i n_i E_p \delta u_p \, \mathrm{d}a - (1/2) \int_a D_n E_n n_k \delta u_k \, \mathrm{d}a \\
& + (1/2) \int_V (D_n E_n)_{,k} \delta u_k \, \mathrm{d}V - \int_V \left(D_i E_p\right)_{,i} \delta u_p \, \mathrm{d}V + \int_{a^\mathrm{int}} \sigma_{jk} n_j \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta \varphi \, \mathrm{d}a \\
= & \int_{a_\sigma} \left(S_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a - \int_V \left(S_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V - \int_V \left(D_{i,i} - \rho_\mathrm{e}\right) \delta \varphi \, \mathrm{d}V \\
& + \int_{a_D} (D_i n_i + \sigma^*) \delta \varphi \, \mathrm{d}a + \int_{a^\mathrm{int}} S_{jk} n_j \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta \varphi \, \mathrm{d}a
\end{aligned} \tag{2.11}$$

In Eq. (2.11), we have

$$\begin{aligned}
& \sigma_{ik}^\mathrm{M} = D_i E_k - (1/2) D_n E_n \delta_{ik}, \\
& S_{kl} = \sigma_{kl} + \sigma_{kl}^\mathrm{M} = C_{ijkl} \varepsilon_{ij} - e_{jkl} E_j - (1/2) l_{ijkl} E_i E_j + \epsilon_{km} E_i E_m + \epsilon_{lm} E_m E_k = S_{kl}
\end{aligned} \tag{2.12}$$

where σ_{ik}^M is the Maxwell stress, $\boldsymbol{S}$ is the pseudo total stress (Jiang and Kuang 2003, 2004) and $\epsilon_{km} E_m = D_k$ has been used. $\boldsymbol{S}$ is a symmetric tensor, but $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\mathrm{M}$ are not. Though the adding term $\int_a D_k n_k E_p \delta u_p \, \mathrm{d}a + \int_a D_k n_k \delta_u \varphi \, \mathrm{d}a$ is zero, the first will be combined with terms of virtual displacements and the second will be combined with terms containing the local variation $\delta_\varphi \varphi$.

Equation (2.10) can also be written as

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} \left(\sigma_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} \sigma_{jk} n_j \delta u_k \, \mathrm{d}a - \int_V \left(\sigma_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V \\
& - (1/2) \int_a D_n E_n n_k \delta u_k \, \mathrm{d}a + (1/2) \int_V (D_n E_n)_{,k} \delta u_k \, \mathrm{d}V - \int_V \left(D_i E_p\right)_{,i} \delta u_p \, \mathrm{d}V \\
& + \int_{a_D} (D_i n_i + \sigma^*) \delta \varphi \, \mathrm{d}a + \int_{a_\varphi + a^\mathrm{int}} D_i n_i \delta_\varphi \varphi \, \mathrm{d}a - \int_{a_D} D_i n_i \delta_u \varphi \, \mathrm{d}a - \int_V \left(D_{i,i} - \rho_\mathrm{e}\right) \delta \varphi \, \mathrm{d}V
\end{aligned} \tag{2.13a}$$

Due to the arbitrariness of $\delta \varphi$, it is obtained:

$$D_i n_i + \sigma^* = 0, \quad \text{on} \quad a_D; \quad D_{i,i} - \rho_\mathrm{e} = 0, \quad \text{in} \quad V \tag{2.13b}$$

Substitution of Eq. (2.13b) into Eq. (2.13a) yields

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} \left(\sigma_{jk} n_j - T_k^*\right) \delta u_k \, \mathrm{d}a + \int_{a^\mathrm{int}} \sigma_{jk} n_j \delta u_k \, \mathrm{d}a - \int_V \left(S_{jk,j} + f_k - \rho \ddot{u}_k\right) \delta u_k \, \mathrm{d}V \\
& - (1/2) \int_a D_n E_n n_k \delta u_k \, \mathrm{d}a + \int_{a_\varphi + a^\mathrm{int}} D_i n_i \delta_\varphi \varphi \, \mathrm{d}a - \int_{a_D} D_i n_i \delta_u \varphi \, \mathrm{d}a \\
= & \int_{a_\sigma} (S_{ij} n_i - T_j^*) \delta u_j \, \mathrm{d}a - \int_V (S_{ij,i} + f_j - \rho \ddot{u}_j) \delta u_j \, \mathrm{d}V + \int_{a^\mathrm{int}} S_{ij} n_i \delta u_j \, \mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta \varphi \, \mathrm{d}a
\end{aligned} \tag{2.13c}$$

In Eq. (2.13c), the following relation was used:

$$\int_{a_\varphi} D_i n_i \delta_\varphi \varphi \,\mathrm{d}a - \int_{a_D} D_i n_i \delta_u \varphi \,\mathrm{d}a = \int_{a_\varphi} D_i n_i \delta\varphi \,\mathrm{d}a - \int_a D_i n_i \delta_u \varphi \,\mathrm{d}a = \int_a D_i n_i E_p \delta \mathrm{u}_p \,\mathrm{d}a$$

Due to the arbitrariness of $\delta\boldsymbol{u}$ and $\delta\varphi$ from Eq. (2.11) or (2.13), it is obtained:

$$\begin{aligned} & S_{jk,j} + f_k = \rho \ddot{u}_k, \quad D_{i,i} = \rho_\mathrm{e}, \quad \text{in} \quad V \\ & S_{jk} n_j = T_k^*, \quad \text{on} \quad a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D \\ & \delta\Pi_1 = \int_{a^\mathrm{int}} S_{ij} n_i \delta u_j \,\mathrm{d}a + \int_{a^\mathrm{int}} D_i n_i \delta\varphi \,\mathrm{d}a \end{aligned} \tag{2.14a}$$

The momentum equation in Eq. (2.14a) in terms of generalized displacements is

$$\begin{aligned} & C_{ijkl} u_{i,jk} + e_{jkl} \varphi_{,jk} - l_{ijkl} \varphi_{,i} \varphi_{,jk} + \left(\epsilon_{km} \varphi_{,l} \varphi_{,m} + \epsilon_{lm} \varphi_{,m} \varphi_{,k}\right)_{,k} + f_l = \rho \ddot{u}_l, \\ & \left[C_{ijkl} u_{i,j} + e_{jkl} \varphi_{,j} - (1/2) l_{ijkl} \varphi_{,i} \varphi_{,j} + \epsilon_{lm} \varphi_{,k} \varphi_{,m} + \epsilon_{km} \varphi_{,m} \varphi_{,l}\right] n_l = T_k^*, \quad \text{on} \quad a_\sigma; \\ & \epsilon_{kl} \varphi_{,lk} = -\rho_\mathrm{e}; \quad \epsilon_{kl} \varphi_{,l} n_k = -\sigma^*, \quad \text{on} \quad a_D \end{aligned} \tag{2.14b}$$

where the terms containing $\boldsymbol{\varepsilon}$ in φ are neglected. Similarly for the environment, we have

$$\begin{aligned} & S_{ij,i}^\mathrm{env} + f_j^\mathrm{env} = \rho^\mathrm{env} \ddot{u}_j^\mathrm{env}, \qquad D_{i,i}^\mathrm{env} = \rho_\mathrm{e}^\mathrm{env}, \quad \text{in} \quad V^\mathrm{env} \\ & S_{ij}^\mathrm{env} n_i^\mathrm{env} = T_j^{*\mathrm{env}}, \quad \text{on} \quad a_\sigma^\mathrm{env}; \quad D_i^\mathrm{env} n_i^\mathrm{env} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^\mathrm{env} \\ & \delta\Pi_2 = \int_{a^\mathrm{int}} S_{ij}^\mathrm{env} n_i^\mathrm{env} \delta u_j^\mathrm{env} \,\mathrm{d}a + \int_{a^\mathrm{int}} D_i^\mathrm{env} n_i^\mathrm{env} \delta\varphi^\mathrm{env} \,\mathrm{d}a \\ & S_{jk}^\mathrm{env} = \sigma_{jk}^\mathrm{env} + \sigma_{jk}^{\mathrm{M\,env}}; \sigma_{jk}^{\mathrm{M\,env}} = D_j^\mathrm{env} E_k^\mathrm{env} - (1/2) D_n^\mathrm{env} E_n^\mathrm{env} \delta_{jk} \end{aligned} \tag{2.15}$$

Using $n_i = -n_i^\mathrm{env}, u_i = u_i^\mathrm{env}, \varphi = \varphi^\mathrm{env}$ and $\delta\Pi_1 + \delta\Pi_2 = \delta W^{*\mathrm{int}}$, we get

$$\left(S_{ij} - S_{ij}^\mathrm{env}\right) n_i = T_j^{*\mathrm{int}}, \quad \left(D_i - D_i^\mathrm{env}\right) n_i = -\sigma^{*\mathrm{int}}, \quad \text{on} \quad a^\mathrm{int} \tag{2.16}$$

The above variational principle requests prior that the generalized displacements satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations:

$$\begin{aligned} & u_i = u_i^*, \quad \text{on} \quad a_u; \quad \varphi = \varphi^*, \quad \text{on} \quad a_\varphi \\ & u_i^\mathrm{env} = u_i^{*\mathrm{env}}, \quad \text{on} \quad a_u^\mathrm{env}; \quad \varphi^\mathrm{env} = \varphi^{*\mathrm{env}}; \quad \text{on} \quad a_\varphi^\mathrm{env} \\ & u_i = u_i^\mathrm{env}, \quad \varphi = \varphi^\mathrm{env}; \quad \text{on} \quad a^\mathrm{int} \end{aligned} \tag{2.17}$$

Equations (2.14), (2.15), (2.16), and (2.17) are the governing equations for the electroelastic analysis.

2.1.3 A Note of the Maxwell Stress

In the books of Stratton (1941) and Landau and Lifshitz (1959), the formula of the stress in an isotropic electrostrictive material was

$$\sigma_{ik} = \partial g_0/\partial \varepsilon_{ik} + \sigma_{ik}^L, \quad \sigma_{ik}^L = (1/2)(2\epsilon - a_1)E_k E_i - (1/2)(\epsilon + a_2)E_m E_m \delta_{ik}$$

where $\partial g_0/\partial \varepsilon_{ik}$ is the stress in the media without the electromagnetic field. This formula is just the pseudo total stress $\boldsymbol{S}$ in Eq. (2.12) for the electrostrictive material. For the Maxwell stress and its related problems in literatures, different author had different understanding as shown in Sect. 1.2.7. McMeeking et al. (2005, 2007) considered that in the electroelastic theory the constitutive model can be simplified to one that embraces simultaneously the Cauchy, Maxwell, electrostrictive and electrostatic stresses, which in any case cannot be separately identified from any experiment. In their method authors did not distinguish the local and migratory variations.

From Eq. (2.12), it is known that the Maxwell stress is related to the square of $\boldsymbol{E}$, i.e., $|\boldsymbol{\sigma}| \propto |\boldsymbol{E}|^2$, but the stress introduced by the piezoelectric effect is related to $\boldsymbol{E}$. So for the piezoelectric material when the electric field is not too large and the piezoelectric coefficient is not too small, the Maxwell stress can be neglected. But the isotropic electrostrictive materials do not have the piezoelectric effect, so in this and similar cases, the Maxwell stress should be considered.

Because the strain is accompanied by the change of the distance between the electric particles, the attraction between electric charges or the Maxwell stress and the stress introduced by strains in the material is produced simultaneously. Though they are produced together, their difference is obvious and important. The strength problem in engineering is determined by the Cauchy stress, which is connected with the constitutive equation. However, the Maxwell stress is an external effective electromagnetic force applied to the body and can be obtained by using the migratory variation of φ in the PVP or in the usual energy principle.

Using $\boldsymbol{D} = D_{\mathrm{n}}\boldsymbol{n} + D_{\mathrm{t}}\boldsymbol{t}$ and similar expressions for $\boldsymbol{E}$ and the continuity conditions in Eqs. (2.16) and (2.17) for the $\boldsymbol{D}, \boldsymbol{E}$ on the interface, the mechanical continuous condition on the interface can also be rewritten as

$$\begin{aligned}
&\boldsymbol{n}\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{env}}) = \tilde{\boldsymbol{T}}^{*\mathrm{int}}, \quad \tilde{\boldsymbol{T}}^{*\mathrm{int}} = \boldsymbol{T}^{*\mathrm{int}} + \boldsymbol{n}\cdot\left(\boldsymbol{\sigma}^{\mathrm{M\,env}} - \boldsymbol{\sigma}^{\mathrm{M}}\right) \\
&\boldsymbol{n}\cdot\left(\boldsymbol{\sigma}^{\mathrm{M\,env}} - \boldsymbol{\sigma}^{\mathrm{M}}\right) = [\boldsymbol{n}\cdot(\boldsymbol{D}^{\mathrm{env}}\otimes\boldsymbol{E}^{\mathrm{env}}) - (1/2)(\boldsymbol{D}^{\mathrm{env}}\cdot\boldsymbol{E}^{\mathrm{env}})\boldsymbol{n}] \\
&\quad - [\boldsymbol{n}\cdot(\boldsymbol{D}\otimes\boldsymbol{E}) - (1/2)(\boldsymbol{D}\cdot\boldsymbol{E})\boldsymbol{n}] = (1/2)\left[D_{\mathrm{n}}\left(E_{\mathrm{n}}^{\mathrm{env}} - E_{\mathrm{n}}\right) - \left(D_{\mathrm{t}}^{\mathrm{env}} - D_{\mathrm{t}}\right)E_{\mathrm{t}}\right]\boldsymbol{n} \\
&\quad = [(\epsilon - \epsilon^{env})/2\epsilon\epsilon^{env}]\left(D_n^2 + \epsilon\epsilon^{env}E_t^2\right)\boldsymbol{n}
\end{aligned} \tag{2.18}$$

where $\boldsymbol{n}$ is the unit normal, subscripts n and t mean the normal and tangential direction respectively; and there is no sum on n and t. Equation (2.18) shows that in the small strain case, the boundary traction produced by the Maxwell stress is along the normal direction. The Maxwell stress can be naturally obtained by the migratory variation of φ in PVP.

2.2 Alternative Forms of the Physical Variational Principles

2.2.1 First Alternative Form of the PVP

From Eqs. (2.14), (2.15), (2.16), and (2.17), it is found that if we use $\boldsymbol{S}$ instead of $\boldsymbol{\sigma}$ and $\boldsymbol{S}^{\text{env}}$ instead of $\boldsymbol{\sigma}^{\text{env}}$ in the governing equations, then the form of governing equations of the physical nonlinear dielectrics is just the same as that in the physical linear electric problem. Therefore, a simpler principle can be obtained: the first alternative form of the variational principle is

$$\begin{aligned}
&\delta\hat{\Pi} = \delta\hat{\Pi}_1 + \delta\hat{\Pi}_2 - \delta W^{\text{int}} = 0 \\
&\delta\hat{\Pi}_1 = \int_V \delta\hat{g}\,\mathrm{d}V - \delta W, \quad \delta\hat{\Pi}_2 = \int_{V^{\text{env}}} \delta\hat{g}^{\text{env}}\,\mathrm{d}V - \delta W^{\text{env}} \\
&\delta\hat{g} = S_{ji}\delta u_{i,j} + D_i\delta\varphi_{,i}, \quad \delta\hat{g}^{\text{env}} = S_{ji}^{\text{env}}\delta u_{i,j}^{\text{env}} + D_i^{\text{env}}\delta\varphi_{,i}^{\text{env}} \\
&S_{kl} = \sigma_{kl} + \sigma_{kl}^{\text{M}}
\end{aligned} \tag{2.19}$$

In Eq. (2.19), the variations of δu and $\delta\varphi$ are all local variations or completely independent, i.e., the migratory variations $\delta_u\varphi$ produced by $\delta\mathbf{u}$ are not needed. $\delta W, \delta W^{\text{env}},$ and δW^{int} are still expressed by Eq. (2.7). An analogous theory was also discussed by Bustamante et al. (2008).

2.2.2 Second Alternative Form of the Physical Variational Principle

Introduce the electric body force $f_k^{\text{e}}, f_k^{\text{e env}}$ and traction $T_k^{\text{e}}, T_k^{\text{e env}}$ in media as

$$\begin{aligned}
f_k^{\text{e}} = \sigma_{jk,j}^{\text{M}}, \quad T_k^{\text{e}} = -\sigma_{jk}^{\text{M}} n_j; \quad f_k^{\text{e env}} = \sigma_{jk,j}^{\text{M env}}, \\
T_k^{\text{e env}} = -\sigma_{jk}^{\text{M env}} n_j^{\text{env}} = \sigma_{jk}^{\text{M env}} n_j
\end{aligned} \tag{2.20}$$

The second alternative form of the variational principle is

$$\begin{aligned}
&\delta\Pi' = \delta\Pi_1' + \delta\Pi_2' - \delta W'^{\text{int}} = 0 \\
&\delta\Pi_1' = \int_V \delta g\,\mathrm{d}V - \delta W', \quad \delta\Pi_2' = \int_{V^{\text{env}}} \delta g^{\text{env}}\,\mathrm{d}V - \delta W'^{\text{env}} \\
&\delta W' = \int_V (f_k + f_k^{\text{e}} - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\text{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} (T_k^* + T_k^{\text{e}})\delta u_k\,\mathrm{d}a - \int_{a_D} \sigma^*\delta\varphi\,\mathrm{d}a \\
&\delta W'^{\text{env}} = \int_{V^{\text{env}}} (f_k^{\text{env}} + f_k^{\text{e env}} - \rho^{\text{env}}\ddot{u}_k^{\text{env}})\delta u_k^{\text{env}}\,\mathrm{d}V - \int_{V^{\text{env}}} \rho_{\text{e}}^{\text{env}}\delta\varphi^{\text{env}}\,\mathrm{d}V \\
&\qquad + \int_{a_\sigma^{\text{env}}} \left(T_k^{*\,\text{env}} + T_k^{\text{e env}}\right)\delta u_k^{\text{env}}\,\mathrm{d}a - \int_{a_D^{\text{env}}} \sigma^{*\,\text{env}}\delta\varphi^{\text{env}}\,\mathrm{d}a \\
&\delta W'^{\text{int}} = \int_{a^{\text{int}}} \left(T_k^{*\,\text{int}} + T_k^{\text{e env}} + T_k^{\text{e}}\right)\delta u_k\,\mathrm{d}a - \int_{a^{\text{int}}} \sigma^{*\,\text{int}}\delta\varphi\,\mathrm{d}a
\end{aligned} \tag{2.21}$$

In Eq. (2.21), the variations of δu and $\delta\varphi$ are also completely independent, i.e., it is also not needed to consider the migratory variation $\delta_u\varphi$. Equation (2.21) is the original form of the PVP Eq. (1.78). The governing equations from Eq. (2.21) are

$$\begin{aligned}
&\sigma_{ij,i} + (f_j + f_j^{\mathrm{e}}) = \rho\ddot{u}_j; \quad D_{i,i} = \rho_{\mathrm{e}}; \quad \text{in} \quad V \\
&\sigma_{ij}n_i = T_j^* + T_j^{\mathrm{e}}, \quad \text{on} \quad a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \\
&\sigma_{ji,j}^{\mathrm{env}} + \left(f_i^{\mathrm{env}} + f_i^{\mathrm{e\,env}}\right) = \rho\ddot{u}_i^{\mathrm{env}}; \quad D_{i,i}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}; \quad \text{in} \quad V^{\mathrm{env}} \\
&\sigma_{ji}^{\mathrm{env}} n_j^{\mathrm{env}} = T_i^{*\mathrm{env}} + T_i^{\mathrm{e\,env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}}; \\
&\left(\sigma_{lk} - \sigma_{lk}^{\mathrm{env}}\right)n_l = T_k^{*\mathrm{int}} + T_k^{\mathrm{e}} + T_k^{\mathrm{e\,env}}, \quad D_k n_k - D_k^{\mathrm{env}} n_k = -\sigma^{*\mathrm{int}}; \quad \text{on} \quad a^{\mathrm{int}}
\end{aligned} \tag{2.22}$$

In many literatures (Pao 1978; Maugin 1988; Moon 1984), the governing equations were expressed in the form of Eq. (2.22) and the electromagnetic force was derived from other methods different with the variational method. In different literatures, $\boldsymbol{f}^{\mathrm{e}}$ and $\boldsymbol{T}^{\mathrm{e}}$ may be different.

2.2.3 The Medium Fully Surrounded by the Air

An important engineering problem is that the medium with symmetric material coefficients is fully surrounded by the air. In air, the mechanical stresses and mechanical energy can be neglected and only the electric field and electric energy should be considered. The physical variational formula (2.7) in this case becomes

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0 \\
&\delta\Pi_1 = \delta\int_V g\,\mathrm{d}V - \delta W, \quad \delta\Pi_2 = \delta\int_{V^{\mathrm{env}}} g^{\mathrm{env}}\,\mathrm{d}V - \delta W^{\mathrm{env}} \\
&\delta W^{\mathrm{int}} = \int_{a_\sigma^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a_q^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a, \quad \delta W = -\int_V \rho\ddot{u}_k\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V \\
&\delta W^{\mathrm{env}} = -\int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\mathrm{d}V + \int_{a_q^{\mathrm{env}}} D_i{}^{*\mathrm{env}} n_i^{env}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a
\end{aligned} \tag{2.23}$$

where the body force is neglected and

$$\begin{aligned}
&g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij} - (1/2)l_{ijkl}E_iE_j\varepsilon_{kl} \\
&g^{\mathrm{env}} = -(1/2)\epsilon_{kl}^{\mathrm{env}}E_k^{\mathrm{env}}E_l^{\mathrm{env}}
\end{aligned} \tag{2.24}$$

2.2.4 Isotropic Materials

For isotropic materials, the constitutive equations are (Kuang 2012)

$$l_{ijkl} = l_1\delta_{ij}\delta_{kl} + l_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \quad \epsilon_{ij} = \epsilon\delta_{ij}, \quad \mu_{ij} = \mu\delta_{ij}, \quad \alpha_{ij} = -2\epsilon\delta_{ij}, \; e_{kij} = 0 \tag{2.25}$$

In isotropic materials, variables $\boldsymbol{S}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}^{\mathrm{M}}$ are all symmetric. So Eqs. (2.2) and (2.3) are reduced to

$$g = (1/2)\lambda\varepsilon_{ii}\varepsilon_{kk} + G\varepsilon_{ij}\varepsilon_{ij} - (1/2)\epsilon E_k E_k - (1/2)(l_1 - \epsilon)E_i E_i \varepsilon_{kk} - (l_2 - \epsilon)E_i E_j \varepsilon_{ij} \tag{2.26}$$

$$\begin{aligned}
&\sigma_{kl} = \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(l_1 - \epsilon)E_i E_i \delta_{kl} - (l_2 - \epsilon)E_k E_l, \\
&D_k = \epsilon E_k + (l_1 - \epsilon)\varepsilon_{ii} E_k + 2(l_2 - \epsilon)\varepsilon_{kl} E_l \approx \epsilon E_k \\
&S_{kl} = \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)l_1 E_i E_i \delta_{kl} - (l_2 - 2\epsilon)E_k E_l = \sigma_{kl} + \sigma_{kl}^{\mathrm{M}}
\end{aligned} \tag{2.27a}$$

If we let $l_1 - \epsilon = a_2, l_2 - \epsilon = (1/2)a_1$, then Eq. (2.27a) is reduced to

$$\begin{aligned}
&\sigma_{kl} = \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(a_2 E_i E_i \delta_{kl} + a_1 E_k E_l) \\
&D_k = \tilde{\epsilon}_{kl} E_l, \quad \tilde{\epsilon}_{kl} = \epsilon\delta_{kl} + a_1\varepsilon_{kl} + a_2\varepsilon_{ii}\delta_{kl} \approx \epsilon\delta_{kl} \\
&S_{kl} = \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(a_2 + \epsilon)E_i E_i \delta_{kl} + (1/2)(2\epsilon - a_1)E_k E_l = \sigma_{kl} + \sigma_{kl}^{\mathrm{M}}
\end{aligned} \tag{2.27b}$$

The first formula in Eq. (2.27b) is just the usual form of the constitutive equation, where a_1 and a_2 are known as electrostrictive coefficients. From Eqs. (2.14), (2.15), (2.16), and (2.17), it is known that solving $\boldsymbol{S}$ is easier than that for $\boldsymbol{\sigma}$, so in experiments, the measured variables usually are $(\boldsymbol{S}, \boldsymbol{\varepsilon}, \boldsymbol{E})$. If the constitutive equation (2.27a) is used, the measured material coefficients are l_1 and $l_2 - 2\epsilon$. If the constitutive equation (2.27b) is used, the measured material coefficients are $2\epsilon - a_1$ and $a_2 + \epsilon$. Therefore, in experiments, the entire system including the dielectric medium, its environment, and their common boundary should be considered together, and appropriate governing equations should be selected.

2.2.5 The Static Electric Force Acting on the Medium by the Electric Field

Comparing Eqs. (2.7) and (2.21), it is found that the difference between them is that in Eq. (2.7), the local variation and the migratory variation are used simultaneously, however in Eq. (2.21), only the local variation is used, but the electric force introduced by electric field is introduced:

$$\delta W^{\mathrm{e}} = \int_V f_k^{\mathrm{e}} \delta u_k \mathrm{d}V + \int_{a_\sigma} T_k^{\mathrm{e}} \delta u_k \mathrm{d}a + \int_{V^{\mathrm{env}}} f_k^{\mathrm{e\ env}} \delta u_k^{\mathrm{env}} \,\mathrm{d}V + \int_{a_\sigma^{\mathrm{env}}} T_k^{\mathrm{e\ env}} \delta u_k^{\mathrm{env}} \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \left(T_k^{\mathrm{e}} + T_k^{\mathrm{e\ env}}\right) \delta u_k \,\mathrm{d}a \tag{2.28}$$

In Eqs. (2.7) and (2.10), the part related to the migratory variations of potential is

$$\begin{aligned}\delta_u \Pi = {} & \int_V g_{,E} \cdot \delta_u \boldsymbol{E} \,\mathrm{d}V + \int_V g^{\mathrm{e}} \delta u_{k,k} \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta_u \varphi \,\mathrm{d}V + \int_{a_D} \sigma^* \delta_u \varphi \,\mathrm{d}a + \int_{V^{\mathrm{env}}} g_{,E}^{\mathrm{env}} \cdot \delta_u E^{\mathrm{env}} \,\mathrm{d}V \\ & + \int_{V^{\mathrm{env}}} g^{\mathrm{e\ env}} \delta u_{i,i}^{\mathrm{env}} \,\mathrm{d}V + \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}V + \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta_u \varphi \,\mathrm{d}a \\ = {} & -\int_V D_i E_{p,i} \delta u_p \,\mathrm{d}V - \int_V (1/2) D_k E_k \delta u_{j,j} \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta_u \varphi \,\mathrm{d}V + \int_{a_D} \sigma^* \delta_u \varphi \,\mathrm{d}a \\ & - \int_{V^{\mathrm{env}}} D_i^{\mathrm{env}} \delta_u E_i^{\mathrm{env}} \,\mathrm{d}V - \int_{V^{\mathrm{env}}} (1/2) D_k^{\mathrm{env}} E_k^{\mathrm{env}} \delta u_{j,j}^{\mathrm{env}} \,\mathrm{d}V + \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}V \\ & + \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta_u \varphi^{\mathrm{env}} \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta_u \varphi \,\mathrm{d}a \end{aligned} \tag{2.29a}$$

Using Eqs. (2.13b) and (2.16) and adding terms $\int_a D_i n_i \left(E_p \delta u_p + \delta_u \varphi\right) \mathrm{d}a = 0$ and $\int_{a^{\mathrm{env}}} D_i^{\mathrm{env}} n_i^{\mathrm{env}} \left(E_p^{\mathrm{env}} \delta u_p^{\mathrm{env}} + \delta_u \varphi^{\mathrm{env}}\right) \mathrm{d}a = 0$ to Eq. (2.29a), then Eq. (2.29a) can be reduced to

$$\begin{aligned}\delta_u \Pi = {} & \int_{a_\sigma} \sigma_{ij}^{\mathrm{M}} n_i \delta u_j \,\mathrm{d}a - \int_V \sigma_{ij,i}^{\mathrm{M}} \delta u_j \,\mathrm{d}V + \int_{\mathrm{a}_\sigma^{\mathrm{env}}} \sigma_{ij}^{\mathrm{M\ env}} n_i^{\mathrm{env}} \delta u_j^{\mathrm{env}} \,\mathrm{d}a \\ & - \int_{V^{env}} \sigma_{ij,i}^{\mathrm{M\ env}} \delta u_j^{\mathrm{env}} \,\mathrm{d}V + \int_{a^{\mathrm{int}}} \sigma_{ij}^{\mathrm{M}} n_i \delta u_j \,\mathrm{d}a + \int_{a^{\mathrm{int}}} \sigma_{ij}^{\mathrm{M\ env}} n_i^{\mathrm{env}} \delta u_j^{\mathrm{env}} \mathrm{d}a \end{aligned} \tag{2.29b}$$

Comparing Eqs. (2.20), (2.28), and (2.29), it is found that the static electric force acting on the medium can be obtained from the general energy migratory variational principle (Kuang 2012):

$$\delta W^{\mathrm{e}} = -\delta_u \Pi \tag{2.30a}$$

If the environment is neglected, it is obtained:

$$\int_V f_k^{\mathrm{e}} \delta u_k \,\mathrm{d}V + \int_{a_\sigma} T_k^{\mathrm{e}} \delta u_k \,\mathrm{d}a = -\left(\int_V g_{,E} \cdot \delta_u \boldsymbol{E} \,\mathrm{d}V + \int_V g^{\mathrm{e}} \delta u_{k,k} \,\mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta_u \varphi \,\mathrm{d}V + \int_{a_D} \sigma^* \delta_u \varphi \,\mathrm{d}a\right) \tag{2.30b}$$

2.2.6 Hamilton Principle

In order to use the PVP for moving electroelastic materials, the D'Alembert principle should be used to make the moving media in a state of relative rest. Using D'Alembert principle, the Hamilton principle can easily be obtained from the PVP. Let δu_{k0} and $\delta u_{k\mathrm{f}}$ be displacements at the initial and final times, respectively, in time interval $[t_0, t_\mathrm{f}]$ and assume $\delta u_{k0} = \delta u_{k\mathrm{f}} = 0$, using

$$\begin{aligned}\delta\int_{t_0}^{t_\mathrm{f}}\int_V K\,\mathrm{d}V\,\mathrm{d}t &= \delta\int_{t_0}^{t_\mathrm{f}}\int_V (1/2)\rho\dot{u}_k\dot{u}_k\,\mathrm{d}V\,\mathrm{d}t = \int_{t_0}^{t_\mathrm{f}}\int_V \rho\dot{u}_k\delta\dot{u}_k\,\mathrm{d}V\,\mathrm{d}t \\ &= \int_V\int_{t_0}^{t_\mathrm{f}}[\rho\mathrm{d}(\dot{u}_k\delta u_k)/\mathrm{d}t - \rho\ddot{u}_k\delta u_k]\mathrm{d}t\,\mathrm{d}V = -\int_{t_0}^{t_\mathrm{f}}\int_V \rho\ddot{u}_k\delta u_k\mathrm{d}V\,\mathrm{d}t \end{aligned} \quad (2.31)$$

where $K = \rho\dot{u}_k\dot{u}_k/2$ and $K^{\mathrm{env}} = \rho^{\mathrm{env}}\dot{u}_k^{\mathrm{env}}\dot{u}_k^{\mathrm{env}}/2$ are the kinetic energies in the material and its environment, respectively. Substituting Eq. (2.31) into (2.7) and integrating it from t_0 to t_f then we get the Hamilton principle:

$$\begin{aligned}&\delta\Pi_H = \delta\Pi_{H1} + \delta\Pi_{H2} - \int_{t_0}^{t_\mathrm{f}}\delta W^{\mathrm{int}}\,\mathrm{d}t = 0\\ &\delta\Pi_{H1} = \int_{t_0}^{t_\mathrm{f}}\int_V \delta(g-K)\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_\mathrm{f}}\int_V g^{\mathrm{e}}\mathrm{d}V\,\mathrm{d}t - \int_{t_0}^{t_\mathrm{f}}\delta W\,\mathrm{d}t\\ &\delta\Pi_{H2} = \int_{t_0}^{t_\mathrm{f}}\int_{V^{\mathrm{env}}}\delta(g^{\mathrm{env}} - K^{\mathrm{env}})\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_\mathrm{f}}\int_V g^{\mathrm{e\ env}}\mathrm{d}V\,\mathrm{d}t - \int_{t_0}^{t_\mathrm{f}}\delta W^{\mathrm{env}}\mathrm{d}t\\ &\delta W = \int_V f_k\delta u_k\,\mathrm{d}V - \int_V \rho_\mathrm{e}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a\\ &\delta W^{\mathrm{env}} = \int_{V^{\mathrm{env}}} f_k^{\mathrm{env}}\delta u_k^{\mathrm{env}}\,\mathrm{d}V - \int_{V^{\mathrm{env}}}\rho_\mathrm{e}^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}V + \int_{a_\sigma^{\mathrm{env}}} T_k^{*\mathrm{env}}\delta u_k^{\mathrm{env}}\,\mathrm{d}a - \int_{a_D^{\mathrm{env}}}\sigma^{*\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a\\ &\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}}\sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a\end{aligned} \quad (2.32)$$

However the energy conservation law is:

$$g + K + g^{env} + K^{env} = \int_{t_0}^{t_\mathrm{f}}(\mathrm{d}W + dW^{env} + dW^{\mathrm{int}})\mathrm{d}t \quad (2.33)$$

Equation (2.33) is held for any time interval. It is noted that the energy principle is held in a real process, but the PVP gives a true process for all virtual possible process satisfying the natural constrained conditions, and it is equivalent to the momentum equation. It is also noted that the Hamilton principle is held in four-dimensional space $(\boldsymbol{x}, t)$ and the time boundary conditions should be added. But the PVP is held in three-dimensional (3D) space $(\boldsymbol{x})$ and does not consider the time variation.

It is obvious that Hamilton principle is also a fundamental principle in the physics and continuum mechanics. Using the local and migratory variation theory, the Maxwell stress can also be obtained automatically.

2.2.7 Physical Variational Principle in Electromagnetic Materials

In this section, the PVP is extended to electromagnetic materials under static electromagnetic field, without the current and the body electromagnetic couple (Kuang 2011a, b, c).

Let constitutive equations be

$$\begin{aligned}
\sigma_{lk} &= C_{ijkl}\varepsilon_{ij} - e^{\mathrm{e}}_{jkl}E_j - e^{\mathrm{m}}_{jkl}H_j - (1/2)l^{\mathrm{e}}_{ijkl}E_iE_j - (1/2)l^{\mathrm{m}}_{ijkl}H_iH_j \\
&\quad - \epsilon_{km}E_mE_l - \mu_{km}H_mH_l - \beta_{km}H_mE_l - \beta_{km}E_mH_l \\
D_k &= \left[\epsilon_{kl} + l^{\mathrm{e}}_{ijkl}\varepsilon_{ij} + (\epsilon_{ml}\varepsilon_{mk} + \epsilon_{mk}\varepsilon_{ml})\right]E_l + e^{\mathrm{e}}_{kij}\varepsilon_{ij} + \beta_{kl}H_l + (\beta_{lm}H_m + \beta_{km}H_l)\varepsilon_{kl} \\
B_k &= \left[\mu_{kl} + l^{\mathrm{m}}_{ijkl}\varepsilon_{ij} + 2\left(\alpha^{\mathrm{m}}_{ml}\varepsilon_{mk} + \alpha^{\mathrm{m}}_{mk}\varepsilon_{ml}\right)\right]H_l + e^{\mathrm{m}}_{kij}\varepsilon_{ij} + \beta_{kl}E_l + (\beta_{lm}E_m + \beta_{km}E_l)\varepsilon_{kl}
\end{aligned} \tag{2.34}$$

where e^{e}_{jkl} and e^{m}_{jkl} are piezoelectric and piezomagnetic coefficients, respectively, l^{e}_{ijkl} and l^{m}_{ijkl} are electrostrictive and magnetostrictive coefficients, respectively, and $\beta_{ij} = \beta_{ji}$ is the magnetoelectric coupling coefficient. The electromagnetic body couple is still balanced by the asymmetric stress. In this case, the electromagnetic Gibbs free energy g is

$$\begin{aligned}
g &= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \left(e^{\mathrm{e}}_{kij}E_k + e^{\mathrm{m}}_{kij}H_k\right)\varepsilon_{ij} - (1/2)\left(\epsilon_{ij}E_iE_j + \mu_{ij}H_iH_j\right) - \beta_{kl}E_kH_l \\
&\quad - (1/2)\left(l^{\mathrm{e}}_{ijkl}E_iE_j + l^{\mathrm{m}}_{ijkl}H_iH_j\right)\varepsilon_{kl} - (\epsilon_{km}E_mE_l + \mu_{km}H_mH_l)\varepsilon_{kl} - \beta_{km}(H_mE_l + E_mH_l)\varepsilon_{kl} \\
&= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mathrm{em}} \\
g^{\mathrm{em}} &= -(1/2)(D_kE_k + B_kH_k + \Delta_{kl}\varepsilon_{lk}), \quad \Delta_{kl} = e^{\mathrm{e}}_{mkl}E_m + e^{\mathrm{m}}_{mkl}H_m
\end{aligned} \tag{2.35}$$

For the small deformation $\boldsymbol{\Delta} : \boldsymbol{\varepsilon}$ can still be neglected. The PVP is

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0 \\
&\delta\Pi_1 = \int_V \delta g\,\mathrm{d}V + \int_V g^{\mathrm{em}}\delta u_{i,i}\,\mathrm{d}V - \delta W \\
&\delta\Pi_2 = \int_{V^{\mathrm{env}}} \delta g^{\mathrm{env}}\,\mathrm{d}V + \int_{V^{\mathrm{env}}} g^{\mathrm{em\ env}}\delta u^{\mathrm{env}}_{i,i}\,\mathrm{d}V - \delta W^{\mathrm{env}} \\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} T^*_k\delta u_k\,\mathrm{d}a - \int_{a_D} \sigma^*\delta\varphi\,\mathrm{d}a + \int_{a_\mu} B^*_i n_i\delta\psi\,\mathrm{d}a \\
&\delta W^{\mathrm{env}} = \int_{V^{\mathrm{env}}} (f^{\mathrm{env}}_k - \rho\ddot{u}^{\mathrm{env}}_k)\delta u^{\mathrm{env}}_k\,\mathrm{d}V - \int_{V^{\mathrm{env}}} \rho^{\mathrm{env}}_{\mathrm{e}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}V \\
&\qquad + \int_{a^{\mathrm{env}}_\sigma} T^{*\mathrm{env}}_k\delta u^{\mathrm{env}}_k\,\mathrm{d}a - \int_{a^{\mathrm{env}}_D} \sigma^{*\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a + \int_{a_\mu} B^{*\mathrm{env}}_i n^{\mathrm{env}}_i\delta\psi^{\mathrm{env}}\,\mathrm{d}a \\
&\delta W^{\mathrm{int}} = \int_{a^{\mathrm{int}}} T^{*\mathrm{int}}_k\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a + \int_{a^{\mathrm{int}}_\mu} B^{*\mathrm{int}}_i n_i\delta\psi^{\mathrm{env}}\,\mathrm{d}a
\end{aligned} \tag{2.36}$$

where $E_i = -\varphi_i, H_i = -\psi_i$. Finishing the variational calculation finally yields

$$
\begin{aligned}
& S_{jk,j} + f_k = \rho \ddot{u}_k, \quad D_{i,i} = \rho_{\mathrm{e}}, \quad B_{i,i} = 0, \quad \text{in} \quad V \\
& S_{jk} n_j = T_k^*, \quad \text{on} \quad a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \quad \left(B_i - B_i^*\right) n_i = 0, \quad \text{on} \quad a_\mu \\
& S_{ij,i}^{\mathrm{env}} + f_j^{\mathrm{env}} = \rho^{\mathrm{env}} \ddot{u}_j^{\mathrm{env}}, \quad D_{i,i}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}, \quad B_{i,i} = 0, \quad \text{in} \quad V^{\mathrm{env}} \\
& S_{ij}^{\mathrm{env}} n_i^{\mathrm{env}} = T_j^{*\mathrm{env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}}; \quad B_i^{\mathrm{env}} n_i^{\mathrm{env}} = B_i^{*\mathrm{env}} n_i^{\mathrm{env}}, \\
& \quad \text{on} \quad a_\mu^{\mathrm{env}} \\
& (S_{ij} - S_{ij}^{\mathrm{env}}) n_i = T_j^{*\mathrm{int}}, \quad (D_i - D_i^{\mathrm{env}}) n_i = -\sigma^{*\mathrm{int}}, \quad (B_i - B_i^{\mathrm{env}}) n_i = B_i^{*\mathrm{int}} n_i, \quad \text{on} \quad a^{\mathrm{int}} \\
& S_{ik} = \sigma_{ik} + \sigma_{ik}^{\mathrm{M}}; \quad \sigma_{ik}^{\mathrm{M}} = D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n)\delta_{ik} \\
& S_{ik}^{\mathrm{env}} = \sigma_{ik}^{\mathrm{env}} + \sigma_{ik}^{\mathrm{M\,env}}; \quad \sigma_{ik}^{\mathrm{M\,env}} = D_i^{\mathrm{env}} E_k^{\mathrm{env}} + B_i^{\mathrm{env}} H_k^{\mathrm{env}} - (1/2)\left(D_n^{\mathrm{env}} E_n^{\mathrm{env}} + B_n^{\mathrm{env}} H_n^{\mathrm{env}}\right)\delta_{ik}
\end{aligned}
\tag{2.37}
$$

2.3 General Variational Principle

2.3.1 General Variational Principle Not Satisfying Boundary Conditions

This principle does not ask $\boldsymbol{u}, \varphi$ and $\boldsymbol{u}^{\mathrm{env}}, \varphi^{\mathrm{env}}$ to satisfy boundary conditions on their own boundaries a_u, a_φ and $a_u^{\mathrm{env}}, a_\varphi^{\mathrm{env}}$, respectively, and continuity conditions on the interface prior. For small deformation, this principle is

$$
\begin{aligned}
\delta\Pi &= \delta\Pi_1 + \delta\Pi_2 - \delta W^{\mathrm{int}} = 0 \\
\delta\Pi_1 &= \int_V \delta g \, \mathrm{d}V + \int_V g \delta u_{k,k} \, \mathrm{d}V - \int_V (f_k - \rho \ddot{u}_k) \delta u_k \, \mathrm{d}V + \int_V \rho_{\mathrm{e}} \delta\varphi \, \mathrm{d}V - \int_{a_\sigma} T_k^* \delta u_k \, \mathrm{d}a \\
&\quad + \int_{a_D} \sigma^* \delta\varphi \, \mathrm{d}a - \delta \int_{a_u} T_k^{\mathrm{S}} (u_k - u_k^*) \, \mathrm{d}a + \delta \int_{a_\varphi} \sigma (\varphi - \varphi^*) \, \mathrm{d}a \\
\delta\Pi_2 &= \int_{V^{\mathrm{env}}} \delta g^{\mathrm{env}} \, \mathrm{d}V + \int_{V^{\mathrm{env}}} g^{\mathrm{env}} \delta u_{k,k}^{\mathrm{env}} \, \mathrm{d}V - \int_{V^{\mathrm{env}}} (f_k^{\mathrm{env}} - \rho^{\mathrm{env}} \ddot{u}_k^{\mathrm{env}}) \delta u_k^{\mathrm{env}} \, \mathrm{d}V \\
&\quad + \int_{V^{\mathrm{env}}} \rho_{\mathrm{e}}^{\mathrm{env}} \delta\varphi^{\mathrm{env}} \, \mathrm{d}V - \int_{a_\sigma^{\mathrm{env}}} T_k^{*\mathrm{env}} \delta u_k^{\mathrm{env}} \, \mathrm{d}a + \int_{a_D^{\mathrm{env}}} \sigma^{*\mathrm{env}} \delta\varphi^{\mathrm{env}} \, \mathrm{d}a \\
&\quad - \delta \int_{a_u^{\mathrm{env}}} T_k^{\mathrm{S\,env}} (u_k^{\mathrm{env}} - u_k^{*\mathrm{env}}) \, \mathrm{d}a + \delta \int_{a_\varphi^{\mathrm{env}}} \sigma^{\mathrm{env}} (\varphi^{\mathrm{env}} - \varphi^{*\mathrm{env}}) \, \mathrm{d}a \\
\delta W^{\mathrm{int}} &= \int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}} \delta u_k \, \mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}} \delta\varphi \, \mathrm{d}a \\
&\quad + \delta \int_{a^{\mathrm{int}}} T_k^{\mathrm{S}} (u_k - u_k^{\mathrm{env}}) \, \mathrm{d}a - \delta \int_{a^{\mathrm{int}}} \sigma (\varphi - \varphi^{\mathrm{env}}) \, \mathrm{d}a
\end{aligned}
\tag{2.38}
$$

In Eq. (2.38), some additional virtual work done on the boundary and interface is added, because the displacements and potentials do not satisfy the boundary conditions and the continuity conditions on the interface. As an example, $\delta \int_{a^{\mathrm{int}}} T_k^{\mathrm{S\,int}}(u_k - u_k^{\mathrm{env}})\,\mathrm{d}a$ is the virtual work introduced by the difference of the virtual displacement $(u_k - u_k^{\mathrm{env}})$ on two sides and the unknown pseudo total traction $\boldsymbol{T}^{\mathrm{S}}$ on the interface. In Eq. (2.38), $\boldsymbol{T}^{\mathrm{S}}$ and σ may also be considered as Lagrange multipliers in the mathematical sense (Kuang 1964, 2002).

Equation (2.38) can be proved as follows. Analogous to the derivation of Eq. (2.11), it is obtained:

$$\begin{aligned}\delta \Pi_1 = &\int_{a_\sigma} \left(S_{jk}n_j - T_k^*\right)\delta u_k\,\mathrm{d}a - \int_V \left(S_{jk,j} + f_k - \rho \ddot{u}_k\right)\delta u_k\,\mathrm{d}V - \int_V \left(D_{i,i} - \rho_{\mathrm{e}}\right)\delta\varphi\,\mathrm{d}V \\ &+ \int_{a_D} (D_i n_i + \sigma^*)\delta\varphi\,\mathrm{d}a + \int_{a^{\mathrm{int}}} S_{jk}n_j\delta u_k\,\mathrm{d}a + \int_{a^{\mathrm{int}}} D_i n_i \delta\varphi\,\mathrm{d}a - \int_{a_u} (u_k - u_k^*)\delta T_k^{\mathrm{S}}\,\mathrm{d}a \\ &+ \int_{a_u} \left(S_{jk}n_j - T_k^{\mathrm{S}}\right)\delta u_k\,\mathrm{d}a + \int_{a_\varphi} (\varphi - \varphi^*)\delta\sigma\,\mathrm{d}a + \int_{a_\varphi} (D_i n_i + \sigma)\delta\varphi\,\mathrm{d}a\end{aligned} \tag{2.39}$$

Due to the arbitrariness of $\delta\boldsymbol{u}$ and $\delta\varphi$ from Eq. (2.39), we get

$$\begin{aligned}&S_{jk,j} + f_k = \rho\ddot{u}_k, \quad D_{i,i} = \rho_{\mathrm{e}}, \quad \text{in} \quad V \\ &S_{jk}n_j = T_k^*, \quad \text{on} \quad a_\sigma; \quad u_k = u_k^*, \quad S_{jk}n_j = T_k^{\mathrm{S}}, \quad \text{on} \quad a_u \\ &D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \quad \varphi = \varphi^*, \quad D_i n_i = -\sigma, \quad \text{on} \quad a_\varphi \\ &\delta\Pi_1 = \int_{a^{\mathrm{int}}} S_{ij}n_i\delta u_j\,\mathrm{d}a + \int_{a^{\mathrm{int}}} D_i n_i\delta\varphi\,\mathrm{d}a\end{aligned} \tag{2.40}$$

Similarly for the environment we get

$$\begin{aligned}&S_{ij,i}^{\mathrm{env}} + f_j^{\mathrm{env}} = \rho^{\mathrm{env}}\ddot{u}_j^{\mathrm{env}}, \quad D_{i,i}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}, \quad \text{in} \quad V^{\mathrm{env}} \\ &S_{ij}^{\mathrm{env}}n_i^{\mathrm{env}} = T_j^{*\,\mathrm{env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad u_k^{\mathrm{env}} = u_k^{*\,\mathrm{env}}, \quad S_{jk}^{\mathrm{env}}n_j^{\mathrm{env}} = T_k^{\mathrm{S\,env}}, \quad \text{on} \quad a_u^{\mathrm{env}} \\ &D_i^{\mathrm{env}}n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}} \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad D_i^{\mathrm{env}}n_i^{\mathrm{env}} = -\sigma^{\mathrm{env}}, \quad \text{on} \quad a_\varphi \\ &\delta\Pi_2 = \int_{a^{\mathrm{int}}} S_{ij}^{\mathrm{env}}n_i^{\mathrm{env}}\delta u_j^{\mathrm{env}}\,\mathrm{d}a + \int_{a^{\mathrm{int}}} D_i^{\mathrm{env}}n_i^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}a\end{aligned} \tag{2.41}$$

For δW^{int} we have

$$\begin{aligned}\delta W^{\mathrm{int}} = &\int_{a^{\mathrm{int}}} T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a + \int_{a^{\mathrm{int}}} T_k^{\mathrm{S}}\delta(u_k - u_k^{\mathrm{env}})\,\mathrm{d}a \\ &+ \int_{a^{\mathrm{int}}} (u_k - u_k^{\mathrm{env}})\delta T_k^{\mathrm{S}}\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \sigma\delta(\varphi - \varphi^{\mathrm{env}})\,\mathrm{d}a - \int_{a^{\mathrm{int}}} (\varphi - \varphi^{\mathrm{env}})\delta\sigma\,\mathrm{d}a\end{aligned} \tag{2.42}$$

Due to $\boldsymbol{n} = -\boldsymbol{n}^{\text{env}}$ and arbitrariness of $\delta\boldsymbol{u}, \delta\varphi,\ \delta\boldsymbol{T}^{\text{S}}, \delta\sigma$ from $\delta\Pi = \delta\Pi_1 + \delta\Pi_2 = \delta W^{\text{int}}$ we get

$$(S_{ij} - S_{ij}^{\text{env}})n_i = T_j^{*\text{int}}, \quad (D_i - D_i^{\text{env}})n_i = -\sigma^{*\text{int}}, \quad u_k = u_k^{\text{env}}, \quad \varphi = \varphi^{\text{env}}; \quad \text{on} \quad a^{\text{int}} \tag{2.43}$$

2.3.2 General Variational Principle in Linear Piezoelectric Materials

Kuang (1964, 2002) proposed a Lagrange multiplier method to derive general variational principle (Hu 1981) from the potential energy principle in linear elasticity. This method is easy extended to the linear electroelastic theory where the Maxwell stress is not considered. So the migratory variation of the electric potential is not needed. Using the Lagrange multiplier method, the general variational principle with independent variables $\boldsymbol{u}, \varphi, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{E}$ in the small deformation case is easy obtained. The boundary conditions and continuity conditions on the interface do not satisfied prior. The electric Gibbs free energy for the linear piezoelectric material under the small deformation is

$$\begin{aligned} g &= (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij} \\ g^{\text{env}} &= (1/2)C_{ijkl}^{\text{env}}\varepsilon_{ij}^{\text{env}}\varepsilon_{kl}^{\text{env}} - (1/2)\epsilon_{kl}^{\text{env}}E_k^{\text{env}}E_l^{\text{env}} - e_{kij}^{\text{env}}E_k^{\text{env}}\varepsilon_{ij}^{\text{env}} \end{aligned} \tag{2.44}$$

Omitting the derivation process, the PVP is

$$\begin{aligned} &\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0 \\ &\delta\Pi_1 = \delta\int_V \left[g - \sigma_{lk}\varepsilon_{kl} + D_kE_k + (1/2)\sigma_{lk}\left(u_{k,l} + u_{l,k}\right) + D_k\varphi_{,k}\right]\mathrm{d}V - \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V \\ &\qquad + \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V - \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a + \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a - \delta\int_{a_u} T_k(u_k - u_k^*)\,\mathrm{d}a + \delta\int_{a_\varphi}\sigma(\varphi - \varphi^*)\,\mathrm{d}a \\ &\delta\Pi_2 = \delta\int_{V^{\text{env}}}\left[g^{\text{env}} - \sigma_{lk}^{\text{env}}\varepsilon_{kl}^{\text{env}} + D_k^{\text{env}}E_k^{\text{env}} + (1/2)\sigma_{lk}^{\text{env}}\left(u_{k,l}^{\text{env}} + u_{l,k}^{\text{env}}\right) + D_k^{\text{env}}\varphi_{,k}^{\text{env}}\right]\mathrm{d}V \\ &\qquad - \int_{V^{\text{env}}}(f_k^{\text{env}} - \rho\ddot{u}_k^{\text{env}})\delta u_k^{\text{env}}\,\mathrm{d}V + \int_{V^{\text{env}}}\rho_e^{\text{env}}\delta\varphi^{\text{env}}\,\mathrm{d}V - \int_{a_\sigma^{\text{env}}} T_k^{*\text{env}}\delta u_k^{\text{env}}\,\mathrm{d}a \\ &\qquad + \int_{a_D^{\text{env}}}\sigma^{*\text{env}}\delta\varphi_k^{\text{env}}\,\mathrm{d}a - \delta\int_{a_u^{\text{env}}} T_k^{\text{env}}(u_k^{\text{env}} - u_k^{*\text{env}})\,\mathrm{d}a + \delta\int_{a_\varphi^{\text{env}}}\sigma^{\text{env}}(\varphi^{\text{env}} - \varphi^{*\text{env}})\,\mathrm{d}a \\ &\delta W^{\text{int}} = \int_{a^{\text{int}}} T_k^{*\text{int}}\delta u_k\,\mathrm{d}a - \int_{a^{\text{int}}}\sigma^{*\text{int}}\delta\varphi\,\mathrm{d}a + \delta\int_{a^{\text{int}}} T_k(u_k - u_k^{\text{env}})\mathrm{d}a - \delta\int_{a^{\text{int}}}\sigma(\varphi - \varphi^{\text{env}})\mathrm{d}a \end{aligned} \tag{2.45}$$

It is easy to show that $\delta\Pi_1$ can be reduced to

$$\begin{aligned}\delta\Pi_1 = \delta\int_V & \left\{ \left(C_{ijkl}\varepsilon_{ij} - e_{kij}E_k - \sigma_{kl}\right)\delta\varepsilon_{kl} + \left(D_k - \epsilon_{kl}E_l - e_{kij}\varepsilon_{ij}\right)\delta E_k + \left(E_k + \varphi_{,k}\right)\delta D_k \right.\\ & \left. + \left[(1/2)\left(u_{k,l} + u_{l,k}\right) - \varepsilon_{kl}\right]\delta\sigma_{kl} - \left(\sigma_{kl,l} + f_k - \rho\ddot{u}_k\right)\delta u_k - \left(D_{k,k} - \rho_e\right)\delta\varphi \right\} \mathrm{d}V \\ & + \int_{a_\sigma}\left(\sigma_{kl}n_l - T_k^*\right)\delta u_k\,\mathrm{d}a + \int_{a_D}\left(D_k n_k + \sigma^*\right)\delta\varphi\,\mathrm{d}a \\ & + \int_{a_u}\left[\left(\sigma_{kl}n_l - T_k\right)\delta u_k - \left(u_k - u_k^*\right)\delta T_k\right]\mathrm{d}a \\ & + \int_{a_\varphi}\left[\left(D_k n_k + \sigma\right)\delta\varphi + \left(\varphi - \varphi^*\right)\delta\sigma\right]\mathrm{d}a + \int_{a^{\mathrm{int}}}\sigma_{kl}n_l\delta u_k\,\mathrm{d}a + \int_{a^{\mathrm{int}}}D_k n_k\delta\varphi\,\mathrm{d}a\end{aligned} \tag{2.46}$$

Completing the variational calculation and considering the arbitrariness of $\delta\boldsymbol{u}, \delta\varphi, \delta\boldsymbol{u}^{\mathrm{env}}, \delta\varphi^{\mathrm{env}}$ and $\boldsymbol{T}, \sigma$ finally we get

$$\begin{aligned}& \varepsilon_{kl} = (1/2)\left(u_{k,l} + u_{l,k}\right), \quad E_k = -\varphi_{,k}; \quad D_k = \epsilon_{kl}E_l + e_{kij}\varepsilon_{ij}, \quad \sigma_{kl} = C_{ijkl}\varepsilon_{ij} - e_{ikl}E_i \\ & \sigma_{kl,l} + f_k = \rho\ddot{u}_k; \quad \sigma_{kl}n_l = T_k^*, \quad \text{on} \quad a_\sigma; \quad u_k = u_k^*, \quad \sigma_{kl}n_l = T_k, \quad \text{on} \quad a_u \\ & D_{k,k} = \rho_{\mathrm{e}}; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_D; \quad \varphi = \varphi^*, \quad D_k n_k = -\sigma, \quad \text{on} \quad a_\varphi \\ & \varepsilon_{kl}^{\mathrm{env}} = (1/2)\left(u_{k,l}^{\mathrm{env}} + u_{l,k}^{\mathrm{env}}\right), \quad E_k^{\mathrm{env}} = -\varphi_{,k}^{\mathrm{env}}; \quad D_k^{\mathrm{env}} = \epsilon_{kl}E_l^{\mathrm{env}} - e_{kij}\varepsilon_{ij}^{\mathrm{env}}, \\ & \sigma_{kl}^{\mathrm{env}} = C_{ijkl}\varepsilon_{ij}^{\mathrm{env}} - e_{ikl}E_i^{\mathrm{env}} \\ & \sigma_{kl,l}^{\mathrm{env}} + f_k^{\mathrm{env}} = \rho\ddot{u}_k^{\mathrm{env}}; \quad \sigma_{kl}^{\mathrm{env}}n_l^{\mathrm{env}} = T_k^{*\mathrm{env}} \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad u_k^{\mathrm{env}} = u_k^{*\mathrm{env}}, \quad \sigma_{kl}^{\mathrm{env}}n_l^{\mathrm{env}} = T_k^{\mathrm{env}}, \\ & \quad \text{on} \quad a_u^{\mathrm{env}} \\ & D_{k,k}^{\mathrm{env}} = \rho_{\mathrm{e}}^{\mathrm{env}}; \quad D_k^{\mathrm{env}}n_k^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_D^{\mathrm{env}}; \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad D_k^{\mathrm{env}}n_k^{\mathrm{env}} = -\sigma^{\mathrm{env}}, \\ & \quad \text{on} \quad a_\varphi^{\mathrm{env}} \\ & \left(\sigma_{kl} - \sigma_{kl}^{\mathrm{env}}\right)n_l = T_k^{*\mathrm{int}}, \quad \left(D_k - D_k^{\mathrm{env}}\right)n_k = -\sigma^{*\mathrm{int}}, \quad u_k = u_k^{\mathrm{env}}, \quad \varphi = \varphi^{\mathrm{env}}; \quad \text{on} \quad a^{\mathrm{int}}\end{aligned} \tag{2.47}$$

Equation (2.47) is the complete governing equation.

2.4 Variational Principle in Piezoelectric Materials Under Finite Deformation

2.4.1 The Electric Gibbs Free Energy in Initial Configuration

Some fundamental formulas and notations for finite deformation shown in Sect. 1.3.4 will be used in this chapter. It is emphasized that the same coordinate system is used in the current and initial configurations. Since the isothermal electric Gibbs free energy $\bar{g}$ in the finite deformation state must be invariant in a rigid body rotation, so the $\bar{g}$ for materials without the electric couple problem should be taken in the following form:

$$\begin{gathered}\bar{g}=(1/2)\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}\bar{\varepsilon}_{KL}-(1/2)\bar{\epsilon}_{kl}\bar{E}_K\bar{E}_L-\bar{e}_{KIJ}\bar{E}_K\bar{\varepsilon}_{IJ}-(1/2)\bar{l}_{IJKL}\bar{E}_I\bar{E}_J\bar{\varepsilon}_{KL}\\ \bar{C}_{IJKL}=\bar{C}_{JIKL}=\bar{C}_{IJLK}=\bar{C}_{KLIJ},\quad \bar{\epsilon}_{KL}=\bar{\epsilon}_{LK},\quad \bar{e}_{KIJ}=\bar{e}_{KJI},\quad \bar{l}_{IJKL}=\bar{l}_{JIKL}=\bar{l}_{IJLK}=\bar{l}_{KLIJ}\end{gathered} \tag{2.48}$$

where $\bar{C}_{IJKL},\bar{\epsilon}_{KL},\bar{e}_{KIJ},\bar{l}_{IJKL}$ are the material coefficients in the initial configuration. It is noted that coefficients in the initial and current configurations are different. From the thermodynamic theory, the constitutive equations are

$$\begin{aligned}\bar{\sigma}_{LK}&=\partial\bar{g}/\partial\bar{\varepsilon}_{KL}=\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}-\bar{e}_{JKL}\bar{E}_J-(1/2)\bar{l}_{IJKL}\bar{E}_I\bar{E}_J\\ \bar{D}_K&=-\partial\bar{g}/\partial\bar{E}_K=(\bar{\epsilon}_{kl}+\bar{l}_{IJKL}\bar{\varepsilon}_{JI})\bar{E}_L+\bar{e}_{KIJ}\bar{\varepsilon}_{IJ}\end{aligned} \tag{2.49}$$

Using Eq. (2.49), Eq. (2.48) can be reduced to

$$\begin{gathered}\bar{g}=(1/2)\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}\bar{\varepsilon}_{KL}+\bar{g}^{\mathrm{e}},\quad \bar{g}^{\mathrm{e}}=-(1/2)\bar{\Gamma}_N\bar{E}_N=-(1/2)\bar{\Gamma}_N\varphi_{,N},\\ \bar{\Gamma}_N=\bar{D}_N+\bar{e}_{NKL}\bar{\varepsilon}_{KL}\end{gathered} \tag{2.50}$$

In $\bar{g}$, the term $(1/2)\bar{C}_{IJKL}\bar{\varepsilon}_{IJ}\bar{\varepsilon}_{KL}$ is the mechanical deformation energy, $(1/2)\bar{D}_K\bar{E}_K$ is the electromagnetic energy, $(1/2)\bar{e}_{NKL}E_N\bar{\varepsilon}_{KL}$ is the mechanical and electromagnetic coupling energy, and $\bar{g}^{\mathrm{e}}$ is the sum of the electromagnetic energy and coupling energy. For the small deformation case, $(1/2)\bar{e}_{NKL}E_N\bar{\varepsilon}_{KL}$ can be neglected, so the coupling energy can also be neglected.

2.4.2 Variational Principle with the Electric Gibbs Free Energy Under Finite Deformation

As in Eq. (2–8), variations of $\varphi,\bar{\boldsymbol{E}}$ are divided into local variation $\delta_\varphi\varphi,\delta_\varphi\bar{\boldsymbol{E}}$ and migratory variation $\delta_u\varphi,\ \delta_u\bar{\boldsymbol{E}}$, i.e.,

$$\begin{aligned}\delta\varphi&=\delta_\varphi\varphi+\delta_u\varphi,\quad \delta_u\varphi=\varphi_{,p}\delta u_p=-E_p\delta u_p=-\bar{E}_LX_{L,p}\delta u_p\\ \delta\bar{E}_I&=\delta_\varphi\bar{E}_I+\delta_u\bar{E}_I,\quad \delta_u\bar{E}_I=\bar{E}_{I,p}\delta u_p=\bar{E}_{I,L}X_{L,p}\delta u_p=\bar{E}_{L,I}X_{L,p}\delta u_p\end{aligned} \tag{2.51}$$

Let the displacement **u** and the potential φ satisfy their boundary conditions on their own boundaries $\bar{a}_u,\bar{a}_\varphi,\bar{a}_\psi$ and the continuity conditions on their interface $\bar{a}^{\mathrm{int}}$ (Fig. 2.1). The variational principle with the electric Gibbs free energy under finite deformation for electroelastic media can be expressed in the following form (Kuang 2008b, 2011a):

$$
\begin{aligned}
&\delta\bar{\Pi} = \delta\bar{\Pi}_1 + \delta\bar{\Pi}_2 - \delta\bar{W}^{*\mathrm{int}} = 0 \\
&\delta\bar{\Pi}_1 = \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V} - \delta\bar{W}^{*} \\
&\delta\bar{\Pi}_2 = \int_{\bar{V}^{\mathrm{env}}} \delta\bar{g}^{\mathrm{env}}\,\mathrm{d}\bar{V} + \int_{\bar{V}^{\mathrm{env}}} \bar{g}^{\mathrm{e\ env}}\delta u_{i,i}^{\mathrm{env}}\,\mathrm{d}\bar{V} - \delta\bar{W}^{*\mathrm{env}} \\
&\delta\bar{W}^{*} = \int_{\bar{V}} (\bar{f}_k - \bar{\rho}\ddot{u}_k)\delta u_k\,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{\rho}_{\mathrm{e}}\delta\varphi\,\mathrm{d}\bar{V} + \int_{\bar{a}_\sigma} \bar{T}_k^{*}\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}_D} \bar{\sigma}^{*}\delta\varphi\,\mathrm{d}\bar{a} \\
&\delta\bar{W}^{*\mathrm{env}} = \int_{\bar{V}^{\mathrm{env}}} (\bar{f}_k^{\mathrm{env}} - \bar{\rho}\ddot{u}_k^{\mathrm{env}})\delta u_k^{\mathrm{env}}\,\mathrm{d}\bar{V} - \int_{\bar{V}^{\mathrm{env}}} \bar{\rho}_{\mathrm{e}}^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{V} \\
&\qquad\qquad + \int_{\bar{a}_\sigma^{\mathrm{env}}} \bar{T}_k^{*\mathrm{env}}\delta u_k^{\mathrm{env}}\,\mathrm{d}\bar{a} - \int_{\bar{a}_D^{\mathrm{env}}} \bar{\sigma}^{*\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a} \\
&\delta\bar{W}^{*\mathrm{int}} = \int_{\bar{a}^{\mathrm{int}}} \bar{T}_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}^{\mathrm{int}}} \bar{\sigma}^{*\mathrm{int}}\delta\varphi\,\mathrm{d}\bar{a}
\end{aligned}
\tag{2.52}
$$

where $\bar{T}_k^{*}, \bar{\sigma}^{*}; \bar{T}_k^{*\mathrm{env}}, \bar{\sigma}^{*\mathrm{env}}; \bar{T}_k^{*\mathrm{int}}, \bar{\sigma}^{*\mathrm{int}}$ are the given values on the corresponding surfaces.

Using relations $g^{\mathrm{e}}\,\mathrm{d}V = \bar{g}^{\mathrm{e}}\,\mathrm{d}\bar{V}$, $\int_V g^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}V = \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V}$ and $\int_V \delta g\,\mathrm{d}V = \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V}$ (Kuang 2008b, 2009a) yields

$$
\begin{aligned}
&\int_{\bar{V}} \delta\bar{g}\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V} = \int_{\bar{V}} (\bar{\sigma}_{JI}\delta\bar{\varepsilon}_{IJ} - \bar{D}_I\delta\bar{E}_I)\,\mathrm{d}\bar{V} + (1/2)\int_{\bar{V}} \bar{\Gamma}_N\varphi_{,N}\delta u_{k,k}\,\mathrm{d}\bar{V} \\
&\quad = \int_{\bar{V}} [\bar{\sigma}_{JI}x_{k,I}\delta u_{k,J} - \bar{D}_I(-\delta_\varphi\varphi_{,I} + \bar{E}_{L,I}X_{L,p}\delta u_p)]\,\mathrm{d}\bar{V} + (1/2)\int_{\bar{V}} \bar{\Gamma}_N\varphi_{,N}X_{J,k}\delta u_{k,J}\,\mathrm{d}\bar{V} \\
&\quad = \int_{\bar{a}} \left[\bar{\sigma}_{JI}x_{k,I} + (1/2)\bar{\Gamma}_N\varphi_{,N}X_{J,k}\right]\bar{n}_J\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{V}} \left[\bar{\sigma}_{JI}x_{k,I} + (1/2)\bar{\Gamma}_N\varphi_{,N}X_{J,k}\right]_{,J}\delta u_k\,\mathrm{d}\bar{V} \\
&\qquad + \int_{\bar{a}} \bar{D}_I\bar{n}_I\delta_\varphi\varphi\,\mathrm{d}\bar{a} - \int_{\bar{V}} \bar{D}_{I,I}\delta_\varphi\varphi\,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{D}_I\bar{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\bar{V}
\end{aligned}
\tag{2.53}
$$

where $\delta u_{k,k} = \delta u_{k,J}X_{J,k}$ was used. Substitution of Eq. (2.53) into Eq. (2.52) yields

$$
\begin{aligned}
\delta\bar{\Pi}_1 = &\int_{\bar{a}} [(\bar{\sigma}_{JI}x_{k,I} + (1/2)(\bar{\Gamma}_N\varphi_{,N}X_{J,k})\bar{n}_J]\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}_\sigma} \bar{T}_k^{*}\delta u_k\,\mathrm{d}\bar{a} \\
&- \int_{\bar{V}} [(\bar{\sigma}_{JI}x_{k,I} + (1/2)\bar{\Gamma}_N\varphi_{,N}X_{J,k})_{,J} + \bar{f}_k - \bar{\rho}\ddot{u}_k]\delta u_k\,\mathrm{d}\bar{V} + \int_{\bar{a}_D} (\bar{D}_I\bar{n}_I + \bar{\sigma}^{*})\delta_\varphi\varphi\,\mathrm{d}\bar{a} \\
&+ \int_{\bar{a}^{\mathrm{int}}+a_\varphi} \bar{D}_I\bar{n}_I\delta_\varphi\varphi\,\mathrm{d}\bar{a} - \int_{\bar{V}} (\bar{D}_{I,I} - \bar{\rho}_{\mathrm{e}})\delta_\varphi\varphi\,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{D}_I\bar{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\bar{V} \\
&- \int_{\bar{V}} \bar{\rho}_{\mathrm{e}}E_p\delta u_p\,\mathrm{d}\bar{V} - \int_{\bar{a}_D} \bar{\sigma}^{*}E_p\delta u_p\,\mathrm{d}\bar{a}
\end{aligned}
\tag{2.54a}
$$

The last three terms in (2.54a) can be reduced to

$$
\begin{aligned}
&-\int_{\bar{V}} \bar{D}_I \bar{E}_{L,I} X_{L,p} \delta u_p \,\mathrm{d}\bar{V} - \int_{\bar{V}} \bar{\rho}_{\mathrm{e}} E_p \delta u_p \,\mathrm{d}\bar{V} - \int_{\bar{a}_D} \bar{\sigma}^* E_p \delta u_p \,\mathrm{d}\bar{a} \\
&\quad = -\int_{\bar{a}} \bar{D}_I \bar{E}_L X_{L,p} \bar{n}_I \delta u_p \,\mathrm{d}\bar{a} + \int_{\bar{V}} (\bar{D}_I X_{L,p} \delta u_p)_{,I} \bar{E}_L \,\mathrm{d}\bar{V} \\
&\qquad - \int_{\bar{V}} \bar{\rho}_{\mathrm{e}} E_p \delta u_p \,\mathrm{d}\bar{V} - \int_{\bar{a}_D} \bar{\sigma}^* E_p \delta u_p \,\mathrm{d}\bar{a} = -\int_{\bar{a}_D} (\bar{D}_I \bar{n}_I + \bar{\sigma}^*) E_p \delta u_p \,\mathrm{d}\bar{a} \\
&\qquad - \int_{\bar{a}^{\mathrm{int}} + \bar{a}_\varphi} \bar{D}_I \bar{n}_I E_p \delta u_p \,\mathrm{d}\bar{a} + \int_{\bar{V}} \bar{D}_I \bar{E}_L X_{L,p} \delta u_{p,I} \,\mathrm{d}\bar{V} + \int_{\bar{V}} (\bar{D}_{I,I} - \bar{\rho}_{\mathrm{e}}) E_p \delta u_p \,\mathrm{d}\bar{V}
\end{aligned}
$$

where $X_{L,p} \delta u_p \bar{E}_L = E_p \delta u_p$ was used. So Eq. (2.54a) can be reduced to

$$
\begin{aligned}
\delta \bar{\Pi}_1 = &\int_{\bar{a}_\sigma} [(\bar{\sigma}_{JI} x_{k,I} + (1/2)\bar{\Gamma}_N \varphi_{,N} X_{J,k}) \bar{n}_J - \bar{T}_k^*] \delta u_k \,\mathrm{d}\bar{a} \\
&+ \int_{\bar{a}^{\mathrm{int}} + a_u} (\bar{\sigma}_{JI} x_{k,I} + \tfrac{1}{2} \bar{D}_N \varphi_{,N} X_{J,k}) \bar{n}_J \delta u_k \,\mathrm{d}\bar{a} \\
&- \int_{\bar{V}} [(\bar{\sigma}_{JI} x_{k,I} + (1/2)\bar{\Gamma}_N \varphi_{,N} X_{J,k})_{,J} + \bar{f}_k - \bar{\rho} \ddot{u}_k] \delta u_k \,\mathrm{d}\bar{V} + \int_{\bar{a}_D} (\bar{D}_I \bar{n}_I + \bar{\sigma}^*) \delta_\varphi \varphi \,\mathrm{d}\bar{a} \\
&- \int_{\bar{a}_D} (\bar{D}_I \bar{n}_I + \bar{\sigma}^*) E_p \delta u_p \,\mathrm{d}\bar{a} + \int_{\bar{a}^{\mathrm{int}} + \bar{a}_\varphi} \bar{D}_I \bar{n}_I \delta_\varphi \varphi \,\mathrm{d}\bar{a} - \int_{\bar{a}^{\mathrm{int}} + \bar{a}_\varphi} \bar{D}_I \bar{n}_I E_p \delta u_p \,\mathrm{d}\bar{a} \\
&- \int_{\bar{V}} (\bar{D}_{I,I} - \bar{\rho}_{\mathrm{e}}) \delta_\varphi \varphi \,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{D}_I \bar{E}_L X_{L,p} \delta u_{p,I} \,\mathrm{d}\bar{V} + \int_{\bar{V}} (\bar{D}_{I,I} - \bar{\rho}_{\mathrm{e}}) E_p \delta u_p \,\mathrm{d}\bar{V} \\
= &\int_{\bar{a}_\sigma} (\bar{S}_{IJ} \bar{n}_I - \bar{T}_J^*) \delta u_J \,\mathrm{d}\bar{a} + \int_{\bar{a}^{\mathrm{int}}} \bar{S}_{IJ} \bar{n}_I \delta u_J \,\mathrm{d}\bar{a} - \int_{\bar{V}} (\bar{S}_{IJ,I} + \bar{f}_J - \bar{\rho} \ddot{u}_J) \delta u_J \,\mathrm{d}\bar{V} \\
&+ \int_{\bar{a}_D} (\bar{D}_I \bar{n}_I + \bar{\sigma}^*) \delta \varphi \,\mathrm{d}\bar{a} + \int_{\bar{a}^{\mathrm{int}}} \bar{D}_I \bar{n}_I \delta \varphi \,\mathrm{d}\bar{a} - \int_{\bar{V}} (\bar{D}_{I,I} - \bar{\rho}_{\mathrm{e}}) \delta \varphi \,\mathrm{d}\bar{V} = 0
\end{aligned} \tag{2.54b}
$$

where

$$
\begin{aligned}
&\bar{S}_{Jk} = \bar{\sigma}_{JI} x_{k,I} + X_{L,k} \bar{\sigma}_{JL}^{\mathrm{M}}, \\
&\bar{\sigma}_{JL}^{\mathrm{M}} = \bar{D}_J \bar{E}_L - \frac{1}{2} \bar{\Gamma}_N \bar{E}_N \delta_{JL} = \bar{D}_J \bar{E}_L - \frac{1}{2} (\bar{D}_N + \bar{e}_{NML} \bar{\varepsilon}_{ML}) \bar{E}_N \delta_{JL}
\end{aligned} \tag{2.55}
$$

$\bar{S}_{IJ}$ is called the pseudo total stress in the initial configuration, $\bar{\sigma}_{IJ}^{\mathrm{M}}$ may be called the second kind of the Maxwell stress defined in initial configurations, and $X_{L,k} \bar{\sigma}_{JL}^{\mathrm{M}}$ may be called the first kind of the Maxwell stress defined in current and initial configurations. From Eq. (2.55), it is known that when the initial configuration is used as the reference configuration, the Maxwell stress is related to strain. But for isotropic materials, the Maxwell stress is still not related to strain due to $\bar{e}_{NML} = 0$.

Due to the arbitrariness of $\delta\bar{u}_i,\ \delta\bar{\varphi}$, from Eq. (2.54b) we get

$$\begin{array}{l}\bar{S}_{Jk,J}+\bar{f}_k=\bar{\rho}\ddot{u}_k,\quad \bar{D}_{I,I}=\bar{\rho}_{\mathrm{e}}\quad \text{in}\quad \bar{V}\\ \bar{S}_{Jk}\bar{n}_J=\bar{T}_k^*\quad \text{on}\quad \bar{a}_\sigma,\quad \bar{D}_I\bar{n}_I=-\bar{\sigma}^*\quad \text{on}\quad \bar{a}_D\end{array}\tag{2.56}$$

and

$$\delta\bar{\Pi}_1=\int_{\bar{a}^{\mathrm{itf}}}\bar{S}_{IJ}\bar{n}_I\delta u_J\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{itf}}}\bar{D}_I\bar{n}_I\delta\varphi\,\mathrm{d}\bar{a}\tag{2.57a}$$

Similarly for the environment, we have

$$\begin{aligned}\delta\bar{\Pi}_2=&\int_{\bar{V}^{\mathrm{env}}}\delta \mathrm{g}^{\mathrm{env}}\,\mathrm{d}\bar{V}+\int_{\bar{V}^{env}}\mathrm{g}^{\mathrm{env}}\delta u_{k,k}\,\mathrm{d}\bar{V}-\delta\bar{W}_1^{*\mathrm{env}}=\int_{\bar{a}_\sigma^{\mathrm{env}}}(\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}-\bar{T}_J^{*\mathrm{env}})\delta u_i^{\mathrm{env}}\,\mathrm{d}\bar{a}\\&+\int_{\bar{a}^{\mathrm{int}}}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta u_i^{\mathrm{env}}\,\mathrm{d}\bar{a}-\int_{\bar{V}^{\mathrm{env}}}(\bar{S}_{IJ,I}^{\mathrm{env}}+\bar{f}_J^{\mathrm{env}}-\bar{\rho}\ddot{u}_J^{\mathrm{env}})\delta\ddot{u}_J^{\mathrm{env}}\,\mathrm{d}\bar{V}\\&+\int_{\bar{a}_D^{\mathrm{env}}}(\overline{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}+\bar{\sigma}^{*\mathrm{env}})\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{itf}}}\overline{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}\\&-\int_{\bar{V}^{env}}(\overline{D}_{I,I}^{\mathrm{env}}-\bar{\rho}_{\mathrm{e}}^{\mathrm{env}})\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{V}=0\end{aligned}\tag{2.58}$$

Due to the arbitrariness of $\delta\bar{u}_i^{\mathrm{env}},\delta\bar{\varphi}^{\mathrm{env}}$, from Eq. (2.58), we get

$$\begin{array}{l}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}=\bar{T}_J^{*\mathrm{env}}\quad \text{on}\quad \bar{a}_\sigma^{\mathrm{env}},\quad \bar{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}=-\bar{\sigma}^{*\mathrm{env}}\quad \text{on}\quad \bar{a}_{DE}^{\mathrm{env}}\\ \bar{S}_{IJ,I}^{\mathrm{env}}+\bar{f}_J^{\mathrm{env}}=\bar{\rho}^{\mathrm{env}}\ddot{u}_J^{\mathrm{env}},\quad \bar{D}_{I,I}^{\mathrm{env}}=\bar{\rho}_{\mathrm{e}}^{\mathrm{env}}\quad \text{in}\quad \bar{V}^{\mathrm{env}}\end{array}\tag{2.59}$$

and

$$\delta\bar{\Pi}_2=\int_{\bar{a}^{\mathrm{itf}}}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta u_J^{\mathrm{env}}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{itf}}}\bar{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}\tag{2.57b}$$

Noting $\bar{n}_I=-\bar{n}_I^{\mathrm{env}},\bar{u}_I=\bar{u}_I^{\mathrm{env}},\varphi=\varphi^{\mathrm{env}}$ on the interface, we get

$$\begin{aligned}\delta\bar{\Pi}=\delta\bar{\Pi}_1+\delta\bar{\Pi}_2-\delta\bar{W}^{*\mathrm{int}}=&\int_{\bar{a}^{\mathrm{int}}}\bar{S}_{IJ}\bar{n}_I\delta u_J\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{int}}}\bar{D}_I\bar{n}_I\delta\varphi\,\mathrm{d}\bar{a}\\&+\int_{\bar{a}^{\mathrm{int}}}\bar{S}_{IJ}^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta u_J^{\mathrm{env}}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{int}}}\bar{D}_I^{\mathrm{env}}\bar{n}_I^{\mathrm{env}}\delta\varphi^{\mathrm{env}}\,\mathrm{d}\bar{a}-\int_{\bar{a}^{\mathrm{int}}}\bar{T}_K^{*\mathrm{int}}\delta u_k\,\mathrm{d}\bar{a}+\int_{\bar{a}^{\mathrm{int}}}\bar{\sigma}^{*\mathrm{int}}\delta\varphi\,\mathrm{d}\bar{a}=0\end{aligned}$$

So on the interface, it is obtained:

$$(\bar{S}_{IJ}-\bar{S}_{IJ}^{\mathrm{env}})\bar{n}_I=\bar{T}_J^{*\mathrm{int}},\quad (\bar{D}_I-\bar{D}_I^{\mathrm{env}})\bar{n}_I=-\bar{\sigma}^{*\mathrm{int}};\quad \text{on}\quad \bar{a}^{\mathrm{int}}\tag{2.60}$$

The above variational principle requests prior that the displacements and the potential satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations:

$$
\begin{aligned}
&u_i = u_i^*, \quad \text{on} \quad a_u; \quad \varphi = \varphi^*, \quad \text{on} \quad a_\varphi \\
&u_i^{\text{env}} = u_i^{*\text{env}}, \quad \text{on} \quad a_u^{\text{env}}; \quad \varphi^{\text{env}} = \varphi^{*\text{env}}; \quad \text{on} \quad a_\varphi^{\text{env}} \\
&u_i = u_i^{\text{env}}, \quad \varphi = \varphi^{\text{env}}; \quad \text{on} \quad a^{\text{int}}
\end{aligned} \tag{2.61}
$$

Equations (2.55), (2.56), (2.59), (2.57b), (2.60), and (2.61) are the governing equations under the finite deformation. It is noted that for the elastic material, these formulas are reduced to the usual elastic governing equations for elasticity. If in Eq. (2.52) we use $\delta \int_{\bar{V}} \bar{g}\,\mathrm{d}\bar{V}$ instead of $\int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{\mathrm{e}}\delta u_{i,i}\,\mathrm{d}\bar{V}$, Eq. (2.52) cannot be reduced to the usual elastic variational formula.

2.5 Internal Energy Variational Principle in Piezoelectric Materials

2.5.1 Internal Energy

It is noted that the constitutive equations of the general electroelastic materials are linear in the elastic part, but are nonlinear in the electric part for small deformation. The internal energy $\mathfrak{A}$ for materials without electric couple is assumed in the following form under small deformation:

$$
\begin{aligned}
&\mathfrak{A}(\varepsilon_{kl}, D_k) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + (1/2)\beta_{kl}D_kD_l - h_{kij}D_k\varepsilon_{ij} - (1/2)k_{ijkl}D_iD_j\varepsilon_{kl} + \cdots \\
&\beta_{kl} = \beta_{lk}, \quad k_{ijkl} = k_{jikl} = k_{ijlk} = k_{klij}, \quad h_{kij} = h_{kji} \quad C_{ijkl} = C_{jikl} = C_{ijlk} = C
\end{aligned} \tag{2.62a}
$$

where $h_{kij}, \beta_{kl}, k_{ijkl},$ and C_{ijkl} are material constants. The constitutive equations are

$$
\begin{aligned}
&\sigma_{lk} = \partial\mathfrak{A}/\partial\varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - h_{kij}D_k - (1/2)k_{ijkl}D_iD_j \\
&E_k = \partial\mathfrak{A}/\partial D_k = (\beta_{kl} - k_{klij}\varepsilon_{ji})D_l - h_{kij}\varepsilon_{ij}
\end{aligned} \tag{2.63}
$$

Equation (2.62a) can be rewritten as

$$
\mathfrak{A}(\varepsilon_{kl}, D_k) = (1/2)C_{ijkl}\varepsilon_{ij} + \mathfrak{A}^e, \quad \mathfrak{A}^e = (1/2)\left(E_kD_k - \Delta_{kl}^{\mathfrak{A}}\varepsilon_{kl}\right); \quad \Delta_{kl}^{\mathfrak{A}} = h_{kij}D_k \tag{2.62b}
$$

2.5.2 Internal Energy Variational Principle under Small Deformation

Let $\mathbf{u}, \boldsymbol{D}, \mathbf{u}^{\text{env}}, \boldsymbol{D}^{\text{env}}$ satisfy their boundary conditions on their own boundaries a_u, $a_D, a_u^{\text{env}}, a_D^{\text{env}}$ and the continuity conditions on the interface a^{int}, i.e.,

$$\begin{aligned}
&u_i = u_i^*, \quad \text{on} \quad a_u; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D \\
&u_i^{\text{env}} = u_i^{*\text{env}}, \quad \text{on} \quad a_u^{\text{env}}; \quad D_i^{\text{env}} n_i^{\text{env}} = -\sigma^{*\text{env}}, \quad \text{on} \quad a_D^{\text{env}} \\
&u_i = u_i^{\text{env}}, \quad (D_i - D_i^{\text{env}}) n_i = -\sigma^{*\text{int}}, \quad \text{on} \quad a^{\text{int}}
\end{aligned} \tag{2.64}$$

where $\boldsymbol{n}$ is the outward normal of the body. Inside the body and environment, it is assumed that

$$\begin{aligned}
&\rho_{\text{e}} = D_{i,i}, \varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2 \quad \text{in} \quad V; \quad \rho_{\text{e}}^{\text{env}} = D_{i,i}^{\text{env}}, \\
&\varepsilon_{ij}^{\text{env}} = \left(u_{i,j}^{\text{env}} + u_{j,i}^{\text{env}}\right)/2 \quad \text{in} \quad V^{\text{env}}
\end{aligned} \tag{2.65}$$

Under the above conditions, given the displacement and electric charge virtual increments, the PVP in term of the internal energy is (Kuang 2009a)

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0 \\
&\delta\Pi_1 = \delta\int_V \mathfrak{A}\,\mathrm{d}V - \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_V \varphi\delta(\rho_{\text{e}}\,\mathrm{d}V) \\
&\qquad - \int_{a_D} \varphi\delta(\sigma^*\,\mathrm{d}a) - \int_{a_\varphi} \varphi^*\delta(\sigma\,\mathrm{d}a) \\
&\delta\Pi_2 = \delta\int_{V^{\text{env}}} \mathfrak{A}^{\text{env}}\,\mathrm{d}V - \int_{V^{\text{env}}} (f_k^{\text{env}} - \rho^{\text{env}}\ddot{u}_k^{\text{env}})\delta u_k^{\text{env}}\,\mathrm{d}V - \int_{a_\sigma^{\text{env}}} T_k^{*\text{env}}\delta u_k^{\text{env}}\,\mathrm{d}a \\
&\qquad - \int_{V^{\text{env}}} \varphi^{\text{env}}\delta(\rho_{\text{e}}^{\text{env}}\,\mathrm{d}V) - \int_{a_D^{\text{env}}} \varphi^{\text{env}}\delta(\sigma^{*\text{env}}\,\mathrm{d}a) - \int_{a_\varphi^{\text{env}}} \varphi^{*\text{env}}\delta(\sigma^{\text{env}}\,\mathrm{d}a) \\
&\delta W^{\text{int}} = \int_{a^{\text{int}}} T_k^{*\text{int}}\delta u_k\,\mathrm{d}a + \int_{a^{\text{int}}} \varphi^{*\text{int}}\delta(\sigma\,\mathrm{d}a)
\end{aligned} \tag{2.66}$$

where $a = a_\sigma + a_u + a^{\text{int}} = a_D + a_\varphi + a^{\text{int}}$, f_k, T_k^*, φ, and φ^* are given body force, traction, potential, and surface potential, respectively. It is noted that the work done by the electric field in Eq. (2.66) has the form $\varphi\delta q$, i.e. the potential is kept constant, but the electric charge $\rho_{\text{e}}\,\mathrm{d}V$, σda etc. have virtual increment. σ^* and $\sigma^{*\text{env}}$ are given constants and do not change when virtual displacements happen, so terms $\sigma^*\mathrm{d}a$ and $\sigma^{*\text{env}}\,\mathrm{d}a^{\text{env}}$ will not be constants. Thus, terms $\int_{a_D} \varphi\delta(\sigma^*\mathrm{d}a)$ and $\int_{a_D^{env}} \varphi^{\text{env}}\delta(\sigma^{*\text{env}}\mathrm{d}a)$ etc. should be added to the variational formula. $T_k^{*\text{int}}$ and $\varphi^{*\text{int}}$ are given surface force and the jump of electric potential on the interface, respectively. Similar to Eq. (2.8),

$$\begin{aligned}&\delta D_i = \delta_D D_i + \delta_u D_i, \quad \delta_u D_i = D_{i,p}\delta u_p, \quad E_{k,j} = E_{j,k} = -\varphi_{,jk}\\&\delta\rho_{\mathrm{e}} = \delta_D\rho_{\mathrm{e}} + \delta_u\rho_{\mathrm{e}}, \quad \delta_u\rho_{\mathrm{e}} = D_{i,ip}\delta u_p = D_{i,pi}\delta u_p\end{aligned} \tag{2.67}$$

The variation of the differential volume and area, etc. are

$$\begin{aligned}&\delta(\mathrm{d}V) = \delta u_{k,k}\mathrm{d}V, \quad \delta(n_k\,\mathrm{d}a) = (n_k\delta u_{p,p} - n_p\delta u_{p,k})\,\mathrm{d}a,\\&\delta(\mathrm{d}a) = (\delta u_{p,p} - \delta u_{p,k}n_p n_k)\,\mathrm{d}a\end{aligned} \tag{2.68}$$

Neglecting terms containing $(\sigma_{kl}\varepsilon_{kl} + k_{ijkl}D_iD_j\varepsilon_{kl})/2$, it is obtained:

$$\begin{aligned}\delta\int_V \mathfrak{A}\,\mathrm{d}V &= \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V + \int_V E_j\delta D_j\,\mathrm{d}V + \int_V \mathfrak{A}^e\delta u_{k,k}\,\mathrm{d}V\\&= \int_a (\sigma_{ji} + E_mD_m\delta_{ij}/2)n_j\delta u_i\,\mathrm{d}a - \int_V (\sigma_{ji} + E_mD_m\delta_{ij}/2)_{,j}\delta u_i\,\mathrm{d}V + \int_V E_j\delta D_j\,\mathrm{d}V\end{aligned} \tag{2.69a}$$

$$\begin{aligned}&\int_V \varphi\delta(\rho_{\mathrm{e}}\,\mathrm{d}V) = \int_V \varphi\delta(D_{i,i}\,\mathrm{d}V) = \int_V \varphi\delta D_{i,i}\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_V \varphi\delta_D D_{i,i}\,\mathrm{d}V\\&+ \int_V \varphi D_{i,ip}\delta u_p\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_a \varphi\delta_D D_i n_i\,\mathrm{d}a - \int_V \varphi_{,i}\delta_D D_i\,\mathrm{d}V\\&+ \int_a \varphi\delta_u D_i n_i\,\mathrm{d}a - \int_V \varphi_{,i}\delta_u D_i\,\mathrm{d}V - \int_V \varphi D_{i,p}\delta u_{p,i}\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_a \varphi\delta D_i n_i\,\mathrm{d}a\\&- \int_V \varphi_{,i}\delta D_i\,\mathrm{d}V - \int_a \varphi D_{i,p}n_i\delta u_p\,\mathrm{d}a + \int_V (\varphi D_{i,p})_{,i}\delta u_p\,\mathrm{d}V + \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V\end{aligned} \tag{2.69b}$$

$$\begin{aligned}\int_{a_D} \varphi\delta(\sigma^*\,\mathrm{d}a) &= -\int_{a_D} \varphi D_i n_i\delta(\mathrm{d}a) = -\int_{a_D} \varphi D_i n_i(\delta u_{p,p} - \delta u_{p,k}n_pn_k)\,\mathrm{d}a\\&= -\int_{a_D} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a - \int_{a_D} \varphi(D_p\delta u_{i,p} - D_i\delta u_{p,k}n_pn_k)n_i\,\mathrm{d}a\\&= -\int_a \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a + \int_{a_\varphi + a^{\mathrm{int}}} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a\\&\quad - \int_{a_D} \varphi(D_p\delta u_{i,p} - D_i\delta u_{p,k}n_pn_k)n_i\,\mathrm{d}a\end{aligned} \tag{2.69c}$$

$$\begin{aligned}&\int_{a_\varphi} \varphi^*\delta(\sigma\,\mathrm{d}a) = -\int_{a_\varphi} \varphi^*\delta(D_in_i\mathrm{d}a) = -\int_{a_\varphi} \varphi^*\delta D_i n_i\,\mathrm{d}a - \int_{a_\varphi} \varphi^* D_i\delta(n_i\,\mathrm{d}a)\\&\qquad = -\int_{a_\varphi} \varphi^*\delta D_i n_i\,\mathrm{d}a - \int_{a_\varphi} \varphi^*(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a\\&\int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}\delta(\sigma\,\mathrm{d}a) = -\int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}\delta D_i^{\mathrm{int}} n_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}(D_i^{\mathrm{int}}\delta u_{p,p} - D_p^{\mathrm{int}}\delta u_{i,p})n_i\,\mathrm{d}a\end{aligned} \tag{2.69d}$$

and

$$
\begin{aligned}
&\int_a \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a - \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_V \varphi_{,i}(D_i\delta u_{p,p} - D_p\delta u_{i,p})\,\mathrm{d}V \\
&\quad + \int_V \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})_{,i}\,\mathrm{d}V - \int_V \varphi D_{i,i}\delta u_{p,p}\,\mathrm{d}V = \int_V \varphi_{,i}D_i\delta u_{p,p}\,\mathrm{d}V \\
&\quad - \int_V \varphi_{,i}D_p\delta u_{i,p}\mathrm{d}V - \int_V \varphi D_{p,i}\delta u_{i,p}\,\mathrm{d}V = \int_V \varphi_{,i}D_i\delta u_{p,p}\,\mathrm{d}V - \int_V (\varphi D_p)_{,i}\delta u_{i,p}\,\mathrm{d}V \\
&\quad = \int_a (\varphi_{,i}D_i\delta u_p)n_p\,\mathrm{d}a - \int_V (\varphi_{,i}D_i)_{,p}\delta u_p\,\mathrm{d}V - \int_a \left[(\varphi D_p)_{,i}\delta u_i\right]n_p\,\mathrm{d}a + \int_V (\varphi D_p)_{,ip}\delta u_i\,\mathrm{d}V \\
&\quad = -\int_a (\varphi D_p)_{,i}n_p\delta u_i\,\mathrm{d}a + \int_V (\varphi D_p)_{,pi}\delta u_i\,\mathrm{d}V + \int_a \varphi_{,i}D_i n_p\delta u_p\,\mathrm{d}a - \int_V (\varphi_{,i}D_i)_{,p}\delta u_p\,\mathrm{d}V
\end{aligned}
\tag{2.70a}
$$

$$
\begin{aligned}
\int_{a_D} \varphi\delta D_i n_i\,\mathrm{d}a &= \int_{a_D} \varphi\delta(D_i n_i)\,\mathrm{d}a - \int_{a_D} \varphi D_i\delta n_i\,\mathrm{d}a \\
&= -\int_{a_D} \varphi(D_i\delta u_{p,l}n_l n_p n_i - D_i\delta u_{p,i}n_p)\,\mathrm{d}a
\end{aligned}
\tag{2.70b}
$$

In Eq. (2.70b), $D_i n_i = -\sigma^*$ is a given value on a_D, so its variation vanishes on a_D.

Substituting above equations into $\delta\Pi_1$ in Eq. (2.66), it is obtained:

$$
\begin{aligned}
\delta\Pi_1 = &\int_{a_\sigma} \left[\left(\sigma_{ji} + \frac{1}{2}E_m D_m\delta_{ij} + \varphi D_{j,i} - (\varphi D_j)_{,i} + \varphi_{,p}D_p\delta_{ij}\right)n_j - T_i^*\right]\delta u_i\mathrm{d}a \\
&- \int_V \left[\left(\sigma_{ji} + \frac{1}{2}E_m D_m\delta_{ij} + \varphi D_{j,i} - (\varphi D_j)_{,i} + \varphi_{,p}D_p\delta_{ij}\right)_{,j} + f_i - \rho\ddot{u}_i\right]\delta u_i\mathrm{d}V \\
&+ \int_V (E_j + \varphi_{,j})\delta D_j\mathrm{d}V + \int_{a_\varphi} (\varphi^* - \varphi)n_i\delta D_i\mathrm{d}a \\
&+ \int_{a_\varphi} (\varphi^* - \varphi)(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\mathrm{d}a \\
&+ \int_{a^{\mathrm{int}}} \left[\left(\sigma_{ji} + \frac{1}{2}E_m D_m\delta_{ij} + \varphi D_{j,i} - (\varphi D_j)_{,i} + \varphi_{,p}D_p\delta_{ij}\right)n_j\right]\delta u_i\mathrm{d}a \\
&- \int_{a^{\mathrm{int}}} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi\delta D_i n_i\mathrm{d}a
\end{aligned}
\tag{2.71}
$$

From Eqs. (2.66)and (2.71) and the arbitrariness of $\delta u_i,\ \delta D_i$ we get

$$
\begin{aligned}
&S_{ji,j} + f_i = \rho\ddot{u}_i, \quad E_j = -\varphi_{,i}, \quad \text{in} \quad V \\
&S_{ji}n_j = T_i^*, \quad \text{on} \quad a_\sigma; \quad \varphi = \varphi^*, \quad \text{on} \quad a_\varphi; \quad S_{ij} = \sigma_{ij} + \sigma_{ij}^{\mathrm{M}} \\
&\sigma_{ji}^{\mathrm{M}} = \varphi D_{j,i} + \frac{1}{2}E_m D_m\delta_{ij} - (\varphi D_j)_{,i} + \varphi_{,p}D_p\delta_{ij} = D_j E_i - \frac{1}{2}E_p D_p\delta_{ij}
\end{aligned}
\tag{2.72}
$$

where $\boldsymbol{\sigma}^{\mathrm{M}}$ is the Maxwell stress. Using Eq. (2.72), $\delta\Pi_1$ is reduced to

$$\begin{aligned}\delta\Pi_1 &= \int_{a^{\mathrm{int}}} S_{ji}n_j\delta u_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi\delta D_i n_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a \\ &= \int_{a^{\mathrm{int}}} S_{ji}n_j\delta u_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi\delta D_i n_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi D_i\delta(n_i\,\mathrm{d}a)\end{aligned} \tag{2.73}$$

Similarly for the environment, we have

$$\begin{aligned}&S^{\mathrm{env}}_{ji,j} + f^{\mathrm{env}}_i = \rho\ddot{u}^{\mathrm{env}}_i, \quad E^{\mathrm{env}}_i = -\varphi^{\mathrm{env}}_{,i}, \quad \text{in} \quad V^{\mathrm{env}} \\ &S^{\mathrm{env}}_{ji}n^{\mathrm{env}}_j = T^{*\mathrm{env}}_i, \quad \text{on} \quad a^{\mathrm{env}}_\sigma; \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad \text{on} \quad a^{\mathrm{env}}_\varphi, \quad S^{\mathrm{env}}_{ij} = \sigma^{\mathrm{env}}_{ij} + \sigma^{\mathrm{M}}_{ij} \\ &\delta\Pi_2 = \int_{a^{\mathrm{int}}} S^{\mathrm{env}}_{ij}n^{\mathrm{env}}_j\delta u^{\mathrm{env}}_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{\mathrm{env}}\delta D^{\mathrm{env}}_i n^{\mathrm{env}}_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{\mathrm{env}} D^{\mathrm{env}}_i\delta\left(n^{\mathrm{env}}_i\mathrm{d}a\right)\end{aligned} \tag{2.74}$$

δW^{int} can be reduced to

$$\begin{aligned}\delta W^{\mathrm{int}} &= \int_{a^{\mathrm{int}}} T^{*\mathrm{int}}_k\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}} n_i\delta D_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}}(D_i\delta u_{p,p} - D_p\delta u_{i,p})n_i\,\mathrm{d}a \\ &= \int_{a^{\mathrm{int}}} T^{*\mathrm{int}}_k\delta u_k\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{*\mathrm{int}} n_i\delta D_i\,\mathrm{d}a - \int_{a^{\mathrm{int}}} \varphi^{int} D_i\delta(n_i\,\mathrm{d}a)\end{aligned} \tag{2.75}$$

Substituting Eqs. (2.73), (2.74) and (2.75) into Eq. (2.66) and noting $\boldsymbol{n}^{\mathrm{env}} = -\boldsymbol{n}$, $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{env}}$ we have

$$\begin{aligned}\delta\Pi = &\int_{a^{\mathrm{int}}} [(S_{ij} - S^{\mathrm{env}}_{ij})n_j - T^{*\mathrm{int}}_i]\delta u_i\,\mathrm{d}a \\ &- \int_{a^{\mathrm{int}}} (\varphi - \varphi^{\mathrm{env}} - \varphi^{*\mathrm{int}})[\delta(D_i n_i) + D_i n_i\delta(\mathrm{d}a)]\end{aligned} \tag{2.76}$$

Due to the arbitrariness of $\delta u_i,\ \delta D_i$, we get

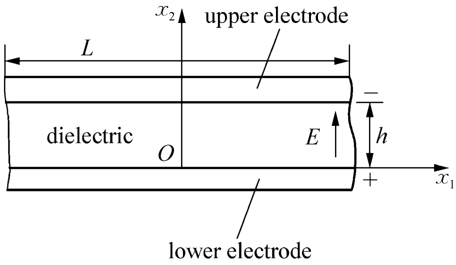
$$(S_{ij} - S^{\mathrm{env}}_{ij})n_j = T^{*\mathrm{int}}_i, \quad \varphi - \varphi^{\mathrm{env}} = \varphi^{*\mathrm{int}}, \quad \text{on} \quad a^{\mathrm{int}} \tag{2.77}$$

Equations (2.72), (2.74), (2.77), (2.64), and (2.65) form the complete governing equations.

2.5.3 The Force Acting on the Dielectric in a Plate Capacitor

As an application of the PVP, we discuss the force acting on the dielectric in a plate capacitor filled the dielectric with permittivity ϵ as shown in Fig. 2.2. Assume both

Fig. 2.2 A plate capacitor



the length and width of the plates are infinitely long, the distance h between two electrodes is small. There is no external mechanical force. The electric field inside the dielectric of the capacitor is homogeneous and $\boldsymbol{E} = E_2\boldsymbol{n}$, where $\boldsymbol{n}$ is along the positive direction of the axis x_2. The electric field inside the electrode is zero. In this simple case, the static electric force can be directly derived from the Maxwell stress and the general equation of the PVP.

1. *The Maxwell Stress Method*

Using the Maxwell stress in Eqs. (2.12) and (2.18), the force acting on the dielectric is

$$\boldsymbol{T} = \boldsymbol{n} \cdot \left(\boldsymbol{\sigma}^{\mathrm{M\ plate}} - \boldsymbol{\sigma}^{\mathrm{M}}\right) = -\boldsymbol{\sigma}^{\mathrm{M}} \cdot \boldsymbol{n} = -(1/2)D_n E_n \boldsymbol{n} = -(1/2)\epsilon E_2^2 \boldsymbol{n} \tag{2.78}$$

2. *The Internal Energy Variational Principle*

Let the upper plate electrode possesses negative charge and the lower electrode possesses positive charge. Because on the electrobe the electric charge is given in the internal energy variational principle, the boundary conditions on the whole boundary of the dielectric are known. In Eq. (2.66) we only need to discuss $\delta\Pi_1$. Given the upper electrode a virtual displacement $\delta u_2 = \delta h$, the virtual strain of the dielectric is $\varepsilon_{22} = u_{2,2} = \delta h/h$. Because only δu_2 is considered, the surface integrals in $\delta\Pi_1$ can be neglected due to that the surface area keeps constant in the virtual displacement process. Therefore the variational principle for the volume between unit surfaces of electrodes is

$$\delta\Pi_{\mathfrak{A}} = \delta\Pi_1 = \delta\int_V \mathfrak{A}\,\mathrm{d}V = h(\sigma_{22}\delta u_{2,2} + E_2\delta D_2 + (1/2)E_2 D_2 \delta u_{2,2}) = 0$$

Because electric charge q on the electrode is constant, so $\delta_D D_2 = 0$. In volume $D_2 = \text{const.}$ due to $D_{2,p} = 0$, so $\delta_u D_2 = 0$. Therefore, it is obtained:

$$\begin{aligned}\delta\Pi_{\mathfrak{A}} &= h(\sigma_{22}\delta u_{2,2} + E_2\delta D_2 + (1/2)E_2 D_2\delta u_{2,2}) = h\left[\sigma_{22}\frac{\delta h}{h} + \frac{1}{2}\epsilon\left(\frac{\varphi}{h}\right)^2\frac{\delta h}{h}\right] = 0\\ &\Rightarrow \quad T_2 = \sigma_{22} = -(\epsilon/2)(\varphi/h)^2\end{aligned}$$

The result is identical with that in Eq. (2.78).

3. *The Electric Gibbs Free Energy Variational Principle*

Let the lower electrode possesses positive potential and the upper plate electrode grounded. Analogous to the above problem, but on the electrode the electric potential is given in the electric Gibbs free energy variational principle now. In Eq. (2.7) we only need to discuss $\delta \Pi_1$.

Give a virtual displacement under the constant electric potential on the electrode plate. It is noted that though φ is constant on the plate, but after virtual displacement, φ is changed for the point inside the dielectric. For a fixed $\boldsymbol{x}$, the change of the electric field due to changed φ is

$$\delta_\varphi E_2 = \varphi/(h+\delta h) - \varphi/h = -\varphi \delta h/h^2, \quad -D_2 \delta E_2 = D_2 E_2 \delta h/h.$$

The potential φ on the electrode is constant, $E_{2,p} = 0$, so $\delta E_2 = \delta_\varphi E_2$. Therefore, we have

$$\begin{aligned}\delta \Pi_g &= h\left[\sigma_{22}\delta u_{2,2} - D_2 \delta E_2 - (1/2)E_2 D_2 \delta u_{2,2}\right] \\ &= hD_2\left[\sigma_{22}(\delta h/h) + (1/2)\epsilon E_2^2(\delta h/h)\right] = 0 \quad \Rightarrow \quad T_2 = \sigma_{22} = -D_2^2/2\epsilon\end{aligned}$$

The result is identical with that in Eq. (2.78).

Equation (2.78) shows that the force acting on the dielectric is compressive. It is just the attractive force between two electrodes. This result is identical with that in usual textbooks.

2.6 Constitutive Equations in Electroelasticity

2.6.1 Constitutive Equations

In this section, we only discuss the case with symmetric stresses. When the thermal effect is omitted in Eq. (1.59), there are only four thermodynamic character functions: the internal energy $\mathfrak{A}(\boldsymbol{\varepsilon}, \boldsymbol{D})$ is equivalent to the free energy f, the electric Gibbs function $g(\boldsymbol{\varepsilon}, \boldsymbol{E})$ is equivalent to the electric enthalpy h^{e}, the enthalpy $h(\boldsymbol{\sigma}, \boldsymbol{E})$ is equivalent to the Gibbs function g^g, and the elastic Gibbs function $g^{\mathrm{el}}(\boldsymbol{\sigma}, \boldsymbol{D})$ is equivalent to the elastic enthalpy h^{el}. In general case, there are two groups with four variables: $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$, $(\boldsymbol{E}, \boldsymbol{D})$ in electroelasticity. Because each variable in two groups can be used as the independent variable, there are four group constitutive equations corresponding to four thermodynamic character functions $\mathfrak{A}$ (f), g (h^{e}), h (g^g), and h^{el} (g^{el}). Constitutive equation (2.3) is derived from g; Eq. (2.63) is derived from $\mathfrak{A}$. The enthalpy h and the elastic Gibbs function g^{el} can, respectively, be assumed in the following forms:

$$h = -(1/2)s_{ijkl}\sigma_{ij}\sigma_{kl} - (1/2)\epsilon_{kl}E_k E_l - d_{kij}E_k\sigma_{ij} - (1/2)p_{ijkl}E_i E_j \sigma_{kl} \tag{2.79}$$

$$g^{\mathrm{el}} = -(1/2)s_{ijkl}\sigma_{ij}\sigma_{kl} + (1/2)\beta_{kl}D_k D_l - \mathrm{g}_{kij}D_k\sigma_{ij} - (1/2)q_{ijkl}D_i D_j \sigma_{kl} \tag{2.80}$$

where $\boldsymbol{s}$ is the flexibleness coefficient tensor. From Eqs. (2.79) and (2.80), the following constitutive equations are obtained, respectively:

$$\begin{aligned} \varepsilon_{ij} &= -\partial h/\partial \sigma_{ij} = s_{ijkl}\sigma_{kl} + d_{kij}E_k + (1/2)p_{ijkl}E_kE_l \\ D_i &= -\partial h/\partial E_i = \epsilon_{ij}E_j + d_{ijk}\sigma_{jk} + p_{ijkl}E_j\sigma_{kl} \end{aligned} \tag{2.81}$$

$$\begin{aligned} \varepsilon_{ij} &= -\partial g^{\mathrm{el}}/\partial \sigma_{ij} = s_{ijkl}\sigma_{kl} + \mathrm{g}_{kij}D_k + (1/2)q_{ijkl}D_kD_l \\ E_i &= \partial g^{\mathrm{el}}/\partial D_i = \beta_{ij}D_j - \mathrm{g}_{ijk}\sigma_{jk} - q_{ijkl}D_i\sigma_{kl} \end{aligned} \tag{2.82}$$

Equations (2.3), (2.63), (2.81), and (2.82) are four kinds of constitutive equations for general ferroelectric materials. In these equations, it has been assumed that $\boldsymbol{\sigma} = \boldsymbol{\varepsilon} = \boldsymbol{E} = \boldsymbol{D} = \mathbf{0}$ at the natural state. Constitutive equations of some simpler materials are as follows.

Linear piezoelectric materials. The constitutive equations of the first, second, third, and fourth types of linear piezoelectric materials are

$$\begin{aligned} &\varepsilon_{ij} = s^E_{ijkl}\sigma_{kl} + d^\sigma_{kij}E_k \quad (or\ d^\sigma_{ijk}E_k), \quad D_i = d^E_{ijk}\sigma_{jk} + \epsilon^\sigma_{ij}E_j \\ &\sigma_{ij} = C^E_{ijkl}\varepsilon_{kl} - e^\varepsilon_{kij}E_k, \quad (or\ e^\varepsilon_{ijk}E_k) \quad D_i = e^E_{ikl}\varepsilon_{kl} + \epsilon^\varepsilon_{ij}E_j \\ &\varepsilon_{ij} = s^D_{ijkl}\sigma_{kl} + \mathrm{g}^\sigma_{kij}D_k, \quad (or\ \mathrm{g}^\sigma_{ijk}E_k) \quad E_i = -\mathrm{g}^D_{ijk}\sigma_{jk} + \beta^\sigma_{ij}D_j \\ &\sigma_{ij} = C^D_{ijkl}\varepsilon_{kl} - h^\varepsilon_{kij}D_k, \quad (or\ h^\varepsilon_{ijk}E_k) \quad E_i = -h^D_{ikl}\varepsilon_{kl} + \beta^\varepsilon_{ij}D_j \end{aligned} \tag{2.83}$$

where the superscript letter "ς" of a material constant means that the constant is measured at ς = const. As an example, $\boldsymbol{C}^E_{ijkl}$ means that the constant $\boldsymbol{C}^E_{ijkl}$ is measured at E_i = const. Usually the coefficient measured at constant $\boldsymbol{E}$ is called the closed circuit coefficient, and the coefficient measured at constant $\boldsymbol{D}$ is called the open circuit coefficient. Usually $e \cdot E = e^\varepsilon_{ijk}E_k$ is more convenient than $E \cdot e = e^\varepsilon_{kij}E_k$ in use. If the Voigt notation (see Eq. (1.37) is used, Eq. (2.83) can be rewritten as

$$\begin{aligned} &\boldsymbol{\varepsilon} = \boldsymbol{s} : \boldsymbol{\sigma} + \boldsymbol{d}^{\mathrm{T}} \cdot \boldsymbol{E}, \quad \boldsymbol{D} = \boldsymbol{d} : \boldsymbol{\sigma} + \boldsymbol{\epsilon} \cdot \boldsymbol{E}; \quad \boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} - \boldsymbol{e}^{\mathrm{T}} \cdot \boldsymbol{E}, \quad \boldsymbol{D} = \boldsymbol{e} : \boldsymbol{\varepsilon} + \boldsymbol{\epsilon} \cdot \boldsymbol{E}; \\ &\boldsymbol{\varepsilon} = \boldsymbol{s} : \boldsymbol{\sigma} + \mathbf{g}^{\mathrm{T}} \cdot \boldsymbol{D}, \quad \boldsymbol{E} = -\mathbf{g} : \boldsymbol{\sigma} + \boldsymbol{\beta} \cdot \boldsymbol{D}; \quad \boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} - \boldsymbol{h}^{\mathrm{T}} \cdot \boldsymbol{D}, \quad \boldsymbol{E} = -\boldsymbol{h} : \boldsymbol{\varepsilon} + \boldsymbol{\beta} \cdot \boldsymbol{D} \end{aligned} \tag{2.84}$$

It is noted that though some coefficients have the same notation in different kind of constitutive equations, they should be measured in different conditions.

Electrostrictive materials with symmetric center. For all electrostrictive materials with symmetric center, the material coefficients with odd number subscript are all zero, so the piezoelectric effect disappeared. In Eq. (2.3), if terms containing $\boldsymbol{\alpha}$ are omitted, the constitutive equations have following forms:

$$\begin{aligned} &\varepsilon_{ij} = S^E_{ijkl}\sigma_{kl} + (1/2)p_{ijkl}E_kE_l, \quad D_i = \epsilon^\sigma_{ij}E_j + p_{ijkl}E_j\sigma_{kl} \approx \epsilon^\sigma_{ij}E_j \\ &\sigma_{ij} = C^E_{ijkl}\varepsilon_{kl} - (1/2)l_{ijkl}E_kE_l, \quad D_i = \epsilon^\varepsilon_{ij}E_j + l_{ijkl}E_j\varepsilon_{kl} \approx \epsilon^\varepsilon_{ij}E_j \\ &\varepsilon_{ij} = S^D_{ijkl}\sigma_{kl} + (1/2)q_{ijkl}D_kD_l, \quad E_i = \beta^\sigma_{ij}D_j - q_{ijkl}D_j\varepsilon_{kl} \approx \beta^\sigma_{ij}D_j \\ &\sigma_{ij} = C^D_{ijkl}\varepsilon_{kl} - (1/2)k_{ijkl}D_kD_l, \quad E_i = \beta^\varepsilon_{ij}D_j - k_{ijkl}D_j\varepsilon_{kl} \approx \beta^\varepsilon_{ij}D_j \end{aligned} \tag{2.85}$$

Under the high electric field, usually the electric hysteretic loop of the electrostrictive material, like PMN, is smaller than that of the piezoelectric material, like PZT.

2.6.2 Relations Between Material Constants of the Linear Piezoelectric Materials

Equation (2.83) is the four kinds of constitutive equations for the linear piezoelectric materials. Substitution of ***D*** in the second equation into the fourth equation in Eq. (2.83) yields

$$
\begin{aligned}
\sigma_{ij} &= C^D_{ijkl}\varepsilon_{kl} - h^E_{kij}\left(e^E_{kmn}\varepsilon_{mn} + \epsilon^\varepsilon_{km}E_m\right) = \left(C^D_{ijkl} - h^E_{pij}e^E_{pkl}\right)\varepsilon_{kl} - h^E_{kij}\epsilon^\varepsilon_{km}E_m = C^E_{ijkl}\varepsilon_{kl} - e^\varepsilon_{mij}E_m \\
E_i &= -h^D_{ikl}\varepsilon_{kl} + \beta^\varepsilon_{ij}\left(e^E_{jmn}\varepsilon_{mn} + \epsilon^\varepsilon_{jm}E_m\right) = \left(-h^D_{ikl} + \beta^\varepsilon_{ij}e^E_{jkl}\right)\varepsilon_{kl} + \beta^\varepsilon_{ij}\epsilon^\varepsilon_{jm}E_m
\end{aligned}
$$

and in the similar discussion, we finally get

$$
\begin{aligned}
& C^D_{ijkl}s^D_{klmn} = C^E_{ijkl}s^E_{klmn} = \delta_{im}\delta_{jn}, \quad \beta^\varepsilon_{ij}\epsilon^\varepsilon_{jm} = \beta^\sigma_{ij}\epsilon^\sigma_{jm} = \delta_{im}, \quad \epsilon^\sigma_{ip} - \epsilon^\varepsilon_{ip} = e^E_{ikl}d^\sigma_{pmn}, \\
& \beta^\varepsilon_{ip} - \beta^\sigma_{ij} = h^D_{ikl}g^\sigma_{pkl}, \quad d^\sigma_{mij} = g^\sigma_{pij}\epsilon^\sigma_{pm}, \quad g^D_{ikl} = \beta^\sigma_{ip}d^D_{pkl}, \quad e^\varepsilon_{mij} = h^E_{kij}\epsilon^\varepsilon_{km}, \quad h^D_{ikl} = e^E_{jkl}\beta^\varepsilon_{ij}, \\
& \beta^\varepsilon_{ip}\epsilon^\sigma_{pm} - h^D_{ikl}d^\sigma_{mkl} = \beta^\sigma_{ip}\epsilon^\varepsilon_{pm} + g^D_{ikl}e^\varepsilon_{mkl} = \delta_{im}, \quad C^D_{ijkl} - C^E_{ijkl} = h^E_{pij}e^E_{pkl}, \quad s^D_{ijkl} - s^E_{ijkl} = -g^\sigma_{pij}d^D_{pkl}, \\
& C^D_{ijkl}g^\sigma_{pkl} = h^E_{pij}, \quad C^D_{ijkl}d^\sigma_{mkl} = h^E_{pij}\epsilon^\sigma_{pm}, \quad C^E_{ijkl}d^\sigma_{pmn} = e^\varepsilon_{pij}, \quad g^D_{ikl}C^E_{klmn} = \beta^\sigma_{ip}e^E_{pmn}, \\
& h^D_{ikl}s^D_{klmn} = g^D_{imn}, \quad e^E_{ikl}s^E_{klmn} = d^D_{imn}, \quad h^D_{ikl}s^E_{klmn} = \beta^\varepsilon_{ip}d^D_{pmn}, \quad s^D_{ijkl}e^\varepsilon_{mkl} = g^\sigma_{pij}\epsilon^\varepsilon_{pm}, \\
& C^D_{ijkl}s^E_{klmn} - h^E_{pij}d^D_{pmn} = s^D_{ijkl}C^E_{klmn} + g^\sigma_{pij}e^E_{pmn} = \delta_{im}\delta_{nj}
\end{aligned} \tag{2.86}
$$

For the nonlinear ferroelectric materials, relations between material coefficients of different constitutive equations are difficult expressed in simple unique forms.

2.7 Variational Principle in Pyroelectric Materials and Its Governing Equations

2.7.1 Internal Energy and Electric Gibbs Function

According to the continuum thermodynamics in Sect. 1.5, the electric Gibbs function g, the electric complementary dissipative energy rate $\dot{h}_g$, the internal energy $\mathfrak{A}$, and the dissipative energy rate $\dot{h}_{\mathfrak{A}}$ can be assumed as

$$
\begin{aligned}
& g(\varepsilon_{kl}, E_k, \vartheta) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}E_k\varepsilon_{ij} - (1/2)\epsilon_{ij}E_iE_j - \alpha_{ij}\varepsilon_{ij}\vartheta - \tau_iE_i\vartheta - (1/2T_0)C\vartheta^2 \\
& \delta h_g = \eta_j\delta\vartheta_{,j} = -\left(\int_0^t \lambda_{ij}T^{-1}\vartheta_{,i}\,\mathrm{d}\tau\right)\delta\vartheta_{,j} \\
& C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad e_{kij} = e_{kji}, \quad \epsilon_{kl} = \epsilon_{lk}, \quad \alpha_{ij} = \alpha_{ji}, \quad \lambda_{ij} = \lambda_{ji}
\end{aligned} \tag{2.87}
$$

$$\mathfrak{A}(\varepsilon_{kl}, D_k, s) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - h_{kij}D_k\varepsilon_{ij} + (1/2)\beta_{ij}D_iD_j - \tilde{\alpha}_{ij}\varepsilon_{ji}s - \tilde{\tau}_iD_is + (T_0/2)\tilde{C}s^2$$
$$\delta h_{\mathfrak{A}} = \tilde{\lambda}_{ij}T\dot{\eta}_j\delta\eta_i (= T\delta s^{(\mathrm{i})} = -T_{,i}\dot{\eta}_i)$$
$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad h_{kij} = h_{kji}, \quad \beta_{kl} = \beta_{lk}, \quad \tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}, \quad \tilde{\lambda}_{ij} = \tilde{\lambda}_{ji} \tag{2.88}$$

Where $\vartheta = T - T_0$, T_0 is the temperature of the environment. It is noted that in Eq. (2.87), $s = 0$ when $T = T_0$, if $\varepsilon_{ij} = E_i = 0$, but in Eq. (2.88), $s = 0$ when $T = 0$ and $s = s_0$ when $T = T_0$; if $\varepsilon_{ij} = D_i = 0$; $\tilde{\alpha}_{ij}, \tilde{\tau}_i, \tilde{C}, \tilde{\lambda}_{ij}$ are all material constants. In the later sections, this rule will be adopted. Constitutive and evolution equations corresponding to Eq. (2.87) are

$$\begin{aligned}
&\sigma_{ji} = \partial g/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta \\
&D_i = -\partial g/\partial E_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta \\
&s = -\partial g/\partial\vartheta = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0 \\
&\eta_i = -\partial h_g/\partial T_{,i} = -\int_0^t T^{-1}\lambda_{ij}\vartheta_{,j}\,\mathrm{d}\tau, \quad T\dot{\eta}_i = q_i = -\lambda_{ij}\vartheta_{,j}
\end{aligned} \tag{2.89}$$

where the evolution equation of temperature has been shown in Eq. (1.71). Corresponding to Eq. (2.88) the constitutive and evolution equations are

$$\begin{aligned}
&\sigma_{ji} = \partial\mathfrak{A}/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - h_{kij}D_k - \tilde{\alpha}_{ij}s \\
&E_i = \partial\mathfrak{A}/\partial D_i = \beta_{ij}D_j - h_{ikl}\varepsilon_{kl} - \tilde{\tau}_is \\
&T = \partial\mathfrak{A}/\partial s = -\tilde{\alpha}_{ij}\varepsilon_{ji} - \tilde{\tau}_iD_i + T_0\tilde{C}s \\
&T_{,i} = -\partial h_{\mathfrak{A}}/\partial\eta_i = -\tilde{\lambda}_{ij}T\dot{\eta}_j = -\tilde{\lambda}_{ij}q_j, \quad \int_0^t T_{,i}\,\mathrm{d}\tau = -T\int_0^t \tilde{\lambda}_{ij}\dot{\eta}_j\,\mathrm{d}\tau
\end{aligned} \tag{2.90}$$

It is obvious that there is $T_{,j} = \vartheta_{,j}, \dot{T} = \dot{\vartheta}$. Using Eqs. (2.89) and (2.90), Eqs. (2.87) and (2.88) can be rewritten, respectively, as

$$g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{E\,T}, \quad g^{E\,T} = -(1/2)(D_kE_k + s\vartheta + \Delta_{kl}\varepsilon_{kl}), \quad \Delta_{kl} = e_{mkl}E_m + \alpha_{kl}\vartheta$$
$$\mathfrak{A} = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \mathfrak{A}^{E\,T}, \quad \mathfrak{A}^{E\,T} = (1/2)(D_kE_k + sT + \Delta'_{kl}\varepsilon_{kl}), \quad \Delta'_{kl} = h_{mkl}D_m + \alpha_{kl}s \tag{2.91}$$

In Eq. (2.91), $\Delta_{kl}\varepsilon_{kl}$ and $\Delta'_{kl}\varepsilon_{kl}$ can be neglected for the case of small strain.

Using the inertial entropy theory given in Sect. 1.7.2, from Eqs. (2.89) and (1.74), the thermal conductive or energy equation can be obtained:

$$-q_{i,i} = T\dot{s} + T\dot{s}^{(\mathrm{a})} - \dot{r}, \quad \lambda_{ij}T_{,ji} = T(\alpha_{ij}\dot{\varepsilon}_{ij} + \tau_i\dot{E}_i + T_0^{-1}C\dot{\vartheta} + T_0^{-1}C\rho_{s0}\ddot{\vartheta}) - \dot{r} \tag{2.92}$$

If ϑ is much less than T_0, $\vartheta \ll T_0$, then the above equation is reduced to

$$\lambda_{ij}\vartheta_{,ji} = T_0\alpha_{ij}\dot{\varepsilon}_{ij} + T_0\tau_i\dot{E}_i + C(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}) - \dot{r} \tag{2.93}$$

Equations (2.92) and (2.93) are temperature wave equations with finite phase velocity.

2.7.2 Electric Gibbs Function Variational Principle

In this section, we only discuss the PVP of the pyroelectric material with linear elasticity under small deformation. For simplicity, it is assumed that the environment is air. It is also assumed that in the air, the temperature is constant or $\vartheta^{\text{env}} = 0$ and at infinity, $\boldsymbol{E}^{\infty} = \mathbf{0},\ \sigma^{*\infty} = 0$. The interface is heat insulated. The heat input and heat output by heat conduction may be occured at some internal boundaries. Analogous to Eq. (2.8), the variation of the temperature ϑ can also be divided into $\delta_\vartheta \vartheta$ and $\delta_u \vartheta$, but it is not needed because the final result shows that terms containing $\delta_u \vartheta$ are countervailed each other. So the body and air only have electric connection. However, the contribution of the heat due to the variation of the volume seems to be considered.

Under the assumption that $\boldsymbol{u}, \varphi, \vartheta$ satisfy their own boundary conditions $u_i = u_i^*, \varphi = \varphi^*$ and $\vartheta = \vartheta^*$ on a_u, a_φ and a_T, respectively. $\varphi = \varphi^{\text{env}}, \vartheta = \vartheta^{\text{env}} = 0$ on the interface except at some heat source and sink places. In the medium $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, E_i = -\varphi_{,i}, T\dot{\eta}_j = -\lambda_{ij}\vartheta_{,i}$ and the constitutive equation (2.89) are held. Noting Eq. (1.59), $g = \mathfrak{A} - \boldsymbol{E} \cdot \boldsymbol{D} - Ts$, the PVP in terms of the electric Gibbs function for the pyroelectricity can be written as (Kuang 2009b)

$$
\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0\\
&\delta\Pi_1 = \int_V \delta(g + h_g)\,\mathrm{d}V + \int_V g^{E\,T}\delta u_{i,i}\,\mathrm{d}V - \delta Q' - \delta W\\
&\delta Q' = -\int_V \int_0^t (\dot{r}/T)\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V + \int_V s^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V + \int_{a_q}\int_0^t \dot{\eta}^*\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}a - \int_V\int_0^t \dot{s}^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V\\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a\\
&\delta\Pi_2 = \int_{V^{\text{env}}} \delta g^{\text{env}}\,\mathrm{d}V + \int_{V^{\text{env}}} g^{E\,T\,\text{env}}\delta u_{i,i}^{\text{env}}\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}^{\text{env}}\delta\varphi\,\mathrm{d}V\\
&\delta W^{\text{int}} = \int_{a^{\text{int}}} T_k^{*\text{int}}\delta u_k\,\mathrm{d}a - \int_{a_D^{\text{int}}}\sigma^{*\text{int}}\delta\varphi\,\mathrm{d}a
\end{aligned}
\tag{2.94}
$$

where $f_k, T_k^*, \rho_{\mathrm{e}}, \sigma^*, \rho_{\mathrm{e}}^{\text{env}}$ and $\dot{\eta}_i^*$ $(\dot{\eta}^* = \dot{\eta}_i^* n_i)$ are the given mechanical body force, traction, body electric charge density, surface electric charge density, body electric charge density in the air, and surface entropy flow, respectively, and a_q is the surface given thermal flow, $g^{\text{env}} = g^{E\,T\,\text{env}} = -(1/2)D_k^{\text{env}}E_k^{\text{env}}, D_k^{\text{env}} = \epsilon_0 E_k^{\text{env}}$. In Eq. (2.94), the term $\int_0^t \dot{s}^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}\tau$ is the electric complement heat rate per unit volume corresponding to the inner electric complement dissipation energy rate δh_g. This is consistent with the laws of the thermodynamics. In order to obtain the heat conduction equation and the boundary condition of the heat flow from the variational principle, the electric complement dissipation energy $\int_V \delta h_g\,\mathrm{d}V$ in $\delta\Pi$ and the inner irreversible electric complement heat $\int_V\int_0^t \dot{s}^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V$ in $\delta Q'$ should be simultaneously included in the variational functional. In Eq. (2.94), the integrands contain the time derivatives of variables and need to integrate with time t, because in the irreversible process the integral is dependent to the integral path. But the time is a parameter and does not join the virtual variation.

It is noted that

$$\begin{aligned}
\int_V \delta g \,\mathrm{d}V &= \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V - \int_V D_k \delta E_k \,\mathrm{d}V - \int_V s\delta\vartheta \,\mathrm{d}V = \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V \\
&\quad + \int_V D_k \delta_\varphi \varphi_{,\,i}\,\mathrm{d}V - \int_V D_i E_{p,i}\delta u_p \,\mathrm{d}V - \int_V s\delta\vartheta \,\mathrm{d}V \\
&= \int_a \sigma_{ji} n_j \delta u_i \,\mathrm{d}a - \int_V \sigma_{ji,j}\delta u_i \,\mathrm{d}V + \int_a D_k n_k \delta_\varphi \varphi \,\mathrm{d}a - \int_V D_{k,k}\delta_\varphi \varphi \,\mathrm{d}V \\
&\quad - \int_V \left(D_i E_p\right)_{,i}\delta u_p \,\mathrm{d}V + \int_V D_{i,i} E_p \delta u_p \,\mathrm{d}V - \int_V s\delta\vartheta \,\mathrm{d}V \\
\int_V g^{E\,T}\delta u_{k,k}\,\mathrm{d}V &= -(1/2)\int_a (D_k E_k + s\vartheta) n_k \delta u_k \,\mathrm{d}V + (1/2)\int_V (D_k E_k + s\vartheta)_{,k}\delta u_k \,\mathrm{d}V \\
\int_V \delta h_g \,\mathrm{d}V &= \int_a \eta_j n_j \delta\vartheta \,\mathrm{d}a - \int_V \eta_{j,j}\delta\vartheta \,\mathrm{d}V, \quad \eta_j = -\int_0^t \lambda_{ij} T^{-1}\vartheta_{,i}\,\mathrm{d}\tau
\end{aligned} \tag{2.95}$$

Substituting Eq. (2.95) into $\delta\Pi_1$ of Eq. (2.94) and adding a term $\int_a D_k n_k \left(E_p \delta u_p + \delta_u \varphi\right)\mathrm{d}a$ to it, similar to the derivation in Sect. 2.1.2, finally we get

$$\begin{aligned}
\delta\Pi_1 = &\int_{a_\sigma} \left(S_{ji} n_j - T_i^*\right)\delta u_i \,\mathrm{d}a - \int_V \left(S_{ji,j} + f_k - \rho \ddot{u}_k\right)\delta u_i \,\mathrm{d}V \\
&+ \int_{a_D} (D_k n_k + \sigma^*)\delta\varphi \,\mathrm{d}a - \int_V \left(D_{k,k} - \rho_\mathrm{e}\right)\delta\varphi \,\mathrm{d}V + \int_{a_q} \left(\eta_j n_j - \eta^*\right)\delta\vartheta \,\mathrm{d}a \\
&+ \int_V \left\{-s - s^{(\mathrm{a})} + \eta_{j,j} + \int_0^t \left(T^{-1}\dot{r} + \dot{s}^{(\mathrm{i})}\right)\mathrm{d}\tau\right\}\delta\vartheta \,\mathrm{d}V \\
&+ \int_{a^\mathrm{int}} S_{ji} n_j \delta u_i \,\mathrm{d}a + \int_{a^\mathrm{int}} D_k n_k \delta\varphi \,\mathrm{d}a + \int_{a^\mathrm{int}} \eta_j n_j \delta\vartheta \,\mathrm{d}a
\end{aligned} \tag{2.96}$$

where

$$\begin{aligned}
\sigma_{ij}^{\mathrm{M}\,T} &= D_i E_j - (1/2)(D_n E_n + s\vartheta)\delta_{ij} \\
S_{ij} &= \sigma_{ij} + \sigma_{ij}^{\mathrm{M}\,T} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta + D_i E_j - (1/2)(D_n E_n + s\vartheta)\delta_{ij}
\end{aligned} \tag{2.97}$$

where $\boldsymbol{\sigma}^{\mathrm{M}\,T}$ is the general Maxwell stress. Whether $\boldsymbol{\sigma}^{\mathrm{M}\,T}$ includes the term $s\vartheta$, it should still be proved by experiments; $\boldsymbol{S}$ is the pseudo total stress (Jiang and Kuang 2003, 2004). In pyroelectric materials, when the electric field and temperature are not too large and the piezoelectric coefficient is not too small, the general Maxwell stress can be neglected.

Due to the arbitrariness of $\delta u_i,\ \delta\varphi$ and $\delta\vartheta$, from Eq. (2.96), it is obtained:

$$\begin{aligned}
&S_{jk,j}+f_k=\rho\ddot{u}_k,\quad D_{k,k}=\rho_{\mathrm{e}};\quad \text{in}\quad V\\
&s+s^{(\mathrm{a})}+\eta_{j,j}=\int_0^t\left(T^{-1}\dot{r}+\dot{s}^{(\mathrm{i})}\right)\mathrm{d}\tau\quad\text{or}\quad T\left(\dot{s}+\rho_{\mathrm{s}}\ddot{\vartheta}\right)=\dot{r}-q_{i,i};\quad\text{in}\quad V\\
&S_{ji}n_j=T_i^*,\quad\text{on}\quad a_\sigma;\quad D_kn_k=-\sigma^*,\quad\text{on}\quad a_D;\quad \eta_jn_j=\eta^*,\quad\text{on}\quad a_q\\
&\delta\Pi_1=\int_{a^{\mathrm{int}}}S_{ji}n_j\delta u_i\,\mathrm{d}a+\int_{a^{\mathrm{int}}}D_kn_k\delta\varphi\,\mathrm{d}a+\int_{a^{\mathrm{int}}}\eta_jn_j\delta\vartheta\,\mathrm{d}a
\end{aligned}\tag{2.98}$$

Analogously in the air,

$$\begin{aligned}
&D_{i,i}^{\mathrm{env}}=\rho_{\mathrm{e}}^{\mathrm{env}},\quad\text{in}\quad\text{air};\quad \delta\Pi_2=\int_{a^{\mathrm{int}}}S_{ji}^{\mathrm{env}}n_j^{\mathrm{env}}\delta u_i\,\mathrm{d}a+\int_{a^{\mathrm{int}}}D_i^{\mathrm{env}}n_i^{\mathrm{env}}\delta\varphi\,\mathrm{d}a\\
&S_{ij}^{\mathrm{env}}=\sigma_{ij}^{\mathrm{M\ air}}=D_i^{\mathrm{air}}E_j^{\mathrm{air}}-(1/2)D_n^{\mathrm{air}}E_n^{\mathrm{air}}\delta_{ij}
\end{aligned}\tag{2.99}$$

Substituting Eqs. (2.98) and (2.99) into Eq. (2.94) and noting $\boldsymbol{n}^{\mathrm{env}}=-\boldsymbol{n}$ we get

$$\left(S_{ij}-S_{ij}^{\mathrm{env}}\right)n_i=T_j^{*\mathrm{int}},\quad\text{on}\quad a_\sigma^{\mathrm{int}};\quad (D_i-D_i^{\mathrm{env}})n_i-\sigma^{*\mathrm{int}},\quad\text{on}\quad a_D^{\mathrm{int}}\tag{2.100}$$

The governing equations must contain the prior conditions of the variational principle:

$$\begin{aligned}
&\mathbf{u}=\mathbf{u}^*,\quad\text{on}\quad a_u;\quad \varphi=\varphi^*;\quad\text{on}\quad a_\varphi;\quad \vartheta=\vartheta^*,\quad\text{on}\quad a_T\\
&\varphi=\varphi^{\mathrm{env}},\quad\text{on}\quad a_\varphi^{\mathrm{int}};\quad \vartheta=\vartheta^{\mathrm{env}}(=0);\quad\text{on}\quad a_\vartheta^{\mathrm{int}}
\end{aligned}\tag{2.101}$$

If $\vartheta\ll T_0$, the integral can be integrated in Eq. (2.94), and $\delta\Pi_1$ in Eq. (2.94) is reduced to

$$\begin{aligned}
&\delta\Pi_1=\int_V\left(\delta g+\eta_j\delta\vartheta_{,j}\right)\mathrm{d}V+\int_V g^{E\,T}\delta u_{i,i}dV-\delta Q'-\delta W=0\\
&\delta Q'=-T_0^{-1}\int_V r\delta\vartheta\,\mathrm{d}V+\int_{a_q}\eta_0^*\delta\vartheta\,\mathrm{d}a-\int_V S^{(\mathrm{i})}\delta\vartheta\,\mathrm{d}V+\int_V S^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V\\
&\delta W=\int_V(f_k-\rho\ddot{u}_k)\delta u_k\,\mathrm{d}V-\int_V\rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V+\int_{a_\sigma}T_k^*\delta u_k\,\mathrm{d}a-\int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a
\end{aligned}\tag{2.102}$$

where $\eta_0^*=(1/T_0)\int_0^t q^*\,\mathrm{d}t$.

There are eight thermodynamic character functions in pyroelectric materials, so there are eight fundamental variational principles. However, the electric Gibbs function only contains five independent variables $\boldsymbol{u},\varphi,T$ and is convenient in practical application.

2.7.3 An Example for Purely Thermal Conduction

When $\vartheta \ll T_0$ for the purely thermal conduction problem without the internal heat source in an isotropic material, Eq. (2.93) is reduced to

$$\lambda\,\vartheta_{,ii} = C\big(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}\big) \tag{2.103a}$$

Now discuss a simple problem in which the wave propagates along the x_1 direction, i.e.,

$$\begin{aligned} &\lambda\frac{\partial^2\vartheta}{\partial x_1^2} = C\left(\frac{\partial\vartheta}{\partial t} + \rho_{s0}\frac{\partial^2\vartheta}{\partial t^2}\right), \quad \text{or} \quad \frac{\partial^2\vartheta}{\partial x^2} = \frac{\partial\vartheta}{\partial\tau} + \frac{\partial^2\vartheta}{\partial\tau^2} \\ &x = x_1\sqrt{\frac{C}{\lambda\rho_{s0}}} = \frac{x_1}{c\rho_{s0}}, \quad \tau = \frac{t}{\rho_{s0}}; \quad c = \sqrt{\frac{\lambda}{\rho_{s0}C}} \end{aligned} \tag{2.103b}$$

where x is the dimensionless coordinate, τ is the dimensionless time, and c is the phase velocity. Let

$$\begin{aligned} &\text{Boundary conditions:} \quad \vartheta(0,t) = \Theta_0 H(t), \quad \vartheta(\infty,t) = 0; \quad t > 0 \\ &\text{Initial conditions:} \quad \vartheta(x,0) = 0, \quad \dot{\vartheta}(x,0) = 0; \quad x > 0 \end{aligned} \tag{2.104}$$

where $H(t)$ is the Heaviside function and Θ_0 is a constant. The solution of the above problem is

$$\vartheta(x,t) = \Theta_0 H(x-t)\left[e^{-x/2} + x\int_x^\tau e^{-\varsigma/2}\frac{I_1\left(\sqrt{\varsigma^2 - x^2}/2\right)}{2\sqrt{\varsigma^2 - x^2}}\,\mathrm{d}\varsigma\right] \tag{2.105}$$

where $I_1[\cdot]$ is the modified first kind of the Bessel function. Equation (2.105) shows that ϑ is an attenuated advanced wave. At the wave front $x = \tau$ or $x_1 = ct$, ϑ is interrupted with value $e^{-x/2} = e^{-x_1/2c\rho_{s0}}$ which is decreased with time.

For a problem without initial conditions, let

$$\vartheta = \Theta_0 \exp(kx - \omega t)$$

where Θ_0 is the amplitude of the wave. Substituting the above equation into Eq. (2.103) yields

$$\begin{aligned} &k^2 = C\lambda^{-1}\omega^2(i\omega^{-1} + \rho_{s0}) \\ &k = \pm(C\lambda^{-1}\rho_{s0})^{\frac{1}{2}}\omega\left[\sqrt{\frac{1}{2}(1 + \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}})} + i\sqrt{\frac{1}{2}(1 - \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}})}\right] \end{aligned}$$

so

$$
\begin{aligned}
&\vartheta = \Theta exp[i(kx - \omega t)] = \Theta \exp[ik_1 x - k_2 x - \omega t)] = \Theta \exp(-k_2 x)\exp[ik_1 x - \omega t)] \\
&k_1 = \pm (C\lambda^{-1}\rho_{s0})^{\frac{1}{2}}\omega\sqrt{\frac{1}{2}(1 + \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}})}, \\
&k_2 = (C\lambda^{-1}\rho_{s0})^{\frac{1}{2}}\omega\sqrt{\frac{1}{2}(1 - \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}})} \\
&c = \frac{\omega}{k_1} = \sqrt{\frac{\lambda}{C\rho_{s0}}}\Big/\sqrt{\frac{1}{2}(1 + \sqrt{1 + \omega^{-2}\rho_{s0}{}^{-2}})}
\end{aligned}
\tag{2.106}
$$

Equation (2.106) shows that the temperature wave is an attenuated dispersive wave. When $\rho_{s0} \to 0$, $c \to \sqrt{2\omega\lambda/C}$ which is just the result of the classical heat conduction theory. It shows that when ρ_{s0} is small, the heat inertial effect can be neglected for the problem without initial conditions.

2.8 Variational Principle and Governing Equations in Pyroelectric Materials with Diffusion

2.8.1 Internal Energy, Electrochemical Gibbs Function, and Electric Gibbs Function

In the diffusion theory, mechanical and electrical processes are reversible, but thermal and diffuse processes are irreversible. The internal energy and entropy are all state functions. The Gibbs equation and evolution equation are still expressed by Eqs. (1.72) and (1.77), respectively. According to Eqs. (1.72), (1.77) and (1.69) the internal energy can be given by

$$
\begin{aligned}
&\dot{\mathfrak{A}} = \sigma : \dot{\boldsymbol{\varepsilon}} + +\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + T\dot{s} + \mu\dot{c} \\
&\dot{h}_{\mathfrak{A}} = T\sigma \approx \boldsymbol{X}_T\cdot\dot{\boldsymbol{\eta}} + \boldsymbol{X}_\mu\cdot\dot{\boldsymbol{\xi}} = -T_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i \geq 0 \\
&g_c = \mathfrak{A} - \boldsymbol{E}\cdot\boldsymbol{D} - Ts - \mu c, \quad \dot{g}_c = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}\cdot\dot{\boldsymbol{E}} - s\dot{\vartheta} - c\dot{\mu} \\
&\dot{h}_{gc} \approx -\vartheta_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i + \left(\vartheta_{,i}\eta_i + \mu_{,i}\xi_i\right)^{\cdot} = \eta_i\dot{\vartheta}_{,i} + \xi_i\dot{\mu}_{,i} \\
&g = \mathfrak{A} - Ts - \boldsymbol{E}\cdot\boldsymbol{D}, \quad \dot{g} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}\cdot\dot{\boldsymbol{E}} - s\dot{T} + \mu\dot{c} \\
&\dot{h}_g = \eta_j\dot{\vartheta}_{,j} - \mu_{,k}\dot{\xi}_k
\end{aligned}
\tag{2.107}
$$

where $\mathfrak{A}$, g_c, and g are the internal energy, electrochemical Gibbs function, and electric Gibbs function. $\dot{h}_{\mathfrak{A}}, \dot{h}_{gc}$ and $\dot{h}_g$ are the corresponding dissipative or complementary dissipative energy rates. In this section, we only discuss electrochemical Gibbs function and electric Gibbs function variational principles. $g_{\rm c}, \dot{h}_{g{\rm c}}$ and $g, \dot{h}_g$ can be assumed as

$$\begin{aligned}
&g_{\rm c}(\varepsilon_{kl}, E_k, \vartheta, \mu) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}E_k\varepsilon_{ij} - (1/2)\epsilon_{ij}E_iE_j - \alpha_{ij}\varepsilon_{ij}\vartheta - \tau_iE_i\vartheta \\
&\quad - (1/2T_0)C\vartheta^2 - (1/2)b\mu^2 - b_{ij}\varepsilon_{ij}\mu - b_iE_i\mu - a\mu\vartheta \\
&\dot{h}_{\rm c} = T\dot{s}^{\rm (i)} - \left(Ts^{\rm (i)}\right)^{\cdot} = -s^{\rm (i)}\dot{T} = \boldsymbol{X}_T\cdot\dot{\boldsymbol{\eta}} + \boldsymbol{X}_\mu\cdot\dot{\boldsymbol{\xi}} - \left(\boldsymbol{X}_T\cdot\boldsymbol{\eta} + \boldsymbol{X}_\mu\cdot\boldsymbol{\xi}\right)^{\cdot} = \eta_i\dot{\vartheta}_{,i} + \xi_i\dot{\mu}_{,i}, \\
&e_{kij} = e_{kji}, \quad \epsilon_{kl} = \epsilon_{lk}, \quad \alpha_{ij} = \alpha_{ji}, \quad b_{ij} = b_{ji}, \quad \lambda_{ij} = \lambda_{ji}, \quad D_{ij} = D_{ji}, \quad L_{ij} = L_{ji}
\end{aligned} \tag{2.108}$$

$$\begin{aligned}
&g(\varepsilon_{kl}, E_k, \vartheta, c) = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}E_k\varepsilon_{ij} - (1/2)\epsilon_{ij}E_iE_j - \alpha_{ij}\varepsilon_{ij}\vartheta - \tau_iE_i\vartheta \\
&\quad - (1/2T_0)C\vartheta^2 + (1/2)\hat{b}c^2 - \hat{b}_{ij}\varepsilon_{ij}c - \hat{b}_iE_ic + \hat{a}c\vartheta \\
&h_g = \eta_j\dot{\vartheta}_{,j} - \mu_{,j}\dot{\xi}_j
\end{aligned} \tag{2.109}$$

where $C, C_{ijkl}, e_{kij}, \epsilon_{ij}, \alpha_{ij}, \tau_i, b, b_{ij}, b_i, a, \hat{b}, \hat{b}_{ij}, \hat{b}_i, \hat{a}$ are all material constants.

Constitutive and evolution equations corresponding to $g_{\rm c}$ and g are, respectively,

$$\begin{aligned}
&\sigma_{ji} = \frac{\partial g_{\rm c}}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta - b_{ij}\mu, \quad D_i = -\frac{\partial g_{\rm c}}{\partial E_i} = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta + b_i\mu \\
&s = -\frac{\partial g_{\rm c}}{\partial \vartheta} = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0 + a\mu, \quad c = -\frac{\partial g_{\rm c}}{\partial \mu} = b\mu + b_{ij}\varepsilon_{ij} + b_iE_i + a\vartheta \\
&\eta_i = \partial h_{\rm c}/\partial\vartheta_{,i} = -\int_0^t \left(\lambda_{ij}T^{-1}\vartheta_{,j} + L_{ij}T^{-1}\mu_{,j}\right)\mathrm{d}\tau, \quad \xi_i = \partial h_{\rm c}/\partial\mu_{,i} = -\int_0^t \left(L_{ij}\vartheta_{,j} + D_{ij}\mu_{,j}\right)\mathrm{d}\tau
\end{aligned} \tag{2.110}$$

$$\begin{aligned}
&\sigma_{ji} = \frac{\partial g}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta - \hat{b}_{ij}c, \quad D_i = -\frac{\partial g}{\partial E_i} = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta + \hat{b}_ic \\
&s = -\frac{\partial g}{\partial \vartheta} = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0 - \hat{a}c, \quad \mu = \frac{\partial g}{\partial c} = \hat{b}c - \hat{b}_{ij}\varepsilon_{ij} - \hat{b}_iE_i + \hat{a}\vartheta \\
&\eta_i = \partial h_g/\partial\vartheta_{,i} = -\int_0^t \left(\lambda_{ij}T^{-1}\vartheta_{,j} + L_{ij}T^{-1}\mu_{,j}\right)\mathrm{d}\tau, \quad \mu_{,j} = -\partial h_g/\partial\xi_j = -\hat{L}_{ij}T\dot{\eta}_i - \hat{D}_{ij}\dot{\xi}_i
\end{aligned} \tag{2.111}$$

where the evolution equations of temperature and concentration have been given in Eq. (1.77).

Using Eqs. (2.110) and (2.111), $g_{\rm c}$ and g can be rewritten as

$$\begin{aligned}
&g_{\rm c} = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + g^{\mu\,T}, \quad g_{\rm c}^{\mu\,T} = -(1/2)\left(D_kE_k + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right) \\
&\Delta_{ij}^{\mu}\varepsilon_{ij} = \left(e_{kij}E_k + \alpha_{ij}\vartheta + b_{ij}\mu\right)\varepsilon_{ij} \approx 0
\end{aligned} \tag{2.112}$$

$$
\begin{aligned}
&g=(1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+g^{c\,T},\quad g^{c\,T}=-(1/2)\left(D_kE_k+s\vartheta-c\mu+\Delta^{c}_{ij}\varepsilon_{ij}\right)\\
&\Delta^{c}_{ij}\varepsilon_{ij}=\left(e_{kij}E_k+\alpha_{ij}\vartheta+\hat{b}_{ij}c\right)\varepsilon_{ij}\approx 0
\end{aligned}
\tag{2.113}
$$

2.8.2 The Electrochemical Gibbs Function Variational Principle

In this section and the following Sect. 2.8.3, we only discuss the pyroelectric material with linear elasticity under small deformation and small variation of the temperature; the environment is air. It is assumed that on the interface, there is no diffusion and heat flow, but the electric coupling is allowed, i.e. $\varphi=\varphi^{\mathrm{env}}, q=q^{\mathrm{env}}=0$ and $d=d^{\mathrm{env}}=0$. The heat and input and output may be occurred at some internal boundaries. The temperature and concentration problems do not considered in air, but the electric field is discussed and at infinity $\varphi^{*\,\mathrm{env}}=0$.

Under assumptions that $\boldsymbol{u}$, φ, ϑ, and μ satisfy their own boundary conditions $\boldsymbol{u}=\boldsymbol{u}^*$, $\varphi=\varphi^*$, $\vartheta=\vartheta^*$, and $\mu=\mu^*$ on a_u, a_φ, a_T, and a_μ, respectively and on the interface $\varphi=\varphi^{\mathrm{env}}$ are satisfied prior. Analogous to Sect. 2.7, the PVP in terms of the electro-chemical Gibbs function is (Kuang 2010, 2011c)

$$
\begin{aligned}
&\delta\Pi=\delta\Pi_1+\delta\Pi_2-\delta W^{\mathrm{int}}=0\\
&\delta\Pi_1=\int_V\delta(g_{\mathrm{c}}+h_{\mathrm{c}})\,\mathrm{d}V+\int_V g_{\mathrm{c}}^{\mu\,T}\delta u_{k,k}\,\mathrm{d}V-\delta Q'-\delta\Phi-\delta W=0\\
&\delta Q'=-\int_V\left(\int_0^t T^{-1}\dot{r}\,\mathrm{d}\tau\right)\delta\vartheta\,\mathrm{d}V+\int_V s^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V+\int_{a_q}\eta^*\delta\vartheta\,\mathrm{d}a\\
&\qquad+\int_V\int_0^t T^{-1}\left(T_{,i}\dot{\eta}_i+\mu_{,i}\dot{\xi}_i\right)\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V-\int_a\int_0^t T^{-1}\mu\dot{\xi}_in_i\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}a\\
&\delta\Phi=\int_V c^{(\mathrm{a})}\delta\mu\,\mathrm{d}V+\int_{a_d}\xi^*\delta\mu\,\mathrm{d}a\\
&\delta W=\int_V(f_k-\rho\ddot{u}_k)\delta u_k\,\mathrm{d}V+\int_{a_\sigma}T_k^*\delta u_k\,\mathrm{d}a-\int_V\rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V-\int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a\\
&\delta\Pi_2=\int_{V^{\mathrm{env}}}\delta g_{\mathrm{c}}^{\mathrm{env}}\,\mathrm{d}V-\int_V\rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi\,\mathrm{d}V\\
&\delta W^{\mathrm{int}}=\int_{a^{\mathrm{int}}}T_k^{*\mathrm{int}}\delta u_k\,\mathrm{d}a-\int_{a_D^{\mathrm{int}}}\sigma^{*\mathrm{int}}\delta\varphi\,\mathrm{d}a
\end{aligned}
\tag{2.114}
$$

In Eq. (2.114), $f_k, T_k^*, T_k^*, \sigma^*, \rho_{\mathrm{e}}^{\mathrm{env}}, \sigma^{*\mathrm{env}}, T_k^{*\mathrm{int}}, \sigma^{*\mathrm{int}}, \dot{\eta}^*=\dot{\eta}_i^*n_i$, and $\dot{\xi}^*=\dot{\xi}_i^*n_i$ are given values; $\delta Q'$ is related to heat (including the irreversible heat produced by the irreversible process in the material and the inertial heat); $\delta\Phi$ is related to the diffusion energy. Equation (2.108) shows that there is no term in $\int_V\delta h_{\mathrm{c}}\,\mathrm{d}V$

corresponding to the term $-\int_a \int_0^t T^{-1}\mu\dot{\xi}_i n_i \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}a$, so it should not be included in $\delta Q'$, as shown in Eq. (2.114). It is also noted that

$$\int_V \int_0^t T^{-1}\mu_{,i}\dot{\xi}_i \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}V - \int_a \int_0^t T^{-1}\mu\dot{\xi}_i n_i \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}a = -\int_V \int_0^t T^{-1}\mu\dot{\xi}_{i,i} \delta\vartheta \,\mathrm{d}\tau \,\mathrm{d}V$$

$$\begin{aligned}
\delta\int_V \mathrm{g_c}\,\mathrm{d}V &= \int_V \sigma_{ji}\delta u_{i,j}\,\mathrm{d}V - \int_V D_k\delta E_k\,\mathrm{d}V - \int_V s\delta\vartheta\,\mathrm{d}V - \int_V c\delta\mu\,\mathrm{d}V \\
&= \int_a \sigma_{ji}n_j\delta u_i\,\mathrm{d}a - \int_V \sigma_{ji,j}\delta u_i\,\mathrm{d}V + \int_a D_k n_k \delta_\varphi\varphi\,\mathrm{d}a - \int_V D_{k,k}\delta_\varphi\varphi\,\mathrm{d}V \\
&\quad - \int_V \left(D_i E_p\right)_{,i}\delta u_p\,\mathrm{d}V + \int_V D_{i,i}E_p\delta u_p\,\mathrm{d}V - \int_V s\delta\vartheta\,\mathrm{d}V - \int_V c\delta\mu\,\mathrm{d}V \\
\int_V g_{\mathrm{c}}^{\mu\,T}\delta u_{k,k}\,\mathrm{d}V &= -(1/2)\int_a \left(D_k E_k + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right) n_k \delta u_k\,\mathrm{d}V \\
&\quad + (1/2)\int_V \left(D_k E_k + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right)_{,k}\delta u_k\,\mathrm{d}V \\
\delta\int_V h_{\mathrm{c}}\,\mathrm{d}V &= \int_V \left(\eta_j\delta\vartheta_{,j} + \xi_j\delta\mu_{,j}\right)\mathrm{d}V = \int_a \left(\eta_j n_j\delta\vartheta + \xi_j n_j \delta\mu\right)\mathrm{d}a - \int_V \left(\eta_{j,j}\delta\vartheta + \xi_{j,j}\delta\mu\right)\mathrm{d}V \\
\eta_j &= -\int_0^t \left(\lambda_{ij}T^{-1}\vartheta_{,i} + L_{ij}T^{-1}\mu_{,i}\right)\mathrm{d}\tau, \quad \xi_j = -\int_0^t \left(L_{ij}\vartheta_{,i} + D_{ij}\mu_{,i}\right)\mathrm{d}\tau
\end{aligned} \tag{2.115}$$

Finishing the variational calculation yields

$$\begin{aligned}
&S_{ik,i} + f_k = \rho\ddot{u}_k, \quad D_{k,k} = \rho_{\mathrm{e}}; \quad \text{in} \quad V \\
&\int_0^t \left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right)\mathrm{d}\tau = \int_0^t \left(T^{-1}\dot{r} - T^{-1}q_{j,j} + T^{-1}\mu\dot{\xi}_{i,i}\right)\mathrm{d}\tau, \quad \text{or} \quad T\left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right) = \dot{r} - q_{i,i} + \mu\dot{\xi}_{i,i} \\
&\int_0^t \left(\dot{c} + \rho_\mu\ddot{\mu}\right)\mathrm{d}\tau = \int_0^t \dot{\xi}_{j,j}\,\mathrm{d}\tau, \quad \text{or} \quad \dot{c} + \rho_\mu\ddot{\mu} = -\dot{\xi}_{j,j}; \quad \text{in} \quad V \\
&S_{ji}n_j = T_i^*, \quad \text{on} \quad a_\sigma; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_D; \\
&\eta_j n_j = \eta^*, \quad \text{or} \quad q_i = q_i^* \quad \text{on} \quad a_q; \quad \dot{\xi}_j n_j = \dot{\xi}^*, \quad \text{or} \quad d_i = d_i^* \quad \text{on} \quad a_d
\end{aligned} \tag{2.116}$$

where

$$\begin{aligned}
&\sigma_{ij}^{\mathrm{M}\,\mu T} = D_i E_j - (1/2)\left(D_n E_n + s\vartheta + c\mu + \Delta_{ij}^{\mu}\varepsilon_{ij}\right)\delta_{ij} \\
&S_{ij} = \sigma_{ij} + \sigma_{ij}^{\mu\,T} \approx C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta + D_i E_j - (1/2)(D_n E_n + s\vartheta + c\mu)\delta_{ij}
\end{aligned} \tag{2.117}$$

From the second and third equations of Eq. (2.116) we find $T\left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right) + \mu\left(\dot{c} + \rho_\mu\ddot{\mu}\right) = \dot{r} - q_{i,i}$, which is identical with Eq. (1.62).

In the air and on the interface, there are

$$\begin{aligned}&D_{i,i}^{\text{env}}=\rho_{\text{e}}^{\text{env}},\quad \text{in}\quad V^{\text{env}}\\&\left(S_{ij}-S_{ij}^{\text{env}}\right)n_i=T_j^{*\text{int}},\quad \text{on}\quad a_\sigma^{\text{int}};\quad (D_i-D_i^{\text{env}})n_i=-\sigma^{*\text{int}},\quad \text{on}\quad a_D^{\text{int}}\\&S_{ij}^{\text{env}}=\sigma_{ij}^{\text{M air}}=D_i^{\text{air}}E_j^{\text{air}}-(1/2)D_n^{\text{air}}E_n^{\text{air}}\delta_{ij}\end{aligned}\tag{2.118}$$

The above variational principle requests prior that the $\boldsymbol{u},\varphi,\vartheta$ and μ satisfy their own boundary conditions, so in governing equations, the following equations should also be added:

$$\begin{aligned}&\boldsymbol{u}=\boldsymbol{u}^*,\quad \text{on}\quad a_u;\ \varphi=\varphi^*,\quad \text{on}\quad a_\varphi;\ \vartheta=\vartheta^*,\quad \text{on}\quad a_T;\\&\mu=\mu^*,\quad \text{on}\quad a_\mu;\ \varphi=\varphi^{\text{env}},\quad \text{on}\quad a^{\text{int}}\end{aligned}\tag{2.119}$$

Equations (2.116), (2.117), (2.118), and (2.119) are the governing equations of the generalized thermodiffusion theory.

If we neglect the term $\mu\left(\dot{c}+\dot{c}^{(\text{a})}\right)$ in Eq. (1.77), or let $T\left(\dot{s}+\dot{s}^{(\text{a})}\right)=\dot{r}-q_{i,i}$, then we get

$$\begin{aligned}&T\left(\dot{s}+\rho_{\text{s}}\ddot{\vartheta}\right)=\dot{r}-q_{j,j},\quad \dot{c}+\rho_{\text{c}}\ddot{\mu}=-\dot{\xi}_{j,j};\quad \text{In medium}\\&\dot{\eta}_j n_j=\dot{\eta}^*,\quad \text{or}\quad q_n=q_n^*\quad \text{on}\quad a_q\\&\dot{\xi}_j n_j=\dot{\xi}^*,\quad \text{or}\quad d_n=d_n^*\quad \text{on}\quad a_d\quad \text{and}\quad a_q\end{aligned}\tag{2.120}$$

If we also assume that $T_{,i}$ and $\mu_{,j}$ are not dependent with each other, for $\dot{r}=0$, Eq. (2.120) becomes

$$\begin{aligned}&T\left(\alpha_{ij}\dot{u}_{i,j}+C\dot{\vartheta}/T_0+a\dot{\mu}+\rho_{\text{s}}\ddot{\vartheta}\right)=\lambda_{ij}\vartheta_{,j}\\&b\dot{\mu}+b_{ij}\dot{u}_{i,j}+a\dot{\vartheta}+\rho_{\text{c}}\ddot{\mu}=D_{ij}\mu_{,j};\quad \text{In medium}\end{aligned}\tag{2.121}$$

The formulas in literatures analogous to Eq. (2.121) can be found, such as in the paper of Sherief et al. (2004), where they used the Maxwell-Cattaneo formula. Genin and Xu (1999) discussed the thermoelastic plastic metals with mass diffusion.

2.8.3 The Electric Gibbs Function Variational Principle

Under assumptions that $\boldsymbol{u}$, φ, ϑ, and c satisfy their own boundary conditions $\boldsymbol{u}=\boldsymbol{u}^*$, $\varphi=\varphi^*$, $\vartheta=\vartheta^*$, and $c=c^*$ on a_u, a_φ, a_T, and a_{c}, respectively. The PVP in terms of the electric Gibbs function for the thermo-electro-elasto-diffusive problem is (Kuang 2010)

$$
\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0 \\
&\delta\Pi_1 = \int_V \delta(g + h_g)\,\mathrm{d}V + \int_V g^{\mathrm{c}\,T}\delta u_{k,k}\mathrm{d}V - \delta Q' + \delta\Phi - \delta W = 0 \\
&\delta Q' = -\int_V \left(\int_0^t T^{-1}\dot{r}\,\mathrm{d}\tau\right)\delta\vartheta \mathrm{d}V + \int_V s^{(\mathrm{a})}\delta\vartheta\,\mathrm{d}V + \int_{a_q}\eta^*\delta\vartheta\,\mathrm{d}a \\
&\qquad + \int_V\int_0^t T^{-1}\left(\vartheta_{,i}\dot{\eta}_i + \mu_{,i}\dot{\xi}_i\right)\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}V - \int_a\int_0^t T^{-1}\mu\dot{\xi}_i n_i\delta\vartheta\,\mathrm{d}\tau\,\mathrm{d}a \\
&\delta\Phi = \int_V \mu_{,j}^{(\mathrm{a})}\delta\xi_j\,\mathrm{d}V + \int_{a_d}\mu^*\delta\xi\,\mathrm{d}a \\
&\delta W = \int_V (f_k - \rho\ddot{u}_k)\delta u_k\,\mathrm{d}V + \int_{a_\sigma} T_k^*\delta u_k\,\mathrm{d}a - \int_V \rho_{\mathrm{e}}\delta\varphi\,\mathrm{d}V - \int_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a \\
&\delta\Pi_2 = \int_{V^{\mathrm{env}}}\delta g^{\mathrm{env}}\,\mathrm{d}V - \int_V \rho_{\mathrm{e}}^{\mathrm{env}}\delta\varphi\,\mathrm{d}V \\
&\delta W^{\text{int}} = \int_{a^{\text{int}}} T_k^{*\text{int}}\delta u_k\,\mathrm{d}a - \int_{a_D^{\text{int}}}\sigma^{*\text{int}}\delta\varphi\,\mathrm{d}a
\end{aligned}
\tag{2.122}
$$

where the symbols are the same as that in Sect. 2.8.2, but the gradient of the inertial chemical potential $\mu_{,i}^{(\mathrm{a})} = \rho_{\mathrm{c}}\ddot{\xi}_i$ is introduced, and μ^* is given value.

Finishing the variational calculation finally yields

$$
\begin{aligned}
&S_{ik,i} + f_k = \rho\ddot{u}_k, \quad D_{k,k} = \rho_{\mathrm{e}}; \quad \text{in} \quad V \\
&\int_0^t \left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right)\mathrm{d}\tau = \int_0^t \left(T^{-1}\dot{r} - T^{-1}q_{j,j} + T^{-1}\mu\dot{\xi}_{i,i}\right)\mathrm{d}\tau, \quad \text{or} \quad T\left(\dot{s} + \rho_{\mathrm{s}}\ddot{\vartheta}\right) = \dot{r} - q_{i,i} + \mu\dot{\xi}_{i,i} \\
&\mu_{,j} + \rho_\mu\ddot{\xi}_j = -\hat{D}_{ij}\dot{\xi}_i - \hat{L}_{ij}T\dot{\eta}_i; \quad \text{in} \quad V \\
&S_{ji}n_j = T_i^*, \quad \text{on} \quad a_\sigma; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_D; \\
&\eta_j n_j = \eta^*, \quad \text{or} \quad q_i = q_i^* \quad \text{on} \quad a_q;
\end{aligned}
\tag{2.123}
$$

If differentiating the equation of the chemical potential with $\boldsymbol{x}$ in Eq. (2.123), it is obtained:

$$
\begin{aligned}
&\mu_{,jj} + \rho_\mu\ddot{\xi}_{j,j} + \hat{D}_{ij}\dot{\xi}_{i,j} + \hat{L}_{ij}(T\dot{\eta}_i)_{,j} = 0; \\
&\left(\hat{b}c - \hat{b}_{ij}\varepsilon_{ij} - \hat{b}_i E_i + \hat{a}\vartheta\right)_{,jj} + \rho_\mu\ddot{\xi}_{j,j} + \hat{D}_{ij}\dot{\xi}_{i,j} + \hat{L}_{ij}(T\dot{\eta}_i)_{,j} = 0; \quad \text{in} \quad V
\end{aligned}
\tag{2.124}
$$

If $\hat{D}_{ij} = \hat{D}\delta_{ij}, \lambda_{ij} = \lambda\delta_{ij}, \hat{L}_{ij} = 0$ from Eq. (2.124), a simpler diffusion equation can be obtained:

$$
\rho_{\mathrm{c}}\ddot{c} + \hat{D}\dot{c} = \left(bc - b_{ik}\varepsilon_{ik} + b_i\varphi_{,i} + a\vartheta\right)_{,jj}; \quad \text{in} \quad V \tag{2.125}
$$

Governing equations in the air are the same as that in Eq. (2.118).

2.8.4 Constitutive Equations

In general case, there are three groups with six variables: $(\boldsymbol{\sigma},\boldsymbol{\varepsilon}),(\boldsymbol{E},\boldsymbol{D}),(\vartheta,s)$ for pyroelectric materials. Because each variable in three groups can be used as the independent variable, there are eight group constitutive equations which just correspond to eight thermodynamic character functions in Eq. (1.59). Equations (2.89) and (2.90) are the constitutive equations corresponding to electric Gibbs function g and internal energy $\mathfrak{A}$. However for pyroelectric materials with diffusion there are four groups with eight variables: $(\boldsymbol{\sigma},\boldsymbol{\varepsilon}),(\boldsymbol{E},\boldsymbol{D}),(\vartheta,s),(\mu,c)$. So there are sixteen group constitutive equations. Equations (2.110) and (2.111) are the constitutive equations corresponding to electrochemical Gibbs function g_{c} and electric Gibbs function g.

2.9 Conservation Integrals in Piezoelectric Materials

2.9.1 Noether Theory

In previous sections of this chapter, it is found that the electroelastic governing equations can be obtained from the extreme value of a variational functional. The governing equation is just the Euler-Lagrange equation of that functional. Based on the theory of Noether's invariant variational problem (1918), conservation laws (integrals) can be easily obtained (Fletcher 1976; Honein and Herrmann 1997). These conservation integrals are very useful in fracture mechanics due to their path independence property. Here, some conservation laws for inhomogeneous materials (Shi and Kuang 2003) will be obtained by using the Noether's invariant variational principle.

Let the variational functional in the continuum mechanics be

$$\mathfrak{I}=\int_{\mathrm{V}}\mathrm{L}(\mathrm{x_i},\psi_{\alpha,\mathrm{j}})\,\mathrm{dV} \tag{2.126}$$

where $L(x_i,\psi_{\alpha,j})$ is the Lagrange density function and $\boldsymbol{x},\boldsymbol{\psi}$ are the independent and dependent variables, respectively. The Euler- Lagrange equation of $\mathfrak{I}$ is

$$\frac{\partial}{\partial x_j}\frac{\partial L(x_i,\psi_{\alpha,j})}{\partial\psi_{\alpha,j}}=0 \tag{2.127}$$

Give an infinitesimal transform as

$$\begin{aligned}
&x_i\to x'_i=x_i+\delta x_i(x_j,\psi_\alpha),\quad \psi_\alpha(x_i)\to\psi'_\alpha(x'_i)=\psi_\alpha(x_i)+\delta\psi_\alpha(x_i,\psi_\beta)\\
&\delta\psi_\alpha=\psi'_\alpha(x'_i)-\psi_\alpha(x_i)=[\psi_\alpha(x_i+\delta u_i)+\delta_\psi\psi_\alpha(x_i)]-\psi_\alpha(x_i)\\
&\qquad=\delta_\psi\psi_\alpha+\delta_u\psi_\alpha=\delta_\psi\psi_\alpha+\psi_{\alpha,i}\delta u_i
\end{aligned} \tag{2.128}$$

Using

$$\frac{\partial x'_i}{\partial x_j} \approx \delta_{ij} + \frac{\partial \delta x_i}{\partial x_j}, \quad \frac{\partial x_i}{\partial x'_j} \approx \delta_{ij} - \frac{\partial \delta x_i}{\partial x_j}, \quad j = \left| \frac{\partial x'_i}{\partial x_j} \right| \approx 1 + \frac{\partial \delta x_i}{\partial x_i} \tag{2.129}$$

from Eq. (2.128) yields

$$\begin{aligned} \delta(\psi_{\alpha,j}) &= \frac{\partial \psi'_\alpha(x'_i)}{\partial x'_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_j} = \frac{\partial[\psi_\alpha(x_i) + \delta\psi_\alpha(x_i, \psi_\beta)]}{\partial x_k} \frac{\partial x_k}{\partial x'_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_j} \\ &= \frac{\partial \delta\psi_\alpha(x_i, \psi_\beta)}{\partial x_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_k} \frac{\partial \delta x_k}{\partial x_j} \end{aligned} \tag{2.130a}$$

Equation (2.130a) can also be reduced to

$$\begin{aligned} \delta(\psi_{\alpha,j}) &= \frac{\partial\left[\delta_\psi \psi_\alpha + \psi_{\alpha,i}\delta x_i\right]}{\partial x_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_k} \frac{\partial \delta x_k}{\partial x_j} \\ &= \frac{\partial(\delta_\psi \psi_\alpha)}{\partial x_j} + \frac{\partial(\psi_{\alpha,i}\delta x_i)}{\partial x_j} - \frac{\partial \psi_\alpha(x_i)}{\partial x_k} \frac{\partial \delta x_k}{\partial x_j} = \frac{\partial(\delta_\psi \psi_\alpha)}{\partial x_j} + \frac{\partial(\psi_{\alpha,i})}{\partial x_j} \delta x_i \\ \frac{\partial}{\partial x_i}(\delta\psi_\alpha) &= \frac{\partial}{\partial x_i}\left(\delta_\psi \psi_\alpha + \psi_{\alpha,i}\delta u_i\right) = \frac{\partial \delta_\psi \psi_\alpha}{\partial x_i} + \psi_{\alpha,ij}\delta u_i + \psi_{\alpha,i}\delta u_{i,j} = \delta(\psi_{\alpha,j}) + \psi_{\alpha,i}\delta u_{i,j} \end{aligned} \tag{2.130b}$$

Equation (2.130b) is identical with that in Eq. (2.8) in Sect. 2.1.2.

Because $\delta\psi_\alpha(x_i, \psi_\beta)$ is the function of x_i, ψ_β, so

$$\frac{\partial \delta\psi_\alpha(x_i, \psi_\beta)}{\partial x_j} = \frac{\bar{\partial} \delta\psi_\alpha(x_i, \psi_\beta)}{\partial x_j} + \frac{\partial \delta\psi_\alpha(x_i, \psi_\beta)}{\partial \psi_\beta} \frac{\partial \psi_\beta}{\partial x_j} \tag{2.131}$$

where the notation $\bar{\partial}/\partial x_i$ is the partial derivative with respect to explicit x_i in ψ_α. If under the transform, Eq. (2.128), on the accuracy of the first order of $\delta x_i, \delta\psi_\alpha$, $\delta\psi_{\alpha,j}$, the following equality holds:

$$\int_{V'} L'(x'_i, \psi'_{\alpha,j})\,\mathrm{d}V' = \int_V L'(x'_i, \psi'_{\alpha,j}) j\,\mathrm{d}V = \int_V L(x_i, \psi_{\alpha,j})\,\mathrm{d}V \tag{2.132}$$

the group of transform Eq. (2.128) is called the symmetric group of a system. From Eq. (2.132), some conservation laws can be found.

Appling Eqs. (2.128) and (2.130), the following relation can be obtained:

$$\begin{aligned} &\left(\frac{\partial L}{\partial \psi_{\alpha,j}} \delta\psi_\alpha - \frac{\partial L}{\partial \psi_{\alpha,j}} \psi_{\alpha,i}\delta x_i\right)_{,j} + \left(\frac{\partial L}{\partial \psi_{\alpha,j}} \psi_{\alpha,i}\right)_{,j} \delta x_i = \frac{\partial L}{\partial \psi_{\alpha,j}} \left(\frac{\partial \delta\psi_\alpha}{\partial x_j} - \psi_{\alpha,i} \frac{\partial \delta x_i}{\partial x_j}\right) \\ &= \frac{\partial L}{\partial \psi_{\alpha,j}} \delta(\psi_{\alpha,j}) \end{aligned}$$

So that

$$\begin{aligned} L'(x'_i, \psi'_{\alpha,j}) &= L\left[x_i + \delta x_i, \psi_{\alpha,j} + \delta(\psi_{\alpha,j})\right] = L(x_i, \psi_{\alpha,j}) + \frac{\bar{\partial} L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \psi_{\alpha,j}}\delta(\psi_{\alpha,j}) \\ &= L(x_i, \psi_{\alpha,j}) + \frac{\bar{\partial} L}{\partial x_i}\delta x_i + \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha - \frac{\partial L}{\partial \psi_{\alpha,j}}\psi_{\alpha,i}\delta x_i\right)_{,j} + \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\psi_{\alpha,i}\right)_{,j}\delta x_i \end{aligned} \tag{2.133}$$

Substituting the identity

$$\left(\frac{\partial L}{\partial \psi_{\alpha,j}}\psi_{\alpha,i}\right)_{,j}\delta x_i = (L\delta x_i)_{,i} - L(\delta x_i)_{,i} - \frac{\bar{\partial} L}{\partial x_i}\delta x_i$$

into Eq. (2.133) yields

$$\begin{aligned} &L'(x'_i, \psi'_{\alpha,j}) = L(x_i, \psi_{\alpha,j}) + \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right)_{,j} - L(\delta x_i)_{,i} \\ &P_{ij} = L\delta_{ij} - \left(\partial L/\partial \psi_{\alpha,j}\right)\psi_{\alpha,i} \end{aligned} \tag{2.134}$$

P_{ij} is called the energy-momentum tensor of matter. Substitution of Eq. (2.134) into Eq. (2.132) yields

$$\int_V \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right)_{,j} \mathrm{d}V = 0, \quad \text{or} \quad \int_a \left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right) n_j\, \mathrm{d}a = 0 \tag{2.135}$$

Equations (2.134) and (2.135) are the invariant conditions under the infinitesimal transform. The second equation in Eq. (2.135) is a path independence integral. Due to the arbitrariness of the volume from Eq. (2.135), the invariant condition in the differential form is

$$\left(\frac{\partial L}{\partial \psi_{\alpha,j}}\delta\psi_\alpha + P_{ij}\delta x_i\right)_{,j} = 0 \tag{2.136}$$

In the above discussion, it is assumed that there is no body force, body electric charge, etc.

For the electroelastic problem without body couple let $\boldsymbol{\psi} = [u_i, \varphi]^T, L \equiv g$, we have

$$\begin{aligned}
&L \equiv g = (1/2)C_{ijkl}u_{i,j}u_{k,l} - (1/2)\epsilon_{kl}\varphi_{,k}\varphi_{,l} + e_{kij}u_{i,j}\varphi_{,k} = (1/2)\,\Sigma_{aj}\,\psi_{a,j} \\
&\Sigma_{\alpha j} = \partial g/\partial \psi_{\alpha,j} = \begin{cases} \sigma_{ij} = \partial g/\partial u_{i,j} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j, & i,j,k,l = 1,2,3 \\ \sigma_{4j} = D_j = -\partial g/\partial \psi_{4,j} = \partial g/\partial \varphi_{,j} = \epsilon_{jl}E_l + e_{jkl}\varepsilon_{kl}, & \alpha = 1,2,3,4 \end{cases} \\
&P_{ij} = g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i} = g\delta_{ij} - \sigma_{mj}u_{m,i} - D_j\varphi_{,i} \\
&\left(\Sigma_{\alpha j}\delta\psi_\alpha + P_{ij}\delta u_i\right)_{,j} = \left(\sigma_{ij}\delta u_i + D_j\delta\varphi + P_{ij}\delta u_i\right)_{,j} = 0
\end{aligned} \tag{2.137}$$

where $\Sigma_{\alpha j}$ is the generalized stress and $\psi_{\alpha,j}$ is the generalized strain and the Greek indices take 1–4.

2.9.2 *Conservation Integral in a Homogeneous Material*

For a homogeneous material, L is independent to $\boldsymbol{x}$, so $L = L(\psi_{\alpha,j}),\ \bar{\partial}/\partial x_i = 0$.

1. *Infinitesimal translation of general displacement*. Let

$$\delta x_i = 0, \quad \delta\psi_\alpha = \varepsilon c_\alpha \tag{2.138}$$

where c_α is a constant and ε is an infinitesimal parameter. Substitution of Eq. (2.138) into Eq. (2.136) yields the invariant condition:

$$\left(\Sigma_{\alpha j}\delta\psi_\alpha\right)_{,j} = \Sigma_{\alpha j,j}\delta\psi_\alpha = 0 \quad \Rightarrow \quad \Sigma_{\alpha j,j} = 0 \tag{2.139}$$

Equation (2.139) is just the generalized momentum equation.

2. *Infinitesimal translation of coordinate*. $\boldsymbol{x}$ Let

$$\delta x_i = \varepsilon c_i, \quad \delta\psi_\alpha = 0 \tag{2.140}$$

Substitution of Eq. (2.140) into Eq. (2.136) yields

$$\left(P_{ij}\delta x_i\right)_{,j} = P_{ij,j}\delta x_i = g_{,j}\delta_{ij}\delta x_i = g_{,i}\delta x_i = \delta g = 0 \quad \Rightarrow \quad g = \text{const.} \tag{2.141}$$

Equation (2.141) is just the energy conservative equation.

3. *Infinitesimal translation of coordinate and generalized displacement*. Let

$$\delta x_i = \varepsilon c_i, \quad \delta\psi_\alpha = \varepsilon\Omega_\alpha \tag{2.142}$$

where c_i, Ω_α are constants and ε is an infinitesimal parameter. Substituting Eq. (2.142) into Eq. (2.136) and noting $\Sigma_{\alpha j,j} = 0$ we find

$$\varepsilon P_{ij,j}c_i = 0, \quad \text{or} \quad \int_V P_{ij,j}\,\mathrm{d}V = \int_a \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)n_j\,\mathrm{d}a = 0 \tag{2.143}$$

Equation (2.143) shows that the integral value is zero along a closed surface for the integrated function $\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)n_j$. For two open surfaces initiated from a same closed curve, Eq. (2.143) shows that the integral values for two open surfaces are the same. In the two-dimensional (2D) problem, it represents the path independence J integral, J_i. Equation (2.143) can also be obtained by taking the divergence of g. In fact using $\nabla g = (\partial g/\partial x_i)\boldsymbol{e}_i$ and the equilibrium equation, we have

$$\frac{\partial g}{\partial x_i} = \frac{\bar{\partial} g}{\partial x_i} + \frac{\partial g}{\partial \psi_{\alpha,j}}\psi_{\alpha,ji} = \Sigma_{\alpha j}\psi_{\alpha,ji}, \quad \text{or} \quad g_{,i} - \Sigma_{\alpha j}\psi_{\alpha,ji} = \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)_{,j} = 0 \tag{2.144}$$

This method was adopted by many authors (Delph 1982; Pak 1992; Wang and Shen 1996).

4. *Infinitesimal expansion of coordinate and generalized displacement*. Let

$$\delta x_i = \varepsilon x_i, \ \ \delta\psi_\alpha = -(1/2)\varepsilon\psi_\alpha \tag{2.145}$$

where ε is an infinitesimal parameter. Substitution of Eq. (2.145) into (2.136) yields

$$\begin{aligned} &\varepsilon\left[-(1/2)\Sigma_{\alpha j}\psi_\alpha + P_{ij}x_i\right]_{,j} = 0, \quad \text{or} \\ &\int_a \left[\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)x_i - (1/2)\Sigma_{\alpha j}\psi_\alpha\right]n_j\,\mathrm{d}a = 0 \end{aligned} \tag{2.146}$$

In the two-dimensional problem Eq. (2.146) represents the path independence M integral

$$\begin{aligned} M &= \int_{\Gamma_1} \left[\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)x_i - (1/2)\Sigma_{\alpha j}\psi_\alpha\right]n_j\,\mathrm{d}l \\ &= \int_{\Gamma_1} \left[\left(g\delta_{ij} - \sigma_{ij}u_{i,j} - D_j\varphi_{,i}\right)x_i - (1/2)\left(\sigma_{ij}u_i + D_j\varphi\right)\right]n_j\,\mathrm{d}l \end{aligned} \tag{2.147}$$

5. *Infinitesimal rotation about the axis* x_3. Let

$$\delta x_1 = \varepsilon x_2, \quad \delta x_2 = -\varepsilon x_1, \quad \delta\psi_1 = \varepsilon\psi_2, \quad \delta\psi_2 = -\varepsilon\psi_1, \quad \delta x_3 = \delta\psi_3 = \delta\psi_4 = 0 \tag{2.148}$$

Substitution of Eq. (2.148) into Eq. (2.136) yields

$$\begin{aligned} &\left(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1\right)_{,k} = 0, \quad \text{or} \\ &\int_a \left(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1\right)n_k\,\mathrm{d}a = 0 \end{aligned} \tag{2.149}$$

6. *The conservative integral in pyroelectric material*. Wang and Kuang (2001) discussed the conservative integral in pyroelectric material by Noether theory and got

$$
\begin{aligned}
J_i &= \int \left(g\delta_{ij} - \sigma_{ij}u_{i,j} - D_j\varphi_{,i} - s_j\vartheta_{,i}\right)n_j\,\mathrm{d}l \\
M &= \int \left[\left(g\delta_{ij} - \sigma_{ij}u_{i,j} - D_j\varphi_{,i} - \dot{\eta}_j\vartheta_{,i}\right)x_i + \frac{1}{2}\left(\sigma_{ij}u_i + D_j\varphi + s_j\vartheta\right)\right]n_j\,\mathrm{d}l
\end{aligned}
\tag{2.150}
$$

where $\dot{\eta}_j = q_j/T_0, \vartheta = T - T_0$.

2.9.3 The Force Acting on a Defect in an Inhomogeneous Material

For an inhomogeneous material, L is dependent to $\boldsymbol{x}$, so $L = L(x_i, \psi_{\alpha,j}), \bar{\partial}/\partial x_i \neq 0$.

1. *Infinitesimal translation of* $\boldsymbol{x}$ *and generalized displacement*. $\delta x_i, \delta u_\alpha$ are also given in Eq. (2.142). The invariant condition under infinitesimal transformation is still $\varepsilon P_{ij,j}c_i = 0$, but

$$
P_{ij,j} = \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)_{,j} = \frac{\bar{\partial} g}{\partial x_i} + \frac{\partial g}{\partial \psi_{\alpha,j}}\psi_{\alpha,ji} - \Sigma_{\alpha j}\psi_{\alpha,ij} = \frac{\bar{\partial} g}{\partial x_i}
$$

So the integral in Eq. (2.143) in an inhomogeneous material becomes

$$
\int_V P_{ij,j}\mathrm{d}V = \int_a \left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)n_j\,\mathrm{d}a = \int_V \left(\bar{\partial} g/\partial x_i\right)\mathrm{d}V \tag{2.151}
$$

Though Eq. (2.151) is not a conservative integral, it still has important meaning. Eshelby (1956, 1975) pointed out that $P_{ij,j} - \bar{\partial} g/\partial x_i = 0$, so the negative derivative of the electric Gibbs function with $\boldsymbol{x}$, $-\bar{\partial} g/\partial x_i$, is the so-called material inhomogeneity force with the dimension of force.

2. *Infinitesimal expansion of coordinate and general displacement*. $\delta x_i, \delta u_\alpha$ are also given in Eq. (2.145). The invariant condition under infinitesimal transformation is still $\varepsilon\left[-(1/2)\Sigma_{\alpha j}\psi_\alpha + P_{ij}x_i\right]_{,j} = 0$, but

$$
\left[-(1/2)\Sigma_{\alpha j}\psi_\alpha + P_{ij}x_i\right]_{,j} = P_{ij,j}x_i + P_{ij}x_{i,j} - (1/2)\Sigma_{\alpha j}\psi_{\alpha,j} = x_i\bar{\partial} g/\partial x_i
$$

So the integral in Eq. (2.146) in an inhomogeneous material becomes

$$
\int_a \left[\left(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i}\right)x_i - (1/2)\Sigma_{\alpha j}\psi_\alpha\right]n_j\,\mathrm{d}a = \int_V x_i\bar{\partial} g/\partial x_i\,\mathrm{d}V \tag{2.152}
$$

where $-x_i\bar{\partial} g/\partial x_i$ is the so-called material inhomogeneity moment.

3. *Infinitesimal rotation about the axis.* x_3 $\delta x_i, \delta u_\alpha$ are also given in Eq. (2.148). The invariant condition under the infinitesimal transformation is still expressed by $(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1)_{,k} = 0$, but

$$(P_{1k}x_2 - P_{2k}x_1 + \sigma_{1k}u_2 - \sigma_{2k}u_1)_{,k} = x_1\bar{\partial}g/\partial x_2 - x_2\bar{\partial}g/\partial x_1 \tag{2.153}$$

2.9.4 Conservation Integral in an Inhomogeneous Material

1. *Infinitesimal translation of coordinate and general displacement is related to an undetermined function.* Let

$$\delta x_i = \varepsilon c_i, \quad \delta\psi_\alpha = \varepsilon\Omega_\alpha \tag{2.154}$$

where c_i is a constant, Ω_α is an undetermined function, and ε is an infinitesimal parameter. Substituting Eq. (2.154) into Eq. (2.136) and noting $P_{ij,j} = (g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i})_{,j} = (\bar{\partial}g/\partial x_i)$, $\Sigma_{\alpha j,j} = 0$, we find

$$\Sigma_{\alpha j}\Omega_{\alpha,i} + c_i(g\delta_{ij} - \Sigma_{\alpha j}\psi_{\alpha,i})_{,j} = \Sigma_{\alpha j}\Omega_{\alpha,i} + c_i\bar{\partial}g/\partial x_i = 0 \tag{2.155}$$

2. *Infinitesimal translation of coordinate, infinitesimal translation, and expansion of general displacement.* Let

$$\delta x_i = \varepsilon x_i, \quad \delta\psi_\alpha = \varepsilon[-(1/2)\psi_\alpha + \Omega_\alpha] \tag{2.156}$$

where Ω_α is an undetermined function and ε is an infinitesimal parameter. Substituting Eq. (2.156) into (2.136) yields

$$\Sigma_{\alpha j}\Omega_{\alpha,j} + \left[P_{ij}x_i - (1/2)\Sigma_{\alpha j}\Omega_\alpha\right]_{,j} = 0 \tag{2.157}$$

From Eqs. (2.156) and (2.157), it is known that when material constants obey definite distribution and select appropriate c_i, Ω_α, the conservative integrals can be obtained; otherwise, the conservative integrals do not exist. In the following sections, some examples will be given to illustrate the above theory.

2.9.5 One-Directional Gradient Material

In engineering, the combined material is often used to improve the material behavior, such as on the substrate covering a surface heat-resisting layer to defense the high temperature environment. In order to reduce the stress on the interface between the substrate and heat-resisting layer, a transient layer constituted of the gradient material is added. One-directional gradient material is often used.

1. *Material constants varied as exponential function*. Assume material constants varied as $C^0_{ijkl}e^{\lambda x_1}, e^0_{kij}e^{\lambda x_1}, \epsilon^0_{ij}e^{\lambda x_1}$, where $C^0_{ijkl}, e^0_{kij}, \epsilon^0_{ij}, \lambda$ are all constants. Let

$$\delta x_1 = \varepsilon, \quad \delta x_2 = \delta x_3 = 0, \quad \delta \psi_\alpha = \varepsilon b \psi_\alpha, \quad \alpha = 1,2,3,4 \tag{2.158}$$

Substituting Eq. (2.158) into (2.136) yields

$$(\Sigma_{\alpha j} b \psi_\alpha + P_{1j})_{,j} = b \Sigma_{\alpha j} \psi_{\alpha,j} + P_{1j,j} = b \Sigma_{\alpha j} \psi_{\alpha,j} + \bar{\partial} g / \partial x_1 = 0 \tag{2.159}$$

Because $\Sigma_{\alpha j}\psi_{\alpha,j} = 2g$, $\bar{\partial} g/\partial x_1 = \lambda g$, from Eq. (2.159), we get $2b + \lambda = 0$ or $b = -\lambda/2$. Substituting these results into Eq. (2.158) and then into Eq. (2.159) we get

$$(-\lambda \Sigma_{\alpha j} \psi_\alpha / 2 + P_{1j})_{,j} = 0 \tag{2.160}$$

Using the relation

$$(P_{1j} - \lambda \Sigma_{\alpha j} \psi_\alpha/2)_{,j} = (g\delta_{1j} - \Sigma_{\alpha j}\psi_{\alpha,1} - \lambda \Sigma_{\alpha j}\psi_\alpha/2)_{,j} = g_{,1} - \Sigma_{\alpha j}\psi_{\alpha,1j} - \lambda \Sigma_{\alpha j}\psi_{\alpha,j}/2$$

It is easy to get the path independence integral

$$\int_a (g\delta_{j1} - \Sigma_{\alpha j}\psi_{\alpha,1} - \lambda \Sigma_{\alpha j}\psi_\alpha/2) n_j \mathrm{d}a = 0 \tag{2.161}$$

2. *Material constants varied as power function*. Assume material constants varied as $C_{ijkl} = C^0_{ijkl}(1 + px_1)^q$, $e_{kij} = e^0_{kij}(1 + px_1)^q$, and $\epsilon_{ij} = \epsilon^0_{ij}(1 + px_1)^q$, where $C^0_{ijkl}, e^0_{kij}, \epsilon^0_{ij}, p, q$ are constants. Let

$$\delta x_i = \varepsilon(1 + px_i), \quad \delta \psi_\alpha = \varepsilon[(1 + p)\Omega_\alpha + p\psi_\alpha/2] \tag{2.162}$$

Substitution of Eq. (2.162) into Eq. (2.136) yields

$$\{\Sigma_{\alpha j}[(1 + p)\Omega_\alpha + p\psi_\alpha/2] + P_{ij}(1 + px_i)\}_{,j} = 0 \tag{2.163}$$

The relations between material constants are

$$\begin{aligned} &\frac{\partial C_{ijkl}}{\partial x_1} = \frac{pq}{1 + px_1} C_{ijkl}, \quad \frac{\partial e_{kij}}{\partial x_1} = \frac{pq}{1 + px_1} e_{kij}, \quad \frac{\partial \epsilon_{ij}}{\partial x_1} = \frac{pq}{1 + px_1} \epsilon_{ij} \\ &\frac{\bar{\partial} g}{\partial x_1} = \frac{pq}{1 + px_1} g, \quad \frac{\bar{\partial} g}{\partial x_2} = \frac{\bar{\partial} g}{\partial x_3} = 0 \end{aligned} \tag{2.164}$$

Substitution of Eq. (2.164) into Eq. (2.163) and using (2.164), we find

$$(1+p)\Sigma_{\alpha j}\Omega_{\alpha,j} + \frac{p}{2}\Sigma_{\alpha j}\psi_{\alpha,j} + (1+px_1)\frac{\bar{\partial} g}{\partial x_1} + pP_{ij}\frac{\partial x_i}{\partial x_j} \tag{2.165}$$

Using the relation $pP_{ij}\partial x_i/\partial x_j = -pg$ and Eq. (2.161), Eq. (2.165) is reduced to

$$(1+p)\Sigma_{\alpha j}\Omega_{\alpha,j} + pq\mathrm{g} = 0 \quad \Rightarrow \quad \Omega_\alpha = -\frac{pq}{2(1+p)}\psi_\alpha \tag{2.166}$$

Finally, the path independence integral is obtained:

$$\begin{aligned} &[P_{ij} + px_i P_{ij} + (1/2)p(1-q)\Sigma_{\alpha j}\psi_\alpha]_{,j} = 0 \\ &\int_a [(1+px_i)P_{ij} + (1/2)p(1-q)\Sigma_{\alpha j}\psi_\alpha] n_j \,\mathrm{d}a = 0 \end{aligned} \tag{2.167}$$

2.9.6 Transversely Isotropic Materials

For transversely isotropic materials, the infinitesimal transformation is taken as

$$\delta x_i = \varepsilon\omega_i(x_j), \quad \delta\psi_\alpha = \varepsilon W_\alpha(\psi_\beta) = \varepsilon A_{\alpha\beta}\psi_\beta \tag{2.168}$$

where ω_i is undetermined function, $A_{\alpha\beta}$ is an undetermined constant. Substitution of Eq. (2.168) into (2.136) yields

$$\omega_j\frac{\bar{\partial} g}{\partial x_j} + g\frac{\bar{\partial}\omega_j}{\partial x_j} + \Sigma_{\alpha j}\left(A_{\alpha\beta}\psi_{\beta,j} - \psi_{\alpha,i}\frac{\bar{\partial}\omega_i}{\partial x_j}\right) = 0 \tag{2.169}$$

From Eq. (2.169), we obtain

$$K_{ijkl}u_{i,j}u_{k,l} + M_{kij}\varphi_{,k}u_{i,j} + N_{ij}\varphi_{,i}\varphi_{,j} = 0 \tag{2.170}$$

where

$$\begin{aligned} &K_{ijkl} = (1/2)\left[\partial(\omega_n C_{ijkl})/\partial x_n\right] + A_{mi}C_{mjkl} - \omega_{j,m}C_{imkl} + A_{4i}e_{jkl} \\ &M_{kij} = \partial(\omega_n e_{kij})/\partial x_n + A_{44}e_{kij} + A_{mi}e_{kmj} - \omega_{j,m}e_{kim} - \omega_{k,m}e_{mij} + A_{m4}C_{mkij} - A_{4i}\epsilon_{jk} \\ &N_{ij} = -(1/2)\partial(\omega_n\epsilon_{ij})/\partial x_n - A_{44}\epsilon_{ij} + \omega_{i,m}\epsilon_{mj} + A_{m4}e_{imj} \end{aligned} \tag{2.171}$$

From Eq. (2.170) we have

$$K_{ijkl} + K_{klij} = 0, \quad M_{kij} = 0, \quad N_{ij} + N_{ji} = 0 \tag{2.172}$$

As an example, we discuss the transversely isotropic piezoelectric ceramic (such as PZT). Assume x_3 is the poling direction, so x_1x_2 is the isotropic plane. Now the plane x_1x_3 is discussed. Appling Voigt notation, the constitutive equation is

$$\{\sigma\} = [C]\{\varepsilon\} - [e]^{\mathrm{T}}\{E\}, \quad \{D\} = [\epsilon]\{E\} + [e]\{\varepsilon\}$$
$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2C_{66} \end{bmatrix},$$
$$[e] = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad [\epsilon] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \tag{2.173}$$

where $C_{11} = C_{1111}, C_{12} = C_{1122}, C_{13} = C_{1133}, C_{33} = C_{3333}, C_{44} = C_{1313}; e_{15} = e_{113}$, $e_{31} = e_{311}, e_{33} = e_{333}$.

Though the number of the undetermined constants in Eq. (2.168) is less than the number of the equation in Eq. (2.170), the undetermined constants can still be determined by special selection of constants. Finally, we get

$$\begin{aligned} &\omega_1 = (b - A_{11})x_1 + A_{12}x_2 + C_1, \quad \omega_2 = -A_{12}x_1 + (b - A_{11})x_2 + C_2 \\ &\omega_3 = (b - A_{33})x_3 + C_3, \quad W_1 = A_{11}u_1 + A_{12}u_2, \quad W_2 = -A_{12}u_1 + A_{11}u_2, \\ &W_3 = A_{33}u_3 + A_{34}\varphi, \quad W_4 = A_{44}\varphi \end{aligned} \tag{2.174}$$

where C_i is a new arbitrary constant. When coefficients in Eq. (2.168) take values given in Eq. (2.174), we can get a group of linear partial differential equation to determine the unknown coefficients by using the invariant conditions Eq. (2.172). This group linear partial differential equation is

$$\begin{aligned} &\omega_n \frac{\partial C_{11}}{\partial x_n} + (2A_{11} - A_{33} + b)C_{11} = 0, \quad \omega_n \frac{\partial C_{33}}{\partial x_n} + (-2A_{11} + 3A_{33} + b)C_{33} = 0, \\ &\omega_n \frac{\partial C_{12}}{\partial x_n} + (2A_{11} - A_{33} + b)C_{12} = 0, \quad \omega_n \frac{\partial C_{13}}{\partial x_n} + (A_{33} + b)C_{13} = 0, \\ &\omega_n \frac{\partial C_{44}}{\partial x_n} + (A_{33} + b)C_{44} = 0, \quad \omega_n \frac{\partial \epsilon_{11}}{\partial x_n} + (2A_{44} - A_{33} + b)\epsilon_{11} - 2A_{34}e_{15} = 0, \\ &\omega_n \frac{\partial \epsilon_{33}}{\partial x_n} + (2A_{44} + A_{33} - 2A_{11} + b)\epsilon_{33} - 2A_{34}e_{33} = 0, \\ &\omega_n \frac{\partial e_{15}}{\partial x_n} + (A_{44} + b)e_{15} + A_{34}C_{44} = 0, \quad \omega_n \frac{\partial e_{31}}{\partial x_n} + (A_{44} + b)e_{31} + A_{34}C_{13} = 0, \\ &\omega_n \frac{\partial e_{33}}{\partial x_n} + (A_{44} + 2A_{33} - 2A_{11} + b)e_{33} + A_{34}C_{33} = 0 \end{aligned} \tag{2.175}$$

If Eq. (2.175) has solution, the infinitesimal transform given by Eq. (2.168) can be obtained. Substitution of Eq. (2.168) into Eq. (2.136) yields the conservative integral:

$$(P_{ij}\omega_i + \sigma_{ij}W_i + D_jW_4)_{,j} = 0, \quad \int_a (P_{ij}\omega_i + \sigma_{ij}W_i + D_jW_4)n_j\,\mathrm{d}a = 0 \tag{2.176}$$

For a homogeneous material, Eq. (2.175) is reduced to linear equations and its solutions are

$$A_{34} = 0, \quad A_{11} = A_{33} = A_{44} \tag{2.177}$$

where $C_1, C_2, C_3, A_{12},$ and A_{11} are arbitrary constants. In this case, Eq. (2.176) is reduced to

$$\begin{aligned}
&\{C_1P_{ij} + C_2P_{2j} + C_3P_{3j} - 2A_{11}[P_{ij}x_1 - (1/2)(\sigma_{ij}u_i + D_j\varphi)] \\
&\quad + A_{12}(P_{1j}x_2 - P_{2j}x_1 + \sigma_{1j}u_2 - \sigma_{2j}u_1)\}_{,j} = 0, \\
&\int_a \{C_1P_{ij} + C_2P_{2j} + C_3P_{3j} + 2A_{11}\left[P_{ij}x_1 - \frac{1}{2}(\sigma_{ij}u_i + D_j\varphi)\right] \\
&\quad + A_{12}(P_{1j}x_2 - P_{2j}x_1 + \sigma_{1j}u_2 - \sigma_{2j}u_1)\}n_j\,\mathrm{d}a = 0
\end{aligned} \tag{2.178}$$

Equation (2.178) can be divided into five group independent conservative integrals due to the arbitrariness of constants. The independent conservative integrals corresponding to C_1, A_{11}, A_{12} are identical with Eqs. (2.143), (2.147), and (2.149). There are no new conservative integrals corresponding C_2, C_3 because $P_{2j,j} = P_{3j,j} = 0$ are the special cases of $P_{ij,j} = 0$.

References

Bustamante R, Dorfmann A, Ogden RW (2008) On variational formulations in nonlinear magnetoelastostatics. Math Mech Solids 13:725–745

Delph TJ (1982) Conservation laws in linear elasticity based upon divergence transformations. J Elast 12:385–393

Eringen AC, Maugin GA (1989) Electrodynamics of continuua. Springer-Verlag, New York

Eshelby JD (1956) The continuum theory of lattice defects. Solid State Phys 3:79–144// Edited by Seitz and TuruBull

Eshelby JD (1975) The elastic energy-momentum tensor. J Elast 5:321–335

Fletcher DC (1976) Conservation laws in linear elastodynamics. J Ration Mech Anal 60:330–353

Genin J, Xu W (1999) Thermoelastic plastic metals with mass diffusion. Z Angew Math Phys 50:511–528

Guillot FM, Jarzynski J, Balizer JE (2001) Measurement of electrostrictive coefficients of polymer films. J Acoust Soc Am 110:2980–2990

Honein T, Herrmann G (1997) Conservation laws in nonhomogeneous plane elastostatics. J Mech Phys Solids 45:789–805

Hu HC (1981) Variational principle and its applications in elasticity. Science Press, Beijing
Jiang Q, Kuang Z-B (2003) Stress analysis of electrostrictive material with an elliptic defect. Sci China Ser G 46:492–500
Jiang Q, Kuang Z-B (2004) Stress analysis in two dimensional electrostrictive material with an elliptic rigid conductor. Eur J Mech A/Solids 23:945–956
Kuang Z-B (1964) General variational principle and the theory of thin plate. Scientific Report of Xian Jiaotong University, Xian (in Chinese)
Kuang Z-B (2002) Nonlinear continuum mechanics. Shanghai Jiaotong University Press, Shanghai (in Chinese)
Kuang Z-B (2007) Some problems in electrostrictive and magnetostrictive materials. Acta Mech Solida Sin 20:219–227
Kuang Z-B (2008a) Some variational principles in elastic dielectric and elastic magnetic materials. Eur J Mech A/Solids 27:504–514
Kuang Z-B (2008b) Some variational principles in electroelastic media under finite deformation. Sci China Ser G 51:1390–1402
Kuang Z-B (2009a) Internal energy variational principles and governing equations in electroelastic analysis. Int J Solids Struct 46:902–911
Kuang Z-B (2009b) Variational principles for generalized dynamical theory of thermopiezoelectricity. Acta Mech 203:1–11
Kuang Z-B (2010) Variational principles for generalized thermodiffusion theory in pyroelectricity. Acta Mech 214:275–289
Kuang Z-B (2011a) Theory of electroelasticity. Shanghai Jiaotong University Press, Shanghai (in Chinese)
Kuang Z-B (2011b) Physical variational principle and thin plate theory in electro-magneto-elastic analysis. Int J Solids Struct 48:317–325
Kuang Z-B (2011c) Some thermodynamic problems in continuum mechanics. In: Piraján JCM (ed) Thermodynamics – kinetics of dynamic systems. Intech Open Access Publisher, Croatia
Kuang Z-B (2012) Static electric force and measurement principle of material constants in electrostrictive material. Smart Mater Res. Article ID 712103, 8 p, doi:10.1155/2012/712103
Kuang Z-B (2013) Two theoretical problems in electro-magneto-elastic analysis. Acta Mech 224:1213–1224
Landau LD, Lifshitz EM, Pitaevskii LP (1984) Electrodynamics of continuum media, 2nd edn. Pergamon Press, Oxford; Ландау Л Д, Лифшиц Е М. Электродинамика сплошных сред. Москва: ФИЗМАТГИЗ, 1959
Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
Mcmeeking RM, Landis CM (2005) Electrostatic forces and stored energy for deformable dielectric materials. J Appl Mech 72:581–590
Mcmeeking RM, Landis CM, Jimenez A (2007) Principle of virtual work for combined electrostatic and mechanical loading of materials. Int J Nonlinear Mech 42:831–838
Moon EC (1984) Magneto-solid mechanics. A Wiley-Interscience Publication, John Wiley & Sons. New York, Chichester, Brisbane, Toronto, Singapore
Nother E (1918) Invariant variations problem. Göttinger Nachrichten Mathematisch-physicalische Klasse 2:235–253
Pak YE (1992) Linear electro-elastic fracture mechanics of piezoelectric materials. Int J Fract 54:79–100
Pao YH (1978) Electromagnetic forces in deformable continua // Nemat-nasser. Mechanics today. Bath: printed in Great Britain by Pitman
Sherief HH, Hamza FA, Saleh HA (2004) The theory of generalized thermoelastic diffusion. Int J Eng Sci 42:591–608
Shi W-C, Kuang Z-B (2003) Conservation laws in non-homogeneous electro-magneto-elastic materials. Eur J Mech A/Solids 22:217–230
Shkel YM, Klingenberg DJ (1996) Material parameters for electrostriction. J Appl Phys 80:4566–4571

Stratton JA (1941) Electromagnetic theory. McGraw-Hill, New York
Wang J-W, Kuang Z-B (2001) The conservative law of the non-conservative dynamic system in the pyroelectric material. Q Mech 22:154–161 (in Chinese)
Wang X-M, Shen Y-P (1996) The conservation laws and path-independent integrals with an application for linear electro-magneto-elastic media. Int J Solids Struct 33:865–878
Zhang QM, Su J, Kim CH (1997) An experimental investigation of electromechanical responses in a polyurethane elastomer. J Appl Phys 81:2770–2776

Chapter 3
Generalized Two-Dimensional Electroelastic Problem

Abstract In this chapter the fundamental theory of the generalized two-dimensional (2D) linear electroelastic analyses is discussed. The generalized 2D Stroh method and the extended generalized Lekhnitskii stress function method are studied. The linear electroelastic analyses in an infinite transversely isotropic material with the permeable, impermeable, and conducting elliptic hole; crack; and the rigid elliptic inclusion under plane strain are discussed in detail. Singularities, including generalized dislocation, generalized force, and electric couple, in homogeneous material and bimaterial are researched. Interaction of an elliptic inclusion with a singularity is discussed, and some numerical examples are also given. In this chapter the asymptotic fields near a line inclusion tip in a homogeneous material and Eshelby's eigenstrain problem are also discussed.

Keywords Generalized 2D electroelastic problem • Stroh method • Extended Lekhnitskii method • Transversely isotropic material • Elliptic hole • Crack • Elliptic inclusion • Singularity

3.1 Generalized Two-Dimensional Linear Electroelastic Problem

The generalized two-dimensional (2D) electroelastic problem means that the generalized displacements $(u_i, \varphi;\ i = 1, 2, 3)$ exactly or the generalized stresses $(\sigma_{ij}, D_i;\ i, j = 1, 2, 3)$ approximately depend only on two of the coordinates (x_1, x_2, x_3). It is seen that the generalized 2D problem is a special three-dimensional (3D) problem, which is different with the plane problem (plane strain and generalized plane stress problems). For the linear electroelastic problem with small electric field, the Maxwell stress can be neglected because $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{D})$ depend on $\boldsymbol{E}$ linearly and the Maxwell stress is depended on the square of $\boldsymbol{E}$. The method to solve the electroelastic problem is directly the extension of that in the anisotropic elastic materials, but the problem is more complex.

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In engineering the extensive applied constitutive equations are the second kind and the third kind of the constitutive equations in Eq. (2.83) for the piezoelectric materials. The governing equations are the generalized momentum equations, constitutive equations, and generalized geometric equations. They are, respectively,

$$\sigma_{ij,i} + \left(f_j^{\mathrm{m}} + f_j^{\mathrm{e}}\right) = \rho \ddot{u}_j, \quad D_{i,i} = \rho_{\mathrm{e}} \tag{3.1}$$

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad D_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} \quad \text{or} \\ \varepsilon_{ij} &= s_{ijkl}\sigma_{kl} + \mathrm{g}_{kij}D_k, \quad E_i = -\mathrm{g}_{ijk}\sigma_{jk} + \beta_{ij}D_j \end{aligned} \tag{3.2}$$

$$\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2, \quad E_i = -\varphi_{,i} \tag{3.3}$$

where $\boldsymbol{f}^{\mathrm{m}}$ is the mechanical force per volume and $\boldsymbol{f}^{\mathrm{e}}$ is the static electric force. The boundary conditions and connective conditions on the interface are, respectively,

$$\begin{array}{llll} \sigma_{ij}n_j = T_i^*, & \text{on} \quad a_\sigma; & u_i = u_i^*, & \text{on} \quad a_u \\ D_i n_i = -\sigma^*, & \text{on} \quad a_D; & \varphi = \varphi^*, & \text{on} \quad a_\varphi \end{array} \tag{3.4}$$

$$\sigma_{ij}^+ n_j = \sigma_{ij}^- n_j, \quad u_i^+ = u_i^-, \quad D_i^+ = D_i^-; \quad \varphi^+ = \varphi^-, \quad \text{on} \quad L \tag{3.5}$$

where $\boldsymbol{T}^*, \sigma^*$ are the traction and electric charge per area and the superscripts "+" and " − " denote the values approached from the upper and lower half planes, respectively. For the linear problem, $\boldsymbol{f}^{\mathrm{e}}$ can be neglected. For the static case without the body force and body electric charge, the governing equations in $(\boldsymbol{u}, \varphi)$ are

$$\left(C_{ijlk}u_l + e_{kij}\varphi\right)_{,ik} = 0; \quad \left(-\epsilon_{ik}\varphi + e_{ijk}u_j\right)_{,ik} = 0 \tag{3.6}$$

For the multi-connected domain, the displacement, electric potential must satisfy the uniqueness conditions

$$\oint_L \mathrm{d}U_i = 0 \quad \text{or} \quad \oint_L \mathrm{d}u_i = 0, \quad \oint_L \mathrm{d}\varphi = 0 \tag{3.7}$$

where L is a closed contour and there is no source inside it.

Sometimes the constitutive equations are written in a more compact form:

$$\begin{aligned} &\Sigma_{iJ} = E_{iJKn}Z_{Kn}; \quad \Sigma_{iJ} = \begin{cases} \sigma_{ij}, & J = 1,2,3 \\ D_i, & J = 4 \end{cases}; \quad \Sigma_{iJ,i} = 0 \\ &U_K = \begin{cases} u_k, & K = 1,2,3 \\ \varphi, & K = 4 \end{cases}; \quad Z_{Kn} = \begin{cases} \varepsilon_{kn}, & K = 1,2,3 \\ -E_n, & K = 4 \end{cases}; \\ &E_{iJKn} = \begin{cases} C_{ijkn}, & J,K = 1,2,3 \\ e_{nij}, & J = 1,2,3; \quad K = 4 \\ e_{ikn}, & J = 4; \quad K = 1,2,3 \\ -\epsilon_{in}, & J = K = 4 \end{cases} \end{aligned} \tag{3.8}$$

where a subscript in upper case takes the value 1, 2, 3, or 4 and a subscript in lower case takes the value 1, 2, or 3. U_K, Σ_{iJ}, Z_{Kl}, and E_{iJKl} are the generalized displacement, generalized stress, generalized strain, and generalized stiffness coefficient, respectively. It is noted that the rule of the subscript used here does not hold everywhere and the meaning of the subscript given in corresponding places.

3.2 Generalized Displacement Method in the Piezoelectric Materials

3.2.1 Generalized Displacement Method

For the generalized 2D problem, the Stroh method (Stroh 1958; Suo 1990; Suo et al. 1992; Ting 1996) is often applied. Let

$$\begin{aligned}
&\boldsymbol{U} = \boldsymbol{a}f(z), \quad \text{or} \quad U_K = a_K f(z), \quad \text{or} \quad u_i = a_i f(z), \quad \varphi = a_4 f(z) \\
&\boldsymbol{U} = \{U_K\}^{\mathrm{T}} = [u_i, \varphi]^{\mathrm{T}}, \quad \boldsymbol{a} = \{a_K\}^{\mathrm{T}} = [a_i, a_4]^{\mathrm{T}} \\
&U_{K,\alpha} = a_K f'(z)(\delta_{\alpha 1} + \mu \delta_{\alpha 2}), \quad z = x_1 + \mu x_2; \quad z_{,1} = 1, \quad z_{,2} = \mu
\end{aligned} \tag{3.9}$$

where the right upper superscript T denotes transpose of a matrix. Substituting Eq. (3.9) into Eq. (3.6) in generalized 2D case yields

$$\begin{aligned}
&\left(C_{\alpha jl\beta} a_l + e_{\beta j\alpha} a_4\right) z_{,\alpha} z_{,\beta} = 0, \quad \left(-\epsilon_{\alpha\beta} a_4 + e_{\beta j\alpha} a_j\right) z_{,\alpha} z_{,\beta} = 0; \quad \text{or} \\
&\left(C_{\alpha jl\beta} a_l + e_{\beta\alpha j} a_4\right) z_{,\alpha} z_{,\beta} = 0, \quad \left(-\epsilon_{\alpha\beta} a_4 + e_{\beta\alpha j} a_j\right) z_{,\alpha} z_{,\beta} = 0
\end{aligned} \tag{3.10}$$

where a Greek subscript takes values 1 and 2 and an English subscript takes values 1, 2, and 3. Equation (3.10) can be written in detail as

$$\begin{aligned}
&\left[C_{i1k1} + \mu(C_{i1k2} + C_{i2k1}) + \mu^2 C_{i2k2}\right] a_k + \left[e_{1i1} + \mu(e_{2i1} + e_{1i2}) + \mu^2 e_{2i2}\right] a_4 = 0 \\
&\left[e_{1k1} + \mu(e_{2k1} + e_{1k2}) + \mu^2 e_{2k2}\right] a_k - \left[\epsilon_{11} + \mu(\epsilon_{12} + \epsilon_{21}) + \mu^2 \epsilon_{22}\right] a_4 = 0
\end{aligned} \tag{3.11}$$

where the subscripts i and k denote row and column, respectively. In order to obtain nontrivial solutions for (a_k, a_4), the coefficient determinant must be zero, i.e.,

$$|\boldsymbol{D}(\mu)| = \begin{vmatrix} C_{i1k1} + \mu(C_{i1k2} + C_{i2k1}) + \mu^2 C_{i2k2} & e_{1i1} + \mu(e_{2i1} + e_{1i2}) + \mu^2 e_{2i2} \\ e_{1k1} + \mu(e_{2k1} + e_{1k2}) + \mu^2 e_{2k2} & -\epsilon_{11} - \mu(\epsilon_{12} + \epsilon_{21}) - \mu^2 \epsilon_{22} \end{vmatrix} = 0 \tag{3.12}$$

$\boldsymbol{D}(\mu)$ is called the character matrix. If introducing 4×4 matrixes $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}$

$$\boldsymbol{Q} = \begin{bmatrix} C_{i1k1} & e_{1i1} \\ e_{1k1} & -\epsilon_{11} \end{bmatrix},\ \boldsymbol{R} = \begin{bmatrix} C_{i1k2} & e_{2i1} \\ e_{1k2} & -\epsilon_{12} \end{bmatrix},\ \boldsymbol{R}^{\mathrm{T}} = \begin{bmatrix} C_{i2k1} & e_{1i2} \\ e_{2k1} & -\epsilon_{12} \end{bmatrix},\ \boldsymbol{T} = \begin{bmatrix} C_{i2k2} & e_{2i2} \\ e_{2k2} & -\epsilon_{22} \end{bmatrix}$$
$$C_{i2k1} = C_{k1i2}; \quad Q_{JK} = E_{1JK1}, \quad R_{JK} = E_{1JK2}, \quad T_{JK} = E_{2JK2} \tag{3.13}$$

then Eqs. (3.11) and (3.12) can also be written as

$$\begin{aligned} &\boldsymbol{D}(\mu)\boldsymbol{a} = \left[\boldsymbol{Q} + \mu\left(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}\right) + \mu^2\boldsymbol{T}\right]\boldsymbol{a} = \boldsymbol{0}, \quad \text{or} \\ &(\boldsymbol{Q} + \mu\boldsymbol{R})\boldsymbol{a} = -\mu\left(\boldsymbol{R}^{\mathrm{T}} + \mu\boldsymbol{T}\right)\boldsymbol{a}, \quad \left(\boldsymbol{R}^{\mathrm{T}} + \mu\boldsymbol{T}\right)\boldsymbol{a} = -\left(\mu^{-1}\boldsymbol{Q} + \boldsymbol{R}\right)\boldsymbol{a} \\ &|\boldsymbol{D}(\mu)| = \left|\boldsymbol{Q} + \mu\left(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}\right) + \mu^2\right| = 0 \end{aligned} \tag{3.14}$$

$|\boldsymbol{D}(\mu)|$ is a 4×4 determinant, $|\boldsymbol{D}(\mu)| = 0$ is the eighth-order equation of μ, so eigenvalue μ has eight roots. Equation (3.11) or (3.14) is used to determine eigenvector $\boldsymbol{a}$. Because μ is complex (Suo et al 1992; Ting 1996), let

$$\begin{aligned} &\mu_P = \alpha_P + \mathrm{i}\beta_P, \quad \beta_P > 0; \quad \mu_{P+4} = \bar{\mu}_P; \quad (P = 1, 2, 3, 4) \\ &z_P = x_1 + \mu_P x_2; \quad x_1 = (\mu_P \bar{z}_P - \bar{\mu}_P z_P)/(\mu_P - \bar{\mu}_P), \quad x_2 = (z_P - \bar{z}_P)/(\mu_P - \bar{\mu}_P) \end{aligned} \tag{3.15}$$

In fact if we multiply the first equation in (3.10) by a_j and sum over j, multiply the second equation in (3.10) by a_4, the difference of these two results is

$$\left(C_{\alpha jl\beta}a_j a_l + \epsilon_{\alpha\beta}a_4^2\right)(\delta_{\alpha 1} + \mu\delta_{\alpha 2})\left(\delta_{\beta 1} + \mu\delta_{\beta 2}\right) = 0$$

If μ is real, we can choose

$$\begin{aligned} &u_{j,\alpha} = (\delta_{\alpha 1} + \mu\delta_{\alpha 2})a_j, \quad u_{l,\beta} = \left(\delta_{\beta 1} + \mu\delta_{\beta 2}\right)a_l; \quad \varphi_{,\alpha} = (\delta_{\alpha 1} + \mu\delta_{\alpha 2})a_4, \\ &\varphi_{,\beta} = \left(\delta_{\beta 1} + \mu\delta_{\beta 2}\right)a_4 \end{aligned}$$

The expression of the strain energy is

$$C_{\alpha jl\beta}u_{j,\alpha}u_{l,\beta} + \epsilon_{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} = 0$$

However, the strain energy is positive definite and cannot equal zero, so μ must be complex.

3.2.2 Eigenvalues $\boldsymbol{\mu}$'s Are All Distinct

When the eigenvalues $\boldsymbol{\mu}$'s in Eq. (3.12) are all distinct, the matrix $\boldsymbol{D}(\mu)$ is called simple. In this case for each μ_P, an independent eigenvector $\boldsymbol{a}_P = [a_{P1}, a_{P2}, a_{P3}, a_{P4}]^{\mathrm{T}}$

can be solved from Eq. (3.11). Corresponding to $\boldsymbol{a}_P$, an arbitrary function $f_P(z_P)$, $z_P = x_1 + \mu_P x_2$, can be assumed. Noting $\boldsymbol{U}$ is real, so the general solution is

$$\begin{aligned}\boldsymbol{U} &= [u_i, \varphi]^{\mathrm{T}} = 2\mathrm{Re}\sum_{P=1}^{4}\boldsymbol{a}_P f_P(z_P) = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z_P)] \\ U_K &= 2\mathrm{Re}\sum_{P=1}^{4}a_{PK}f_P(z_P) = 2\mathrm{Re}\sum_{P=1}^{4}A_{KP}f_P(z_P)\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\boldsymbol{a} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]; \quad \boldsymbol{A} = [A_{KP}], \quad A_{KP} = a_{PK} \\ &\boldsymbol{f}(z_P) = [f_P(z_P)]^{\mathrm{T}} = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^{\mathrm{T}}\end{aligned} \tag{3.17}$$

where symbol Re means the real part of a complex function, $\boldsymbol{A}$ is a 4×4 matrix, and $\boldsymbol{f}(z_P)$ is a vector function and may be called the displacement generation function. It is noted that matrix $\boldsymbol{A}$ and matrix $\boldsymbol{a}$ are identical, but the notations of their components are different. When the number of a summation dummy subscript is larger than 2, we shall directly use the notation Σ as shown in Eq. (3.16). For most engineering problem, $f_P(z_P)$ in Eq. (3.16) can be simplified as $f(z_P)V_P$, where $\boldsymbol{V}$ is a constant vector. So Eq. (3.16) can be reduced to

$$\boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\langle f(z_P)\rangle \boldsymbol{V}], \quad \langle f(z_P)\rangle = \mathrm{diag}[f(z_P)], \quad \boldsymbol{V} = \left[V_j, V_4\right]^{\mathrm{T}} \tag{3.18}$$

Analogous to Eq. (3.10), for any subscript "P," we have

$$\begin{aligned}&\left(C_{i\alpha k\beta}A_{kP} + e_{\beta i\alpha}A_{4P}\right)z_{P,\alpha}z_{P,\beta} = 0, \quad \left(e_{\alpha k\beta}A_{kP} - \epsilon_{\alpha\beta}A_{4P}\right)z_{P,\alpha}z_{P,\beta} = 0; \quad \text{or} \\ &\left(C_{i1k\beta}A_{kP} + e_{\beta i1}A_{4P}\right)z_{P,\beta} = -\mu_P\left(C_{i2k\beta}A_{kP} + e_{\beta i2}A_{4P}\right)z_{P,\beta} \\ &\left(e_{1k\beta}A_{kP} - \epsilon_{\beta 1}A_{4P}\right)z_{P,\beta} = -\mu_P\left(e_{2k\beta}A_{kP} - \epsilon_{\beta 2}A_{4P}\right)z_{P,\beta}\end{aligned} \tag{3.19}$$

Substitution of Eq. (3.16) into Eq. (3.2) yields

$$\begin{aligned}\sigma_{ij} &= 2\mathrm{Re}\sum_{P=1}^{4}\left(C_{ijk\beta}A_{kP} + e_{\beta ij}A_{4P}\right)z_{P,\beta}F_P(z_P) \\ D_i &= 2\mathrm{Re}\sum_{P=1}^{4}\left(e_{ik\beta}A_{kP} - \epsilon_{i\beta}A_{4P}\right)z_{P,\beta}F_P(z_P)\end{aligned} \tag{3.20}$$

where $F_p(z_p) = df_p/dz_p = f_p'(z_p)$ is the derivative of $f_p(z_p)$ with z_p. Substitution of Eq. (3.18) into Eq. (3.2) yields

$$\begin{aligned}\sigma_{ij} &= 2\mathrm{Re}\sum_{P=1}^{4}\left(C_{ijk\beta}A_{kP} + e_{\beta ij}A_{4P}\right)z_{P,\beta}F(z_P)V_P \\ D_i &= 2\mathrm{Re}\sum_{P=1}^{4}\left(e_{ik\beta}A_{kP} - \epsilon_{i\beta}A_{4P}\right)z_{P,\beta}F(z_P)V_P\end{aligned} \tag{3.21}$$

Using Eq. (3.19) from Eq. (3.20), we can get

$$
\begin{aligned}
\sigma_{i1} &= 2\mathrm{Re}\sum_{P=1}^{4}\left(C_{i1k\beta}A_{kP} + e_{\beta i1}A_{4P}\right)z_{P,\beta}F_P(z_P) = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P\left(C_{i2k\beta}A_{kP} + e_{\beta i2}A_{4P}\right)z_{P,\beta}F_P(z_P)\\
D_1 &= 2\mathrm{Re}\sum_{P=1}^{4}\left(e_{1k\beta}A_{kP} - \epsilon_{1\beta}A_{4P}\right)z_{P,\beta}F_P(z_P) = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P\left(e_{2k\beta}A_{kP} - \epsilon_{\beta 2}A_{4P}\right)z_{P,\beta}F_P(z_P)\\
\sigma_{i2} &= 2\mathrm{Re}\sum_{P=1}^{4}\left(C_{i2k\beta}A_{kP} + e_{\beta i2}A_{4P}\right)z_{P,\beta}F_P(z_P) = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P^{-1}\left(C_{i1k\beta}A_{kP} + e_{\beta i1}A_{4P}\right)z_{P,\beta}F_P(z_P)\\
D_2 &= 2\mathrm{Re}\sum_{P=1}^{4}\left(e_{2k\beta}A_{kP} - \epsilon_{2\beta}A_{4P}\right)F_P(z_P)z_{P,\beta} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P^{-1}\left(e_{1k\beta}A_{kP} - \epsilon_{\beta 1}A_{4P}\right)z_{P,\beta}F_P(z_P)
\end{aligned}
\tag{3.22}
$$

Introduce the generalized stress function $\boldsymbol{\Phi}$ satisfying the equilibrium equation automatically:

$$
\begin{aligned}
&\boldsymbol{\Phi} = [\Phi_1, \Phi_2, \Phi_3, \Phi_4]^{\mathrm{T}} = [\Phi_i, \Phi_4]^{\mathrm{T}} = 2\mathrm{Re}\sum_{P=1}^{4}\boldsymbol{b}_P f_P(z_P) = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)]\\
&\boldsymbol{\Sigma}_1 = -\boldsymbol{\Phi}_{,2} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P\boldsymbol{b}_P F_P(z_P), \quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Phi}_{,1} = 2\mathrm{Re}\sum_{P=1}^{4}\boldsymbol{b}_P F_P(z_P)\\
&\sigma_{i1} = \Sigma_{i1} = -\Phi_{i,2} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P b_{Pi}F_P(z_P), \quad D_1 = \Sigma_{41} = -\Phi_{4,2} = -2\mathrm{Re}\sum_{P=1}^{4}\mu_P b_{P4}F_P(z_P)\\
&\sigma_{i2} = \Sigma_{i2} = \Phi_{i,1} = 2\mathrm{Re}\sum_{P=1}^{4}b_{Pi}F_P(z_P), \quad D_2 = \Sigma_{42} = \Phi_{4,1} = 2\mathrm{Re}\sum_{P=1}^{4}b_{P4}F_P(z_P)
\end{aligned}
\tag{3.23}
$$

Comparing Eqs. (3.22) and (3.23), it is easily found that

$$
\begin{aligned}
b_{Pi} &= B_{iP} = \left(C_{i2k\beta}A_{kP} + e_{\beta i2}A_{4P}\right)z_{P,\beta} = -\mu_P^{-1}\left(C_{i1k\beta}A_{kP} + e_{\beta i1}A_{4P}\right)z_{P,\beta}\\
b_{P4} &= B_{4P} = \left(e_{2k\beta}A_{kP} - \epsilon_{\beta 2}A_{4P}\right)z_{P,\beta} = -\mu_P^{-1}\left(e_{1k\beta}A_{kP} - \epsilon_{\beta 1}A_{4P}\right)z_{P,\beta}\\
\boldsymbol{b} &= [\boldsymbol{b}_P] = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4] = [b_{PK}] = [B_{KP}] = B
\end{aligned}
\tag{3.24}
$$

Combining Eqs. (3.23) and (3.24), we get

$$
\begin{aligned}
\boldsymbol{b}_P &= \left(\boldsymbol{R}^{\mathrm{T}} + \mu_P\boldsymbol{T}\right)\boldsymbol{a}_P = -\mu_P^{-1}(\boldsymbol{Q} + \mu_P\boldsymbol{R})\boldsymbol{a}_P\\
\boldsymbol{B} &= \left(\boldsymbol{R}^{\mathrm{T}} + \mu_{P,row}\boldsymbol{T}\right)\boldsymbol{A} = -\mu_{P,row}^{-1}\left(\boldsymbol{Q} + \mu_{P,row}\boldsymbol{R}\right)\boldsymbol{A}\\
\boldsymbol{\Sigma}_1 &= -\boldsymbol{\Phi}_{,2} = -2\mathrm{Re}[\boldsymbol{B}\mu_P\boldsymbol{F}(z_P)], \quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Phi}_{,1} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{F}(z_P)]
\end{aligned}
\tag{3.25}
$$

where $\mu_{P,row}$ is a special symbol, the subscript P in $\mu_{P,row}$ takes the value of the row number of the matrix $\boldsymbol{A}$ or $\boldsymbol{B}$ under matrix calculation. Similar to $\boldsymbol{a}_P$, components of $\boldsymbol{b}_P$ are $b_{PK}, K = 1, 2, 3, 4$. Because $\sigma_{12} = \sigma_{21}$, we get $\Phi_{1,1} + \Phi_{2,2} = 0$.

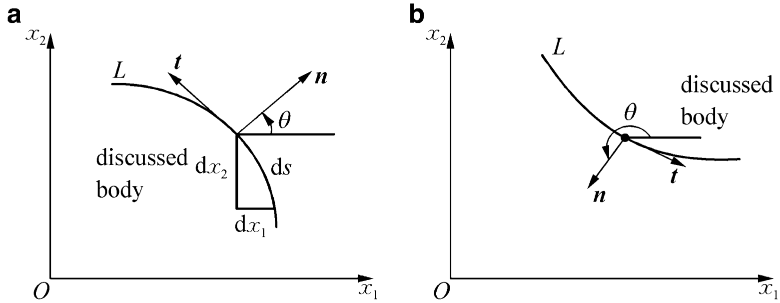


Fig. 3.1 First kind of natural coordinate system on a curve L

Similarly for the general solution Eq. (3.18), we have

$$\begin{aligned}&\boldsymbol{\Phi} = [\Phi_1, \Phi_2, \Phi_3, \Phi_4]^{\mathrm{T}} = [\Phi_i, \Phi_4]^{\mathrm{T}} = 2\mathrm{Re}\sum_{P=1}^{4} \boldsymbol{b}_P f(z_P)\boldsymbol{V}_P = 2\mathrm{Re}[\boldsymbol{B}\langle f(z_P)\rangle \boldsymbol{V}]\\ &\boldsymbol{\Sigma}_1 = -\boldsymbol{\Phi}_{,2} = -2\mathrm{Re}[\boldsymbol{B}\langle \mu_P F(z_P)\rangle \boldsymbol{V}], \quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Phi}_{,1} = 2\mathrm{Re}[\boldsymbol{B}\langle F(z_P)\rangle \boldsymbol{V}]\end{aligned} \tag{3.26}$$

The generalized stress σ_{33} can be obtained by the condition of the generalized plain strain $\varepsilon_{33} = 0$. In the 2D, $D_3 = 0$ is assumed.

Now the physical meaning of $\boldsymbol{\Phi}$ is discussed. Usually the first natural coordinate system at a point on a curve L is used. Let $\boldsymbol{n}$ be the outward normal to L; when an observer moves along the positive direction of the tangent $\boldsymbol{t}$ around L, the discussed body is located in the left side. θ is directed counterclockwise from the positive x_1-axis to the positive direction of $\boldsymbol{n}$ (Fig. 3.1). Therefore

$$\begin{aligned}&n_1 = t_2 = \cos\theta = \mathrm{d}x_2/\mathrm{d}s, \quad n_2 = -t_1 = \sin\theta = -\mathrm{d}x_1/\mathrm{d}s;\\ &n = n_1 + \mathrm{i}n_2 = -\mathrm{i}\mathrm{d}z/\mathrm{d}s = -\mathrm{i}\mathrm{d}z/|\mathrm{d}z|, \quad t = t_1 + \mathrm{i}t_2 = \mathrm{d}z/\mathrm{d}s = \mathrm{d}z/|\mathrm{d}z| = \mathrm{i}n\end{aligned} \tag{3.27}$$

where ds is the arc length of an infinitesimal element. The traction $\boldsymbol{T}$ on L is

$$\begin{aligned}&T_i = \sigma_{ij}n_j = \sigma_{i1}\mathrm{d}x_2/\mathrm{d}s - \sigma_{i2}\mathrm{d}x_1/\mathrm{d}s = -\Phi_{i,2}\mathrm{d}x_2/\mathrm{d}s - \Phi_{i,1}\mathrm{d}x_1/\mathrm{d}s = -\mathrm{d}\Phi_i/\mathrm{d}s\\ &-\sigma = D_i n_i = D_n = -\Phi_{4,2}\mathrm{d}x_2/\mathrm{d}s - \Phi_{4,1}\mathrm{d}x_1/\mathrm{d}s = -\mathrm{d}\Phi_4/\mathrm{d}s\\ &\boldsymbol{T} = [T_i, -\sigma] = -\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s, \quad \boldsymbol{\Phi}|_0^s = -\int_0^s \boldsymbol{T}\mathrm{d}s, \quad \Phi_i|_0^s = -\int_0^s T_i\mathrm{d}s, \quad \Phi_4|_0^s = -\int_0^s D_n\mathrm{d}s\end{aligned} \tag{3.28}$$

So $-\Delta\boldsymbol{\Phi}$ represents the increased resultant force on Δs of the boundary.

In literatures authors also adopted the second natural coordinate system. In this system authors take the tangent $\boldsymbol{t}'$ and $\boldsymbol{t}' = -\boldsymbol{t}$. This system is often used for a hole or inclusion in a multiply connected region. For this system in literatures, there are

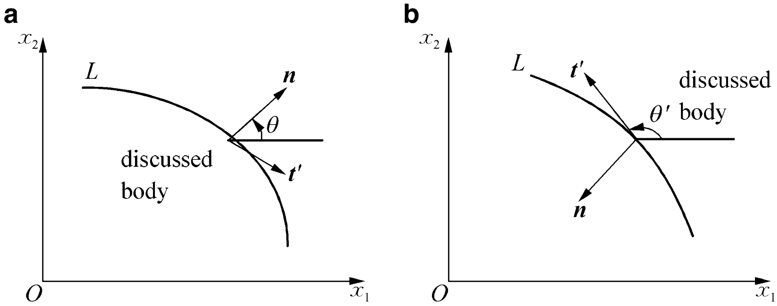


Fig. 3.2 Second kind of natural coordinate system on a curve L

two kinds. The first is that θ is directed counterclockwise from the positive x_1-axis to the direction of $\boldsymbol{n}$ (Fig. 3.2a), so

$$n_1 = -t_2' = \cos\theta = -\mathrm{d}x_2/\mathrm{d}s, \quad n_2 = t_1' = \sin\theta = \mathrm{d}x_1/\mathrm{d}s; \quad n = \mathrm{i}t' = \mathrm{i}\mathrm{d}z/\mathrm{d}s$$
$$\boldsymbol{T} = [t_1, t_2, t_3, -\sigma] = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s, \boldsymbol{\Phi}|_0^s = \int_0^s \boldsymbol{T}\mathrm{d}s, \quad \Phi_i|_0^s = \int_0^s T_i \mathrm{d}s, \quad \Phi_4|_0^s = -\int_0^s \sigma \mathrm{d}s \tag{3.29a}$$

The second is that θ' is directed counterclockwise from the positive x_1-axis to the direction of $\boldsymbol{t}'$ (Fig. 3.2b). In this case we have $\theta = \pi/2 + \theta'$, so we have

$$\boldsymbol{n} = (-\sin\theta', \cos\theta'), \quad \boldsymbol{t}'(= \cos\theta', \sin\theta'); \quad \boldsymbol{T} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s \tag{3.29b}$$

3.2.3 Orthogonality of A and B

From Eq. (3.14) we can get (Ting 1996; Kuang 2011)

$$\begin{bmatrix} -\boldsymbol{Q} & \mathbf{0} \\ -\boldsymbol{R}^{\mathrm{T}} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} = \mu \begin{bmatrix} \boldsymbol{R} & \mathbf{I} \\ \boldsymbol{T} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} \tag{3.30}$$

Multiply on both sides of Eq. (3.30) from left by the following matrix:

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{T}^{-1} \\ \mathbf{I} & -\boldsymbol{R}\boldsymbol{T}^{-1} \end{bmatrix}$$

Equation (3.30) can be reduced to the standard 8×8 eigen-equation

$$\boldsymbol{N}\boldsymbol{\xi} = \mu_P \boldsymbol{\xi}, \quad \boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 & \boldsymbol{N}_1^T \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix}$$
$$\boldsymbol{N}_1 = -\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}}, \quad \boldsymbol{N}_2 = \boldsymbol{T}^{-1}, \quad \boldsymbol{N}_3 = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}} - \boldsymbol{Q} \tag{3.31}$$

where ξ is the right eigenvector. By using Eq. (3.25), from Eq. (3.31), it yields $\left[\boldsymbol{U}_{,2}\ \boldsymbol{\Phi}_{,2}\right] = \left[\boldsymbol{U}_{,1}\ \boldsymbol{\Phi}_{,1}\right]\boldsymbol{N}^{\mathrm{T}}$. If multiply on both sides of Eq. (3.31) from left by the matrix

$$\boldsymbol{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{J} = \boldsymbol{J}^{\mathrm{T}} = \boldsymbol{J}^{-1}, \quad \boldsymbol{JN} = (\boldsymbol{JN})^{\mathrm{T}} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{J} \tag{3.32}$$

Eq. (3.30) can be reduced to

$$\boldsymbol{JN\xi} = \boldsymbol{N}^{\mathrm{T}}(\boldsymbol{J\xi}) = \mu(\boldsymbol{J\xi}), \quad \text{or} \quad \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\eta} = \mu\boldsymbol{\eta} = \quad \boldsymbol{\eta} = \boldsymbol{J\xi} = [\boldsymbol{b}, \boldsymbol{a}]^{\mathrm{T}} \tag{3.33}$$

where $\boldsymbol{\eta}$ is the left eigenvector. According to the mathematical theory (Ting 1996), the left and right eigenvectors associated with different eigenvalues are orthogonal to each other. So for the normalized ξ and $\boldsymbol{\eta}$, we have

$$\boldsymbol{\eta}_i^{\mathrm{T}}\boldsymbol{\xi}_j = \delta_{ij}, \quad \text{or} \quad \boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{a}_j + \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{b}_j = \delta_{ij}, \quad \text{when} \quad \mu_i \neq \mu_j \tag{3.34a}$$

From Eq. (3.34a) the following identities can be obtained:

$$\begin{gathered}
\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \bar{\boldsymbol{B}}^{\mathrm{T}}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{B} = \mathbf{I}, \quad \boldsymbol{B}^{\mathrm{T}}\bar{\boldsymbol{A}} + \boldsymbol{A}^{\mathrm{T}}\bar{\boldsymbol{B}} = \bar{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{A} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{B} = \mathbf{0} \\
\boldsymbol{AB}^{\mathrm{T}} + \bar{\boldsymbol{A}}\bar{\boldsymbol{B}}^{\mathrm{T}} = \boldsymbol{BA}^{\mathrm{T}} + \bar{\boldsymbol{B}}\bar{\boldsymbol{A}}^{\mathrm{T}} = \mathbf{I}, \quad \boldsymbol{AA}^{\mathrm{T}} + \bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^{\mathrm{T}} = \boldsymbol{BB}^{\mathrm{T}} + \bar{\boldsymbol{B}}\bar{\boldsymbol{B}}^{\mathrm{T}} = \mathbf{0}; \quad \text{or} \\
\begin{pmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} & \bar{\boldsymbol{A}}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}; \quad \begin{pmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} & \bar{\boldsymbol{A}}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}
\end{gathered} \tag{3.34b}$$

From above equations it is known that $\boldsymbol{AA}^{\mathrm{T}}$ and $\boldsymbol{BB}^{\mathrm{T}}$ are pure imaginary. Let

$$\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} = 2\mathrm{i}\boldsymbol{AA}^{\mathrm{T}}, \quad \boldsymbol{L} = \boldsymbol{L}^{\mathrm{T}} = -2\mathrm{i}\boldsymbol{BB}^{\mathrm{T}}, \quad \boldsymbol{S} = \mathrm{i}\left(2\boldsymbol{AB}^{\mathrm{T}} - \mathbf{I}\right) \tag{3.35}$$

where $\boldsymbol{M}$ and $\boldsymbol{L}$ are real positive definite symmetric matrixes and $\boldsymbol{S}$ is a real matrix. It is easy to prove that there are the following relations:

$$\boldsymbol{LS} + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{L} = \mathbf{0}, \quad \boldsymbol{MS}^{\mathrm{T}} + \boldsymbol{SM} = \mathbf{0}, \quad \boldsymbol{ML} - \boldsymbol{S}\,\boldsymbol{S} = \mathbf{I} \tag{3.36}$$

From Eqs. (3.35) and (3.36), it is known that $\boldsymbol{SL}^{-1}$ and $\boldsymbol{M}^{-1}\boldsymbol{S}$ are antisymmetric matrix. Using $\boldsymbol{AB}^{-1} = \boldsymbol{AB}^{\mathrm{T}}\left(\boldsymbol{BB}^{\mathrm{T}}\right)^{-1}, \boldsymbol{BA}^{-1} = \left(\boldsymbol{AB}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\boldsymbol{AA}^{\mathrm{T}}\right)^{-1}$ and the above equations we get

$$\begin{aligned}
\boldsymbol{Y} &= \mathrm{i}\boldsymbol{AB}^{-1} = -\mathrm{i}(\boldsymbol{S} + \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1} = \mathrm{i}\boldsymbol{L}^{-1}\left(\boldsymbol{S}^{\mathrm{T}} - \mathrm{i}\mathbf{I}\right), \quad \bar{\boldsymbol{Y}}^{\mathrm{T}} = \mathrm{i}\left(\boldsymbol{SL}^{-1}\right)^{\mathrm{T}} + \boldsymbol{L}^{-1} \\
&= -\mathrm{i}\boldsymbol{SL}^{-1} + \boldsymbol{L}^{-1} = \boldsymbol{Y} \\
\boldsymbol{Y}^{-1} &= -\mathrm{i}\boldsymbol{BA}^{-1} = -\mathrm{i}\left(\boldsymbol{S}^{\mathrm{T}} + \mathrm{i}\mathbf{I}\right)\boldsymbol{M}^{-1} = \mathrm{i}\boldsymbol{M}^{-1}(\boldsymbol{S} - \mathrm{i}\mathbf{I}) = \boldsymbol{M}^{-1} + \mathrm{i}\boldsymbol{M}^{-1}\boldsymbol{S}
\end{aligned} \tag{3.37}$$

It is obvious that $\boldsymbol{Y}$ is a Hermite matrix, i.e., $\boldsymbol{Y} = \mathrm{i}\boldsymbol{AB}^{-1} = \bar{\boldsymbol{Y}}^{\mathrm{T}}, \boldsymbol{Y}^{-1} = \bar{\boldsymbol{Y}}^{-\mathrm{T}}$.

3.2.4 Semisimple and Degenerate Matrixes

If the eigenvectors in Eq. (3.11) corresponding to each repeated root λ of multiplicity r in Eq. (3.12) have r independent eigenvectors $\lambda_\nu, \nu = 1, \ldots r$, the corresponding matrix $\boldsymbol{D}$ is called semisimple. The eigen-space is complete for the semisimple matrix. For a semisimple matrix, the eigenvectors associated with a repeated eigenvalue are not unique; however, it is possible to establish a set of eigenvectors such that the orthogonality relations hold and normalized. In this case the general solutions Eqs. (3.16) and (3.18) are also held. A real and symmetric matrix or a complex Hermite matrix is always either simple or semisimple, and their eigenvalues are all real. If the number of the independent eigenvectors is less than the multiplicity of a repeat root, the corresponding matrix is called nonsemisimple or degenerate matrix. The eigen-space is not complete for the degenerate matrix. In order to make the eigen-space of the degenerate matrix complete, we can establish the generalized eigenvectors to provide the missing eigenvectors (Ting 1996). Sometimes in the practical calculation, a very small difference between the repeated roots is assumed to approximately satisfy the eigen-equation.

3.2.5 A General Theory of the Generalized Eigenvectors

The general theory of the generalized eigenvectors for the simple, semisimple, and degenerated matrixes is expressed in the following theorem (Dempsey and Sinclair 1979; Yang et al. 1997):

Theorem *Let μ be the eigenvalue of a square matrix $\boldsymbol{D}(\mu)$ of order n (here $n = 4$) and $\boldsymbol{a}$ be the corresponding eigenvector. If rank of matrix $\boldsymbol{D}$ is $m = n - r < n$, where r is the number of the eigenvectors corresponding to a repeated eigenvalue, and if at $\mu = \mu_p$, we have*

$$\boldsymbol{D}\boldsymbol{a} = 0 \tag{3.38}$$

$$\mathrm{d}(\boldsymbol{D}\boldsymbol{a})/\mathrm{d}\mu = (\mathrm{d}\boldsymbol{D}/\mathrm{d}\mu)\boldsymbol{a} + \boldsymbol{D}(\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu) = \mathbf{0} \tag{3.39}$$

$$\mathrm{d}^2(\boldsymbol{D}\boldsymbol{a})/\mathrm{d}\mu^2 = \left(\mathrm{d}^2\boldsymbol{D}/\mathrm{d}\mu^2\right)\boldsymbol{a} + 2(\mathrm{d}\boldsymbol{D}/\mathrm{d}\mu)(\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu) + \mathbf{D}\boldsymbol{D}\left(\mathrm{d}^2\boldsymbol{a}/\mathrm{d}\mu^2\right) = \mathbf{0} \tag{3.40}$$

In order to get nontrivial solution for $\boldsymbol{a}$ in Eq. (3.38), it must be

$$|\boldsymbol{D}| = 0 \tag{3.41}$$

In order to get nontrivial solution for $\boldsymbol{a}$ and $\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu$ in Eq. (3.39), it must be

$$|\boldsymbol{D}| = \mathrm{d}^{n-m}|\boldsymbol{D}|/\mathrm{d}\mu^{n-m} = 0 \tag{3.42}$$

In order to get nontrivial solution for $\boldsymbol{a}$, $\mathrm{d}\boldsymbol{a}/\mathrm{d}\mu$, and $\mathrm{d}^2\boldsymbol{a}/\mathrm{d}\mu^2$ in Eq. (3.40), it must be

$$|\boldsymbol{D}| = \mathrm{d}^{n-m}|\boldsymbol{D}|/\mathrm{d}\mu^{n-m} = \mathrm{d}^{2(n-m)}|\boldsymbol{D}|/\mathrm{d}\mu^{2(n-m)} = 0 \tag{3.43}$$

1. $\mu_1, \mu_2, \mu_3, \mu_4$ are all single roots: $|\boldsymbol{D}|$ is a polynomial containing first power of μ_j, so $\mathrm{d}|\boldsymbol{D}|/\mathrm{d}\mu_j \neq 0$. In this case $r = 1, m = n - 1 = 3$, and Eq. (3.42) is not satisfied. Equation (3.38) has four independent eigenvectors. The general solution of $\boldsymbol{U}$ is expressed by Eq. (3.16).

2. μ_1 is a repeated root with multiplicity 2 and μ_3, μ_4 are single roots: $|\boldsymbol{D}|$ is a polynomial containing second power of μ_j, so $\mathrm{d}^2|\boldsymbol{D}|/\mathrm{d}\mu_1^2 \neq 0$.

(a) There are two independent eigenvectors corresponding to $\mu_1, m = n - 2 = 2$. In this case Eq. (3.42) is not satisfied. The general solution of $\boldsymbol{U}$ is still expressed by Eq. (3.16).

(b) There is only one independent eigenvectors corresponding to μ_1, $m = n - 1 = 3$. In this case Eq. (3.42) can be satisfied; $\boldsymbol{a}_1$ and $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ in Eq. (3.39) all have nontrivial solutions. The general solution of $\boldsymbol{U}$ can be expressed by

$$\begin{aligned} &\boldsymbol{U} = 2\mathrm{Re}\left[\boldsymbol{A}'f(z_*) + x_2\boldsymbol{a}_1 f_1'(z_1)\right]; \quad \boldsymbol{A}' = [\boldsymbol{a}_1, \mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1, \boldsymbol{a}_3, \boldsymbol{a}_4] \\ &\boldsymbol{f}(z_*) = [f_1(z_1), f_1(z_1), f_3(z_3), f_4(z_4)]^{\mathrm{T}} \end{aligned} \tag{3.44}$$

where $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ is solved from Eq. (3.39).

3. μ_1 is a repeated root with multiplicity 3 and μ_3 is a single root: $|\boldsymbol{D}|$ is a polynomial containing third power of μ_j, so $\mathrm{d}^3|\boldsymbol{D}|/\mathrm{d}\mu_1^3 \neq 0$.

(a) There are three independent eigenvectors corresponding to $\mu_1, m = n - 3 = 1$. In this case it is still that only Eq. (3.41) has nontrivial solution. The general solution of $\boldsymbol{U}$ is still expressed by Eq. (3.16).

(b) There are two independent eigenvectors corresponding to $\mu_1, m = n - 2 = 2$. In this case

Eq. (3.42) can be satisfied; $\boldsymbol{a}_1$ and $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ in Eq. (3.39) have nontrivial solutions. The general solution of $\boldsymbol{U}$ can still be expressed by (3.44).

(c) There is only one independent eigenvector corresponding to $\mu_1, m = n - 1 = 3$. In this case Eq. (3.43) is satisfied. $\boldsymbol{a}_1$, $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$, and $\mathrm{d}^2\boldsymbol{a}_1/\mathrm{d}\mu_1^2$ all have nontrivial solutions. The general solution of $\boldsymbol{U}$ can be expressed by

$$\begin{aligned} &\boldsymbol{U} = 2\mathrm{Re}\left[\boldsymbol{A}''f(z_*) + x_2\boldsymbol{a}_1 f_1'(z_1) + 2x_2(\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1)f_1'(z_1) + x_2^2\boldsymbol{a}_1 f_1''(z_1)\right]; \\ &\boldsymbol{A}'' = \left[\boldsymbol{a}_1, \mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1, \boldsymbol{a}_3, \mathrm{d}^2\boldsymbol{a}_1/\mathrm{d}\mu_1^2\right], \quad \boldsymbol{f}(z_P) = [f_1(z_1), f_1(z_1), f_3(z_3), f_1(z_1)]^{\mathrm{T}} \end{aligned} \tag{3.45}$$

where $\mathrm{d}\boldsymbol{a}_1/\mathrm{d}\mu_1$ is solved from Eq. (3.40).

3.2.6 Electric Displacement Tensor Method

The fourth kind of the constitutive equations in Eq. (2.83) is

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - h_{nij}D_n, \quad E_i = -h_{imn}\varepsilon_{mn} + \beta_{in}D_n \tag{3.46}$$

Shen and Kuang (1999a) introduced an antisymmetric tensor $\boldsymbol{G}$ of second order and a vector potential $\boldsymbol{\psi}$ of the electric displacement to satisfy $\nabla \cdot \boldsymbol{D} = 0$ automatically and let

$$D_i = \varpi_{imn}G_{mn}, \quad G_{ij} = \left(\psi_{i,j} - \psi_{j,i}\right)/2, \quad G_{ij} = -G_{ji}; \quad \psi_{i,i} = 0 \tag{3.47}$$

where $\nabla \cdot \boldsymbol{\Psi} = 0$ is the condition to make $\boldsymbol{\Psi}$ unique and $\boldsymbol{\varpi}$ is a permutation tensor:

$$\varpi_{123} = \varpi_{231} = \varpi_{312} = 1, \quad \varpi_{213} = \varpi_{132} = \varpi_{321} = -1, \quad \text{otherwise} \quad \varpi_{ijk} = 0 \tag{3.48}$$

Introduce the electric tensor $\boldsymbol{L}$:

$$E_i = (1/2)\varpi_{imn}L_{mn}, \quad L_{mn} = \varpi_{imn}E_i = -L_{nm} \tag{3.49}$$

Using $E_{i,j} = E_{j,i}$ from Eq. (3.49), we get

$$L_{ij,j} = \varpi_{mij}E_{m,j} = \varpi_{ijm}E_{m,j} = \varpi_{imj}\varphi_{,mj} = 0 \tag{3.50}$$

Using Eqs. (3.47) and (3.49), the constitutive equations Eq. (3.46) can be written as

$$\begin{aligned}
&\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - h_{tij}\varpi_{tmn}G_{mn} = C_{ijmn}\varepsilon_{mn} - \overline{h}_{mnij}G_{mn} \\
&L_{ij} = \varpi_{tij}E_t = \varpi_{tij}\left(-h_{tmn}\epsilon_{mn} + \beta_{tn}\varpi_{npq}\varepsilon_{npq}G_{pq}\right) = -\overline{h}_{ijmn}\varepsilon_{mn} + \overline{\beta}_{ijmn}G_{mn} \\
&\overline{h}_{mnij} = \overline{h}_{mnji} = -\overline{h}_{nmij} = \varpi_{tmn}h_{tij} \\
&\overline{\beta}_{ijmn} = \overline{\beta}_{mnij} = -\overline{\beta}_{jimn} = -\overline{\beta}_{ijnm}\varpi_{tij}\varpi_{npq}\beta_{tn}
\end{aligned} \tag{3.51}$$

Using Eqs. (3.47) and (3.51), the equations $\nabla \cdot \sigma = 0$, $\boldsymbol{E} = -\nabla\varphi$ can be written as

$$\left(C_{ijkl}u_{k,l} - \overline{h}_{klij}\psi_{k,l}\right)_{,j} = 0, \quad \left(\overline{h}_{ijkl}u_{k,l} - \overline{\beta}_{ijkl}\psi_{k,l}\right)_{,j} = 0 \tag{3.52a}$$

When material coefficients are all constants for the general plane problem, Eq. (3.52a) becomes

$$U_{i\alpha\beta,\alpha\beta} = 0, \quad L_{i\alpha\beta,\alpha\beta} = 0; \quad U_{i\alpha\beta} = C_{i\beta k\alpha}u_k - \overline{h}_{k\alpha i\beta}\psi_k, \quad L_{i\alpha\beta} = \overline{h}_{i\beta k\alpha}u_k - \overline{\beta}_{i\beta k\alpha}\psi_k \tag{3.52b}$$

Equation (3.52) is a pretty equation. How to use it in engineering should be studied in the future.

3.3 Stress Function Method

3.3.1 Solution for a General Piezoelectric Material

Using the Voigt notation, the third kind of the constitutive equations in Eq. (2.84) is

$$\varepsilon_i = s_{ij}\sigma_j + g_{\alpha i}D_\alpha, \quad E_\alpha = -g_{\alpha j}\sigma_j + \beta_{\alpha\beta}D_\beta; \quad i,j = 1-6;\ \alpha,\beta = 1-3 \tag{3.53}$$

In this section a subscript in English letter takes the values $1-6$ and a subscript in Greek letter takes the values $1-3$. In the general plane strain problem, $u_{\alpha,3} = 0$ and

$$\varepsilon_3 = u_{3,3} = s_{3j}\sigma_j + g_{\alpha 3}D_\alpha = 0, \quad E_3 = -\varphi_{,3} = -g_{3j}\sigma_j + \beta_{3\alpha}D_\alpha = 0 \tag{3.54}$$

Solving σ_3 and D_3 from Eq. (3.54) yields

$$\begin{aligned}
&\sigma_3 = F_j\sigma_j + G_\alpha D_\alpha, \quad D_3 = H_j\sigma_j + J_\alpha D_\alpha, \quad (j,\alpha \neq 3)\\
&F_j = -\left(g_{33}g_{3j} + s_{3j}\beta_{33}\right)M, \quad G_\alpha = (g_{33}\beta_{3\alpha} - g_{\alpha 3}\beta_{33})M,\\
&H_j = \left(s_{33}g_{3j} - s_{3j}g_{33}\right)M, \quad J_\alpha = -(s_{33}\beta_{3\alpha} + g_{33}g_{\alpha 3})M, \quad M = 1/\left(g_{33}^2 + s_{33}\beta_{33}\right)
\end{aligned} \tag{3.55}$$

Substitution of Eq. (3.55) into Eq. (3.53) yields

$$\begin{aligned}
&u_{1,1} = \kappa_{1j}\sigma_j + \eta_{1\alpha}D_\alpha, \quad u_{2,2} = \kappa_{2j}\sigma_j + \eta_{2\alpha}D_\alpha, \quad u_{3,2} = \kappa_{4j}\sigma_j + \eta_{4\alpha}D_\alpha,\\
&u_{3,1} = \kappa_{5j}\sigma_j + \eta_{5\alpha}D_\alpha, \quad u_{2,1} + u_{1,2} = \kappa_{6j}\sigma_j + \eta_{6\alpha}D_\alpha,\\
&E_1 = -h_{1j}\sigma_j + \xi_{1\alpha}D_\alpha, \quad E_2 = -h_{2j}\sigma_j + \xi_{2\alpha}D_\alpha \quad (j,\alpha \neq 3)
\end{aligned} \tag{3.56}$$

where the reduced constants $\kappa_{ij}, \eta_{i\alpha}, h_{\alpha j}, \xi_{\beta\alpha}$ are

$$\begin{aligned}
&\kappa_{ij} = s_{ij} + s_{i3}F_j + g_{3i}H_j = \kappa_{ji}, \quad \eta_{j\alpha} = g_{\alpha j} + s_{j3}G_\alpha + g_{3j}J_\alpha,\\
&h_{\alpha j} = g_{\alpha j} + g_{\alpha 3}F_j - \beta_{\alpha 3}H_j = \eta_{j\alpha}, \quad \xi_{\beta\alpha} = \beta_{\beta\alpha} - g_{\beta 3}G_\alpha + \beta_{\beta 3}J_\alpha
\end{aligned} \tag{3.57}$$

Applying Lekhnitskii method (1987, 1957), Kosmodamianskii and Lozhkin (1975) discussed the plane stress state of thin piezoelectric plates and gave the expressions with complex potentials. Hao and Shen (1994) and Huang and Kuang (2000a) discussed the general generalized plane problem. They introduced the stress functions Λ, Ψ and the electric potential V to satisfy the generalized equilibrium equations automatically:

$$\sigma_1 = \Lambda_{,22}, \quad \sigma_2 = \Lambda_{,11}, \quad \sigma_6 = -\Lambda_{,12}, \quad \sigma_4 = -\Psi_{,1}, \quad \sigma_5 = \Psi_{,2}; \quad D_1 = V_{,2}, \quad D_2 = -V_{,1} \tag{3.58}$$

Substitution of Eq. (3.58) into Eq. (3.56) finally yields the general compatibility equation

$$L_4\Lambda + L_3\Psi + L_5V = 0, \quad L_3\Lambda + L_2\Psi + L_6V = 0, \quad L_5\Lambda + L_6\Psi - L_9V = 0 \tag{3.59}$$

where

$$\begin{aligned}
L_2 &= \kappa_{55}\frac{\partial^2}{\partial x_2^2} - 2\kappa_{45}\frac{\partial^2}{\partial x_2\partial x_1} + \kappa_{44}\frac{\partial^2}{\partial x_1^2}\\
L_3 &= \kappa_{15}\frac{\partial^3}{\partial x_2^3} - (\kappa_{14}+\kappa_{56})\frac{\partial^3}{\partial x_2^2\partial x_1} + (\kappa_{55}+\kappa_{46})\frac{\partial^3}{\partial x_2\partial x_1^2} - \kappa_{24}\frac{\partial^3}{\partial x_1^3}\\
L_4 &= \kappa_{11}\frac{\partial^4}{\partial x_2^4} - 2\kappa_{16}\frac{\partial^4}{\partial x_2^3\partial x_1} + (2\kappa_{12}+\kappa_{66})\frac{\partial^4}{\partial x_2^2\partial x_1^2} - 2\kappa_{26}\frac{\partial^4}{\partial x_2\partial x_1^3} + \kappa_{22}\frac{\partial^4}{\partial x_1^4}\\
L_5 &= \eta_{11}\frac{\partial^3}{\partial x_2^3} - (\eta_{12}+\eta_{61})\frac{\partial^3}{\partial x_2^2\partial x_1} + (\eta_{21}+\eta_{62})\frac{\partial^3}{\partial x_2\partial x_1^2} + \eta_{22}\frac{\partial^3}{\partial x_1^3}\\
L_6 &= \eta_{51}\frac{\partial^2}{\partial x_2^2} - (\eta_{52}+\eta_{41})\frac{\partial^2}{\partial x_2\partial x_1} + \eta_{42}\frac{\partial^2}{\partial x_1^2}\\
L_7 &= -L_5 = -h_{11}\frac{\partial^3}{\partial x_2^3} + (h_{16}+h_{21})\frac{\partial^3}{\partial x_2^2\partial x_1} - (h_{12}+h_{26})\frac{\partial^3}{\partial x_2\partial x_1^2} - h_{22}\frac{\partial^3}{\partial x_1^3}\\
L_8 &= -L_6 = -h_{15}\frac{\partial^2}{\partial x_2^2} + (h_{14}+h_{25})\frac{\partial^2}{\partial x_2\partial x_1} - h_{24}\frac{\partial^2}{\partial x_1^2}\\
L_9 &= \xi_{11}\frac{\partial^2}{\partial x_2^2} - (\xi_{12}+\xi_{21})\frac{\partial^2}{\partial x_2\partial x_1} + \xi_{22}\frac{\partial^2}{\partial x_1^2}
\end{aligned} \tag{3.60}$$

Eliminating Ψ and V from Eq. (3.59) yields an eighth-order differential equation of Λ:

$$\begin{aligned}
&\left(L_6L_8L_4 - L_9L_4L_2 + L_9L_3^2 - L_5L_3L_8 + L_2L_5L_7 - L_3L_6L_7\right)\Lambda = 0, \quad \text{or}\\
&\left(L_6L_8L_4 - L_9L_4L_2 + L_9L_3^2 + 2L_5L_3L_6 + L_2L_5L_7\right)\Lambda = 0
\end{aligned} \tag{3.61}$$

Its solution is

$$\Lambda = 2\mathrm{Re}\sum_{P=1}^{4}\tilde{f}_P(z_P), \quad z_P = x_1 + \mu_P x_2 \tag{3.62}$$

where $\tilde{f}_P(z_P)$ is an analytic function of z_P and μ_P is the root of the following eigen-equation

$$\begin{aligned}
&l_6l_8l_4 - l_9l_4l_2 + l_9l_3^2 - l_5l_3l_8 + l_2l_5l_7 - l_3l_6l_7 = 0, \quad \text{or}\\
&l_4l_6^2 + l_2l_4l_9 - l_3^2l_9 - 2l_3l_5l_6 + l_2l_5^2 = 0
\end{aligned} \tag{3.63}$$

where $l_i \equiv l_i(\mu)$ can be obtained by using μ instead of the differential operator $\partial/\partial x_2$ in L_i of Eq. (3.60) and the power of μ is the same as the power of $\partial/\partial x_2$, such as

$$l_3(\mu) = \kappa_{15}\mu^3 - (\kappa_{14} + \kappa_{56})\mu^2 + (\kappa_{55} + \kappa_{46})\mu - \kappa_{24}$$

where μ_P is the same as that in Stroh's formula. From Eq. (3.59), it is obtained that

$$\begin{gathered}
\Psi = 2\mathrm{Re}\sum_{p=1}^{4} a_P f_P(z_P), \quad V = 2\mathrm{Re}\sum_{p=1}^{4} b_P f_P(z_P), \quad f_P(z_P) = d\tilde{f}'_p/dz_P \\
a_P = -\frac{l_5 b_P + l_4}{l_3} = -\frac{l_6 b_P + l_3}{l_2} = -\frac{l_9 b_P + l_7}{l_8}, \quad b_P = \frac{l_3^2 - l_4 l_2}{l_2 l_5 - l_3 l_6} = \frac{l_8 l_3 - l_2 l_7}{l_2 l_9 - l_8 l_6}
\end{gathered} \tag{3.64}$$

where $l_i \equiv l_i(\mu_P)$. Substitution of Λ, Ψ and V into Eq. (3.58) yields the generalized stress; then substitution of the result into Eq. (3.56) yields the generalized displacements. Comparing the generalized stress and displacement with that in Stroh's formula, the explicit forms of $\boldsymbol{B}$ and $\boldsymbol{A}$ are obtained:

$$B = \begin{bmatrix} -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 \\ 1 & 1 & 1 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 \\ -b_1 & -b_2 & -b_3 & -b_4 \end{bmatrix}, \quad \mathbf{A} = [A_{ij}] \tag{3.65}$$

$$\begin{aligned}
A_{1j} &= \kappa_{11}\mu_j^2 + \kappa_{12} - \kappa_{16}\mu_j + a_j(\kappa_{15}\mu_j - \kappa_{14}) + (\eta_{11}\mu_j - \eta_{12})b_j \\
A_{2j} &= \left[\kappa_{21}\mu_j^2 + \kappa_{22} - \kappa_{26}\mu_j + a_j(\kappa_{25}\mu_j - \kappa_{24}) + (\eta_{21}\mu_j - \eta_{22})b_j\right]\Big/\mu_j \\
A_{3j} &= \left[\kappa_{41}\mu_j^2 + \kappa_{42} - \kappa_{46}\mu_j + a_j(\kappa_{45}\mu_j - \kappa_{44}) + (\eta_{41}\mu_j - \eta_{42})b_j\right]\Big/\mu_j \\
A_{4j} &= h_{11}\mu_j^2 + h_{12} - h_{16}\mu_j + a_j(h_{15}\mu_j - h_{14}) + (\xi_{11}\mu_j - \xi_{12})b_j
\end{aligned} \tag{3.66}$$

The above results are obtained for the generalized plane strain. For the generalized plane stress, the constants should be simply replaced by

$$\kappa_{ij} = s_{ij} = \kappa_{ji}, \quad \eta_{j\alpha} = g_{\alpha j}, \quad h_{\alpha j} = g_{\alpha j}, \quad \xi_{\beta\alpha} = \beta_{\beta\alpha} \quad (j, \alpha, \beta \neq 3) \tag{3.67}$$

It is also noted that the plane stress deformation can be existed only in the materials with at least one material symmetric plane such as monoclinic material.

From Eq. (3.58), the stress functions can be obtained as

$$\begin{aligned}
\Phi_1 &= -\Lambda_{,2} = -2\mathrm{Re}\sum_{P=1}^{4} \mu_P f_P(z_P), \quad \Phi_2 = \Lambda_{,1} = 2\mathrm{Re}\sum_{P=1}^{4} f_P(z_P) \\
\Phi_3 &= -\Psi = -2\mathrm{Re}\sum_{P=1}^{4} a_P f_P(z_P), \quad \Phi_4 = -V = -2\mathrm{Re}\sum_{P=1}^{4} b_P f_P(z_P)
\end{aligned} \tag{3.68}$$

Equation (3.68) shows that $\boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)]$ where $\boldsymbol{B}$ is shown in Eq. (3.65). This is consistent with the Stroh's formula Eq. (3.23).

3.3.2 The Transversely Isotropic Material in Plane Strain

Usually in engineering the coordinate system x-y-z is used, and the material constants given in the handbooks are under the condition that the poling direction is along the axis z. For a general piezoelectric material, there are 45 independent material constants: 21 elastic constants, 18 piezoelectric constants, and 6 permittivity constants. For the orthogonal materials in the material principle coordinate system with poling axis z, the plane x-y is an isotropic plane. For an isotropic plane x-y, the in-plane electric field couples only with the out-plane mechanical stress. In the anisotropic plane x-z, y-z, the in-plane electric field couples with the in-plane mechanical stress, and the mechanical behaviors in x-z and y-z planes are the same. In this case when the axis z is taken as the poling axis, there are 17 independent material constants: 9 elastic constants, $s_{11}, s_{12}, s_{13}, s_{22}, s_{23}, s_{33}, s_{44}, s_{55}, s_{66}$ or $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55}, C_{66}$; 5 piezoelectric constants, g_{15}, $g_{24}, g_{31}, g_{32}, g_{33}$ or $e_{15}, e_{24}, e_{31}, e_{32}, e_{33}$; and 3 electric constants, $\beta_{11}, \beta_{22}, \beta_{33}$ or ϵ_{11}, $\epsilon_{22}, \epsilon_{33}$. The second kind of the constitutive equation in Eq. (2.84) is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \\ D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 & 0 & 0 & -e_{32} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & -e_{24} & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 & -e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 & 0 & \epsilon_{22} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 & 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ E_x \\ E_y \\ E_z \end{Bmatrix} \tag{3.69a}$$

The third kind of the constitutive equation in Eq. (2.84) is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ E_x \\ E_y \\ E_z \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 & 0 & 0 & g_{31} \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 & 0 & 0 & g_{32} \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 & 0 & 0 & g_{33} \\ 0 & 0 & 0 & s_{44} & 0 & 0 & 0 & g_{24} & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 & g_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_{15} & 0 & \beta_{11} & 0 & 0 \\ 0 & 0 & 0 & -g_{24} & 0 & 0 & 0 & \beta_{22} & 0 \\ -g_{31} & -g_{32} & -g_{33} & 0 & 0 & 0 & 0 & 0 & \beta_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \\ D_x \\ D_y \\ D_z \end{Bmatrix} \tag{3.69b}$$

It is noted that $[\boldsymbol{s}] = [\boldsymbol{C}]^{-1}$. For the transversely isotropic material, such as piezoelectric ceramic PZT and many other materials, in the material principle coordinate system, there are ten independent material constants because there are relations between material constants:

$$\begin{aligned} & s_{13}=s_{23}, s_{11}=s_{22}, s_{44}=s_{55}, s_{66}=2(s_{11}-s_{12}); \quad g_{31}=g_{32}, g_{15}=g_{24}; \quad \beta_{11}=\beta_{22} \\ & C_{13}=C_{23}, C_{11}=C_{22}, C_{44}=C_{55}, C_{66}=(C_{11}-C_{12})/2; \quad e_{31}=e_{32}, e_{15}=e_{24}; \quad \epsilon_{11}=\epsilon_{22} \end{aligned} \tag{3.70}$$

In this section the plane strain problem is discussed and adopted the third kind of the constitutive equation, Eq. (3.69b). Let

$$\varepsilon_x = \gamma_{zx} = \gamma_{xy} = E_x = 0 \tag{3.71}$$

From Eq. (3.71), it can be obtained that

$$D_x = 0, \quad \tau_{zx} = \tau_{yz} = 0, \quad \sigma_x = -\left(s_{12}\sigma_y + s_{13}\sigma_z + g_{31}D_z\right)/s_{11} \tag{3.72}$$

Analogous to the Voigt expression of the stress and strain in 3D case, we introduce the vector expression of the stress and strain in plane strain case. Let

$$\begin{aligned} & x_1 = y,\ x_2 = z,\ x_3 = x; \quad \sigma_1 = \sigma_y,\ \sigma_2 = \sigma_z,\ \sigma_3 = \tau_{yz}; \\ & \varepsilon_1 = \varepsilon_y,\ \varepsilon_2 = \varepsilon_z,\ \varepsilon_3 = \gamma_{yz}; \quad D_1 = D_y,\ D_2 = D_z; \quad E_1 = E_y, E_2 = E_z \end{aligned} \tag{3.73}$$

Substitution of Eqs. (3.71), (3.72), and (3.73) into Eq. (3.69) yields

$$\begin{aligned} & \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ E_1 \\ E_2 \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & b_{21} \\ a_{12} & a_{22} & 0 & 0 & b_{22} \\ 0 & 0 & a_{33} & b_{13} & 0 \\ 0 & 0 & -b_{13} & k_{11} & 0 \\ -b_{21} & -b_{22} & 0 & 0 & k_{22} \end{pmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ D_1 \\ D_1 \end{Bmatrix} \\ & a_{11} = s_{11} - s_{12}^2/s_{11}, \quad a_{12} = s_{13} - s_{12}s_{13}/s_{11}, \quad a_{22} = s_{33} - s_{13}^2/s_{11}, \quad a_{33} = s_{44} \\ & b_{21} = (1 - s_{12}/s_{11})g_{31}, \quad b_{22} = g_{33} - g_{31}s_{13}/s_{11}, \quad b_{13} = g_{15}, \quad k_{11} = \beta_{11}, \\ & k_{22} = \beta_{33} + g_{31}^2/s_{11} \end{aligned} \tag{3.74}$$

where a_{ij}, b_{ij}, k_{ij} are reduced material constants and $s_{ij}g_{ij}, \beta_{ij}$ are material constants as shown in Eq. (3.69b). In the plane strain problem, $\sigma_{13} = \sigma_{23} = 0$, so the stress function Ψ in Eq. (3.58) is not needed. The eighth-order differential equation (3.61) is reduced to sixth-order differential equation, and the eighth-order eigen-equation (3.63) is reduced to sixth-order eigen-equation. Repeating the process analogous to Sect. 3.3.1 finally yields (Sosa 1991; Sosa and Khutoryansky 1996; Kuang and Ma 2002)

$$\boldsymbol{U}=\boldsymbol{U}(u_1,u_2,\varphi)=2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z_P)],\quad \boldsymbol{\Phi}=\boldsymbol{\Phi}(\Phi_1,\Phi_2,\Phi_4)=2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)]$$
$$\boldsymbol{A}=\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 \end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix} -\mu_1 & -\mu_2 & -\mu_3 \\ 1 & 1 & 1 \\ -\eta_1 & -\eta_2 & -\eta_3 \end{pmatrix} \tag{3.75}$$

where μ_P is the root of the following eigen-equation:

$$a_{11}k_{11}\mu^6+\left(a_{11}k_{22}+2a_{12}k_{11}+a_{33}k_{11}+b_{21}^2+b_{13}^2+2b_{21}b_{13}\right)\mu^4$$
$$+\left(a_{22}k_{11}+2a_{12}k_{22}+a_{33}k_{22}+2b_{21}b_{22}+2b_{22}b_{13}\right)\mu^2+a_{22}k_{22}+b_{22}^2=0 \tag{3.76}$$

and

$$\begin{aligned} p_P &= a_{11}\mu_P^2+a_{12}-b_{21}\eta_P,\quad q_P=\left(a_{12}\mu_P^2+a_{22}-b_{22}\eta_P\right)/\mu_P \\ \lambda_P &= (b_{13}+k_{11}\eta_P)\mu_P,\quad \lambda_P\mu_P=-\left(b_{21}\mu_P^2+b_{22}+k_{22}\eta_P\right) \\ \eta_P &= -\left[(b_{21}+b_{13})\mu_P^2+b_{22}\right]/\left(k_{11}\mu_P^2+k_{22}\right) \end{aligned} \tag{3.77}$$

If the rigid rotation angle ω is considered, we have

$$\begin{aligned} &u_1=2\mathrm{Re}\sum_{P=1}^{3}p_Pf_P(z_P)-\omega x_3,\quad u_2=2\mathrm{Re}\sum_{P=1}^{3}q_Pf_P(z_P)+\omega x_1, \\ &\varphi=-2\mathrm{Re}\sum_{P=1}^{3}\lambda_Pf_P(z_P);\quad \Phi_1=-2\mathrm{Re}\sum_{P=1}^{3}\mu_Pf_P(z_P), \\ &\Phi_2=2\mathrm{Re}\sum_{P=1}^{3}f_P(z_P),\quad \Phi_4=-2\mathrm{Re}\sum_{P=1}^{3}\eta_Pf_P(z_P) \\ &\sigma_1=2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2F_j(z_j),\quad \sigma_2=2\mathrm{Re}\sum_{j=1}^{3}F_j(z_j),\quad \sigma_3=-2\mathrm{Re}\sum_{j=1}^{3}\mu_jF_j(z_j) \\ &D_1=2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_jF_j(z_j),\quad D_2=-2\mathrm{Re}\sum_{j=1}^{3}\eta_jF_j(z_j) \end{aligned} \tag{3.78}$$

3.4 An Elliptic Hole or Inclusion in a Transversely Isotropic Piezoelectric Material

3.4.1 Electrical Permeable Hole

Let a transversely isotropic piezoelectric material with an elliptic hole of semiaxes a and b directed along the material principle axes x_1 and x_2, respectively, be subjected to the uniform generalized stresses at infinity. The hole is filled with air with

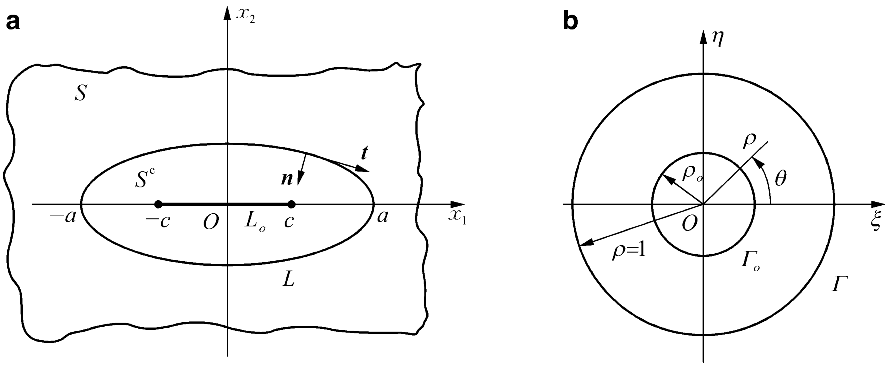


Fig. 3.3 An infinite plane with an elliptic hole: (**a**) physical plane z and (**b**) mapping plane ζ

permittivity ϵ_0 and is mechanically free (Fig. 3.3) (Parton 1976; Sosa and Khutoryansky 1996; Chung and Ting 1996; Zhang et al. 1998; Gao and Fan 1999; Kuang and Ma 2002). The potential electric field and electric displacement in the air region S^c are denoted by $\varphi^c, \boldsymbol{E}^c$ and $\boldsymbol{D}^c$, respectively, and in the piezoelectric material S are denoted by $\varphi, \boldsymbol{E}$ and $\boldsymbol{D}$, respectively. On the interface L, the outward normal is denoted by $\boldsymbol{n}$ directed from the material into the hole, and the first natural coordinate system (see Eqs. (3.27), (3.28)) is adopted. In the air only electric field is researched. Therefore the boundary conditions at infinity are

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{D} = \boldsymbol{D}^\infty \tag{3.79}$$

On the interface the connective conditions are

$$T_1 = T_2 = 0; \quad D_n = D_n^c = -\epsilon_0 \partial\varphi^c/\partial n, \quad \varphi = \varphi^c \quad \text{on} \quad L \tag{3.80}$$

The method solving this problem is the direct extension for the inclusion problem in an elastic anisotropic material (Mura 1987).

3.4.2 Electric Field Inside the Hole Filled with Air

It is assumed that there is free of charge in the air; from $\nabla \cdot \boldsymbol{D}^c = 0$, $\boldsymbol{D}^c = \epsilon_0 \boldsymbol{E}^c = -\epsilon_0 \boldsymbol{\nabla}\varphi^c$, the governing equation is

$$\nabla^2 \varphi^c = 0, \quad \text{in} \quad S^c \tag{3.81}$$

The conformal mapping function $\omega(\varsigma)$, transforming an ellipse L in the physical plane $z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta}$ into a unit circle Γ in the mapping plane $\varsigma = \xi_1 + \mathrm{i}\xi_2 = \rho\mathrm{e}^{\mathrm{i}\psi}$, is

$$z=\omega(\varsigma)=R\left(\varsigma+\frac{m}{\varsigma}\right),\quad \varsigma=\frac{z+\sqrt{z^2-4mR^2}}{2R};\quad R=\frac{a+b}{2},\quad m=\frac{a-b}{a+b}\quad \text{or}$$
$$z=R(1+m)\cos\psi+\mathrm{i}R(1-m)\sin\psi=a\cos\psi+\mathrm{i}b\sin\psi;\quad \text{or}\quad x_1=a\cos\psi,$$
$$x_2=b\sin\psi \tag{3.82a}$$

where (r,θ) and (ρ,ψ) are the polar axes and polar angles in the z and ς planes, respectively. Mapping function $\omega(\varsigma)$ transforms L into Γ and one to one for points outside the ellipse L into the outside of Γ, however, only one to one for points inside the ellipse L with a cut L_0 from $-c$ to c on the major axis in z plane into the inside of Γ with a circular cut L_0 of radius ρ_0 in ς plane (Fig. 3.3), where $\rho_0=\sqrt{m}<|z|<1,\ 0\le\theta<2\pi$ and $c=\sqrt{a^2-b^2}$ is the half of the focal length.

From Eq. (3.82a), it is known that the arc lengths on Γ and on L are respectively

$$\mathrm{d}l^2=\mathrm{d}\varsigma\mathrm{d}\bar{\varsigma}=\mathrm{d}e^{\mathrm{i}\psi}\mathrm{d}e^{-\mathrm{i}\psi}=\mathrm{d}\psi^2;\quad \text{on } \Gamma$$
$$\mathrm{d}s^2=\mathrm{d}z\mathrm{d}\bar{z}=\omega'(\varsigma)\overline{\omega'(\varsigma)}\mathrm{d}\varsigma\mathrm{d}\bar{\varsigma}=\rho^2\mathrm{d}\psi^2,\quad \rho^2=a^2\sin^2\psi+b^2\cos^2\psi;\quad \text{on}\quad L$$
$$\mathrm{d}x_2/\mathrm{d}s=b\cos\psi/\rho,\ \mathrm{d}x_1/\mathrm{d}s=-a\sin\psi/\rho \tag{3.82b}$$

Because φ^c is a harmonic function, it can be expressed by an analytic function $\phi(z)$ as

$$\varphi^c(x_1,x_2)=\phi(z)+\overline{\phi(z)},\quad \text{in } z \text{ plane};\quad \varphi^c(\rho,\psi)=\phi(\varsigma)+\overline{\phi(\varsigma)},\quad \text{in } \zeta \text{ plane}$$
$$E_1^c=-\left[\phi'(z)+\overline{\phi'(z)}\right]=-2\mathrm{Re}\phi'(z),\quad E_2^c=-\mathrm{i}\left[\phi'(z)-\overline{\phi'(z)}\right]=2\mathrm{Im}\phi'(z) \tag{3.83}$$

where $\phi(\varsigma)=\phi[\omega(\varsigma)]$. Because $\phi(z)$ is analytic inside $L-L_0$ and continuous on L_0, so

$$\phi(\rho_0 e^{\mathrm{i}\psi})=\phi(\rho_0 e^{-\mathrm{i}\psi}) \tag{3.84}$$

The solution of $\phi(\varsigma)$ in the annular region $L-L_0$ can be expressed in the Laurent series

$$\phi(\varsigma)=\sum_{k=-\infty}^{\infty}a_k^c\varsigma^k,\quad a_{-k}^c=\rho_0^{2k}a_k^c\ (\text{no sum on } k),\quad \rho_0\le|\varsigma|\le 1 \tag{3.85}$$

3.4.3 Generalized Stress Field in the Piezoelectric Material

The general solution in a transversely isotropic material has been given in Sect. 3.3.2. The mapping function $\omega_j(\varsigma), j=1,2,3$ transforming an ellipse L_j in the physical plane $z_j=x_1+\mu_j x_2$ into a unit circle Γ_j in a mapping plane $\varsigma_j=\xi_1+\mu_j\xi_2$ is

$$\begin{aligned}
&z_j = \omega_j(\varsigma_j) = c_j\varsigma_j + d_j\varsigma_j^{-1} = R_j\left(\varsigma_j + m_j\varsigma_j^{-1}\right)\\
&R_j = c_j = \left(a - \mathrm{i}\mu_j b\right)/2, \quad d_j = \left(a + \mathrm{i}\mu_j b\right)/2, \quad m_j = d_j/c_j\\
&\varsigma_j = \frac{z_j + \sqrt{z_j^2 - \left(a^2 + \mu_j^2 b^2\right)}}{a - \mathrm{i}\mu_j b}, \quad \frac{1}{\varsigma_j} = \frac{z_j - \sqrt{z_j^2 - (a^2 + \mu_j^2 b^2)}}{a + \mathrm{i}\mu_j b}
\end{aligned} \tag{3.86}$$

When $\mu_j = \mathrm{i}$, Eq. (3.86) is reduced to Eq. (3.82), and the mapping function is conformal. If $\mu_j \neq \mathrm{i}$, the mapping function is not conformal. Because a function $f_j(z_j)$ outside L is analytically transformed into the region outside Γ in the ς plane, so

$$f_j(z_j) = C_j z_j + f_j^0(\varsigma_j), \quad f_j^0(\varsigma_j) = a_{j0} + \sum_{k=1}^{\infty} a_{jk}\varsigma_j^{-k} \quad (\text{not summed on } k) \tag{3.87}$$

where $f_j^0(z_j) = f_j^0[\omega(\varsigma_j)]$ is an analytic function in ς_j plane. C_j is determined by the boundary conditions at infinity and a_{jk} is undetermined coefficient. From Eq. (3.78),

$$\begin{aligned}
&\sigma_1^\infty = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2 C_j, \quad \sigma_2^\infty = 2\mathrm{Re}\sum_{j=1}^{3} C_j, \quad \sigma_3^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j C_j\\
&D_1^\infty = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_j C_j, \quad D_2^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j C_j\\
&\left(E_1 = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j C_j, \quad E_2 = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j\mu_j C_j\right)
\end{aligned} \tag{3.88}$$

In Eq. (3.88) if one real constant is selected arbitrarily, such as let Im $C_1 = 0$, it does not affect the stresses. So Eq. (3.88) is solvable.

3.4.4 The Connective Conditions on the Interface L

Equations (3.75) and (3.28) yield

$$\begin{aligned}
&\Phi_1 = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j f_j(\sigma) = -\int_0^s T_1^*\mathrm{d}s = 0, \quad \Phi_2 = 2\mathrm{Re}\sum_{j=1}^{3} f_j(\sigma) = -\int_0^s T_2^*\mathrm{d}s = 0\\
&\Phi_4 = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j(\sigma) = -\int_0^s D_n\mathrm{d}s
\end{aligned} \tag{3.89}$$

where $\sigma = \mathrm{e}^{\mathrm{i}\psi}$ is a point on Γ. The mechanical connective conditions in Eq. (3.89) can be reduced to

$$\sum_{j=1}^{3}\left\{f_j^0(\sigma)+\overline{f_j^0(\sigma)}\right\}=\bar{l}_1\sigma+l_1\bar{\sigma},\quad \sum_{j=1}^{3}\left\{\mu_j f_j^0(\sigma)+\overline{\mu_j f_j^0(\sigma)}\right\}=\bar{l}_2\sigma+l_2\bar{\sigma}$$
$$l_1=-(1/2)\left(a\sigma_2^\infty-\mathrm{i}b\sigma_3^\infty\right),\quad l_2=(1/2)\left(a\sigma_3^\infty-\mathrm{i}b\sigma_1^\infty\right) \tag{3.90}$$

Using Eq. (3.27) and $\mathrm{d}s=|\mathrm{d}z|=|\omega'(\varsigma)||\mathrm{d}\varsigma|,\mathrm{d}\varsigma/|\mathrm{d}\varsigma|=\mathrm{e}^{\mathrm{i}\psi}$ we have

$$\begin{aligned}n&=n_1+\mathrm{i}n_2=\mathrm{d}x_2/\mathrm{d}s-\mathrm{i}\mathrm{d}x_1/\mathrm{d}s=-\mathrm{i}\mathrm{d}z/\mathrm{d}s=\mathrm{e}^{\mathrm{i}\psi}\omega'(\varsigma)/|\omega'(\varsigma)|\\ t&=t_1+\mathrm{i}t_2=\mathrm{d}x_1/\mathrm{d}s+\mathrm{i}\mathrm{d}x_2/\mathrm{d}s=\mathrm{d}z/\mathrm{d}s=\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}\omega'(\varsigma)/|\omega'(\varsigma)|\end{aligned} \tag{3.91}$$

where n is the outward normal on the interface of S. Using Eqs. (3.91) and (3.83) yields

$$\begin{aligned}\frac{\partial\varphi^c}{\partial n}&=\frac{\partial\varphi^c}{\partial x_1}n_1+\frac{\partial\varphi^c}{\partial x_2}n_2=\left(\frac{\partial\varphi^c}{\partial z}\frac{\partial z}{\partial x_1}+\frac{\partial\varphi^c}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x_1}\right)n_1+\left(\frac{\partial\varphi^c}{\partial z}\frac{\partial z}{\partial x_2}+\frac{\partial\varphi^c}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x_2}\right)n_2\\ &=\left(\frac{\partial\varphi^c}{\partial z}n+\frac{\partial\varphi^c}{\partial\bar{z}}\bar{n}\right)=\phi'(z)n+\overline{\phi'(z)n}\end{aligned} \tag{3.92}$$

If let $\phi(1)=0$ which is not effect on the stress, on L we have

$$\begin{aligned}-\int_0^s\frac{\partial\varphi^c}{\partial n}\mathrm{d}s&=-\int_0^\psi\left[\frac{\phi'(\varsigma)}{\omega'(\varsigma)}\frac{\omega'(\varsigma)}{|\omega'(\varsigma)|}\mathrm{e}^{\mathrm{i}\psi}+\frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}\frac{\overline{\omega'(\varsigma)}}{|\omega'(\varsigma)|}\mathrm{e}^{-\mathrm{i}\psi}\right]|\omega'(\varsigma)|\mathrm{d}\psi\\ &=-\int_0^\psi\left\{\mathrm{e}^{\mathrm{i}\psi}\phi'\left(\mathrm{e}^{\mathrm{i}\psi}\right)+\mathrm{e}^{-\mathrm{i}\psi}\overline{\phi'(\mathrm{e}^{\mathrm{i}\psi})}\right\}\mathrm{d}\psi=\mathrm{i}\left\{\phi(\sigma)-\overline{\phi(\sigma)}\right\}\end{aligned} \tag{3.93}$$

Substituting Eqs. (3.78), (3.83) and (3.87) and the third equation in Eq. (3.89) and (3.93) into Eq. (3.80), the electric connective conditions are reduced to

$$\begin{aligned}&\sum_{j=1}^{3}\left\{\eta_j f_j^0(\sigma)+\bar{\eta}_j\overline{f_j^0(\sigma)}\right\}=\bar{l}'_3\sigma+l'_3\bar{\sigma}+\mathrm{i}\epsilon^c\left[\phi(\sigma)-\overline{\phi(\sigma)}\right]\\ &\sum_{j=1}^{3}\left\{\lambda_j f_j^0(\sigma)+\bar{\lambda}_j\overline{f_j^0(\sigma)}\right\}=\bar{l}_4\sigma+l_4\bar{\sigma}-\left[\phi(\sigma)+\overline{\phi(\sigma)}\right]\\ &l'_3=(1/2)\left(aD_2^\infty-\mathrm{i}bD_1^\infty\right),l_4=-(1/2)\left(aE_1^\infty+\mathrm{i}bE_2^\infty\right)\end{aligned} \tag{3.94}$$

3.4.5 Solutions in the Air and Piezoelectric Material

Substituting Eqs. (3.85) and (3.87) into Eqs. (3.90) and (3.94) and neglecting some useless constants yield enough equations to determine the undetermined constants. It is found that only four complex constants $a_{11},\ a_{21},\ a_{31}, a_1^c\ \left(a_{-1}^c = \rho_0^2 a_1^c\right)$ are not zero and they obey the following equations:

$$\begin{aligned}
&\sum_{j=1}^{3} a_{j1} = l_1, \quad \sum_{j=1}^{3} \mu_j a_{j1} = l_2 \\
&\sum_{j=1}^{3} \eta_j a_{j1} + \mathrm{i}\epsilon^c\left(\bar{a}_1^c - \rho_0^2 a_1^c\right) = l_3', \quad \sum_{j=1}^{3} \lambda_j a_{j1} + \bar{a}_1^c + \rho_0^2 a_1^c = l_4
\end{aligned} \tag{3.95}$$

Finally we get

1. *The electric field inside the hole filled with air*

$\varphi^c,\ D_1^c,\ D_2^c$ are constants and obtained from the following equations:

$$\begin{aligned}
&\varphi^c = -E_1^c x_1 - E_2^c x_2 = 2\left(a_1^c z + \bar{a}_1^c \bar{z}\right)/(a+b); \quad D_1^c = \epsilon_0 E_1^c, \quad D_2^c = \epsilon_0 E_2^c \\
&\left(a - \mathrm{i}b\epsilon_0 \sum_{j=1}^{3} \lambda_j \alpha_{j3}\right) D_1^c + \left(a\epsilon_0 \sum_{j=1}^{3} \lambda_j \alpha_{j3} + \mathrm{i}b\right) D_2^c \\
&\quad = \epsilon_0 \left\{ 2 \sum_{j=1}^{3} \sum_{k=1}^{2} \lambda_j \alpha_{jk} l_k + 2 \sum_{j=1}^{3} \lambda_j \alpha_{j3} l_3' + aE_1^\infty + \mathrm{i}bE_2^\infty \right\}
\end{aligned} \tag{3.96}$$

2. *Solutions in the piezoelectric material*

$$\begin{aligned}
&f_j(z_j) = C_j z_j + a_{j1}/\varsigma_j = C_j z_j + \sum_{k=1}^{3} \alpha_{jk} l_k/\varsigma_j, \quad a_{j1} = \sum_{k=1}^{3} \alpha_{jk} l_k \\
&F_j(z_j) = C_j + \left(\alpha_{j1} l_1 + \alpha_{j2} l_2 + \alpha_{j3} l_3\right)\left(a + i\mu_j b\right)^{-1} \left\{ 1 - z_j \left[z_j^2 - \left(a^2 + \mu_j^2 b^2\right)\right]^{-1/2} \right\}
\end{aligned} \tag{3.97}$$

where

$$\boldsymbol{\alpha} = \frac{1}{N} \begin{pmatrix} \mu_2\eta_3 - \mu_3\eta_2 & \eta_2 - \eta_3 & \mu_3 - \mu_2 \\ \mu_3\eta_1 - \mu_1\eta_3 & \eta_3 - \eta_1 & \mu_1 - \mu_3 \\ \mu_1\eta_2 - \mu_2\eta_1 & \eta_1 - \eta_2 & \mu_2 - \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix}^{-1} \tag{3.98}$$

$$N = (\eta_2 - \eta_3)\mu_1 + (\eta_3 - \eta_1)\mu_2 + (\eta_1 - \eta_2)\mu_3$$

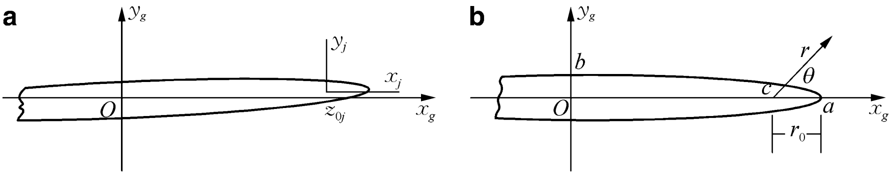


Fig. 3.4 Local coordinate system for a blunt crack

3.4.6 Electroelastic Asymptotic Field Near a Blunt (Slender) Crack Tip

Analogous to the elastic blunt crack (Kuang 1982; Kuang and Ma 2002), Huang and Kuang (2000b) discussed the electroelastic asymptotic field near a blunt (slender) crack tip. Take the global coordinate system $z_g(x_g, y_g)$ and the local coordinate system $z_j(x_j, y_j)$ whose origin is at z_{0j} ($z_0 = x_{01} = c = \sqrt{a^2 - b^2}$; $2c$ is the focal length) (Fig. 3.4). z_{0j} is the point in z_j plane corresponding to the branch point ς_{0j} in ς_j plane with $\omega'(\varsigma_{0j}) = 0$. It has the relation

$$\begin{aligned} &z_{0j} = x_{0j} + \mu_j y_{0j} = \sqrt{a^2 + \mu_j^2 b^2} \approx a + \mu_j^2 r_0, \quad r_0 = b^2/2a \\ &x_{0j} = a^2 - \mu^2 \rho, \quad y_{0j} = 2\alpha_j \rho; \quad \mu_j = \alpha_j + i\beta_j, \quad \mu^2 = \alpha_j^2 + \beta_j^2 \end{aligned} \tag{3.99}$$

where $2r_0$ is the curvature radius at the major end of the slender ellipse. At the local coordinate system, we have

$$z_j = x_j + \mu_j y_j = z_{gj} - z_{0j} = (x_g - x_{0j}) + \mu_j (y_g - y_{0j})$$

From the knowledge of the analytical geometry, it is known that

$$x_g = c + r\cos\theta, \quad y_g = r\sin\theta, \quad a = r_0 + \sqrt{c^2 + r_0^2} \approx c + r_0$$

where r, θ are the polar axis and angle, respectively. Therefore it is easy to derive

$$\begin{aligned} &z_k = r\Theta_k - (1 + \mu_k^2)\rho + 0(\rho^2/c) \approx r\Theta_k\{1 - [(1 + \mu_k^2)\rho/\Theta_k r]\} \\ &\Theta_k = \cos\theta + \mu_k \sin\theta \end{aligned} \tag{3.100}$$

In the local coordinate system, Eq. (3.97) becomes

$$F_j(z_j) = -\sum_{k=1}^{3} \alpha_{jk} l_k \frac{1}{a + \mathrm{i}\mu_j b} \sqrt{\frac{c}{2r\Theta_j}} \left[1 - \frac{1 + \mu_j^2}{\Theta_j} \frac{\rho}{r}\right] + C_j + \frac{1}{a + \mathrm{i}\mu_j b} \sum_{k=1}^{3} \alpha_{jk} l_k \tag{3.101}$$

It is seen that the singularity of the stress near the crack tip not only depends on $1/\sqrt{r}$ but also depends on ρ/r. It can also be seen that the electric field at infinity affects the stress near the blunt crack tip.

3.4.7 Impermeable and Conductive Elliptic Holes

Impermeable elliptic hole. Comparing to a piezoelectric material, in many cases the air is approximately considered as an insulated material, i.e., $\epsilon_0 = 0$ or $D_1^c = D_2^c = 0$, so $D_n^c = 0$ in Eq. (3.80), i.e. the piezoelectric material can be considered alone. On the interface, Eq. (3.94) is reduced to

$$\begin{aligned} &\sum_{j=1}^{3}\left\{\eta_j f_j^0(\sigma) + \overline{\eta}_j \overline{f_j^0(\sigma)}\right\} = \overline{l_3'}\sigma + l_3'\overline{\sigma} \\ &l_3' = -\sum_{j=1}^{3}\left\{a\mathrm{Re}(C_j\eta_j) + \mathrm{i}b\mathrm{Re}(C_j\eta_j\mu_j)\right\} = \frac{1}{2}\left(aD_2^\infty - \mathrm{i}bD_1^\infty\right) = l_3 \end{aligned} \tag{3.102}$$

Correspondingly Eq. (3.95) becomes

$$\sum_{j=1}^{3} a_{j1} = l_1, \quad \sum_{j=1}^{3}\mu_j a_{j1} = l_2 \quad \sum_{j=1}^{3}\eta_j a_{j1} = l_3' \tag{3.103}$$

The solution in the piezoelectric material is still formally expressed by Eqs. (3.97) and (3.98).

Conductive elliptic hole. If the hole is filled with an ideal conductive liquid or on the boundary of the hole deposited a thin flexible layer metal, it can be assumed that the electric potential is equal zero, i.e., $\varphi = 0$ in Eq. (3.80). On interface, Eq. (3.94) is reduced to

$$\begin{aligned} &\sum_{j=1}^{3}\left\{\lambda_j f_j^0(\sigma) + \overline{\lambda}_j \overline{f_j^0(\sigma)}\right\} = \overline{l}_4\sigma + l_4\overline{\sigma} \\ &l_4 = -\sum_{j=1}^{3}\left\{a\mathrm{Re}(C_j\lambda_j) + \mathrm{i}b\mathrm{Re}(C_j\lambda_j\mu_j)\right\} = -\frac{1}{2}\left(aE_1^\infty + \mathrm{i}bE_2^\infty\right) \end{aligned} \tag{3.104}$$

If we let $1/\epsilon^c = 0$ in Eq. (3.94), Eq. (3.104) can also be obtained. Correspondingly Eq. (3.103) becomes

$$\sum_{j=1}^{3} a_{j1} = l_1, \quad \sum_{j=1}^{3}\mu_j a_{j1} = l_2 \quad \sum_{j=1}^{3}\lambda_j a_{j1} = l_4 \tag{3.105}$$

In this case Eqs. (3.97) and (3.98) become

$$f_j(z_j) = C_j z_j + a_{j1}/\varsigma_j = C_j z_j + \sum_{k=1}^{3} \tilde{\alpha}_{jk} l_k/\varsigma_j, \quad \tilde{a}_{j1} = \sum_{k=1}^{3} \tilde{\alpha}_{jk} l_k \tag{3.106}$$

and

$$\tilde{\boldsymbol{\alpha}} = \frac{1}{\tilde{N}} \begin{pmatrix} \mu_2\lambda_3 - \mu_3\lambda_2 & \lambda_2 - \lambda_3 & \mu_3 - \mu_2 \\ \mu_3\lambda_1 - \mu_1\lambda_3 & \lambda_3 - \lambda_1 & \mu_1 - \mu_3 \\ \mu_1\lambda_2 - \mu_2\lambda_1 & \lambda_1 - \lambda_2 & \mu_2 - \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}^{-1}$$
$$N = (\lambda_2 - \lambda_3)\mu_1 + (\lambda_3 - \lambda_1)\mu_2 + (\lambda_1 - \lambda_2)\mu_3 \tag{3.107}$$

3.4.8 Crack Problem

1. *Permeable crack*. When the length of the minor axis approaches zero, i.e., $b \to 0$, for a permeable crack the solution can be obtained from a permeable elliptic hole. Neglecting terms containing b/a yields

$$l_1 = -\frac{1}{2}a\sigma_2^\infty, \quad l_2 = -\frac{1}{2}a\sigma_3^\infty, \quad l_3' = \frac{1}{2}aD_2^\infty, \quad l_3 = \frac{1}{2}a\left(D_2^\infty - D_2^c\right), \quad l_4 = -\frac{1}{2}aE_1^\infty \tag{3.108}$$

(a) *Electric field in the air*. Equation (3.96) becomes

$$D_1^c + \epsilon_0 \sum_{j=1}^{3} \lambda_j \alpha_{j3} D_2^c = \epsilon_0 \left\{ \sum_{j=1}^{3} \left[-\lambda_j\left(\alpha_{j1}\sigma_2^\infty + \alpha_{j2}\sigma_3^\infty + \alpha_{j3}D_2^\infty\right)\right] + E_1^\infty \right\} \tag{3.109a}$$

Noting $\sum_{j=1}^{3} \lambda_j \alpha_{j2}$ is real (Gao and Fan 1999), separating the real and imaginary parts from Eq. (3.109a) yields

$$D_2^\infty - D_2^c = \mathrm{Im}\left(\sum_{j=1}^{3} \lambda_j \alpha_{j1}\right)\sigma_2^\infty \Big/ \mathrm{Im}\left(\sum_{j=1}^{3} \lambda_j \alpha_{j3}\right)$$
$$E_1^c = E_1^\infty + \mathrm{Re}\left\{ \sum_{j=1}^{3} \lambda_j \left[-a_{j1}\sigma_2^\infty + a_{j2}\sigma_3^\infty + a_{j3}\left(D_2^\infty - D_2^c\right)\right] \right\} \tag{3.109b}$$

(b) *Generalized stress in the piezoelectric material*. Equation (3.97) is reduced to

$$F_j(z_j) = C_j - \frac{1}{2}\left[\alpha_{j1}\sigma_2^\infty - \alpha_{j2}\sigma_3^\infty - \alpha_{j3}\left(D_2^\infty - D_2^c\right)\right]\left[1 - z_j\left(z_j^2 - a^2\right)^{-1/2}\right] \tag{3.110}$$

It is noted that ϵ_0 is not included in Eqs. (3.109) and (3.110).

The stress intensities at the crack tip $x_1 = a$ are

$$(K_I, K_{II}, K_e) = \sqrt{2\pi} \lim_{x_1 \to a} \sqrt{x_1 - a} (\sigma_2, \sigma_3, D_2)_{x_2=0}$$
$$= \sqrt{2\pi} \mathrm{Re} \lim_{x_1 \to a} \sqrt{x_1 - a} \sum_{j=1}^{3} f_j'(x_1) \left(1, -\mu_j, -\eta_j\right)_{x_2=0} \tag{3.111a}$$

or

$$K_I = \sqrt{\pi a} \sigma_2^\infty, \quad K_{II} = \sqrt{\pi a} \sigma_3^\infty, \quad K_e = \sqrt{\pi a} \left(D_2^\infty - D_2^c\right) \tag{3.111b}$$

From Eq. (3.111), it is seen that the electric field at infinity does not affect the stress intensity and the mechanical stress at infinity does not affect the electric displacement intensity. This result is obtained from the linear theory.

2. *Impermeable (or insulated) crack*. The correct solution of an impermeable crack can be obtained from the degenerate solution from the insulated elliptic hole, i.e., let $\epsilon_0 = 0$ or $D_2 = 0$ at first and then let $b/a \to 0$. In order to study the electroelastic asymptotic field near a sharp crack tip, the right crack tip should be taken as the origin of the local polar coordinate system, i.e.,

$$x_1 = a + r\cos\theta, \quad x_2 = a + r\sin\theta \tag{3.112}$$

When $r \ll 1$, we have

$$z_j \approx a, \quad \sqrt{z_j^2 - a^2} \approx \sqrt{2ar} \sqrt{\cos\theta + \mu_j \sin\theta} \tag{3.113}$$

Equation (3.97) is reduced to

$$F_j(z_j) \approx \left(\alpha_{j1}\sigma_2^\infty - \alpha_{j2}\sigma_3^\infty - \alpha_{j3} D_2^\infty\right) \frac{\sqrt{a}}{2\sqrt{2r}\sqrt{\cos\theta + \mu_j \sin\theta}} \tag{3.114}$$

Let

$$C_j = \alpha_{j1} K_I - \alpha_{j2} K_{II} - \alpha_{j3} K_e \tag{3.115}$$

where K_{I}, K_{II}, K_e is defined by Eq. (3.111). Substituting Eqs. (3.114) and (3.115) into Eq. (3.26) in the Cartesian coordinate system yields (Hoenig 1982)

$$\sigma_1 = \left(1\Big/\sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \mu_j^2 \Big/ \sqrt{\boldsymbol{\Theta}_j}, \quad \sigma_2 = \left(1\Big/\sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \Big/ \sqrt{\boldsymbol{\Theta}_j},$$
$$\sigma_3 = -\left(1\Big/\sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \mu_j \Big/ \sqrt{\boldsymbol{\Theta}_j}, \quad D_1 = \left(1\Big/\sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \mu_j \eta_j \Big/ \sqrt{\boldsymbol{\Theta}_j},$$
$$D_2 = -\left(1\Big/\sqrt{2\pi r}\right) \mathrm{Re} \sum_{j=1}^{3} C_j \eta_j \Big/ \sqrt{\boldsymbol{\Theta}_j}, \quad \boldsymbol{\Theta}_j = \cos\theta + \mu_j \sin\theta \tag{3.116}$$

or in the polar coordinate system yields

$$\begin{aligned}
&\sigma_\theta = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \boldsymbol{\Theta}_j^{3/2}, \quad \sigma_r = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \tilde{\boldsymbol{\Theta}}_j^2 \Big/ \sqrt{\boldsymbol{\Theta}_j}, \\
&\sigma_{\theta r} = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \tilde{\boldsymbol{\Theta}}_j \sqrt{\boldsymbol{\Theta}_j}, \quad D_r = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \eta_j \tilde{\boldsymbol{\Theta}}_j \Big/ \sqrt{\boldsymbol{\Theta}_j}, \\
&D_\theta = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{3} C_j \eta_j \sqrt{\boldsymbol{\Theta}_j}, \quad \tilde{\Theta}_j = \cos\theta - \mu_j \sin\theta
\end{aligned} \tag{3.117}$$

It is seen that the stresses have the singularity $1/\sqrt{r}$ and σ_1^∞, D_1^∞ do not affect the electroelastic asymptotic field. Xu and Rajapakse (1999) discussed an arbitrarily oriented void/crack.

3. *Conducting crack*

The solution of a conducting crack can be obtained from the conducting elliptic hole directly when $b/a \to 0$ or from the general solution when $\epsilon_0 \to \infty$ at first and then when $b/a \to 0$.

3.4.9 Eshelby's Elliptic Inclusion Problem in a Piezoelectric Material

Now discuss the Eshelby's elliptic inclusion problem in a piezoelectric material (Ru 1997). In a piezoelectric material, there is a region S^c. In S^c there are the generalized eigenstrains $(\boldsymbol{\varepsilon}^*, E^*)$ and the corresponding additional generalized displacement:

$$\begin{aligned}
\boldsymbol{U}^* &= (\boldsymbol{u}^*, \varphi^*) \\
&= \left[\varepsilon_{11}^* x_1 + \varepsilon_{12}^* x_2, \varepsilon_{12}^* x_1 + \varepsilon_{22}^* x_2, 2\left(\varepsilon_{13}^* x_1 + \varepsilon_{23}^* x_2\right), -\left(E_1^* x_1 + E_2^* x_2\right)\right]^{\mathrm{T}}
\end{aligned} \tag{3.118}$$

The connective conditions on the interface L are

$$\boldsymbol{u} = \boldsymbol{u}^c + \boldsymbol{u}^*, \quad \varphi = \varphi^c + \varphi^*; \quad z \in L \tag{3.119}$$

Substituting Eqs. (3.16) and (3.23) into Eq. (3.119), the connective conditions become

$$\left.\begin{aligned}
\boldsymbol{Af}(z_P) + \overline{\boldsymbol{Af}(z_P)} = \boldsymbol{Af}^c(z_P) + \overline{\boldsymbol{Af}^c(z_P)} + \boldsymbol{u}^* \\
\boldsymbol{Bf}(z_P) + \overline{\boldsymbol{Bf}(z_P)} = \boldsymbol{Bf}^c(z_P) + \overline{\boldsymbol{Bf}^c(z_P)}
\end{aligned}\right\}, \quad z \in L \tag{3.120}$$

where $\boldsymbol{f}^c(z_P)$ is the solution in S^c. Multiplying the first equation and second equation by $\boldsymbol{B}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}$, respectively, then adding them, and using Eq. (3.34) yield

$$\boldsymbol{f}(z_P) = \boldsymbol{f}^c(z_P) + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}^*, \quad z \in L \tag{3.121}$$

Using the relations between x_1, x_2 and $z_P, \bar{z}_P$ in Eq. (3.15), for $z \in L$ we have

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}^* = [\xi_P z_P + \eta_P \bar{z}_P]^{\mathrm{T}} = [\xi_1 z_1 + \eta_1 \bar{z}_1, \xi_2 z_2 + \eta_2 \bar{z}_2, \xi_3 z_3 + \eta_3 \bar{z}_3, \xi_4 z_4 + \eta_4 \bar{z}_4]^{\mathrm{T}} \tag{3.122}$$

where ξ_P, η_P are constants determined by $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}^*$. Therefore Eq. (3.121) can be separated into four independent scalar equations:

$$f_P(z_P) = f_P^c(z_P) + \xi_P z_P + \eta_P \bar{z}_P, \quad z_P \in L_P; \quad P = 1 \sim 4 \tag{3.123}$$

Using the mapping function described in Eq. (3.86) yields

$$\begin{aligned} &\bar{z}_P = \bar{\omega}_P(1/\sigma_P) = \bar{c}_P \sigma_P^{-1} + \bar{d}_P \sigma_P, \quad c_P = (a - \mathrm{i}\mu_P b)/2, \\ &d_P = (a + \mathrm{i}\mu_P b)/2; \quad z_P \in L_P \end{aligned}$$

So $\bar{z}_P$ is the boundary value of an analytic function $D_P(\varsigma_P)$ in S_P or in the exterior of S_P^c

$$\begin{aligned} &D_P(\varsigma_P) = \bar{c}_P \varsigma_P^{-1} + \bar{d}_P \varsigma_P \\ &D_P(z_P) = \bar{c}_P \frac{z_P - \sqrt{z_P^2 - (a^2 + \mu_P^2 b^2)}}{a + \mathrm{i}\mu_P b} + \bar{d}_P \frac{z_P + \sqrt{z_P^2 - (a^2 + \mu_P^2 b^2)}}{a - \mathrm{i}\mu_P b}; \quad z_P \in S_P \\ &D_P(z_P) \to h_P z_P, \quad h_P = (a - \mathrm{i}\bar{\mu}_P b)/(a - \mathrm{i}\mu_P b); \quad \text{when} \quad z_P \to \infty \end{aligned} \tag{3.124}$$

Substitution of Eq. (3.124) into Eq. (3.123) yields

$$f_P(z_P) - \xi_P z_P - \eta_P D_P(z_P) = f_P^c(z_P), \quad z_P \in L_P; \quad P = 1 \sim 4 \tag{3.125}$$

Usually the boundary conditions at infinity are $f_P(z_P) \to 0$, when $|z_P| \to \infty$, so the functions in the left- and right-hand side of Eq. (3.125) are all analytic. Therefore we have

$$\left.\begin{aligned} &f_P(z_P) = \eta_P[D_P(z_P) - h_P z_P], \quad z_P \in S_P \\ &f_P^c(z_P) = -(\xi_P + \eta_P h_P) z_P, \quad z_P \in S_P^c \end{aligned}\right\}; P = 1 \sim 4, \ \text{not summation on } P \tag{3.126}$$

From Eq. (3.126), it is known that the generalized stress field is uniform in the elliptic inclusion.

In Ru's paper (1997), he also discussed the inclusion with arbitrary shape by the mapping function (Muskhelishvili 1954, 1975; Kantorovich and Krylov 1958)

$$z = \omega(\varsigma) = \lambda\varsigma + \sum_{k=0}^{\infty} \lambda_k \varsigma^{-k} \tag{3.127}$$

In many cases the truncation of the infinite series to finite terms $k = N$ offers good approximation (Savin 1961).

Zeng and Rajapakse (2003) discussed the Eshelby's elliptic inclusion problem with specified generalized eigenstrains $(\boldsymbol{\varepsilon}^*, \boldsymbol{D}^*)$.

3.5 Rigid Elliptic Inclusion in Transversely Piezoelectric Material

3.5.1 Basic Theory

Though we can use the theory obtained in Sect. 3.3.2, but in this section we rather use the first kind of the constitutive equation in Eq. (2.83) to discuss the problem, i.e.,

$$\varepsilon_{ij} = s^E_{ijkl}\sigma_{kl} + d^\sigma_{kij}E_k, \quad D_i = d^D_{ijk}\sigma_{jk} + \epsilon^\sigma_{ij}E_j \tag{3.128}$$

Analogous to the derivation in Sect. 3.3.2, for the generalized plane problem in the transversely isotropic material, we have

$$\begin{aligned}
&\varepsilon_x = \gamma_{zx} = \gamma_{xy} = \tau_{zx} = \tau_{yz} = E_x = D_x = 0, \quad \sigma_x = -\left(s_{12}\sigma_y + s_{13}\sigma_z + d_{31}E_z\right)/s_{11} \\
&x_1 = y,\ x_2 = z,\ x_3 = x; \quad \sigma_1 = \sigma_y,\ \sigma_2 = \sigma_z,\ \sigma_3 = \tau_{yz}; \quad \varepsilon_1 = \varepsilon_y,\ \varepsilon_2 = \varepsilon_z,\ \varepsilon_3 = \gamma_{yz} \\
&D_1 = D_y, D_2 = D_z;\ E_1 = E_y, E_2 = E_z \\
&s_{13} = s_{23}, s_{11} = s_{22}, s_{44} = s_{55}, s_{66} = 2(s_{11} - s_{12});\ \ d_{31} = d_{32}, d_{15} = d_{24};\ \ \epsilon_{11} = \epsilon_{22}
\end{aligned} \tag{3.129}$$

Analogous to Eq. (3.74), the constitutive equation in terms of the reduced material constants is

$$\begin{aligned}
&\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ D_1 \\ D_2 \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & b_{21} \\ a_{12} & a_{22} & 0 & 0 & b_{22} \\ 0 & 0 & a_{33} & b_{13} & 0 \\ 0 & 0 & b_{13} & k_{11} & 0 \\ b_{21} & b_{22} & 0 & 0 & k_{22} \end{pmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ E_1 \\ E_2 \end{Bmatrix} \\
&a_{11} = s_{11} - s_{12}^2/s_{11}, \quad a_{12} = s_{13} - s_{12}s_{13}/s_{11}, \quad a_{22} = s_{33} - s_{13}^2/s_{11}, \quad a_{33} = s_{44} \\
&b_{21} = (1 - s_{12}/s_{11})d_{31},\ b_{22} = d_{33} - d_{31}s_{13}/s_{11},\ b_{13} = d_{15},\ k_{11} = \epsilon_{11},\ k_{22} = \epsilon_{33} - d_{31}^2/s_{11}
\end{aligned} \tag{3.130}$$

In the present case the generalized equilibrium and compatibility equations are, respectively,

$$\sigma_{1,1}+\sigma_{3,2}=0,\quad \sigma_{3,1}+\sigma_{2,2}=0,\quad E_{1,2}-E_{2,1}=0 \tag{3.131}$$

$$\varepsilon_{1,22}+\varepsilon_{2,11}-\varepsilon_{3,12}=0,\quad D_{1,1}+D_{2,2}=0 \tag{3.132}$$

Introduce the stress function Λ and the electric potential φ:

$$\sigma_1=\Lambda_{,22},\quad \sigma_2=\Lambda_{,11},\quad \sigma_3=-\Lambda_{,12},\quad E_1=-\varphi_{,1},\quad E_2=-\varphi_{,2} \tag{3.133}$$

Equation (3.131) is satisfied automatically and Eq. (3.132) becomes

$$\begin{aligned}
&L_4\Lambda-L_3\varphi=0,\quad L_3\Lambda-L_2\varphi=0\\
&L_4=a_{22}\frac{\partial^4}{\partial x_1^4}+a_{11}\frac{\partial^4}{\partial x_2^4}+(2a_{12}+a_{33})\frac{\partial^4}{\partial x_1^2\partial x_2^2}\\
&L_3=b_{21}\frac{\partial^3}{\partial x_2^3}+(b_{22}-b_{13})\frac{\partial^3}{\partial x_2\partial x_1^2},\quad L_2=k_{11}\frac{\partial^2}{\partial x_1^2}+k_{22}\frac{\partial^2}{\partial x_2^2}
\end{aligned} \tag{3.134}$$

Eliminating Λ or φ from Eq. (3.134) yields

$$\left(L_4L_2-L_3^2\right)\Lambda=0,\quad \left(L_4L_2-L_3^2\right)\varphi=0 \tag{3.135}$$

The solution of Eq. (3.135) is

$$\begin{aligned}
&\Lambda=2\mathrm{Re}\sum_{j=1}^{3}\tilde{f}_j(z);\quad \varphi=2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j(z_j);\quad f_j(z_j)=\tilde{f}_j'(z_j)\\
&\eta_j=l_3/l_2=\mu_j\left[b_{21}\mu_j^2+(b_{22}-b_{13})\right]\Big/\left(k_{11}+k_{22}\mu_j^2\right);\quad z=x_1+\mu x_3
\end{aligned} \tag{3.136}$$

where μ_j is the root of the following eigen-equation

$$\begin{aligned}
&l_4(\mu)l_2(\mu)-l_3^2(\mu)=0;\quad l_4=a_{22}+a_{11}\mu^4+(2a_{12}+a_{33})\mu^2\\
&l_3=\mu\left[b_{21}\mu^2+(b_{22}-b_{13})\right],\quad l_2=k_{11}+k_{22}\mu^2
\end{aligned} \tag{3.137}$$

Equation (3.137) has 6 roots and $\mu_k=\alpha_k+\mathrm{i}\beta_{k,}$ $\alpha_1=0,$ $\mu_3=-\bar{\mu}_2,$ $\mu_{k+3}=\bar{\mu}_k$. From Eqs. (3.134) and (3.137), we get

$$\varphi=2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j(z_j),\quad \eta_j=\frac{l_3}{l_2}=\frac{\mu_j[d_{21}\mu^2+(d_{22}-d_{13})]}{\epsilon_{11}+\epsilon_{22}\mu^2} \tag{3.138}$$

where $f_j(z_j) = \tilde{f}_j'(z_j)$. Substituting Eqs. (3.136) and (3.138) into Eqs. (3.133) and (3.130) yields, respectively,

$$\begin{aligned}
&\sigma_1 = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2 F_j(z_j), \quad \sigma_2 = 2\mathrm{Re}\sum_{j=1}^{3} F_j(z_j), \quad \sigma_3 = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j F_j(z_j) \\
&E_1 = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j F_j(z_j), \quad E_2 = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_j F_j(z_j), \quad F_j(z_j) = f_j'(z_j)
\end{aligned} \tag{3.139}$$

$$\begin{aligned}
&u_1 = 2\mathrm{Re}\sum_{j=1}^{3} p_j f_j(z_j) - \omega x_2, \quad p_j = a_{11}\mu_j^2 + a_{12} - b_{21}\mu_j\eta_j \\
&u_2 = 2\mathrm{Re}\sum_{j=1}^{3} q_j f_j(z_j) + \omega x_1, \quad q_j = \left(a_{12}\mu_j^2 + a_{22} - b_{22}\mu_j\eta_j\right)\Big/\mu_j \\
&D_1 = -2\mathrm{Re}\sum_{j=1}^{3}\lambda_j\mu_j F_j(z_j), \quad D_2 = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j F_j(z_j) \\
&\lambda_j\mu_j = b_{13}\mu_j + k_{11}\eta_j, \quad \lambda_j = b_{21}\mu_j^2 + b_{22} - k_{22}\mu_j\eta_j
\end{aligned} \tag{3.140}$$

where ω is the rigid rotation angle.

3.5.2 *Rigid Elliptic Inclusion*

The discussed problem can also be shown in (Fig. 3.3) as that in the Sect. 3.4.1, but here S^c is not a hole, rather a rigid inclusion. The notations are the same as that in Sect. 3.4.1, except the permittivity in the inclusion is denoted by ϵ^c instead of ϵ_0 in the air. The boundary conditions at infinity are assumed $\omega = 0$ and

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\infty}, \quad \boldsymbol{E} = \boldsymbol{E}^{\infty} \tag{3.141}$$

On the interface the connective equation is

$$\begin{aligned}
&u_1 = u_1^c = -\omega^c x_2, \quad u_2 = u_2^c = \omega^c x_1; \quad \varphi = \varphi^i, \quad \int_0^s D_n \mathrm{d}s = \int_0^s D_n^c \mathrm{d}s; \quad \text{on} \quad L \\
&\int_0^s D_n \mathrm{d}s = \int_0^s (D_1 n_1 + D_2 n_2)\mathrm{d}s = \int_0^s (D_1 \mathrm{d}x_2 - D_2 \mathrm{d}x_1) = 2\mathrm{Re}\sum_{j=1}^{3}\left(\lambda_j f_j\right)\Big|_0^s
\end{aligned} \tag{3.142}$$

Similar to Sect. 3.4, the complex function method is used. The mapping function is shown in Eq. (3.86). Assume that the function $f_j(z_j)$ can be expanded as that in Eq. (3.87) and

$$\begin{aligned} &\sigma_1^\infty = 2\mathrm{Re}\sum_{j=1}^{3}\mu_j^2 C_j, \quad \sigma_2^\infty = 2\mathrm{Re}\sum_{j=1}^{3} C_j, \quad \sigma_3^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j C_j \\ &E_1^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\eta_j C_j, \quad E_2^\infty = -2\mathrm{Re}\sum_{j=1}^{3}\mu_j\eta_j C_j, \quad \mathrm{Im}C_1 = 0 \end{aligned} \tag{3.143}$$

Substitution of Eqs. (3.140) and (3.87) into first two equations in (3.142) yields

$$\begin{aligned} &2\mathrm{Re}\sum_{j=1}^{3} p_j\left[C_j\left(x_1+\mu_j x_2\right)+a_{j0}+\sum_{k=1}^{\infty}a_{jk}\bar{\sigma}^k\right] = -\omega^c x_2 \\ &2\mathrm{Re}\sum_{j=1}^{3} q_j\left[C_j\left(x_1+\mu_j x_2\right)+a_{j0}+\sum_{k=1}^{\infty}a_{jk}\bar{\sigma}^k\right] = \omega^c x_1, \quad \text{on} \quad \Gamma \end{aligned} \tag{3.144}$$

Noting on Γ, $\varsigma_j = \sigma = \mathrm{e}^{\mathrm{i}\vartheta}(j=0,1,2,3)$, $x_1 = a(\sigma+\bar{\sigma})/2$, and $x_2 = -\mathrm{i}b(\sigma-\bar{\sigma})/2$. From Eqs. (3.144), we have

$$\begin{aligned} &a_{j0} = 0; \quad a_{jk} = 0, \quad k \geq 2 \\ &\sum_{j=1}^{3} 2p_j a_{j1} + p_j C_j\left(a+\mathrm{i}\mu_j b\right) + \bar{p}_j\bar{C}_j\left(a+\mathrm{i}\bar{\mu}_j b\right) = -\mathrm{i}b\omega^c \\ &\sum_{j=1}^{3} 2q_j a_{j1} + q_j C_j\left(a+\mathrm{i}\mu_j b\right) + \bar{q}_j\bar{C}_j\left(a+\mathrm{i}\bar{\mu}_j b\right) = a\omega^c, \quad \text{on} \quad \Gamma \end{aligned} \tag{3.145}$$

According to the knowledge of the elastic inclusion (Mura 1987) and the solution, Eq. (3.96), it is assumed that the electric field in the inclusion S^c is constant. Let

$$\begin{aligned} &\varphi^c = 2\mathrm{Re}\phi^c(z_0) = 2\mathrm{Re}C_0^c z_0, \\ &z_0 = x_1+\mu_0 x_3 = (1/2)\left[(a-\mathrm{i}\mu_0 b)\varsigma_0 + (a+\mathrm{i}\mu_0 b)\varsigma_0^{-1}\right] \end{aligned} \tag{3.146}$$

where C_0^c is a constant. From $D_{1,1}^c + D_{2,2}^c = 0$, it can be obtained that

$$\epsilon_{11}^c\varphi_{,11}^c + \epsilon_{22}^c\varphi_{,22}^c = 0, \quad \Rightarrow \quad \epsilon_{11}^c\phi''^c + \epsilon_{22}^c\mu_0^2\phi''^c = 0; \quad \phi''^c = \mathrm{d}^2\phi^c/\mathrm{d}z_0^2 \tag{3.147}$$

From Eqs. (3.146) and (3.147), we find

$$\epsilon_{11}^c + \epsilon_{22}^c\mu_0^2 = 0, \quad \mu_0 = \mathrm{i}\sqrt{\epsilon_{11}^c/\epsilon_{22}^c}, \quad \epsilon_{11}^c = -\mu_0^2\epsilon_{22}^c \tag{3.148}$$

$$\int_0^s D_n^c \mathrm{d}s = \int_0^s \left(D_1^c n_1^c + D_2^c n_2^c\right)\mathrm{d}s = 2\mathrm{Re}\int_0^s \epsilon_{22}^c \mu_0 \phi'^c (\mu_0 \mathrm{d}x_2 + \mathrm{d}x_1) = 2\mathrm{Re}\left[\epsilon_{22}^c \mu_0 \phi^c\right]_0^s \tag{3.149}$$

Substituting Eqs. (3.148) and (3.149) into the last two equations in Eq. (3.140), the electric connective conditions on the interface are reduced to

$$\begin{aligned} &2\mathrm{Re}\sum_{j=1}^{3}\eta_j f_j(z_j) = 2\mathrm{Re}\sum_{j=1}^{3}\eta_j\left[C_j z_j + a_{j1}\varsigma_j^{-1}\right] = 2\mathrm{Re}\left[C_0^c z_0\right] \\ &2\mathrm{Re}\sum_{j=1}^{3}\lambda_j f_j(z_j) = 2\mathrm{Re}\sum_{j=1}^{3}\lambda_j\left[C_j z_j + a_{j1}\varsigma_j^{-1}\right] = 2\mathrm{Re}\left[\epsilon_{22}^c \mu_0 \phi^i\right] \end{aligned} \tag{3.150}$$

or

$$\begin{aligned} &\sum_{j=1}^{3}\left\{2\eta_j a_{j1} + \eta_j C_j\left(a + \mathrm{i}b\mu_j\right) + \bar{\eta}_j \bar{C}_j\left(a + \mathrm{i}b\bar{\mu}_j\right)\right\} = C_0^c(a + \mathrm{i}b\mu_0) + \bar{C}_0^c(a + \mathrm{i}\bar{\mu}_0 b) \\ &\sum_{j=1}^{3}\left\{2\lambda_j a_{j1} + \lambda_j C_j\left(a + \mathrm{i}b\mu_j\right) + \bar{\lambda}_j \bar{C}_j\left(a + \mathrm{i}b\bar{\mu}_j\right)\right\} = \epsilon_{22}^c[\mu_0(a + \mathrm{i}b\mu_0) + \bar{\mu}_0(a + \mathrm{i}\bar{\mu}_0 b)] \end{aligned} \tag{3.151}$$

According to Eqs. (3.28) and (3.71), we know that

$$T_1 = -\mathrm{d}\Phi_1/\mathrm{d}s = (\mathrm{d}\Lambda/\mathrm{d}s)_{,2}, \quad T_2 = -\mathrm{d}\Phi_2/\mathrm{d}s = -(\mathrm{d}\Lambda/\mathrm{d}s)_{,1} \tag{3.152}$$

The ω^c is determined by the condition that there is no moment acting on the inclusion, i.e.,

$$\begin{aligned} &M_n = \oint (-T_1 x_2 + T_2 x_1)\,\mathrm{d}s = -\oint \left(\mathrm{d}\left(\Lambda_{,2}\right)x_2 + \mathrm{d}\left(\Lambda_{,1}\right)x_1\right) = 0 \\ &\sigma_1 = \Lambda_{,22}, \quad \sigma_2 = \Lambda_{,11}, \quad \sigma_3 = -\Lambda_{,12}, \quad E_1 = -\varphi_{,1}, \quad E_2 = -\varphi_{,2} \end{aligned} \tag{3.153}$$

Using Eqs. (3.136) and (3.145) and the residual theorem, we finally get

$$\sum_{j=1}^{3}\left[\left(a - \mathrm{i}b\mu_j\right)a_{j1} - \left(a + \mathrm{i}b\bar{\mu}_j\right)\bar{a}_{j1}\right] = 0 \tag{3.154}$$

The undetermined constants $C_j, a_{j1}, C_0^c, \omega^c$ can be determined by Eqs. (3.143), (3.145), (3.151), and (3.154). If ω^c is given, the moment acting on the inclusion is determined by Eq. (3.153).

3.6 Singularity

3.6.1 Singularity in a Homogeneous Material

Let a generalized singularity load be located at a point $z_0(x_{10}, x_{20})$ in an infinite homogeneous material. A generalized singularity load means a generalized dislocation $\boldsymbol{b}(b_1, b_2, b_3, b_4)$ and a generalized force $\boldsymbol{p}(p_1, p_2, p_3, p_4)$, where (b_1, b_2, b_3) are the Burgers vectors representing the displacement increment around the dislocation line and b_4 is the potential increment around the dislocation line. (p_1, p_2, p_3) are the concentrate forces and p_4 is the point electric charge or the electric displacement flux. Let

$$\begin{aligned}
&\boldsymbol{g}(z_j) = \langle \ln(z_j - z_{0j}) \rangle c, \quad g_j(z_j) = c_j \ln(z_j - z_{0j}), \quad z_{0j} = x_{01} + \mu_j x_{02} \\
&\boldsymbol{G}(z_j) = \boldsymbol{g}'(z_j) = \left\langle (z_j - z_{0j})^{-1} \right\rangle \boldsymbol{c}, \quad G_j(z_j) = c_j (z_j - z_{0j})^{-1}
\end{aligned} \tag{3.155}$$

where $\boldsymbol{c}(c_1, c_2, c_3, c_4)$ is an undetermined constant vector, $\langle \ln(z_j - z_{0j}) \rangle = \mathrm{diag}\left[\ln(z_j - z_{0j}\right]$. Obviously $\ln(z_j - z_{0j})$ is a multivalued function and z_{0j} is a branch point. According to Eqs. (3.16), (3.23), and (3.155), the solutions are assumed as

$$\begin{aligned}
&\boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{g}(z_P)], \quad U_i = 2\mathrm{Re}\sum_{j=1}^{4} A_{ij} c_j \ln\left(z_j - z_{0j}\right) \\
&\boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{g}(z_P)], \quad \Phi_i = 2\mathrm{Re}\sum_{j=1}^{4} B_{ij} c_j \ln\left(z_j - z_{0j}\right)
\end{aligned} \tag{3.156a}$$

where z_{0j} is the branch point of the ln-function (usually the branch cut is chosen in the negative x_1 direction, from z_{j0} to $-\infty$) and select a single-valued branch that the polar angle is measured from the positive x_1 direction . On the two sides of the cut, it is defined

$$\boldsymbol{b} = \boldsymbol{U}^+ - \boldsymbol{U}^- = 2\pi\mathrm{i}(\boldsymbol{A}\boldsymbol{c} - \bar{\boldsymbol{A}}\bar{\boldsymbol{c}}), \quad \boldsymbol{p} = \boldsymbol{T}^- - \boldsymbol{T}^+ = \boldsymbol{\Phi}^+ - \boldsymbol{\Phi}^- = 2\pi\mathrm{i}(\boldsymbol{B}\boldsymbol{c} - \bar{\boldsymbol{B}}\bar{\boldsymbol{c}}) \tag{3.157}$$

where the superscript "+" and " − " denote the values approached from the upper and lower half planes, respectively. Using the identities (3.34) from Eqs. (3.156a) and (3.157) yields

$$\begin{aligned}
&2\pi\mathrm{i}\begin{bmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{c} \\ -\bar{\boldsymbol{c}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{b} \\ \boldsymbol{p} \end{Bmatrix}, \quad \begin{Bmatrix} \boldsymbol{c} \\ -\bar{\boldsymbol{c}} \end{Bmatrix} = \frac{1}{2\pi\mathrm{i}}\begin{bmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} & \bar{\boldsymbol{A}}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{b} \\ \boldsymbol{p} \end{Bmatrix}, \quad \text{or} \\
&\boldsymbol{c} = (1/2\pi)\left\{\boldsymbol{B}^{-1}(\boldsymbol{Y} + \bar{\boldsymbol{Y}})^{-1}\boldsymbol{b} - \boldsymbol{A}^{-1}\left(\boldsymbol{Y}^{-1} + \bar{\boldsymbol{Y}}^{-1}\right)^{-1}\boldsymbol{p}\right\} = (1/2\pi\mathrm{i})\boldsymbol{V}, \\
&\boldsymbol{V} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{p}
\end{aligned} \tag{3.158}$$

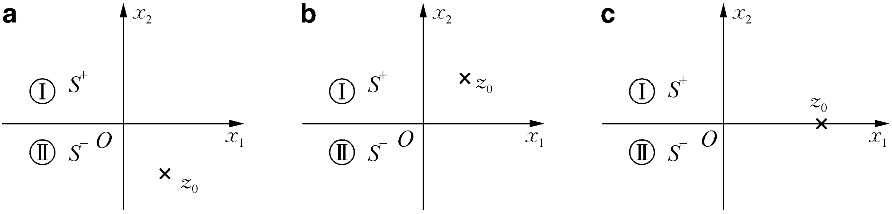


Fig. 3.5 Singularity in a bimaterial, singularity located (**a**) at lower plane, (**b**) at upper plane, and (**c**) on interface

So the solutions in Eq. (3.156a) become

$$\begin{aligned} &\boldsymbol{g}(z_j) = (1/2\pi \mathrm{i})\langle \ln(z_j - z_{0j})\rangle \boldsymbol{V}, \quad \boldsymbol{G}(z_j) = (1/2\pi \mathrm{i})\left\langle (z_j - z_{0j})^{-1}\right\rangle \boldsymbol{V} \\ &\boldsymbol{U} = (1/\pi)\mathrm{Im}\left[\boldsymbol{A}\langle \ln(z_j - z_{0j})\rangle \boldsymbol{V}\right], \quad \boldsymbol{\Phi} = (1/\pi)\mathrm{Im}\left[\boldsymbol{B}\langle \ln(z_j - z_{0j})\rangle \boldsymbol{V}\right] \end{aligned} \tag{3.156b}$$

The solution of the singularity problem can be used as the source function of a general problem.

3.6.2 *Singularity in a Bimaterial*

Let the material I be located at the upper half-plane $S^+, x_2 > 0$, and the material II be located at the lower half-plane $S^-, x_2 < 0; x_1 = 0$ is the interface L; a singularity load is located at $z_0(x_{10}, x_{20})$ in the material II (Fig. 3.5a). At first the problem is discussed in the z plane. Let (Tucker 1969; Barnett and Lothe 1975)

$$\boldsymbol{f}(z, z_0) = \begin{cases} \boldsymbol{f}_{\mathrm{I}}(z, z_0) & z \in S^+ \\ \boldsymbol{f}_{\mathrm{II}}(z, z_0) + \boldsymbol{g}_{\mathrm{II}}(z, z_0) & z \in S^- \end{cases} \tag{3.159}$$

where $\boldsymbol{g}_{\mathrm{II}}(z, z_0) = \boldsymbol{g}_{\mathrm{II}}(z)$ is the solution in a homogeneous material, i.e., the solution when the material II is extended to whole infinite plane, so $\boldsymbol{g}_{\mathrm{II}}(z, z_0)$ is analytic in the material I.

$$\boldsymbol{g}_{\mathrm{II}}(z_j) = \boldsymbol{c}_{\mathrm{II}}\langle \ln(z - z_0)\rangle; \quad \boldsymbol{c}_{\mathrm{II}} = (1/2\pi \mathrm{i})\boldsymbol{V}_{\mathrm{II}}, \quad \boldsymbol{V}_{\mathrm{II}} = \left(\boldsymbol{B}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{p}\right) \tag{3.160}$$

On the interface $x_2 = 0$, we have $\boldsymbol{U}_{\mathrm{I}} = \boldsymbol{U}_{\mathrm{II}}, \boldsymbol{\Phi}_{\mathrm{I}} = \boldsymbol{\Phi}_{\mathrm{II}}$ or

$$\begin{aligned} &\boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{I}}\overline{\boldsymbol{f}_{\mathrm{I}}(x_1)} = \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{II}}\overline{\boldsymbol{f}_{\mathrm{II}}(x_1)} + \boldsymbol{A}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01}) + \bar{\boldsymbol{A}}_{\mathrm{II}}\overline{\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01})} \\ &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}}\overline{\boldsymbol{f}_{\mathrm{I}}(x_1)} = \boldsymbol{B}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{II}}\overline{\boldsymbol{f}_{\mathrm{II}}(x_1)} + \boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01}) + \bar{\boldsymbol{B}}_{\mathrm{II}}\overline{\boldsymbol{g}_{\mathrm{II}}(x_1, x_{01})} \end{aligned} \tag{3.161}$$

It is noted (Muskhelishvili 1954, 1975) that

$$\overline{\boldsymbol{f}^{+}(x_1)}=\bar{\boldsymbol{f}}^{-}(x_1),\ \overline{\boldsymbol{f}^{-}(x_1)}=\bar{\boldsymbol{f}}^{+}(x_1);\quad \boldsymbol{f}_{\mathrm{I}}(x_1)=\boldsymbol{f}^{+}(x_1),\ \boldsymbol{f}_{\mathrm{II}}(x_1)=\boldsymbol{f}^{-}(x_1) \tag{3.162}$$

where the superscripts " + " and " − " indicate the limit values taken from the upper and lower half planes, respectively. By using Eq. (3.162), Eq. (3.161) can be reduced to

$$\begin{aligned}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}^{+}(x_1)-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}^{+}(x_1)-\boldsymbol{A}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1,x_{01})=\boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}^{-}(x_1)-\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}^{-}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(x_1,x_{01})\\ \boldsymbol{B}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}^{+}(x_1)-\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}^{+}(x_1)-\boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(x_1,x_{01})=\boldsymbol{B}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}^{-}(x_1)-\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}^{-}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(x_1,x_{01})\end{aligned} \tag{3.163}$$

It is known that the functions at the left side in Eq. (3.163) are analytic in the upper half-plane $x_2 > 0$, whereas those on the right side are analytic in the lower half-plane $x_2 < 0$, and they are continuous on $x_1 = 0$. So according to Liouville theorem, these functions are analytic in whole plane and must be constants. If there are no generalized external forces and displacements acting at infinite, these constants must be zero. So we have

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(z)-\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}(z)-\boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(z)=\boldsymbol{0},\quad \boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(z)-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{f}}_{\mathrm{II}}(z)-\boldsymbol{A}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(z)=\boldsymbol{0};\quad z\in S^{+}\\ &\boldsymbol{B}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(z)-\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}(z)+\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z)]=\boldsymbol{0},\quad \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(z)-\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{f}}_{\mathrm{I}}(z)+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z)]=\boldsymbol{0};\quad z\in S^{-}\end{aligned} \tag{3.164}$$

From Eq. (3.164) we get

$$\begin{aligned}\boldsymbol{f}_{\mathrm{I}}(z)&=\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}+\boldsymbol{Y}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{II}}\boldsymbol{g}_{\mathrm{II}}(z),\quad z\in S^{+}\\ \boldsymbol{f}_{\mathrm{II}}(z)&=\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z),\quad z\in S^{-}\end{aligned} \tag{3.165}$$

where $\boldsymbol{H}=\boldsymbol{Y}_{\mathrm{I}}+\bar{\boldsymbol{Y}}_{\mathrm{II}},\quad \boldsymbol{Y}=\mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1}$. It is noted that the above theory will be often used in the following sections and we only give a simple illustration there.

Finally the solution of the problem is

$$\begin{aligned}\boldsymbol{U}_{\mathrm{I}}&=2\mathrm{Re}[\boldsymbol{A}_{\mathrm{I}}\langle\boldsymbol{f}_{\mathrm{I}}(z_P)\rangle],\quad z\in S^{+};\quad \boldsymbol{U}_{\mathrm{II}}=2\mathrm{Re}[\boldsymbol{A}_{\mathrm{II}}\langle\boldsymbol{f}_{\mathrm{II}}(z_P)+\boldsymbol{g}_{\mathrm{II}}(z_P)\rangle],\quad z\in S^{-}\\ \boldsymbol{\Phi}_{\mathrm{I}}&=2\mathrm{Re}[\boldsymbol{B}_{\mathrm{I}}\langle\boldsymbol{f}_{\mathrm{I}}(z_P)\rangle],\quad z\in S^{+};\quad \boldsymbol{\Phi}_{\mathrm{II}}=2\mathrm{Re}[\boldsymbol{B}_{\mathrm{II}}\langle\boldsymbol{f}_{\mathrm{II}}(z_P)+\boldsymbol{g}_{\mathrm{II}}(z_P)\rangle],\quad z\in S^{-}\end{aligned} \tag{3.166}$$

Some special cases are discussed as follows:

1. *Semi-infinite material*. If the material I is not existed, i.e., $x_2 = 0$ is a free plane, i.e., $\boldsymbol{f}_{\mathrm{I}}(z_j)=\boldsymbol{0},\boldsymbol{\Phi}_{\mathrm{II}}(x_1,0)=\boldsymbol{0}$. Let $\boldsymbol{A}_{\mathrm{II}}=\boldsymbol{A},\ \boldsymbol{B}_{\mathrm{II}}=\boldsymbol{B},$ then

$$\boldsymbol{f}(z_j)=\boldsymbol{g}_{\mathrm{II}}(z_j)-\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z_j) \tag{3.167}$$

2. *Material I is rigid.* $x_2 = 0$ is a fixed plane, i.e., $\boldsymbol{f}_{\mathrm{I}}(z_j) = \mathbf{0}, \boldsymbol{U}_{\mathrm{II}}(x_1, 0) = \boldsymbol{0}$. Let $\boldsymbol{A}_{\mathrm{II}} = \boldsymbol{A}, \quad \boldsymbol{B}_{\mathrm{II}} = \boldsymbol{B},$ then

$$\boldsymbol{f}(z_j) = \boldsymbol{g}_{\mathrm{II}}(z_j) - \boldsymbol{A}^{-1}\bar{\boldsymbol{A}}\bar{\boldsymbol{g}}_{\mathrm{II}}(z_j) \tag{3.168}$$

3. *Singularity at the upper semi-infinite plane*. If a singularity $z_0(x_{10}, x_{20})$ is located in the material I (Fig. 3.5b), then

$$\begin{aligned} &\boldsymbol{F}(z_j) = \boldsymbol{f}'(z_j) = \begin{cases} \boldsymbol{F}_{\mathrm{I}}(z_j) + \boldsymbol{G}_{\mathrm{I}}(z_j) & z \in S^+ \\ \boldsymbol{F}_{\mathrm{II}}(z_j) & z \in S^- \end{cases} \\ &\boldsymbol{G}_{\mathrm{I}}(z_j) = \boldsymbol{c}_{\mathrm{I}}\left\langle (z_j - z_{0j})^{-1} \right\rangle, \quad \boldsymbol{c}_{\mathrm{I}} = (1/2\pi \mathrm{i})\boldsymbol{V}_{\mathrm{I}}, \quad \boldsymbol{V}_{\mathrm{I}} = \left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{p}\right) \end{aligned} \tag{3.169}$$

$$\begin{aligned} \boldsymbol{F}_{\mathrm{I}}(z) &= \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{I}} - \bar{\boldsymbol{Y}}_{\mathrm{II}})\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z) = \left(\bar{\boldsymbol{A}}_{\mathrm{II}}^{-1}\boldsymbol{A}_{\mathrm{I}} - \bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\right)^{-1}\left(\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\bar{\boldsymbol{B}}_{\mathrm{I}} - \bar{\boldsymbol{A}}_{\mathrm{II}}^{-1}\bar{\boldsymbol{A}}_{\mathrm{I}}\right)\bar{\boldsymbol{G}}_{\mathrm{I}}(z) \\ \boldsymbol{F}_{\mathrm{II}}(z) &= \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{I}} + \boldsymbol{Y}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(z) = \left(\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}} - \bar{\boldsymbol{A}}_{\mathrm{I}}^{-1}\boldsymbol{A}_{\mathrm{II}}\right)^{-1}\left(\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{I}} - \bar{\boldsymbol{A}}_{\mathrm{I}}^{-1}\boldsymbol{A}_{\mathrm{I}}\right)\boldsymbol{G}_{\mathrm{I}}(z) \end{aligned} \tag{3.170}$$

3.6.3 Singularity on the Interface in a Bimaterial

Let a singularity $z_0(x_{01}, x_{02} = 0)$ be located on the interface in a biomaterial (Fig. 3.5c). Wang and Kuang (2000, 2002) took the following solution:

$$\begin{aligned} \boldsymbol{U}_{\alpha d} &= 2\mathrm{Re}\left[\boldsymbol{A}_\alpha \left\langle \ln(z_{\alpha j} - x_{01j})\right\rangle \boldsymbol{V}_\alpha\right], \quad \boldsymbol{V}_\alpha = (1/\pi)\left(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{l}_\alpha + \boldsymbol{B}_\alpha^{\mathrm{T}}\mathbf{g}_\alpha\right) \\ \boldsymbol{\Phi}_{\alpha d} &= 2\mathrm{Re}\left[\boldsymbol{B}_\alpha \left\langle \ln(z_{\alpha j} - x_{01j})\right\rangle \boldsymbol{V}_\alpha\right], \quad \alpha = \mathrm{I}, \mathrm{II} \end{aligned} \tag{3.171}$$

where $\boldsymbol{l}_\alpha, \mathbf{g}_\alpha$ are undetermined vectors. Draw a cut from x_{01} to $-\infty$; the jump value on the cut (between crack surfaces) of the generalized displacement and traction are

$$\boldsymbol{U}_{\mathrm{I}}(x_1, 0^+) - \boldsymbol{U}_{\mathrm{II}}(x_1, 0^-) = \boldsymbol{b}, \quad x_1 < 0; \quad \boldsymbol{\Phi}_{\mathrm{I}}(x_1, 0^+) - \boldsymbol{\Phi}_{\mathrm{II}}(x_1, 0^-) = \boldsymbol{p}\delta(x_{01}) \tag{3.172}$$

where $\delta(x_1)$ is the Dirac function. Using the following result (Qu and Li 1991),

$$\begin{aligned} &\lim_{x_2 \to \pm 0} \ln(x_1 + \mu x_2) = \ln|x_1| \pm \mathrm{i}\pi H(x_1), \\ &\lim_{x_2 \to \pm 0} \frac{1}{x_1 + \mu x_2} = \frac{1}{x_1} \mp \mathrm{i}\pi\delta(x_1), \quad \text{if } Im\,\mu > 0 \end{aligned} \tag{3.173}$$

where $H(x_1)$ is the Heaviside unit step function. Substituting Eqs. (3.171) and (3.173) into Eq. (3.172) and using Eq. (3.34) we get

$$\begin{aligned} \boldsymbol{b} &= (2/\pi)\mathrm{Re}\left\{\left(\left[\boldsymbol{A}_{\mathrm{I}}\langle \ln|x_1 - x_{01}| + \mathrm{i}\pi H(x_1)\rangle\left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\mathbf{g}_1 + \boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{l}_1\right)\right]\right.\right. \\ &\quad \left.\left. -\left[\boldsymbol{A}_{\mathrm{II}}\langle \ln|x_1 - x_{01}| - \mathrm{i}\pi H(x_1)\rangle\left(\boldsymbol{B}_{\mathrm{II}}^{\mathrm{T}}\mathbf{g}_2 + \boldsymbol{A}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{l}_2\right)\right]\right\} \\ &= (1/\pi)\ln|x_1 - x_{01}|(\mathbf{g}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}}) + \boldsymbol{S}_{\mathrm{I}}\mathbf{g}_{\mathrm{I}} + \boldsymbol{S}_{\mathrm{II}}\mathbf{g}_{\mathrm{II}} + \boldsymbol{M}_{\mathrm{I}}\boldsymbol{l}_{\mathrm{I}} + \boldsymbol{M}_{\mathrm{II}}\boldsymbol{l}_{\mathrm{II}} \\ \boldsymbol{p} &= (1/\pi x_1)(\boldsymbol{l}_{\mathrm{I}} - \boldsymbol{l}_{\mathrm{II}}) + \left(\boldsymbol{S}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{l}_{\mathrm{I}} + \boldsymbol{S}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{l}_{\mathrm{II}} - \boldsymbol{L}_{\mathrm{I}}\mathbf{g}_{\mathrm{I}} - \boldsymbol{L}_{\mathrm{II}}\mathbf{g}_{\mathrm{II}}\right) \end{aligned} \tag{3.174}$$

where $\boldsymbol{S}, \boldsymbol{M}$ and $\boldsymbol{L}$ are shown in Eq. (3.35) and all real matrixes. From Eq. (3.174) we get

$$\begin{aligned} &\mathbf{g}_{I} = \mathbf{g}_{\mathrm{II}} = \mathbf{g}, \quad \boldsymbol{l}_{\mathrm{I}} = \boldsymbol{l}_{\mathrm{II}} = \boldsymbol{l}, \quad \begin{Bmatrix} \boldsymbol{l} \\ \mathbf{g} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Omega_1} & \boldsymbol{\Omega_2} \\ \boldsymbol{\Omega_3} & \boldsymbol{\Omega_4} \end{bmatrix} \begin{Bmatrix} \boldsymbol{b} \\ \boldsymbol{p} \end{Bmatrix} \\ &\boldsymbol{\Omega_1} = \left\{(\boldsymbol{M}_1 + \boldsymbol{M}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)(\boldsymbol{L}_1 + \boldsymbol{L}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}\right\}^{-1} \\ &\boldsymbol{\Omega_2} = \left\{(\boldsymbol{M}_1 + \boldsymbol{M}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)(\boldsymbol{L}_1 + \boldsymbol{L}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}\right\}(\boldsymbol{S}_1 + \boldsymbol{S}_2)(\boldsymbol{L}_1 + \boldsymbol{L}_2)^{-1} \\ &\boldsymbol{\Omega_3} = \left\{(\boldsymbol{L}_1 + \boldsymbol{L}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}(\boldsymbol{M}_1 + \boldsymbol{M}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)\right\}^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}(\boldsymbol{M}_1 + \boldsymbol{M}_2)^{-1} \\ &\boldsymbol{\Omega_4} = -\left\{(\boldsymbol{L}_1 + \boldsymbol{L}_2) + (\boldsymbol{S}_1 + \boldsymbol{S}_2)^{\mathrm{T}}(\boldsymbol{M}_1 + \boldsymbol{M}_2)^{-1}(\boldsymbol{S}_1 + \boldsymbol{S}_2)\right\}^{-1} \end{aligned} \tag{3.175}$$

where $\mathbf{\Omega_1}, \mathbf{\Omega_2}, \mathbf{\Omega_3}, \mathbf{\Omega}_4$ are all real matrix. Substitution of Eq. (3.175) into Eq. (3.171) yields

$$\boldsymbol{V}_\alpha = \boldsymbol{M}_\alpha \boldsymbol{b} + \boldsymbol{N}_\alpha \boldsymbol{p}, \quad \boldsymbol{M}_\alpha = (1/\pi)(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{\Omega_1} + \boldsymbol{B}_\alpha^{\mathrm{T}}\boldsymbol{\Omega_3}), \quad \boldsymbol{N}_\alpha = (1/\pi)(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{\Omega_2} + \boldsymbol{B}_\alpha^{\mathrm{T}}\boldsymbol{\Omega_4}) \tag{3.176}$$

Zhou et al. (2007) discussed a generalized screw dislocation in a piezoelectric tri-material body.

3.6.4 Electric Dipole

Wang and Kuang (2000, 2002) discussed the electric dipole in a piezoelectric material. The electric dipole may be useful in the discussion on the electric switching wake. Let a generalized concentrate load $\boldsymbol{p} = -q_{\mathrm{e}}\mathbf{I}_4, \mathbf{I}_4 = [0, 0, 0, 1]^{\mathrm{T}}$ be acted at z_0 and $\boldsymbol{p} = q_{\mathrm{e}}\mathbf{I}_4$ acted at z_1. Solutions of these problems are $\boldsymbol{U}_0, \boldsymbol{\Phi}_0$ and $\boldsymbol{U}_1, \boldsymbol{\Phi}_1$, respectively:

$$\begin{aligned} &\boldsymbol{U}_0 = \mathrm{Re}\left[-(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{A}\left\langle \ln(z_j - z_{0j})\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right], \quad \boldsymbol{\Phi}_0 = \mathrm{Re}\left[-(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{B}\left\langle \ln(z_j - z_{0j})\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right] \\ &\boldsymbol{U}_1 = \mathrm{Re}\left[(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{A}\left\langle \ln(z_j - z_{1j})\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right], \quad \boldsymbol{\Phi}_1 = \mathrm{Re}\left[(q_{\mathrm{e}}/\pi\mathrm{i})\boldsymbol{B}\left\langle \ln(z_j - z_{1j})\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right] \end{aligned}$$

Using the relation,

$$\begin{aligned}&z_1 - z_0 = d(\cos\theta + \mathrm{i}\sin\theta) \to 0, \quad z_{1j} - z_{0j} = d\left(\cos\theta + \mu_j \sin\theta\right) \to 0 \\ &\lim_{d\to 0, q_e d \to p} \left\{q_e \ln(z_j - z_{1j}) - q_e \ln(z_j - z_{0j})\right\} = \lim_{d\to 0, q_e d\to p} q_e \ln\left[\left(z_j - z_{1j}\right)/\left(z_j - z_{0j}\right)\right] \\ &= -p_e\left[\Theta_j/\left(z_j - z_{0j}\right)\right]; \quad \lim_{q_e\to\infty, d\to 0} q_e d = p_e, \quad \Theta_j = \cos\theta + \mu_j \sin\theta\end{aligned} \tag{3.177}$$

where p_e is the electric pole couple and d is the distance from the negative charge to the positive charge. Thus the solution of an electric dipole in a homogeneous material is

$$\begin{aligned}\boldsymbol{U}_p &= \boldsymbol{U}_1 - \boldsymbol{U}_0 = \mathrm{Re}\left[\mathrm{i}(p_e/\pi)\boldsymbol{A}\left\langle \Theta_j\left(z_j - z_0\right)^{-1}\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right] \\ \boldsymbol{\Phi}_p &= \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_0 = \mathrm{Re}\left[\mathrm{i}(p_e/\pi)\boldsymbol{B}\left\langle \Theta_j\left(z_j - z_0\right)^{-1}\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right]\end{aligned} \tag{3.178}$$

$$\begin{aligned}\boldsymbol{\Sigma}_2 &= \boldsymbol{\Phi}_{,1} = \mathrm{Re}\left[(p_e/\pi\mathrm{i})\boldsymbol{B}\left\langle \Theta_j\left(z_j - z_0\right)^{-2}\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right] \\ \boldsymbol{\Sigma}_1 &= -\boldsymbol{\Phi}_{,2} = -\mathrm{Re}\left[(p_e/\pi\mathrm{i})\boldsymbol{B}\left\langle \mu_j\Theta_j\left(z_j - z_0\right)^{-2}\right\rangle \boldsymbol{A}^{\mathrm{T}}\mathbf{I}_4\right]\end{aligned} \tag{3.179}$$

For an electric dipole on the interface in a bimaterial consisted of materials I and II, the solution can be obtained from Eqs. (3.171) and (3.176):

$$\begin{aligned}\boldsymbol{U}_{\alpha d} &= 2\mathrm{Re}\left[\boldsymbol{A}_\alpha\left\langle \ln\left(z_{\alpha j} - x_{01} - d\right) - \ln\left(z_{\alpha j} - x_{01}\right)\right\rangle_\alpha \boldsymbol{N}_\alpha q_e\right]\mathbf{I}_4 \\ &= -2p_e\mathrm{Re}\left[\boldsymbol{A}_\alpha\left\langle \left(z_{\alpha j} - x_{01}\right)^{-1}\right\rangle \boldsymbol{N}_\alpha\right]\mathbf{I}_4 \\ \boldsymbol{\Phi}_{\alpha d} &= 2\mathrm{Re}\left[\boldsymbol{B}_\alpha\left\langle \ln\left(z_{\alpha_j} - x_{01} - d\right) - \ln\left(z_{\alpha j} - x_{01}\right)\right\rangle_\alpha \boldsymbol{N}_\alpha q_e\right]\mathbf{I}_4 \\ &= -2p_e\mathrm{Re}\left[\boldsymbol{B}_\alpha\left\langle \left(z_{\alpha j} - x_{01}\right)^{-1}\right\rangle \boldsymbol{N}_\alpha\right]\mathbf{I}_4; \ \alpha = \mathrm{I.II}\end{aligned} \tag{3.180}$$

3.6.5 General Case

Now we discuss a multiply connected plate. Let the kth singularity be located at $z_k^{(0)}$ and its total number be N, the kth inclusion occupy the region S_k and its total number be M, the region occupied by the piezoelectric material be denoted by S with the outer profile L_0, and the interface between S_k and S be L_k (Fig. 3.6). The complex stress function $\boldsymbol{\Phi}(z_P)$ can beassumed in the following form and complete determined by the boundary conditions:

$$\begin{aligned}&f_P(z_P) = C_P z_P + \sum_{k=1}^{N} \alpha_{Pk} \ln\left(z_P - z_{Pk}^{(0)}\right) + \sum_{k=1}^{M} \beta_{Pk} \ln(z_P - z_{Pk}) + f_{0P}(z_P) \\ &\boldsymbol{\alpha}_k = (1/2\pi\mathrm{i})\boldsymbol{V}_k, \quad \boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z_P)], \quad \boldsymbol{\Phi}_k - \boldsymbol{\Phi}_0 = -\oint_{L_k} \boldsymbol{T}\mathrm{d}s, \quad \oint_{L_k} \mathrm{d}\boldsymbol{U} = \boldsymbol{b}_k - \boldsymbol{b}_0\end{aligned} \tag{3.181}$$

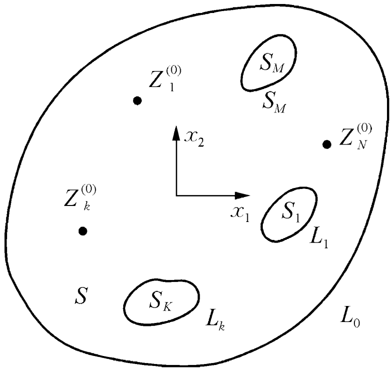


Fig. 3.6 General multiply connected plane zone

where z_{Pk} is a point inside the contour L_k and can be selected arbitrarily, $f_{0P}(z_P)$ is a single-valued function analytic in S, and $\boldsymbol{\Phi}_0, \boldsymbol{b}_0$ are constant vectors. If the singularity is considered as an infinitesimal inclusion, the terms containing singularity can be omitted. C_P can be determined by the stress condition at infinity and for a finite body $C_P = \mathbf{0}$; β_{Pk} can be expressed by the external generalized resultant force and the generalized dislocation acted on S_k. When we use the stress function method given in Sect. 3.3, the generalized stress function and displacement are expressed by $\boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{Bf}(z_P)]$ and $\boldsymbol{U} = (1/\pi)\mathrm{Im}[\boldsymbol{Af}(z_P)]$, respectively, where $\boldsymbol{B}$ and $\boldsymbol{A}$ are expressed by Eqs. (3.65) and (3.66).

3.7 Interaction of an Elliptic Inclusion with a Singularity

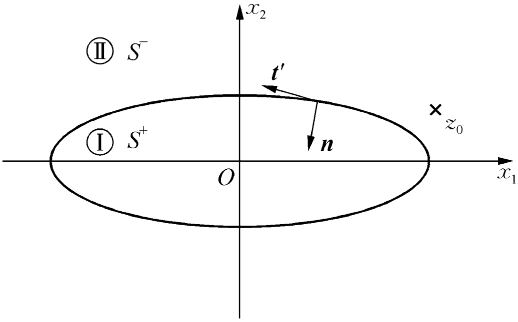
3.7.1 Green Function for a Singularity Outside the Elliptic Inclusion

Let an elliptic inclusion I with major axis 2a– and minor axis 2b–occupied S^+ be imbedded in an infinite piezoelectric material matrix II–occupied S^-. L is their interface. A singularity with strength $(\boldsymbol{b}, \boldsymbol{p})$ is acted at $z_0 = x_{01} + ix_{02}$ located in the matrix (Fig. 3.7). Huang and Kuang (2001b) discussed this problem under the conditions

$$\begin{aligned} &\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{\infty} = \mathbf{0}, \quad |z| \to \infty \\ &\boldsymbol{U}^{\mathrm{I}} = \boldsymbol{U}^{\mathrm{II}}, \quad \boldsymbol{\Phi}^{\mathrm{I}} = \boldsymbol{\Phi}^{\mathrm{II}}, \quad z \in L \end{aligned} \tag{3.182}$$

In this problem the second natural coordinate system is used, i.e., use $(\boldsymbol{n}, \boldsymbol{t}')$ in (3.29b) and $\boldsymbol{T} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s$. The transform method is used and the transform functions $\omega(\varsigma)$ and $\omega_j(\varsigma_j)$ are shown in Eqs. (3.82) and (3.86) respectively.

Fig. 3.7 An elliptic inclusion



In this section for clarity, the notations I, II will be written as superscripts. According to Ting (1996) and Huang and Kuang (2001b), the solution for a singularity outside the elliptic inclusion is assumed in the following form:

$$\begin{aligned}\boldsymbol{U}^{\mathrm{II}} &= (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \varsigma_{0j}^{\mathrm{II}}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] + (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left[\left(1/\varsigma_j^{\mathrm{II}}\right) - \bar{\varsigma}_{0\beta}^{\mathrm{II}}\right]\right\rangle \boldsymbol{V}_\beta''\right] \\ &+ (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle\left(1/k\varsigma_j^{\mathrm{II}\,k}\right)\right\rangle \mathbf{g}_k\right]\end{aligned} \tag{3.183}$$

$$\boldsymbol{U}^{\mathrm{I}} - \boldsymbol{U}_0^{\mathrm{I}} = (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle \ln\left(z_j^{\mathrm{I}} - y_{j\beta}^{\mathrm{I}}\right)\right\rangle \boldsymbol{V}_\beta'\right] + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{A}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle \boldsymbol{h}_k\right] \tag{3.184}$$

where $\boldsymbol{u}_0^{\mathrm{I}}, \boldsymbol{V}_\beta', \boldsymbol{V}_\beta'', \mathbf{g}_\delta, \boldsymbol{h}_\delta$ are undetermined vectors and

$$\begin{aligned}&\boldsymbol{V}^{\mathrm{II}} = \boldsymbol{B}^{\mathrm{IIT}}\boldsymbol{b} + \mathbf{A}^{\mathrm{IIT}}\boldsymbol{p}, \quad f_{jk}^{\mathrm{I}} = \left(\varsigma_j^{\mathrm{I}}\right)^k + \left(m_j^{\mathrm{I}}\right)^k\left(\varsigma_j^{\mathrm{I}}\right)^{-k} \\ &z_{0j}^{\mathrm{II}} = x_{01} + \mu_j^{\mathrm{II}}x_{02} = c_j^{\mathrm{II}}\varsigma_{0j}^{\mathrm{II}} + d_j^{\mathrm{II}}\left(\varsigma_{0j}^{\mathrm{II}}\right)^{-1}, \quad c_j^{\mathrm{II}} = R_j^{\mathrm{II}}, \quad d_j^{\mathrm{II}} = R_j^{\mathrm{II}}m_j^{\mathrm{II}} \\ &y_{j\beta}^{\mathrm{I}} = y_{j\beta1} + \mu_j^{\mathrm{I}}y_{j\beta2} = c_j^{\mathrm{I}}\varsigma_{0\beta}^{\mathrm{II}} + d_j^{\mathrm{I}}\left(\varsigma_{0\beta}^{\mathrm{II}}\right)^{-1}, \quad c_j^{\mathrm{I}} = R_j^{\mathrm{I}}, \quad d_j^{\mathrm{I}} = R_j^{\mathrm{I}}m_j^{\mathrm{I}}\end{aligned} \tag{3.185}$$

where f_{jk}^{I} is analytic in an annular region $\sqrt{m} < |\varsigma_i| < 1,\ 0 \le \theta < 2\pi$ (see Sect. 3.4.2).

Now Eqs. (3.183) and (3.184) will be explained in detail. In Eq. (3.183) the first term is the solution of a singularity when matrix II extended to whole space. It is noted that $\varsigma_{0\beta}^{\mathrm{II}}$ and $1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}$ are mirror images of each other with respect to the unit circle γ in ς plane, so $|\varsigma_{0\beta}^{\mathrm{II}}||1/\bar{\varsigma}_{0\beta}^{\mathrm{II}}| = 1$. From $(1/\varsigma_j^{\mathrm{II}}) - \bar{\varsigma}_{0\beta}^{\mathrm{II}} = 0$, it is known that this singularity is located at $(\varsigma_j^{\mathrm{II}}) = (1/\bar{\varsigma}_{0\beta}^{\mathrm{II}})$, so the second term represents solutions of 4 image singularities located at $1/\bar{\varsigma}^{\mathrm{II}}_{0\beta}$, $|1/\bar{\varsigma}^{\mathrm{II}}_{0\beta}| < 1$ inside the inclusion I, in ς plane, or total 16 image singularities located at $z^{\mathrm{I}}_{0j} = [c_j(1/\bar{\varsigma}^{\mathrm{II}}_{0\beta}) + d_j\bar{\varsigma}^{\mathrm{II}}_{0\beta}]$ in four z_j plane. Similarly the first term in Eq. (3.184) represents the solution for material I, its 16 image singularities located at $(y_{j\beta 1}, y_{j\beta 2})$ in the matrix II. For an impermeable hole and conductive rigid inclusion, $\boldsymbol{\Phi}^{\mathrm{I}}$ is not needed, so the third term in Eq. (3.183) can be omitted. This source function method is often used in the static electromagnetic field and stationary ideal fluid mechanics but here more complex. Using the relation

$$z^{\mathrm{I}}_j - y^{\mathrm{I}}_{j\beta} = c^{\mathrm{I}}_j\left(\varsigma^{\mathrm{I}}_j - \varsigma^{\mathrm{II}}_{0\beta}\right) + d^{\mathrm{I}}_j\left[\left(\varsigma^{\mathrm{I}}_j\right)^{-1} - \left(\varsigma^{\mathrm{II}}_{0\beta}\right)^{-1}\right] = c^{\mathrm{I}}_j\left(\varsigma^{\mathrm{I}}_j - \varsigma^{\mathrm{II}}_{0\beta}\right)\left[1 - \tau^{\mathrm{I}}_{j\beta}\left(\varsigma^{\mathrm{I}}_j\right)^{-1}\right]$$
$$\tau^{\mathrm{I}}_{j\beta} = m^{\mathrm{I}}_j\left(\varsigma^{\mathrm{II}}_{0\beta}\right)^{-1} \tag{3.186}$$

and $\mathrm{Im}(F) = -\mathrm{Im}(\bar{F})$, Eqs. (3.183) and (3.184), respectively, take

$$\begin{aligned}\boldsymbol{U}^{\mathrm{II}} &= \frac{1}{\pi}\mathrm{Im}\Big[-\bar{\boldsymbol{A}}^{\mathrm{II}}\left\langle \ln\left(\mathrm{e}^{-\mathrm{i}\psi} - \bar{\varsigma}^{\mathrm{II}}_{0j}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}} + \sum_{\beta=1}^{4}\ln\left(\mathrm{e}^{-\mathrm{i}\psi} - \bar{\varsigma}^{\mathrm{II}}_{0\beta}\right)\boldsymbol{A}^{\mathrm{II}}\boldsymbol{V}''_{\beta} \\ &\quad + \sum_{k=1}^{\infty}(1/k)\mathrm{e}^{-\mathrm{i}k\psi}\boldsymbol{A}^{\mathrm{II}}\mathbf{g}_k\Big]\end{aligned} \tag{3.187}$$

$$\begin{aligned}\boldsymbol{U}^{\mathrm{I}} - \boldsymbol{U}^{\mathrm{I}}_0 &= (1/\pi)\mathrm{Im}\Bigg\{\sum_{\beta=1}^{4}\Big[\boldsymbol{A}^{\mathrm{I}}\left\langle \ln c^{\mathrm{I}}_j\right\rangle \boldsymbol{V}'_{\beta}\Big) - \ln\left(\mathrm{e}^{-\mathrm{i}\psi} - \bar{\varsigma}^{\mathrm{II}}_{0\beta}\right)\bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{V}}'_{\beta} \\ &\quad + \boldsymbol{A}^{\mathrm{I}}\left\langle \ln\left(1 - \tau^{\mathrm{I}}_{j\beta}\mathrm{e}^{-\mathrm{i}\psi}\right)\boldsymbol{V}'_{\beta}\right\rangle\Big] + \sum_{k=1}^{\infty}(1/k)\mathrm{e}^{-\mathrm{i}k\psi}\left[-\bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k + \boldsymbol{A}^{\mathrm{I}}\left\langle\left(m^{\mathrm{I}}_j\right)^k\right\rangle \boldsymbol{h}_k\right]\Bigg\}\end{aligned} \tag{3.188}$$

In Eqs. (3.183) and (3.187) replacing $\boldsymbol{A}^{\mathrm{II}}$ by $\boldsymbol{B}^{\mathrm{II}}$, we get $\boldsymbol{\Phi}^{\mathrm{II}}$, and in Eqs. (3.184) and (3.188) replacing $\boldsymbol{A}^{\mathrm{I}}$ by $\boldsymbol{B}^{\mathrm{I}}$ and $\boldsymbol{u}^{\mathrm{I}}_0$ by $\boldsymbol{\Phi}^{\mathrm{I}}_0$, we get $\boldsymbol{\Phi}^{\mathrm{I}}$:

$$\begin{aligned}\boldsymbol{\Phi}^{\mathrm{II}} &= (1/\pi)\mathrm{Im}\left[\boldsymbol{B}^{\mathrm{II}}\left\langle \ln\left(\varsigma^{\mathrm{II}}_j - \varsigma^{\mathrm{II}}_{0j}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] + (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{B}^{\mathrm{II}}\left\langle \ln\left(1\big/\varsigma^{\mathrm{II}}_j - \bar{\varsigma}^{\mathrm{II}}_{0\beta}\right)\right\rangle \boldsymbol{V}''_{\beta}\right] \\ &\quad + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left(\boldsymbol{B}^{\mathrm{II}}\left\langle 1\big/k{\varsigma^{\mathrm{II}}_j}^{k}\right\rangle \mathbf{g}_k\right) \\ \boldsymbol{\Phi}^{\mathrm{I}} - \boldsymbol{\Phi}^{\mathrm{I}}_0 &= (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{B}^{\mathrm{I}}\left\langle \ln\left(z^{\mathrm{I}}_j - y^{\mathrm{I}}_{j\beta}\right)\right\rangle \boldsymbol{V}'_{\beta}\right] + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{B}^{\mathrm{I}}\left\langle f^{\mathrm{I}}_{jk}\right\rangle \boldsymbol{h}_k\right]\end{aligned} \tag{3.189}$$

Substituting these equations into the continuity conditions (3.182) on the interface and noting $\ln(1-x) = -\sum_{k=1}^{\infty} x^k/k$, the following equations to determine unknown functions are obtained:

$$\boldsymbol{U}_0^{\mathrm{I}} = -(1/\pi)\mathrm{Im}\left(\boldsymbol{A}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}'\right), \quad \boldsymbol{\Phi}_0^{\mathrm{I}} = -(1/\pi)\mathrm{Im}\left(\boldsymbol{B}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}'\right), \quad \boldsymbol{V}' = \sum_{\beta=1}^{4} \boldsymbol{V}'_\beta \tag{3.190a}$$

$$\begin{aligned} &\boldsymbol{A}^{\mathrm{II}}\boldsymbol{V}''_\beta + \bar{\boldsymbol{A}}^{\mathrm{I}}\boldsymbol{V}'_\beta = \bar{\boldsymbol{A}}^{\mathrm{II}}\mathbf{I}_\beta \bar{\boldsymbol{V}}^{\mathrm{II}}, \quad \boldsymbol{B}^{\mathrm{II}}\boldsymbol{V}''_\beta + \bar{\boldsymbol{B}}^{\mathrm{I}}\boldsymbol{V}'_\beta = \bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_\beta \bar{\boldsymbol{V}}^{\mathrm{II}} \\ &\mathbf{I}_1 = \langle 1,0,0,0\rangle,\ \mathbf{I}_2 = \langle 0,1,0,0\rangle,\ \mathbf{I}_3 = \langle 0,0,1,0\rangle,\ \mathbf{I}_4 = \langle 0,0,0,1\rangle \end{aligned} \tag{3.190b}$$

$$\begin{aligned} \boldsymbol{A}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k &= \boldsymbol{A}^{\mathrm{I}}\left[\left\langle \left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4}\left\langle \left(\tau_{j\beta}^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{V}'_\beta\right] \\ \boldsymbol{B}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{B}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k &= \boldsymbol{B}^{\mathrm{I}}\left[\left\langle \left(m_j^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4}\left\langle \left(\tau_{j\beta}^{\mathrm{I}}\right)^k\right\rangle \boldsymbol{V}'_\beta\right] \end{aligned} \tag{3.190c}$$

From Eq. (3.190b) we can get

$$\begin{aligned} &\boldsymbol{B}^{\mathrm{II}}\boldsymbol{V}'' = \bar{\boldsymbol{H}}^{-1}\left(\bar{\boldsymbol{Y}}^{\mathrm{I}} - \bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_\beta \bar{\boldsymbol{V}}^{\mathrm{II}}, \quad \bar{\boldsymbol{B}}^{\mathrm{I}}\boldsymbol{V}'_\beta = \bar{\boldsymbol{H}}^{-1}\left(\boldsymbol{Y}^{\mathrm{II}} + \bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_\beta \bar{\boldsymbol{V}}^{\mathrm{II}} \\ &\boldsymbol{Y}^\alpha = \mathrm{i}\boldsymbol{A}^\alpha(\boldsymbol{B}^\alpha)^{-1}, \quad \boldsymbol{H} = \boldsymbol{Y}^{\mathrm{I}} + \bar{\boldsymbol{Y}}^{\mathrm{II}} \end{aligned} \tag{3.191}$$

3.7.2 Green Function for a Singularity Inside the Elliptic Inclusion

When a singularity is located inside the elliptic inclusion, the solution can be assumed:

$$\begin{aligned} \boldsymbol{U}^{\mathrm{II}} = {} & (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \varsigma_{0j}^{\mathrm{I}}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] \\ & + (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left[\left(1\big/\varsigma_j^{\mathrm{II}}\right) - \left(1\big/\varsigma_{0\beta}^{\mathrm{I}}\right)\right]\right\rangle \boldsymbol{V}''_\beta\right] \\ & + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \left(1\big/k\varsigma_j^{\mathrm{II}\,k}\right)\right\rangle \mathbf{g}_k\right] \end{aligned} \tag{3.192}$$

$$\begin{aligned} \boldsymbol{U}^{\mathrm{I}} - \boldsymbol{U}_0^{\mathrm{I}} = {} & (1/\pi)\mathrm{Im}\left\{\boldsymbol{A}^{\mathrm{I}}\ln\left(z_j^{\mathrm{I}} - z_{0j}^{\mathrm{I}}\right)\boldsymbol{V}^{\mathrm{I}}\right\} \\ & + (1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle \ln\left(z_j^{\mathrm{I}} - \hat{y}_{j\beta}^{\mathrm{I}}\right)\right\rangle \left(\boldsymbol{V}'_\beta - \mathbf{I}_\beta \boldsymbol{V}^{\mathrm{I}}\right)\right] \\ & + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{A}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle \boldsymbol{h}_k\right] \end{aligned} \tag{3.193}$$

where

$$\begin{aligned} &\boldsymbol{V}^{\alpha} = \boldsymbol{B}^{\alpha\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}^{\alpha\mathrm{T}}\boldsymbol{p}, \quad \alpha = \mathrm{I}, \mathrm{II} \\ &z_{0j}^{\mathrm{I}} = c_j^{\mathrm{I}}\varsigma_{0j}^{\mathrm{I}} + d_j^{\mathrm{I}}\Big/\varsigma_{0j}^{\mathrm{I}}, \quad \hat{y}_{j\beta}^{\mathrm{I}} = \hat{y}_{j\beta 1} + \mu_j^{\mathrm{I}}\hat{y}_{j\beta 2} = c_j^{\mathrm{I}}\Big/\bar{\varsigma}_{0\beta}^{\mathrm{I}} + d_j^{\mathrm{I}}\bar{\varsigma}_{0\beta}^{\mathrm{I}} \end{aligned} \tag{3.194}$$

When z_0 is located in the inclusion, the single-valued cut from z_0 to $-\infty$ goes through the material II. So the first terms in Eqs. (3.192) and (3.193) are all discontinuous through this branch cut. The second term in Eq. (3.192) represents solutions for material II of 16 image singularities located at $\varsigma_{0\beta}^{\mathrm{I}}$ with $|\varsigma_{0\beta}^{\mathrm{I}} < 1|$ in the z_j plane. Similarly the second term in Eq. (3.193) represents solutions for material I of 16 image singularities located at $(\hat{y}_{j1}^{\mathrm{I}}, \hat{y}_{j2}^{\mathrm{I}})$ outside the elliptic inclusion.

In Eq. (3.192) replacing $\boldsymbol{A}^{\mathrm{II}}$ by $\boldsymbol{B}^{\mathrm{II}}$, we get $\boldsymbol{\Phi}^{\mathrm{II}}$, and in Eq. (3.193) replacing $\boldsymbol{A}^{\mathrm{I}}$ by $\boldsymbol{B}^{\mathrm{I}}$ and $\boldsymbol{u}_0^{\mathrm{I}}$ by $\boldsymbol{\Phi}_0^{\mathrm{I}}$, we get $\boldsymbol{\Phi}^{\mathrm{I}}$.

Substituting the solutions into the continuity conditions in Eq. (3.182) on the interface yields

$$\begin{aligned} &\boldsymbol{U}_0^{\mathrm{I}} = -(1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left\{\boldsymbol{A}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}' + \ln\left(-\varsigma_{0\beta}^{\mathrm{I}}\right)\left(\boldsymbol{A}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{A}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right)\right\} \\ &\boldsymbol{\Phi}_0^{\mathrm{I}} = -(1/\pi)\mathrm{Im}\sum_{\beta=1}^{4}\left\{\boldsymbol{B}^{\mathrm{I}}\left\langle \ln c_j^{\mathrm{I}}\right\rangle \boldsymbol{V}' + \ln\left(-\varsigma_{0\beta}^{\mathrm{I}}\right)\left(\boldsymbol{B}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{B}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right)\right\} \end{aligned} \tag{3.195a}$$

$$\begin{aligned} \boldsymbol{A}^{\mathrm{II}}\boldsymbol{V}_{\beta}'' + \bar{\boldsymbol{A}}^{\mathrm{I}}\boldsymbol{V}_{\beta}' = \bar{\boldsymbol{A}}^{\mathrm{II}}\mathbf{I}_{\beta}\bar{\boldsymbol{V}}^{\mathrm{II}} + 2\mathrm{Re}\left(\boldsymbol{A}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{A}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right) \\ \boldsymbol{B}^{\mathrm{II}}\boldsymbol{V}_{\beta}'' + \bar{\boldsymbol{B}}^{\mathrm{I}}\boldsymbol{V}_{\beta}' = \bar{\boldsymbol{B}}^{\mathrm{II}}\mathbf{I}_{\beta}\bar{\boldsymbol{V}}^{\mathrm{II}} + 2\mathrm{Re}\left(\boldsymbol{B}^{\mathrm{I}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}} - \boldsymbol{B}^{\mathrm{II}}\mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{II}}\right) \end{aligned} \tag{3.195b}$$

$$\begin{aligned} &\boldsymbol{A}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{A}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k = \boldsymbol{A}^{\mathrm{I}}\left[\left\langle\left(m_j^{\mathrm{I}}\right)^k\right\rangle\boldsymbol{h}_k - \left\langle\left(\tau_{0j}^{\mathrm{I}}\right)^k\right\rangle\boldsymbol{V}^{\mathrm{I}} - \sum_{\beta=1}^{4}\left\langle\left(\hat{\tau}_{j\beta}^{\mathrm{I}}\right)^k\right\rangle\left(\boldsymbol{V}_{\beta}' - \mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}}\right)\right] \\ &\boldsymbol{B}^{\mathrm{II}}\mathbf{g}_k + \bar{\boldsymbol{B}}^{\mathrm{I}}\bar{\boldsymbol{h}}_k = \boldsymbol{B}^{\mathrm{I}}\left[\left\langle\left(m_j^{\mathrm{I}}\right)^k\right\rangle\boldsymbol{h}_k - \left\langle\left(\tau_{0j}^{\mathrm{I}}\right)^k\right\rangle\boldsymbol{V}^{\mathrm{I}} - \sum_{\beta=1}^{4}\left\langle\left(\hat{\tau}_{j\beta}^{\mathrm{I}}\right)^k\right\rangle\left(\boldsymbol{V}_{\beta}' - \mathbf{I}_{\beta}\boldsymbol{V}^{\mathrm{I}}\right)\right] \\ &\tau_{0j}^{\mathrm{I}} = m_j^{\mathrm{I}}\left(1\Big/\varsigma_{0j}^{\mathrm{I}}\right), \quad \hat{\tau}_{j\beta}^{\mathrm{I}} = m_j^{\mathrm{I}}\bar{\varsigma}_{0\beta}^{\mathrm{II}} \end{aligned} \tag{3.195c}$$

3.7.3 Green Function for a Singularity on the Interface

When a singularity is located at $z_0 = a\cos\psi_0 + \mathrm{i}b\sin\psi_0$ on the elliptic boundary, Eqs. (3.183) and (3.184) become

$$\begin{aligned} \boldsymbol{U}^{\mathrm{II}} = {} & (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left(\varsigma_j^{\mathrm{II}} - \mathrm{e}^{\mathrm{i}\psi_0}\right)\right\rangle \boldsymbol{V}^{\mathrm{II}}\right] + (1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{II}}\left\langle \ln\left[\left(1\Big/\varsigma_j^{\mathrm{II}}\right) - \mathrm{e}^{-\mathrm{i}\psi_0}\right]\right\rangle \boldsymbol{V}''\right] \\ & + (1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left(\boldsymbol{A}^{\mathrm{II}}\left\langle 1\Big/k\varsigma_j^{\mathrm{II}\,k}\right\rangle \mathbf{g}_k\right) \end{aligned} \tag{3.196}$$

$$\boldsymbol{U}^{\mathrm{I}}-\boldsymbol{U}_{0}^{\mathrm{I}}=(1/\pi)\mathrm{Im}\left[\boldsymbol{A}^{\mathrm{I}}\left\langle\ln\left(z_{j}^{\mathrm{I}}-z_{0j}^{\mathrm{I}}\right)\right\rangle\boldsymbol{V}'\right]+(1/\pi)\mathrm{Im}\sum_{k=1}^{\infty}\left[(1/k)\boldsymbol{A}^{\mathrm{I}}\left\langle f_{jk}^{\mathrm{I}}\right\rangle\mathbf{h}_{k}\right] \tag{3.197}$$

where

$$\begin{aligned}\boldsymbol{V}''&=\sum_{\beta=1}^{4}\boldsymbol{V}_{\beta}''=\left(\boldsymbol{B}^{\mathrm{II}}\right)^{-1}\left(\boldsymbol{Y}^{\mathrm{II}}+\bar{\boldsymbol{Y}}^{\mathrm{I}}\right)^{-1}\left(\bar{\boldsymbol{Y}}^{\mathrm{I}}-\bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\bar{\boldsymbol{V}}^{\mathrm{II}}\\ \boldsymbol{V}'&=\sum_{\beta=1}^{4}\boldsymbol{V}_{\beta}'=\left(\boldsymbol{B}^{\mathrm{I}}\right)^{-1}\left(\boldsymbol{Y}^{\mathrm{I}}+\bar{\boldsymbol{Y}}_{\mathrm{II}}\right)^{-1}\left(\boldsymbol{Y}^{\mathrm{II}}+\bar{\boldsymbol{Y}}^{\mathrm{II}}\right)\bar{\boldsymbol{B}}^{\mathrm{II}}\bar{\boldsymbol{V}}^{\mathrm{II}}\end{aligned} \tag{3.198}$$

In Eq. (3.196) replacing $\boldsymbol{A}^{\mathrm{II}}$ by $\boldsymbol{B}^{\mathrm{II}}$, we get $\boldsymbol{\Phi}^{\mathrm{II}}$, and in Eq. (3.197) replacing $\boldsymbol{A}^{\mathrm{I}}$ by $\boldsymbol{B}^{\mathrm{I}}$ and $\boldsymbol{u}_{0}^{\mathrm{I}}$ by $\boldsymbol{\Phi}_{0}^{\mathrm{I}}$, we get $\boldsymbol{\Phi}^{\mathrm{I}}$.

From Eqs. (3.192) and (3.193), we still get the same result.

3.7.4 Material Force Between the Singularity and the Elliptic Inclusion

Eshelby (1956) defined the material force as the negative gradient of the total mechanical and electrical energy with respect to the position variation of the defect. For a linear electroelasticity, we can also use the total electric enthalpy (Eq. (1.55)) instead of total energy. The general method calculating the material force is given in many literatures, such as Lardner (1974), Pak (1990), Wen and Hwu (1994), and Kuang et al. (1998). The total electric enthalpy of the system for a dislocation at (x_{01}, x_{02}) can be defined as the work required to introduce the dislocation in the material, i.e.,

$$H=(1/2)\int_{x_{01}+\delta}^{\Lambda}(\sigma_{2i}b_{i}+D_{2}b_{4})\mathrm{d}x_{1},\quad \delta\to0,\quad \Lambda\to\infty \tag{3.199}$$

1. *Dislocation is inside the matrix*. Equation (3.199) becomes

$$\begin{aligned}&H=(1/2)\boldsymbol{b}^{\mathrm{T}}\cdot\boldsymbol{\Phi}^{\mathrm{II}}\Big|_{z_{0j}^{\mathrm{II}}+\delta_{j}}^{z_{0j}^{\mathrm{II}}=\Lambda_{j}}\\ &\boldsymbol{\Phi}^{\mathrm{II}}\Big|_{z_{0j}^{\mathrm{II}}+\delta_{j}}^{z_{0j}^{\mathrm{II}}=\Lambda_{j}}=(1/\pi)\mathrm{Im}\Big[\boldsymbol{B}^{\mathrm{II}}\langle\ln(\Lambda_{j}/\delta_{j})\rangle\boldsymbol{V}^{\mathrm{II}}+\boldsymbol{B}^{\mathrm{II}}\left\langle\ln\left(1-m_{j}^{\mathrm{II}}\Big/\varsigma_{0j}^{\mathrm{II}}\varsigma_{0j}^{\mathrm{II}}\right)\right\rangle\boldsymbol{V}^{\mathrm{II}}\\ &\qquad-\sum_{\beta=1}^{4}\boldsymbol{B}^{\mathrm{II}}\left\langle\ln\left(1-1\Big/\varsigma_{0j}^{\mathrm{II}}\bar{\varsigma}_{0\beta}^{\mathrm{II}}\right)\right\rangle\boldsymbol{V}_{\beta}''-\sum_{k=1}^{\infty}\boldsymbol{B}^{\mathrm{II}}\left\langle1\Big/k\varsigma_{0j}^{\mathrm{II}\,k}\right\rangle\mathbf{g}_{k}\Big]\end{aligned} \tag{3.200}$$

By excluding singular part of the dislocation enthalpy itself, the interaction enthalpy part of the media with the dislocation is obtained as

$$\begin{aligned} H_{\text{int}}^{\text{II}} = (1/2\pi)\boldsymbol{b}^{\text{T}} \cdot \text{Im}\Big[&\boldsymbol{B}^{\text{II}}\Big\langle \ln\Big(1 - m_j^{\text{II}} \Big/ \varsigma_{0j}^{\text{II}}\varsigma_{0j}^{\text{II}}\Big)\Big\rangle \boldsymbol{V}^{\text{II}} \\ &- \sum_{\beta=1}^{4} \boldsymbol{B}^{\text{II}}\Big\langle \ln\Big(1 - 1\Big/\varsigma_{0j}^{\text{II}}\bar{\varsigma}_{0\beta}^{\text{II}}\Big)\Big\rangle \boldsymbol{V}'' - \sum_{k=1}^{\infty} \boldsymbol{B}^{\text{II}}\Big\langle 1\Big/k\varsigma_{0j}^{\text{II}\,k}\Big\rangle \mathbf{g}_k\Big] \end{aligned} \tag{3.201}$$

2. *Dislocation is inside the inclusion*. Equation (3.199) becomes

$$\begin{aligned} H = (1/2)\boldsymbol{b}^{\text{T}} \cdot \left(\boldsymbol{\Phi}^{\text{II}}\Big|_{z_{0j}^{\text{II}*}}^{z_{0j}^{\text{II}}=\Lambda_j} + \boldsymbol{\Phi}^{\text{I}}\Big|_{z_{0j}^{\text{I}}+\delta_j}^{z_{0j}^{\text{I}*}}\right) = \frac{1}{2\pi}\boldsymbol{b}^{\text{T}} \cdot \text{Im}\Big[&\boldsymbol{B}^{\text{II}}\langle \ln \Lambda_j\rangle \boldsymbol{V}^{\text{II}} - \boldsymbol{B}^{\text{I}}\langle \ln \delta_j\rangle \boldsymbol{V}^{\text{I}} \\ - \sum_{k=1}^{\infty} (1/k)\boldsymbol{B}^{\text{I}}\Big\langle f_{jk}^{\text{I}}\Big\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4} \boldsymbol{B}^{\text{I}}\Big\langle \ln\Big(z_{0j}^{\text{I}} - \hat{y}_{j\beta}^{\text{I}}\Big)\Big\rangle \Big(\boldsymbol{V}'_\beta - \text{I}_\beta \boldsymbol{V}^{\text{I}}\Big) \\ - \sum_{\beta=1}^{4} \ln\Big(-\varsigma_{0\beta}^{\text{I}}\Big)\Big(-\boldsymbol{B}^{\text{II}}\boldsymbol{V}''_\beta + \boldsymbol{B}^{\text{I}}\text{I}_\beta \boldsymbol{V}^{\text{I}} - \boldsymbol{B}^{\text{II}}\text{I}_\beta \boldsymbol{V}^{\text{II}}\Big)\Big] \end{aligned} \tag{3.202}$$

where $z_{0j}^{\text{II}*} = z_{0j}^{\text{I}*}$ is the same point on the interface. By excluding singular part of the dislocation enthalpy itself, the interaction enthalpy part of the media with the dislocation is obtained as

$$\begin{aligned} H_{\text{int}}^{\text{I}} = \frac{1}{2\pi}\mathbf{b}^{\text{T}} \cdot \text{Im}\Big[&- \sum_{k=1}^{\infty} (1/k)\boldsymbol{B}^{\text{I}}\Big\langle f_{jk}^{\text{I}}\Big\rangle \boldsymbol{h}_k - \sum_{\beta=1}^{4} \boldsymbol{B}^{\text{I}}\Big\langle \ln\Big(z_{0j}^{\text{I}} - \hat{y}_{j\beta}^{\text{I}}\Big)\Big\rangle \Big(\boldsymbol{V}'_\beta - \text{I}_\beta \boldsymbol{V}^{\text{I}}\Big) \\ &- \sum_{\beta=1}^{4} \ln\Big(-\varsigma_{0\beta}^{\text{I}}\Big)\Big(-\boldsymbol{B}^{\text{II}}\boldsymbol{V}''_\beta + \boldsymbol{B}^{\text{I}}\text{I}_\beta \boldsymbol{V}^{\text{I}} - \boldsymbol{B}^{\text{II}}\text{I}_\beta \boldsymbol{V}^{\text{II}}\Big)\Big] \end{aligned} \tag{3.203}$$

The generalized interaction force per unit length $\boldsymbol{F}$ along the direction $\boldsymbol{s}$ on the dislocation is

$$\boldsymbol{F} = -\partial H_{\text{int}}/\partial \boldsymbol{s} \tag{3.204}$$

which is usually obtained by numerical calculation.

3.7.5 Numerical Example

Let the matrix be PZT-5H and the inclusions be epoxy, insulated void, and rigid conductor, respectively. Usually material constants are given in the material principle coordinate system (X_1, X_2, X_3) with poling axis X_3. For PZT-5H matrix,

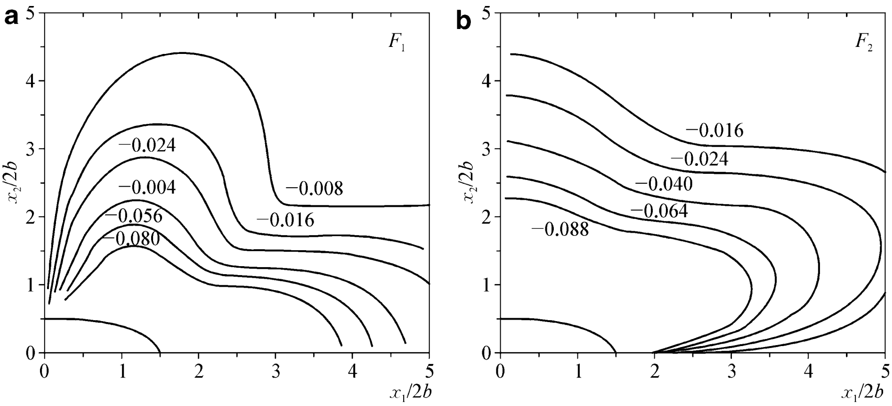


Fig. 3.8 PZT-5H/epoxy under loading $\mathbf{b}/2b = (1,0,0,0)$: (**a**) contour plots of the dimensionless glide force F_1 and (**b**) contour plots of the dimensionless climb force F_2

$$
\begin{aligned}
&C_{11}^{\mathrm{II}} = 126, \quad C_{33}^{\mathrm{II}} = 117, \quad C_{44}^{\mathrm{II}} = 35.3, \quad C_{12}^{\mathrm{II}} = 55, \quad C_{13}^{\mathrm{II}} = 53(\mathrm{GPa}) \\
&e_{31}^{\mathrm{II}} = -6.5, \quad e_{33}^{\mathrm{II}} = 23.3, \quad e_{15}^{\mathrm{II}} = 17.0\left(\mathrm{C/m^2}\right) \\
&\epsilon_{11}^{\mathrm{II}} = 15.1 \times 10^{-9}, \quad \epsilon_{33}^{\mathrm{II}} = 13.0 \times 10^{-9}\left(\mathrm{C^2/Nm^2}\right)
\end{aligned}
$$

For inclusion epoxy,

$$
\begin{aligned}
&C_{11}^{\mathrm{I}} = 6.43, \quad C_{33}^{\mathrm{I}} = 6.429, \quad C_{44}^{\mathrm{I}} = 1.07, \quad C_{12}^{\mathrm{I}} = 4.29, \quad C_{13}^{\mathrm{I}} = 4.289(\mathrm{GPa}) \\
&e_{31}^{\mathrm{I}} = e_{33}^{\mathrm{I}} = e_{15}^{\mathrm{I}} = 0\left(\mathrm{C/m^2}\right), \quad \epsilon_{11}^{\mathrm{I}} = 5.0 \times 10^{-9}, \quad \epsilon_{33}^{\mathrm{I}} = 5.001 \times 10^{-9}\left(\mathrm{C^2/Nm^2}\right)
\end{aligned}
$$

Material constants for epoxy were be modified slightly to avoid repeated eigenvalues. It is noted that in above analyses of this section, the coordinates (x_1, x_2, x_3) with polarized x_2-axis are used, so in numerical calculation, materials should be converted. The corresponding relation between (X_1, X_2, X_3) and (x_1, x_2, x_3) is $X_3 \to x_2, X_1 \to x_1, X_2 \to x_3$.

The dimensionless glide force F_1 and climb force F_2 of the interaction between the inclusion and dislocation are defined as

$$
F_1 = -\left(\partial H_{\mathrm{int}}^e / \partial x_1\right) / \left(L_{11} \times 10^{-9} / 4\pi\right), \quad F_2 = -\left(\partial H_{\mathrm{int}}^e / \partial x_2\right) / \left(L_{11} \times 10^{-9} / 4\pi\right) \tag{3.205}
$$

where L_{11} is shown in Eq. (3.35). F_1 and F_2 will be numerically studied. The positive glide and climb forces show that the dislocation is repelled from x_2- and x_1-axes, respectively. Figures 3.8 and 3.9 show the contour plots of F_1 and F_2 under two cases: (1) $\mathbf{b}/2b = (1,0,0,0)$, only mechanical dislocation b_1, and (2) $\mathbf{b}/2b = (0,0,0,10^9\ \mathrm{V/m})$, only electric dislocation b_4.

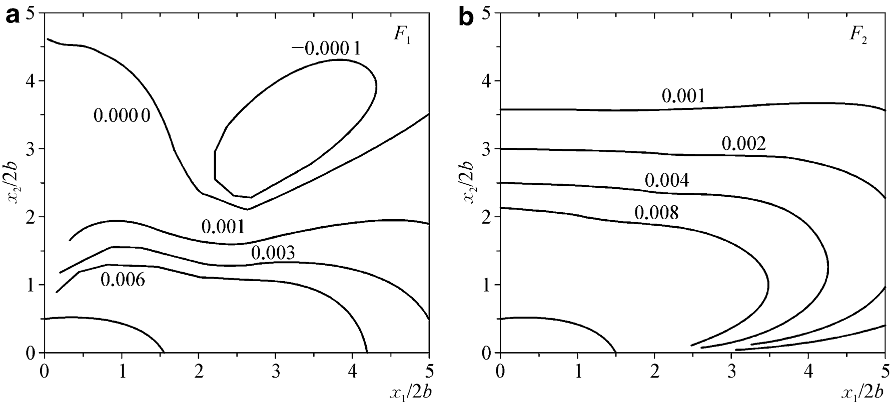


Fig. 3.9 PZT-5H/epoxy under loading $\mathbf{b}/2b = (0, 0, 0, 10^9\ \mathrm{V/m})$: (**a**) contour plots of the dimensionless glide force F_1 and (**b**) contour plots of the dimensionless climb force F_2

Interaction of an elliptic inclusion with a singularity was discussed in many literatures, such as Meguid and Deng (1998), Deng and Meguid (1999), Liu et al. (1999), and Fan et al. (2005).

3.8 Asymptotic Fields near a Line Inclusion Tip in a Homogeneous Material

3.8.1 A General Form of the Asymptotic Fields near a Line Inclusion Tip

Discuss a homogeneous material with a line inclusion. It is assumed that the size of the line inclusion is much smaller than that of the material. The region near the tip to be suitable for an asymptotic analysis is much smaller than that of the line inclusion, so the asymptotic fields near a line inclusion tip in a practical structure are almost the same as that in a semi-infinite line inclusion. Let a semi-infinite line inclusion be along the axis x_1 from the origin to negative infinite, i.e., the region Ω of the material is $0 \le r \le \infty, -\pi < \theta \le \pi$, where θ is the polar angle (Fig. 3.10). The asymptotic fields near the right tip can be assumed in the following form (Ting 1996; Kuang and Ma 2002):

$$\begin{aligned} f_j(z_j) &= V_j z_j^{\lambda+1}/(\lambda+1), \quad z_j = x_1 + \mu_j x_2 = r(\cos\theta + \mu_j \sin\theta) \\ F_j(z_j) &= f_j'(z_j) = V_j z_j^{\lambda}, \quad \mu_j = \alpha_j + i\beta_j \end{aligned} \tag{3.206}$$

where $\boldsymbol{V}$ is an undetermined complex constant, λ is an undetermined singular index, and $\lambda > -1/2$ to keep a finite strain energy density at the tip region. In plane problem we often use $\boldsymbol{f}(z_j), \boldsymbol{F}(z_j)$ instead of $\boldsymbol{f}(z_P), \boldsymbol{F}(z_P)$, where $j = 1, 2, 3$ and $j = 3$

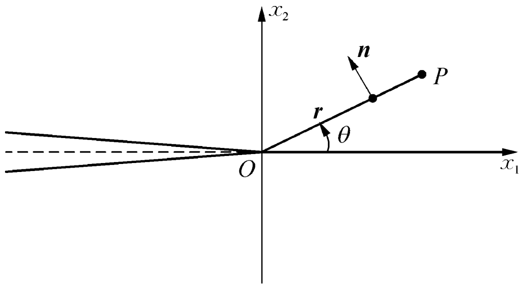


Fig. 3.10 Local coordinate system at a crack tip

represent the electric variables, as shown in Eq. (3.206). From Eqs. (3.26) and (3.206), it is known that

$$\begin{aligned}
\boldsymbol{U} &= (\lambda+1)^{-1}\sum_{j=1}^{4}\left(\boldsymbol{a}_j z_j^{\lambda+1}V_j + \bar{\boldsymbol{a}}_j \bar{z}_j^{\lambda+1}\bar{V}_j\right) = (\lambda+1)^{-1}\left[\boldsymbol{A}\left\langle z_j^{\lambda+1}\right\rangle\boldsymbol{V} + \bar{\boldsymbol{A}}\left\langle \bar{z}_j^{\lambda+1}\right\rangle\bar{\boldsymbol{V}}\right]\\
\Phi &= (\lambda+1)^{-1}\sum_{j=1}^{4}\left(\boldsymbol{b}_j z_j^{\lambda+1}V_j + \bar{\boldsymbol{b}}_j \bar{z}_j^{\lambda+1}\bar{V}_j\right) = (\lambda+1)^{-1}\left[\boldsymbol{B}\left\langle z_j^{\lambda+1}\right\rangle\boldsymbol{V} + \bar{\boldsymbol{B}}\left\langle \bar{z}_j^{\lambda+1}\right\rangle\bar{\boldsymbol{V}}\right]\\
\boldsymbol{\Sigma}_1 &= -\boldsymbol{\Phi}_{,2} = -\sum_{j=1}^{4}\left(\mu_j\boldsymbol{b}_j z_j^{\lambda}V_j + \bar{\mu}_j\bar{\boldsymbol{b}}_j \bar{z}_j^{\lambda}\bar{V}_j\right) = \boldsymbol{B}\left\langle \mu_j z_j^{\lambda}\right\rangle\boldsymbol{V} + \bar{\boldsymbol{B}}\left\langle \bar{\mu}_j\bar{z}_j^{\lambda}\right\rangle\bar{\boldsymbol{V}}\\
\boldsymbol{\Sigma}_2 &= \boldsymbol{\Phi}_{,1} = \sum_{j=1}^{4}\left(\boldsymbol{b}_j z_j^{\lambda}V_j + \bar{\boldsymbol{b}}_j \bar{z}_j^{\lambda}\bar{V}_j\right) = \boldsymbol{B}\left\langle z_j^{\lambda}\right\rangle\boldsymbol{V} + \bar{\boldsymbol{B}}\left\langle \bar{z}_j^{\lambda}\right\rangle\bar{\boldsymbol{V}}
\end{aligned} \tag{3.207}$$

In the polar coordinate system, the normal of a radial plane is $\boldsymbol{n}\ (-\sin\theta, \cos\theta)$ which is identical with the tangent $\boldsymbol{t}$ of a circle with the center at the coordinate origin (Fig. 3.10). The traction $\boldsymbol{T}$ on the radial plane is

$$\begin{aligned}
T_i &= \sigma_{i1}n_1 + \sigma_{i2}n_2 = -\sigma_{i1}\sin\theta + \sigma_{i2}\cos\theta = \Phi_{i,2}\sin\theta + \Phi_{i,1}\cos\theta\\
&= \Phi_i'(z_j)\left(z_j/r\right) = 2\mathrm{Re}\sum_{j=1}^{4}\left(B_{ij}z_j^{\lambda+1}V_j r^{-1}\right), \quad \boldsymbol{T} = 2\mathrm{Re}\left\{r^{-1}\boldsymbol{B}\left\langle z_j^{\lambda+1}\right\rangle\boldsymbol{V}\right\}
\end{aligned} \tag{3.208}$$

where $\Phi_i'(z_j) = \mathrm{d}\Phi_i(z_j)/\mathrm{d}z_j$. Equation (3.208) can be used to discuss the asymptotic field near a wedge, but in this book we only discuss the line inclusion.

3.8.2 The Stress Singularity

The stress singularity near a tip is related to the boundary conditions of the inclusion.

1. *Two sides of the line crack are free*. The boundary conditions are

$$\boldsymbol{T}(r, \pm\pi) = \boldsymbol{0}, \quad \text{or} \quad \boldsymbol{\Sigma}_2(r, \pm\pi) = \boldsymbol{0} \tag{3.209}$$

Substituting Eq. (3.207) or (3.208) into Eq. (3.209) and noting $x_{1j} = x_1; z_j(r,0) = r, z_j(r,\pm\pi) = re^{\pm i\pi}$ on axis x_1 yield

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{BV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\mathbf{B}\boldsymbol{V} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \text{or} \\ &\sum_{j=1}^{4}\left(\mathrm{e}^{\mathrm{i}\lambda\pi}V_j\boldsymbol{b}_j + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{b}}_j\right) = \boldsymbol{0}, \quad \sum\nolimits_{j=1}^{4}\left(\mathrm{e}^{-\mathrm{i}\lambda\pi}V_j\boldsymbol{b}_j + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{b}}_j\right) = \boldsymbol{0} \end{aligned} \tag{3.210}$$

Equation (3.210) yields

$$\left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right)^4\boldsymbol{BV} = \boldsymbol{0}, \quad \text{or} \quad \left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right)\boldsymbol{b}_jV_j = \boldsymbol{0}; \quad j = 1-4 \tag{3.211}$$

Because B is not singular, so we have the eigenvalue equation

$$\left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right)^4 = 0, \quad \text{or} \quad \left(1 - \mathrm{e}^{4\mathrm{i}\lambda\pi}\right) = 0 \tag{3.212}$$

From Eq. (3.212) it is known that $\lambda = -1/2, 0, m/2$, where m is an integer. When $\lambda = -1/2$, the generalized stresses are singular with the singular index $-1/2$.

2. *Two sides of the line crack are fixed (rigid inclusion with zero electric potential)*. The boundary conditions are

$$\boldsymbol{U}(\pm\pi) = \boldsymbol{0} \tag{3.213}$$

We have

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \text{or} \\ &\sum_{j=1}^{4}\left(\mathrm{e}^{\mathrm{i}\lambda\pi}V_j\boldsymbol{a}_j + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{a}}_j\right) = \boldsymbol{0}, \quad \sum_{j=1}^{4}\left(\mathrm{e}^{-\mathrm{i}\lambda\pi}V_j\boldsymbol{a}_j + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{V}_j\bar{\boldsymbol{a}}_j\right) = \boldsymbol{0} \end{aligned} \tag{3.214}$$

It is also found that the eigenvalue equation is Eq. (3.212), so we also have $\lambda = -1/2, 0, m/2$; m is an integer.

3. *One side free and one side fixed*. The boundary conditions are

$$\boldsymbol{T}(r,\pi) = \boldsymbol{0}, \quad \boldsymbol{U}(-\pi) = \boldsymbol{0} \tag{3.215}$$

We have

$$\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{BV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0} \tag{3.216}$$

The eigenvalue equation is

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\lambda\pi}\boldsymbol{BV} + \mathrm{e}^{-\mathrm{i}\lambda\pi}\bar{\boldsymbol{B}}\bar{\boldsymbol{V}} = \boldsymbol{0}, \quad \mathrm{e}^{-\mathrm{i}\lambda\pi}\boldsymbol{AV} + \mathrm{e}^{\mathrm{i}\lambda\pi}\bar{\boldsymbol{A}}\bar{\boldsymbol{V}} = \boldsymbol{0}; \quad \text{or} \\ &\left(\mathrm{e}^{-2\mathrm{i}\lambda\pi}\boldsymbol{Y} + \mathrm{e}^{2\mathrm{i}\lambda\pi}\bar{\boldsymbol{Y}}\right)\boldsymbol{BV} = \boldsymbol{0}; \quad \boldsymbol{Y} = \mathrm{i}\boldsymbol{AB}^{-1}, \quad \bar{\boldsymbol{Y}} = -\mathrm{i}\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}^{-1} \end{aligned} \tag{3.217}$$

Substituting Eq. (3.37), $\boldsymbol{Y} = -\mathrm{i}(\boldsymbol{S} + \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1}, \boldsymbol{S} = \mathrm{i}\left(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} - \mathbf{I}\right)$, into Eq. (3.217), the eigen-equation in a more convenient form is obtained:

$$\left[-\mathrm{i}e^{-2\mathrm{i}\lambda\pi}(\boldsymbol{S} + \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1} + \mathrm{i}e^{2\mathrm{i}\lambda\pi}(\boldsymbol{S} - \mathrm{i}\mathbf{I})\boldsymbol{L}^{-1}\right]\boldsymbol{B}\boldsymbol{V} = \mathbf{0} \quad \Rightarrow \quad (\boldsymbol{S} + \cot 2\lambda\mathbf{I})\boldsymbol{L}^{-1}\boldsymbol{B}\boldsymbol{V} = \mathbf{0} \tag{3.218}$$

From the above analyses, it is known that the singular index λ is independent to the selected z_j plane.

3.8.3 The Stress Asymptotic Field near a Crack Tip

From Eqs. (3.212) and(3.214), it is known that when $\lambda = -1/2$, the stresses are singular. Substituting it into Eqs. (3.206) and (3.207) yields

$$\begin{aligned}
&F_j(z_j) = V_j\big/\sqrt{z_j} = V_j\big/\sqrt{r\Theta_j}, \quad \Theta_j = \cos\theta + \mu_j \sin\theta \\
&\Sigma_{1i} = -2\mathrm{Re}\sum_{j=1}^{4}\left(\mu_j b_{ji} V_j\big/\sqrt{z_j}\right) = -2\mathrm{Re}\sum_{j=1}^{4} B_{ij}\mu_j V_j\big/\sqrt{r\Theta_j} \\
&\Sigma_{2i} = 2\mathrm{Re}\sum_{j=1}^{4}\left(b_{ji} V_j\big/\sqrt{z_j}\right) = 2\mathrm{Re}\sum_{j=1}^{4} B_{ij} V_j\big/\sqrt{r\Theta_j} \\
&\boldsymbol{\Sigma}_1 = -2\mathrm{Re}\boldsymbol{B}\left\langle \mu_j\big/\sqrt{z_j}\right\rangle\boldsymbol{V}, \quad \boldsymbol{\Sigma}_2 = 2\mathrm{Re}\boldsymbol{B}\left\langle 1\big/\sqrt{z_j}\right\rangle\boldsymbol{V}
\end{aligned} \tag{3.219}$$

Define the stress intensities as

$$\boldsymbol{K} = (K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_D)^T = \lim_{r\to 0}\sqrt{2\pi r}(\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2)^{\mathrm{T}}\big|_{\theta=0} = \lim_{r\to 0}\sqrt{2\pi r}\,\boldsymbol{\Sigma}_2\big|_{\theta=0} \tag{3.220}$$

Let

$$\boldsymbol{V} = 1\Big/\left(2\sqrt{2\pi}\right)\boldsymbol{B}^{-1}\boldsymbol{K}, \quad V_i = 1\Big/\left(2\sqrt{2\pi}\right)B_{ij}^{-1}K_j, \quad B_{ij}^{-1} = \left[\boldsymbol{B}^{-1}\right]_{ij} \tag{3.221}$$

Substitution of Eqs. (3.220) and (3.221) into Eq. (3.219) yields

$$\begin{aligned}
&\Sigma_{1i} = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{4} B_{ij}\mu_j B_{jl}^{-1}K_l\big/\sqrt{\Theta_j}, \quad \Sigma_{2i} = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\sum_{j=1}^{4} B_{ij}B_{jl}^{-1}K_l\big/\sqrt{\Theta_j} \\
&\boldsymbol{\Sigma}_1 = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\boldsymbol{B}\left\langle \mu_j\big/\sqrt{\Theta_j}\right\rangle\boldsymbol{B}^{-1}\boldsymbol{K}, \quad \boldsymbol{\Sigma}_2 = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\boldsymbol{B}\left\langle 1\big/\sqrt{\Theta_j}\right\rangle\boldsymbol{B}^{-1}\boldsymbol{K}
\end{aligned} \tag{3.222}$$

It is noted that in general situation $B_{ij}B_{jl}^{-1}K_l\big/\sqrt{\Theta_j} \neq K_l\big/\sqrt{\Theta_j}$. But when $\theta = 0$ and $\Theta_j = 1$, $B_{ij}B_{jl}^{-1}K_l\big/\sqrt{\Theta_j} = B_{ij}B_{jl}^{-1}K_l = K_l$ and $\boldsymbol{\Sigma}_2 = \boldsymbol{K}\big/\sqrt{2\pi r}$.

References

Barnett DM, Lothe J (1975) Dislocations and line charges in anisotropic piezoelectric insulators. Physica Status Solidi (b) 67:105–111

Chung MY, Ting TCT (1996) Piezoelectric solid with an elliptic inclusion or hole. Int J Solid Struct 33:3343–3361

Dempsey JP, Sinclair GB (1979) On the stress singularities in the plane elasticity of the composite wedge. J Elast 9:373–391

Deng W, Meguid SA (1999) Analysis of a screw dislocation inside an elliptical inhomogeneity in piezoelectric solids. Int J Solid Struct 36:1449–1469

Eshelby JD (1956) The continuum theory of lattice defects. Solid State Phys 3:79–144. Edited by Seitz and TuruBull

Fan QH, Liu YW, Jiang CP (2005) Electroelastic interaction between a piezoelectric screw dislocation and an elliptical inclusion with interfacial cracks. Physica Status Solidi (b) 242:2775–2794

Gao C-F, Fan W-X (1999) Exact solutions for the plane problem in piezoelectric material with an elliptic or a crack. Int J Solid Struct 36:2527–2540

Hao TH, Shen ZY (1994) A new electric boundary condition of electric fracture mechanics and its applications. Eng Fract Mech 47:793–802

Hoenig A (1982) Near tip behavior of a crack in a plane anisotropic elastic body. Eng Fract Mech 16:393–403

Huang Z-Y, Kuang Z-B (2000a) Explicit expression of the A and B matrices for piezoelectric media. Mech Res Commun 27:575–581

Huang Z-Y, Kuang Z-B (2000b) Asymptotic electro-elastic field near a blunt crack tip in a transversely isotropic piezoelectric material. Mech Res Commun 27:601–606

Huang Z-Y, Kuang Z-B (2001) Dislocation inside a piezoelectric media with an elliptic inhomogeneity. Int J Solid Struct 38:8459–8479

Kantorovich LV, Krylov VI (1958) Approximate methods of higher analysis. Interscience, New York

Kosmodamianskii AS, Lozhkin VN (1975) Generalized two-dimensional stressed state of thin piezoelectric plates. Prikl Mekh (Prikladnaya Mekhanika) 1:45–53

Kuang Z-B (1982) The stress field near the blunt crack tip and the fracture criterion. Eng Frac Mech 16:19–33

Kuang Z-B (2011) Theory of electroelasticity. Shanghai Jiaotong University Press, Shanghai (in Chinese)

Kuang Z-B, Gu H-C, Li Z-H (1998) The mechanical behavior of materials. Higher Education Publishing House, Beijing (in Chinese)

Kuang Z-B, Ma F-S (2002) Crack tip field. Xian Jiaotong University Press, Xian (in Chinese)

Lardner RW (1974) Mathematical theory of dislocations and fracture. University of Toronto Press, Toronto

Lekhnitskii SG (1987) Anisotropic plates, 2nd edn transl by Tsai SW, Cheron T. Gordon and Breach, New York; Лехницкий СГ (1957) Анизотропные пластинки Государственные издадельство технико-теоретической литературы. Москва

Liu JX, Du SY, Wang B (1999) A screw dislocation interaction with a piezoelectric bimaterial interface. Mech Res Commun 26:415–420

Meguid SA, Deng W (1998) Electro-elastic interaction between a screw dislocation and an elliptical inhomogeneity in piezoelectric materials. Int J Solid Struct 35:1467–1482

Mura T (1987) Micromechanics of defects in solids, 2nd Rev Edn. Martinus-Nijhoff, Deventer

Muskhelishvili NI (1975) Some basic problems of mathematical theory of elasticity. Noordhoof, Leyden; Мусхелишвили НЕ (1954) Некоторые осноные задачи математической теории упругости. Издательство академии наук СССР, Масква

Pak YE (1990) Force on a piezoelectric screw dislocation. J Appl Mech 57:863–869

Parton VZ (1976) Fracture mechanics of piezoelectric materials. Acta Astronaut 3:671–683

Qu J, Li Q (1991) Interfacial dislocation and its applications to interface cracks in anisotropic materials. J Elast 26:169–195
Ru CQ (1997) Eshelby's problem for two-dimensional piezoelectric inclusions of arbitrary shape. Proc Math Phys Eng Sci 156:1051–1068
Savin GN (1961) Stress concentration around holes. Pergamon, New York
Shen S-P, Kuang Z-B (1999) An alternative expression of piezoelectric representation and its application in inclusion problem. Int J Appl Electrmagn Mech 10:279–292
Sosa H (1991) Plane problems in piezoelectric media with defects. Int J Solid Struct 28:491–505
Sosa H, Khutoryansky N (1996) New developments concerning piezoelectric materials with defects. Int J Solid Struct 33:3399–3414
Stroh AN (1958) Dislocations and cracks in anisotropic elasticity. Philos Mag 3:625–646
Suo Z (1990) Singularities interfaces and cracks in dissimilar anisotropic media. Proc R Soc Lond A 427:331–358
Suo Z, Kuo CM, Barnett DM, Willis JR (1992) Fracture mechanics for piezoelectric ceramics. J Mech Phys Solid 40:739–765
Ting TCT (1996) Anisotropic elasticity, theory and applications. Oxford University Press, New York/Oxford
Tucker MO (1969) Plane boundaries and straight dislocations in elastically anisotropic materials. Philos Mag 19:1141–1159
Wang J-W, Kuang Z-B (2000) The interaction between crack and electric dipole of piezoelectricity. Acta Mech Solida Sin 13:283–289
Wang J-W, Kuang Z-B (2002) The electric dipole and crack on the interface in a bi-piezoelectric material. Acta Mech Sin 34:192–199 (in Chinese)
Wen JY, Hwu CB (1994) Interactions between dislocations and anisotropic elastic elliptical inclusions, Trans ASME J Appl Mech 61:548–554
Xu XL, Rajapakse RKND (1999) Analytical solution for an arbitrarily oriented void/crack and fracture of piezoceramics. Acta Mater 47:1735–1747
Yang X-X, Shen S-P, Kuang Z-B (1997) The degenerate solution for piezothermoelastic materials. Eur J Mech A Solid 16:779–793
Zeng X, Rajapakse RKND (2003) Eshelby tensor for piezoelectric inclusion and application to modeling of domain switching and evolution. Acta Mater 51:4121–4134
Zhang T-Y, Qian C-F, Tong P (1998) Linear electro-elastic analysis of a cavity or a crack in a piezoelectric material. Int J Solid Struct 35:2121–2149
Zhou Z-D, Nishioka T, Kuang Z-B (2007) A generalized screw dislocation in a piezoelectric tri-material body. Int J Appl Electr Mech 26:21–36

Chapter 4
Linear Inclusion and Related Problems

Abstract In this chapter, the linear cracks and inclusions are discussed. These problems are mainly reduced to vector Riemann-Hilbert boundary problem with many variables at first, and then the standard method to solve the Riemann-Hilbert boundary problem is used. In general case, the numerical computation is used to get the final results due to its complexity, but for some simpler problems, the analytical solutions can also be obtained. The interface cracks, rigid inclusion, and electrodes in piezoelectric bimaterials are discussed in detail. Some special problems, such as partly insulated and partly conducting crack, the nonideal crack and some other models in a homogeneous piezoelectric material, and contact zone model for interface cracks in a piezoelectric bimaterial, are also discussed shortly. Some interesting problems in engineering, such as interaction of collinear inclusions with singularity loading, interaction of an elliptic hole and a vice-crack, strip electric saturation model of an impermeable crack in a homogeneous material and a strip electric saturation model for mode-III interface crack in a bimaterial, and mode-III problem for a circular inclusion with interface cracks, are also discussed.

Keywords Linear interface crack and inclusion • Singularity • Strip electric saturation model • Circular inclusion

4.1 Vector Riemann-Hilbert Boundary-Value Problem in the z Plane

4.1.1 Fundamental Solution of the Homogeneous Equation

Let n non-intersect line segments $L_k, k = 1 \sim n$, be in the complex z plane, and its assemble is denoted by L. The end points of L_k are a_k, b_k and from a_k to b_k is its positive direction; the left region of $a_k b_k$ is the region S^+, and the right region is S^-.

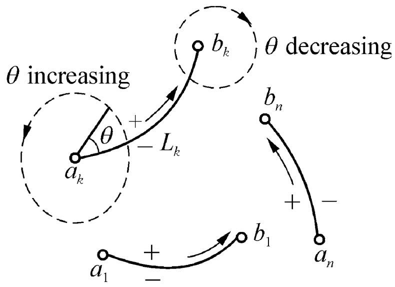


Fig. 4.1 Riemann-Hilbert boundary problem on smooth non-intersecting curves

On L functions $\mathbf{g}(t)$ and $\boldsymbol{\Sigma}_0(t)$ satisfied Hölder condition are given (Fig. 4.1). Now discuss thesolution of the following vector Riemann-Hilbert equation on L (Muskhelishvili 1954, 1975; Hou et al. 1990):

$$\boldsymbol{h}^+(t) - \mathbf{g}\boldsymbol{h}^-(t) = \boldsymbol{\Sigma}_0(t), \quad h_j^+(t) - g_{ij}h_j^-(t) = \Sigma_{0i}(t), \quad i,j = 1 - m; \quad t \in L \tag{4.1}$$

where $\mathbf{g}$ is an $m \times m$ order Hermite matrix and $\det \mathbf{g} \neq 0$ and t is a point on L. The superscripts " + " and " − " indicate the limit values taken from the left and right sides along $a_k b_k$, respectively. The corresponding homogeneous equation is

$$\boldsymbol{h}^+(t) - \mathbf{g}\boldsymbol{h}^-(t) = \mathbf{0}, \quad h_j^+(t) - g_{ij}h_j^-(t) = 0, \quad t \in L \tag{4.2}$$

Let the fundamental solution of the homogeneous equation be

$$\begin{aligned} &\boldsymbol{X}_0(z) = [X_{01}(z), X_{02}(z), \ldots X_{0m}(z))]^{\mathrm{T}} = \boldsymbol{\omega} Y_0(z) \\ &X_{0j}(z) = \omega_j Y_0(z), \quad Y_0(z) = \prod_{k=1}^{n} (z - a_k)^{-\gamma}(z - b_k)^{\gamma - 1} \\ &\boldsymbol{X}_0^-(t) = \mathrm{e}^{-2\pi \mathrm{i}\gamma}\boldsymbol{X}_0^+(t), \quad \text{or} \quad \boldsymbol{X}_0^+(t) = \mathrm{e}^{2\pi \mathrm{i}\gamma}\boldsymbol{X}_0^-(t) \end{aligned} \tag{4.3a}$$

Usually, the single-valued branch of the multi-value function $Y_0(z)$ is selected such that $Y_0(z) \to z^{-n}$ when $z \to \infty$. Substitution of Eq. (4.3a) into Eq. (4.2) yields

$$\mathrm{e}^{2\pi \mathrm{i}\gamma}\boldsymbol{X}_0^-(t) - \mathbf{g}\boldsymbol{X}_0^-(t) = 0 \quad \Rightarrow \quad (\mathrm{e}^{2\pi \mathrm{i}\gamma}\mathbf{I} - \mathbf{g})\boldsymbol{\omega} = 0 \tag{4.4}$$

In order to have nontrivial solution for $\boldsymbol{\omega}$, it must be

$$\left|\mathrm{e}^{2\pi \mathrm{i}\gamma}\mathbf{I} - \mathbf{g}\right| = 0, \quad \mathbf{I} = \mathrm{diag}[1, 1, \ldots, 1]_{m \times m} \tag{4.5}$$

From Eqs. (4.5) and (4.4), we can get m eigenvectors $\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \ldots, \boldsymbol{\omega}^{(m)}$ corresponding to m eigenvalues, $\mathrm{e}^{2\pi \mathrm{i}\gamma_1}, \mathrm{e}^{2\pi \mathrm{i}\gamma_2}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\gamma_m}$, where γ_k is limited within the semi-open interval $[0, 2\pi)$. For a eigenvalue $e^{2\pi i\gamma_k}$, there is only one component of $\boldsymbol{\omega}^{(i)}$ is undetermined. The fundamental solution Eq. (4.3a) becomes

$$\begin{aligned} &\mathbf{X}_0^{(i)}(z) = \boldsymbol{\omega}^{(i)} Y_0^{(i)}(z), \quad X_{0j}^{(i)}(z) = \omega_j^{(i)} Y_0^{(i)}(z) \\ &Y_0^{(i)}(z) = \prod_{k=1}^{n} (z - a_k)^{-\gamma_i} (z - b_k)^{\gamma_i - 1}, \quad i, j = 1, 2, \ldots, m \end{aligned} \tag{4.3b}$$

The complete fundamental solutions form a square matrix $\boldsymbol{P}(z)$:

$$\begin{aligned} &\boldsymbol{P}(z) = \left[\boldsymbol{X}_0^{(1)}(z), \boldsymbol{X}_0^{(2)}(z), \ldots, \boldsymbol{X}_0^{(n)}(z)\right] = \boldsymbol{\Omega Q}(z), \quad P_{ij}(z) = X_{0j}^{(i)}(z) \\ &\boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z) \right\rangle = \operatorname{diag}\left[Y_0^{(1)}(z), \ldots, Y_0^{(m)}(z)\right], \quad \boldsymbol{\Omega} = \left[\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \ldots, \boldsymbol{\omega}^{(m)}\right] \end{aligned} \tag{4.6}$$

4.1.2 First Solving Method

From the behavior of the fundamental solution, it is known that

$$\boldsymbol{P}^+(t) - \mathbf{g}\boldsymbol{P}^-(t) = \mathbf{0}, \quad \mathbf{g} = \boldsymbol{P}^+(t)[\boldsymbol{P}^-(t)]^{-1}, \quad t \in L \tag{4.7}$$

Substitution of Eq. (4.7) into Eq. (4.2) yields

$$[\boldsymbol{P}^+(t)]^{-1}\boldsymbol{h}^+(t) = [\boldsymbol{P}^-(t)]^{-1}\boldsymbol{h}^-(t), \quad t \in L \tag{4.8}$$

The function at the left side in Eq. (4.8) is analytic in S^+, whereas those on the right side are analytic in S^-, and they are continuous on L. So these functions are analytic in whole plane and must be constants. The general solution $\boldsymbol{h}_0(z)$ of Eq. (4.2) is

$$\begin{aligned} &[\boldsymbol{P}(z)]^{-1}\boldsymbol{h}_0(z) = \boldsymbol{C}(z), \quad \boldsymbol{h}_0(z) = \boldsymbol{P}(z)\boldsymbol{C}(z) \\ &\boldsymbol{C}(z) = \mathbf{c}_n z^n + \mathbf{c}_{n-1} z^{n-1} + \cdots + \boldsymbol{c}_0, \quad \boldsymbol{c}_k = [c_k^{(1)}, c_k^{(2)}, \ldots, c_k^{(m)}]^{\mathrm{T}}, \quad \text{or} \\ &\boldsymbol{C}(z) = [C^{(1)}(z), C^{(2)}(z), \ldots, C^{(m)}(z)]^{\mathrm{T}}, \quad C^{(k)}(z) = \left[c_n^{(k)} z^n + c_{n-1}^{(k)} z^{n-1} + \cdots + c_0^{(k)}\right] \end{aligned} \tag{4.9}$$

If the infinite point is a pole in order p, $\boldsymbol{C}(z)$ is a vector polynomial less than order $n + p$.

Substitution of Eq. (4.7) into the inhomogeneous equation (4.1) yields

$$[\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{h}^{+}(t)-[\boldsymbol{P}^{-}(t)]^{-1}\boldsymbol{h}^{-}(t)=[\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{\Sigma}_0(t),\quad t\in L \tag{4.10}$$

Equation (4.10) is a decoupling Riemann-Hilbert boundary problem of $[\boldsymbol{P}(z)]^{-1}\boldsymbol{h}(z)$. By using the Cauchy formula, the special solution $\boldsymbol{h}_{\mathrm{sp}}$ is

$$[\boldsymbol{P}(z)]^{-1}\boldsymbol{h}_{\mathrm{sp}}(z)=\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\boldsymbol{\Sigma}_0(t)\mathrm{d}t}{\boldsymbol{P}^{+}(t)(t-z)},\quad \boldsymbol{h}_{\mathrm{sp}}(z)=\frac{\boldsymbol{P}(z)}{2\pi\ \mathrm{i}}\int_L\frac{\boldsymbol{\Sigma}_0(t)\mathrm{d}t}{\boldsymbol{P}^{+}(t)(t-z)} \tag{4.11}$$

The general solution of the inhomogeneous equation (4.1) is

$$\begin{aligned}
\boldsymbol{h}(z)&=\boldsymbol{h}_0(z)+\boldsymbol{h}_{\mathrm{sp}}(z)\\
&=\boldsymbol{P}(z)\left[\boldsymbol{C}(z)+\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\tilde{\boldsymbol{\Sigma}}(t)\mathrm{d}t}{(t-z)}\right]=\sum_{k=1}^{m}\mathbf{X}_0^{(k)}(z)\left[\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\tilde{\Sigma}_k(t)}{(t-z)}\mathrm{d}t+C^{(k)}(z)\right]\\
\tilde{\boldsymbol{\Sigma}}(t)&=[\tilde{\Sigma}_1(t),\tilde{\Sigma}_2(t),\ldots,\tilde{\Sigma}_m(t)]^{\mathrm{T}}=[\boldsymbol{P}^{+}(t)]^{-1}\boldsymbol{\Sigma}_0(t)
\end{aligned} \tag{4.12}$$

4.1.3 Second Solving Method

Because $\mathbf{g}$ is a Hermite matrix, the eigenvectors corresponding to the different eigenvalues are orthogonal to each other in the complex space. Form a square matrix $\boldsymbol{\Omega}$ consisted of $\boldsymbol{\omega}$ and

$$\begin{aligned}
&\boldsymbol{\Omega}=\left[\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\ldots,\boldsymbol{\omega}^{(m)}\right],\quad \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\boldsymbol{\Lambda},\quad \bar{\boldsymbol{\Omega}}^{-\mathrm{T}}=\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\\
&\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\mathbf{g}\Omega=\boldsymbol{M},\quad \boldsymbol{M}=\mathrm{diag}[\mathrm{e}^{2\pi\ \mathrm{i}\gamma_1}\Lambda_1^2,\ldots,\mathrm{e}^{2\pi\ \mathrm{i}\gamma_m}\Lambda_m^2]\\
&\boldsymbol{\Lambda}=\mathrm{diag}[\Lambda_1^2,\Lambda_2^2,\ldots,\Lambda_m^2],\quad \Lambda_i^2=\bar{\omega}_1^{(i)}\omega_1^{(i)}+\bar{\omega}_2^{(i)}\omega_2^{(i)}+\cdots+\bar{\omega}_m^{(i)}\omega_m^{(i)}
\end{aligned} \tag{4.13}$$

In most cases $\boldsymbol{\Omega}$ is assumed normalized, i.e., $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\boldsymbol{\Lambda}=\mathbf{I}$.

Multiplying on both sides of Eq. (4.1) from left by $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}$ and using Eq. (4.13) we get

$$\begin{aligned}
&\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}^{+}(t)-(\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\mathbf{g}\Omega)\boldsymbol{\Omega}^{-1}\boldsymbol{h}^{-}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}^{+}(t)-\boldsymbol{M}\boldsymbol{\Lambda}^{-1}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}^{-}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(t)\\
&\boldsymbol{M}\boldsymbol{\Lambda}^{-1}=\mathrm{diag}[\mathrm{e}^{2\pi\ \mathrm{i}\gamma_1},\mathrm{e}^{2\pi\ \mathrm{i}\gamma_2},\ldots,\mathrm{e}^{2\pi\ \mathrm{i}\gamma_m}]=\left\langle\mathrm{e}^{2\pi\ \mathrm{i}\gamma_j}\right\rangle
\end{aligned} \tag{4.14}$$

Equation (4.14) can be expressed in the following decoupling form:

$$\begin{aligned}
&\boldsymbol{\Psi}^{+}(t)-\boldsymbol{M}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Psi}^{-}(t)=\boldsymbol{\Sigma}^{*}(t),\quad \Psi_i^{+}(t)-\mathrm{e}^{2\pi\ \mathrm{i}\gamma_i}\Psi_i^{-}(t)=\Sigma_i^{*}(t)\\
&\boldsymbol{\Psi}(z)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z),\quad \boldsymbol{\Sigma}^{*}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(t)
\end{aligned} \tag{4.15}$$

Equation (4.15) is the scalar Riemann-Hilbert boundary-value problem of the component Ψ_i of $\boldsymbol{\Psi}$, soits solution is

$$\begin{aligned}
&\boldsymbol{\Psi}(z)=\boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda C}(z)+\frac{1}{2\pi \mathrm{i}}\int_L\frac{[\boldsymbol{Q}^+(t)]^{-1}\boldsymbol{\Sigma}^*(t)\mathrm{d}t}{(t-z)}\right\};\quad \boldsymbol{h}\ \ (z)=\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z)\\
&\Psi_i(z)=Y_0^{(i)}(z)\left\{\Lambda_i^2C^{(i)}(z)+\frac{1}{2\pi \mathrm{i}}\int_L\frac{\Sigma_i^*(t)\mathrm{d}t}{Y_0^{(i)+}(t)(t-z)}\right\}\\
&\boldsymbol{Q}(z)=\left\langle Y_0^{(i)}(z)\right\rangle
\end{aligned}\tag{4.16}$$

Solving $\boldsymbol{\Psi}(z)$, $\boldsymbol{h}(z)$ is obtained by $\boldsymbol{h}(z)=\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z)$, where $\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}=[\bar{\boldsymbol{\Omega}}^{\mathrm{T}}]^{-1}=\boldsymbol{\Omega\Lambda}^{-1}$. If we assume $[\boldsymbol{Q}^+(t)]_{ij}^{-1}\Sigma_j^*(t)\to\boldsymbol{\alpha}_qt^q+\cdots+\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_{-1}/t+\cdots$, when $t\to\infty$ and it is single valued, Eq. (4.16) is reduced to

$$\begin{aligned}
&\boldsymbol{\Psi}(z)=\boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda C}(z)+\left\langle 1-\mathrm{e}^{2\pi\mathrm{i}\gamma_i}\right\rangle^{-1}\left[[\boldsymbol{Q}(z)]^{-1}\boldsymbol{\Sigma}^*(z)-\left(\boldsymbol{\alpha}_qz^q+\cdots+\boldsymbol{\alpha}_0\right)\right]\right\}\\
&\Psi_i(z)=Y_0^{(i)}(z)\left\{\Lambda_i^2C^{(i)}(z)+\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}\gamma_i}}\left[\frac{\boldsymbol{\Sigma}_i^*(z)}{Y_0^{(i)}(z)}-\left(\alpha_q^{(i)}z^q+\cdots+\alpha_0^{(i)}\right)\right]\right\}
\end{aligned}\tag{4.17}$$

where the following integral formula has been used (Shen and Kuang 1998):

$$\begin{aligned}
&\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{G^+(t)\mathrm{d}t}{X^+(t)(t-z)}=\frac{1}{1-\mathrm{g}\mathrm{g}^*}\left[\frac{G(z)}{X(z)}-\alpha_qz^q-\cdots-\alpha_0\right]\\
&X^+(t)-\mathrm{g}X^-(t)=0,\quad G^-(t)=\mathrm{g}^*G^+(t),\quad \mathrm{g}=\mathrm{e}^{2\pi\mathrm{i}\gamma},\quad t\in L
\end{aligned}\tag{4.18}$$

For a single-valued function $G(z)$, $\mathrm{g}^*=1$, Eq. (4.18) is just the formula given by Muskhelishvili (1954).

The two methods are equivalent. In fact by using $\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}=\boldsymbol{\Omega\Lambda}^{-1}$, Eq. (4.15) can be reduced to

$$\begin{aligned}
\boldsymbol{h}(z)=\bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z)&=\boldsymbol{\Omega\Lambda}^{-1}\boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda C}(z)+\frac{1}{2\pi \mathrm{i}}\int_L\frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(x_1)\mathrm{d}t}{\boldsymbol{Q}^+(x_1)(x_1-z)}\right\}\\
&=\boldsymbol{\Omega Q}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\boldsymbol{\Sigma}_0(x_1)\mathrm{d}t}{\boldsymbol{\Omega Q}^+(x_1)(x_1-z)}\right\}\\
&=\boldsymbol{P}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\ \mathrm{i}}\int_L\frac{\tilde{\boldsymbol{\Sigma}}(x_1)\mathrm{d}x_1}{(x_1-z)}\right\}
\end{aligned}$$

which is identical with Eq. (4.12).

4.2 Interface Cracks in Piezoelectric Bimaterials

4.2.1 General Discussion of an Impermeable Interface Cracks

Discuss a piezoelectric bimaterial with collinear impermeable cracks without generalized loading at infinity (Suo 1990; Suo et al. 1992; Kuang and Ma 2002). Let the material I be located at the upper half plane S^+, $x_2 > 0$; the material II is located at the lower half plane S^-, $x_2 < 0$; $x_2 = 0$ is the interface L, there are collinear cracks, the left end point of the crack L_k is denoted by a_k, and the right end point b_k and its assemble is denoted by L_c. $L - L_c$ is the connected surface (Fig. 4.2). For a single crack with length $2a$, we always let the coordinate origin be selected at the center of the crack. These notations will be used in this whole chapter. Assuming the generalized forces $\boldsymbol{\Sigma}_0(x_1) = [t_1^*, t_2^*, t_3^*, -\sigma^*]^{\mathrm{T}}$ acting on the crack surfaces are self-equilibrium, at infinity, generalized forces are equal to zero, i.e.,

$$\begin{aligned}&\boldsymbol{\Sigma}(x_1) = \boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) = \boldsymbol{\Sigma}_0(x_1), \quad x \in L_{\mathrm{c}}\\ &\boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) = \mathbf{0}, \quad \text{at infinity;} \quad \boldsymbol{\Sigma}_\beta(x_1) = 2\mathrm{Re}[\boldsymbol{B}_\beta \mathbf{F}_\beta(x_1)], \quad \beta = \mathrm{I}, \mathrm{II}\end{aligned} \tag{4.19}$$

On the connected surface, the generalized displacements and traction are continuous:

$$\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = \mathbf{0}, \quad \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1), \quad x \in L - L_{\mathrm{c}} \tag{4.20}$$

where $\hat{\boldsymbol{d}}(x_1)$ is the displacement disconnected value between crack surfaces and the crack opening displacement. Because for any subscript j, $x_{1j} = x_1$ is held on the axis x_1, so

$$\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = 2\mathrm{Re}[\boldsymbol{A}_{\mathrm{I}}\boldsymbol{f}_{\mathrm{I}}(x_1) - \boldsymbol{A}_{\mathrm{II}}\boldsymbol{f}_{\mathrm{II}}(x_1)] \tag{4.21}$$

According to the given conditions, the generalized tractions are continuous on the whole axis x_1, i.e., $\boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1)$, or

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}}\overline{\boldsymbol{F}_{\mathrm{I}}(x_1)} = \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{II}}\overline{\boldsymbol{F}_{\mathrm{II}}(x_1)}, \quad -\infty < x_1 < \infty, \quad \text{or}\\ &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^+(x_1) - \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^+(x_1) = \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^-(x_1) - \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^-(x_1)\end{aligned} \tag{4.22}$$

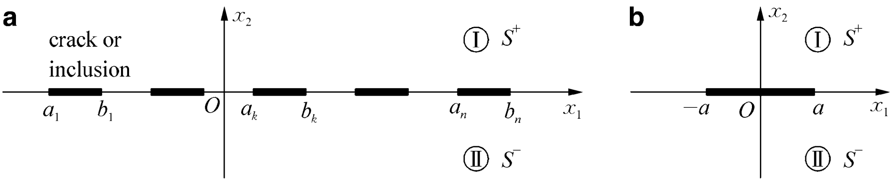


Fig. 4.2 Collinear interface cracks or inclusions: (**a**) general case and (**b**) one crack or inclusion

where the superscripts " + " and " − " indicate the limit values taken from the upper and lower half -planes, respectively. It is known that the functions at the left side in Eq. (4.22) are analytic in the upper half plane $x_2 > 0$, whereas those on the right side are analytic in the lower half plane $x_2 < 0$, and they are continuous on $x_1 = 0$. So, according to Liouville theorem (Lavrenchive and Shabat 1951), these functions are analytic in whole plane and must be constants and equal to zero due to $\boldsymbol{\Sigma}^\infty = \mathbf{0}$. So,

$$\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) = \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z), \quad x_2 > 0; \quad \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) = \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z), \quad x_2 < 0 \tag{4.23}$$

From Eqs. (4.21) and (4.23), the dislocation density $\hat{\boldsymbol{d}}'(x_1)$ can be written as

$$\begin{aligned} \mathrm{i}\hat{\boldsymbol{d}}'(x_1) = \mathrm{i}\mathrm{d}\hat{\boldsymbol{d}}(x_1)/\mathrm{d}x_1 &= [\mathrm{i}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{I}}\overline{\mathbf{F}_{\mathrm{I}}(x_1)}] - [\mathrm{i}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{II}}\overline{\mathbf{F}_{\mathrm{II}}(x_1)}] \\ &= [\mathrm{i}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{B}_{\mathrm{I}}^{-1} - \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}]\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) - [\mathrm{i}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{II}}^{-1} - \mathrm{i}\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}]\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) \\ &= \boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) - \bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) \end{aligned} \tag{4.24}$$

where

$$\boldsymbol{H} = \boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}}, \quad \boldsymbol{Y}_\alpha = \mathrm{i}\boldsymbol{A}_\alpha\boldsymbol{B}_\alpha^{-1}, \quad \boldsymbol{Y}_\alpha = \begin{pmatrix} \boldsymbol{Y}_{\alpha 11} & \boldsymbol{Y}_{\alpha 14} \\ \boldsymbol{Y}_{\alpha 41} & \boldsymbol{Y}_{\alpha 44} \end{pmatrix}, \quad \alpha = \mathrm{I}, \mathrm{II} \tag{4.25}$$

It is easy shown that $\boldsymbol{Y}_\alpha$ and $\boldsymbol{H}$ are all Hermite matrixes. $\boldsymbol{Y}_{\alpha 11}$ is a 3×3 positive definite matrix, $\boldsymbol{Y}_{\alpha 14} = \bar{\boldsymbol{Y}}_{\alpha 41}^{\mathrm{T}}$ is a piezoelectric matrix, and $Y_{\alpha 44}$ is an element of dielectric coefficient. For a stable material, $Y_{\alpha 44} < 0$.

4.2.2 A Simple Method to Get $\boldsymbol{F}_\beta(z_j)$

Because on the interface $z_j = x_1$, a simple method to solve the problem can be adopted (Suo 1990; Kuang and Ma 2002). At first we discuss two auxiliary complex functions $\boldsymbol{F}_{\mathrm{I}}(z)$ and $\boldsymbol{F}_{\mathrm{II}}(z)$ in z plane with complex variable z which also satisfy Eqs. (4.19) and (4.24) on the interface and solve the problem in z plane. According to Eq. (4.24), we can construct an auxiliary function $\boldsymbol{h}(z)$ analytic in whole plane except cracks by standard analytic continuation through the connected part on the interface:

$$\boldsymbol{h}(z) = \begin{cases} \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z), & x_2 \geq 0 \\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z), & x_2 \leq 0 \end{cases}, \quad z \notin L_{\mathrm{c}}, \quad z = x_1 + \mathrm{i}x_2 \tag{4.26}$$

The standard analytic continuation will be often used in the following sections. It is obvious that at points $x_1 \notin L_{\mathrm{c}}$ on axis x_1, $\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) = \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)$ is held. Solving $\boldsymbol{h}(z)$, the $\boldsymbol{F}_\beta(z)$ can be obtained by the following equations:

$$
\begin{aligned}
&\boldsymbol{F}_{\mathrm{I}}(z)=\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z), \quad x_2\geq 0; \quad \boldsymbol{F}_{\mathrm{II}}(z)=\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}(z), \quad x_2\leq 0\\
&F_{\mathrm{I}j}(z)=B_{\mathrm{I}jl}^{-1}h_l(z), \quad x_2\geq 0; \quad F_{\mathrm{II}j}(z)=B_{\mathrm{II}jm}^{-1}\overline{H}_{\mathrm{m}\,n}^{-1}H_{nl}h_l(z), \quad x_2\leq 0
\end{aligned}
\tag{4.27}
$$

where $B_{\beta jl}^{-1}=\left[\boldsymbol{B}_{\beta}^{-1}\right]_{jl}; \beta=\mathrm{I},\mathrm{II}; j,l=1-4$.

Substituting Eqs. (4.26) and (4.23) into Eq. (4.19) and noting on x_1 axis all $z_j=x_1$ we find

$$
\boldsymbol{h}^{+}(x)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x)=\boldsymbol{\Sigma}_0(x_1), \quad x_1\in L_{\mathrm{c}} \tag{4.28}
$$

If let $\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}=-\mathbf{g}$, Eq. (4.28) is identical with Eq. (4.1), which is solved as shown in Sect. 4.2.

A simple method to get $\boldsymbol{F}_{\beta}(z_j)$, for the original piezoelectric problem is replacing z by z_j in $\boldsymbol{F}_{\beta}(z)$. In fact the solution $F_{\beta}(z_j)$ solved by this method are still satisfy Eqs. (4.19) and (4.20) due to on the axis x_1, $\boldsymbol{F}_{\mathrm{I}}(z_j)=\boldsymbol{F}_{\mathrm{I}}(x_1)$, $\boldsymbol{F}_{\mathrm{II}}(z_j)=\boldsymbol{F}_{\mathrm{II}}(x_1)$, and $\boldsymbol{h}(z_j)=\boldsymbol{h}(x_1)$. Outside axis x_1, $\boldsymbol{Af}(z_j)$ and $\boldsymbol{Bf}(z_j)$ satisfy the generalized equilibrium equations due to they are selected as the general solutions given in Eqs. (3.18) and (3.23).

4.2.3 General Solution of the Homogeneous Equation

From Eq. (4.28), the homogeneous vector Riemann-Hilbert equation in the z plane is

$$
\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x_1)=\mathbf{0}, \quad x_1\in L_{\mathrm{c}} \tag{4.29}
$$

If let $\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}=-\mathbf{g}$, Eq. (4.29) is identical with (4.1). So the solution of the homogeneous equation is still expressed by Eqs. (4.3) and (4.6), but Eqs. (4.4) and (4.5) are changed to

$$
\left(\mathrm{e}^{2\pi\mathrm{i}\gamma}\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{\omega}=\mathbf{0}, \quad \left|\mathrm{e}^{2\pi\mathrm{i}\gamma}\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right|=0 \tag{4.30}
$$

Let $\gamma=1/2+\mathrm{i}\varepsilon$. Using $\mathrm{e}^{2\pi\mathrm{i}(1/2+\mathrm{i}\varepsilon)}=-\mathrm{e}^{-2\pi\varepsilon}$, Eq. (4.30) and its conjugate equation can be reduced to

$$
\begin{aligned}
&\left(\mathrm{e}^{-2\pi\varepsilon}\mathbf{I}-\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{\omega}=\mathbf{0}, \quad (\mathrm{e}^{-2\pi\varepsilon}\mathbf{I}-\boldsymbol{H}^{-1}\bar{\boldsymbol{H}})\bar{\boldsymbol{\omega}}=\mathbf{0}\\
&\left|\bar{\boldsymbol{H}}-\mathrm{e}^{2\pi\varepsilon}\boldsymbol{H}\right|=0, \quad \left|\bar{\boldsymbol{H}}-\mathrm{e}^{-2\pi\varepsilon}\boldsymbol{H}\right|=0
\end{aligned}
\tag{4.31}
$$

It is obvious that ε and $-\varepsilon$ are all the solutions of Eq. (4.31). Because $\boldsymbol{H}$ is a 4×4 order Hermite matrix, it can be decomposed to

$$\boldsymbol{H} = \boldsymbol{\Lambda}_1 + \mathrm{i}\boldsymbol{\Lambda}_2, \quad \bar{\boldsymbol{H}} = \boldsymbol{\Lambda}_1 - \mathrm{i}\boldsymbol{\Lambda}_2 \tag{4.32}$$

where $\boldsymbol{\Lambda}_1$ is a real symmetric matrix and $\boldsymbol{\Lambda}_2$ is an antisymmetric matrix. Let

$$\beta = \tanh(\pi\varepsilon) = \frac{\mathrm{e}^{\pi\varepsilon} - \mathrm{e}^{-\pi\varepsilon}}{\mathrm{e}^{\pi\varepsilon} + \mathrm{e}^{-\pi\varepsilon}} = \frac{\mathrm{e}^{2\pi\varepsilon} - 1}{\mathrm{e}^{2\pi\varepsilon} + 1}, \quad \text{or} \quad \varepsilon = \frac{\mathbf{1}}{2\pi} \ln \frac{1+\beta}{1-\beta} \tag{4.33}$$

Substitution of Eqs. (4.32) and (4.33) into Eq. (4.31) yields

$$\left|\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2 + \mathrm{i}\beta\mathbf{I}\right| = 0, \quad \left|\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2 - \mathrm{i}\beta\mathbf{I}\right| = 0 \tag{4.34}$$

It is known that $\beta, -\beta$ are all roots of the above equation. Expanding above equation, we get

$$\beta^4 + 2b\beta^2 + c = 0, \quad b = (1/4)\mathrm{tr}[(\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2)^2], \quad c = \left|\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_2\right| \tag{4.35a}$$

Because $\boldsymbol{\Lambda}_2$ is an even antisymmetric matrix, $|\boldsymbol{\Lambda}_2| \geq 0$, $Y_{\alpha 11}$ positive definite, $Y_{\alpha 44} < 0$, it is derived that $\left|\boldsymbol{\Lambda}_1^{-1}\right| < 0$ and $c < 0$. Therefore

$$\beta_{1,2} = \pm\sqrt{(b^2 - c)^{1/2} - b}, \quad \beta_{3,4} = \pm\mathrm{i}\sqrt{(b^2 - c)^{1/2} + b} \tag{4.35b}$$

Corresponding ε is denoted as

$$\begin{aligned} &\varepsilon_1 = -\varepsilon_2 = \varepsilon_0, \quad \varepsilon_0 = \frac{1}{\pi} \operatorname{ar\,tanh} \sqrt{(b^2 - c)^{1/2} - b} \\ &\varepsilon_4 = -\varepsilon_3 = \mathrm{i}\kappa, \quad \kappa = \frac{1}{\pi} \operatorname{ar\,tan} \sqrt{(b^2 - c)^{1/2} + b} \end{aligned} \tag{4.36}$$

where ε_0, κ are real. From Eqs. (4.31), (4.32), (4.33), (4.34), (4.35a), (4.35b) and (4.36), it is known that $\boldsymbol{\omega}^{(1)}$ and $\bar{\boldsymbol{\omega}}^{(2)}$, $\boldsymbol{\omega}^{(3)}$ and $\bar{\boldsymbol{\omega}}^{(3)}$, and $\boldsymbol{\omega}^{(4)}$ and $\bar{\boldsymbol{\omega}}^{(4)}$ satisfy the same eigen-equation, so we have $\boldsymbol{\omega}^{(1)} = c\bar{\boldsymbol{\omega}}^{(2)}$, where c is a real constant and $\boldsymbol{\omega}^{(3)}$ and $\boldsymbol{\omega}^{(4)}$ are real vectors.

The fundamental solution of Eqs. (4.3), (4.6), and (4.13) can be rewritten in ε as

$$\begin{aligned} &\boldsymbol{P}(z) = \left[\boldsymbol{X}_0^{(i)}(z)\right] = \boldsymbol{\Omega}\boldsymbol{Q}(z), \quad \boldsymbol{\Omega} = \left[\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \boldsymbol{\omega}^{(3)}, \boldsymbol{\omega}^{(4)}\right], \quad \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega} = \boldsymbol{\Lambda} = \left\langle \Lambda_j^2 \right\rangle \\ &\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{\Omega} = -\boldsymbol{M}, \quad \boldsymbol{M} = \left\langle \mathrm{e}^{2\pi\,\mathrm{i}\gamma_j}\Lambda_j^2 \right\rangle, \quad \boldsymbol{M}\boldsymbol{\Lambda}^{-1} = \left\langle \mathrm{e}^{2\pi\,\mathrm{i}\gamma_j} \right\rangle = \left\langle -\mathrm{e}^{2\pi\,\mathrm{i}\varepsilon_j} \right\rangle \\ &\boldsymbol{Q}(z) = \left\langle Y_0^{(i)}(z) \right\rangle, \quad Y_0^{(i)}(z) = \prod_{k=1}^{n} \frac{1}{\sqrt{(z-a_k)(z-b_k)}} \left(\frac{z-b_k}{z-a_k}\right)^{\mathrm{i}\varepsilon_i}, \quad i = 1,2,3,4 \end{aligned} \tag{4.37}$$

In practice $\boldsymbol{\Omega}$ is normalized, i.e., $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega} = \mathbf{I}$.

For a homogeneous material, $\boldsymbol{H}$, $\boldsymbol{\Omega}$ are real, so $\varepsilon_j = \varepsilon = 0, \gamma_j = \gamma = 1/2$, $\boldsymbol{Q}(z) = \langle Y_0(z) \rangle$, $Y_0(z) = Y_0^{(i)}(z) = \prod_{k=1}^{n} [(z - a_k)(z - b_k)]^{-1/2}, \; i = 1 - m$.

4.2.4 General Solution of the Inhomogeneous Equation for Impermeable Cracks

First method. For the inhomogeneous equation (4.1) in z plane, the solution of $\boldsymbol{h}(z)$ is Eq. (4.12), i.e.,

$$\boldsymbol{h}(z) = \boldsymbol{P}(z)\left[\boldsymbol{C}(z) + \frac{1}{2\pi\,\mathrm{i}} \int_L \frac{\tilde{\boldsymbol{\Sigma}}(t)\mathrm{d}t}{(t - z)}\right] = \sum_{k=1}^{4} \mathbf{X}_0^{(k)}(z)\left[\frac{1}{2\pi\,\mathrm{i}} \int_L \frac{\tilde{\Sigma}_k(t)}{(t - z)}\mathrm{d}t + C^{(k)}(z)\right]$$
$$\tilde{\boldsymbol{\Sigma}}(t) = [\tilde{\Sigma}_1(t), \tilde{\Sigma}_2(t), \ldots, \tilde{\Sigma}_4(t)]^{\mathrm{T}} = [\boldsymbol{P}^+(t)]^{-1}\boldsymbol{\Sigma}_0(t) \tag{4.38}$$

Solving $\boldsymbol{h}(z)$, according to Sect. 4.2.2, $\boldsymbol{F}_\beta(z_j)$ can be solved by the following equations:

$$\boldsymbol{F}_{\mathrm{I}}(z) = \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z), \quad F_{\mathrm{I}j}(z_j) = B_{\mathrm{I}jl}^{-1}h_l(z_j), \quad \boldsymbol{F}_{\mathrm{I}}(z_j) = \left[F_{\mathrm{I}j}(z_j)\right]^{\mathrm{T}}, \quad x_2 \geq 0;$$
$$\boldsymbol{F}_{\mathrm{II}}(z) = \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}(z), \quad F_{\mathrm{II}j}(z_j) = B_{\mathrm{II}jm}^{-1}\overline{H}_{mn}^{-1}H_{nl}h_l(z_j), \quad \boldsymbol{F}_{\mathrm{II}}(z_j) = \left[F_{\mathrm{II}j}(z_j)\right]^{\mathrm{T}}, \quad x_2 \leq 0 \tag{4.39}$$

The stresses are

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathrm{I}1} &= -2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{I}}\mu_j\boldsymbol{F}_{\mathrm{I}}(z_j)\right], \quad \boldsymbol{\Sigma}_{\mathrm{I}2} = 2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z_j)\right], \quad x_2 \geq 0 \\ \boldsymbol{\Sigma}_{\mathrm{II}1} &= -2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{II}}\mu_j\boldsymbol{F}_{\mathrm{II}}(z_j)\right], \quad \boldsymbol{\Sigma}_{\mathrm{II}2} = 2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z_j)\right], \quad x_2 \leq 0 \end{aligned} \tag{4.40}$$

Second method. The solution $\boldsymbol{\Psi}(z)$ of Eq. (4.1) is shown in Eqs. (4.16) or (4.17), i.e.,

$$\begin{aligned} \boldsymbol{\Psi}(z) &= \boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda}\boldsymbol{C}(z) + \frac{1}{2\pi\mathrm{i}} \int_L \frac{[\boldsymbol{Q}^+(x_1)]^{-1}\boldsymbol{\Sigma}^*(t)\mathrm{d}t}{(x_1 - z)}\right\} \\ &= \boldsymbol{Q}(z)\left\{\boldsymbol{\Lambda}\boldsymbol{C}(z) + \left\langle 1 - \mathrm{e}^{2\pi\mathrm{i}\gamma_i}\right\rangle^{-1}\left[[\boldsymbol{Q}(z)]^{-1}\boldsymbol{\Sigma}^*(z) - \left(\boldsymbol{\alpha}_q z^q + \cdots + \boldsymbol{\alpha}_0\right)\right]\right\} \\ \boldsymbol{\Psi}(z) &= \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z), \quad \boldsymbol{\Sigma}^*(t) = \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0(t) \end{aligned} \tag{4.41a}$$

where $[\boldsymbol{Q}^+(t)]_{ij}^{-1}\Sigma_j^*(t) \to \boldsymbol{\alpha}_q t^q + \cdots + \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_{-1}/t + \cdots$, when $t \to \infty$ is assumed. Combining the similar terms in Eq. (4.41a) yields

$$\boldsymbol{\Psi}(z) = \boldsymbol{Q}(z)\boldsymbol{C}(z) + \left(\mathbf{I} - \boldsymbol{M}\boldsymbol{\Lambda}^{-1}\right)^{-1}\boldsymbol{\Sigma}^*(z), \quad \boldsymbol{h}(z) = \bar{\boldsymbol{\Omega}}^{-\mathrm{T}}\boldsymbol{\Psi}(z) \tag{4.41b}$$

$\boldsymbol{F}_\beta(z_j)$ can be obtained from Eq. (4.39).

The closed solutions of the displacements and stresses are difficult obtained, usually adopted numerical method. But the stress intensity can be expressed analytically.

4.2.5 The Stress Asymptotic Field and the Stress Intensity Factors

Discuss a crack of length $2a$ and its center is selected as the origin (Fig. 4.2b). From Eqs. (4.37) and (4.38), the fundamental solution can be written as

$$\begin{aligned} & Y_0^{(i)}(z) = \frac{1}{\sqrt{z^2 - a^2}} \left(\frac{z-a}{z+a} \right)^{\mathrm{i}\varepsilon_i}, \quad \hat{\boldsymbol{C}}'(z) = \boldsymbol{C}(z) + \frac{1}{2\pi\,\mathrm{i}} \int_L \frac{\tilde{\boldsymbol{\Sigma}}(x_1)\mathrm{d}x_1}{x_1 - z} \\ & \boldsymbol{h}(z) = \boldsymbol{\Omega} \left\langle Y_0^{(i)}(z) \right\rangle \hat{\boldsymbol{C}}'(z), \quad \tilde{\boldsymbol{\Sigma}}(t) = [\boldsymbol{P}^+(t)]^{-1} \boldsymbol{\Sigma}_0(t), \quad i = 1,2,3,4 \end{aligned} \tag{4.42}$$

Near the right crack tip $x_1 = a$, the asymptotic form of $\boldsymbol{h}(z)$ and $\boldsymbol{F}_\beta(z_j)$ is, respectively,

$$\begin{aligned} & \lim_{z \to a} \boldsymbol{h}(z) = \boldsymbol{\Omega} \left\langle (z-a)^{-(1/2)+\mathrm{i}\varepsilon_j} \right\rangle \hat{\boldsymbol{C}}(a), \quad \hat{\boldsymbol{C}}(a) = \left\langle (z+a)^{-(1/2)-\mathrm{i}\varepsilon_j} \right\rangle \hat{\boldsymbol{C}}'(a) \\ & \boldsymbol{F}_{\mathrm{I}}(z) = \boldsymbol{B}_{\mathrm{I}}^{-1} \boldsymbol{h}(z), \quad F_{\mathrm{I}j}(z_j) = B_{\mathrm{I}jl}^{-1} h_l(z_j), \quad x_2 \geq 0; \\ & \boldsymbol{F}_{\mathrm{II}}(z) = \boldsymbol{B}_{\mathrm{II}}^{-1} \bar{\boldsymbol{H}}^{-1} \boldsymbol{H} \boldsymbol{h}(z), \quad F_{\mathrm{II}j}(z_j) = B_{\mathrm{II}jm}^{-1} \bar{H}_{\mathrm{m}n}^{-1} H_{nl} h_l(z_j), \quad x_2 \leq 0 \end{aligned} \tag{4.43}$$

Combining Eqs. (4.40) and (4.43) yields the asymptotic stresses near the right crack tip, but they are complex. However when the stress intensity factors are discussed only, the general expressions of the stress asymptotic field are not needed. Using all $z_j = x_1$ on the axis x_1 yields

$$\begin{aligned} & \boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1) = \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{B}_{\mathrm{I}} \boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{B}}_{\mathrm{I}} \bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1) \\ & \boldsymbol{h}^+(x_1) = \boldsymbol{h}^-(x_1) = \boldsymbol{h}(x_1), \quad \text{or} \quad \boldsymbol{H}\boldsymbol{B}_{\mathrm{I}} \boldsymbol{F}_{\mathrm{I}}(x_1) = \bar{\boldsymbol{H}} \boldsymbol{B}_{\mathrm{II}} \boldsymbol{F}_{\mathrm{II}}(x_1) \end{aligned}$$

Using Eqs. (4.23), (4.26), and (4.31) yields

$$\begin{aligned} \boldsymbol{\Sigma}_2(x_1) &= \boldsymbol{h}^+(x_1) + \bar{\boldsymbol{H}}^{-1} \boldsymbol{H} \boldsymbol{h}^-(x_1) = (\mathbf{I} + \bar{\boldsymbol{H}}^{-1} \boldsymbol{H}) \boldsymbol{h}(x_1) \\ &= \boldsymbol{\Omega} \left\langle \left(1 + \mathrm{e}^{-2\pi\varepsilon_j}\right) (x_1 - a)^{-(1/2)+\mathrm{i}\varepsilon_j} \right\rangle \hat{\boldsymbol{C}}(a) \end{aligned} \tag{4.44}$$

If $\boldsymbol{H}$ is complex, from Eq. (4.44), it is seen that the stresses are oscillated near the crack tip. The stress intensity factors $\boldsymbol{K}$ of the bimaterial are defined in the way that they can be reduced to the definition in a homogeneous material. According to Eq. (4.44), the $\boldsymbol{K}$ can be defined as

$$\boldsymbol{K} = \sqrt{2\pi}\boldsymbol{\Omega}\left\langle 1 + \mathrm{e}^{-2\pi\varepsilon_j}\right\rangle\hat{\boldsymbol{C}}, \quad \hat{\boldsymbol{C}} = \left(1\Big/\sqrt{2\pi}\right)\left\langle 1 + \mathrm{e}^{-2\pi\varepsilon_j}\right\rangle^{-1}\boldsymbol{\Omega}^{-1}\boldsymbol{K} \tag{4.45}$$

The stress asymptotic field can be written as

$$\lim_{x_1\to a}\boldsymbol{\Sigma}_2(x_1) = \frac{1}{\sqrt{2\pi(x_1-a)}}\boldsymbol{\Omega}\left\langle (x_1-a)^{\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{K} \tag{4.46}$$

According to Eq. (4.46) $\boldsymbol{K}$ can be expressed by the generalized stresses as

$$\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Omega}\left\langle (x_1-a)^{-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1) \tag{4.47}$$

where $\boldsymbol{K}$ is real and does not effect by the constant in $\boldsymbol{\Omega}$. For a homogeneous material, $\left\langle (x_1-a)^{-i\varepsilon_j}\right\rangle = \mathbf{I}$ and $\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Sigma}_2(x_1)$ which is identical with that in Eq. (3.220). In some literatures, the following definition is also used:

$$\begin{gathered}\boldsymbol{K} = [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Sigma}_2(x_1)\left\langle (x_1-a)^{-\mathrm{i}\varepsilon_j}\right\rangle \\ \lim_{x_1\to a}\boldsymbol{\Sigma}_2(x_1) = \frac{1}{\sqrt{2\pi(x_1-a)}}\left\langle (x_1-a)^{\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{K}\end{gathered} \tag{4.48}$$

Beom and Atluri (1996), Shen et al. (1999, 2007), and many other literatures discussed many interesting problems.

4.2.6 Permeable Crack

Discuss a permeable crack in an infinite bimaterial. The boundary condition at infinity is

$$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\infty}(x_1), \quad \text{at infinity} \tag{4.49}$$

The mixed boundary conditions on the crack surface and the continuity conditions on the connective interface are

$$\begin{aligned}&\sigma_{\mathrm{I}\,2\,i}(x_1) = \sigma_{\mathrm{II}\,2\,i}(x_1) = 0; \quad E_{\mathrm{I}\,1} = E_{\mathrm{II}\,1}, \quad D_{\mathrm{I}2} = D_{\mathrm{II}2} = D_2; \quad x_1 \in L_{\mathrm{c}}; \quad i = 1, 2 \\ &u_{\mathrm{I}\,2\,i}(x_1) = u_{\mathrm{II}\,2\,i}(x_1), \ \sigma_{\mathrm{I}\,2\,i}(x_1) = \sigma_{\mathrm{II}\,2\,i}(x_1); \quad E_{\mathrm{I}\,1} = E_{\mathrm{II}\,1}, \quad D_{\mathrm{I}2} = D_{\mathrm{II}2} = D_2; \\ &\quad x_1 \in L - L_{\mathrm{c}}\end{aligned} \tag{4.50}$$

The main different of the permeable crack with the impermeable crack is that the electric displacement on an impermeable crack is given and the potential is unknown, but on a permeable crack D_2 is undetermined and the potential is given. Because the generalized stresses are continuous on the whole axis x_1, so

$$\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{x}_1)=\boldsymbol{\Sigma}_2(x_1),\quad -\infty<x_1<\infty \tag{4.51}$$

Noting $\boldsymbol{\Sigma}_2=\boldsymbol{\Sigma}_2^{\infty}(x_1)\neq\mathbf{0}$ at infinity, like Eqs. (4.22) and (4.23), from Eq. (4.51) we get

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{x}_1)=\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{\Delta}^{\infty}\\&\boldsymbol{\Delta}^{\infty}=(1/2)[(\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{\infty}+\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^{\infty})-(\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^{\infty}+\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty})];\quad \boldsymbol{F}_{\alpha}(z),\quad \alpha=\mathrm{I},\mathrm{II}\end{aligned} \tag{4.52}$$

where $\boldsymbol{\Delta}^{\infty}$ is a pure imaginary vector. Analogous to Eq. (4.24),

$$\mathrm{i}\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty} \tag{4.53}$$

Analogous to Eq. (4.26) let,

$$\boldsymbol{h}(z)=\begin{cases}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z), & x_2\geq 0\\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)+\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty}, & x_2\leq 0\end{cases},\quad z\notin L_{\mathrm{c}} \tag{4.54}$$

Using Eq. (4.54), Eq. (4.53) is reduced to

$$\mathrm{i}\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{H}[\boldsymbol{h}^{+}(x_1)-\boldsymbol{h}^{-}(x_1)] \tag{4.55}$$

Substituting Eq. (4.54) into (4.51) and using Eq. (4.52) we get

$$\begin{aligned}\boldsymbol{\Sigma}_2(x_1)&=\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{h}^{+}(x_1)+\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\boldsymbol{\Delta}^{\infty}\\&=\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x_1)-\boldsymbol{\Delta}_1^{\infty},\quad \boldsymbol{\Delta}_1^{\infty}=\bar{\boldsymbol{H}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}-\bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty}\end{aligned} \tag{4.56}$$

According to Eq. (4.50), on the crack surface $\sigma_{2j}=0$, but D_2 is unknown, so on the crack surface, Eq. (4.56) is reduced to

$$\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^{-}(x_1)=\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(x_1),\quad \mathbf{i}_4=[0,0,0,1]^{\mathrm{T}},\quad z\in L_{\mathrm{c}} \tag{4.57}$$

According to Eq. (4.50), E_1 is continuous on whole axis x_1, so according to Eq. (4.55), we have

$$\boldsymbol{H}_4[\boldsymbol{h}^{+}(x_1)-\boldsymbol{h}^{-}(x_1)]=\mathbf{0},\quad \boldsymbol{H}_4=[H_{41},H_{42},H_{43},H_{44}],\quad |x_1|<\infty \tag{4.58}$$

The solution of Eq. (4.58) in the z plane is

$$\boldsymbol{H}_4\boldsymbol{h}(z)=\boldsymbol{H}_4\boldsymbol{h}(\infty),\quad h_4(z)=-H_{44}^{-1}\sum_{j=1}^{3}H_{4j}h_j(z)+H_{44}^{-1}\boldsymbol{H}_4\boldsymbol{h}(\infty) \tag{4.59}$$

Multiplying both sides of Eq. (4.56) by $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}$, noting on connective surface $\boldsymbol{h}^{+}(x_1)=\boldsymbol{h}^{-}(x_1)$, and when $x_1\to\infty$ we get

$$\boldsymbol{h}(\infty)=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left(\boldsymbol{\Delta}_1^{\infty}+\boldsymbol{\Sigma}_2^{\infty}\right) \tag{4.60}$$

Now, the problem is reduced to solve Eqs. (4.57) and (4.59).

The homogeneous equation corresponding to Eq. (4.57) is identical with Eq. (4.29), so its solution is still expressed by Eq. (4.37). We shall use the second method to solve the inhomogeneous equation (4.57) and adopt the normalized matrix $\boldsymbol{\Omega}$, i.e., $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\mathbf{I}$. Multiplying on both sides of Eq. (4.57) from left by $\bar{\boldsymbol{\Omega}}^{\mathrm{T}}$,

$$\begin{aligned}&\boldsymbol{\Psi}^{+}(x_1)-\boldsymbol{M}\boldsymbol{\Psi}^{-}(x_1)=\boldsymbol{\Sigma}^{*}(x_1),\quad \Psi_i^{+}(x_1)-\mathrm{e}^{2\pi\mathrm{i}\gamma_i}\Psi_i^{-}(x_1)=\Sigma_i^{*}(x_1)\\&\boldsymbol{\Psi}(z)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z),\quad \boldsymbol{M}=\left\langle\mathrm{e}^{2\pi\mathrm{i}\gamma_1},\ldots\mathrm{e}^{2\pi\mathrm{i}\gamma_4}\right\rangle,\quad \boldsymbol{\Sigma}^{*}(t)=\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left[\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(x_1)\right]\end{aligned} \tag{4.61}$$

Analogous to Eqs. (4.14), (4.15), (4.16), and (4.17), the solution of Eq. (4.61) in the z plane is

$$\begin{aligned}&\boldsymbol{h}(z)=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left\{\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(z)\right\}+\boldsymbol{\Omega}\left\langle Y_0^{(j)}(z)\right\rangle\boldsymbol{C}(z)\\&\boldsymbol{C}(z)=\boldsymbol{c}_nz^n+\boldsymbol{c}_{n-1}z^{n-1}+\cdots+\boldsymbol{c}_0\end{aligned} \tag{4.62}$$

Using the condition at infinity yields

$$\boldsymbol{h}(\infty)=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left\{\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(\infty)\right\}+\boldsymbol{\Omega}\boldsymbol{C}_n \tag{4.63}$$

Substituting Eq. (4.62) into Eq. (4.59) yields the equation to determine $D_2(z)$:

$$\boldsymbol{H}_4\left\{\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left\{\boldsymbol{\Delta}_1^{\infty}+\mathbf{i}_4D_2(z)\right\}+\boldsymbol{\Omega}\left\langle Y_0^{(j)}(z)\right\rangle\boldsymbol{C}(z)\right\}=\boldsymbol{H}_4\boldsymbol{h}(\infty) \tag{4.64}$$

Comparing Eqs. (4.60) and (4.63) yields

$$\boldsymbol{C}_n=\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_j}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_2^{\infty},\quad \tilde{\boldsymbol{\Sigma}}_2^{\infty}=\boldsymbol{\Sigma}_2^{\infty}-\mathbf{i}_4D_2(\infty)=\left[\boldsymbol{\sigma}_2^{\infty},0\right]^{\mathrm{T}} \tag{4.65}$$

Other unknowns in $\boldsymbol{C}(z)$ are determined by the single-valued condition. Using Eq. (4.55) yields

$$\oint_{L_c} h(z)\mathrm{d}z = 0, \quad \text{or} \quad \int_{-a}^{a} (\boldsymbol{U}^{+} - \boldsymbol{U}^{-})\,\mathrm{d}x_1 = 0 \text{ (for one crack)} \tag{4.66}$$

Equation (4.54) yields

$$\boldsymbol{F}_{\mathrm{I}}(z) = \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{h}(z), \quad x_2 \geq 0; \quad \boldsymbol{F}_{\mathrm{II}}(z) = \boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}[\boldsymbol{H}\boldsymbol{h}(z) - (\bar{\boldsymbol{Y}}_{\mathrm{II}} - \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{\Delta}^{\infty}], \quad x_2 \leq 0 \tag{4.67}$$

Solving $\boldsymbol{h}(z)$, $\boldsymbol{F}_{\beta}(z_j)$ can be obtained. From Eq. (4.56), the stress on the axis x_1 is

$$\boldsymbol{\Sigma}_2(x_1) = \left(\mathbf{I} + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{h}(x_1) - \boldsymbol{\Delta}_1^{\infty} \tag{4.68}$$

For one crack in a homogeneous material, we have

$$\boldsymbol{A}_{\mathrm{I}} = \boldsymbol{A}_{\mathrm{II}} = \boldsymbol{A}, \boldsymbol{B}_{\mathrm{I}} = \boldsymbol{B}_{\mathrm{II}} = \boldsymbol{B}, \boldsymbol{H}_{\mathrm{I}} = \boldsymbol{H}_{\mathrm{II}} = \boldsymbol{H} = \bar{\boldsymbol{H}}, \boldsymbol{\Delta}_1^{\infty} = \mathbf{0}, \varepsilon_j = 0, \gamma = 1/2$$

and

$$\boldsymbol{h}(z) = (1/2)\mathbf{i}_4 D_2(z) + \boldsymbol{\Omega}\left\langle Y_0^{(j)}(z)\right\rangle \boldsymbol{C}, \quad \boldsymbol{C} = (1/2)\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_2^{\infty}, \quad \tilde{\boldsymbol{\Sigma}}_2^{\infty} = \left[\boldsymbol{\sigma}_2^{\infty}, 0\right]^{\mathrm{T}} \tag{4.69}$$

Gao and Wang (2000, 2001) discussed the collinear permeable cracks and the mutual effect of a crack with a point singularity.

4.3 Other Line Inclusions

4.3.1 Rigid Line Inclusion

Discuss a nonconductive rigid line inclusion in an infinite bimaterial (Zhou et al. 2008). In Fig. 4.2 the crack is replaced by a rigid inclusion. The boundary condition at infinity is

$$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\infty}(x_1), \quad \text{at infinity} \tag{4.70}$$

The mixed boundary conditions on the surface of the rigid line inclusion and the continuity conditions on the connective interface are

$$u_{j,1} = u_{\mathrm{I}j,1} = u_{\mathrm{II}j,1} = \omega_0 \delta_{j2}, \quad x \in L_{\mathrm{c}}; \quad E_{\mathrm{I}\,1} = E_{\mathrm{II}\,1}, \quad D_{\mathrm{I}2} = D_{\mathrm{II}2}, \quad x \in L_{\mathrm{c}}$$
$$\boldsymbol{U}_{\mathrm{I}}(x_1) = \boldsymbol{U}_{\mathrm{II}}(x_1), \quad \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1), \quad x \in L - L_{\mathrm{c}} \tag{4.71}$$

where ω_0 is the rotation angle about axis x_3 of the inclusion. The main difference between the rigid line inclusion and a permeable crack is that in a permeable crack surfaces, the stresses are given, but for a rigid line inclusion, the rotational angles or moments are given.

According to Stroh's formula we have

$$\begin{aligned} &\boldsymbol{U}_{\alpha,\,1} = \boldsymbol{A}_\alpha \boldsymbol{F}_\alpha(z) + \overline{\boldsymbol{A}_\alpha \boldsymbol{F}_\alpha(z)}; \quad \boldsymbol{\Phi}_{\alpha,\,1} = \boldsymbol{B}_\alpha \boldsymbol{F}_\alpha(z) + \overline{\boldsymbol{B}_\alpha \boldsymbol{F}_\alpha(z)}, \\ &\boldsymbol{F}_\alpha(z) = \boldsymbol{f}'_\alpha(z), \quad \alpha = \mathrm{I}, \mathrm{II} \end{aligned} \tag{4.72}$$

The generalized displacements are continuous on the whole axis x_1, so analogous to Eq. (4.52) it yields

$$\begin{aligned} &\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1) = \boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{x}_1), \quad -\infty < x_1 < \infty \\ &\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1) = \boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) - \boldsymbol{\Delta}^\infty, \quad \boldsymbol{\Delta}^\infty = (1/2)[(\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^\infty + \boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^\infty) - (\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^\infty + \bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^\infty)], \\ &\quad \alpha = \mathrm{I}, \mathrm{II} \end{aligned} \tag{4.73}$$

Analogous to previous sections, we have

$$\begin{aligned} \boldsymbol{\Delta\Phi}_{,1}(x_1) &= \boldsymbol{\Phi}_{\mathrm{I},1}(x_1) - \boldsymbol{\Phi}_{\mathrm{II},1}(x_1) = \left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \overline{\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)}\right] - \left[\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \overline{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)}\right] \\ &= \mathrm{i}\boldsymbol{R}\left[\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) - \boldsymbol{R}^{-1}\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) - \boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1} - \bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^\infty\right] \\ \boldsymbol{Y}_\alpha^{-1} &= -\mathrm{i}\boldsymbol{B}_\alpha\boldsymbol{A}_\alpha^{-1}, \quad \boldsymbol{Y}_\alpha = \mathrm{i}\boldsymbol{A}_\alpha\boldsymbol{B}_\alpha^{-1}, \quad \boldsymbol{R} = \boldsymbol{Y}_{\mathrm{I}}^{-1} + \bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1} \end{aligned} \tag{4.74}$$

On the connective surface, Eq. (4.74) is zero, so by standard analytic continuation, we can construct a function $\boldsymbol{h}(z)$ analytic in whole plane except the rigid inclusions:

$$\boldsymbol{h}(z) = \begin{cases} \boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) & z \in S^+ \\ \boldsymbol{R}^{-1}\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) + \boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1} - \bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^\infty & z \in S^- \end{cases} \tag{4.75}$$

Equations (4.74) and (4.75) yield

$$\boldsymbol{\Delta\Phi}_{,1}(x_1) = \mathrm{i}\boldsymbol{R}[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)], \quad x_1 \in L_{\mathrm{c}}; \quad \boldsymbol{\Delta\Phi}_{,1}(x_1) = \boldsymbol{0}, \quad x_1 \notin L_{\mathrm{c}} \tag{4.76}$$

From Eq. (4.71), it is known that $D_2(x_1)$ is continuous on whole axis x_1, so $\Delta\Phi_{4,1}(x_1) = 0$, or

$$\boldsymbol{R}_4[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] = \boldsymbol{0}, \quad -\infty < x_1 < \infty, \quad \boldsymbol{R}_4 = [R_{41}, R_{42}, R_{43}, R_{44}] \tag{4.77}$$

where $\boldsymbol{R}_4$ is the fourth row of $\boldsymbol{R}$ and $\boldsymbol{R}_4^{\mathrm{T}}$ can be seen as a vector. The solution of Eq. (4.77) is

$$\boldsymbol{R}_4\boldsymbol{h}(z)=\boldsymbol{R}_4\boldsymbol{h}^{\infty},\quad \boldsymbol{h}^{\infty}=\boldsymbol{h}(\infty) \tag{4.78}$$

Using Eq. (4.73) it is easy get

$$\begin{aligned}&\boldsymbol{U}_{\mathrm{I},1}(x_1)=\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)=\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)-\boldsymbol{\Delta}_1^{\infty}\\&\boldsymbol{\Delta}_1^{\infty}=\bar{\boldsymbol{R}}^{-1}\left(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\right)\boldsymbol{\Delta}^{\infty}\end{aligned} \tag{4.79}$$

From Eq. (4.71), it is known that on the inclusion surface, we have

$$\boldsymbol{U}_{\mathrm{I},1}(x_1)=\omega_0\mathbf{i}_2-E_1(x_1)\mathbf{i}_4,\quad x_1\in L_{\mathrm{c}};\quad \mathbf{i}_2=[0,1,0,0]^{\mathrm{T}},\quad \mathbf{i}_4=[0,0,0,1]^{\mathrm{T}} \tag{4.80}$$

where $E_1(x_1)$ is the boundary value of $E_1(z)$ on the inclusion surface and is unknown. So Eq. (4.79) can be reduced to a vector Riemann-Hilbert equation:

$$\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)=\boldsymbol{\Delta}_1^{\infty}+\omega_0\mathbf{i}_2-E_1(x_1)\mathbf{i}_4,\quad x_1\in L_{\mathrm{c}} \tag{4.81}$$

Equation (4.81) is identical with (4.28) except using $\boldsymbol{R}$ and $\boldsymbol{\Delta}_1^{\infty}+\omega_0\mathbf{i}_2-E_1(x_1)\mathbf{i}_4$ instead of $\boldsymbol{H}$ and $\boldsymbol{\Sigma}_0(x_1)$, respectively, but $E_1(x_1)\mathbf{i}_4$ is undetermined, and $\omega_0\mathbf{i}_2$ is given or determined by given moment on the inclusion. The homogeneous equation of Eq. (4.81) is

$$\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)=\mathbf{0},\quad x_1\in L_{\mathrm{c}} \tag{4.82}$$

The difference of the homogeneous equation Eqs. (4.82) and (4.29) is only using $\boldsymbol{R}$ instead of $\boldsymbol{H}$. So the fundamental solution of Eq. (4.82) is still expressed by Eq. (4.37), but the eigen-equation is changed to

$$\begin{aligned}&\left(\mathrm{e}^{-2\pi\varepsilon}\mathbf{I}-\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\right)\boldsymbol{\omega}=\mathbf{0},\quad \left(\mathrm{e}^{2\pi\varepsilon}\mathbf{I}-\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\right))\bar{\boldsymbol{\omega}}=\mathbf{0},\quad \left|\bar{\boldsymbol{R}}-\mathrm{e}^{2\pi\varepsilon}\boldsymbol{R}\right|=\mathbf{0},\\&\left|\bar{\boldsymbol{R}}-\mathrm{e}^{-2\pi\varepsilon}\boldsymbol{R}\right|=\mathbf{0}\end{aligned} \tag{4.83}$$

From Eqs. (4.78) and (4.81), the solution of the inhomogeneous problem is

$$\begin{aligned}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z)&=\frac{\boldsymbol{Q}(z)}{2\pi\,\mathrm{i}}\int_L\frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\{\boldsymbol{\Delta}_1^{\infty}+\omega_0\mathbf{i}_2-E_1(x_1)\mathbf{i}_4\}}{\boldsymbol{Q}^{+}(x_1)(x_1-z)}\mathrm{d}x_1+\boldsymbol{Q}(z)\boldsymbol{C}(z)\\&=\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\left\{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left[\boldsymbol{\Delta}_1^{\infty}+\omega_0\mathbf{i}_2-E_1(z)\mathbf{i}_4\right]\right\}+\boldsymbol{Q}(z)\boldsymbol{C}(z)\\\boldsymbol{h}(z)&=\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left[\boldsymbol{\Delta}_1^{\infty}+\omega_0\mathbf{i}_2-E_1(z)\mathbf{i}_4\right]+\boldsymbol{\Omega}\left\langle Y_0^{(i)}(z)\right\rangle\boldsymbol{C}(z)\\\boldsymbol{Q}(z)&=\left\langle Y_0^{(j)}(z)\right\rangle,\quad Y_0^{(j)}(z)=\frac{1}{\sqrt{z^2-a^2}}\left(\frac{z-a}{z+a}\right)^{\mathrm{i}\varepsilon_j};\quad \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Omega}=\mathbf{I}\end{aligned} \tag{4.84}$$

where $\boldsymbol{C}(z) = \boldsymbol{C}_1 z + \boldsymbol{C}_0$. $E_1(z)$ can be obtained from Eqs. (4.78) and (4.84):

$$\begin{aligned}&\boldsymbol{R}_4\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\mathbf{i}_4 E_1(z)\\&\quad=\boldsymbol{R}_4\boldsymbol{\Omega}\left\langle\left(1+\mathrm{e}^{2\pi\varepsilon_i}\right)^{-1}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left(\boldsymbol{\Delta}_1^{\infty}+\omega_0\mathbf{i}_2\right)+\boldsymbol{\Omega}\left\langle Y_0^{(i)}(z)\right\rangle\boldsymbol{C}(z)-\boldsymbol{R}_4\boldsymbol{h}^{\infty}\end{aligned} \quad (4.85)$$

The unknown constants are obtained by using the conditions at infinity and the single-valued conditions and the moment condition:

$$\int_L \Delta\boldsymbol{\Phi}_{,1}\mathrm{d}x_1 = \int_{-a}^{a}\Delta\boldsymbol{\Phi}_{,1}\mathrm{d}x_1 = \mathbf{0}, \quad \int_{-a}^{a}\Delta\Phi_{2,\,1}(x_1-x_0)\mathrm{d}x_1 = M \quad (4.86)$$

The rigid line inclusion is discussed in many literatures (Shi 1997; Deng and Meguid 1998).

4.3.2 A Bimaterial with an Electrode on the Interface

Discuss a thin soft electrode of length $2a$ occupied L_c and let the coordinate origin be located at the center of the electrode (Ru 2000). In Fig. 4.2 the crack is changed to an electrode. The connective surface is denoted by $L - L_\mathrm{c}$. Assume the boundary conditions are

$$\begin{aligned}&\sigma_{\mathrm{I}2i}=\sigma_{\mathrm{II}2i},\quad u_{\mathrm{I}i}=u_{\mathrm{II}i},\quad E_{\mathrm{I}1}=E_{\mathrm{II}1},\quad D_{\mathrm{I}2}=D_{\mathrm{II}2},\quad x_1\notin L_\mathrm{c}\\&\sigma_{\mathrm{I}2i}=\sigma_{\mathrm{II}2i},\quad u_{\mathrm{I}i}=u_{\mathrm{II}i},\quad E_{\mathrm{I}1}=E_{\mathrm{II}1}=0,\quad \int_{L_\mathrm{c}}\delta(x_1)\mathrm{d}x_1=q,\quad x_1\in L_\mathrm{c}\\&\sigma_{ij}\to 0,\quad D_j\to 0,\quad |z|\to\infty\end{aligned} \quad (4.87)$$

where $\delta(x_1) = D_{\mathrm{I}2}(x_1) - D_{\mathrm{II}2}(x_1)$ and q is the total electric charge on the electrode.

Because the generalized displacements are continuous on whole axis x_1, analogous to Eqs. (4.23) and (4.73) and noting $\sigma_{ij}, D_j \to 0$ at infinity, we have

$$\begin{aligned}&\boldsymbol{A}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(x_1)+\bar{\boldsymbol{A}}_\mathrm{I}\bar{\boldsymbol{F}}_\mathrm{I}(\bar{x}_1)=\boldsymbol{A}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(x_1)+\bar{\boldsymbol{A}}_\mathrm{II}\bar{\boldsymbol{F}}_\mathrm{II}(\bar{x}_1),\quad -\infty<x_1<\infty\\&\boldsymbol{A}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(z)=\bar{\boldsymbol{A}}_\mathrm{II}\bar{\boldsymbol{F}}_\mathrm{II}(z),\quad \text{or}\quad \boldsymbol{Y}_\mathrm{I}\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(z)=-\bar{\boldsymbol{Y}}_\mathrm{II}\bar{\boldsymbol{B}}_\mathrm{II}\bar{\boldsymbol{F}}_\mathrm{II}(z);\quad x_2>0\\&\boldsymbol{A}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(z)=\bar{\boldsymbol{A}}_\mathrm{I}\bar{\boldsymbol{F}}_\mathrm{I}(z),\quad \text{or}\quad \boldsymbol{Y}_\mathrm{II}\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(z)=-\bar{\boldsymbol{Y}}_\mathrm{I}\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{F}}_\mathrm{I}(z);\quad x_2<0\end{aligned} \quad (4.88)$$

According to Eqs. (4.87) and noting $\boldsymbol{\Sigma}_{\mathrm{I}2} - \boldsymbol{\Sigma}_{\mathrm{II}2} = \boldsymbol{\Phi}_{\mathrm{I},1} - \boldsymbol{\Phi}_{\mathrm{II},1}$ yield

$$[\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(x_1)-\bar{\boldsymbol{B}}_\mathrm{II}\bar{\boldsymbol{F}}_\mathrm{II}(x_1)]^+-[\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(x_1)-\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{F}}_\mathrm{I}(x_1)]^-=\begin{cases}\mathbf{0}, & z\notin L_\mathrm{c}\\ [0,0,0,\delta(x_1)]^{\mathrm{T}}, & z\in L_\mathrm{c}\end{cases} \quad (4.89)$$

From Eq. (4.89), we can construct a function $\boldsymbol{h}(z)$ analytic in whole plane except L_c by the analytic continuation through $L - L_c$. Using the Sokhotski (Сохоцкий)-Plemelj formula of the Cauchy-type integral, its solution is

$$\boldsymbol{h}(z) = [0,0,0,\chi(z)]^{\mathrm{T}} = \begin{cases} \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) - \bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z), & z \in S^+ \\ \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) - \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z), & z \in S^- \end{cases}; \quad \chi(z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\delta(x_1)}{x_1 - z} \mathrm{d}x_1 \tag{4.90}$$

Using Eq. (4.88), Eq. (4.90) can be reduced to

$$\begin{aligned} (\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) &= \bar{\boldsymbol{Y}}_{\mathrm{II}}[0,0,0,\chi(z)]^{\mathrm{T}}, \quad z \in S^+ \\ (\boldsymbol{Y}_{\mathrm{II}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) &= \bar{\boldsymbol{Y}}_{\mathrm{I}}[0,0,0,\chi(z)]^{\mathrm{T}}, \quad z \in S^- \end{aligned} \tag{4.91}$$

Using Eq. (4.88) from $E_1^+ = 0$ on L_c, see Eq. (4.87), yields

$$[\boldsymbol{Y}_{\mathrm{I}}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)]^+ + [\boldsymbol{Y}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)]^- = [*,*,*,0]^{\mathrm{T}}, \quad z \in L_c \tag{4.92}$$

where "$*$" is not an applied variable and omitted. Substitution of Eq. (4.91) into Eq. (4.92) yields

$$\begin{aligned} &\boldsymbol{Y}_{\mathrm{I}}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}})^{-1}\bar{\boldsymbol{Y}}_{\mathrm{II}}[0,0,0,\chi^+(x_1)]^{\mathrm{T}} + \boldsymbol{Y}_{\mathrm{II}}(\boldsymbol{Y}_{\mathrm{II}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})^{-1}\bar{\boldsymbol{Y}}_{\mathrm{I}}[0,0,0,\chi^-(x_1)]^{\mathrm{T}} \\ &\quad = [*,*,*,0]^{\mathrm{T}}, \quad z \in L_c \end{aligned} \tag{4.93}$$

The fourth component of Eq. (4.93) is

$$\chi^+(x_1) - \mathrm{g}\chi^-(x_1) = 0, \quad \mathrm{g} = -\left[\boldsymbol{Y}_{\mathrm{II}}(\boldsymbol{Y}_{\mathrm{II}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})^{-1}\bar{\boldsymbol{Y}}_{\mathrm{I}}\right]_{44} \Big/ \left[\boldsymbol{Y}_{\mathrm{I}}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}})^{-1}\bar{\boldsymbol{Y}}_{\mathrm{II}}\right]_{44} \tag{4.94}$$

Equation (4.94) is identical with (4.1) in form, so its solution is

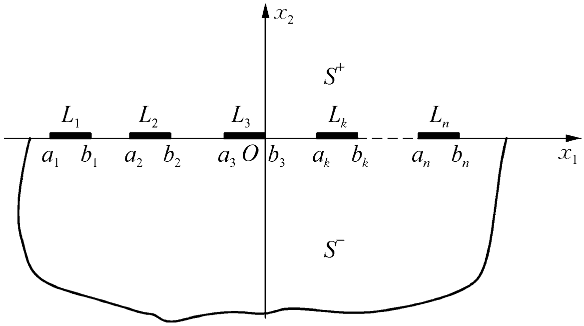
$$\begin{aligned} &\chi(z) = c(z+a)^{-\gamma}(z-a)^{\gamma-1}, \\ &\gamma = \frac{1}{2\pi\,\mathrm{i}} \ln \mathrm{g} = \frac{1}{2\pi\,\mathrm{i}} \ln \frac{-\left[\boldsymbol{Y}_{\mathrm{I}}(\boldsymbol{Y}_{\mathrm{I}} + \bar{\boldsymbol{Y}}_{\mathrm{II}})^{-1}\bar{\boldsymbol{Y}}_{\mathrm{II}}\right]_{44}}{\left[\boldsymbol{Y}_{\mathrm{II}}(\boldsymbol{Y}_{\mathrm{II}} + \bar{\boldsymbol{Y}}_{\mathrm{I}})^{-1}\bar{\boldsymbol{Y}}_{\mathrm{I}}\right]_{44}} \end{aligned} \tag{4.95}$$

For a homogeneous material, we have $\gamma = 1/2$. Comparing Eqs. (4.90) and (4.95) at infinity, it is found that

$$c = -(1/2\pi \mathrm{i}) \int_{L_c} \delta(x_1)\mathrm{d}x_1 = \mathrm{i}q/2\pi \tag{4.96}$$

Substituting Eq. (4.96) into Eq. (4.91) yields $\boldsymbol{F}_{\mathrm{I}}(z), \boldsymbol{F}_{\mathrm{II}}(z)$. Replacing z by z_j in $\boldsymbol{F}_\beta(z)$, the stress potential $\boldsymbol{F}_\beta(z_j)$ is obtained. Ru (2000) discussed the collinear cracks also.

Fig. 4.3 Collinear surface electrodes



4.3.3 Surface Electrodes

In this section, we shall discuss surface electrodes (Fig. 4.3) in details (Zhou et al. 2005a, b; Kuang et al. 2004). In this case, air occupies S^+ and it is assumed that in the air only the electric variables need to be considered; the dielectric occupies S^-. The boundary conditions are

$$\begin{aligned}
&\sigma_{ij} \to 0, \quad D_i \to 0, \quad |z| \to \infty; \quad \sigma_{2i} = 0, \quad D_2 = 0, \quad z \in L - L_c; \quad i,j = 1,2,3 \\
&\sigma_{2i} = 0, \quad E_1 = 0, \quad \text{and} \quad \int_{L_{ck}} D_2^-(x_1)\mathrm{d}x_1 = -q_k, \quad \text{or} \\
&\varphi_k = V_k, \quad \int_{L_c} D_2^-(x_1)\mathrm{d}x_1 = -Q = -\sum_{k=1}^{n} q_k, \quad k = 1,2,\ldots n; \quad z \in L_c
\end{aligned} \tag{4.97}$$

where $D_2^-(x_1)$ is an undetermined function. According to Eq. (4.97), it is known that $\boldsymbol{\Sigma}_2 = \mathbf{0}$ or $\boldsymbol{B}\boldsymbol{F}^-(x_1) + \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^-(\bar{x}_1) = \mathbf{0}$ on $L - L_c$, so we can construct a function $\boldsymbol{h}(z)$ analytic in whole z plane except L_c by the standard analytic continuation method:

$$\boldsymbol{h}(z) = \begin{cases} -\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\bar{\boldsymbol{F}}(z), & z \in S^+ \\ \boldsymbol{F}(z), & z \in S^- \end{cases} \tag{4.98}$$

From Eqs. (4.97) and (4.98) and using $\overline{\boldsymbol{F}^+(x_1)} = \bar{\boldsymbol{F}}^-(x_1), \overline{\boldsymbol{F}^-(x_1)} = \bar{\boldsymbol{F}}^+(x_1)$ we get

$$\begin{aligned}
&\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1) = -\boldsymbol{B}^{-1}\left[\boldsymbol{B}\boldsymbol{F}^-(x_1) + \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^+(x_1)\right] = \mathbf{0}, \quad z \in L - L_c \\
&\boldsymbol{B}\boldsymbol{F}^-(x_1) + \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^+(x_1) = \boldsymbol{\Sigma}_D, \quad \boldsymbol{\Sigma}_D = \left[0,0,0,D_2^-(x_1)\right]^{\mathrm{T}} \\
&\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1) = -\boldsymbol{B}^{-1}\boldsymbol{\Sigma}_D, \quad h_j^+ - h_j^- = -B_{j4}^{-1}D_2^-(x_1), \quad j = 1-4, \quad z \in L_c
\end{aligned} \tag{4.99}$$

Equation (4.99) is a decoupling Riemann-Hilbert boundary problem, and its solution is

$$\boldsymbol{h}(z) = -\boldsymbol{B}^{-1}\frac{1}{2\pi\,\mathrm{i}}\int_{L''}\frac{\boldsymbol{\Sigma}_D(x_1)}{x_1 - z}\mathrm{d}x_1, \quad \boldsymbol{F}(z) = \boldsymbol{h}(z), \quad z \in S^- \tag{4.100}$$

From the known knowledge, it is assumed

$$\begin{aligned} &D_2(z) = D_2^-(z) = P(z) \Big/ \prod_{i=1}^{n}\sqrt{(z-a_i)(z-b_i)}, \quad z \in S^- \\ &P(z) = \mathrm{i}\left(\gamma_{n-1}z^{n-1} + \cdots + \gamma_1 z + \gamma_0\right) \end{aligned} \tag{4.101}$$

where γ_i is a complex constant. Usually, select function $\sqrt{(z-a_i)(z-b_i)} \to z$ when $z \to \infty$ as its single-valued branch. Substitution of Eq. (4.101) into Eq. (4.100) yields

$$\begin{aligned} &F_j(z_j) = (1/2)B_{j4}^{-1}P(z_j)\left(\prod_{i=1}^{n}\sqrt{(z_j-a_i)(z_j-b_i)}\right)^{-1} \\ &f_j(z_j) = \int F_j(z_j)\mathrm{d}z_j + (1/2)\mathrm{i}CB_{j4}^{-1}, \quad z \in S^- \end{aligned} \tag{4.102}$$

where C is a constant. According to Eq. (4.97), it has $E_1 = 0$ on L_c, so

$$\boldsymbol{A}\boldsymbol{F}^-(x_1) + \overline{\boldsymbol{A}\boldsymbol{F}^-(x_1)} = [*,*,*,0], \quad x_1 \in L_\mathrm{c} \tag{4.103}$$

Substituting Eq. (4.102) into Eq. (4.103), on ith electrode, yields

$$\begin{aligned} &A_{4j}B_{j4}^{-1}\frac{\mathrm{i}P(x_1)}{\sqrt{(x_1-a_i)(b_i-x_1)}\prod_{k=1,k\neq i}^{n}\sqrt{(x_1-a_k)(x_1-b_k)}} \\ &\quad - \bar{A}_{4j}\bar{B}_{j4}^{-1}\frac{\mathrm{i}\bar{P}(\bar{x}_1)}{\sqrt{(x_1-a_i)(b_i-x_1)}\prod_{k=1,k\neq i}^{n}\sqrt{(x_1-a_k)(x_1-b_k)}} = 0, \quad x_1 \in L_\mathrm{c} \end{aligned} \tag{4.104}$$

Using $H_{44} = \mathrm{i}A_{4j}B_{j4}^{-1}$ is real, $A_{4j}B_{j4}^{-1}$ is pure imaginary number, and Eq. (4.104) can be reduced to $P(x_1) + \bar{P}(\bar{x}_1) = 0$, it is concluded that all γ_i in $P(z)$ are real.

The generalized stress Σ_{2k} and the generalized displacement U_k are, respectively,

$$\begin{aligned} &\Sigma_{2k} = \mathrm{Re}\sum_{j=1}^{4}{}_{kj}B_{j4}^{-1}P(z_j)\left(\prod_{i=1}^{n}\sqrt{(z_j-a_i)(z_j-b_i)}\right)^{-1} \\ &U_k = 2\mathrm{Re}\left[A_{kj}f_j(z_j)\right] = \mathrm{Re}\left[A_{kj}B_{j4}^{-1}\int\frac{P(z_j)\mathrm{d}z_j}{\prod_{i=1}^{n}\sqrt{(z_j-a_i)(z_j-b_i)}}\right] + H_{44}C \end{aligned} \tag{4.105}$$

If the electric charge on the electrode i is given, we have

$$\begin{aligned}\int_{a_i}^{b_i} \Sigma_{24}(x_1)\mathrm{d}x_1 &= \int_{a_i}^{b_i} D_2^-(x_1)\mathrm{d}x_1 = \int_{a_i}^{b_i} \mathrm{Re}\frac{P(x_1)\mathrm{d}x_1}{\prod_{m=1}^n \sqrt{(z_j - a_m)(z_j - b_m)}} \\ &= \int_{a_i}^{b_i} \frac{\mathrm{i}P(x_1)\mathrm{d}x_1}{\sqrt{(x_1 - a_i)(b_i - x_1)}\prod_{k=1,k\neq i}^n \sqrt{(x_1 - a_k)(x_1 - b_k)}} = -q_i, \quad i = 1 - n\end{aligned} \tag{4.106}$$

where n unknowns $\gamma_i(i = 0, 1, \ldots, n-1)$ are just determined by n equations. Especially when $z \to \infty$, we have

$$\lim_{z\to\infty} F_4(z_4) = \frac{1}{2\pi\,\mathrm{i}z_4} B_{44}^{-1}\int_{L''} D_2(x_1)\mathrm{d}x_1, \quad \lim_{z\to\infty} F_4(z_4) = \frac{\mathrm{i}}{2z_4}B_{44}^{-1}\gamma_{n-1}$$

so

$$\gamma_{n-1} = -\frac{1}{\pi}\int_{L_c} D_2(x_1)\mathrm{d}x_1 = -\frac{1}{\pi}(-Q) = \frac{Q}{\pi} \tag{4.107}$$

If the electric potential on the electrode i is given, we have

$$\begin{aligned}(U_4)_i = \varphi_i &= \mathrm{Re}\left[\sum_{j=1}^4 A_{4j}B_{j4}^{-1}\int_{a_i}^{b_i}\frac{P(x_1)\mathrm{d}x_1}{\prod_{i=1}^n \sqrt{(x_1 - a_i)(x_1 - b_i)}}\right] + H_{44}C \\ &= H_{44}\mathrm{Im}\int_{a_i}^{b_i}\frac{P(x_1)\mathrm{d}x_1}{\prod_{i=1}^n \sqrt{(x_1 - a_i)(x_1 - b_i)}} + H_{44}C = V_i \\ \int_{L_c} D_2^-(x_1)\mathrm{d}x_1 &= -Q = \sum_{i=1}^n \mathrm{Re}\int_{a_k}^{b_k}\frac{P(x_1)\mathrm{d}x_1}{\prod_{i=1}^n \sqrt{(x_1 - a_i)(x_1 - b_i)}}\end{aligned} \tag{4.108}$$

where $n+1$ unknowns $\gamma_i(i = 0, 1, \ldots, n-1)$ and C are just determined by $n+1$ equations.

For only one electrode located in $(-a, a)$ case, from Eq. (4.102) by using Eq. (4.107) we get

$$F_j(z_j) = B_{j4}^{-1}\frac{q\mathrm{i}}{2\pi\sqrt{z_j^2 - a^2}}, \quad f_j(z_j) = \frac{q\mathrm{i}}{2\pi}B_{j4}^{-1}\left\{\ln\left(z_j + \sqrt{z_j^2 - a^2}\right) + \ln\tilde{C}\right\} \tag{4.109}$$

where $\tilde{C}$ is a real constant. Let $\varphi = V_0$ on the electrode, then we have

$$\varphi = 2\mathrm{Re}\{A_{4j}f_j(x_1)\} = H_{44}(q/\pi)\mathrm{Re}\left\{\ln\left(x_1 - i\sqrt{a^2 - x_1^2}\right) + \ln\tilde{C}\right\} = V_0$$

Because $H_{44} = iA_{4j}B_{j4}$ is real, $\mathrm{Re}\ln\left(x_1 - \mathrm{i}\sqrt{a^2 - x_1^2}\right) = \ln a$, from the above equation we get

$$H_{44}(q/\pi)\ln a\tilde{C} = V_0, \quad \text{or} \quad \tilde{C} = (1/a)\exp((\pi V_0/qH_{44})) \tag{4.110}$$

The electric potential and generalized stresses are, respectively,

$$\begin{aligned}
&\varphi = H_{44}(q/\pi)\mathrm{Re}\left[\ln\left(z_j + \sqrt{z_j^2 - a^2}\right)\Big/a\right] + V_0 \\
&\Sigma_{2k} = -(q/\pi)\mathrm{Im}\sum_{j=1}^{4} B_{kj}B_{j4}^{-1}\left(z_j^2 - a^2\right)^{-1/2}, \quad \Sigma_{1k} = \frac{q}{\pi}\mathrm{Im}\sum_{j=1}^{4} B_{kj}B_{j4}^{-1}\mu_j\left(z_j^2 - a^2\right)^{-1/2}
\end{aligned} \tag{4.111}$$

For the dielectric without the piezoelectric effect, we have

$$\begin{aligned}
&Q_{44} = -\epsilon_{11}, \quad R_{44} = -\epsilon_{12}, \quad T_{44} = -\epsilon_{22}; \quad \mu_4 = \left(-\epsilon_{12} + \mathrm{i}\sqrt{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2}\right)\Big/\epsilon_{22} \\
&A_{44} = -\mathrm{i}\sqrt{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2}, \quad B_{44} = -(\epsilon_{12} + \mu_4\epsilon_{22}), \quad H_{44} = -\left(\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2\right)^{-1/2} < 0 \\
&F_4(z_4) = B_{44}^{-1}\frac{q\mathrm{i}}{2\pi\sqrt{z_4^2 - a^2}}, \quad \varphi = V_0 - H_{44}\frac{q}{\pi\sqrt{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2}}\mathrm{Re}\left\{\ln\frac{1}{a}\left(z_4 + \sqrt{z_4^2 - a^2}\right)\right\}
\end{aligned} \tag{4.112}$$

For an isotropic dielectric $\epsilon_{ij} = \epsilon\delta_{ij}$, so it is obtained

$$\varphi = V_0 - (q/\pi\epsilon)\mathrm{Re}\left\{\ln\left[\left(z + \sqrt{z^2 - a^2}\right)\Big/a\right]\right\} \tag{4.113}$$

which is identical with the result in usual textbooks. Kuang et al. (2004) gave numerical examples for the case of two electrodes. Shindo et al. (1998) discussed the surface electrode also.

4.4 Short Discussions on Some Special Problems

4.4.1 Partly Insulated and Partly Conducted Crack in a Homogeneous Material

The impermeable or conducting electric boundary conditions are idealization case. Breakdown of the dielectric inside the crack was observed in experiments, especially near the crack tip region. The local electric discharge may make an impermeable crack conducting electrically and change the failure behavior of piezoelectric materials (Lynch et al. 1995; Zhang et al. 2001). The discharge process at the gap near a crack tip is complex dynamic process. When the electric

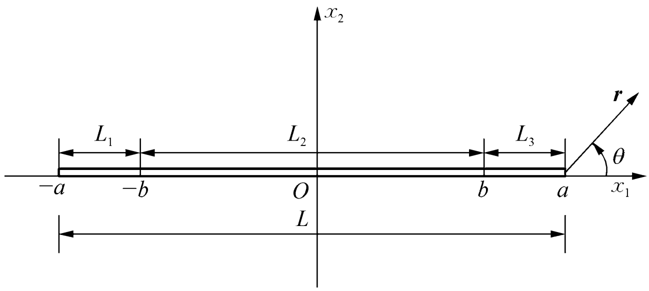


Fig. 4.4 Partly insulated and partly conducted crack

field approaches the critical value, the air breaks down and becomes conducting gas, but after air breakdown, the electric field diminishes quickly and air becomes insulated again. This process will be repeated and form discontinuous electric sparks. For the homogeneous material, Huang and Kuang (2003) proposed an ideal static model: partly insulated and partly conducted crack. Near the crack tip, the conducting boundary condition is adopted, but in the middle part of the crack, it is considered insulated (Fig. 4.4). The boundary conditions are

$$\begin{aligned} &\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_1^{\infty}, \quad \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^{\infty}, \quad \text{at infinity} \\ &\sigma_{2j}^{\pm}(x_1, 0) = 0, \quad E_1^{\pm}(x_1, 0) = 0, \quad x_1 \in L_1 \cup L_3 \\ &\sigma_{2j}^{\pm}(x_1, 0) = 0, \quad D_2^{\pm}(x_1, 0) = 0, \quad x_1 \in L_2 \end{aligned} \tag{4.114}$$

where $L_2(-b, b)$ is the insulated region and $L_1(-a, -b)$ and $L_3(b, a)$ are the conducting region. For an electric free crack the single-valued conditions are

$$\begin{aligned} &\int_L \left[u_{j,1}^{+}(x_1, 0) - u_{j,1}^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0, \quad \int_{L_2} \left[\varphi_{,1}^{+}(x_1, 0) - \varphi_{,1}^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0 \\ &\int_{L_1} \left[D_2^{+}(x_1, 0) - D_2^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0, \quad \int_{L3} \left[D_2^{+}(x_1, 0) - D_2^{-}(x_1, 0)\right] \mathrm{d}x_1 = 0 \end{aligned} \tag{4.115}$$

Equation (4.114) can be reduced to the following inhomogeneous Riemann-Hilbert equations:

$$\begin{aligned} &\sum\nolimits_k A_{4k} F_k^{\pm}(x_1) + \sum\nolimits_k \overline{A}_{4k} \overline{F}_k^{\mp}(x_1) = 0, \quad x_1 \in L_1 \cup L_3 \\ &\Sigma_{2j}^{+}(x_1) + \Sigma_{2j}^{-}(x_1) = \sum\nolimits_k \left[B_{jk} F_k^{+} + \overline{B}_{jk} \overline{F}_k^{+} + \overline{B}_{jk} \overline{F}_k^{-} + B_{jk} F_k^{-}\right] = s_1(x_1) \delta_{4j} \\ &\Sigma_{2j}^{+}(x_1) - \Sigma_{2j}^{-}(x_1) = \sum\nolimits_j \left[B_{jk} F_k^{+} - \overline{B}_{jk} \overline{F}_k^{+} + \overline{B}_{jk} \overline{F}_k^{-} - B_{jk} F_k^{-}\right] = s_2(x_1) \delta_{4j} \\ &s_1(x_1) = \begin{cases} 0, & x_1 \in L_2 \\ D_2^{+} + D_2^{-}, & x_1 \in L_1 \cup L_3 \end{cases}; \quad s_2(x_1) = \begin{cases} 0, & x_1 \in L_2 \\ D_2^{+} - D_2^{-}, & x_1 \in L_1 \cup L_3 \end{cases} \end{aligned} \tag{4.116}$$

Because D_2 is unknown on $x_1 \in L_1 \cup L_3$, so $s_1(x_1)$ and $s_2(x_1)$ in Eq. (4.116) are undetermined functions. Eq. (4.116) can be solved as an inhomogeneous Riemann-Hilbert problem by using the analytic continuation method. Finally Huang and Kuang (2003) obtained the solution in z plane

$$\begin{aligned} F_j(z) = &\frac{1}{2}B_{j4}^{-1}\left[\gamma_6 z\{X_b(z) - X_a(z)\} + \mathrm{i}\left(\gamma_0 + \gamma_2 z^2\right)X_{ab}(z) - \mathrm{i}\gamma_2\right] \\ &+ \frac{1}{2}B_{jk}^{-1}[\beta_{1k} z X_a(z) + \mathrm{i}\beta_{2k}]; \quad j,k = 1,2,3,4 \end{aligned} \tag{4.117}$$

It is known that an impermeable crack intensifies an electric field perpendicular to it, but does not perturb an electric field parallel to it. The effect of a conducting crack is just conversely. The singular parts of the generalized stresses are

$$\begin{aligned} \sigma_{2j}(x_1) &= \beta_{1j} x_1 X_a(x_1) = \sigma_{2j}^{\infty} x_1 X_a(x_1) \\ D_2(x_1) &= \left(H_{4j}/H_{44}\right)\sigma_{2j}^{\infty} x_1 [X_b(x_1) - X_a(x_1)] + D_2^{\infty} x_1 X_b(x_1) \\ E_1(x_1) &= \left(\beta_{2j} H_{4j}/2\right)\left(I_2/I_1 - x_1^2\right)X_{ab}(x_1) - \sigma_{2j}^{\infty}\mathrm{Im}\left(Y_{4j}\right)x_1 X_a(x_1) \end{aligned} \tag{4.118}$$

where

$$\begin{aligned} &X_a(z) = 1\Big/\sqrt{z^2 - a^2}, \quad X_b(z) = 1\Big/\sqrt{z^2 - b^2}, \quad X_{ab}(z) = X_a(z)X_b(z) \\ &\Sigma_{2j}^{\infty} = \beta_{1j}, \quad \Sigma_{1j}^{\infty} = -\mathrm{Re}\left[\Sigma_{k=1}^{4} B_{jk}\mu_k B_{km}^{-1}(\beta_{1m} + \mathrm{i}\beta_{2m})\right] \\ &\gamma_2 = H_{4j}\beta_{2j}/H_{44}, \quad \gamma_6 = H_{4j}\beta_{1j}/H_{44}, \quad \gamma_0 = -\gamma_2 I_2/I_1 \\ &I_2 = \int_b^a \left[x_1^2/X_{ab}(x_1)\right]\mathrm{d}x_1, \quad I_1 = \int_b^a [1/X_{ab}(x_1)]\mathrm{d}x_1 \end{aligned} \tag{4.119}$$

The limit analysis shows that $\gamma_0 = 0$ for $b = 0$ and $\gamma_0 = -a^2\gamma_2$ for $b = a$. These show that the present solution is consistent with solutions of the conventional conducting crack and impermeable crack. For the general situation $0 \ll b < a$ at the tip region, where r and $a - b$ is in the same order, the generalized stresses are related to both r and $a - b$.

In electroelastic fracture mechanics, the energy release rate and $J-$integral (Pak 1990; Suo et al. 1992) is often used. Because there are two singular points, crack tip $x_1 = a, x_2 = 0$ and the tip of the conductive part $x_1 = b, x_2 = 0$, so two $J-$integrals expressed with electric enthalpy are defined as

$$J_1 = \int_{L_a}\left(gn_1 - n_i\sigma_{ip}u_{p,1} - n_i D_i\varphi_{,1}\right)\mathrm{d}l, \quad J_2 = \int_{L_{a+b}}\left(gn_1 - n_i\sigma_{ip}u_{p,1} - n_i D_i\varphi_{,1}\right)\mathrm{d}l \tag{4.120}$$

where L_a is the contour only enclosed the crack tip, L_{a+b} is the contour enclosed two singular points, g is the electric enthalpy, and $\boldsymbol{n}$ is the outward normal of the contour.

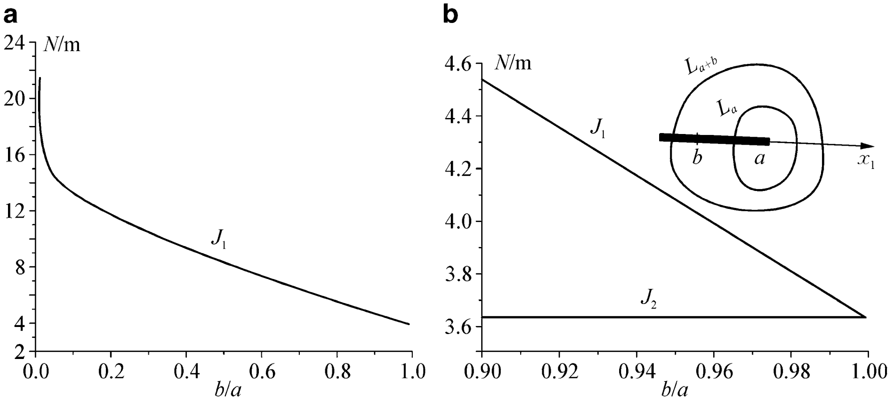


Fig. 4.5 Variation of J-integral value with respect to b/a under loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_1^{\infty} = 0.1\text{MV/m}$: (**a**) J_1 and (**b**) J_1 and J_2

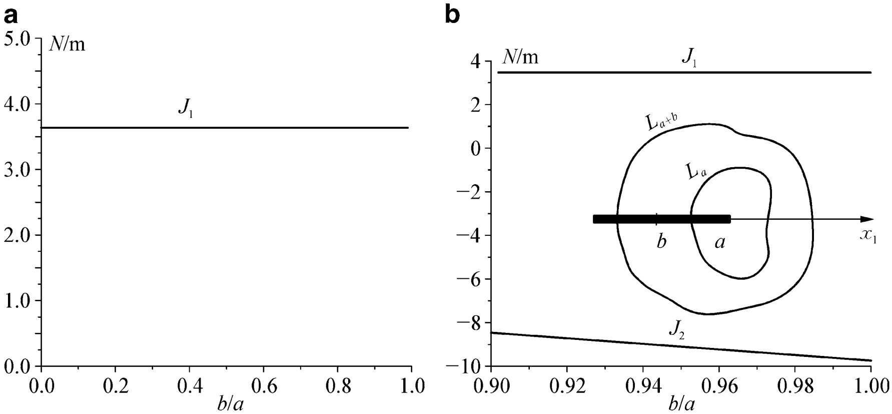


Fig. 4.6 Variation of J-integral value with respect to b/a under loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_2^{\infty} = 0.1\text{MV/m}$: (**a**) J_1 and (**b**) J_1 and J_2

Now give a numerical example. When the poling direction is along axis x_3, the material constants of PZT-4 are

$$
\begin{aligned}
&C_{11} = 13.9 \times 10^{10}, \quad C_{12} = 7.78 \times 10^{10}, \quad C_{13} = 7.43 \times 10^{10}, \quad C_{33} = 11.3 \times 10^{10},\\
&C_{44} = 2.56 \times 10^{10}\left(\text{N/m}^2\right); \quad e_{31} = -6.98, \quad e_{33} = 13.84, \quad e_{15} = 13.44\left(\text{C/m}^2\right)\\
&\epsilon_{11} = 6.00 \times 10^{-9}, \quad \epsilon_{33} = 5.47 \times 10^{-9}(\text{C/Vm})
\end{aligned}
$$

In the above theoretical analyses, the poling direction is along axis x_2, so the material constants need to be transformed. Figure 4.5 gives the variation of J_1 and J_2 values with respect to b/a under the loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_1^{\infty} = 0.1\text{MV/m}$. Figure 4.6 gives the variation of J_1 and J_2 values with respect to b/a under the

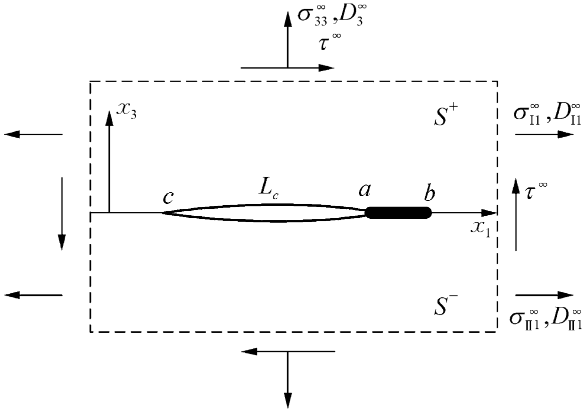


Fig. 4.7 Contact zone model in a bimaterial

loading $\sigma_{22}^{\infty} = 1\,\text{MPa}$ and $E_2^{\infty} = 0.1\text{MV/m}$. A completely conducting crack can be obtained from J_1 when $b/a \to 0$, while completely impermeable crack can be obtained from J_2 when $b/a \to 1$.

4.4.2 Contact Zone Model for Interface Cracks in a Piezoelectric Bimaterial

Figure 4.7 shows a contact zone model in a bimaterial (in $x_1 - x_3$ plane) for an electrically permeable interface crack (Herrmann and Loboda 2000; Loboda 1993). Let material I is located in the upper half space S^+ and material II is located in the lower half space S^-. Let c the left end of the crack, a the right end, and ab the contact zone. The boundary conditions are

$$\begin{aligned}
&\boldsymbol{\Sigma}_{\mathrm{I}} = \boldsymbol{\Sigma}_{\mathrm{II}} = \boldsymbol{\Sigma}^{\infty}, \quad \text{at infinity} \\
&\hat{\boldsymbol{d}}(x_1) = [\![u_3]\!] = u_3^+ - u_3^- = \mathbf{0}, \quad [\![\boldsymbol{\Sigma}_3]\!] = \boldsymbol{\Sigma}_{\mathrm{I}3}(x_1) - \boldsymbol{\Sigma}_{\mathrm{II}3}(x_1) = \mathbf{0}, \quad x_1 \notin (c, b) \\
&\sigma_{13}^{\pm} = 0, \ \ \sigma_{33}^{\pm} = 0, \quad [\![\varphi]\!] = 0, \quad [\![D_3]\!] = 0, \quad x_1 \in (c, a) \\
&\sigma_{13}^{\pm} = 0, \ \ [\![\sigma_{33}]\!] = 0, \quad [\![u_3]\!] = 0, \quad [\![\varphi]\!] = 0, \quad [\![D_3]\!] = 0, \quad x_1 \in (b, a)
\end{aligned} \tag{4.121}$$

It is assumed that only normal unknown contact stress σ_{33} is acted on the contact zone and no tangential frictional force. Because on whole axis x_1, $\boldsymbol{\Sigma}_{\mathrm{I}3}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}3}(x_1)$, like Eqs. (4.51), (4.52), (4.53), (4.54), and (4.55), of Sect. 4.2.6 or Eqs. (4.72), (4.73), (4.74), and (4.75) of Sect. 4.3.1, but different notations are adopted, we have

$$\begin{aligned}
&\bar{\boldsymbol{F}}_{\mathrm{II}}(z)=\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}(\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)-\boldsymbol{\Delta}^{\infty}),\quad x_3>0;\quad \bar{\boldsymbol{F}}_{\mathrm{I}}(z)=\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}(\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\boldsymbol{\Delta}^{\infty}),\quad x_3<0\\
&\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)-\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(x_1)=\boldsymbol{M}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{M}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)+\boldsymbol{\Delta}_1^{\infty}\\
&\boldsymbol{M}=\boldsymbol{A}_{\mathrm{I}}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}=(\boldsymbol{A}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1})\boldsymbol{B}_{\mathrm{I}}=-\mathrm{i}\boldsymbol{H}\boldsymbol{B}_{\mathrm{I}},\quad \boldsymbol{\Delta}_1^{\infty}=\left(-\boldsymbol{A}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{II}}^{-1}+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\right)\boldsymbol{\Delta}^{\infty}
\end{aligned}\tag{4.122}$$

and

$$\begin{aligned}
&\boldsymbol{W}(z)=\begin{cases}\boldsymbol{M}\boldsymbol{F}_{\mathrm{I}}(z), & x_2\geq 0\\ -\bar{\boldsymbol{M}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z)-\boldsymbol{\Delta}_1^{\infty}, & x_2\leq 0\end{cases}\\
&\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{W}_{\mathrm{I}}(x_1)-\boldsymbol{W}_{\mathrm{II}}(x_1),\quad \boldsymbol{\Sigma}_0(x_1)=\boldsymbol{G}\boldsymbol{W}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{G}}\boldsymbol{W}_{\mathrm{II}}(x_1)-\bar{\boldsymbol{M}}^{-1}\boldsymbol{\Delta}_1^{\infty}\\
&\boldsymbol{G}=\boldsymbol{B}_{\mathrm{I}}\boldsymbol{M}^{-1}=\boldsymbol{B}_{\mathrm{I}}\left\{(\boldsymbol{A}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1})\boldsymbol{B}_{\mathrm{I}}\right\}^{-1}=\left(\boldsymbol{A}_{\mathrm{I}}\bar{\boldsymbol{B}}_{\mathrm{I}}^{-1}-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{B}}_{\mathrm{II}}^{-1}\right)^{-1}=\mathrm{i}\boldsymbol{H}^{-1}=-\bar{\boldsymbol{G}}^{\mathrm{T}}
\end{aligned}\tag{4.123}$$

where $\boldsymbol{H}$ is shown in Eq. (4.25), $\boldsymbol{\Delta}^{\infty}$ is shown in Eq. (4.52), $\boldsymbol{W}(z)$ is a vector function analytic in whole plane except cracks. For a kind of $6mm$ piezoelectric materials poling along axis x_3, $\boldsymbol{G}$ possesses the following behavior:

$$\begin{aligned}
&\boldsymbol{G}=\begin{bmatrix}G_{11}&G_{13}&G_{14}\\G_{31}&G_{33}&G_{34}\\G_{41}&G_{43}&G_{44}\end{bmatrix}=\begin{bmatrix}\mathrm{i}g_{11}&g_{13}&g_{14}\\g_{31}&\mathrm{i}g_{33}&\mathrm{i}g_{34}\\g_{41}&\mathrm{i}g_{43}&\mathrm{i}g_{44}\end{bmatrix},\quad\begin{bmatrix}g_{11}&g_{13}\\g_{31}&g_{33}\end{bmatrix}\text{ positive definite}\\
&g_{13}=-g_{31},\ g_{14}=-g_{41},\ g_{34}=g_{43},\ g_{44}<0,\quad \text{all } g_{ij} \textit{ is real}
\end{aligned}\tag{4.124}$$

and the eigen-equation Eq. (3.12) becomes

$$\begin{bmatrix}C_{11}+C_{44}\mu^2 & (C_{13}+C_{44})\mu & (e_{31}+e_{15})\mu\\ (C_{13}+C_{44})\mu & C_{44}+C_{33}\mu^2 & e_{15}+e_{33}\mu^2\\ (e_{31}+e_{15})\mu & e_{15}+e_{33}\mu^2 & -\epsilon_{11}-\epsilon_{33}\mu^2\end{bmatrix}\begin{Bmatrix}a_1\\a_3\\a_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}\tag{4.125}$$

The roots of Eq. (4.125) are $\mu_1=\alpha_1+\mathrm{i}\beta_1,\ \mu_3=-\alpha_1+\mathrm{i}\beta_1,\ \mu_4=\mathrm{i}\beta_4$ where α_1, β_1,β_4 are all real:

$$\begin{aligned}
&B_{1j}=C_{44}\left(\mu_jA_{1j}+A_{3j}\right)+e_{15}A_{4j},\quad B_{3j}=C_{13}A_{1j}+C_{33}\mu_jA_{3j}+e_{33}\mu_jA_{4j}\\
&B_{4j}=e_{31}A_{1j}+e_{33}\mu_jA_{3j}-\epsilon_{33}\mu_jA_{4j};\quad j=1,3,4,
\end{aligned}\tag{4.126}$$

Finally they get

$$\begin{aligned}
&K_{\mathrm{I}}=\sqrt{\pi l/2a}\left[\sqrt{1-\lambda}\left(\sigma_{33}^{\infty}\cos\delta+m\sigma_{13}^{\infty}\sin\delta\right)-2\varepsilon\left(\sigma_{33}^{\infty}\sin\delta-m\sigma_{13}^{\infty}\cos\delta\right)\right]\\
&K_{\mathrm{II}}=-\sqrt{\pi l/2m^2}\left[\left(\sigma_{33}^{\infty}\sin\delta-m\sigma_{13}^{\infty}\cos\delta\right)+2\varepsilon\sqrt{1-\lambda}\left(\sigma_{33}^{\infty}\cos\delta+m\sigma_{13}^{\infty}\sin\delta\right)\right]\\
&K_{\mathrm{D}}=g_{33}^{-1}\left[g_{43}-(g_{31}g_{43}-g_{41}g_{33})\left(\gamma^2-1\right)/2\lambda\rho\right]K_{\mathrm{I}}
\end{aligned}\tag{4.127}$$

where

$$\gamma = -(g_{31} + mg_{11})/t, \quad s = (g_{33} + mg_{13})/t, \quad m = \sqrt{-g_{33}/g_{11}}, \quad \rho = t(1+\gamma)$$
$$\delta = \varepsilon \ln\left[\left(1 - \sqrt{1-\lambda}\right)\Big/\left(1 + \sqrt{1-\lambda}\right)\right], \quad \lambda = (a-b)/l, \quad \varepsilon = (1/2\pi)\ln\gamma, \quad l = a - c \tag{4.128}$$

The contact point b (or the parameter λ) is determined by $K_{\mathrm{I}} = 0$, i.e., under the conditions

$$\sigma_{\mathrm{I}33}(x_1, 0) \le 0, \quad x_1 \in (a, b); \quad [\![u_3(x_1, 0)]\!] \ge 0, \quad x_1 \in (c, a) \tag{4.129}$$

Select the maximum λ_0 from the following equation:

$$\tan\delta = \left[\left(\sqrt{1-\lambda}\sigma_{33}^{\infty} + 2\varepsilon m\sigma_{13}^{\infty}\right)\Big/\left(2\varepsilon\sigma_{33}^{\infty} - \sqrt{1-\lambda}m\sigma_{13}^{\infty}\right)\right] \tag{4.130}$$

For the bimaterial CTS-19(S^+)/PZT-4(S^-) and cadmium sulfide/barium sodium niobate, numerical results show that $\lambda_0 \sim 0.3$, when $\sigma_{13}^{\infty}/\sigma_{33}^{\infty} \to \infty$, and $\lambda_0 \sim 1\big/\mathrm{e}^{100}$, $1\big/\mathrm{e}^{50}$ when $\sigma_{13}^{\infty}/\sigma_{33}^{\infty} \to 0, 1$, respectively.

Herrmann and Loboda (2000) considered that $\boldsymbol{\Delta}^{\infty}$ can be included in undetermined functions $\boldsymbol{F}_{\mathrm{I}}(z), \boldsymbol{F}_{\mathrm{II}}(z)$, so they let $\boldsymbol{\Delta}^{\infty} = \boldsymbol{0}$. However if let $\boldsymbol{\Delta}^{\infty} = \boldsymbol{0}$, then $\boldsymbol{\Sigma}_{\beta} - 2\mathrm{Re}\left(\boldsymbol{B}_{\beta}\boldsymbol{F}_{\beta}\right) = \boldsymbol{C}_{\beta} \ne \boldsymbol{0}$, where $\boldsymbol{C}_{\beta}$ is a known constant vector. But this does not influence the stress intensity factors and the length of the contact zone.

Herrmann et al. (2001) discussed also the contact zone model of the impermeable crack.

4.4.3 Nonideal Crack in a Homogeneous Piezoelectric Material

In practical structure, the crack cannot be ideal. Now discuss a simple free nonideal crack in a homogeneous piezoelectric material subjected $\boldsymbol{\Sigma}_1^{\infty}, \boldsymbol{\Sigma}_2^{\infty}$ at infinity. Figure 4.8 shows a nonideal symmetric crack expressed by the equation:

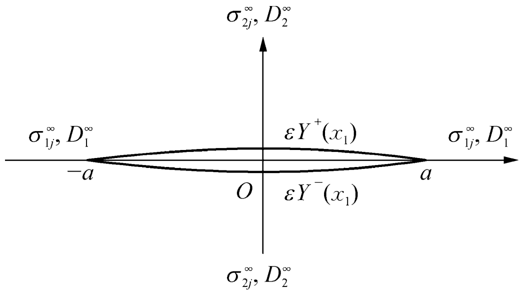
$$\begin{aligned} &x_2 = \varepsilon Y_{\pm}(x_1), \quad Y_{+}(x_1) - Y_{-}(x_1) > 0, \quad |x_1| < a \\ &Y'_{+}(\pm a) - Y'_{-}(\pm a) = 0 \end{aligned} \tag{4.131a}$$

where ε is a small parameter and $2a$ is the length of the crack. The last equation in Eq. (4.131a) ensures the crack tip idealization.

Huang and Kuang (2001) applied the small parameter method to solve this problem. According to Eq. (4.131a), the points on the crack surfaces in z and z_j planes are denoted respectively by

$$z^0 = x_1 + \mathrm{i}\varepsilon Y_{\pm}(x_1), \quad z_j^0 = x_1 + \varepsilon\mu_j Y_{\pm}(x_1), \quad |x_1| \le a \tag{4.132}$$

Fig. 4.8 Nonideal crack



Expand the complex potential in the piezoelectric material in the series of ε

$$f_j(z_j) = f_j(z_j;\varepsilon) = \sum_{n=0}^{\infty} (\varepsilon^n/n!) f_j^{(n)}(z_j) = f_j^{(0)}(z_j) + \varepsilon f_j^{(1)}(z_j) + \cdots \tag{4.133}$$

On the crack surfaces, we have

$$f_j^{(n)}\left(z_j^0\right) = f_j^{(n)\pm}(x_1) + \varepsilon\mu_j Y_\pm(x_1) f_j'(n)\pm(x_1) + \cdots \tag{4.134}$$

where $f_j^{(n)\pm}(x_1)$ is the value at z_j^0 of $f_j^{(n)}(z_j)$ and $f_j'^{(n)\pm}(z)$ is the derivative of $f_j^{(n)\pm}(z)$ with z. The complex electric potential $\phi(z)$ in the air can be expressed in the same way:

$$\begin{aligned} &\phi(z) = \phi(z;\varepsilon) = \phi^{(0)}(z) + \varepsilon\phi^{(1)}(z) + \cdots; \quad \varphi^c(z;\varepsilon) = \phi(z;\varepsilon) + \bar{\phi}(\bar{z};\varepsilon) \\ &\phi^{(n)}\left(z^0\right) = \phi^{(n)\pm}(x_1) + \mathrm{i}\varepsilon Y_\pm(x_1)\phi'^{(n)\pm}(x_1) + \cdots \\ &E_1^c = -\varphi_{,1}^c(z) = -2\mathrm{Re}\phi'(z), \quad E_2^c = -\varphi_{,2}^c(z) = 2\mathrm{Im}\phi'(z) \end{aligned} \tag{4.135}$$

The boundary conditions on a permeable crack surfaces are

$$\begin{aligned} &2\mathrm{Re}\sum_j B_{kj} f_j\left(z_j^0\right) = 0, \quad k = 1,2,3 \\ &2\mathrm{Re}\sum_{j=1}^{4} A_{4j} f_j\left(z_j^0\right) = 2\mathrm{Re}\phi\left(z^0\right), \quad 2\mathrm{Re}\sum_{j=1}^{4} B_{4j} f_j\left(z_j^0\right) = 2\epsilon_0 \mathrm{Im}\phi\left(z^0\right) \end{aligned} \tag{4.136}$$

The zero-order approximation on the crack surfaces $|x_1| \le a, x_2 = 0$ is

$$\begin{aligned} &2\mathrm{Re}\sum_{j=1}^{4} B_{Pj} f_j^{(0)\pm}(x_1) = T_P^{(0)}(x_1), \quad 2\mathrm{Re}\sum_j A_{4j} f_j^{(0)\pm}(x_1) = 2\mathrm{Re}\phi^{(0)}(x_1) \\ &T_P^{(0)}(x_1) = \left[0,0,0,2\epsilon_0 \mathrm{Im}\phi^{(0)}(x_1)\right]^{\mathrm{T}}, \quad P = 1,2,3,4 \end{aligned} \tag{4.137}$$

The first-order approximation on the crack surfaces $|x_1| \leq a, x_2 = 0$ is

$$
\begin{aligned}
&2\,\mathrm{Re}\sum_{j=1}^{4} B_{Pj} f_j^{(1)\pm}(x_1) = T_P^{(1)\pm}(x_1) \\
&2\,\mathrm{Re}\sum_{j=1}^{4} A_{4j}\left[\mu_j Y_\pm(x_1) f_j'^{(0)\pm}(x_1) + f_j^{(1)\pm}(x_1)\right] = 2\mathrm{Re}\left[\mathrm{i}Y_\pm(x_1)\phi'^{(0)}(x_1) + \phi^{(1)}(x_1)\right] \\
&T_P^{(1)\pm}(x_1) = -2Y_\pm(x_1)\mathrm{Re}\left[\sum_j B_{Pj}\mu_j f_j'^{(0)\pm}(x_1)\right] \\
&\qquad\qquad + 2\delta_{4P}\epsilon_0\,\mathrm{Im}\left[\mathrm{i}Y_\pm(x_1)\phi'^{(0)}(x_1) + \phi^{(1)}(x_1)\right]
\end{aligned}
\tag{4.138}
$$

The zero-order and first-order approximations at infinity are, respectively,

$$
\begin{aligned}
&\lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj}\mu_j f_j'^{(0)}(z_j)\right] = -\Sigma_{1P}^{\infty}, \quad \lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj}\mu_j f_j'^{(1)}(z_j)\right] = 0 \\
&\lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(0)}(z_j)\right] = \Sigma_{2P}^{\infty}, \quad \lim_{x\to\infty} 2\mathrm{Re}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(1)}(z_j)\right] = 0
\end{aligned}
\tag{4.139}
$$

The single-valued conditions are

$$
\begin{aligned}
&\int_{-a}^{a}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(0)+}(x_1) - \sum_{j=1}^{4} B_{Pj} f_j'^{(0)-}(x_1)\right] = 0 \\
&\int_{-a}^{a}\left[\sum_{j=1}^{4} B_{Pj} f_j'^{(1)+}(x_1) - \sum_{j=1}^{4} B_{Pj} f_j'^{(1)+}(x_1)\right] = 0
\end{aligned}
\tag{4.140}
$$

In Eqs. (4.137), (4.138), (4.139), and (4.140), the subscript P takes the values $1, 2, 3, 4$.

From Eqs. (4.137) and (4.138), an inhomogeneous Riemann-Hilbert equations can be obtained.

According to previous sections, it is easy to get their solutions. Finally the stress asymptotic fields near the crack tip are obtained. For a specific symmetric perturbed crack surface configuration,

$$
Y_\pm(x_1) = \pm Y(x_1) = \pm\left(a^2 - x_1^2\right)^{3/2} / 3a^2 \tag{4.131b}
$$

The singular term of the generalized stress fields on the x-axis in piezoelectric material for the zero-order approximation are

$$
\begin{aligned}
&\Sigma_{2P}^{(0)}(r,\theta)=\sqrt{a/2r}\,\mathrm{Re}\left[\sum_{j=1}^{4}B_{Pj}R_j\Big/\sqrt{\Theta_j}\right]+\delta_{4P}C,\quad \Theta_j=\cos\theta+\mu_j\sin\theta\\
&R_j=\left(B_{jP}^{-1}-B_{j4}^{-1}H_{4P}/H_{44}\right)\Sigma_{2P}^{\infty},\quad C=(H_{4J}/H_{44})\Sigma_{2J}^{\infty}\\
&K_{\mathrm{I}}^{(0)}=\sqrt{\pi a}\sigma_{22}^{\infty},\quad K_{\mathrm{II}}^{(0)}=\sqrt{\pi a}\sigma_{21}^{\infty},\quad K_{\mathrm{III}}^{(0)}=\sqrt{\pi a}\sigma_{23}^{\infty},\quad K_{\mathrm{D}}^{(0)}=-\sqrt{\pi a}\sigma_{2j}^{\infty}H_{4j}/H_{44}
\end{aligned}
\tag{4.141}
$$

and the electric fields in the air are

$$
E_2^{(0)c}(x_1,0)=\frac{D_2^{\infty}}{\epsilon^c}+\frac{H_{4j}\sigma_{2j}^{\infty}}{H_{44}\epsilon^c},\quad E_1^{(0)c}(x_1,0)=E_1^{\infty}+\mathrm{Re}\sum_{j=1}^{4}A_{4j}R_j
\tag{4.142}
$$

From Eqs. (4.141) and (4.142), it is seen that the zero-order approximate solution of a permeable crack is consistent with the conducting crack.

The singular term of the generalized stress fields on the x_1-axis in piezoelectric material for the first-order approximation are

$$
\begin{aligned}
&\Sigma_{2P}^{(1)}(r,0)=\frac{1}{6}\sqrt{\frac{a}{2r}}\left\{\mathrm{Re}\left(\sum_{j=1}^{4}\mathrm{i}B_{Pj}\mu_jR_j\right)-\frac{\delta_{4P}H_{4N}}{H_{44}}\mathrm{Re}\left(\sum_{j=1}^{4}\mathrm{i}B_{Nj}\mu_jR_j\right)\right\}\\
&K_{\mathrm{I}}^{(1)}=\frac{\sqrt{\pi a}}{6}\mathrm{Re}\left(\sum_{j=1}^{4}\mathrm{i}B_{2j}\mu_jR_j\right),\quad K_{\mathrm{II}}^{(1)}=\frac{\sqrt{\pi a}}{6}\mathrm{Re}\left(\sum_{j=1}^{4}\mathrm{i}B_{1j}\mu_jR_j\right)\\
&K_{\mathrm{III}}^{(1)}=\frac{\sqrt{\pi a}}{6}\mathrm{Re}\left(\sum_{j=1}^{4}\mathrm{i}B_{3j}\mu_jR_j\right),\quad K_{\mathrm{D}}^{(1)}=-\frac{\sqrt{\pi a}H_{4N}}{6H_{44}}\mathrm{Re}\left(\sum_{j}\mathrm{i}B_{Nj}\mu_jR_j\right)
\end{aligned}
\tag{4.143}
$$

and the electric fields in the air are

$$
\begin{aligned}
&E_1^{(1)c}(x_1,0)=-\frac{1}{2}\left(3\Pi_1\frac{x_1^2}{a^2}+\Pi_2\right),\quad E_2^{(1)c}(x_1,0)=-\frac{1}{\pi\mathrm{i}A_1}\left[(A_2+A_3)\frac{x_1^2}{a^2}-\left(\frac{A_3}{2}+\frac{A_2}{3}\right)\right]\\
&\varphi[x_1,Y_+(x_1)]-\varphi[x_1,Y_-(x_1)]=-2\varepsilon[Y_+(x_1)-Y_-(x_1)]\left(E_2^{(0)c}+E_2^{(1)c}\right)
\end{aligned}
\tag{4.144}
$$

where A,A_2,A_3 and Π_1,Π_2 are known complex constants and functions, respectively. It is found that the generalized stress intensity factors of the zero- and first-order approximations have the same singularity $1/\sqrt{r}$, but the stress angular distributions are different. The future research finds that for an isotropic material, $K_{\mathrm{I}}^{(1)}=K_{\mathrm{D}}^{(1)}=0$. The electric fields are inhomogeneous in the air gap and the electric potential discontinuity is also inhomogeneous.

In Huang and Kuang's paper (2001), they also discussed the insulated and conducted cracks.

4.4.4 Other Crack Models

Hao and Shen (1994) proposed a model that the electric displacement is dependent on the crack opening displacement. They assumed that the boundary conditions on the crack surfaces are

$$D_2^+ = D_2^-, \quad D_2^+\left(u_2^+ - u_2^-\right) = \epsilon_0(\varphi^- - \varphi^+) \tag{4.145}$$

and discussed a single crack located on the $ox_1(-a, a)$ under the boundary conditions:

$$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2^\infty, \quad \text{at infinity;} \quad \boldsymbol{\Sigma}_2 = \mathbf{0}, \quad |x_1| \le a,\ x_2 = 0 \tag{4.146}$$

At first it is assumed $\epsilon_0(\varphi^- - \varphi^+)/\left(u_2^+ - u_2^-\right) = D_2^0$ prior and D_2^0 is a constant determined in the solving process. They applied the stress function method as shown in Sect. 3.3 in the transform planes to solve this problem. The transform function is the same as shown in Eqs. (3.82) and (3.86). Finally they get

$$K_{\mathrm{I}} = \sigma_{22}^\infty\sqrt{\pi a}, \quad K_{\mathrm{II}} = \sigma_{21}^\infty\sqrt{\pi a}, \quad K_{\mathrm{III}} = \sigma_{23}^\infty\sqrt{\pi a}, \quad K_{\mathrm{D}} = \left(D_2^\infty - D_2^0\right)\sqrt{\pi a} \tag{4.147}$$

Their numerical example showed that the smaller external force, the smaller K_{D}. The maximum K_{D} is equal to the electric displacement intensity factor of the insulated crack. It is interest that the boundary conditions Eq. (4.145) can be derived from Eq. (4.144).

Zhang et al. (1998) proposed a self-consistent calculation of a crack profile. They considered that the profile of the opened crack is an elliptic cavity and the ratio of the minor semiaxis to the major semiaxis $\alpha_{\mathrm{s}} = [\![\boldsymbol{A}\boldsymbol{f}(\alpha_{\mathrm{s}}) + \bar{\boldsymbol{A}}\bar{\boldsymbol{f}}(\alpha_{\mathrm{s}})]\!]_2$ (component along x_2) at $x_1 = x_2 = 0$. In the solving process, the current crack profile is used by numerical calculation.

4.5 Interaction of Collinear Inclusions with Singularity

4.5.1 Interaction of an Interface Permeable Crack with a Singularity in a Bimaterial

Let a generalized mechanical singular load with strength $(\boldsymbol{b},\ \boldsymbol{p})$ be located at z_0 in material I occupied the upper half plane $S^+, x_2 > 0$. A permeable crack $(-a, a)$ is located on the interface $x_2 = 0$ (Suo 1990; Gao and Wang 2001; Kuang and Ma 2002). The boundary conditions are

$$\begin{aligned}
&\Sigma_{ij} = \Sigma_{ij}^\infty = 0; \quad |z| \to \infty \\
&\sigma_{2j}^+ = \sigma_{2j}^- = 0; \quad D_2^+ = D_2^- = D_2, \quad E_1^+ = E_1^-; \quad x_1 \in L_{\mathrm{c}} = (-a, a) \\
&\sigma_{2j}^+ = \sigma_{2j}^-, \quad u_j^+ = u_j^-; \quad D_2^+ = D_2^- = D_2, \quad E_1^+ = E_1^-; \quad x_1 \notin L_{\mathrm{c}}
\end{aligned} \tag{4.148}$$

Assume the solution takes the following form:

$$\begin{aligned}&\boldsymbol{F}_\alpha(z)=\boldsymbol{F}_{\alpha\,0}(z)+\boldsymbol{G}_\mathrm{I}(z)\delta_{\alpha\mathrm{I}},\quad \alpha=\mathrm{I},\mathrm{II}\\&\boldsymbol{G}_\mathrm{I}(z)=(1/2\pi\mathrm{i})\boldsymbol{V}\left\langle\left(z_j-z_{0j}\right)^{-1}\right\rangle,\quad \boldsymbol{V}=\left(\boldsymbol{B}_\mathrm{I}^\mathrm{T}\boldsymbol{b}+\boldsymbol{A}_\mathrm{I}^\mathrm{T}\boldsymbol{p}\right)\end{aligned} \tag{4.149}$$

where $\boldsymbol{G}_\mathrm{I}(z)$ is the solution of a singularity in an infinite material I, see Eq. (3.165b). $\boldsymbol{F}_{\alpha\,0}(z)$ is the analytic function in the material α and is zero at infinity, because the generalized stress $\boldsymbol{\Sigma}_2$ is continuous in whole axis x_1. Similar to Eqs. (4.22) and (4.23), it can be obtained:

$$\begin{aligned}&\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_{\mathrm{I}\,0}(z)-\bar{\boldsymbol{B}}_\mathrm{II}\bar{\boldsymbol{F}}_{\mathrm{II}\,0}(z)+\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{G}}_\mathrm{I}(z)=\boldsymbol{0},\quad z\in S^+\\&\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_{\mathrm{II}\,0}(z)-\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{F}}_{\mathrm{I}\,0}(z)-\boldsymbol{B}_\mathrm{I}\boldsymbol{G}_\mathrm{I}(z)=\boldsymbol{0},\quad z\in S^-\end{aligned} \tag{4.150}$$

Equations (4.21), (4.24), and (4.150) yield

$$\begin{aligned}&\hat{\boldsymbol{d}}(x_1)=\boldsymbol{U}_\mathrm{I}(x_1)-\boldsymbol{U}_\mathrm{II}(x_1)=2\mathrm{Re}[\boldsymbol{A}_\mathrm{I}\boldsymbol{f}_\mathrm{I}(x_1)-\boldsymbol{A}_\mathrm{II}\boldsymbol{f}_\mathrm{II}(x_1)]\\&\mathrm{i}\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{H}\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I0}(x_1)+(\bar{\boldsymbol{Y}}_\mathrm{II}-\bar{\boldsymbol{Y}}_\mathrm{I})\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{G}}_\mathrm{I}(x_1)-\bar{\boldsymbol{H}}\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II0}(x_1)+(\boldsymbol{Y}_\mathrm{I}+\bar{\boldsymbol{Y}}_\mathrm{I})\boldsymbol{B}_\mathrm{I}\boldsymbol{G}_\mathrm{I}(x_1)\end{aligned} \tag{4.151}$$

Because the generalized displacements are continuous on the connective interface, using analytic continuation, a function $\boldsymbol{h}(z)$ analytic in whole z plane except the crack can be constructed:

$$\boldsymbol{h}(z)=\begin{cases}\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I0}(z)+\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_\mathrm{II}-\bar{\boldsymbol{Y}}_\mathrm{I})\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{G}}_\mathrm{I}(z), & z\in S^+\\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II0}(z)-\boldsymbol{H}^{-1}(\boldsymbol{Y}_\mathrm{I}+\bar{\boldsymbol{Y}}_\mathrm{I})\boldsymbol{B}_\mathrm{I}\boldsymbol{G}_\mathrm{I}(z), & z\in S^-\end{cases};\quad z\notin L_\mathrm{c} \tag{4.152}$$

The stress $\boldsymbol{\Sigma}_\mathrm{I}(x_1)=\boldsymbol{\Sigma}_\mathrm{II}(x_1)=\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(x_1)+\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{F}}_\mathrm{I}(x_1)$ on the axis x_1 can be expressed as

$$\boldsymbol{\Sigma}(x_1)=\boldsymbol{h}^+(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1)+\bar{\boldsymbol{H}}^{-1}(\boldsymbol{Y}_\mathrm{I}+\bar{\boldsymbol{Y}}_\mathrm{I})\boldsymbol{B}_\mathrm{I}\boldsymbol{G}_\mathrm{I}+\boldsymbol{H}^{-1}(\boldsymbol{Y}_\mathrm{I}+\bar{\boldsymbol{Y}}_\mathrm{I})\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{G}}_\mathrm{I} \tag{4.153}$$

According to Eq. (4.148) on the crack surface, we have $\boldsymbol{\Sigma}(x_1)=D_2(x_1)\mathbf{i}_4$, $\mathbf{i}_4=[0,0,0,1]^\mathrm{T}$, where $D_2(x_1)$ is unknown. So a Riemann-Hilbert equation is obtained:

$$\begin{aligned}&\boldsymbol{h}^+(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1)=\tilde{\boldsymbol{\Sigma}}(x_1),\quad x_1\in L_\mathrm{c}\\&\tilde{\boldsymbol{\Sigma}}(x_1)=D_2(x_1)\mathbf{i}_4-\bar{\boldsymbol{H}}^{-1}(\boldsymbol{Y}_\mathrm{I}+\bar{\boldsymbol{Y}}_\mathrm{I})\boldsymbol{B}_\mathrm{I}\boldsymbol{G}_\mathrm{I}-\boldsymbol{H}^{-1}(\boldsymbol{Y}_\mathrm{I}+\bar{\boldsymbol{Y}}_\mathrm{I})\bar{\boldsymbol{B}}_\mathrm{I}\bar{\boldsymbol{G}}_\mathrm{I}\end{aligned} \tag{4.154}$$

Equation (4.154) is identical with Eq. (4.28) except using $\tilde{\boldsymbol{\Sigma}}(x_1)$ instead of $\boldsymbol{\Sigma}_0(x_1)$. The form of the solution is still expressed by Eq. (4.41), i.e.,

$$\begin{aligned}&\boldsymbol{\Psi}(z)=\boldsymbol{Q}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\mathrm{i}}\int_L\frac{[\boldsymbol{Q}^+(x_1)]^{-1}\boldsymbol{\Sigma}^*(x_1)\mathrm{d}x_1}{(x_1-z)}\right\},\quad \boldsymbol{C}(z)=\boldsymbol{C}_1z+\boldsymbol{C}_0\\&\boldsymbol{\Psi}(z)=\bar{\boldsymbol{\Omega}}^\mathrm{T}\boldsymbol{h}(z),\quad \boldsymbol{\Sigma}^*(t)=\bar{\boldsymbol{\Omega}}^\mathrm{T}\tilde{\boldsymbol{\Sigma}}(t),\quad \boldsymbol{Q}(z)=\left\langle Y_0^{(j)}(z)\right\rangle,\quad Y_0^{(i)}(z)=\frac{1}{\sqrt{z^2-a^2}}\left(\frac{z+a}{z-a}\right)^{\mathrm{i}\varepsilon_i}\end{aligned} \tag{4.155}$$

In this problem, it is known that $\boldsymbol{\Psi}(\infty) = \mathbf{0}$ from $\boldsymbol{h}(\infty) = \mathbf{0}$ and $\oint_{\Gamma} \boldsymbol{\Psi}(z)\mathrm{d}z = \mathbf{0}$ from the single-valued condition. So unknown constant vectors $\boldsymbol{C}_1 = \boldsymbol{C}_0 = \mathbf{0}$.

Equations (4.151) and (4.52) yield

$$\mathrm{i}\hat{\boldsymbol{d}}'(x_1) = \boldsymbol{H}[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] \tag{4.156}$$

Because the electric potential is continuous on whole axis, Eq. (4.156) yields

$$\boldsymbol{H}_4[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] = \mathbf{0}, \quad \boldsymbol{H}_4 = [H_{41}, H_{42}, H_{43}, H_{44}], \quad |x_1| < \infty \tag{4.157}$$

Noting $\boldsymbol{h}(\infty) = \mathbf{0}$ the solution of Eq. (4.157) is

$$\boldsymbol{H}_4\boldsymbol{h}(z) = \boldsymbol{H}_4\boldsymbol{\Omega}\boldsymbol{\Psi} = \mathbf{0}, \quad \boldsymbol{\Omega}\bar{\boldsymbol{\Omega}}^{\mathrm{T}} = \mathbf{I} \tag{4.158}$$

From Eq. (4.158), $D_2(z)$ can be determined and then Eq. (4.155) can be solved. Substituting $\boldsymbol{\Psi}(z)$ into Eq. (4.152) yields $\boldsymbol{F}_{\alpha 0}(z_j)$.

4.5.2 Interaction of an Interface Impermeable Crack with an Interface Singularity

Let a generalized singularity load located at $(x_{01}, 0)$ in front of the right tip of a crack $(-a, a)$ (Wang and Kuang 2002). The superposition method is used to solve this problem, i.e., let

$$\boldsymbol{U}_\alpha = \boldsymbol{U}_{\alpha d} + \boldsymbol{U}_{\alpha c}, \quad \boldsymbol{\Phi}_\alpha = \boldsymbol{\Phi}_{\alpha d} + \boldsymbol{\Phi}_{\alpha c} \tag{4.159}$$

where $\boldsymbol{U}_{\alpha d}, \boldsymbol{\Phi}_{\alpha d}$ are expressed in Eqs. (3.171) and (3.176) representing the solutions of an interface singularity in a bimaterial without crack. This solution introduces the traction $\boldsymbol{\Sigma}_{2d}$. $\boldsymbol{U}_{\alpha c}, \boldsymbol{\Phi}_{\alpha c}$ are the solutions of a crack subjected to $-\boldsymbol{\Sigma}_{2d}$ in a bimaterial.

Using the orthogonal relations of $\boldsymbol{A}$ and $\boldsymbol{B}$ from Eq. (3.171) yields

$$-\boldsymbol{\Sigma}_{2d} = -2\mathrm{Re}\left[\boldsymbol{B}_\alpha\left\langle\frac{1}{x_1 - x_{01}}\right\rangle\boldsymbol{V}_\alpha\right] = -(\boldsymbol{B}_{\mathrm{I}}\boldsymbol{V}_{\mathrm{I}} + \bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{V}}_{\mathrm{I}})\frac{1}{x_1 - x_{01}} = -\frac{1}{x_1 - x_{01}}\frac{\boldsymbol{l}}{\pi} \tag{4.160}$$

where $\boldsymbol{l}$ is expressed in Eq. (3.175). The solution of a crack subjected to $-\boldsymbol{\Sigma}_{2d}$ in a bimaterial can be found in Eq. (4.38). From $\boldsymbol{\Sigma}^\infty = \mathbf{0}$ and the single-valued condition of generalized displacement, it yields $\boldsymbol{C}(z) = \mathbf{0}$ in Eq. (4.38). So the solution is

$$\begin{aligned}\boldsymbol{h}_{\mathrm{c}}(z) = \boldsymbol{B}\boldsymbol{F}(z) &= \frac{1}{2\pi\mathrm{i}}\boldsymbol{P}(z)\int_L \frac{-\boldsymbol{\Sigma}_{2d}(x_1)\mathrm{d}x_1}{\boldsymbol{P}^+(x_1)(x_1 - z)} = -\frac{1}{2\pi\mathrm{i}}\boldsymbol{P}(z)\int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}\boldsymbol{l}\mathrm{d}x_1}{\pi(x_1 - z)(x_1 - x_{01})} \\ &= -\frac{1}{2\pi\mathrm{i}}\frac{1}{z - x_{01}}\boldsymbol{P}(z)\left\{\int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}\boldsymbol{l}\mathrm{d}x_1}{\pi(x_1 - x_{01})} - \int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}\boldsymbol{l}\mathrm{d}x_1}{\pi(x_1 - z)}\right\}\end{aligned} \tag{4.161}$$

Through some manipulation, we get

$$\boldsymbol{h}_{\mathrm{c}}(z)=\frac{1}{z-x_{01}}\left(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{\Omega}\boldsymbol{Q}(z)\left\langle z-x_{01}-\frac{1}{Y_0(z)}+\frac{1}{Y_0(x_{01})}\right\rangle\boldsymbol{\Omega}^{-1}\frac{\boldsymbol{l}}{\pi} \tag{4.162}$$

From Eqs. (4.44) and (4.162) in front of the crack, the asymptotic stress is

$$\begin{aligned}\boldsymbol{\Sigma}_{2\mathrm{c}}(x_1)&=\boldsymbol{h}_{\mathrm{c}}^{+}(x_1)+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}_{\mathrm{c}}^{-}(x_1)=(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H})\boldsymbol{h}_{\mathrm{c}}(x_1)\\&=\boldsymbol{\Omega}\boldsymbol{Q}(z)\left\langle 1-\frac{1}{Y_0(z)(x_1-x_{01})}+\frac{1}{Y_0(x_{01})(x_1-x_{01})}\right\rangle\boldsymbol{\Omega}^{-1}\frac{\boldsymbol{l}}{\pi}\end{aligned} \tag{4.163}$$

According to Eqs. (4.47) and (4.163), the stress intensity factor is

$$\begin{aligned}\boldsymbol{K}&=[K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_{\mathrm{D}}]^{\mathrm{T}}=\lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Omega}\left\langle(x_1-a)^{-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1)\\&=\frac{1}{\sqrt{\pi a}}\boldsymbol{\Omega}\left\langle(2a)^{-\mathrm{i}\varepsilon_*}\left[1+\frac{1}{Y_0(x_{01})(a-x_{01})}\right]\right\rangle\boldsymbol{\Omega}^{-1}(\boldsymbol{\Omega}_1\boldsymbol{b}+\boldsymbol{\Omega}_2\boldsymbol{p})=\boldsymbol{W}_1\boldsymbol{b}+\boldsymbol{W}_2\boldsymbol{p}\end{aligned} \tag{4.164}$$

Sometimes $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are called the weight functions.

4.5.3 Interaction of Collinear Rigid Inclusions with a Singularity

Now we discuss the interaction of collinear rigid inclusions with singularity. The singularity is also located at z_0 with strength $(\boldsymbol{b},\ \boldsymbol{p})$ in material I (Zhou et al. 2008). The boundary conditions are assumed:

$$\begin{aligned}&\boldsymbol{\Sigma}_2=\boldsymbol{\Sigma}_2^{\infty}(x_1),\quad |z|\to\infty\\&u_{j,1}=\omega_{\mathrm{r}}\delta_{j2},\quad U_{\mathrm{I}}=U_{\mathrm{II}},\quad E_{\mathrm{I}1}=E_{\mathrm{II}1}=E_{\mathrm{r}1}=-\varphi_{\mathrm{r},1},\quad D_{\mathrm{I}1}=D_{\mathrm{II}1},\quad x_1\in L_{\mathrm{cr}}\\&\boldsymbol{U}_{\mathrm{I}}(x_1)=\boldsymbol{U}_{\mathrm{II}}(x_1),\quad \boldsymbol{\Sigma}_2(x_1)=\boldsymbol{\Sigma}_{\mathrm{I}2}(x_1)=\boldsymbol{\Sigma}_{\mathrm{II}2}(x_1),\quad x\notin L_{\mathrm{c}},\quad L_{\mathrm{c}}=\cup L_{\mathrm{cr}}\end{aligned} \tag{4.165}$$

where ω_{r} is the rotation angle about axis x_3 of the rth inclusion. Comparing with Sect. 4.3.1 (rigid line inclusion) here, only a singularity is added, so the solving process is similar. Assume the solution is in the following form:

$$\begin{aligned}&\boldsymbol{U}_{\alpha,\,1}=2\mathrm{Re}[\boldsymbol{A}_\alpha\boldsymbol{F}_\alpha(z)+\boldsymbol{A}_\alpha\boldsymbol{G}_{\mathrm{I}}(z)\delta_{\alpha\mathrm{I}}];\quad \boldsymbol{\Phi}_{\alpha,\,1}=2\mathrm{Re}[\boldsymbol{B}_\alpha\boldsymbol{F}_\alpha(z)+\boldsymbol{B}_\alpha\boldsymbol{G}_{\mathrm{I}}(z)\delta_{\alpha\mathrm{I}}]\\&\boldsymbol{G}_{\mathrm{I}}(z)=(1/2\pi\mathrm{i})\boldsymbol{V}\left\langle(z_j-z_{0j})^{-1}\right\rangle,\quad \boldsymbol{V}=\left(\boldsymbol{B}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{b}+\boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}}\boldsymbol{p}\right),\quad \alpha=\mathrm{I},\mathrm{II}\end{aligned} \tag{4.166}$$

The generalized displacements are continuous on the whole axis x_1. Like Eqs. (4.73) and (4.150) we have

$$\begin{aligned}&\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(x_1)=\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(x_1)=\boldsymbol{\Delta}^{\infty}\\&\boldsymbol{\Delta}^{\infty}=(1/2)[(\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{\infty}+\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^{\infty})-(\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}^{\infty}+\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty})],\quad \alpha=\mathrm{I},\mathrm{II}\end{aligned}\tag{4.167}$$

Like Eq. (4.74), we have

$$\begin{aligned}\Delta\boldsymbol{\Phi}_{,1}(x_1)=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)-\boldsymbol{\Phi}_{\mathrm{II},1}(x_1)=\mathrm{i}\big[&\boldsymbol{R}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)-\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)-\big(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\big)\boldsymbol{\Delta}^{\infty}\big]\\&+\big(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\big)\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)+\big(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1}\big)\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)\end{aligned}\tag{4.168}$$

where $\boldsymbol{Y}_{\alpha},\boldsymbol{R}$ are also shown in Eq. (4.74). By the standard analytic continuation through the connective interface $L-L_{\mathrm{c}}$, we can construct a function $\boldsymbol{h}(z)$ analytic in whole plane except the rigid inclusions L_{c} and at infinity $\boldsymbol{h}(\infty)=\boldsymbol{h}^{\infty}$:

$$\boldsymbol{h}(z)=\begin{cases}\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)+\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z), & z\in S^{+}\\ \boldsymbol{R}^{-1}\bar{\boldsymbol{R}}\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)+\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^{\infty} & z\in S^{-}\end{cases}\tag{4.169}$$

Equation (4.169) yields

$$\begin{aligned}&\boldsymbol{F}_{\mathrm{I}}(z_j)=\boldsymbol{A}_{\mathrm{I}}^{-1}\big[\boldsymbol{h}(z_j)-\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(z_j)\big]\\&\boldsymbol{F}_{\mathrm{II}}(z_j)=\boldsymbol{A}_{\mathrm{II}}^{-1}\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\big[\boldsymbol{h}(z_j)+\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(z_j)-\boldsymbol{R}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^{\infty}\big]\end{aligned}\tag{4.170}$$

Like Eq. (4.79), we have

$$\begin{aligned}\boldsymbol{U}_{\mathrm{I},1}(x_1)&=\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{x}_1)+\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)+\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)\\&=\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)+\bar{\boldsymbol{R}}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)\\&\quad+\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)-\boldsymbol{\Delta}_{1}^{\infty}\\ \boldsymbol{\Delta}_{1}^{\infty}&=\bar{\boldsymbol{R}}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}}^{-1}-\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{\Delta}^{\infty}\end{aligned}\tag{4.171}$$

On the surfaces of inclusions, like Eq. (4.80), we have

$$\mathbf{U}_{\mathrm{I},1}=\omega(x_1)\mathbf{i}_2-E_1(x_1)\mathbf{i}_4,\quad \omega(x_1)=\omega_{\mathrm{r}},\quad r=1-n,\quad x_1\in L_{\mathrm{c}}\tag{4.172}$$

From Eqs. (4.171) and (4.172), a Riemann-Hilbert equation is obtained:

$$\begin{aligned}&\boldsymbol{h}^{+}(x_1)+\bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^{-}(x_1)=\boldsymbol{N}(x_1),\quad x_1\in L_{\mathrm{c}}\\&\boldsymbol{N}(x_1)=\boldsymbol{\Delta}_{1}^{\infty}+\omega_r(x_1)\mathbf{i}_2-E_1(x_1)\mathbf{i}_4-\bar{\boldsymbol{R}}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\boldsymbol{A}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}}(x_1)-\boldsymbol{R}^{-1}(\boldsymbol{Y}_{\mathrm{I}}^{-1}+\bar{\boldsymbol{Y}}_{\mathrm{I}}^{-1})\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{G}}_{\mathrm{I}}(x_1)\end{aligned}\tag{4.173}$$

Equation (4.173) is identical with (4.81) if we use $\boldsymbol{N}$ instead of $\boldsymbol{\Delta}_1^\infty + \omega_0\mathbf{i}_2 - E_1(x_1)\mathbf{i}_4$. Its solution is

$$\begin{aligned}
&\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z) = \boldsymbol{Q}(z)\boldsymbol{C}(z) + \frac{\boldsymbol{Q}(z)}{2\pi\mathrm{i}}\int_L \frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{N}(x_1)}{\boldsymbol{Q}^+(x_1)(x_1 - z)}\mathrm{d}x_1 \\
&\boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z)\right\rangle, \quad Y_0^{(j)}(z) = \prod_{k=1}^{n}\frac{1}{\sqrt{(z-a_k)(z-b_k)}}\left(\frac{z-b_k}{z-a_k}\right)^{\mathrm{i}\varepsilon_j} \\
&C^{(i)}(z) = C_n^{(i)}z^n + C_{n-1}^{(i)}z^{n-1} + \cdots + C_1^{(i)}z + C_0^{(i)}
\end{aligned} \tag{4.174}$$

Equations (4.168), (4.169), and (4.165) yield

$$\begin{aligned}
&\Delta\boldsymbol{\Phi}_{,1}(x_1) = \mathrm{i}\boldsymbol{R}[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)], \quad x_1 \in L_{\mathrm{c}} \\
&\Delta\boldsymbol{\Phi}_{,1}(x_1) = \boldsymbol{0}, \quad x_1 \notin L_{\mathrm{c}}
\end{aligned} \tag{4.175}$$

According to Eq. (4.165), $D_2(x_1)$ is continuous on whole $x_1 = 0$, so $\Delta\Phi_{4,1}(x_1) = 0$, or

$$\boldsymbol{R}_4[\boldsymbol{h}^+(x_1) - \boldsymbol{h}^-(x_1)] = \boldsymbol{0}, \quad -\infty < x_1 < \infty \tag{4.176}$$

where $\boldsymbol{R}_4$ is the fourth row of $\boldsymbol{R}$. The solution is

$$\boldsymbol{R}_4\boldsymbol{h}(z) = \boldsymbol{R}_4\boldsymbol{h}^\infty, \quad \boldsymbol{h}^\infty = \boldsymbol{h}(\infty) \tag{4.177}$$

Assume $\boldsymbol{\varepsilon}^\infty = \left[\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + w^\infty, \varepsilon_{13}^\infty + w_3^\infty, -E_1^\infty\right]^{\mathrm{T}}$ at infinity and noting $\boldsymbol{h}^+(x_1) = \boldsymbol{h}^-(x_1)$ on the crack surface and $\boldsymbol{h}^+(\infty) = \boldsymbol{h}^-(\infty)$ we can get $\boldsymbol{h}^\infty$:

$$\begin{aligned}
&\mathbf{U}_{I,1}(\infty) = \left[\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + \omega^\infty, \varepsilon_{13}^\infty + \omega_3^\infty, -E_1^\infty\right] = \boldsymbol{\varepsilon}^\infty = \boldsymbol{h}^+(\infty) + \bar{\boldsymbol{R}}^{-1}\boldsymbol{R}\boldsymbol{h}^-(\infty) - \boldsymbol{\Delta}_1^\infty \\
&\boldsymbol{h}^\infty = \boldsymbol{\Omega} < \left(1 + \mathrm{e}^{2\pi\varepsilon_i}\right)^{-1} > \bar{\boldsymbol{\Omega}}^{\mathrm{T}}\left(\boldsymbol{\varepsilon}^\infty + \boldsymbol{\Delta}_1^\infty\right)
\end{aligned} \tag{4.178}$$

From Eqs. (4.177) and (4.178), $E_1(z)$ can be obtained and then Eq. (4.174) can be solved.

If $\boldsymbol{R} = \bar{\boldsymbol{R}}$ is a real matrix, the solution does not oscillate.

4.5.4 Interaction of a Crack with an Electric Dipole in a Homogeneous Piezoelectric Material

Let an impermeable crack $(-a, a)$ in an infinite piezoelectric material and an electric dipole with strength p_{e} located at z_0 formed an angle θ with positive axis x_1. The distance from z_0 to $(a, 0)$ is $\rho = \left|\overrightarrow{z_0 a}\right|$ and $\overrightarrow{z_0 a}$ form an angle ϕ with the positive direction of x_1 (Fig. 4.9).

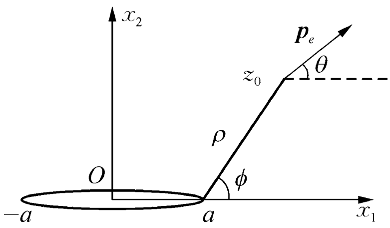


Fig. 4.9 Crack and electric dipole

Wang and Kuang (2000, 2002) discussed the interaction of a crack with an electric dipole in a homogeneous piezoelectric material. Let $\boldsymbol{U}_\mathrm{p}, \boldsymbol{\Phi}_\mathrm{p}$ as shown in Eq. (3.178) are the solutions of an electric dipole in an infinite piezoelectric material. The generalized traction on the line corresponding to the crack surfaces introduced by this electric dipole is $\boldsymbol{\Sigma}_2$ shown in Eq. (3.179). Assuming $\boldsymbol{h}_\mathrm{c}, \boldsymbol{U}_\mathrm{c}, \boldsymbol{\Phi}_\mathrm{c}$ are the solutions when the crack surfaces are subjected to $-\boldsymbol{\Sigma}_2$, the solutions of a piezoelectric material with a crack and an electric dipole are

$$\boldsymbol{U} = \boldsymbol{U}_\mathrm{p} + \boldsymbol{U}_\mathrm{c}, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}_\mathrm{p} + \boldsymbol{\Phi}_\mathrm{c} \tag{4.179}$$

According to Eq. (4.38) and noting $\boldsymbol{\Omega} = \mathbf{I}$ for a homogeneous material, the solution $\boldsymbol{h}_\mathrm{c}$ is

$$\boldsymbol{h}_\mathrm{c}(z) = \boldsymbol{B}\boldsymbol{F}_\mathrm{c}(z) = \boldsymbol{Q}(z)\left\{\boldsymbol{C} + \frac{1}{2\pi \mathrm{i}}\int_L \frac{-\boldsymbol{\Sigma}_2(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1 - z)}\right\}, \quad \boldsymbol{Q}(z) = \langle Y_0(z)\rangle \tag{4.180}$$

where $Y_0(z) = 1\Big/\sqrt{z^2 - a^2}$. Using Eq. (4.18) yields

$$\frac{1}{2\pi \mathrm{i}}\int_{-a}^{a} \frac{1}{Y_0^+(x_1 - z)}\mathrm{d}x_1 = \frac{1}{2}\left[\frac{1}{Y_0(z)} - z\right] = \frac{1}{2}\left[\sqrt{z^2 - a^2} - z\right]$$

Substituting Eq. (3.179) and above equation into Eq. (4.180) yields

$$\begin{aligned}
&\boldsymbol{h}_\mathrm{c}(z) = \frac{1}{2\sqrt{z^2 - a^2}}\mathrm{Re}\left\{\frac{p_\mathrm{e}}{\pi \mathrm{i}}\boldsymbol{B}\left\langle \Theta\left[\frac{\sqrt{z_{0j}^2 - a^2}}{\left(z - z_{0j}\right)^2} - \frac{\sqrt{z^2 - a^2}}{\left(z - z_{0j}\right)^2} + \frac{z_{0j}}{\left(z - z_{0j}\right)\sqrt{z_{0j}^2 - a^2}}\right]\right\rangle \boldsymbol{A}^\mathrm{T}\mathbf{i}_4\right\} \\
&\boldsymbol{\Sigma}_{2\mathrm{c}}(x_1) = 2\mathrm{Re}\boldsymbol{h}_\mathrm{c}(x_1), \quad \Theta = \cos\theta + \mu_j \sin\theta
\end{aligned} \tag{4.181}$$

The stress intensity factor is

$$\boldsymbol{K} = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)}\boldsymbol{\Sigma}_{2\mathrm{c}}(x_1) = p_\mathrm{e}\sqrt{\frac{a}{\pi}}\mathrm{Im}\left\{\boldsymbol{B}\left\langle \Theta\left[\frac{1}{\left(z_{0j} - a\right)\sqrt{z_{0j}^2 - a^2}}\right]\right\rangle \boldsymbol{A}^\mathrm{T}\mathbf{i}_4\right\} \tag{4.182}$$

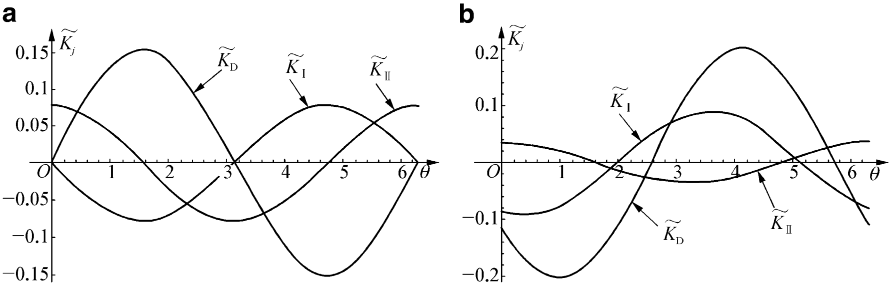


Fig. 4.10 Variation of $\tilde{\boldsymbol{K}}$ with θ: (**a**) dipole located at (2a,0) and (**b**) dipole located at (a,a)

Take a local coordinate system (ρ, ϕ) with the origin at the right crack tip; when $\rho \ll a$, $\boldsymbol{K}$ can be expressed by

$$\boldsymbol{K} = \sqrt{\frac{1}{2\pi}} p_e \frac{1}{\rho\sqrt{\rho}} \operatorname{Im}\left\{ \boldsymbol{B} \left\langle \left[\frac{\Theta}{(\cos\phi + \mu_j \sin\phi)^{3/2}} \right] \right\rangle \boldsymbol{A}^{\mathrm{T}} \mathbf{i}_4 \right\} \tag{4.183}$$

Figure 4.10 gives the variation of the dimensionless stress intensity factor $\tilde{\boldsymbol{K}} = \boldsymbol{K}/p_e a^{-3/2}$ with the electric dipole direction θ: (a) dipole located at (2a,0) and (b) dipole located at (a,a).

4.5.5 Interaction of a Crack with an Electric Dipole on the Interface in a Bimaterial

Let the electric dipole at $(x_{01}, 0)$ with strength p_e on the interface in a bimaterial. The superposition method is used to solve this problem, i.e.,

$$\boldsymbol{U} = \boldsymbol{U}_{ad} + \boldsymbol{U}_{ac}, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}_{ad} + \boldsymbol{\Phi}_{ac} \tag{4.184}$$

where $\boldsymbol{\Phi}_{ad}$ is shown in Eq. (3.180), and $\boldsymbol{\Sigma}_2(x_1)$ on the crack surfaces introduced by $\boldsymbol{\Phi}_{ad}$ is

$$\begin{aligned} \boldsymbol{\Sigma}_2(x_1) &= 2\operatorname{Re}\left[\boldsymbol{B}_\alpha \left\langle \frac{1}{x_1 - x_{01} - d} - \frac{1}{x_1 - x_{01}} \right\rangle \boldsymbol{N}_\alpha q_e \right] \mathbf{i}_4 \\ &= \frac{q_e}{\pi} \left(\frac{1}{x_1 - x_{01} - d} - \frac{1}{x_1 - x_{01}} \right) \boldsymbol{\Omega}_2 \mathbf{i}_4 \end{aligned} \tag{4.185}$$

where $2\operatorname{Re}(\boldsymbol{B}_\alpha \boldsymbol{N}_\alpha) = \boldsymbol{\Omega}_2/\pi$ is used and $\boldsymbol{\Omega}_2$ is shown in Eq. (3.175). Because the generalized stresses are assumed zero at infinity and generalized displacement are single valued, so $\boldsymbol{C}(z) = \mathbf{0}$ in Eq. (4.38). Substituting $\boldsymbol{C}(z) = \mathbf{0}$ and $-\boldsymbol{\Sigma}_2(x_1)$ into Eq. (4.38) yields $\boldsymbol{h}_{\alpha c}$, i.e.,

$$\begin{aligned}
\boldsymbol{h}_{\alpha c}(z) &= \boldsymbol{B}\boldsymbol{F}(z) = \frac{1}{2\pi \mathrm{i}} \boldsymbol{P}(z) \int_L \frac{-\boldsymbol{\Sigma}_2(x_1)\mathrm{d}x_1}{\boldsymbol{P}^+(x_1)(x_1 - z)} \\
&= -\frac{1}{2\pi \mathrm{i}} \boldsymbol{P}(z) \left\{ \int_L \frac{[\boldsymbol{P}^+(x_1)]^{-1}}{(x_1 - z)} \left(\frac{1}{x_1 - x_{01} - d} - \frac{1}{x_1 - x_{01}} \right) \mathrm{d}x_1 \right\} \frac{q_\mathrm{e}}{\pi} \boldsymbol{\Omega}_2 \mathbf{i}_4 \\
\boldsymbol{P}(z) &= \boldsymbol{\Omega}\boldsymbol{Q}(z), \quad \boldsymbol{Q}(z) = \langle Y_0(z) \rangle, \quad \boldsymbol{\Omega} = \left[\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \boldsymbol{\omega}^{(3)}, \boldsymbol{\omega}^{(4)} \right]
\end{aligned} \tag{4.186}$$

Using the theory of the singular integral equation, finishing the integral and noting $\lim_{d \to 0} q_\mathrm{e} d \to p_\mathrm{e}$ we get

$$\begin{aligned}
\boldsymbol{h}_{\alpha c}(z) &= \left(\mathbf{I} + \bar{\boldsymbol{H}}^{-1} \boldsymbol{H} \right)^{-1} \boldsymbol{\Omega}\boldsymbol{Q}(z) \lim_{q_\mathrm{e} d \to p_\mathrm{e}, d \to 0} \\
&\times \left\langle \frac{1}{Y_0(z)} \left(\frac{1}{z - x_{01}} - \frac{1}{z - x_{01} - d} \right) + \frac{1}{Y_0(x_{01} + d)} \frac{1}{z - x_{01} - d} - \frac{1}{Y_0(d)} \frac{1}{z - x_{01}} \right\rangle \boldsymbol{\Omega}^{-1} \frac{q_\mathrm{e}}{\pi} \boldsymbol{\Omega}_2 \mathbf{i}_4
\end{aligned}$$

Taking the approximation in first order, the above equation is reduced to

$$\begin{aligned}
\boldsymbol{h}_{\alpha c}(z) &= \left(\mathbf{I} + \bar{\boldsymbol{H}}^{-1} \boldsymbol{H} \right)^{-1} \boldsymbol{\Omega}\boldsymbol{Q}(z) \\
&\times \left\langle -\frac{1}{Y_0(z)} \frac{1}{(z - x_{01})^2} + \frac{1}{Y_0(x_{01})} \frac{1}{z - x_{01}} \left(\frac{1}{z - x_{01}} + \frac{x_{01} - 2\mathrm{i}a\varepsilon_j}{x_{01}^2 - a^2} \right) \right\rangle \boldsymbol{\Omega}^{-1} \frac{q_\mathrm{e}}{\pi} \boldsymbol{\Omega}_2 \mathbf{i}_4
\end{aligned} \tag{4.187}$$

In front of and near the crack tip, the principle singular term is

$$\begin{aligned}
\boldsymbol{\Sigma}_{2c}(x_1) &= \boldsymbol{h}^+_{\alpha c}(x_1) + \bar{\boldsymbol{H}}^{-1} \boldsymbol{H} \boldsymbol{h}^-_{\alpha c}(x_1) = (\mathbf{I} + \bar{\boldsymbol{H}}^{-1} \boldsymbol{H}) \boldsymbol{h}_{\alpha c}(x_1) \\
&= \boldsymbol{\Omega} \frac{1}{\sqrt{2a(x_1 - a)}} \left(\frac{x_1 - a}{2a} \right)^{\mathrm{i}\varepsilon_j} \left\langle \frac{1}{Y_0(x_{01})} \frac{1}{a - x_{01}} \left(\frac{1}{a - x_{01}} + \frac{x_{01} - 2\mathrm{i}a\varepsilon_j}{x_{01}^2 - a^2} \right) \right\rangle \boldsymbol{\Omega}^{-1} \frac{p_\mathrm{e}}{\pi} \boldsymbol{\Omega}_2 \mathbf{i}_4
\end{aligned} \tag{4.188}$$

The generalized stress intensity factors at the right tip are

$$\begin{aligned}
\boldsymbol{K} &= [K_\mathrm{II}, K_\mathrm{I}, K_\mathrm{III}, K_\mathrm{D}]^\mathrm{T} = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)} \boldsymbol{\Omega} \left\langle (x_1 - a)^{-\mathrm{i}\varepsilon_j} \right\rangle \boldsymbol{\Omega}^{-1} \boldsymbol{\Sigma}_2(x_1) \\
&= \frac{p_\mathrm{e}}{\sqrt{\pi a}} \boldsymbol{\Omega} \left\langle (2a)^{-\mathrm{i}\varepsilon_j} \left[\frac{1}{Y_0(x_{01})(a - x_{01})} \right] \left(\frac{1}{a - x_{01}} + \frac{x_{01} - 2\mathrm{i}a\varepsilon_j}{x_{01}^2 - a^2} \right) \right\rangle \boldsymbol{\Omega}^{-1} \boldsymbol{\Omega}_2 \mathbf{i}_4 \\
&= p_\mathrm{e} \sqrt{\frac{a}{\pi}} \frac{1}{(x_{01} - a)\sqrt{x_{01}^2 - a^2}} \boldsymbol{\Omega} \left\langle (2a)^{-\mathrm{i}\varepsilon_j} (1 + 2\mathrm{i}\varepsilon_j) \left(\frac{x_{01} + a}{x_{01} - a} \right)^{\mathrm{i}\varepsilon_j} \right\rangle \boldsymbol{\Omega}^{-1} \boldsymbol{\Omega}_2 \mathbf{i}_4
\end{aligned} \tag{4.189}$$

When $\rho = x_{01} - a \to 0$, in a region $x_1 - a \ll \rho$, we get

$$\boldsymbol{K} == p_{\mathrm{e}}\left(1\Big/\sqrt{2\pi}\right)\boldsymbol{\Omega}\left\langle\left(1+2\mathrm{i}\varepsilon_j\right)\rho^{-\frac{3}{2}-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_2\mathbf{i}_4 \tag{4.190}$$

4.6 Interaction of an Elliptic Hole and a Vice-Crack

4.6.1 The Solution Method

Figure 4.11 shows an elliptic hole filled air and a vice-crack in an infinity piezoelectric material subjected $\boldsymbol{\Sigma}^{\infty}$ at infinity. The major and minor axes of the ellipse $(2a, 2b)$ are aligned along x_1 and x_2, respectively. The center of the vice-crack of length $2c_0$ is located at $z^{(0)}\left(x_1^{(0)} + \mathrm{i}x_2^{(0)}\right)$ and forms an angle γ with the positive direction of x_1. The distance from $z^{(0)}$ to $(a, 0)$ is d_0 and $\overrightarrow{z^{(0)}a}$ form an angle α with the positive direction of x_1. Zhou et al. (2005b) used the continuous distribution dislocation method to solve this problem. The main steps of this method are: (1) Problem I. A singularity located in an infinite piezoelectric material with an elliptic hole. The solution of problem I is used as the Green function, which does not produces the traction at infinity and on the boundary of the elliptic hole, but produces tractions on an artificial cut corresponding to the original vice-crack. (2) Problem II. An infinite piezoelectric material with an elliptic cavity filled air subjected to $\boldsymbol{\Sigma}^{\infty}$ at infinity. The solution of problem II produces tractions also on an artificial cut corresponding to the original vice-crack. (3) Problem III. The geometric shape of this problem is identical with the original problem, but the vice-crack is replaced by an artificial generalized continuous distribution dislocation with undetermined density. Add the tractions on the vice-crack surface obtained from problems II and III to satisfy the original boundary conditions, and the unknown dislocation density can be obtained. (4) After solving the unknown

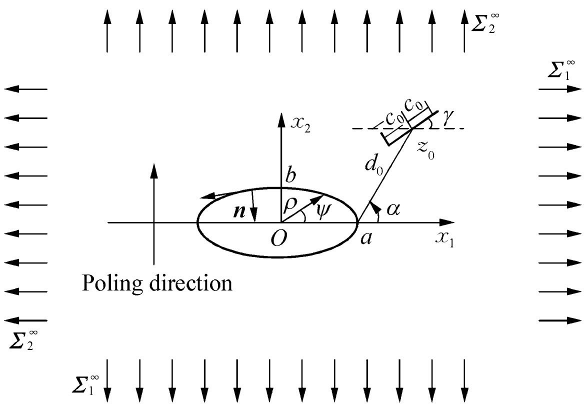


Fig. 4.11 An elliptic hole and a vice-crack

dislocation density, the original problem can be solved. The transform method is used to solve this problem. The transform functions are shown in Eqs. (3.82) and (3.86). The boundary L of the elliptic in the z plane is mapped to the unit circle Γ in the ς plane. In this section, the second natural coordinate system, i.e., use $(\boldsymbol{n}, \boldsymbol{t}')$ in (3.29b) and $\boldsymbol{T} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}s$, is used. Some geometric relations can be seen in Eqs. (3.29b) and (3.82b).

4.6.2 Problem I

In this section, a slightly simpler method to solve this problem is used. The problem is decomposed into two subproblems: (1) Problem Ia, a singularity locates at $z_0(x_{01} + \mathrm{i}x_{02})$ in an infinite homogeneous material, and (2) Problem Ib, a distributed loading acts on the boundary of the elliptic hole. (3) Superpose the solutions of problems Ia and Ib, and let the resultant solution satisfy the boundary conditions of the original problem.

(a) According to Eqs. (3.156) and (3.158), the solution of the problem Ia is

$$\begin{aligned}
&\boldsymbol{U}_{\mathrm{I}}^{(a)} = (1/\pi)\mathrm{Im}\left[\boldsymbol{A}\left\langle \ln(z_j - z_{0j})\right\rangle \boldsymbol{V}\right], \quad \boldsymbol{\Phi}_{\mathrm{I}}^{(a)} = (1/\pi)\mathrm{Im}\left[\boldsymbol{B}\left\langle \ln(z_j - z_{0j})\right\rangle \boldsymbol{V}\right] \\
&\ln(z_j - z_{oj}) = \ln(\varsigma_j - \varsigma_{0j}) + \ln\left[c_j\left(1 - d_j/c_j\varsigma_j\varsigma_{0j}\right)\right], \quad \boldsymbol{V} = \boldsymbol{B}^T\boldsymbol{b} + \boldsymbol{A}^T\boldsymbol{p}
\end{aligned} \tag{4.191}$$

On the unit circle Γ in the ς plane, $\varsigma = \varsigma_j = \sigma = \mathrm{e}^{\mathrm{i}\psi}$, so

$$\boldsymbol{\Phi}_{\mathrm{I}}^{(a)}(\sigma) = (1/\pi)\mathrm{Im}\left\{\boldsymbol{B}\left\langle \ln(\sigma - \varsigma_{0j}) + \ln\left[c_j\left(1 - d_j/c_j\sigma\varsigma_{0j}\right)\right]\right\rangle \boldsymbol{V}\right\} \tag{4.192}$$

Using $\mathrm{d}s = \rho(\psi)\mathrm{d}\psi, \quad \rho^2 = a^2\sin^2\psi + b^2\cos^2\psi$ given in Eq. (3.82b). Eq. (4.192) can be expanded in the following series

$$\begin{aligned}
\boldsymbol{T}_{\mathrm{I}}^{(a)}(\sigma) = \mathrm{d}\boldsymbol{\Phi}^{(a)}(\sigma)/\mathrm{d}s = (1/\pi\rho(\psi))\mathrm{Im}\Bigg\{\boldsymbol{B}\sum_{k=1}^{\infty}\Big\langle &\left[\left(1/\varsigma_{0j}\right)^k + \left(d_j/c_j\varsigma_{0j}\right)^k\right]\sin k\psi \\
&+\mathrm{i}\left[\left(d_j/c_j\varsigma_{0j}\right)^k - \left(1/\varsigma_{0j}\right)^k\right]\cos k\psi\Big\rangle \boldsymbol{V}\Bigg\}
\end{aligned} \tag{4.193}$$

(b) The solution of the problem Ib can be taken as (Chung and Ting 1996)

$$\begin{aligned}
\boldsymbol{U}_{\mathrm{I}}^{(b)} &= 2\mathrm{Re}\sum_{m=1}^{\infty}\left\{\boldsymbol{A}\left\langle \varsigma_j^{-m}\right\rangle\left(\boldsymbol{A}^{\mathrm{T}}\mathbf{g}_m + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_m\right)\right\} \\
\boldsymbol{\Phi}_{\mathrm{I}}^{(b)} &= 2\mathrm{Re}\sum_{m=1}^{\infty}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-m}\right\rangle\left(\boldsymbol{A}^{\mathrm{T}}\mathbf{g}_m + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_m\right)\right\}
\end{aligned} \tag{4.194}$$

where $\mathbf{g}_m, \boldsymbol{h}_m$ are real vectors determined by the boundary conditions. On Γ we have

$$\begin{aligned}
\boldsymbol{U}_{\mathrm{I}}^{(b)}(\sigma) &= \sum_{m=1}^{\infty}\left[\cos(m\psi)\boldsymbol{h}_m - \sin(m\psi)\hat{\boldsymbol{h}}_m\right] \\
\boldsymbol{\Phi}_{\mathrm{I}}^{(b)}(\sigma) &= \sum_{m=1}^{\infty}[\cos(m\psi)\mathbf{g}_m - \sin(m\psi)\hat{\mathbf{g}}_m] \\
\boldsymbol{T}_{\mathrm{I}}^{(b)}(\sigma) &= \mathrm{d}\boldsymbol{\Phi}^{(b)}(\sigma)/\mathrm{d}s = -[1/\rho(\psi)]\sum_{m=1}^{\infty} m[\sin(m\psi)\mathbf{g}_m + \cos(m\psi)\hat{\mathbf{g}}_m] \\
\hat{\boldsymbol{h}}_m &= \boldsymbol{S}\boldsymbol{h}_m + \boldsymbol{M}\mathbf{g}_m, \quad \hat{\mathbf{g}}_m = \boldsymbol{S}^{\mathrm{T}}\mathbf{g}_m - \boldsymbol{L}\boldsymbol{h}_m
\end{aligned} \tag{4.195}$$

where $\boldsymbol{S}, \boldsymbol{M}, \boldsymbol{L}$ are shown in Eq. (3.35).

(c) The solution of the electric potential inside the cavity hole filled air has been discussed in Sect. 3.4.2. Using $\varphi_{\mathrm{I}}(\sigma) = 2\mathrm{Re}\phi_{\mathrm{I}}(\sigma)$ according to Eq. (3.85) we get

$$\begin{aligned}
&\phi_{\mathrm{I}}(\varsigma) = \sum_{m=1}^{\infty} a_m^c[\varsigma^m + (d/c)^m\varsigma^{-m}], \quad \varsigma = \rho e^{\mathrm{i}\psi} \\
&\varphi_{\mathrm{I}}(\sigma) = 2\mathrm{Re}\sum_{m=1}^{\infty} a_m^c\left[\left(1 + \left(\frac{d}{c}\right)^m\right)\cos m\psi + \mathrm{i}\left(1 - \left(\frac{d}{c}\right)^m\right)\sin m\psi\right] \\
&D_{\mathrm{I}}^c(\sigma) = -2\epsilon^c\mathrm{Im}[\mathrm{d}\phi(\sigma)/\mathrm{d}s] = -(2\epsilon^c/\rho^c)\times \\
&\quad \sum_{m=1}^{\infty}\left\{\left[-m(1 + (d/c)^m)\mathrm{Im}a_m^c\right]\sin m\psi + \left[m(1 - (d/c)^m)\mathrm{Re}a_m^c\right]\cos m\psi\right\}
\end{aligned} \tag{4.196}$$

Comparing $\varphi_{\mathrm{I}}(\sigma)$ in Eq. (4.196) and $\left(\boldsymbol{U}_{\mathrm{I}}^{(b)}\right)_4(\sigma)$ in Eq. (4.195) yields

$$(\boldsymbol{h}_m)_4 = 2[1 + (d/c)^m]\mathrm{Re}a_m^c, \quad \left(\hat{\boldsymbol{h}}_m\right)_4 = 2[1 - (d/c)^m]\mathrm{Im}a_m^c, \quad m \geq 1 \tag{4.197}$$

(d) The sum of generalized stresses in problems Ia and Ib on the elliptic boundary must satisfy the original boundary condition:

$$\boldsymbol{T}_{\mathrm{I}}^{(b)} + \boldsymbol{T}_{\mathrm{I}}^{(a)} = D_{\mathrm{I}}^c\mathbf{i}_4, \quad \mathbf{i}_4 = [0,0,0,1]^{\mathrm{T}}; \quad \text{on} \quad \Gamma \tag{4.198}$$

Substitution of Eqs. (4.193), (4.195), and (4.196) into Eq. (4.198) yields

$$\begin{aligned}
&\mathbf{g}_m = \mathbf{g}_{m1} + \mathbf{g}_{m2}, \quad \hat{\mathbf{g}}_m = \hat{\mathbf{g}}_{m1} + \hat{\mathbf{g}}_{m2} \\
&\mathbf{g}_{m1} = (1/m\pi)\mathrm{Im}\left[\boldsymbol{B}\left\langle\left(1/\varsigma_{0j}\right)^m + \left(d_j/c_j\varsigma_{0j}\right)^m\right\rangle\boldsymbol{V}\right], \quad \mathbf{g}_{m2} = -2\epsilon^c[1 + (d/c)^m]\mathrm{Im}a_m^c\mathbf{i}_4 \\
&\hat{\mathbf{g}}_{m1} = (1/m\pi)\mathrm{Im}\left[\boldsymbol{B}\left\langle\left(d_j/c_j\varsigma_{0j}\right)^m - \left(1/\varsigma_{0j}\right)^m\right\rangle\boldsymbol{V}\right], \quad \hat{\mathbf{g}}_{m2} = 2\epsilon^c[1 - (d/c)^m]\mathrm{Re}a_m^c\mathbf{i}_4
\end{aligned} \tag{4.199}$$

From Eqs. (4.191), (4.194), (4.195), (4.199), and

$$\sum_{m=1}^{\infty} \varsigma_j^{-m}\bar{\varsigma}_{0k}^{-m}/m = -\ln\left(1 - \varsigma_j^{-1}\bar{\varsigma}_{0k}^{-1}\right), \quad \text{when} \quad \left|\varsigma_j^{-1}\bar{\varsigma}_{0k}^{-1}\right| < 1$$
$$\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{L}^{-1}\boldsymbol{S}^{\mathrm{T}} = \boldsymbol{B}^{-1}/2, \quad \boldsymbol{B}^{\mathrm{T}}\boldsymbol{L}^{-1} = \mathrm{i}\boldsymbol{B}^{-1}/2$$

the stress functions in the piezoelectric material finally are

$$\begin{aligned}
&\boldsymbol{\Phi}_{\mathrm{I}} = \boldsymbol{\Phi}_{\mathrm{I}}^{(a)} + \boldsymbol{\Phi}_{\mathrm{I}}^{(b)} = \boldsymbol{\Phi}_{\mathrm{I}}^{(1)} + \boldsymbol{\Phi}_{\mathrm{I}}^{(2)} \\
&\boldsymbol{\Phi}_{\mathrm{I}}^{(1)} = (1/\pi)\mathrm{Im}\left\{\boldsymbol{B}\left\langle \ln(\varsigma_j - \varsigma_{0j})\right\rangle \boldsymbol{V}\right\} + (1/\pi)\sum_{k=1}^{4}\mathrm{Im}\left\{\boldsymbol{B}\left\langle \ln(\varsigma_j^{-1} - \bar{\varsigma}_{0k})\right\rangle \boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{V}}\right\} \\
&\boldsymbol{\Phi}_{\mathrm{I}}^{(2)} = 2\epsilon^c\sum_{m=1}^{\infty}\mathrm{Im}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-m}\right\rangle \boldsymbol{B}^{-1}\left[\bar{a}_m^c - (d/c)^m a_m^c\right]\mathbf{i}_4\right\}
\end{aligned} \tag{4.200}$$

where

$$\begin{aligned}
&a_m^c = \alpha_m/\beta_m, \quad C_m = C_{1m} + \mathrm{i}C_{2m} \\
&\alpha_m = \frac{1}{2}\left\{\overline{C}_m(d/c)^m\left(1 - \epsilon^c L_{44}^{-1} + \mathrm{i}\epsilon^c L_{4i}^{-1}S_{4i}\right) - C_m\left(1 + \epsilon^c L_{44}^{-1} + \mathrm{i}\epsilon^c L_{4i}^{-1}S_{4i}\right)\right\} \\
&\beta_m = \left[1 - (d/c)^{2m}\right]\left[1 - \left(\epsilon^c L_{44}^{-1}\right)^2 - \left(\epsilon^c L_{4i}^{-1}S_{4i}\right)^2\right] - 2\epsilon^c\left[1 + (d/c)^{2m}\right]L_{44}^{-1}
\end{aligned} \tag{4.201}$$

$$\begin{aligned}
C_{1m} = &\frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\boldsymbol{S}^{\mathrm{T}}\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1 + (d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right] - \mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m - 1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right]\right\}\boldsymbol{p} \\
&+ \frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\boldsymbol{S}^{\mathrm{T}}\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1 + (d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right] - \mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m - 1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right]\right\}\boldsymbol{b} \\
C_{2m} = &\frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1 + (d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right] + \boldsymbol{S}^{\mathrm{T}}\mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m - 1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{A}^{\mathrm{T}}\right]\right\}\boldsymbol{p} \\
&+ \frac{\boldsymbol{L}_4^{-1}}{m\pi}\left\{\mathrm{Im}\left[\boldsymbol{B}\left\langle\left[1 + (d_j/c_j)^m\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right] + \boldsymbol{S}^{\mathrm{T}}\mathrm{Re}\left[\boldsymbol{B}\left\langle\left[(d_j/c_j)^m - 1\right]\varsigma_{0j}^{-m}\right\rangle\boldsymbol{B}^{\mathrm{T}}\right]\right\}\boldsymbol{b}
\end{aligned} \tag{4.202}$$

where C_{1m}, C_{2m} are real, $\boldsymbol{L}_4^{-1} = \left[L_{41}^{-1}, L_{42}^{-1}, L_{43}^{-1}, L_{44}^{-1}\right]$.

The solution shown in Eq. (4.200) is the solution of the problem I representing a singularity located in an infinite piezoelectric material with an elliptic hole. It is a Green function.

When $b = 0$, the elliptic hole is reduced to a crack and $c = d = c_j = d_j = a/2$. In this case, the Green function is simplified significantly. The stress intensity factor at $x_1 = a$ is

$$\begin{aligned}
\boldsymbol{K}(a) &= \sqrt{2\pi}\lim_{z_j\to a, x_2=0}\sqrt{z_j-a}\boldsymbol{\Phi}_{,1} = \sqrt{\pi/a}\lim_{\varsigma_j\to 1}\partial\boldsymbol{\Phi}/\partial\varsigma_j \\
&= \frac{1}{\sqrt{\pi a}}\left\{\operatorname{Im}\left[\boldsymbol{B}\left\langle 1-\sqrt{\frac{z_{0j}+a}{z_{0j}-a}}\right\rangle\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}\right]-\frac{L_{4j}^{-1}}{L_{44}^{-1}}\operatorname{Im}\left[\boldsymbol{B}\left\langle 1-\sqrt{\frac{z_{0j}+a}{z_{0j}-a}}\right\rangle\boldsymbol{B}^{\mathrm{T}}\boldsymbol{b}\mathbf{i}_4\right]\right\} \\
K_{\mathrm{D}}(a) &= -\left(L_{4m}^{-1}/L_{44}^{-1}\right)K_m(a), \quad m=1,2,3
\end{aligned} \tag{4.203}$$

4.6.3 Problem II

Problem II can be decomposed into two subproblems. Problem IIa: a homogeneous infinite piezoelectric material subjected $\boldsymbol{\Sigma}^{\infty}$ at infinity. Its solution is

$$\boldsymbol{\Phi}_{\mathrm{II}}^{(a)} = \boldsymbol{\Sigma}_2^{\infty}x_1 - \boldsymbol{\Sigma}_1^{\infty}x_2, \quad \boldsymbol{\Sigma}_{\mathrm{II}\,1}^{(a)} = \boldsymbol{\Sigma}_1^{\infty}, \quad \boldsymbol{\Sigma}_{\mathrm{II}\,2}^{(a)} = \boldsymbol{\Sigma}_2^{\infty} \tag{4.204}$$

Remove a piece of material to form an artificial elliptic hole whose size is identical to the hole in the original problem. Using Eqs. (3.29a) and (3.82b) the generalized traction on this artificial elliptic boundary is $\boldsymbol{\Sigma}_n^{\Gamma}$:

$$\begin{aligned}
\boldsymbol{\Sigma}_n^{\Gamma} &= \left(\boldsymbol{\Sigma}_1^{\infty}n_1 + \boldsymbol{\Sigma}_2^{\infty}n_2 - D_n^c\mathbf{i}_4\right) \\
&= -\frac{b}{\rho(\psi)}\cos\psi\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c\mathbf{i}_4\right) - \frac{a}{\rho(\psi)}\sin\psi\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c\mathbf{i}_4\right)
\end{aligned} \tag{4.205}$$

The electric field in the elliptic hole is assumed as unknown constant E_i^c $(i=1,2)$:

$$\varphi_{\mathrm{II}}^c = -E_1^c x_1 - E_2^c x_2, \quad D_i^c = \epsilon^c E_i^c, \quad D_n^c = D_i^c n_i \tag{4.206}$$

Problem IIb: $-\boldsymbol{\Sigma}_n^{\Gamma}$ is applied on the artificial elliptic boundary. The general solution of this problem has been shown in Eq. (4.194) and the expression on Γ is given in Eq. (4.195). Comparing φ_{II}^c with $\boldsymbol{U}_{\mathrm{I}}^{(b)}(\sigma)$ and $-\boldsymbol{\Sigma}_n^{\Gamma}$ with $\boldsymbol{T}_{\mathrm{I}}^{(b)}(\sigma)$, it is find that in present problem,

$$\begin{aligned}
&\mathbf{g}_1 = -a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c\mathbf{i}_4\right), \quad \hat{\mathbf{g}}_1 = -b\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c\mathbf{i}_4\right); \quad \mathbf{g}_m = \hat{\mathbf{g}}_m = \mathbf{0}, \text{ for } m\neq 1 \\
&(\boldsymbol{h}_1)_4 = -aE_1^c, \quad \left(\hat{\boldsymbol{h}}_1\right)_4 = bE_2^c
\end{aligned} \tag{4.207}$$

Using the relations between $\boldsymbol{g}_1, \hat{\boldsymbol{g}}_1, \boldsymbol{h}_1, \hat{\boldsymbol{h}}_1$ in Eq. (4.195) the unknown electric displacements D_1^c, D_2^c in the hole are determined by

$$\begin{aligned}
\left(bL_{44}^{-1} - a/\epsilon^c\right)D_1^c - aL_{4i}^{-1}S_{4i}D_2^c &= bL_{4i}^{-1}\sigma_{1i}^{\infty} - aL_{4i}^{-1}S_{ji}\sigma_{2j}^{\infty} \\
bL_{4i}^{-1}S_{4i}D_1^c + \left(aL_{44}^{-1} - b/\epsilon^c\right)D_2^c &= bL_{4i}^{-1}S_{ji}\sigma_{1j}^{\infty} + aL_{4i}^{-1}\sigma_{2i}^{\infty}
\end{aligned} \tag{4.208}$$

Substituting $\boldsymbol{g}_m, \boldsymbol{h}_m$ into Eq. (4.194), $\boldsymbol{\Phi}_{\text{II}}^{(b)}$ can be obtained. The sum of the solutions of the problems IIa and IIb $\boldsymbol{\Phi}_{\text{II}} = \boldsymbol{\Phi}_{\text{II}}^{(a)} + \boldsymbol{\Phi}_{\text{II}}^{(b)}$ is the solution of the problem II. Finally it yields

$$\boldsymbol{\Phi}_{\text{II}} = \boldsymbol{\Sigma}_2^{\infty} x_1 - \boldsymbol{\Sigma}_1^{\infty} x_2 - \text{Re}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-1}\right\rangle \boldsymbol{B}^{-1}\left[a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c \boldsymbol{i}_4\right)\right] - \text{i}b\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c \boldsymbol{i}_4\right)\right\} \tag{4.209}$$

For a crack, $b = 0$, Eqs. (4.208) and (4.209) respectively reduced to

$$\boldsymbol{\Phi}_{\text{II}} = \boldsymbol{\Sigma}_2^{\infty} x_1 - \boldsymbol{\Sigma}_1^{\infty} x_2 - \text{Re}\left\{\boldsymbol{B}\left\langle \varsigma_j^{-1}\right\rangle \boldsymbol{B}^{-1} a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c \boldsymbol{i}_4\right)\right\}, \quad D_2^c = L_{4i}^{-1}\sigma_{2i}^{\infty}/L_{44}^{-1} \tag{4.210}$$

4.6.4 Problem III

For an artificial generalized continuous distribution dislocation instead of the original vice-crack, the solution can be obtained by integrating the Green function Eq. (4.200) with respect to z_{0j} along the vice-crack or the artificial dislocation line, i.e.,

$$\begin{aligned}\boldsymbol{\Phi}_{\text{III}}(\xi) = \frac{1}{\pi}\int_{-c_0}^{c_0}&\left\{\text{Im}\left[\boldsymbol{B}\langle\ln(\varsigma_j - \varsigma_{0j})\rangle \boldsymbol{V}\right] + \frac{1}{\pi}\sum\nolimits_{k=1}^{4}\text{Im}\left[\boldsymbol{B}\left\langle\ln(\varsigma_j^{-1} - \bar{\varsigma}_{0k})\right\rangle \boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{V}}\right]\right\}\text{d}\xi_0\\ &+ 2\epsilon^c\int_{-c_0}^{c_0}\sum_{m=1}^{\infty}\text{Im}\left\{\boldsymbol{B}\left\langle\varsigma_j^{-m}\right\rangle \boldsymbol{B}^{-1}\left[\bar{a}_m^c - (d/c)^m a_m^c\right]\mathbf{i}_4\right\}\text{d}\xi_0\end{aligned} \tag{4.211}$$

where $2c_0$ is the length of vice-crack and $\text{d}\xi_0$ is the dislocation differentiate element. Assuming the middle point of the vice-crack is at $z_j^0\left(x_1^0 + \mu_j x_2^0\right)$, the angle of the vice-crack with the positive axis x_1 is γ. A certain point on the vice-crack is at $z_j = z_j^0 + \xi\left(\cos\gamma + \mu_j \sin\gamma\right)$ and the position of a dislocation is at $z_{0j} = z_j^0 + \xi_0(\cos\gamma + \mu_j \sin\gamma)$, where ξ, ξ_0 is the algebraic length calculated from z^0. The traction on the crack surface is $\partial\boldsymbol{\Phi}_{\text{II}}/\partial\xi + \partial\boldsymbol{\Phi}_{\text{III}}/\partial\xi = \mathbf{0}$ due to original vice-crack is free. From this condition, it yields

$$-\frac{1}{\pi}\int_{-c_0}^{c_0}\text{Im}\left[\boldsymbol{B}\boldsymbol{B}^{\text{T}}\boldsymbol{b}\frac{1}{\xi_0 - \xi}\right]\text{d}\xi_0 + \int_{-c_0}^{c_0}\boldsymbol{K}_1(\xi,\xi_0)\boldsymbol{b}\text{d}\xi_0 + \int_{-c_0}^{c_0}\boldsymbol{K}_2(\xi,\xi_0)\text{d}\xi_0 = -\boldsymbol{T}^a(\xi) \tag{4.212}$$

where

$$\begin{aligned}\boldsymbol{T}^a(\xi) &= \boldsymbol{\Sigma}_2^{\infty}\cos\alpha - \boldsymbol{\Sigma}_1^{\infty}\sin\alpha + \text{Re}\left\{\boldsymbol{B}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j^2}\right\rangle \boldsymbol{B}^{-1}a\left(\boldsymbol{\Sigma}_2^{\infty} - D_2^c\mathbf{i}_4\right) - \text{i}b\left(\boldsymbol{\Sigma}_1^{\infty} - D_1^c\mathbf{i}_4\right)\right\}\\ \boldsymbol{K}_1(\xi,\xi_0) &= -\frac{1}{\pi}\text{Im}\left[\boldsymbol{B}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j\left[(c_j/d_j)\varsigma_j\varsigma_{0j} - 1\right]}\right\rangle \boldsymbol{B}^{\text{T}}\right] - \frac{1}{\pi}\sum_{l=1}^{4}\text{Im}\left[\boldsymbol{B}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j(1 - \varsigma_j\bar{\varsigma}_{0l})}\right\rangle \boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\mathbf{I}_l\bar{\boldsymbol{B}}^{\text{T}}\right]\\ \boldsymbol{K}_2(\xi,\xi_0) &= -2\epsilon^c\sum_{m=1}^{\infty}\text{Im}\left\{\boldsymbol{B}\left\langle\frac{\delta\partial\varsigma_j/\partial\xi}{\varsigma_j^{m+1}}\right\rangle \boldsymbol{B}^{-1}\left[\bar{a}_m^c - (d/c)^m a_m^c\right]\mathbf{i}_4\right\}\end{aligned} \tag{4.213}$$

For the insulated elliptic hole, $\boldsymbol{K}_2(\xi,\xi_0)=\mathbf{0}$. When the elliptic hole is degenerated into a main crack, the kernel function $\boldsymbol{K}_2(\xi,\xi_0)$ is reduced to

$$\boldsymbol{K}_2=\frac{1}{\pi}\operatorname{Im}\left\{\boldsymbol{B}\left\langle\frac{L_{4m}^{-1}}{L_{44}^{-1}}\left[B_{ml}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j(\varsigma_j\varsigma_{0l}-1)}\right\rangle B_{pl}-\bar{B}_{ml}\left\langle\frac{\partial\varsigma_j/\partial\xi}{\varsigma_j(\varsigma_j\bar{\varsigma}_{0l}-1)}\right\rangle\bar{B}_{pl}\right]\boldsymbol{b}_p\right\rangle\boldsymbol{B}^{-1}\right\}\mathbf{i}_4 \tag{4.214}$$

Adopt the dimensionless length $l'=\xi_0/c_0, l=\xi/c_0$ and noting the singular behavior of the kernel function, Eq. (4.199) (Muskhelishvili 1975; Erdogan and Gupta 1972) is rewritten as

$$\begin{aligned}&-\frac{1}{\pi}\int_{-1}^{1}\operatorname{Im}\left[\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\right]\frac{\hat{\boldsymbol{b}}(l')}{\sqrt{1-l'^2}}\frac{\mathrm{d}l'}{l'-l}+\int_{-1}^{1}\boldsymbol{K}_1(l',l)\frac{\hat{\mathbf{b}}(l')\mathrm{d}l'}{\sqrt{1-l'^2}}+\int_{-1}^{1}\boldsymbol{K}_2(l',l)\mathrm{d}l'=-\boldsymbol{T}^a(l)\\&\hat{b}=b\sqrt{1-l'^2},\quad -1<l'<1\end{aligned} \tag{4.215}$$

where $|l|<1$ and $\hat{\boldsymbol{b}}$ is finite. The generalized displacement single-valued condition is

$$\int_{-1}^{1}\left[\hat{\boldsymbol{b}}(l')\Big/\sqrt{1-l'^2}\right]\mathrm{d}l'=\mathbf{0} \tag{4.216}$$

Equations (4.215) and (4.216) are the singular integral equation system of the original problem and calculated by the numerical method. Here the selected collocation points l_i', l_r in the interval $[-1,1]$ are

$$l_i'=\cos\frac{(2i-1)\pi}{2n},\quad l_r=\cos\frac{r\pi}{n},\quad i=1,2,\ldots,n,\quad r=1,2,\ldots,n-1 \tag{4.217}$$

and Eq. (4.214) is reduced to a set of algebraic equations:

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{n}\hat{\boldsymbol{b}}(l_i')\left\{\operatorname{Im}\left[\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\right]\frac{1}{l_i'-l_r}-\pi\boldsymbol{K}_1(l_i'-l_r)-\pi\hat{\boldsymbol{K}}_2(l_i'-l_r)\right\}=\boldsymbol{T}^a(l_r)\\&\sum_{i=1}^{n}\hat{\boldsymbol{b}}(l_i')=0\\&\hat{\boldsymbol{K}}_2=\frac{1}{\pi}\operatorname{Im}\left\{\boldsymbol{B}\left\langle\frac{L_{4m}^{-1}}{L_{44}^{-1}}\left[B_{ml}\left\langle\frac{\partial\varsigma_*/\partial\xi}{\varsigma_*(\varsigma_*\varsigma_{0l}-1)}\right\rangle B_{pl}-\bar{B}_{ml}\left\langle\frac{\partial\varsigma_*/\partial\xi}{\varsigma_*(\varsigma_*\bar{\varsigma}_{0l}-1)}\right\rangle\bar{B}_{pl}\right]\right\rangle\boldsymbol{B}^{-1}\right\}\mathbf{i}_4\end{aligned} \tag{4.218}$$

Equation (4.218) gives $4(n-1)+4=4n$ equations with $4n$ unknowns. Solving $\hat{\boldsymbol{b}}$, the asymptotic field $\boldsymbol{T}(l)$ near the crack tip is

$$\begin{aligned}
&\boldsymbol{T}(l)=\mathrm{i}\boldsymbol{B}\boldsymbol{B}^T\hat{\boldsymbol{b}}(l)\Big/\sqrt{l^2-1},\quad l=1+\varepsilon,\quad \varepsilon(>0)\to 0\\
&\hat{\boldsymbol{b}}(1)=\frac{1}{n}\sum_{i=1}^{n}\frac{\sin[(2i-1)(2n-1)\pi/4n]}{\sin[(2i-1)\pi/4n]}\hat{\boldsymbol{b}}\left(l_i'\right)\\
&\hat{\boldsymbol{b}}(-1)=\frac{1}{n}\sum_{i=1}^{n}\frac{\sin[(2i-1)(2n-1)\pi/4n]}{\sin[(2i-1)\pi/4n]}\hat{\boldsymbol{b}}\left(l_{n+1-i}'\right)
\end{aligned}\tag{4.219}$$

The stress intensity of the right crack tip of the vice-crack is

$$\begin{aligned}
&[K_{\mathrm{I}},K_{\mathrm{II}},K_{\mathrm{III}},K_{\mathrm{D}}]^{\mathrm{T}}=\lim_{l\to\pm1}\sqrt{2\pi(l-1)}\boldsymbol{Q}\boldsymbol{T}(l)=-\mathrm{i}\sqrt{\pi c}\boldsymbol{Q}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\hat{\boldsymbol{b}}(1)\\
&\boldsymbol{Q}=\begin{bmatrix}\boldsymbol{Q}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix},\quad \boldsymbol{Q}_{11}=\begin{bmatrix}-\sin\alpha & \cos\alpha\\ \cos\alpha & \sin\alpha\end{bmatrix},\quad \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}
\end{aligned}\tag{4.220}$$

If the elliptic is degenerated to a main crack, the stress intensity factor of the main crack is

$$\begin{aligned}
&[K_{\mathrm{I}},K_{\mathrm{II}},K_{\mathrm{III}},K_D]^{\mathrm{T}}=\boldsymbol{K}^0+\hat{\boldsymbol{K}};\quad \boldsymbol{K}^0=\sqrt{\pi a}\left(\boldsymbol{\Sigma}_2^{\infty}-D_2^c\mathbf{i}_4\right)\\
&\hat{\boldsymbol{K}}=\int_{-c}^{c}\mathrm{d}\boldsymbol{K}=\int_{-1}^{1}\boldsymbol{P}(l')\boldsymbol{b}(l')\mathrm{d}l'=(\pi/n)\sum\nolimits_{i=1}^{n}\boldsymbol{P}(l')\hat{\boldsymbol{b}}(l')
\end{aligned}\tag{4.221}$$

where $\boldsymbol{P}$ is complicated and omitted here.

4.6.5 Example

The matrix piezoelectric material is PZT-4 and the material constants are shown in Sect. 4.4.1. In the following examples, let $\gamma=0,\ d_0/c_0=a/c_0=2$ and $K_{\mathrm{I}}^{(0)}=\sigma_2^{\infty}\sqrt{\pi a}, K_{\mathrm{I}}^{(0m)}=\sigma_2^{\infty}\sqrt{\pi c_0}$. Figure 4.12 shows the distributions of the normalized mechanical stress intensity factors at right tips with α under $\gamma=0$ and different electric loading: (a) $K_{\mathrm{I}}\big/K_{\mathrm{I}}^{(0)}$ of the main crack ($b=0$) and (b) $K_{\mathrm{I}}^{(m)}\big/K_{\mathrm{I}}^{(0)}$ of the vice-crack. Figure 4.13 shows (a) the distributions of the normalized stress $\sigma_2\big/\sigma_2^{\infty}$ at right end of the elliptic hole of $b/a=0.1$ with α under $\gamma=0$ and different electric loading and (b) $K_{\mathrm{I}}^{(m)}\big/K_{\mathrm{I}}^{(0m)}$ of the vice-crack with α under $\gamma=0$ and different electric loading.

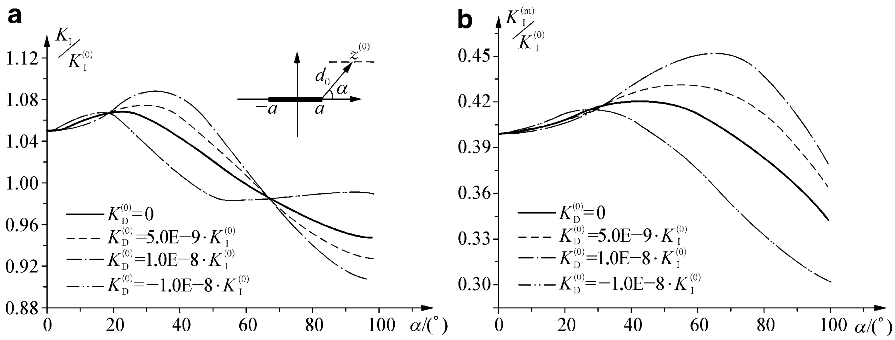


Fig. 4.12 Under $\gamma = 0$, $d_0/c_0 = a/c_0 = 2$: (**a**) variation of K_I/K_{I0} with α at right tip of main crack and (**b**) variation of K_I^m/K_{I0} with α at right tip of vice-crack

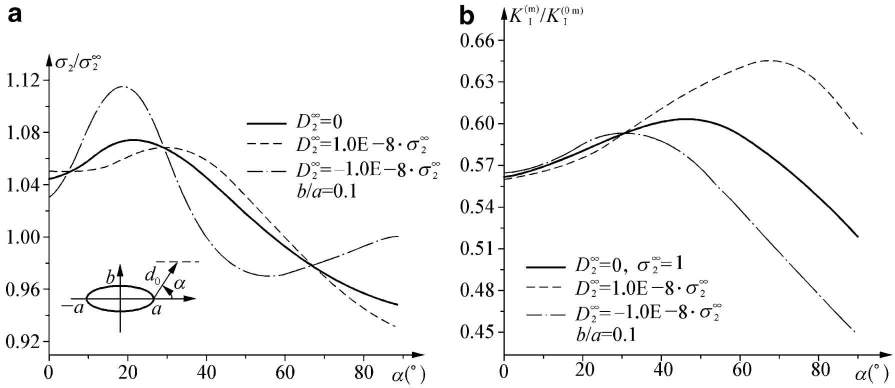


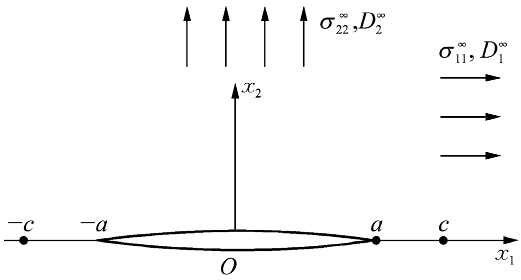
Fig. 4.13 Under $\gamma = 0$: (**a**) variation of σ_2/σ_2^∞ with α at right end of the elliptic hole of $b/a = 0.1$ and (**b**) variation of K_I^m/K_{I0} with α at right tip of vice-crack

4.7 Strip Electric Saturation Model of an Impermeable Crack in a Homogeneous Material

4.7.1 Fundamental Theory

Usually, the mechanical strength of a ceramic is high, and the plastic deformation is very small which can be neglected. Contrarily under high electric field, the crack tip region can be saturated due to the electric field concentration, if breakdown does not happen. Referencing to the Dugdale model in the elastoplastic fracture mechanics, the strip electric saturation model was proposed (Gao et al. 1997; Fulton and Gao 1997; Wang 2000). This model assumes that at crack tip region, the mechanical behavior is elastic, but the electric behavior is saturated. In order to solve this problem, by linear analysis, it is assumed that the electric saturation region is limited on a line segment in front of the tip (Fig. 4.14). The boundary conditions are

Fig. 4.14 Strip electric saturation model



$$\begin{aligned} &\boldsymbol{\Sigma}_2^{\infty} = \mathbf{0}, \quad |z| \to \infty \\ &\boldsymbol{\Sigma}_2^{\pm} = -\boldsymbol{T}, \quad \boldsymbol{T} = \left[\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}\right]^T, \quad |x_1| \le a \\ &\boldsymbol{U}^+ = \boldsymbol{U}^-, \quad \boldsymbol{\Sigma}_2^+ = \boldsymbol{\Sigma}_2^- = -\tilde{\boldsymbol{T}}, \quad \tilde{\boldsymbol{T}} = \left[*, *, *, D_2^{\infty} - D_{\mathrm{s}}\right], \quad a \le |x_1| \le c \end{aligned} \tag{4.222}$$

where "$*$" denotes variable which does not applied and omitted here, D_{s} is the saturation value, $2a$ is the crack length, and $a \le |x_1| \le c$ is the strip electric saturation region.

Because the generalized stress $\boldsymbol{\Sigma}_2(x_1)$ is continuous on whole axis x_1, similar to Eqs. (4.21), (4.22), (4.23), (4.24), (4.25), and (4.26) in Sect. 4.2.1, we can obtain

$$\boldsymbol{BF}^+(z) = \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^-(z), \quad x_2 > 0; \quad \boldsymbol{BF}^-(z) = \bar{\boldsymbol{B}}\bar{\boldsymbol{F}}^+(z), \quad x_2 < 0 \tag{4.223}$$

the displacement jump $\hat{\boldsymbol{d}}(x_1)$, and the dislocation density $\hat{\boldsymbol{d}}'(x_1)$ are

$$\begin{aligned} &\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}^+(x_1) - \boldsymbol{U}^-(x_1) = 2\mathrm{Re}[\boldsymbol{Af}^+(x_1) - \boldsymbol{Af}^-(x_1)] \\ &\mathrm{i}\hat{\boldsymbol{d}}'(x_1)(x_1) = \mathrm{id}\hat{\boldsymbol{d}}(x_1)/\mathrm{d}x_1 = \mathrm{i}2\mathrm{Re}\{\boldsymbol{A}[\boldsymbol{F}(x_1) - \boldsymbol{F}^-(x_1)]\} = \boldsymbol{H}[\boldsymbol{h}^+(x_1) + \boldsymbol{h}^-(x_1)] \end{aligned} \tag{4.224}$$

where the auxiliary function $\boldsymbol{h}(z)$ analytic in whole plane except crack. For a homogeneous material $\boldsymbol{H}$ is real. On the crack surface, we have

$$\boldsymbol{h}^+(x_1) + \boldsymbol{h}^-(x_1) = -\boldsymbol{T}, \quad |x_1| < a; \quad \boldsymbol{h}(z) = \boldsymbol{BF}(z) \tag{4.225}$$

4.7.2 Solution of the Strip Electric Saturation Model for an Impermeable Crack

Introduce a new function $\boldsymbol{\xi}(z)$:

$$\boldsymbol{\xi}(z) = \boldsymbol{Hh}(z), \quad \boldsymbol{h}(z) = \boldsymbol{L}\xi(z), \quad \boldsymbol{L} = \boldsymbol{H}^{-1} \tag{4.226}$$

Substitution of Eq. (4.226) into Eq. (4.225), in terms of component form, yields

$$\begin{aligned} &L_{ik}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+L_{i4}\left[\xi_4^+(x_1)+\xi_4^-(x_1)\right]=-T_i, \quad i,k=1,2,3 \\ &L_{4k}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+L_{44}\left[\xi_4^+(x_1)+\xi_4^-(x_1)\right]=-T_4, \quad |x_1|<a \end{aligned} \tag{4.227}$$

Eliminating $\xi_4^+(x_1)+\xi_4^-(x_1)$ from Eq. (4.227) yields

$$\begin{aligned} &L_{ik}^*\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]=-T_i^*, \quad i,k=1,2,3, \quad |x_1|<a \\ &L_{ik}^*=L_{ik}-L_{i4}L_{4k}/L_{44}, \quad T_i^*=T_i-T_4L_{i4}/L_{44} \end{aligned} \tag{4.228a}$$

Introducing 3D vectors $\boldsymbol{\xi}^*(z), \boldsymbol{T}^*$, etc., the vector form of Eq. (4.228a) is

$$\begin{aligned} &\boldsymbol{L}^*\left[\boldsymbol{\xi}^{*+}(x_1)+\boldsymbol{\xi}^{*-}(x_1)\right]=-\boldsymbol{T}^*, \quad |x_1|<a \\ &\boldsymbol{\xi}^*(z)=\left[\boldsymbol{\xi}_1(z),\boldsymbol{\xi}_2(z),\boldsymbol{\xi}_3(z)\right]^{\mathrm{T}}, \quad \boldsymbol{T}^*=\left[T_1^*,T_2^*,T_3^*\right]^{\mathrm{T}} \end{aligned} \tag{4.228b}$$

The solution of Eq. (4.228) is

$$\boldsymbol{L}^*\boldsymbol{\xi}^*(z)=\boldsymbol{T}^*F_a(z), \quad F_a(z)=(1/2)\left(z\Big/\sqrt{z^2-a^2}-1\right) \tag{4.229}$$

Equations (4.227), (4.222), and (4.226) yield

$$\begin{aligned} &\xi_4^+(x_1)+\xi_4^-(x_1)=-\left\{L_{4k}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+T_4\right\}/L_{44}, \quad |x_1|<a; \quad k=1,2,3 \\ &\xi_4^+(x_1)+\xi_4^-(x_1)=-\left\{L_{4k}\left[\xi_k^+(x_1)+\xi_k^-(x_1)\right]+T_4-D_{\mathrm{s}}\right\}/L_{44}, \quad a\le|x_1|\le c \end{aligned} \tag{4.230}$$

The solution of Eq. (4.230) is

$$\begin{aligned} &\xi_4(z)=\{-L_{4k}\xi_k(z)+T_4F_c(z)+D_{\mathrm{s}}F_D(z)\}/L_{44}; \quad k=1,2,3 \\ &F_c(z)=\frac{1}{2}\left\{\frac{z}{\sqrt{z^2-c^2}}-1\right\} \\ &F_D(z)=\frac{1}{2}-\frac{1}{2\pi\,\mathrm{i}}\ln\frac{z\sqrt{c^2-a^2}+\mathrm{i}a\sqrt{z^2-c^2}}{z\sqrt{c^2-a^2}-\mathrm{i}a\sqrt{z^2-c^2}}-\frac{1}{\pi}\frac{z}{\sqrt{z^2-c^2}}\arccos\frac{a}{c} \end{aligned} \tag{4.231}$$

where $F_c(z), F_D(z)$ is analytic in z plane except a slit $(-c,c)$ and has the following behavior:

$$F_D^+(x_1)+F_D^-(x_1)=\begin{cases}0, & |x_1|<a \\ 1, & a\le|x_1|\le c\end{cases}, \quad F_D(\infty)=0 \tag{4.232}$$

Equations (4.229) and (4.231) give a complete solution of $\boldsymbol{\xi}(z)$.

4.7.3 The Size of the Strip Region and the Stress Intensity Factor

According to $\boldsymbol{\Sigma}_2(x_1) = \boldsymbol{h}^+(x_1) + \boldsymbol{h}^-(x_1)$, the electric displacement in front of the crack is

$$D_2 = L_{4k}\left[\xi_k^+(x_1) + \xi_k^-(x_1)\right] + L_{44}\left[\xi_4^+(x_1) + \xi_4^-(x_1)\right]; \quad |x_1| \geq c \quad k = 1,2,3$$

Substitution of Eqs. (4.229) and (4.231) into the above equation yields

$$\begin{aligned} D_2 = 2T_4 f_c'(x_1) + D_\mathrm{s}\left[F_D^+(x_1) + F_D^-(x_1)\right] = \left(D_2^\infty - \frac{2}{\pi} D_\mathrm{s} \cos^{-1}\frac{a}{c}\right)\frac{x_1}{\sqrt{x_1^2 - c^2}} \\ - D_2^\infty + D_\mathrm{s}\left(1 - \frac{1}{\pi\,\mathrm{i}}\ln\frac{x_1\sqrt{c^2-a^2} + \mathrm{i}a\sqrt{x_1^2-c^2}}{x_1\sqrt{c^2-a^2} - \mathrm{i}a\sqrt{x_1^2-c^2}}\right), \quad |x_1| \geq c \end{aligned} \tag{4.233}$$

In order to make D_2 finite, it is necessary that

$$D_2^\infty - (2/\pi)D_\mathrm{s}\arccos(a/c) = 0, \quad \text{or} \quad a/c = \cos\left(\pi D_2^\infty / 2D_\mathrm{s}\right) \tag{4.234}$$

The size of the strip region is $c - a$.

According to $\boldsymbol{\Sigma}_2(x_1) = \boldsymbol{h}^+(x_1) + \boldsymbol{h}^-(x_1)$, the stress in front of the crack on the axis x_1 is

$$\begin{aligned} \sigma_{2i} &= L_{ik}\left[\xi_k^+(x_1) + \xi_k^-(x_1)\right] + L_{i4}\left[\xi_4^+(x_1) + \xi_4^-(x_1)\right] \\ &= L_{ik}^*\left(\xi_k^+ + \xi_k^-\right) + (L_{i4}/L_{44})\left\{D_2^\infty\left(F_c^+ + F_c^-\right) + D_\mathrm{s}\left(F_D^+ + F_D^-\right)\right\} \\ &= T_i^*\left(x_1 \Big/ \sqrt{x_1^2 - c^2} - 1\right) + (L_{i4}/L_{44})\left(D_\mathrm{s} - D_2^\infty\right) \end{aligned} \tag{4.235}$$

It is noted that adding $\boldsymbol{\Sigma}_2^\infty$ to the solution Eq. (4.231), the solution of a free crack under $\boldsymbol{\Sigma}_2^\infty$ at infinity is obtained. In this case, the stress and stress intensity factors are

$$\begin{aligned} &\sigma_{2i} = T_i^* x_1 \Big/ \sqrt{x_1^2 - c^2} + (L_{i4}/L_{44})D_\mathrm{s} \\ &K_\mathrm{I} = \sqrt{\pi a}\left(\sigma_{22}^\infty - \frac{L_{24}}{L_{44}}D_2^\infty\right), \quad K_\mathrm{II} = \sqrt{\pi a}\left(\sigma_{21}^\infty - \frac{L_{14}}{L_{44}}D_2^\infty\right), \\ &K_\mathrm{III} = \sqrt{\pi a}\left(\sigma_{23}^\infty - \frac{L_{34}}{L_{44}}D_2^\infty\right) \end{aligned} \tag{4.236}$$

and the electric displacement is finite due to electric saturation.

Ru and Mao (1999) discussed the strip electric saturation model for a conducting crack. Their results showed that when the electric loading is parallel to the poling axis, then (1) for a conducting crack perpendicular to the poling axis, in front of the crack tip, a saturation strip is existed and the stresses and electric displacements are all finite. (2) For a conducting crack parallel to the poling axis, behind the crack tip, a saturation strip is existed and the stress intensity factors are identical to those predicted by the linear piezoelectric model and the electric loading does not induce any nonzero stress intensity factor.

4.8 Strip Electric Saturation Model of a Mode-III Interface Crack in a Bimaterial

4.8.1 Fundamental Theory

For a transversely isotropic piezoelectric material with poling direction along axis x_3, plane (x_1, x_2) is isotropic. The mode-III (antiplane shear) problem in a piezoelectric material means that the mechanical loading is applied out of plane (x_1, x_2), but the electric loading is in-plane (x_1, x_2), i.e.,

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2); \quad E_1 = E_1(x_1, x_2), \quad E_2 = E_2(x_1, x_2), \quad E_3 = 0 \tag{4.237}$$

Shen et al. (2000) discussed the strip electric saturation model for a mechanical III-type interface crack. From Eqs. (3.1), (3.2), and (3.3), the governing equations for III-type problem are

$$\begin{aligned} &\sigma_{31,1} + \sigma_{32,2} = 0, \quad D_{1,1} + D_{2,2} = 0 \\ &\sigma_{31} = C_{44} u_{3,1} - e_{15} E_1, \quad \sigma_{32} = C_{44} u_{3,2} - e_{15} E_2, \\ &D_1 = e_{15} u_{3,1} + \epsilon_{11} E_1, \quad D_2 = e_{15} u_{3,2} + \epsilon_{11} E_2 \end{aligned} \tag{4.238}$$

Using $\boldsymbol{E} = -\nabla\varphi$ the equilibrium equation in terms of the generalized displacements is

$$C_{44}\nabla^2 u_3 + e_{15}\nabla^2\varphi = 0, \quad e_{15}\nabla^2 u_3 - \epsilon_{11}\nabla^2\varphi = 0; \quad \text{or} \quad \nabla^2 u_3 = 0, \quad \nabla^2\varphi = 0 \tag{4.239}$$

Figure 4.15 shows a III-type strip electric saturation model for an interface crack of length $2a$ in a bimaterial. The material I and II are located at the upper and lower half planes respectively. Let the boundary conditions are

$$\begin{aligned} &\boldsymbol{\Sigma}_2^\infty = \mathbf{0}, \quad |z| \to \infty \\ &\sigma_{23} = -\tau^\infty, \quad D_2 = -D^\infty, \quad |x_1| \le a, \quad x_2 = 0 \\ &\boldsymbol{U}_{\mathrm{I}}(x_1) = \boldsymbol{U}_{\mathrm{II}}(x_1), \quad \boldsymbol{\Sigma}_{\mathrm{I}2}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1) = \boldsymbol{\Sigma}_2(x_1); \quad |x_1| > a, \quad x_2 = 0 \end{aligned} \tag{4.240}$$

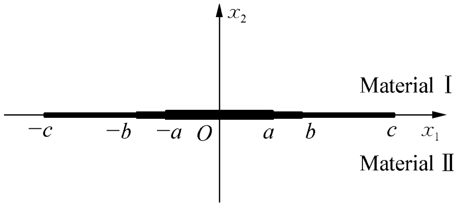


Fig. 4.15 Strip electric saturation model of a mode-III interface crack

where $\boldsymbol{U}_\beta = [U_{\beta 3}, \varphi_\beta]^T$. The single-valued condition is

$$\begin{aligned} &\int_{-a}^{a} \boldsymbol{\Psi}(x_1)\mathrm{d}x_1 = 0, \quad \boldsymbol{\Psi}(x_1) = [\psi_1(x_1), \psi_2(x_1)] \\ &\hat{\boldsymbol{d}}(x_1) = \Delta\boldsymbol{U}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1, 0) - \boldsymbol{U}_{\mathrm{II}}(x_1, 0), \quad \boldsymbol{\Psi}(x_1) = \Delta\boldsymbol{U}'(x_1) \end{aligned} \tag{4.241}$$

where $\boldsymbol{\Psi}(x_1)$ is called the dislocation density. On the connective surface, $\Delta\boldsymbol{U}(x_1) = \mathbf{0}$.

The Fourier transform method is used to solve this problem. For a function $f(x_1, x_2)$, the Fourier transform and the corresponding inverse transform are, respectively,

$$\tilde{f}(s, x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\mathrm{e}^{-\mathrm{i}\, s\, x_1}\mathrm{d}x_1, \quad f(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(s, x_2)\mathrm{e}^{\mathrm{i}\, x_1 s}\mathrm{d}s \tag{4.242}$$

where $f(t)$ is called the original function, $\tilde{f}(s)$ is the image function, and s is a real number. We have

$$\begin{aligned} &\int_{-\infty}^{\infty} f^{(n)}(x_1, x_2)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1 = (\mathrm{i}s)^n\tilde{f}(s, x_2); \quad \text{if} \quad f^{(n)}(x_1, x_2) \to 0, \quad \text{when} \quad \sqrt{x_1^2 + x_2^2} \to \infty \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}^{(n)}(s, x_2)\mathrm{e}^{\mathrm{i}x_1 s}\mathrm{d}s = (-\mathrm{i}x_1)^n f(x_1, x_2); \quad \text{if} \quad \int_{-\infty}^{\infty} \left|x_1^n f(x_1, x_2)\right|\mathrm{d}x_1 < \infty \end{aligned} \tag{4.243}$$

where $f^{(n)} = \partial^n f / \partial x_1^n$. Using Eqs. (4.242) and (4.243), Eq. (3.239) is transformed to

$$\int_{-\infty}^{\infty} \left(\frac{\partial^2 \boldsymbol{U}_\beta}{\partial x_1^2} + \frac{\partial^2 \boldsymbol{U}_\beta}{\partial x_2^2}\right)\mathrm{e}^{-\mathrm{i}sx_2}\mathrm{d}x_2 = -s^2\tilde{\boldsymbol{U}}_\beta(s, x_2) + \frac{\partial^2 \tilde{\boldsymbol{U}}_\beta(s, x_2)}{\partial x_2^2} = 0, \ \beta = \mathrm{I}, \mathrm{II} \tag{4.244}$$

The Fourier transform of the constitutive equation in Eq. (4.238) is

$$\tilde{\boldsymbol{\Sigma}}_{\beta 2} = \left\{ \begin{matrix} \tilde{\sigma}_{\beta 23} \\ \tilde{D}_{\beta 2} \end{matrix} \right\} = \boldsymbol{B}_\beta \frac{\partial \tilde{\boldsymbol{U}}_\beta(s, x_2)}{\partial x_2}, \quad \tilde{\boldsymbol{U}}_\beta = \left\{ \begin{matrix} \tilde{u}_{\beta 3} \\ \tilde{\varphi}_\beta \end{matrix} \right\}, \quad \boldsymbol{B}_\beta = \begin{bmatrix} C_{\beta 44} & e_{\beta 15} \\ e_{\beta 15} & -\epsilon_{\beta 11} \end{bmatrix}, \quad \beta = \mathrm{I}, \mathrm{II} \tag{4.245}$$

Because $\tilde{\boldsymbol{U}}_\beta$ is finite at infinity, the solution of Eq. (4.244) takes the following form:

$$\begin{aligned} &\tilde{\boldsymbol{U}}_{\mathrm{I}}(s,x_2) = \mathrm{e}^{sx_2}\boldsymbol{G}_{\mathrm{I}}(s), \quad \text{if} \quad s<0; \qquad \tilde{\boldsymbol{U}}_{\mathrm{I}}(s,x_2) = \mathrm{e}^{-sx_2}\boldsymbol{F}_{\mathrm{I}}(s), \quad \text{if} \quad s>0 \\ &\tilde{\boldsymbol{U}}_{\mathrm{II}}(s,x_2) = \mathrm{e}^{-sx_2}\boldsymbol{F}_{\mathrm{II}}(s), \quad \text{if} \quad s<0; \quad \tilde{\boldsymbol{U}}_{\mathrm{II}}(s,x_2) = \mathrm{e}^{sx_2}\boldsymbol{G}_{\mathrm{II}}(s), \quad \text{if} \quad s>0 \end{aligned} \tag{4.246}$$

where $G_{\mathrm{I}}\ (s)$, $F_{\mathrm{I}}\ (s)$, $G_{\mathrm{II}}\ (s)$, $F_{\mathrm{II}}\ (s)$ are undetermined functions. The generalized stress can be expressed by

$$\begin{aligned} &\tilde{\boldsymbol{\Sigma}}_{\mathrm{I}2}(s,x_2) = \boldsymbol{R}_{\mathrm{I}}\tilde{\boldsymbol{U}}_{\mathrm{I}}(s,x_2); \quad \boldsymbol{R}_{\mathrm{I}} = s\boldsymbol{B}_{\mathrm{I}}, \quad \text{if} \quad s<0; \quad \boldsymbol{R}_{\mathrm{I}} = -s\boldsymbol{B}_{\mathrm{I}}, \quad \text{if} \quad s>0 \\ &\tilde{\boldsymbol{\Sigma}}_{\mathrm{II}2}(s,x_2) = \boldsymbol{R}_{\mathrm{II}}\tilde{\boldsymbol{U}}_{\mathrm{II}}(s,x_2); \quad \boldsymbol{R}_{\mathrm{II}} = -sB_{\mathrm{II}}, \quad \text{if} \quad s<0; \quad \boldsymbol{R}_{\mathrm{II}} = s\boldsymbol{B}_{\mathrm{II}}, \quad \text{if} \quad s>0 \end{aligned} \tag{4.247}$$

It is known from Eq. (4.240) that on whole axis x_1, the generalized stress is continuous, so

$$\boldsymbol{R}_{\mathrm{I}}(s)\tilde{\boldsymbol{U}}_{\mathrm{I}}(s,0) = \boldsymbol{R}_{\mathrm{II}}(s)\tilde{\boldsymbol{U}}_{\mathrm{II}}(s,0); \quad |x_1|<\infty, \quad x_2=0 \tag{4.248}$$

The Fourier transform of Eq. (4.241) is

$$\begin{aligned} &\int_{-\infty}^{\infty}\boldsymbol{\Psi}(x_1)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1 = \mathrm{i}s\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s), \boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s) = \tilde{\boldsymbol{U}}_{\mathrm{I}}(s) - \tilde{\boldsymbol{U}}_{\mathrm{II}}(s) \\ &\quad = -(\mathrm{i}/s)\int_{-a}^{a}\boldsymbol{\Psi}(x_1)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1 \end{aligned} \tag{4.249}$$

Combining Eqs. (4.248) and (4.249) yields

$$\begin{aligned} &\tilde{\boldsymbol{U}}_{\mathrm{I}} = \boldsymbol{P}_{\mathrm{I}}\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s), \quad \tilde{\boldsymbol{U}}_{\mathrm{II}} = \boldsymbol{P}_{\mathrm{II}}\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s); \quad \boldsymbol{P}_{\mathrm{I}} = \frac{\boldsymbol{R}_{\mathrm{II}}}{\boldsymbol{R}_{\mathrm{II}}-\boldsymbol{R}_{\mathrm{I}}} = \frac{\boldsymbol{B}_{\mathrm{II}}}{\boldsymbol{B}_{\mathrm{II}}+\boldsymbol{B}_{\mathrm{I}}}, \\ &\boldsymbol{P}_{\mathrm{II}} = \frac{\boldsymbol{R}_{\mathrm{I}}}{\boldsymbol{R}_{\mathrm{II}}-\boldsymbol{R}_{\mathrm{I}}} = -\frac{\boldsymbol{B}_{\mathrm{I}}}{\boldsymbol{B}_{\mathrm{II}}+\boldsymbol{B}_{\mathrm{I}}} \end{aligned} \tag{4.250}$$

Combining Eqs. (4.247) and (4.250) and inversely transforming the obtained results yield

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathrm{II}2}(x_1,0) &= (1/2\pi)\int_{-\infty}^{\infty}\tilde{\boldsymbol{\Sigma}}_{\mathrm{II}2}(s,x_2)\mathrm{e}^{\mathrm{i}x_1 s}\mathrm{d}s = (1/2\pi)\int_{-\infty}^{\infty}\boldsymbol{R}_{\mathrm{II}}\boldsymbol{P}_{\mathrm{II}}\boldsymbol{\Delta}\tilde{\boldsymbol{U}}(s)\mathrm{e}^{\mathrm{i}x_1 s}\mathrm{d}s \\ &= -(1/2\pi)\int_{-\infty}^{\infty}\boldsymbol{R}_{\mathrm{II}}\boldsymbol{P}_{\mathrm{II}}\left[(\mathrm{i}/s)\int_{-a}^{a}\boldsymbol{\Psi}(t)\mathrm{e}^{-\mathrm{i}s\xi}\mathrm{d}t\right]\mathrm{e}^{\mathrm{i}x_1 s}\mathrm{d}s \\ &= -(\mathrm{i}/2\pi)\int_{-a}^{a}\left[\int_{-\infty}^{\infty}(1/s)\boldsymbol{R}_{\mathrm{II}}(s)\boldsymbol{P}_{\mathrm{II}}(s)\mathrm{e}^{-\mathrm{i}s(\xi-x_1)}\mathrm{d}s\right]\boldsymbol{\Psi}(\xi)\mathrm{d}\xi \end{aligned} \tag{4.251}$$

And for a certain s, the following relations hold:

$$\frac{1}{s}\boldsymbol{R}_{\mathrm{II}}(s)\boldsymbol{P}_{\mathrm{II}}(s)=\frac{1}{s}\frac{\boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}}{\boldsymbol{R}_{\mathrm{II}}-\boldsymbol{R}_{\mathrm{I}}}=-\frac{s}{|s|}\boldsymbol{V},\quad \boldsymbol{V}=\frac{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{B}_{\mathrm{I}}}{\boldsymbol{B}_{\mathrm{I}}+\boldsymbol{B}_{\mathrm{II}}} \tag{4.252}$$

$\boldsymbol{V}$ is a real symmetric matrix. Using the following formula:

$$\int_{-\infty}^{\infty}\frac{s}{|s|}\mathrm{e}^{-\mathrm{i}s(t-x_1)}\mathrm{d}s=-\frac{2\mathrm{i}}{t-x_1},\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}s(t-x_1)}\mathrm{d}s=\delta(t-x_1) \tag{4.253}$$

we can get the solution of Eq. (4.251) as

$$(1/\pi)\boldsymbol{V}\int_{-a}^{a}[\boldsymbol{\Psi}(\xi)/(\xi-x_1)]\mathrm{d}\xi=\boldsymbol{\Sigma}_{\mathrm{II}2}(x_1,0)=[t_{01}(x_1),t_{02}(x_1)]^{\mathrm{T}},\quad |x_1|<\infty,\quad x_2=0 \tag{4.254}$$

4.8.2 Solution for Longer Electric Saturation Size

The strip electric saturation model of a mode-III interface crack in a bimaterial is that: Let c and b are the right ends of the electric saturation and mechanical yielding regions respectively and $c>b$, the following boundary conditions are assumed (Fig. 4.15):

$$t_{01}(x_1)=\begin{cases}-\tau^{\infty}, & \text{if}\quad |x_1|<a\\ -\tau^{\infty}+\tau_{\mathrm{s}} & \text{if}\quad a<|x_1|<b\end{cases};\quad t_{02}(x_1)=\begin{cases}-D^{\infty}, & \text{if}\quad |x_1|<a\\ -D^{\infty}+D_{\mathrm{s}} & \text{if}\quad a<|x_1|<c\end{cases}$$
$$\psi_1(x_1)=0,\quad |x_1|>b;\quad \psi_2(x_1)=0,\quad |x_1|>c;\quad c>b \tag{4.255}$$

where τ_{s} is the yielding stress, D_{s} is the saturation electric displacement, and they take the smaller values of materials I and II. Equation (4.254) yields

$$\begin{aligned}&(1/\pi)\int_{-b}^{b}[\psi_1(t)/(t-x_1)]\mathrm{d}t=G_{1j}t_{0j}(x_1),\quad |x_1|<b\\ &(1/\pi)\int_{-l}^{l}\left[V_{2j}\psi_j(t)/(t-x_1)\right]\mathrm{d}t=t_{02}(x_1),\quad |x_1|<c,\quad j=1,2\end{aligned} \tag{4.256}$$

where $\boldsymbol{G}=\boldsymbol{V}^{-1}$. Because the stress is not singular at $x_1=\pm b, \psi_1(t)$ must be finite at $x_1=\pm b$. Analogously, the electric displacement is not singular at $x_1=\pm c; H_{2j}\psi_j(t)$ must be finite at $x_1=\pm c$. Since the first and second equations in Eq. (4.256) are solvable, the following conditions should be satisfied, respectively (Muskhelishvili 1975; Hou et al. 1990; Barnett and Asaro 1972):

$$\int_{-b}^{b}\left[G_{1j}t_{0j}(\xi)\Big/\sqrt{b^2-\xi^2}\right]\mathrm{d}\xi=0,\quad \int_{-c}^{c}\left[t_{02}(\xi)\Big/\sqrt{c^2-\xi^2}\right]\mathrm{d}\xi=0 \qquad (4.257)$$

Using

$$\begin{aligned}&\int_{-b}^{b}\frac{G_{1j}t_{0j}(\xi)}{\sqrt{b^2-\xi^2}}\mathrm{d}\xi=[G_{11}(\tau_\mathrm{s}-\tau^\infty)+G_{12}(D_\mathrm{s}-D^\infty)]\left(\int_{-b}^{-a}\frac{\mathrm{d}\xi}{\sqrt{b^2-\xi^2}}+\int_{a}^{b}\frac{\mathrm{d}\xi}{\sqrt{b^2-\xi^2}}\right)\\&\quad-[G_{11}\tau^\infty+G_{12}D^\infty]\int_{-a}^{a}\frac{\mathrm{d}\xi}{\sqrt{b^2-\xi^2}}\\&\quad=[G_{11}(\tau_\mathrm{s}-\tau^\infty)+G_{12}(D_\mathrm{s}-D^\infty)][\pi-2\arcsin(a/b)]-[G_{11}\tau^\infty+G_{12}D^\infty]2\arcsin(a/b)\end{aligned}$$

and $\arcsin(a/b)=\pi/2-\arccos(a/b)$, from the first equation of Eq. (4.257),we get the size of the plastic region:

$$b/a=\sec[\pi(G_{11}\tau^\infty+G_{12}D^\infty)/2(G_{11}\tau_\mathrm{s}+G_{12}D_\mathrm{s})] \qquad (4.258)$$

Analogously, the size of the electric saturation region is

$$c/a=\sec(\pi D^\infty/2D_\mathrm{s}) \qquad (4.259)$$

From Eqs. (4.258) and (4.259), it is known that $c/a>b/a,$ if $D^\infty/D_\mathrm{s}>\tau^\infty/\tau_\mathrm{s}$.

Under condition Eq. (4.257), the solution of the first equation in Eq. (4.256) is

$$\begin{aligned}\psi_1(x_1)&=\frac{1}{\pi}\sqrt{b^2-x_1^2}\int_{-b}^{b}\frac{G_{1j}t_{0j}(\xi)}{\sqrt{b^2-\xi^2}(\xi-x_1)}\mathrm{d}\xi\\&=(1/\pi)(G_{11}\tau_\mathrm{s}+G_{12}D_\mathrm{s})[\omega(x_1,a,b)-\omega(-x_1,a,b)],\quad |x_1|<b\\\omega(x_1,a,b)&=\mathrm{ar\,cosh}\left|\frac{b^2-a^2}{b(a-x_1)}+\frac{a}{b}\right|\end{aligned} \qquad (4.260)$$

and the solution of the second equation in Eq. (4.256) is

$$V_{2j}\psi_j(x_1)=\frac{1}{\pi}\sqrt{c^2-x_1^2}\int_{-c}^{c}\frac{t_{02}(\xi)}{\sqrt{c^2-\xi^2}(\xi-x_1)}\mathrm{d}\xi=\frac{D_\mathrm{s}}{\pi}[\omega(x_1,a,c)-\omega(-x_1,a,c)] \qquad (4.261)$$

Equation (4.261) yields

$$\psi_2(x_1)=\frac{D_\mathrm{s}}{\pi V_{22}}[\omega(x_1,a,c)-\omega(-x_1,a,c)]-\frac{V_{21}}{V_{22}}\psi_1(x_1),\quad x_1<c \qquad (4.262)$$

The generalized crack opening displacements are

$$\Delta u_3(x_1) = -\int_b^{x_1} \psi_1(\xi)\mathrm{d}\xi, \quad \Delta\varphi(x_1) = -\int_b^{x_1} \psi_2(\xi)\mathrm{d}\xi$$
$$\Delta u_3(x_1) = \frac{G_{11}\tau_s + G_{12}D_\mathrm{s}}{\pi}[(a-x_1)\omega(x_1,a,b) + (a+x_1)\omega(-x_1,a,b)], \quad |x_1| < b$$
$$\Delta\varphi(x_1) = \frac{D_\mathrm{s}}{\pi V_{22}}[(a-x_1)\omega(x_1,a,c) + (a+x_1)\omega(-x_1,a,c)] - \frac{V_{21}}{V_{22}}\Delta u_3(x_1), \quad |x_1| < c \tag{4.263}$$

The generalized crack tip opening displacements are

$$\begin{aligned}\Delta u_3(a) &= \frac{2a}{\pi}(G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s}) \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{11}\tau^\infty + G_{12}D^\infty}{G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s}}\right)\right] \\ \Delta\varphi(a) &= \frac{2aD_\mathrm{s}}{\pi V_{22}} \ln\left[\sec\left(\frac{\pi D^\infty}{2D_\mathrm{s}}\right)\right] - \frac{V_{21}}{V_{22}}\Delta u_3(a)\end{aligned} \tag{4.264}$$

The energy release rate is

$$\begin{aligned}J = \tau_\mathrm{s}\Delta u_3(a) + D_\mathrm{s}\Delta\varphi(a) &= \frac{2a}{\pi}\left\{\left(\tau_\mathrm{s} - \frac{V_{21}}{V_{22}}D_\mathrm{s}\right)(G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s})\right. \\ &\left.\times \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{11}\tau^\infty + G_{12}D^\infty}{G_{11}\tau_\mathrm{s} + G_{12}D_\mathrm{s}}\right)\right] + \frac{D_\mathrm{s}^2}{V_{22}} \ln\left[\sec\left(\frac{\pi D^\infty}{2D_\mathrm{s}}\right)\right]\right\}\end{aligned} \tag{4.265}$$

For the small-scale saturation and yielding, we have $c/a \sim b/a \sim 1$, so

$$J = \frac{\pi a}{4}[\tau^\infty \;\; D^\infty][\boldsymbol{G}]\begin{Bmatrix}\tau^\infty \\ D^\infty\end{Bmatrix} \tag{4.266}$$

It is also noted that all the singular integrals are in the sense of the Cauchy principle value.

4.8.3 Solution for Longer Mechanical Yielding Size

In this case, the size of the mechanical yielding region is c and the size of the electric saturation region is b and $c > b$. Equation (4.254) yields

$$\begin{aligned}&(1/\pi)\int_{-b}^{b} [\psi_2(t)/(t-x_1)]\mathrm{d}t = G_{2j}t_{0j}(x_1), \quad |x_1| < b \\ &(1/\pi)\int_{-l}^{l} \left[V_{1j}\psi_j(t)/(t-x_1)\right]\mathrm{d}t = t_{01}(x_1), \quad |x_1| < c, \quad j = 1,2\end{aligned} \tag{4.267}$$

The sizes of the yielding and saturating regions are, respectively,

$$\frac{b}{a} = \sec\left(\frac{\pi}{2}\frac{G_{21}\tau^{\infty} + G_{22}D^{\infty}}{G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}}\right), \quad \frac{c}{a} = \sec\left(\frac{\pi\tau^{\infty}}{2\tau_{\mathrm{s}}}\right) \tag{4.268}$$

The generalized crack tip opening displacements are

$$\begin{aligned} \Delta\varphi(a) &= \frac{2a}{\pi}(G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}) \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{21}\tau^{\infty} + G_{22}D^{\infty}}{G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}}\right)\right] \\ \Delta u_3(a) &= \frac{2a\tau_{\mathrm{s}}}{\pi V_{11}} \ln\left[\sec\left(\frac{\pi\tau^{\infty}}{2\tau_{\mathrm{s}}}\right)\right] - \frac{V_{12}}{V_{11}}\Delta\varphi(a) \end{aligned} \tag{4.269}$$

The energy release rate is

$$\begin{aligned} J = \frac{2a}{\pi}\Bigg\{ &\left(D_{\mathrm{s}} - \frac{V_{12}}{V_{11}}\tau_{\mathrm{s}}\right)(G_{11}\tau_{\mathrm{s}} + G_{12}D_{\mathrm{s}}) \\ &\times \ln\left[\sec\left(\frac{\pi}{2}\frac{G_{21}\tau^{\infty} + G_{22}D^{\infty}}{G_{21}\tau_{\mathrm{s}} + G_{22}D_{\mathrm{s}}}\right)\right] + \frac{\tau_{\mathrm{s}}^2}{V_{11}} \ln\left[\sec\left(\frac{\pi\tau^{\infty}}{2\tau_{\mathrm{s}}}\right)\right]\Bigg\} \end{aligned} \tag{4.270}$$

4.9 Mode-III Problem for a Circular Inclusion with Interface Cracks

4.9.1 Fundamental Equations

The generalized equilibrium and constitutive equations of a mode-III problem (antiplane shear) are shown in Eq. (4.238), and the equilibrium equations in terms of generalized displacements are shown in Eq. (4.239), i.e.,

$$\nabla^2 u_3 = 0, \quad \nabla^2 \varphi = 0 \tag{4.271}$$

where ∇^2 is the 2D Laplace operator. Introduce two analytical functions $\phi_1(z)$ and $\phi_2(z)$. Let

$$\begin{aligned} &u_3(x_1, x_2) = \left[\phi_1(z) + \overline{\phi_1(z)}\right], \quad \varphi = \left[\phi_2(z) + \overline{\phi_2(z)}\right]; \\ &z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta}, \quad z_{,\theta} = \mathrm{i}z \end{aligned} \tag{4.272}$$

Note

$$u_{3,\theta} = u_{3,z}z_{,\theta} + u_{3,\bar{z}}\bar{z}_{,\theta} = \mathrm{i}\left[z\phi_1'(z) - \overline{z\phi_1'(z)}\right], \quad \varphi_{,\theta} = \mathrm{i}\left[z\phi_2'(z) - \overline{z\phi_2'(z)}\right] \tag{4.273}$$

where $\phi_i'(z) = \mathrm{d}\phi_i(z)/\mathrm{d}z$. Equations (4.272) and (4.273) yield

$$\begin{aligned}
&\sigma_{31} - \mathrm{i}\sigma_{32} = 2\left[G\phi_1'(z) + e_{15}\phi_2'(z)\right], \quad D_1 - \mathrm{i}D_2 = 2\left[e_{15}\phi_1'(z) - \epsilon_{11}\phi_2'(z)\right] \\
&\sigma_{3r} - \mathrm{i}\sigma_{3\theta} = 2\mathrm{e}^{\mathrm{i}\theta}\left[G\phi_1'(z) + e_{15}\phi_2'(z)\right], \quad D_r - \mathrm{i}D_\theta = 2\mathrm{e}^{\mathrm{i}\theta}\left[e_{15}\phi_1'(z) - \epsilon_{11}\phi_2'(z)\right] \\
&E_1 - \mathrm{i}E_2 = -2\phi_2'(z), \quad E_r - iE_\theta = -2\mathrm{e}^{\mathrm{i}\theta}\phi_2'(z)
\end{aligned} \tag{4.274}$$

Let

$$\begin{aligned}
&\boldsymbol{f}(z) = \begin{Bmatrix} \phi_1(z) \\ \phi_2(z) \end{Bmatrix}, \quad \boldsymbol{F}(z) = \begin{Bmatrix} \phi_1'(z) \\ \phi_2'(z) \end{Bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} G & e_{15} \\ e_{15} & -\epsilon_{11} \end{bmatrix}, \\
&\boldsymbol{\Sigma}_r = \begin{Bmatrix} \sigma_{3r} \\ D_r \end{Bmatrix}, \quad \boldsymbol{U}_{,\theta} = \begin{Bmatrix} u_{3,\theta}/r \\ -E_\theta \end{Bmatrix}
\end{aligned} \tag{4.275}$$

Notations used in this section may be different with other sections. In Eq. (4.275), $\boldsymbol{B}$ is real, so $\boldsymbol{B} = \bar{\boldsymbol{B}}$. Adopting notations in Eq. (4.275) yields

$$\boldsymbol{\Sigma}_r = \left\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{BF}(z) + \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{BF}(z)}\right\}, \quad \boldsymbol{U}_{,\theta} = \mathrm{i}\left\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{F}(z) - \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{F}(z)}\right\} \tag{4.276}$$

On the interface, Eq. (4.276) is reduced to

$$\boldsymbol{\Sigma}_r = (z/a)\left\{\boldsymbol{BF}(z) + (a/z)^2\overline{\boldsymbol{BF}(z)}\right\}, \quad \boldsymbol{U}_{,\theta} = \mathrm{i}(z/a)\left\{\boldsymbol{F}(z) - (a/z)^2\overline{\boldsymbol{F}(z)}\right\}; \quad z \in L \tag{4.277}$$

4.9.2 Permeable Crack

Figure 4.16a shows an infinite matrix II occupied region S^- including a circular inclusion I of radius a occupied region S^+. Materials I and II are all transversely isotropic. The entire interface is denoted by L and there are n circular arc cracks on it. The ends of cracks are successively counterclockwise denoted by a_k, b_k and its whole is denoted by L_c. The origin of the coordinate system (x_1, x_2) or (r, θ) is selected at the center of the inclusion. The boundary conditions are

$$\begin{aligned}
&\sigma_{31} = \sigma_{31}^\infty, \quad \sigma_{32} = \sigma_{32}^\infty, \quad D_2 = D_2^\infty, \quad D_1 = D_1^\infty; \quad |z| \to \infty \\
&\sigma_{\mathrm{I}r3} = \sigma_{\mathrm{II}r3} = 0, \quad D_{\mathrm{I}r} = D_{\mathrm{II}r}, \quad \varphi_{\mathrm{I}} = \varphi_{\mathrm{II}}\ (E_{\mathrm{I}\theta} = E_{\mathrm{II}\theta}); \quad z \in L_{\mathrm{c}} \\
&\sigma_{\mathrm{I}r3} = \sigma_{\mathrm{II}r3}, \quad D_{\mathrm{I}r} = D_{\mathrm{II}r}, \quad u_{\mathrm{I}3} = u_{\mathrm{II}3}\ \left(u_{\mathrm{I}3,\theta} = u_{\mathrm{II}3,\theta}\right), \quad \varphi_{\mathrm{I}} = \varphi_{\mathrm{II}}(E_{\mathrm{I}\theta} = E_{\mathrm{II}\theta}); \\
&\quad z \in L - L_{\mathrm{c}}
\end{aligned} \tag{4.278}$$

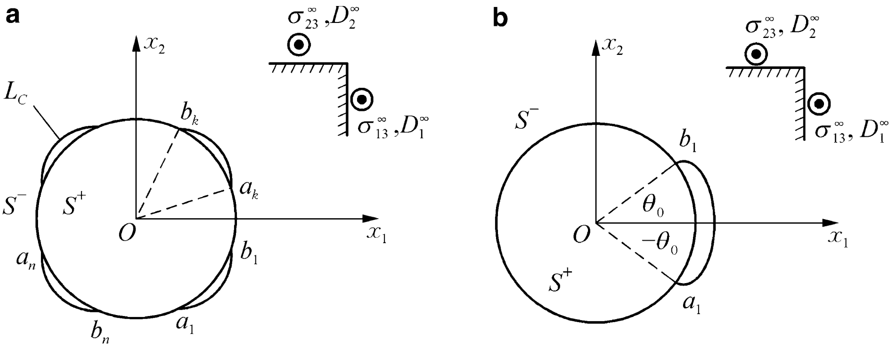


Fig. 4.16 A circular interface inclusions with interface cracks: (**a**) general case and (**b**) one crack

For convenience the following mapping function is used:

$$z = \omega(\varsigma) = a\varsigma, \quad z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta}, \quad \varsigma = \xi + \mathrm{i}\eta = R\mathrm{e}^{\mathrm{i}\theta}; \quad r = aR \tag{4.279}$$

Under this transformation, the circle with radius a in z plane is transformed to a unit circle in ς plane and L, L_c is transformed to $\Gamma, \Gamma_\mathrm{c}$, respectively. In ς plane, the matrix is located in the region S^-, $|\varsigma| > 1$. The inclusion is located in the region S^+, $|\varsigma| < 1$. The ends of cracks are all on the unit circle and denoted by $\sigma_k^{(1)}, \sigma_k^{(2)}$ in the ς plane. It is noted that

$$\boldsymbol{f}(z) = \boldsymbol{f}[\omega(\varsigma)] = \boldsymbol{f}(\varsigma), \quad \boldsymbol{F}(z) = \boldsymbol{f}'(z) = \boldsymbol{f}'(\varsigma)/\omega'(\varsigma) = \boldsymbol{F}(\varsigma)/a \tag{4.280}$$

In ς plane, Eq. (4.276) is reduced to

$$\boldsymbol{\Sigma}_r = (1/a)\left[\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{BF}(\varsigma) + \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{BF}(\varsigma)}\right], \quad \boldsymbol{U}_{,\theta} = (\mathrm{i}/a)\left\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{F}(\varsigma) - \mathrm{e}^{-\mathrm{i}\theta}\overline{\boldsymbol{F}(\varsigma)}\right\} \tag{4.281}$$

On the interface Γ, $\sigma = \mathrm{e}^{\mathrm{i}\theta}$. Equation (4.277) is reduced to

$$\begin{aligned} &\boldsymbol{\Sigma}_r = (1/a)\left[\sigma\boldsymbol{BF}(\sigma) + \bar{\sigma}\overline{\boldsymbol{BF}(\sigma)}\right], \quad \boldsymbol{U}_{,\theta} = (\mathrm{i}/a)\left[\sigma\boldsymbol{F}(\sigma) - \bar{\sigma}\overline{\boldsymbol{F}(\sigma)}\right] \\ &E_r - iE_\theta = -(2/a)\sigma\phi_2'(\sigma), \quad E_r = -(2/a)\mathrm{Re}\left[\sigma\phi_2'(\sigma)\right], \quad \sigma \in \Gamma \end{aligned} \tag{4.282}$$

4.9.3 *Reduced to Riemann-Hilbert Equation*

According to Eq. (4.278) on whole interface, $\boldsymbol{\Sigma}_{\mathrm{I}r} = \boldsymbol{\Sigma}_{\mathrm{II}r}$, so Eq. (4.282) yields

$$\sigma\boldsymbol{B}_\mathrm{I}\boldsymbol{F}_\mathrm{I}(\sigma) + \bar{\sigma}\bar{\boldsymbol{B}}_\mathrm{I}\overline{\mathbf{F}_\mathrm{I}(\sigma)} = \sigma\boldsymbol{B}_\mathrm{II}\boldsymbol{F}_\mathrm{II}(\sigma) + \bar{\sigma}\bar{\boldsymbol{B}}_\mathrm{II}\overline{\mathbf{F}_\mathrm{II}(\sigma)}; \quad \varsigma \in \Gamma \tag{4.283}$$

For a unit circular region, if $\mathrm{g}(\varsigma)$ is analytic in $S^+(S^-)$, $\mathrm{g}_*(\varsigma) = \bar{\mathrm{g}}(1/\varsigma)$ is analytic in $S^-(S^+)$ (Muskhelishvili 1954) and

$$\mathrm{g}_*^-(\sigma) = \bar{\mathrm{g}}^+(\bar{\sigma}), \quad \mathrm{g}_*^+(\sigma) = \bar{\mathrm{g}}^-(\bar{\sigma}), \quad \mathrm{g}_*(\varsigma) = \bar{\mathrm{g}}(1/\varsigma) \tag{4.284}$$

Rewrite Eq. (4.283) as

$$\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^+(\sigma) - \bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}*}^+(\sigma) = \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^-(\sigma) - \bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}^-; \quad \sigma \in \Gamma \tag{4.285}$$

Now research the behavior of $\boldsymbol{F}_\alpha(\varsigma)(\alpha = \mathrm{I}, \mathrm{II})$. Denote $\boldsymbol{F}_{\mathrm{I}}(\varsigma)$ is analytic in S^+ and rewritten as $\boldsymbol{F}_{\mathrm{I}0}(\varsigma)$; $\boldsymbol{F}_{\mathrm{II}}(\varsigma)$ is analytic in S^- except at infinite and can be expressed as

$$\boldsymbol{F}_{\mathrm{II}}(\varsigma) = \boldsymbol{F}_{\mathrm{II}}^\infty + \boldsymbol{F}_{\mathrm{II}0}(\varsigma), \quad \boldsymbol{F}_{\mathrm{II}}^\infty = \boldsymbol{F}_{\mathrm{II}}(\infty), \quad \varsigma \in S^- \tag{4.286}$$

Because there is no generalized force and dislocation in a finite region, from Eqs. (4.274), (4.275), and (4.278), it is easy to obtain

$$\boldsymbol{F}_{\mathrm{II}}(\infty) = \frac{a}{2}\boldsymbol{B}_{\mathrm{II}}^{-1}\begin{Bmatrix} \sigma_{31}^\infty - \mathrm{i}\sigma_{32}^\infty \\ D_1^\infty - \mathrm{i}D_2^\infty \end{Bmatrix} \tag{4.287}$$

In Eq. (4.285), $\bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}*}^+(\sigma)$ is the boundary value on Γ of the function $\left(1/\varsigma^2\right)\bar{\boldsymbol{B}}_{\mathrm{II}}$ $\boldsymbol{F}_{\mathrm{II}*}(\varsigma) = \left(1/\varsigma^2\right)\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(1/\varsigma)$ which is analytic in S^+ except the pole point $\varsigma = 0$. $\bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}^-$ is the boundary value on Γ of the function $\left(1/\varsigma^2\right)\overline{\boldsymbol{B}}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}(\varsigma) = \left(1/\varsigma^2\right)\overline{\boldsymbol{B}}_{\mathrm{I}}\overline{\boldsymbol{F}}_{\mathrm{I}}$ $(1/\varsigma)$ which is analytic in S^-. These two functions can be analytic continuation through the connective parts on Γ. The function after analytic continuation and the original function must possess the same pole points and values at infinity. Let

$$\begin{aligned} &\boldsymbol{G}_{\mathrm{II}}(\varsigma) = \boldsymbol{F}_{\mathrm{II}*}(\varsigma)/\varsigma^2 = \bar{\boldsymbol{F}}_{\mathrm{II}}^\infty/\varsigma^2 + \boldsymbol{G}_{\mathrm{II}0}(\varsigma), \quad \boldsymbol{G}_{\mathrm{II}0}(\varsigma) = \bar{\boldsymbol{F}}_{\mathrm{II}0}(1/\varsigma)/\varsigma^2, \quad \varsigma \in S^+ \\ &\boldsymbol{G}_{\mathrm{I}0}(\varsigma) = \boldsymbol{F}_{\mathrm{I}*}(\varsigma)/\varsigma^2 = \bar{\boldsymbol{F}}_{\mathrm{I}0}(1/\varsigma)/\varsigma^2, \quad \varsigma \in S^- \end{aligned} \tag{4.288}$$

Using $\boldsymbol{B}_{\mathrm{I}} = \bar{\boldsymbol{B}}_{\mathrm{I}},\ \boldsymbol{B}_{\mathrm{II}} = \bar{\boldsymbol{B}}_{\mathrm{II}}$, it can be assumed

$$\begin{aligned} &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(\varsigma) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{G}_{\mathrm{II}}(\varsigma) = \mathbf{g}(\varsigma), \quad \varsigma \in S^+ \\ &\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(\varsigma) - \boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}0}(\varsigma) = \mathbf{g}(\varsigma), \quad \varsigma \in S^- \\ &\mathbf{g}(\varsigma) = -\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^\infty/\varsigma^2 + \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^\infty \end{aligned} \tag{4.289}$$

Substituting Eqs. (4.286) and (4.288) into Eq. (4.289) yield

$$\begin{aligned} &\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(\varsigma) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{G}_{\mathrm{II}0}(\varsigma) - \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}^\infty = 0, \quad \varsigma \in S^+ \\ &\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(\varsigma) - \boldsymbol{B}_{\mathrm{I}}\boldsymbol{G}_{\mathrm{I}0}(\varsigma) + \left(1/\varsigma^2\right)\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^\infty = 0, \quad \varsigma \in S^- \end{aligned} \tag{4.290}$$

On the interface, we have

$$\boldsymbol{G}_{\mathrm{II}0}(\sigma) = -\boldsymbol{F}_{\mathrm{II}}^{\infty} + \boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(\sigma), \quad \boldsymbol{G}_{\mathrm{I}0}(\sigma) = \bar{\sigma}^2\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty} + \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(\sigma) \tag{4.291}$$

According to Eq. (4.282), the jump $\hat{\boldsymbol{d}}'$ of the direction derivative $U_{,\theta}$ is

$$\begin{aligned}
\hat{\boldsymbol{d}}' &= \left(\boldsymbol{U}_{\mathrm{I},\theta} - \boldsymbol{U}_{\mathrm{II},\theta}\right) = \mathrm{i}(\sigma/a)\left\{\left[\boldsymbol{F}_{\mathrm{I}}(\sigma) - \bar{\sigma}^2\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{\sigma})\right] - \left[\boldsymbol{F}_{\mathrm{II}}(\sigma) - \bar{\sigma}^2\bar{\boldsymbol{F}}_{\mathrm{II}}(\bar{\sigma})\right]\right\}, \quad \text{or}\\
(a/\sigma)\hat{\boldsymbol{d}}' &= \mathrm{i}\left[\boldsymbol{F}_{\mathrm{I}}(\sigma) + \bar{\sigma}^2\boldsymbol{F}_{\mathrm{II}*}^{+}(\sigma)\right] - \left[\boldsymbol{F}_{\mathrm{II}}(\sigma) + \bar{\sigma}^2\boldsymbol{F}_{\mathrm{I}*}^{-}(\sigma)\right]\\
&= \mathrm{i}\left[\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma) - 2\boldsymbol{F}_{\mathrm{II}}^{\infty} + \boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma)\right]\\
&\quad - \mathrm{i}\left[\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma) - \bar{\sigma}^2\left(\mathbf{I} - \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty} + \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma)\right]
\end{aligned} \tag{4.292}$$

Construct a function $\boldsymbol{h}(\varsigma)$ analytic in whole plane except cracks and $\varsigma = 0$ by the analytic continuation method through $\Gamma - \Gamma_{\mathrm{c}}$:

$$\boldsymbol{h}(\varsigma) = \begin{cases} \boldsymbol{F}_{\mathrm{I}0}(\varsigma) + \boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}(\varsigma) - 2\,\boldsymbol{F}_{\mathrm{II}}^{\infty} \\ \boldsymbol{F}_{\mathrm{II}0}(\varsigma) + \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}(\varsigma) - \left(1/\varsigma^2\right)\left(\mathbf{I} - \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\,\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty} \end{cases} \tag{4.293}$$

According to Eqs. (4.282) and (4.290), on the crack surface, we have

$$\begin{aligned}
(a/\sigma)\boldsymbol{\Sigma}_{\mathrm{I}\,r} &= \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(\sigma) + \bar{\sigma}^2\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(\bar{\sigma}) = \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{+}(\sigma) + \bar{\sigma}^2\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}*}^{-}(\sigma)\\
&= \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}^{+}(\sigma) + \bar{\sigma}^2\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty} + \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma), \quad \varsigma \in L_{\mathrm{c}}
\end{aligned} \tag{4.294}$$

Equation (4.293) yields

$$\begin{aligned}
\boldsymbol{h}^{+}(\sigma) + \boldsymbol{h}^{-}(\sigma) &= \left(\mathbf{I} + \boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\right)\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma) - 2\boldsymbol{F}_{\mathrm{II}}^{\infty} + \left(\mathbf{I} + \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\boldsymbol{F}_{\mathrm{II}0}^{-} - \bar{\sigma}^2\left(\mathbf{I} - \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\\
&= \boldsymbol{H}\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}0}^{+}(\sigma) + \boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}0}^{-}(\sigma) + \bar{\sigma}^2\boldsymbol{B}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\right] - 2\boldsymbol{F}_{\mathrm{II}}^{\infty} - 2\bar{\sigma}^2\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}
\end{aligned} \tag{4.295}$$

where $\boldsymbol{H} = \boldsymbol{B}_{\mathrm{I}}^{-1} + \boldsymbol{B}_{\mathrm{II}}^{-1}$. Comparing Eqs. (4.294) and (4.295), the Riemann-Hilbert equation on the crack surface is obtained:

$$(a/\sigma)\boldsymbol{\Sigma}_{\mathrm{I}\,r} = \boldsymbol{H}^{-1}[\boldsymbol{h}^{+}(\sigma) + \boldsymbol{h}^{-}(\sigma)] + \boldsymbol{p}^{\infty} + \bar{\boldsymbol{p}}^{\infty}\bar{\sigma}^2, \quad \boldsymbol{p}^{\infty} = 2\boldsymbol{H}^{-1}\boldsymbol{F}_{\mathrm{II}}^{\infty} \tag{4.296}$$

After $\boldsymbol{h}(\varsigma)$ is solved, from Eq. (4.293), $\boldsymbol{F}_{\mathrm{I}0}\left(\varsigma_j\right), \boldsymbol{F}_{\mathrm{II}0}\left(\varsigma_j\right)$ can be obtained:

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{I}0}\left(\varsigma_j\right) &= \left(\mathbf{I} + \boldsymbol{B}_{\mathrm{II}}^{-1}\boldsymbol{B}_{\mathrm{I}}\right)^{-1}\left[\boldsymbol{h}\left(\varsigma_j\right) + 2\boldsymbol{F}_{\mathrm{II}}^{\infty}\right]\\
\boldsymbol{F}_{\mathrm{II}0}\left(\varsigma_j\right) &= \left(\mathbf{I} + \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)^{-1}\left[\boldsymbol{h}\left(\varsigma_j\right) + \left(1\big/\varsigma_j^2\right)\left(\mathbf{I} - \boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{B}_{\mathrm{II}}\right)\bar{\boldsymbol{F}}_{\mathrm{II}}^{\infty}\right]
\end{aligned} \tag{4.297}$$

4.9.4 Solution for Permeable Crack

From Eqs. (4.292) and (4.293), it is found that

$$-\mathrm{i}(a/\sigma)\hat{\boldsymbol{d}}' = \boldsymbol{h}^{+}(\sigma) - \boldsymbol{h}^{-}(\sigma) \tag{4.298}$$

From Eq. (4.278), it is known that on the crack surface, $E_{\mathrm{I}\theta} = E_{\mathrm{II}\theta}$, so on the whole interface L, $\hat{d}_2' = 0$, or $h_2^{+}(\sigma) - h_2^{-}(\sigma) = 0$, $\sigma \in L$. So $h_2(\varsigma)$ is analytic in whole plane. Because $h_2(\infty) = 0$, so $h_2(\varsigma) = 0$.

Using the boundary conditions Eq. (4.278) and $h_2(\varsigma) = 0$, Eq. (4.296) yields

$$\begin{aligned} (a/\sigma)\sigma_{\mathrm{I}r} &= H_{11}^{-1}\left[h_1^{+}(\sigma) + h_1^{-}(\sigma)\right] + p_1^{\infty} + \bar{p}_1^{\infty}\bar{\sigma}^2 = 0 \\ (a/\sigma)D_{\mathrm{I}r} &= H_{21}^{-1}\left[h_1^{+}(\sigma) + h_1^{-}(\sigma)\right] + p_2^{\infty} + \bar{p}_2^{\infty}\bar{\sigma}^2 \end{aligned} \tag{4.299}$$

Equation (4.299) yields

$$D_{\mathrm{I}r} = (1/a)\mathrm{Re}\left[\sigma p_2^{\infty} - \left(H_{21}^{-1}/H_{11}^{-1}\right)\sigma p_1^{\infty}\right] \tag{4.300}$$

Because the traction on the crack surface is zero and $D_{\mathrm{I}r}$ is shown in Eq. (4.300), Eq. (4.296) can be reduced to

$$\left[\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right]^{+} + \left[\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right]^{-} = \boldsymbol{P}^{\infty} + \bar{\boldsymbol{P}}^{\infty}\bar{\sigma}^2; \quad \boldsymbol{P}^{\infty} = -p_1^{\infty}\left(\mathbf{i}_1 + \left(H_{21}^{-1}/H_{11}^{-1}\right)\mathbf{i}_2\right) \tag{4.301}$$

The general solution is

$$\begin{aligned} &\boldsymbol{H}^{-1}\boldsymbol{h}(\varsigma) = (1/2)\left(\boldsymbol{P}^{\infty} + \bar{\boldsymbol{P}}^{\infty}/\varsigma^2\right) + (1/2)X(\varsigma)\left[\boldsymbol{C}(\varsigma) + \boldsymbol{C}_{-1}/\varsigma + \boldsymbol{C}_{-2}/\varsigma^2\right] \\ &X(\varsigma) = \prod_{k=1}^{n}\left(\varsigma - \sigma_k^{(1)}\right)^{-1/2}\left(\varsigma - \sigma_k^{(2)}\right)^{-1/2}, \quad \boldsymbol{C}(\varsigma) = \boldsymbol{C}_n\varsigma^n + \cdots + \boldsymbol{C}_0 \end{aligned} \tag{4.302}$$

where n is the number of cracks. It is noted that

$$\lim_{\varsigma\to 0} X(\varsigma) \approx \left[\prod_{k=1}^{n}(-1)^n\left(1\Big/\sqrt{\sigma_k^{(1)}\sigma_k^{(2)}}\right)\right]\left[1 + (\varsigma/2)\sum_{k=1}^{n}\left(1\Big/\sigma_k^{(1)} + 1\Big/\sigma_k^{(2)}\right)\right] \tag{4.303}$$

Substituting Eq. (4.303) into Eq. (4.302) and comparing the order of ς yield

$$\begin{aligned} \varsigma^{-1}: &\quad \boldsymbol{C}_{-1} = -(1/2)\sum_{k=1}^{n}\left(1\Big/\sigma_k^{(1)} + 1\Big/\sigma_k^{(2)}\right)\boldsymbol{C}_{-2} \\ \varsigma^{-2}: &\quad \boldsymbol{C}_{-2} = (-1)^{n+1}\prod_{k=1}^{n}\sqrt{\sigma_k^{(1)}\sigma_k^{(2)}}\,\bar{\boldsymbol{P}}^{\infty} \end{aligned} \tag{4.304}$$

When $\varsigma \to \infty$, we get

$$\lim_{\varsigma\to\infty} X(\varsigma) \approx 1/\varsigma^n + (1/2\varsigma)\sum_{k=1}^{n}\left(\sigma_k^{(1)} + \sigma_k^{(2)}\right)\left(1/2\varsigma^{n+1}\right)$$
$$\varsigma^0: \ \ \boldsymbol{C}_n = -\boldsymbol{P}^\infty, \quad \varsigma^{-1}: \ \ \boldsymbol{C}_{n-1} = -(1/2)\sum_{k=1}^{n}\left(\sigma_k^{(1)} + \sigma_k^{(2)}\right)\boldsymbol{C}_n \tag{4.305}$$

Other coefficients are determined by single-valued conditions of the generalized displacement:

$$\int_{L_c} \hat{\boldsymbol{d}}' \mathrm{d}\sigma = 0, \quad \text{or} \quad \int_{L_c} [\boldsymbol{h}^+(\sigma) - \boldsymbol{h}^-(\sigma)]\mathrm{d}\sigma = 0 \tag{4.306}$$

4.9.5 Single Crack

Figure 4.16b shows a single crack with $a_1 = a\mathrm{e}^{-\mathrm{i}\theta_0}$, $b_1 = a\mathrm{e}^{\mathrm{i}\theta_0}$, where $2\theta_0$ is the center angle spanning by the crack. In this case we have

$$\sigma_k^{(1)} = \mathrm{e}^{-\mathrm{i}\theta_0}, \quad \sigma_k^{(2)} = \mathrm{e}^{\mathrm{i}\theta_0}, \quad X(\varsigma) = \left(\varsigma - \mathrm{e}^{-\mathrm{i}\theta_0}\right)^{-1/2}\left(\varsigma - \mathrm{e}^{\mathrm{i}\theta_0}\right)^{-1/2}$$
$$\boldsymbol{C}(\varsigma) = \boldsymbol{C}_1\varsigma + \boldsymbol{C}_0, \quad \boldsymbol{C}_1 = -\boldsymbol{P}^\infty, \quad \boldsymbol{C}_0 = \cos\theta_0\boldsymbol{P}^\infty, \quad \boldsymbol{C}_{-2} = \bar{\boldsymbol{P}}^\infty, \quad \boldsymbol{C}_{-1} = -\cos\theta_0\bar{\boldsymbol{P}}^\infty \tag{4.307}$$

The solution is

$$\boldsymbol{H}^{-1}\boldsymbol{h}(\varsigma) = (1/2)\left(\boldsymbol{P}^\infty + \bar{\boldsymbol{P}}^\infty/\varsigma^2\right)$$
$$+ (1/2)\left(\varsigma^2 - 2\cos\theta_0 + 1\right)^{-1/2}\left[-\boldsymbol{P}^\infty\varsigma + \cos\theta_0\boldsymbol{P}^\infty - \cos\theta_0\bar{\boldsymbol{P}}^\infty/\varsigma + \bar{\boldsymbol{P}}^\infty/\varsigma^2\right] \tag{4.308}$$

$\boldsymbol{F}_{\mathrm{I}0}\left(\varsigma_j\right), \boldsymbol{F}_{\mathrm{II}0}\left(\varsigma_j\right)$ can be obtained from Eq. (4.297). So the generalized stress and displacement in any point can also be obtained. It is noted that

$$X(\sigma) = \left(\sigma^2 - 2\cos\theta_0 + 1\right)^{-1/2} = \mathrm{e}^{-\mathrm{i}\theta/2}[2\sin\theta_0(\theta_0 - \theta)]^{-1/2}$$
$$\sigma\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma) = (1/2)\sigma X(\varsigma)\left[-\boldsymbol{P}^\infty\sigma + \cos\theta_0\boldsymbol{P}^\infty - \cos\theta_0\bar{\boldsymbol{P}}^\infty\bar{\sigma} + \bar{\sigma}^2\bar{\boldsymbol{P}}^\infty\right]$$
$$= \frac{\mathrm{e}^{\mathrm{i}\theta/2}}{2\sqrt{2\sin\theta_0(\theta_0 - \theta)}}\left\{-\mathrm{i}\sin\theta\boldsymbol{P}^\infty - \mathrm{i}\bar{\sigma}\sin\theta\bar{\boldsymbol{P}}^\infty\right\} = -\frac{\sqrt{\sin\theta_0}}{2\sqrt{2(\theta - \theta_0)}}\left\{\sqrt{\sigma}\boldsymbol{P}^\infty + \sqrt{\bar{\sigma}}\bar{\boldsymbol{P}}^\infty\right\}$$
$$\boldsymbol{p}^\infty = 2\boldsymbol{H}^{-1}\boldsymbol{F}_{\mathrm{II}}^\infty = a\boldsymbol{H}^{-1}\boldsymbol{B}_{\mathrm{II}}^{-1}\begin{Bmatrix}\sigma_{31}^\infty - \mathrm{i}\sigma_{32}^\infty \\ D_1^\infty - \mathrm{i}D_2^\infty\end{Bmatrix} = a\boldsymbol{M}\begin{Bmatrix}\sigma_{31}^\infty - \mathrm{i}\sigma_{32}^\infty \\ D_1^\infty - \mathrm{i}D_2^\infty\end{Bmatrix}, \quad \boldsymbol{M} = (\boldsymbol{B}_{\mathrm{II}}\boldsymbol{H})^{-1}$$
$$\boldsymbol{P}^\infty = -p_1^\infty\left(\mathbf{i}_1 + \mathbf{i}_2 H_{21}^{-1}/H_{11}^{-1}\right), \quad p_1^\infty = a\left\{M_{11}\left(\sigma_{31}^\infty - \mathrm{i}\sigma_{32}^\infty\right) + M_{12}\left(D_1^\infty - \mathrm{i}D_2^\infty\right)\right\} \tag{4.309}$$

The stress intensity factors can be directly obtained from $\boldsymbol{h}(\sigma)$ and is only related to the singular parts of the generalized stress. Using Eqs. (4.282), (4.296), and (4.309), the stress intensity factor at $\varsigma = \mathrm{e}^{\mathrm{i}\theta_0}\left(\text{or } z = a\mathrm{e}^{\mathrm{i}\theta_0}\right)$ is

$$
\begin{aligned}
\boldsymbol{K} &= [K_{\mathrm{III}}, K_{\mathrm{D}}]^{\mathrm{T}} = \lim_{\theta\to\theta_0}\sqrt{2\pi a(\theta-\theta_0)}\boldsymbol{\Sigma}_r = (2/a)\sqrt{2\pi a(\theta-\theta_0)}\mathrm{Re}\left[\sigma\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right] \\
&= 2\sqrt{\pi a\sin\theta_0}\left[\cos(\theta_0/2)\left(M_{11}\sigma_{31}^{\infty}+M_{12}D_1^{\infty}\right)+\sin(\theta_0/2)\left(M_{11}\sigma_{32}^{\infty}+M_{12}D_2^{\infty}\right)\right] \\
&\quad\times\left[\mathbf{i}_1+\left(H_{21}^{-1}/H_{11}^{-1}\right)\mathbf{i}_2\right] \\
K_{\alpha E} &= \lim_{\theta\to\theta_0}\sqrt{2\pi a(\theta-\theta_0)}E_{\alpha r} = -(2/a)\lim_{\theta\to\theta_0}\sqrt{2\pi a(\theta-\theta_0)}\boldsymbol{B}_{\alpha}^{-1}\mathrm{Re}\left[\sigma\boldsymbol{H}^{-1}\boldsymbol{h}(\sigma)\right]_2
\end{aligned}
\tag{4.310}
$$

For a homogeneous material, $\boldsymbol{M} = \mathbf{I}/2$, so

$$
\begin{aligned}
\boldsymbol{K} &= \sqrt{\pi a\sin\theta_0}\left[\sigma_{31}^{\infty}\cos(\theta_0/2)+\sigma_{32}^{\infty}\sin(\theta_0/2)\right]\left(\mathbf{i}_1+\mathbf{i}_2H_{21}^{-1}/H_{11}^{-1}\right) \\
K_{\alpha E} &= -\sqrt{\pi a\sin\theta_0}\left[\sigma_{31}^{\infty}\cos(\theta_0/2)+\sigma_{32}^{\infty}\sin(\theta_0/2)\right]\left(B_{\alpha 21}^{-1}+B_{\alpha 22}^{-1}H_{21}^{-1}/H_{11}^{-1}\right)
\end{aligned}
\tag{4.311}
$$

From Eq. (4.311), it is known that for a permeable crack, the stress intensity factors do not depend to the external electric field.

4.9.6 Impermeable Crack

For an impermeable crack, $D_{\mathrm{I}r} = D_{\mathrm{II}r} = 0$ on the crack surface are known and $\boldsymbol{P}^{\infty} = -\boldsymbol{p}^{\infty}$. $\boldsymbol{H}^{-1}\boldsymbol{h}(\varsigma)$ is still expressed by Eqs. (4.302), (4.303), (4.304), (4.305), and (4.306). The stress intensity factor is

$$
\begin{aligned}
\boldsymbol{K} &= \begin{Bmatrix} K_{\mathrm{III}} \\ K_{\mathrm{D}} \end{Bmatrix} = \frac{2}{a}\sqrt{\pi a\sin\theta_0}\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta_0/2}p^{\infty}\right) = 2\sqrt{\pi a\sin\theta_0}\boldsymbol{M}\mathrm{Re}\left[\left\{\mathrm{e}^{\mathrm{i}\theta_0/2}\begin{Bmatrix} \sigma_{31}^{\infty}-\mathrm{i}\sigma_{32}^{\infty} \\ D_1^{\infty}-\mathrm{i}D_2^{\infty} \end{Bmatrix}\right\}\right] \\
&= 2\sqrt{\pi a\sin\theta_0}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{Bmatrix} \sigma_{32}^{\infty}\sin(\theta_0/2)+\sigma_{31}^{\infty}\cos(\theta_0/2) \\ D_2^{\infty}\sin(\theta_0/2)+D_1^{\infty}\cos(\theta_0/2) \end{Bmatrix}
\end{aligned}
\tag{4.312}
$$

From Eq. (4.312), it is known that for an impermeable crack, the stress intensity factors are dependent to the external electric field.

Zhong and Meguid (1997), Gao and Balke (2003), and Liu and Fang (2004) et al. discussed the similar problem.

References

Barnett DM, Asaro RJ (1972) The fracture mechanics of slit-like cracks in anisotropic elastic media. J Mech Phys Solids 20:353–366
Beom HG, Atluri SN (1996) Near-tip fields and intensity factors for interfacial cracks in dissimilar anisotropic piezoelectric media. Int J Fract 75:163–183
Chung MY, Ting TCT (1996) Piezoelectric solid with an elliptic inclusion or hole. Int J Solids Struct 33:3343–3361
Deng W, Meguid SA (1998) Analysis of conducting rigid inclusion at the interface of two dissimilar piezoelectric materials. J Appl Mech 65:76–94
Erdogan F, Gupta GD (1972) On the numerical solution of singular integral equations. Q Appl Math 32:525–553
Fulton CC, Gao H (1997) Electrical nonlinearity in fracture of piezoelectric ceramics. Appl Mech Rev 50:556–563
Gao C-F, Balke H (2003) Fracture analysis of circular-arc interface cracks in piezoelectric materials. Int J Solids Struct 40:3507–3522
Gao C-F, Wang M-Z (2000) Collinear permeable cracks between dissimilar piezoelectric materials. Int J Solids Struct 37:4969–4986
Gao C-F, Wang M-Z (2001) Green's functions of an interfacial crack between two dissimilar piezoelectric media. Int J Solids Struct 38:5323–5334
Gao HJ, Zhang TY, Tong P (1997) Local and global energy release rates for an electrically yielded crack in a piezoelectric ceramic. J Mech Phys Solids 45:491–510
Hao TH, Shen ZY (1994) A new electric boundary condition of electric fracture mechanics and its applications. Eng Fract Mech 47:793–802
Herrmann KP, Loboda VV (2000) Fracture-mechanical assessment of electrically permeable interface cracks in piezoelectric bimaterials by consideration of various contact zone models. Arch Appl Mech 70:127–143
Herrmann KP, Loboda VV, Govorukha VB (2001) On contact zone models for an electrically impermeable interface crack in a piezoelectric bimaterial. Int J Fract 111:203–227
Hou Z-Y, Li M-Z, Zhang W-G (1990) Singular integral equations and its applications. Shanghai Science and Technology Publishing House, Shanghai (in Chinese)
Huang Z-Y, Kuang Z-B (2001) A first order perturbation analysis of a non-ideal crack in a piezoelectric material. Int J Solids Struct 38:7261–7281
Huang Z-Y, Kuang Z-B (2003) A mixed electric boundary value problem for a two-dimensional piezoelectric crack. Int J Solids Struct 40:1433–1453
Kuang Z-B, Ma F-S (2002) Crack tip field. Xian Jiaotong University Press, Xian (in Chinese)
Kuang Z-B, Zhou Z-D, Zhou K-L (2004) Electroelastic analysis of a piezoelectric half-plane with finite surface electrodes. Int J Eng Sci 42:1603–1619
Lavrenchive MA, Shabat VA (1951) Method and theory of complex functions. National Technical and Theoretical Literature Press, Moscow; Лаврентьев M A, Шабат B A. Методтеориифукцийкоплекснгопеременого. Государственноеиздадельство. технико-теоретическийлитературы. Москва (1951)
Liu Y-W, Fang Q-H (2004) Electroelastic interaction between a piezoelectric screw dislocation and a circular inhomogeneity with interfacial cracks. Appl Math Mech 25:1428–1436
Loboda VV (1993) The quasi-invariant in the theory of interface cracks. Eng Fract Mech 44:573–580
Lynch CS, Yang W, Collier L, Suo Z, McCMeeking RM (1995) Electric field induced cracking in ferroelectric ceramics. Ferroelectrics 166:11–30
Muskhelishvili NI (1975) Some basic problems of mathematical theory of elasticity. Noordhoof, Leyden; Мусхелишвили H E. Некоторые осноные задачи математической теории упругости, Масква: Издательство академии наук СССР (1954)
Muskhelishvili NI (1975) Singular integral equations. Noordhoff, Gronigen
Pak YE (1990) Force on a piezoelectric screw dislocation. J Appl Mech 57:863–869

Ru CQ (2000) Exact solution for finite electrode layers embedded at the interface of two piezoelectric half-planes. J Mech Phys Solids 48:693–708
Ru CQ, Mao X (1999) Conducting cracks in a piezoelectric ceramic of limited electrical polarization. J Mech Phys Solids 47:2125–2146
Shen S-P, Kuang Z-B (1998) Interface crack in bi-piezothermoelastic media and the interaction with a point heat source. Int J Solids Struct 35:3899–3915
Shen S-P, Kuang Z-B, Hu SL (1999) Interface crack problem of a laminated piezoelectric plate. Eur J Mech A/Solids 18:219–238
Shen S-P, Nishioka T, Kuang Z-B, Liu Z-X (2000) Nonlinear electromechanical interface fracture for piezoelectric materials. Mech Mater 32:57–64
Shen MH, Chen FM, Chen SN (2007) Piezoelectric study on singularities interacting with interfaces in an anisotropic medium. Int J Solids Struct 44:5598–5610
Shi W-C (1997) Rigid line inclusions under anti-plane deformation and in-plane electric field in piezoelectric materials. Eng Fract Mech 56:265–274
Shindo Y, Narita F, Sosa H (1998) Electroelastic analysis of piezoelectric ceramics with surface electrodes. Int J Eng Sci 36:1001–1009
Suo Z (1990) Singularities interfaces and cracks in dissimilar anisotropic media. Proc R Soc Lond A 427:331–358
Suo Z, Kuo CM, Barnett DM, Willis JR (1992) Fracture mechanics for piezoelectric ceramics. J Mech Phys Solids 40:739–765
Wang T-C (2000) Analysis of strip electric saturation model of crack problem in piezoelectric materials. Int J Solids Struct 37:6031–6049
Wang J-W, Kuang Z-B (2000) The interaction between crack and electric dipole of piezoelectricity. Acta Mech Solida Sin 13:283–289
Wang J-W, Kuang Z-B (2002) The electric dipole and crack on the interface in a bi-piezoelectric material. Acta Mech Sin 34:192–199 (in Chinese)
Zhang T-Y, Qian C-F, Tong P (1998) Linear electro-elastic analysis of a cavity or a crack in a piezoelectric material. Int J Solids Struct 35:2121–2149
Zhang TY, Zhao M, Tong P (2001) Fracture of piezoelectric ceramics. Adv Appl Mech 38:147–289
Zhong Z, Meguid SA (1997) Interfacial debonding of a circular inhomogeneity in piezoelectric materials. Int J Eng Sci 34:1965–1984
Zhou K-L, Zhou Z-D, Kuang Z-B (2005a) Surface electrode problems in piezoelectric materials. Adv Mater Res 9:191–198
Zhou Z-D, Zhao S-X, Kuang Z-B (2005b) Stress and electric displacement analysis in piezoelectric media with an elliptic hole and a small crack. Int J Solids Struct 42:2803–2822
Zhou Z-D, Zhao S-X, Kuang Z-B (2008) Electroelastic analysis of bi-piezoelectrics embedded interfacial rigid lines with generalized piezoelectric dislocation. Acta Mech Solida Sin 29:1–8

Chapter 5
Some Problems in More Complex Materials with Defects

Abstract In this chapter some electroelastic problems in more complex materials with defects are discussed. It is pointed out that the electroelastic analysis for electrostrictive materials, the entire system including the dielectric medium, its environment, and their common boundary should be considered together. So the Maxwell stress should be considered. The theory illustrated in this chapter is an important complement for the present theory published in literatures. The electroelastic analyses of an infinite isotropic electrostrictive material containing an elliptic hole, containing a crack with and without local saturation electric field near the crack tip, are carried out. The basic theory of the thermo-electro-elastic analysis is given. An elliptic hole in a homogeneous pyroelectric material, interface crack in dissimilar pyroelectric material, point heat source, and its interaction with cracks are discussed. The electroelastic analyses of a functionally graded piezoelectric material are also introduced. These analyses are useful in engineering applications.

Keywords Electroelastic analysis • Electrostrictive material • Maxwell stress • Pyroelectric material • Functionally graded piezoelectric material

5.1 Isotropic Electrostrictive Material

5.1.1 Governing Equations

Some polyurethane elastomers and perovskite-type ceramics can produce large deformation under applied electric field. Their strains are proportional to the square of electric field and larger than $10^{-4}(\mathrm{m/mV})^2E^2$. The electrostrictive effect can occur in all dielectric, such as the electrostrictive ceramic PMN-PT, electrostrictive polymer EPs, and polyurethane PUE. The constitutive equation has been discussed in Sects. 2.2 and 2.6. In this section we only discuss the isotropic electrostrictive

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material occupying the region S. The environment occupies S^c. According to Eq. (2.27b) the constitutive equation with independent variables $(\boldsymbol{\varepsilon}, \boldsymbol{E})$ is

$$\begin{aligned} &\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - (1/2)(a_1 E_i E_j + a_2 E_k E_k \delta_{ij}) \\ &D_i = \tilde{\epsilon}_{ij} E_j, \quad \tilde{\epsilon}_{ij} = \epsilon\delta_{ij} + a_1\varepsilon_{ij} + a_2\varepsilon_{kk}\delta_{ij} \approx \epsilon\delta_{ij}, \quad E_i = -\varphi_{,i} \end{aligned} \tag{5.1}$$

where a_1, a_2 are electrostrictive coefficients. For electrostrictive materials the entire system including the dielectric medium, its environment, and their common boundary should be considered together, as shown in Sect. 2.2. The governing equations are

$$\begin{aligned} &S_{kl,l} + f_k = \rho\ddot{u}_k, \qquad D_{k,k} = \rho_{\mathrm{e}} \qquad \text{in material} \\ &S_{ij,j}^{\mathrm{env}} + f_i^{\mathrm{env}} = \rho\ddot{u}_i^{\mathrm{env}}, \quad D_{i,i}^{\mathrm{env}} = \rho_e^{\mathrm{env}}, \quad \text{in environment} \\ &S_{kl} = \sigma_{kl} + \sigma_{kl}^{\mathrm{M}} \approx= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - (1/2)(a_2 + \epsilon)E_i E_i \delta_{kl} + (1/2)(2\epsilon - a_1)E_k E_l \\ &\sigma_{ij}^{M} = E_i D_j - (1/2)E_m D_m \delta_{ij} \end{aligned} \tag{5.2}$$

where $\boldsymbol{S}$ is the pseudo total stress (Jiang and Kuang 2003, 2004). In isotropic case $\boldsymbol{S}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^M$ are all symmetric. The boundary conditions are

$$\begin{aligned} &S_{kl}n_l = T_k^*, \quad \text{on } a_\sigma; \quad D_k n_k = -\sigma^*, \quad \text{on} \quad a_{\mathrm{D}}; \quad u_i = u_i^*, \quad \text{on} \quad a_u; \quad \varphi = \varphi^*; \quad \text{on} \quad a_\varphi \\ &S_{ij}^{\mathrm{env}} n_j^{\mathrm{env}} = T_i^{*\mathrm{env}}, \quad \text{on} \quad a_\sigma^{\mathrm{env}}; \quad D_i^{\mathrm{env}} n_i^{\mathrm{env}} = -\sigma^{*\mathrm{env}}, \quad \text{on} \quad a_{\mathrm{D}}^{\mathrm{env}} \\ &u_i^{\mathrm{env}} = u_i^{*\mathrm{env}}, \quad \text{on} \quad a_u^{\mathrm{env}}; \quad \varphi^{\mathrm{env}} = \varphi^{*\mathrm{env}}, \quad \text{on} \quad a_\varphi^{\mathrm{env}} \end{aligned} \tag{5.3}$$

The interface conditions are

$$\left(S_{ij} - S_{ij}^{\mathrm{env}}\right)n_j = T_i^{*\mathrm{int}}, \quad \left(D_i - D_i^{\mathrm{env}}\right)n_i = -\sigma^{*\mathrm{int}}; \quad u_i = u_i^{\mathrm{env}}, \quad \varphi = \varphi^{\mathrm{env}}; \quad \text{on} \quad a^{\mathrm{int}} \tag{5.4}$$

For the ceramic material the difference between $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ is small, but for the electrostrictive polymer ϵ and $a_m\varepsilon_{ij}$ may be in the same order, and the difference between $\boldsymbol{S}$ and $\boldsymbol{\sigma}$ may not be small.

In the case of small strain, it is usually assumed that the electric field is approximately independent to the displacement, i.e., the terms containing strains in $\boldsymbol{D}$ in Eq. (5.1) can be neglected, but the stress field is related to the electric field. So the electric field is decoupled with the elastic field and can be solved independently (Knops 1963; Smith and Warren 1966; McMeeking 1989; Jiang and Kuang 2003, 2004). Assuming the air is charge free, from $\nabla \cdot \boldsymbol{D} = 0$, it is known that φ is a harmonic function, so it can be expressed by the real (or imaginary) part of a complex analytic function $w(z)$, i.e.,

$$w(z) = \varphi(x_1, x_2) + \mathrm{i}A(x_1, x_2), \quad \varphi(x_1, x_2) = \mathrm{Re}\,w(z) = \left[w(z) + \overline{w(z)}\right] \Big/ 2 \tag{5.5}$$

where A is called the stream function. Comparing Eqs. (3.83) and (5.5), it is found that $w(z) = 2\phi(z)$. These two expressions of the complex electric potential can all be found in literatures. It is noted that in this section $\phi(z)$ denotes the complex stress function. Using Cauchy-Riemann condition $\partial\varphi/\partial x_1 = \partial A/\partial x_2$, $\partial\varphi/\partial x_2 = -\partial A/\partial x_1$ yields

$$\begin{aligned}&\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}\varphi}{\mathrm{d}z} + \mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}z} = \left(\frac{\partial\varphi}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial\varphi}{\partial x_2}\frac{\partial x_2}{\partial z}\right) + \mathrm{i}\left(\frac{\partial A}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial A}{\partial x_2}\frac{\partial x_2}{\partial z}\right) = -\overline{E}\\ &E = E_1 + \mathrm{i}E_2 = -\overline{w'(z)}\end{aligned} \tag{5.6}$$

So the solution of the electric field is reduced to seek a function analytic in the region S.

On a boundary we have

$$\begin{aligned}\int D_n \mathrm{d}s &= \int (D_1 n_1 + D_2 n_2)\mathrm{d}s = \epsilon \int (E_1 \mathrm{d}x_2 - E_2 \mathrm{d}x_1)\\ &= (\mathrm{i}\epsilon/2)\int \left[w'(z)\mathrm{d}z - \overline{w'(z)\mathrm{d}z}\right] = (\mathrm{i}\epsilon/2)\left[w(z) - \overline{w(z)}\right]\end{aligned} \tag{5.7}$$

where a trivial integral constant is omitted.

For a plane strain problem, we have $\varepsilon_{i3} = 0, \quad (i = 1, 2, 3)$; the constitutive equation expressed by the pseudo total stress $\boldsymbol{S}$ is

$$\begin{aligned}&S_{\alpha\beta} = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2G\varepsilon_{\alpha\beta} + aE_\alpha E_\beta + bE_\gamma E_\gamma \delta_{\alpha\beta}\\ &a = (2\epsilon - a_1)/2, \quad b = -(a_2 + \epsilon)/2, \quad \alpha, \beta = 1, 2\end{aligned} \tag{5.8}$$

Using $\varepsilon_{\gamma\gamma} = \left[S_{\gamma\gamma} + (a - 2b)E_\gamma E_\gamma\right]/2(\lambda + G)$, Eq. (5.8) can also be written as

$$\begin{aligned}&2G\varepsilon_{\alpha\beta} = S_{\alpha\beta} - aE_\alpha E_\beta - \left[\lambda S_{\gamma\gamma} + (-\lambda a + 2Gb)E_\gamma E_\gamma\right]\delta_{\alpha\beta}/2(\lambda + G)\\ &\varepsilon_{\alpha\beta} = (1 + \nu)\left\{S_{\alpha\beta} - \nu S_{\gamma\gamma}\delta_{\alpha\beta} - aE_\alpha E_\beta + [\nu a - (1 - 2\nu)b]E_\gamma E_\gamma \delta_{\alpha\beta}\right\}/Y\end{aligned} \tag{5.9}$$

where $Y = 2G(1 + \nu)$ is the elastic modulus and ν is the Poisson ratio. Substitution of Eq. (5.9) into the compatible equation $2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}$ finally yields

$$\begin{aligned}2(S_{12} - aE_1E_2)_{,12} &= \left\{S_{11} - aE_1E_1 - \left[\lambda S_{\gamma\gamma} + (-\lambda a + 2Gb)E_\gamma E_\gamma\right]/2(\lambda + G)\right\}_{,22}\\ &+ \left\{S_{22} - aE_2E_2 - \left[\lambda S_{\gamma\gamma} + (-\lambda a + 2Gb)E_\gamma E_\gamma\right]/2(\lambda + G)\right\}_{,11}\end{aligned} \tag{5.10}$$

Let $\tilde{U}$ denote the pseudo total stress function satisfying the equilibrium equation automatically:

$$S_{11} = \tilde{U}_{,22}, \quad S_{22} = \tilde{U}_{,11}, \quad S_{12} = -\tilde{U}_{,12}, \quad \text{or} \quad S_{\alpha\beta} = \nabla^2\tilde{U}\delta_{\alpha\beta} - \tilde{U}_{,\alpha\beta} \tag{5.11}$$

Substituting Eq. (5.11) into Eq. (5.10), after some manipulation, yields

$$\nabla^4\tilde{U} = \kappa\nabla^2\left(E_\gamma E_\gamma\right), \quad \frac{\partial^4\tilde{U}}{\partial z^2\partial\bar{z}^2} = \frac{\kappa}{4}\frac{\partial^4}{\partial z^2\partial\bar{z}^2}[w'(z)\bar{w}'(\bar{z})] = \frac{\kappa}{4}\frac{\partial^2 w'}{\partial z^2}\frac{\partial^2\bar{w}'}{\partial\bar{z}^2}$$
$$\kappa = -G(a_1+2a_2)/2(\lambda+2G) = -(1-2v)(a_1+2a_2)/4(1-v) \tag{5.12}$$

Using the Muskhelishvili's formulas (1975), the general solution of Eq. (5.12) is

$$\tilde{U}(x_1,x_2) = (\kappa/4)w(z)\overline{w(z)} + (1/2)\left[z\overline{\phi(z)} + \bar{z}\phi(z) + \chi(z) + \overline{\chi(z)}\right] \tag{5.13}$$

where $(\kappa/4)w(z)\overline{w(z)}$ is the special solution; $\phi(z),\chi(z)$ are two analytic functions of z. Equation (5.11) yields

$$\begin{aligned} &S_{22}+S_{11} = \kappa w'(z)\overline{w'(z)} + 2\left[\phi'(z)+\overline{\phi'(z)}\right] \\ &S_{22}-S_{11}+2\mathrm{i}S_{12} = \kappa w''(z)\overline{w(z)} + 2[\bar{z}\phi''(z)+\psi'(z)], \quad \psi(z)=\chi'(z) \end{aligned} \tag{5.14}$$

From Eqs. (5.2) and (5.6), it is known that

$$\sigma_{22}^{\mathrm{M}}+\sigma_{11}^{\mathrm{M}} = 0, \quad \sigma_{22}^{\mathrm{M}}-\sigma_{11}^{\mathrm{M}}+2\mathrm{i}\sigma_{12}^{\mathrm{M}} = -\epsilon\Omega'(z), \quad \Omega'(z) = [w'(z)]^2 \tag{5.15}$$

The mechanical stresses are

$$\begin{aligned} &\sigma_{22}+\sigma_{11} = \kappa w'(z)\overline{w'(z)} + 2\left[\phi'(z)+\overline{\phi'(z)}\right] \\ &\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} = \kappa w''(z)\overline{w(z)} + 2[\bar{z}\phi''(z)+\psi'(z)] + \epsilon\Omega'(z) \end{aligned} \tag{5.16}$$

and displacements are

$$\begin{aligned} &2G(u_1+iu_2) = K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - (\kappa/2)w(z)\overline{w'(z)} + \alpha_1\overline{\Omega(z)} \\ &K = (3-4\nu), \quad \alpha_1 = (a_1-2\epsilon)/4; \quad \Omega(z) = \int\Omega'(z)\mathrm{d}z \end{aligned} \tag{5.17}$$

The stress boundary condition is

$$\begin{aligned} \mathrm{i}(P_1+iP_2) &= \mathrm{i}\int_A^B\left(\tilde{T}_1+\mathrm{i}\tilde{T}_2\right)\mathrm{d}s = 2\left[\partial\tilde{U}/\partial\bar{z}\right]_A^B \\ &= \left[z\overline{\phi'(z)} + \phi(z) + \overline{\psi(z)} + (1/2)\kappa w(z)\overline{w'(z)}\right]_A^B \end{aligned} \tag{5.18}$$

where A,B are two points on the boundary; P_1,P_2 are pseudo resultant forces; $\tilde{T}_i = S_{ij}n_j$.

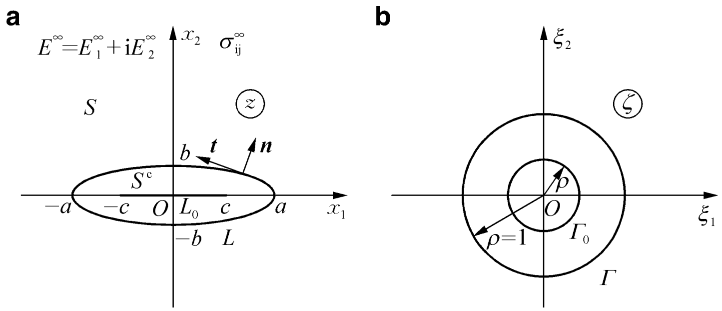


Fig. 5.1 A 2D plane electrostrictive material with an *elliptic hole* or inclusion: (**a**) physical plane z; (**b**) mapping plane ζ

5.1.2 An Impermeable Elliptic Hole in an Isotropic Electrostrictive Material

Let an isotropic electrostrictive material with an elliptic hole of semiaxes a and b directed along the material principle axes x_1 and x_2, respectively, filled by air. The uniform generalized stresses $\boldsymbol{\sigma}^\infty, \boldsymbol{E}^\infty$ are applied at infinity, but the boundary of the hole is free; see Fig. 5.1. A further assumption is that the electric field in the air will be neglected due to the small permittivity comparing with the electrostrictive material. Therefore, in this simple case the Maxwell stress in the hole is neglected, and the electrostrictive material can be studied alone (Jiang and Kuang 2003; Kuang and Jiang 2006). The boundary conditions are

$$
\begin{aligned}
&S_{ij} = S_{ij}^\infty, \quad E = E_1 + \mathrm{i}E_2 = E^\infty, \text{ at infinity}; \quad S_{ij}n_j = 0, \quad D_n = 0, \text{ on interface} \\
&S_{ij}^\infty = \sigma_{ij}^\infty + \sigma_{ij}^{\mathrm{M}\infty}; \quad \sigma_{ij}^{\mathrm{M}\infty} = E_i^\infty D_j^\infty - (1/2)E_m^\infty D_m^\infty \delta_{ij} \\
&E^\infty = E_0 \mathrm{e}^{\mathrm{i}\beta}, \quad E_0 = \sqrt{\left(E_1^\infty\right)^2 + \left(E_2^\infty\right)^2}, \quad \tan\beta = E_2^\infty / E_1^\infty
\end{aligned}
\tag{5.19}
$$

Electric field The mapping function method is used to solve this problem. The mapping function $z = \omega(\varsigma)$ shown in Eq. (3.82a) is still adopted. In ς plane the general solution of $w(\varsigma)$ can be written as

$$
\begin{aligned}
&w(\varsigma) = -\bar{E}^\infty R\left(\varsigma + \alpha\varsigma^{-1}\right) = -R\left(\bar{E}^\infty \varsigma + E^\infty \varsigma^{-1}\right); \quad \alpha = E^\infty / \bar{E}^\infty = \mathrm{e}^{2\mathrm{i}\beta} \\
&E = E_1 + \mathrm{i}E_2 = -\left[\overline{w'(\varsigma)} \Big/ \overline{\omega'(\varsigma)}\right] = E^\infty \frac{1 - \bar{\alpha}\bar{\varsigma}^{-2}}{1 - m\bar{\varsigma}^{-2}} = E^\infty \frac{\bar{\varsigma}^2 - \bar{\alpha}}{\bar{\varsigma}^2 - m} = \frac{E^\infty - \bar{E}^\infty \bar{\varsigma}^{-2}}{1 - m\bar{\varsigma}^{-2}} \\
&z = \omega(\varsigma) = R(\varsigma + m/\varsigma); \quad R = (a+b)/2, \quad m = (a-b)/(a+b)
\end{aligned}
\tag{5.20}
$$

$w(\varsigma)$ expressed by Eq. (5.20) satisfies the boundary at infinity and on the interface. In fact on the interface, we have

$$E_n + \mathrm{i}E_t = \frac{|\mathrm{d}z|}{\mathrm{d}z}(E_1 + \mathrm{i}E_2) = \frac{\bar{\sigma}}{|\sigma|}\frac{\overline{\omega'(\sigma)}}{|\omega'(\sigma)|}\frac{E^\infty - \bar{E}^\infty\sigma^2}{1 - m\sigma^2} = \frac{E^\infty\bar{\sigma} - \bar{E}^\infty\sigma}{|(1 - m\bar{\sigma}^2)|}$$
$$E_n = 0, \quad E_t = 2E_0\sin(\beta - \vartheta)\big/\left|1 - m\mathrm{e}^{-2\mathrm{i}\vartheta}\right|, \quad \Rightarrow \quad D_n = 0$$

Stress field The general solutions of complex potentials $\phi(\varsigma), \psi(\varsigma)$ can be assumed as

$$\phi(\varsigma) = \Gamma_1 R\varsigma + \phi_0(\varsigma), \quad \psi(\varsigma) = \Gamma_2 R\varsigma + \psi_0(\varsigma) \tag{5.21}$$

where $\phi_0(\varsigma), \ \psi_0(\varsigma)$ are undetermined functions analytic in S; Γ_1, Γ_2 are determined by

$$\Gamma_1 = \left(S_{22}^\infty + S_{11}^\infty - \kappa E_k^\infty E_k^\infty\right)/4, \quad \Gamma_2 = \left(S_{22}^\infty - S_{11}^\infty + 2iS_{12}^\infty\right)/2 \tag{5.22}$$

Because the boundary of the hole is free, the boundary condition Eq. (5.18) becomes

$$\omega(\sigma)\overline{\phi'(\sigma)}\Big/\overline{\omega'(\sigma)} + \phi(\sigma) + \overline{\psi(\sigma)} + \kappa w(\sigma)\overline{w'(\sigma)}\Big/2\overline{\omega'(\sigma)} = 0 \tag{5.23}$$

Substitution of Eqs. (5.20) and (5.21) into Eq. (5.23) yields

$$\begin{aligned} &\omega(\sigma)\overline{\phi_0'(\sigma)}\Big/\overline{\omega'(\sigma)} + \phi_0(\sigma) + \overline{\psi_0(\sigma)} + f(\sigma) = 0 \\ &f(\sigma) = R\Gamma_1\frac{\sigma^2 + m}{\sigma(1 - m\sigma^2)} + R\Gamma_1\sigma + \frac{R\bar{\Gamma}_2}{\sigma} + \frac{\kappa RE^\infty\bar{E}^\infty(1 - \bar{\alpha}\sigma^2)(\alpha + \sigma^2)}{2\sigma(1 - m\sigma^2)} \end{aligned} \tag{5.24}$$

Multiplying Eq. (5.24) and its conjugate equation by $\mathrm{d}\sigma/[2\pi\mathrm{i}(\sigma - \varsigma)]$ and using the Cauchy integral formulas we find

$$\begin{aligned} \phi_0(\varsigma) &= \frac{1}{2\pi\mathrm{i}}\int\frac{f(\sigma)\mathrm{d}\sigma}{\sigma - \varsigma} = -\frac{mR\Gamma_1}{\varsigma} - \frac{R\bar{\Gamma}_2}{\varsigma} - \frac{\kappa\alpha RE^\infty\bar{E}^\infty}{2\varsigma} \\ \psi_0(\varsigma) &= \frac{1}{2\pi\mathrm{i}}\int\frac{\overline{f(\sigma)}d\sigma}{\sigma - \varsigma} - \varsigma\frac{1 + m\varsigma^2}{\varsigma^2 - m}\phi_0'(\varsigma) \\ &= -2R\Gamma_1\frac{(1 + m^2)\varsigma}{\varsigma^2 - m} - \frac{R\bar{\Gamma}_2(1 + m\varsigma^2)}{(\varsigma^2 - m)\varsigma} - \frac{\kappa RE^\infty\bar{E}^\infty[1 - \alpha\bar{\alpha} + (\alpha + \bar{\alpha})m]\varsigma}{2(\varsigma^2 - m)} \end{aligned} \tag{5.25}$$

Substitution of Eqs. (5.21), (5.22), and (5.25) into Eq. (5.16) yields the stresses

$$\begin{aligned} \sigma_{22} + \sigma_{11} &= \kappa\frac{w'(\varsigma)}{\omega'(\varsigma)}\frac{\overline{w'(\varsigma)}}{\overline{\omega'(\varsigma)}} + 2\left[\frac{\phi'(\varsigma)}{\omega'(\varsigma)} + \frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}\right] \\ &= \kappa E^\infty\bar{E}^\infty\frac{\varsigma^2 - \alpha}{\varsigma^2 - m}\frac{\bar{\varsigma}^2 - \bar{\alpha}}{\bar{\varsigma}^2 - m} - 2\mathrm{Re}\left[\frac{\kappa\alpha E^\infty\bar{E}^\infty + 2\Gamma_1(\varsigma^2 + m) + 2\bar{\Gamma}_2}{\varsigma^2 - m}\right] \end{aligned} \tag{5.26a}$$

$$
\begin{aligned}
\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} &= \kappa\frac{\mathrm{d}}{\mathrm{d}\varsigma}\left[\frac{w'(\varsigma)}{\omega'(\varsigma)}\right]\overline{w(\varsigma)}+2\left\{\frac{\mathrm{d}}{\mathrm{d}\varsigma}\left[\frac{\phi'(\varsigma)}{\omega'(\varsigma)}\right]\overline{\omega'(\varsigma)}+\frac{\psi'(\varsigma)}{\omega'(\varsigma)}\right\}+\epsilon\left[\frac{w'(\varsigma)}{\omega'(\varsigma)}\right]^2 \\
&= \frac{2\kappa E^\infty\bar{E}^\infty(\alpha-m)\varsigma^3(\bar{\varsigma}^2+\bar{\alpha})}{(\varsigma^2-m)^3\bar{\varsigma}}+2\left\{\frac{-(\kappa\alpha E^\infty\bar{E}^\infty+4m\Gamma_1+2\bar{\Gamma}_2)\varsigma^3(\bar{\varsigma}^2+m)}{(\varsigma^2-m)^3\bar{\varsigma}}\right. \\
&\quad+\frac{\Gamma_2\varsigma^2}{\varsigma^2-m}+\frac{\kappa E^\infty\bar{E}^\infty[1-\alpha\bar{\alpha}+(\alpha+\bar{\alpha})m]\varsigma^2(\varsigma^2+m)}{2(\varsigma^2-m)^3}+\frac{2\Gamma_1(1+m^2)\varsigma^2(\varsigma^2+m)}{(\varsigma^2-m)^3} \\
&\quad\left.+\frac{\bar{\Gamma}_2[m\varsigma^4+(m^2+3)\varsigma^2-m]}{(\varsigma^2-m)^3}\right\}+\frac{\epsilon(E^\infty)^2(\varsigma^2+1)^2}{(\varsigma^2-m)^2}
\end{aligned} \tag{5.26b}
$$

Asymptotic fields near the end of a narrow elliptic hole under the electric load As in Sect. 3.4.6 the asymptotic stress fields near the end of a narrow elliptic hole only under an electric load in the local coordinate system with the origin at the focus of the ellipse are

$$
\begin{aligned}
&\sigma_{22}+\sigma_{11}\approx\kappa E^\infty\bar{E}^\infty(1-\alpha)(1-\bar{\alpha})c/4r \\
&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} \\
&\quad\approx\kappa E^\infty\bar{E}^\infty\left\{(1-\alpha)[2(\epsilon/\kappa)(1-\alpha)/\alpha-(1-\bar{\alpha})]+2(1-\alpha)(1-\bar{\alpha})\sqrt{\rho_0/r}\right\}c/8r
\end{aligned} \tag{5.27a}
$$

The electric asymptotic field is

$$
E_1+\mathrm{i}E_2=(1/4)\left\{\sqrt{2}\bar{E}^\infty(1-\bar{\alpha})\sqrt{c/r}+E^\infty\left((3+\bar{\alpha})+4\bar{\alpha}\sqrt{\rho_0/r}\right)\right. \tag{5.27b}
$$

where $\rho_0=b^2/2a$, $c=\sqrt{a^2-b^2}$.

5.1.3 The Permeable Elliptic Hole

For a permeable elliptic hole, the electric connective conditions in Eq. (5.19) are changed to

$$
\varphi=\varphi^{\mathrm{c}},\quad D_n=D_n^{\mathrm{c}}\quad or\quad \int D_n\mathrm{d}s=\int D_n^{\mathrm{c}}\mathrm{d}s,\quad D_n=D_in_i \tag{5.28}
$$

According to the previous knowledge, it is assumed prior that the electric field in the air is constant (Smith and Warren 1966, 1968; Gao et al. 2010), and the complex electric potential in the media $w(z)$ is in the following form:

$$
\begin{aligned}
&\varphi^{\mathrm{c}}=\mathrm{Re}\,w^{\mathrm{c}}(z)=-E_1^{\mathrm{c}}x_1-E_2^{\mathrm{c}}x_2 \\
&w(z)=\Gamma_3z+w_0(z),\quad \Gamma_3=-\overline{E}^\infty=-\left(E_1^\infty-\mathrm{i}E_2^\infty\right)
\end{aligned} \tag{5.29}
$$

where $w_0(z)$ is an unknown function analytic in S. Substituting Eqs. (5.5), (5.6), and (5.29) into Eq. (5.28) and using Eq. (5.7) we get

$$\begin{aligned} w_0(\sigma)+\overline{w_0(\sigma)} &= 2\left[\left(E_1^{\infty}-E_1^{\mathrm{c}}\right)x_1+\left(E_2^{\infty}-E_2^{\mathrm{c}}\right)x_2\right] \\ w_0(\sigma)-\overline{w_0(\sigma)} &= -2\mathrm{i}\left[\left(D_2^{\infty}-D_2^{\mathrm{c}}\right)x_1-\left(D_1^{\infty}-D_1^{\mathrm{c}}\right)x_2\right]/\epsilon \end{aligned} \tag{5.30}$$

Substituting $x_1=a(\sigma+\sigma^{-1})/2,\ x_2=\mathrm{i}b(\sigma-\sigma^{-1})/2$ and multiplying $\mathrm{d}\sigma/[2\pi\mathrm{i}(\sigma-\varsigma)]$ to two sides of Eq. (5.30) and then integrating the result identity we get

$$\begin{aligned} w_0(\varsigma) &= \left[a\left(E_1^{\infty}-E_1^{\mathrm{c}}\right)+\mathrm{i}b\left(E_2^{\infty}-E_2^{\mathrm{c}}\right)\right]/\varsigma \\ w_0(\varsigma) &= \left[-\mathrm{i}a\left(D_2^{\infty}-D_2^{\mathrm{c}}\right)-b\left(D_1^{\infty}-D_1^{\mathrm{c}}\right)\right]/\epsilon\varsigma \end{aligned} \tag{5.31}$$

Equation (5.31) yields

$$a\left(E_1^{\infty}-E_1^{\mathrm{c}}\right)=-b\left(D_1^{\infty}-D_1^{\mathrm{c}}\right)/\epsilon,\quad b\left(E_2^{\infty}-E_2^{\mathrm{c}}\right)=-a\left(D_2^{\infty}-D_2^{\mathrm{c}}\right)/\epsilon \tag{5.32}$$

So, using $D_1^{\infty}=\epsilon E_1^{\infty}, D_1^{c}=\epsilon^{c} E_1^{c}$ we obtain

$$\begin{aligned} & D_1^{\mathrm{c}}=D_1^{\infty}\bar{\epsilon}^{\mathrm{c}}(1+\bar{b})(1+\bar{\epsilon}^{\mathrm{c}}\bar{b})^{-1},\quad D_2^{\mathrm{c}}=D_2^{\infty}\bar{\epsilon}^{\mathrm{c}}(1+\bar{b})(\bar{\epsilon}^{\mathrm{c}}+\bar{b})^{-1} \\ & w^{\mathrm{c}}(z)=(1+\bar{b})\left[-E_1^{\infty}(1+\bar{\epsilon}^{\mathrm{c}}\bar{b})^{-1}+\mathrm{i}E_2^{\infty}(\bar{\epsilon}^{\mathrm{c}}+\bar{b})^{-1}\right]z,\quad \sqrt{m}\le|\varsigma|\le 1 \\ & w(\varsigma)=-RE_1^{\infty}(\varsigma+A/\varsigma)+\mathrm{i}RE_2^{\infty}(\varsigma-B/\varsigma),\qquad |\varsigma|\ge 1 \\ & \bar{b}=b/a,\quad \bar{\epsilon}^{\mathrm{c}}=\epsilon^{\mathrm{c}}/\epsilon,\quad A=(1-\bar{b}\bar{\epsilon}^{\mathrm{c}})/(1+\bar{b}\bar{\epsilon}^{\mathrm{c}}),\quad B=(\bar{b}-\bar{\epsilon}^{\mathrm{c}})/(\bar{b}+\bar{\epsilon}^{\mathrm{c}}) \end{aligned} \tag{5.33}$$

Especially for a crack ($b=0$) we have

$$E_1^{\infty}=E_1^{\mathrm{c}},\quad D_2^{\infty}=D_2^{\mathrm{c}},\quad w^{\mathrm{c}}(z)=\left(-E_1^{\infty}+\mathrm{i}E_2^{\infty}\epsilon/\epsilon^{\mathrm{c}}\right)z,\quad w(z)=\left(-E_1^{\infty}+\mathrm{i}E_2^{\infty}\right)z$$

It means that the electric fields are homogeneous in the crack, but with different constant values. When $E_1^{\infty}=0$, the electric asymptotic field near the right crack tip is

$$E_2=E_2^{\infty}\frac{\varsigma^2+(1-\bar{\delta})/(1+\bar{\delta})}{\varsigma^2-m}\approx E_2^{\infty}\left[\frac{1}{1+\bar{\delta}}\sqrt{\frac{a}{2r}}e^{-\mathrm{i}\theta/2}+\frac{1+2\bar{\delta}}{2(1+\bar{\delta})}\right] \tag{5.34}$$

where $\bar{\delta}=\bar{\epsilon}^{\mathrm{c}}/\bar{b}=(\epsilon^{\mathrm{c}}a)/(\epsilon b)$ is an important parameter; r,θ are polar coordinates in the local coordinate system with the origin at the focus of the ellipse.

The complex stress functions are still expressed by Eq. (5.21). For $E_1^\infty = 0$ finally we find

$$\begin{aligned}
\phi(\varsigma) &= R\left\{\Gamma_1\varsigma - \frac{m\Gamma_1}{\varsigma} - \frac{\bar{\Gamma}_2}{\varsigma} + \frac{\kappa B E_2^{\infty 2}}{2\varsigma}\right\} \\
\psi(\varsigma) &= R\left\{\Gamma_2\varsigma - 2\Gamma_1\frac{1+m^2}{\varsigma^2-m}\varsigma - \frac{\bar{\Gamma}_2(1+m\varsigma^2)}{(\varsigma^2-m)\varsigma} - \frac{\kappa E_2^{\infty 2}(1-2mB-B^2)\varsigma}{2(\varsigma^2-m)}\right\}
\end{aligned} \tag{5.35}$$

The stress is obtained by Eq. (5.16).

5.1.4 A Rigid Elliptic Conduction Inclusion

In this section we shall discuss a rigid elliptic conducting inclusion with boundary L in an isotropic electrostrictive material (Jiang and Kuang 2004). In this case the problem can be discussed independently in the material region Ω and the boundary conditions are assumed:

$$\begin{aligned}
&\sigma_{ij} = \sigma_{ij}^\infty, \quad E = E_1 + \mathrm{i}E_2 = E^\infty = E_0\mathrm{e}^{\mathrm{i}\beta}, \quad \text{when} \quad x_1^2 + x_2^2 \to \infty \\
&u_1 = u_1^{\mathrm{c}} = -\omega^{\mathrm{c}} x_2, \quad u_2 = u_2^{\mathrm{c}} = \omega^{\mathrm{c}} x_1; \quad \varphi = 0; \quad \text{on} \quad L \\
&E_0 = \sqrt{\left(E_1^\infty\right)^2 + \left(E_2^\infty\right)^2}, \quad \tan\beta = E_2^\infty / E_1^\infty
\end{aligned} \tag{5.36}$$

where ω^{c} is the rotation angle about axis x_3 of the inclusion. The pseudo total moment $\tilde{M}$, Maxwell stress moment M^e, and mechanical moment M are, respectively,

$$\begin{aligned}
\tilde{M} &= \int_A^B \left(-\tilde{T}_1 x_2 + \tilde{T}_2 x_1\right)\mathrm{d}s = -\left[z\left(\partial\tilde{U}/\partial z\right) + \bar{z}\left(\partial\tilde{U}/\partial\bar{z}\right)\right]_A^B + \tilde{U}_A^B \\
&= \mathrm{Re}\left[\chi(z) - z\psi(z) - z\bar{z}\phi'(z) - (1/2)\kappa w'(z)\overline{w(z)}\right]_A^B + \left[(1/4)\kappa w(z)\overline{w(z)}\right]_A^B \\
M^e &= \int_A^B \left(-\sigma_{1j}^M n_j x_2 + \sigma_{2j}^M n_j x_1\right)\mathrm{d}s = \mathrm{Re}\{(1/2)\epsilon[z\Omega(z) - \Omega_1(z)]\} \\
M &= \tilde{M} - M^e = \mathrm{Re}\Big\{\chi(z) - z\psi(z) - z\bar{z}\phi'(z) - (1/2)\kappa w'(z)\overline{w(z)} \\
&\qquad -(1/2)\epsilon[z\Omega(z) - \Omega_1(z)]\Big\}_A^B + \left[(1/4)\kappa w(z)\overline{w(z)}\right]_A^B, \quad \Omega_1(z) = \int \Omega(z)\mathrm{d}z
\end{aligned} \tag{5.37}$$

When there are no body force and free charge, the stress complex potential can be assumed as

$$\begin{aligned}
&w(z) = \Gamma_3 z + w_0(z) \\
&\phi(z) = -[1/8(1-\nu)](\tilde{T}_1 + \mathrm{i}\tilde{T}_2)\ln z + \Gamma_1 z + \phi_0(z) \\
&\psi(z) = [(3-4\nu)/8(1-\nu)]\left(\tilde{T}_1 - \mathrm{i}\tilde{T}_2\right)\ln z + \Gamma_2 z + \psi_0(z) \\
&\Gamma_3 = -\bar{E}^\infty, \quad E^\infty = \left(E_1^\infty + \mathrm{i}E_2^\infty\right) = E_0 \mathrm{e}^{\mathrm{i}\beta} \\
&\Gamma_1 = (1/4)\left(S_{22}^\infty + S_{11}^\infty\right) - (1/4)\kappa E_k^\infty E_k^\infty, \quad \Gamma_2 = (1/2)\left(S_{22}^\infty - S_{11}^\infty + 2\mathrm{i}S_{12}^\infty\right)
\end{aligned} \tag{5.38}$$

where $w_0(z), \phi_0(z), \psi_0(z)$ are complex functions analytic in the region S. $\tilde{T}_i$ is the generalized concentrate force, which is zero in present case, so the terms containing $\ln z$ will be omitted in later.

The conformal mapping method is used to solve the problem. The mapping function is shown in Eq. (3.82). It is easy to prove that the electric field in S can be obtained by changing α to $(-\alpha)$ in Eq. (5.20) discussed in Sect. 5.1.2, i.e.,

$$\begin{aligned}
&w(\varsigma) = -R\bar{E}^\infty\left(\varsigma - \alpha\varsigma^{-1}\right), \quad \overline{w(\varsigma)} = -RE^\infty\left(\bar{\varsigma} - \alpha\bar{\varsigma}^{-1}\right), \quad \alpha = E^\infty/\bar{E}^\infty = \mathrm{e}^{2\mathrm{i}\beta} \\
&E = E_1 + \mathrm{i}E_2 = -\frac{\overline{w'(\varsigma)}}{\overline{\omega'(\varsigma)}} = E^\infty \frac{1+\bar{\alpha}\bar{\varsigma}^{-2}}{1-m\bar{\varsigma}^{-2}} = E^\infty \frac{\bar{\varsigma}^2 + \bar{\alpha}}{\bar{\varsigma}^2 - m} = \frac{E^\infty + \bar{E}^\infty \bar{\varsigma}^{-2}}{1-m\bar{\varsigma}^{-2}}
\end{aligned} \tag{5.39}$$

Using Eq. (5.17) the displacement boundary condition in Eq. (5.36) can be expressed as

$$2\mathrm{i}G\omega^c z = K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - \frac{\kappa}{2}w(z)\overline{w'(z)} + \alpha_1\overline{\Omega(z)} \tag{5.40}$$

On the mapping plane Eq. (5.40) becomes

$$\Lambda\phi(\varsigma) + \omega(\varsigma)\frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}} + \overline{\psi(\varsigma)} + \frac{\kappa}{2}w(\varsigma)\frac{\overline{w'(\varsigma)}}{\overline{\omega'(\varsigma)}} - \alpha_1\overline{\Omega(\varsigma)} = -2\mathrm{i}G\omega^c\omega(\varsigma) \tag{5.41}$$

where $\Lambda = -K = -3 + 4\nu$ and $\phi(\varsigma)$ and $\psi(\varsigma)$ are given in Eq. (5.38). Noting

$$\begin{aligned}
&\Omega(\varsigma) = \int \frac{[w'(\varsigma)]^2}{\omega'(\varsigma)}\mathrm{d}\varsigma = R(\bar{E}^\infty)^2\left[\frac{\alpha^2}{m\varsigma} + \varsigma - \frac{(m+\alpha)^2\arctan(\varsigma/\sqrt{m})}{m^{3/2}}\right] \\
&\frac{1}{2\pi\mathrm{i}}\int \frac{\Omega(\sigma)\mathrm{d}\sigma}{\sigma - \varsigma} = R\left(\overline{E}^\infty\right)^2\left[-\frac{\alpha^2}{m\varsigma} - \frac{(m+\alpha)^2}{2m^{3/2}}\ln\frac{\varsigma - \sqrt{m}}{\varsigma + \sqrt{m}}\right] \\
&\frac{1}{2\pi\mathrm{i}}\int \frac{\overline{\Omega(\sigma)}\mathrm{d}\sigma}{\sigma - \varsigma} = -R\left(\overline{E}^\infty\right)^2\frac{1}{\varsigma}
\end{aligned} \tag{5.42}$$

The future process to solve the problem is fully similar to that in Sect. 5.1.2. Finally we obtain

$$\begin{aligned}
\Lambda\phi(\varsigma) &= \Lambda\Gamma_1 R\varsigma - \frac{mR\Gamma_1}{\varsigma} - \frac{R\bar{\Gamma}_2}{\varsigma} + \frac{\kappa\alpha RE^\infty\bar{E}^\infty}{2\varsigma} + \alpha_1 R(E^\infty)^2\frac{1}{\varsigma} - 2\mathrm{i}RG\omega^{\mathrm{c}}\frac{m}{\varsigma} \\
\psi(\varsigma) &= \Gamma_2 R\varsigma + \frac{\kappa RE^\infty\bar{E}^\infty[\alpha\bar{\alpha} - 1 + (\alpha/\Lambda + \bar{\alpha})m]\varsigma^2 + \alpha/\Lambda - \alpha}{2\varsigma(\varsigma^2 - m)} - \frac{R\bar{\Gamma}_2(1 + m\varsigma^2)}{\Lambda(\varsigma^2 - m)\varsigma} \\
&\quad - \frac{R\Gamma_1[(1 + m^2 + \Lambda + m^2/\Lambda)]\varsigma^2 - \Lambda m + m/\Lambda}{\varsigma(\varsigma^2 - m)} + \frac{\alpha_1 R(E^\infty)^2(1 + m\varsigma^2)}{\Lambda\varsigma(\varsigma^2 - m)} \\
&\quad + \frac{2\mathrm{i}RG\omega^{\mathrm{c}}}{\varsigma} - \frac{2\mathrm{i}RG\omega^{\mathrm{c}}(1 + m\varsigma^2)}{\Lambda\varsigma(\varsigma^2 - m)} + \alpha_1 R(\bar{E}^\infty)^2\left[\frac{\alpha^2}{m\varsigma} + \frac{(m+\alpha)^2}{2m^{3/2}}\ln\frac{\varsigma - \sqrt{m}}{\varsigma + \sqrt{m}}\right]
\end{aligned} \tag{5.43}$$

If ω^{c} is given, the mechanical moment acting on the inclusion can be determined by Eq. (5.37), or

$$\begin{aligned}
M = \mathrm{Re}\Big\{&\chi(\varsigma) - \omega(\varsigma)\psi(\varsigma) - \omega(\varsigma)\overline{\omega(\varsigma)}\phi'(\varsigma)/\omega'(\varsigma) - (1/2)\kappa w'(\varsigma)/\omega'(\varsigma)\overline{w(\varsigma)} \\
&-(1/2)\epsilon[\omega(\varsigma)\Omega(\varsigma) - \Omega_1(\varsigma)]\Big\}_A^B + \left[(1/4)\kappa w(\varsigma)\overline{w(\varsigma)}\right]_A^B
\end{aligned} \tag{5.44}$$

In Eq. (5.44) points A and B are the same point, so only multiple value terms containing $\ln\varsigma$ are not zero, i.e., only should keep terms containing $\chi(\varsigma)$ and $\Omega_1(\varsigma)$. From the second equation in Eq. (5.43) we get

$$\begin{aligned}
\chi(\varsigma) &= \oint \psi(\varsigma)\omega'(\varsigma)\mathrm{d}\varsigma \\
&= \Gamma_2 R^2\left(\frac{1}{2}\varsigma^2 - m\ln\varsigma\right) + \frac{1}{2}\kappa R^2 E^\infty\bar{E}^\infty\left[\left(\alpha\bar{\alpha} - 1 + \frac{\alpha m}{\Lambda} + \bar{\alpha}m\right)\ln\varsigma + \frac{\alpha\Lambda - \alpha}{2\Lambda\varsigma^2}\right] \\
&\quad - \frac{R^2\bar{\Gamma}_2}{\Lambda}\left(m\ln\varsigma - \frac{1}{2\varsigma^2}\right) - R^2\Gamma_1\left[\left(1 + m^2 + \Lambda + \frac{m^2}{\Lambda}\right)\ln\varsigma - \left(-\Lambda m + \frac{m}{\Lambda}\right)\frac{1}{2\varsigma^2}\right] \\
&\quad + \frac{\alpha_1 R^2(E^\infty)^2}{\Lambda}\left(m\ln\varsigma - \frac{1}{2\varsigma^2}\right) + \alpha_1 R^2(E^\infty)^2\Bigg\{\alpha^2\left(\frac{1}{2\varsigma^2} + \frac{\ln\varsigma}{m}\right) \\
&\quad + \frac{(m+\alpha)^2}{2m^{3/2}}\left[2\sqrt{m}\ln\varsigma + \frac{m + \varsigma^2}{\varsigma}\ln\frac{\varsigma - \sqrt{m}}{\varsigma + \sqrt{m}} - 2\sqrt{m}\ln\left(\varsigma^2 - m\right)\right]\Bigg\} \\
&\quad + 2\mathrm{i}R^2 G\omega^{\mathrm{c}}\left(\ln\varsigma + \frac{m}{2\varsigma^2}\right) - \frac{2\mathrm{i}R^2 G\omega^{\mathrm{c}} m}{\Lambda}\left(m\ln\varsigma - \frac{1}{2\varsigma^2}\right)
\end{aligned} \tag{5.45}$$

$$\begin{aligned}\Omega_1(\varsigma) &= \int \Omega(\varsigma)\omega'(\varsigma)\mathrm{d}\varsigma \\ &= \left(R\overline{E}^{\infty}\right)^2\left[\frac{\alpha^2}{2\varsigma}+\frac{\varsigma^2}{2}-\left(\frac{\alpha^2}{m}-m\right)\ln\varsigma\right] \\ &\quad -\left(R\overline{E}^{\infty}\right)^2\frac{(m+\alpha)^2}{2m^{3/2}}\left\{2\sqrt{m}\left[-\ln\varsigma+\ln\left(\varsigma^2-m\right)+\frac{m+\varsigma^2}{\varsigma}\ln\frac{\varsigma+\sqrt{m}}{-\varsigma+\sqrt{m}}\right]\right\}\end{aligned} \tag{5.46}$$

So we have

$$\begin{aligned}M =& -2\pi R^2\mathrm{Im}\{-\varGamma_2 m+(1/2)\kappa E^{\infty}\bar{E}^{\infty}(\alpha\bar{\alpha}-1+\alpha m/\varLambda+\bar{\alpha}m) \\ &-\varGamma_1\left(1+m^2+\varLambda+m^2/\varLambda\right)-m\bar{\varGamma}_2/\varLambda+(E^{\infty})^2\alpha_1 m/\varLambda \\ &-[\alpha_1(2\alpha+m)+\epsilon(\alpha+m)](\bar{E}^{\infty})^2+2\mathrm{i}G\omega^{\mathrm{c}}\left(\varLambda-m^2\right)/\varLambda\}\end{aligned} \tag{5.47}$$

Noting $\varGamma_1, m, \varLambda, G, \omega^{\mathrm{c}}$ are all real, Eq. (5.47) can be reduced to

$$\begin{aligned}M =& -2\pi R^2\mathrm{Im}\{-\varGamma_2 m+(1/2)\kappa E^{\infty}\bar{E}^{\infty}(\alpha\bar{\alpha}-1+\alpha m/\varLambda+\bar{\alpha}m)-m\bar{\varGamma}_2/\varLambda \\ &+\alpha_1 m(E^{\infty})^2/\varLambda-[\alpha_1(2\alpha+m)+\in(\alpha+m)](\bar{E}^{\infty})^2\}-4\pi R^2 G\omega^{\mathrm{c}}\left(\varLambda-m^2\right)/\varLambda\end{aligned} \tag{5.48}$$

If there is no moment acting on the inclusion, the ω^{c} is determined by the following equation:

$$\begin{aligned}\omega^{\mathrm{c}} =& \left[\varLambda/2G\left(m^2-\varLambda\right)\right]\mathrm{Im}\{-\varGamma_2 m+\frac{1}{2}\kappa E^{\infty}\bar{E}^{\infty}(\alpha\bar{\alpha}-1+\alpha m/\varLambda+\bar{\alpha}m)-m\bar{\varGamma}_2/\varLambda \\ &+\alpha_1 m(E^{\infty})^2/\varLambda-[\alpha_1(2\alpha+m)+\in(\alpha+m)](\bar{E}^{\infty})^2\}\end{aligned} \tag{5.49}$$

For a conductor ball $m=0$, from Eq. (5.49), it is seen that $\omega^{\mathrm{c}}=0$, i.e., there is no rotation. It is also noted that for $\beta=n\pi/2, n=1,2,3,4,\ \omega^{\mathrm{c}}=0$ for pure electric loading. ω^{c} is proportional to the square of the electric field and linear of the stress at infinity. Substituting ω^{c} into Eq. (5.43), the stress potentials are obtained and then the stresses are all obtained. The asymptotic field near the right end of a narrow rigid elliptic inclusion under an electric field at infinity is

$$\begin{aligned}&\sigma_{22}+\sigma_{11}=(1/8)[\kappa E^{\infty}\bar{E}^{\infty}(1+\alpha)(1+\bar{\alpha})](c/r) \\ &\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12}=\Big\{(1/8)(1+\alpha)^2\bar{E}^{\infty}[(2\alpha_1+\epsilon)\bar{E}^{\infty}-\kappa E^{\infty}] \\ &\qquad\qquad +(1/4)\kappa E^{\infty}\bar{E}^{\infty}(1+\alpha)(1+\bar{\alpha})\sqrt{\rho_0/r}\Big\}(c/r) \\ &E_1+\mathrm{i}E_2=\bar{E}^{\infty}\left(1/2m^{3/4}\right)\sqrt{R/r}\end{aligned} \tag{5.50}$$

Jiang and Kuang (2005, 2007) discussed a general elliptic inclusion. Liang et al. (1995) discussed piezoelectric materials with a general elliptic inclusion.

5.2 Cracked Infinite Electrostrictive Plate with Local Saturation Electric Field

5.2.1 The Constitutive Equations and Boundary Conditions

For an electrostrictive ceramic with a crack under external high electric field, the mechanical state near the crack tip is elastic, but the electric field may be saturated. Jiang and Kuang (2006) discussed an infinite plate with a central crack of length $2a$, subjected to the electric field $E^{\infty} = E_1^{\infty} + \mathrm{i}E_2^{\infty}$ at infinity. It is assumed that the electric field in the region S_0 of the plate is linear, but two zones S_{R} and S_{L} near the right and left crack tips are local small-scale saturated (Fig. 5.2). The constitutive equations for an isotropic electrostrictive material are

$$\begin{aligned} &\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2G\varepsilon_{ij} - \left(a_1 D_i D_j + a_2 D_k D_k \delta_{ij}\right)/\left(2\hat{\epsilon}^2\right) \\ &D = \hat{\epsilon}E, \quad \hat{\epsilon} = D(E)/E \end{aligned} \tag{5.51a}$$

where $D(E)$ is the uniaxial dielectric response in the absence of stress. Here it is assumed

$$\begin{aligned} &D_i = \left(\epsilon\delta_{ij} + a_1\varepsilon_{ij} + a_2\varepsilon_{kk}\delta_{ij}\right)E_j, \quad \text{when} \quad |E| = \sqrt{E_k E_k} < E_{\mathrm{c}} \\ &D_i = D_{\mathrm{c}}E_i/|E|, \quad \text{when} \quad |E| = \sqrt{E_k E_k} \geq E_{\mathrm{c}} \end{aligned} \tag{5.51b}$$

where D_{c} and E_{c} are the saturation electric displacement and saturation electric field, respectively. For linear case $\hat{\epsilon} = \epsilon$ is constant, but for the nonlinear case $\hat{\epsilon}$ may be dependent to electric field. If the electric field is linear, $\boldsymbol{\sigma}$ in Eq. (5.51a) can also be expressed by

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2G\varepsilon_{ij} - (a_1 E_i E_j + a_2 E_k E_k \delta_{ij})/2 \tag{5.51c}$$

The boundary condition of the problem is

$$\begin{aligned} &\sigma_{ij} = \sigma_{ij}^{\infty}, \quad E = E_1^{\infty} + \mathrm{i}E_2^{\infty} = E_0 e^{\mathrm{i}\beta}, \quad \text{when} \quad x_k x_k \to \infty \\ &D_2 = 0, \quad \text{on} \quad x_2 = 0, \quad -a < x_1 < a \end{aligned} \tag{5.52}$$

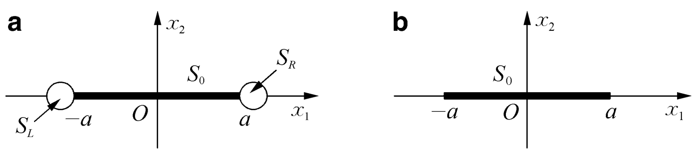


Fig. 5.2 An infinite plane with a *central crack* located at $(-a, a)$: (**a**) local small-scale saturation model at crack tips; (**b**) linear model

5.2.2 The Electric Field in an Electrostrictive Material with an Impermeable Crack

1. *The electric field in a linear plate without local saturation region*
According to Eq. (5.6) and approximately taking $\boldsymbol{D} = \epsilon \boldsymbol{E}$ we have

$$D_1 = -(1/2)\ \epsilon \left[\overline{w'(z)} + w'(z) \right], \quad D_2 = (1/2)\mathrm{i}\ \epsilon \left[\overline{w'(z)} - w'(z) \right] \tag{5.53}$$

where $w(z)$ is a complex potential shown in Eq. (5.5). On the crack surface

$$D_2^+ = D_2^- = 0, \quad \text{or} \quad \bar{w}'^+(x_1) - w'^+(x_1) = \bar{w}'^-(x_1) - w'^-(x_1) = 0 \tag{5.54}$$

Equation (5.54) yields

$$\begin{aligned} &[\bar{w}'(x_1) - w'(x_1)]^+ + [\bar{w}'(x_1) - w'(x_1)]^- = 0, \\ &[\bar{w}'(x_1) - w'(x_1)]^+ - [\bar{w}'(x_1) - w'(x_1)]^- = 0 \end{aligned} \tag{5.55}$$

This is a standard Hilbert problem. Noting Eq. (5.52) its solution is

$$\begin{aligned} &w'(z) = \frac{1}{2}(\Gamma_3 - \bar{\Gamma}_3)\frac{z}{\sqrt{z^2 - a^2}} + \frac{1}{2}(\Gamma_3 + \bar{\Gamma}_3) = \mathrm{i}E_2^\infty \frac{z}{\sqrt{z^2 - a^2}} - E_1^\infty \\ &w(z) = \mathrm{i}E_2^\infty \sqrt{z^2 - a^2} - E_1^\infty z; \quad \Gamma_3 = -\bar{E}^\infty \end{aligned} \tag{5.56}$$

The asymptotic field near the crack tip $z = a$ is

$$w'(z) = \frac{\mathrm{i}K_\mathrm{e}}{\sqrt{2\pi(z-a)}}, \quad E_1 - \mathrm{i}E_2 = -\frac{\mathrm{i}K_\mathrm{e}}{\sqrt{2\pi(z-a)}} = -\frac{\mathrm{i}K_\mathrm{e}}{\sqrt{2\pi r}}\mathrm{e}^{-\mathrm{i}\theta/2} \tag{5.57}$$

where $K_\mathrm{e} = E_2^\infty \sqrt{\pi a}$ is the electric field intensity factor, $z - a = re^{i\theta}$.

2. *The electric field in a plate with local saturation region*

The local saturation model of the electric field at the crack tip is similar to III-type yielding model in an elastoplastic material, so the method used in elastoplastic analysis can also be used here (Cherepanov 1979). The asymptotic solution near a tip of a central crack is the same as that in a semi-infinite crack problem. A local coordinate system Oy_i with the origin located at the crack tip (Fig. 5.3) is also used. A point in it is denoted by $y = y_1 + \mathrm{i}y_2 = z - a$. The boundary value problem is

$$\begin{aligned} &D_{i,i} = 0, \quad \text{when} \quad y \notin (y_2 = 0, -\infty < y_1 \le 0) \\ &D_2^\pm = 0, \quad \text{when} \quad y_2 = 0, \quad -\infty < y_1 \le 0 \\ &\sqrt{D_1^2 + D_2^2} = D_\mathrm{c}, \quad \text{when} \quad y \in S_\mathrm{R}; \quad D_1^2 + D_2^2 = 0, \quad \text{when} \quad y \to \infty \end{aligned} \tag{5.58}$$

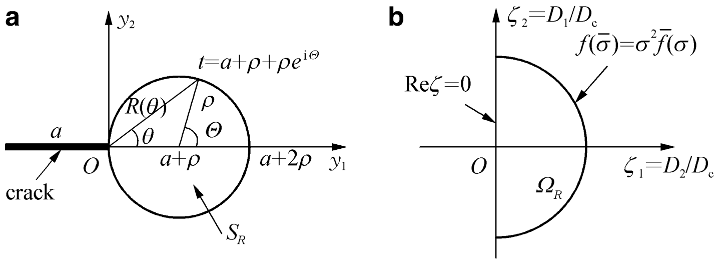


Fig. 5.3 The local saturation zone near crack tip: (**a**) physical plane z; (**b**) mapping plane

where the origin O is not included in S_R. Let

$$\varsigma = \varsigma_1 + \mathrm{i}\varsigma_2 = (D_2 + \mathrm{i}D_1)/D_\mathrm{c}, \quad \text{or} \quad \varsigma = (E_2 + \mathrm{i}E_1)/E_\mathrm{c} = -\mathrm{i}w'(z)/E_\mathrm{c} \tag{5.59}$$

According to Eqs. (5.57) and (5.58), it yields (Fig. 5.3)

$$D_1 = -D_\mathrm{c}\sin\theta, \quad D_2 = D_c\cos\theta, \quad D_2 + \mathrm{i}D_1 = D_\mathrm{c}\mathrm{e}^{-\mathrm{i}\theta}, \quad |\theta| \le \pi/2; \quad \text{in} \quad S_\mathrm{R} \tag{5.60}$$

According to Eqs. (5.58) and (5.60), the crack boundary $y_2 = 0,\ y_1 < 0$ in the y plane is transformed to $\theta = \pm\pi/2$ in the ς plane. Let $R(\theta)$ be the boundary of the saturation zone S_R; a point t on the boundary of S_R can be expressed as

$$t = R(\theta)\mathrm{e}^{\mathrm{i}\theta}; \quad \tan\theta = y_2/y_1, \quad y = y_1 + \mathrm{i}y_2 \tag{5.61}$$

According to Eq. (5.60) in the ς plane, the boundary of Ω_R is $\mathrm{e}^{-\mathrm{i}\theta} = \bar{\sigma}$. In order to simplify the problem, the hodograph transform method is used. The boundary value problem in the ς plane is

$$\begin{array}{lll} y_2 = 0, \quad -\infty < y_1 \le 0, & \text{when} & \mathrm{Re}\varsigma = 0 \\ y \in R, & \text{when} & \varsigma = \mathrm{e}^{-\mathrm{i}\theta} \\ y \to \infty, & \text{when} & \varsigma = 0 \end{array} \tag{5.62}$$

In the ς plane Eq. (5.62) shows that the zone Ω_R is constituted of a unit semicircle and a line segment $-1 \le \xi_2 \le 1$ on the image axis. The zone inside Ω_R is corresponding to the zone outside S_R. Now we shall solve the problem, Eq. (5.62), in the ς plane. Let

$$R(\theta) = \bar{\sigma}f(\bar{\sigma}) \tag{5.63}$$

where $f(\bar{\sigma})$ is an unknown function. Because $R(\theta)$ is real, so

$$\bar{\sigma}f(\bar{\sigma}) - \sigma\bar{f}(\sigma) = 0, \quad \text{or} \quad f(\bar{\sigma}) = \sigma^2\bar{f}(\sigma) \tag{5.64}$$

It is considered that the linear asymptotic solution Eq. (5.57) can approximately be used in the present problem, i.e., outside S_R the following relation is held:

$$y = f(\varsigma) = -K_e^2 \Big/ 2\pi(E_1 - iE_2)^2 = K_e^2 \Big/ 2\pi(E_c\varsigma)^2, \quad (E_c\varsigma)^2 = -(E_1 - iE_2)^2 \tag{5.65}$$

Equation (5.65) also satisfies the condition, $\varsigma = 0$, when $y \to \infty$.

From Eqs. (5.64) and (5.65), it is derived that outside the saturation zone we have

$$y = f(\varsigma) = \frac{K_e^2}{2\pi E_c^2}\left(1 + \varsigma^{-2}\right), \quad w'(z) = \frac{iK_e}{\sqrt{2\pi(z-a) - (K_e/E_c)^2}} \tag{5.66}$$

The boundary of the saturation zone S_R in the ς plane is

$$R(\theta) = \bar{\sigma} f(\bar{\sigma}) = \left(K_e^2/2\pi E_c^2\right)\bar{\sigma}\left(1 + \bar{\sigma}^{-2}\right) = 2\rho\cos\theta; \quad \rho = \left(K_e^2/2\pi E_c^2\right) \tag{5.67}$$

Equation (5.67) shows that the saturation zone S_R in the y plane is a circle with radius ρ. Equation (5.67) can also be obtained if in Eq. (5.57) let $E_1^2 + E_2^2 = E_c^2$.

From Eq. (5.66) it is found that the linear field in S_0 for a material with a saturation zone near the tip is the same as that in a material without a saturation zone, if we use the effective crack length a_{eff} instead of the real crack length a. It is just the method used in the elastoplastic fracture mechanics. The effective crack length is

$$a_{eff} = a + \rho, \quad \rho = K_e^2/2\pi E_c^2, \; K_e = \left(E_2^\infty\right)\sqrt{\pi(a+\rho)} \tag{5.68}$$

Using the above theory the electric field in S_0 for a central crack problem is

$$\begin{aligned} w(z) &= iE_2^\infty\sqrt{z^2 - (a+\rho)^2} - E_1^\infty z \\ E_2 + iE_1 &= E_2^\infty \frac{z}{\sqrt{z^2 - (a+\rho)^2}} - E_1^\infty \approx \frac{K_e}{\sqrt{\pi(a+\rho)}}\frac{z}{\sqrt{z^2 - (a+\rho)^2}} \end{aligned} \tag{5.69}$$

On the boundary of S_R we have $z = a + \rho + \rho e^{i\Theta} (\Theta = 2\theta)$ (Fig. 5.3a). Substituting it into Eq. (5.69) yields

$$E_2 + iE_1 = \frac{E_2^\infty\left(a + \rho + \rho e^{i\Theta}\right)}{\sqrt{(2a + 2\rho + \rho e^{i\Theta})\rho e^{i\Theta}}} \approx \frac{E_2^\infty\sqrt{a+\rho}}{\sqrt{2}}\sqrt{\frac{1}{\rho}}e^{-i\theta} = E_c e^{-i\theta} + \frac{3\rho}{4(a+\rho)}E_c e^{i\theta}$$

It is seen that on the interface the limit values of the electric field taken from S_0 and S_R are equal in the accuracy of $\rho/(a+\rho)$. Usually $\rho/(a+\rho) \ll 1$, so the above solution is reasonable.

5.2.3 The Stress in an Impermeable Crack with Local Saturation

1. *Stress in linear zone* S_0 This problem in S_0 is similar to that in Sect. 5.1.2. Let $b = 0, m = 1, R = a/2$ and use the effective crack length instead of the real crack length; the solution of the central crack problem can be obtained from the solution of an elliptic hole problem. Equations (5.21), (5.22), (5.23), and (5.24) are still appropriate here, but it should be used the electric field Eq. (5.69) instead of Eq. (5.20). According to above discussions in the ς plane, the stress potentials are determined by the following equations:

$$\begin{aligned}
&\omega(\sigma)\left[\overline{\phi'(\sigma)}\Big/\overline{\omega'(\sigma)}\right] + \phi(\sigma) + \overline{\psi(\sigma)} + (1/2)\kappa w(\sigma)\overline{w'(\sigma)} = 0 \\
&\phi(\varsigma) = \Gamma_1 R\varsigma + \phi_0(\varsigma), \quad \psi(\varsigma) = \Gamma_2 R\varsigma + \psi_0(\varsigma) \\
&w(z) = \mathrm{i}E_2^{\infty}\sqrt{z^2 - (a+\rho)^2} = \mathrm{i}E_2^{\infty}(a/2)\sqrt{[(\varsigma + \varsigma^{-1})]^2 - [2(a+\rho)/a]^2}
\end{aligned} \tag{5.70}$$

where $z = \omega(\varsigma)$ is shown in Eq. (5.20) with $m = 1$. Γ_1, Γ_2 are shown in Eq. (5.22).

Multiplying the first equation in Eq. (5.70) and its conjugate equation by $\mathrm{d}\sigma/[2\pi\mathrm{i}(\sigma - \varsigma)]$ and using the Cauchy integral formulas we find

$$\begin{aligned}
&\phi(\varsigma) = \frac{\Gamma_1 a\varsigma}{2} + \phi_0(\varsigma), \quad \phi_0(\varsigma) = -\frac{\Gamma_1 a}{2\varsigma} - \frac{\bar{\Gamma}_2 a}{2\varsigma} + \frac{\kappa a\left(E_2^{\infty}\right)^2}{2\varsigma} \\
&\psi(\varsigma) = \frac{\Gamma_2 a\varsigma}{2} - \frac{\Gamma_1 a}{2\varsigma} - \frac{\Gamma_1 a}{\varsigma^2 - 1}\varsigma - \frac{\varsigma(1+\varsigma^2)}{(\varsigma^2 - 1)}\phi_0'(\varsigma) + \frac{\kappa a\left(E_2^{\infty}\right)^2}{2\varsigma}
\end{aligned} \tag{5.71}$$

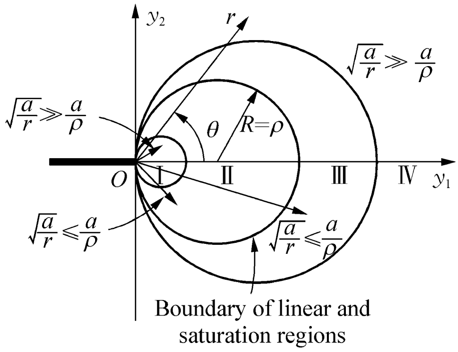
Near the crack tip let $z = a + r\mathrm{e}^{\mathrm{i}\theta}$ (Fig. 5.4); through tedious calculation the pseudo total asymptotic stresses are

$$\begin{aligned}
S_{22} + S_{11} = {} & \left[\left(2\Gamma_1 + \bar{\Gamma}_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta/2} + \left(2\Gamma_1 + \Gamma_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta/2}\right]\sqrt{a/2r} \\
& + \kappa E_2^{\infty 2}(a+\rho)/2l \\
S_{22} - S_{11} + 2\mathrm{i}S_{12} = {} & -\frac{1}{2\sqrt{2}}\mathrm{e}^{-3\mathrm{i}\theta/2}\left\{\left(2\Gamma_1 + \bar{\Gamma}_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{-\mathrm{i}\theta}\right. \\
& \left. -\left(2\Gamma_1 + 2\Gamma_2 - \bar{\Gamma}_2 + \kappa E_2^{\infty 2}\right)\mathrm{e}^{\mathrm{i}\theta}\right\}\sqrt{a/r} - \mathrm{e}^{-2\mathrm{i}\Theta}\kappa E_2^{\infty 2}(a+\rho)/2l \\
l = {} & \left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right| \geq \rho, \quad \Theta = \mathrm{Arg}\left(r\mathrm{e}^{\mathrm{i}\theta} - \rho\right)
\end{aligned} \tag{5.72}$$

2. *Stress in saturation zone* S_R In the saturation zone S_R, the electric displacements are finite; the asymptotic stresses near the crack tip will possess singular behavior like $1/\sqrt{r}$ and relate to the size of the saturation zone, so it is assumed

$$S_{ij} = h_{ij}^{(1)}(\theta)/\sqrt{r} + h_{ij}^{(2)}(\theta)/\rho + 0(r) \tag{5.73}$$

Fig. 5.4 Division regions near the crack tip



Because the electric displacements are continuous on the interface from S_R and S_0, so the Maxwell stress and mechanical and pseudo total stresses are all continuous. So $h_{ij}^{(1)}(\theta), h_{ij}^{(2)}(\theta)$ can be obtained from these continuous conditions:

$$\begin{aligned}
&\frac{h_{11}^{(1)}(\theta)+h_{22}^{(1)}(\theta)}{\sqrt{R(\theta)}}+\frac{h_{11}^{(2)}(\theta)+h_{22}^{(2)}(\theta)}{\rho}=\frac{\kappa E_2^{\infty 2}(a+\rho)}{2l_0} \\
&\quad+\left[\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)e^{-i\theta/2}+\left(2\Gamma_1+\Gamma_2+\kappa E_2^{\infty 2}\right)e^{i\theta/2}\right]\sqrt{\frac{a}{2R(\theta)}} \\
&\frac{h_{22}^{(1)}(\theta)-h_{11}^{(1)}(\theta)+2ih_{12}^{(1)}(\theta)}{\sqrt{R(\theta)}}+\frac{h_{22}^{(2)}(\theta)-h_{11}^{(2)}(\theta)+2ih_{12}^{(2)}(\theta)}{\rho}=-e^{-4i\theta}\frac{\kappa E_2^{\infty 2}(a+\rho)}{2l_0} \\
&\quad-\frac{1}{2\sqrt{2}}e^{-3i\theta/2}\left\{\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)e^{-i\theta}-\left(2\Gamma_1+2\Gamma_2-\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)e^{i\theta}\right\}\sqrt{\frac{a}{R(\theta)}}
\end{aligned} \tag{5.74}$$

where $R(\theta)=2\rho\cos\theta$ and on the interface $\Theta=2\theta$, $l_0=\left|R(\theta)e^{i\theta}-\rho\right|=\rho$. If $\rho/a \ll 1$, $(a+\rho)/\left|R(\theta)e^{i\theta}-\rho\right| \approx a/\rho$. Comparing the coefficients before $\sqrt{R}$ and $1/\rho$ yields

$$\begin{aligned}
S_{22}+S_{11}&=\left[\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)e^{-i\theta/2}+\left(2\Gamma_1+\Gamma_2+\kappa E_2^{\infty 2}\right)e^{i\theta/2}\right]\sqrt{a/2r} \\
&\quad+\kappa E_2^{\infty 2}a/2\rho \\
S_{22}-S_{11}+2iS_{12}&=-\left(1\big/2\sqrt{2}\right)e^{-3i\theta/2}\left\{\left(2\Gamma_1+\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)e^{-i\theta}\right. \\
&\quad\left.-\left(2\Gamma_1+2\Gamma_2-\bar{\Gamma}_2+\kappa E_2^{\infty 2}\right)e^{i\theta}\right\}\sqrt{a/r}-e^{-4i\theta}\kappa E_2^{\infty 2}a/2\rho
\end{aligned} \tag{5.75}$$

It is easy to prove that on the interface, the limit values of the stresses taken from S_0 and S_R are equal in the accuracy of $1/\sqrt{r}$ and $1/\rho$ which is consistent of the electric field.

3. *Division region near the crack tip* According to Eqs. (5.72) and (5.73), the stress can be divided into four regions (Fig. 5.4).

Region I: Region I is located in S_R and very near the crack tip, where $\sqrt{a/r} \gg a/\rho$ and the stresses possess the singularity $1/\sqrt{r}$. Under $\sigma_{22}^{\infty}, E_2^{\infty}$ at infinity we have

$$\begin{aligned}
S_{22} &= \frac{1}{2\sqrt{2\pi r}}\left[(K_{\mathrm{I}} - K_{\mathrm{E}})\left(2\cos\frac{\theta}{2} + \sin\theta\sin\frac{3\theta}{2}\right) + K_{\mathrm{I}}\sin\theta\cos\frac{3\theta}{2}\right] \\
S_{11} &= \frac{1}{2\sqrt{2\pi r}}\left[(K_{\mathrm{I}} - K_{\mathrm{E}})\left(2\cos\frac{\theta}{2} - \sin\theta\sin\frac{3\theta}{2}\right) - K_{\mathrm{I}}\left(4\sin\frac{\theta}{2} + \sin\theta\cos\frac{3\theta}{2}\right)\right] \\
S_{12} &= \frac{1}{2\sqrt{2\pi r}}\left[(K_{\mathrm{I}} - K_{\mathrm{E}})\sin\theta\cos\frac{3\theta}{2} + K_{\mathrm{I}}\left(2\cos\frac{\theta}{2} - \sin\theta\sin\frac{3\theta}{2}\right)\right]
\end{aligned} \tag{5.76}$$

Region II: Region II is located in S_R and $\sqrt{a/r} \sim a/\rho$. The stresses should be calculated by Eq. (5.73). The terms containing $\sqrt{a/r}, a/\rho$ all should be considered.

Region III: Region III is in S_0 but neighboring S_R and $\sqrt{a/r} \sim a/\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right|$. The stresses should be calculated by Eq. (5.75).

Region IV: Region IV is in S_0 and $\sqrt{a/r} \gg a/\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right|$. Terms containing $a/\left|r\mathrm{e}^{\mathrm{i}\theta} - \rho\right|$ can be neglected. If r/a is still small, the stresses can be calculated from Eq. (5.76) also.

5.2.4 Conducting Crack

For the conducting crack or the soft electrode, the boundary conditions are

$$\begin{aligned}
&\sigma_{ij} = 0, \quad E = E_1^{\infty} + \mathrm{i}E_2^{\infty} = E_0\mathrm{e}^{\mathrm{i}\beta}, \quad \text{when} \quad x_k x_k \to \infty \\
&\varphi = 0, \quad \text{or} \quad E_1 = 0, \quad \text{on} \quad x_2 = 0, \quad -a < x_1 < a
\end{aligned} \tag{5.77}$$

1. *The electric field in a linear piezoelectric plate without local saturation zone* According to Eq. (5.6) on the electrode, we have

$$E_1^+ = E_1^- = 0, \quad \text{or} \quad \bar{w}'^+(x_1) + w'^+(x_1) = \bar{w}'^-(x_1) + w'^-(x_1) = 0 \tag{5.78}$$

For the central crack $(-a, a)$ from Eq. (5.78), it can be derived

$$w'(z) = -E_1^{\infty} z\Big/\sqrt{z^2 - a^2} + iE_2^{\infty}, \quad w(z) = -E_1^{\infty}\sqrt{z^2 - a^2} + \mathrm{i}E_2^{\infty} z \tag{5.79}$$

The asymptotic solution near the crack tip $z = a$ is

$$\begin{aligned} & w'(z) = -E_1 + \mathrm{i}E_2 \approx -K_\mathrm{e} \Big/ \sqrt{2\pi(z-a)} = -\left(K_\mathrm{e} \Big/ \sqrt{2\pi r}\right)\mathrm{e}^{-\mathrm{i}\theta} \\ & E_1 = \left(K_\mathrm{e} \Big/ \sqrt{2\pi r}\right)\cos\theta, \quad E_2 = \left(K_\mathrm{e} \Big/ \sqrt{2\pi r}\right)\sin\theta, \quad K_\mathrm{e} = E_1^\infty\sqrt{\pi a} \end{aligned} \tag{5.80}$$

2. *The electric field in a plate with local saturation zone*
Similar to the impermeable crack the boundary value problem in y plane is

$$\begin{aligned} & D_1^\pm = 0, \quad 当 \quad y_2 = 0, \quad -\infty < y_1 \le 0 \\ & \sqrt{D_1^2 + D_2^2} = D_c, \qquad \text{when} \quad y \in R(\theta) \\ & D_1^2 + D_2^2 = 0, \qquad \text{when} \quad y \to \infty \end{aligned} \tag{5.81}$$

where $R = R(\theta)$ is the boundary of the saturation zone in y plane. The hodograph transform method is used. Let

$$\varsigma = (D_1 - \mathrm{i}D_2)/D_\mathrm{c}, \quad \text{or} \quad \varsigma = (E_1 - \mathrm{i}E_2)/E_\mathrm{c} \tag{5.82}$$

According to Eq. (5.80) the electric displacements in the saturation zone is assumed as

$$D_1 = D_\mathrm{c}\cos\theta, \quad D_2 = D_\mathrm{c}\sin\theta \tag{5.83}$$

Obviously Eq. (5.83) satisfies Eq. (5.77). Repeating the discussion in Sect. 5.2.2, the boundary and the radius of the saturation zone are, respectively,

$$R(\theta) = \left(K_\mathrm{e}^2/\pi E_\mathrm{c}^2\right)\cos\theta, \quad \rho = K_\mathrm{e}^2/\left(2\pi E_\mathrm{c}^2\right) \tag{5.84}$$

The remaining discussion is fully similar to Sect. 5.2.3 and omitted here.

5.3 Asymptotic Analysis of a Crack Subjected to Electric Loading

Yang and Suo (1994) and Hao et al. (1996) discussed the ceramic actuators caused by electrostriction; Beom et al. (2006) discussed the asymptotic analysis of an impermeable crack subjected to electric loading. The crack extension criterion in plane strain is mainly determined by the stress field near the crack tip, so they adopted the linear asymptotic solution of a semi-infinite crack as the boundary condition of the asymptotic analysis at infinity (Fig. 5.5). In this analysis the Maxwell stress is not considered. Analogous to Eq. (5.57) we have

$$D_2 + \mathrm{i}D_1 = K_\mathrm{D} \Big/ \sqrt{2\pi y}, \quad y = y_1 + \mathrm{i}y_2 = r\mathrm{e}^{\mathrm{i}\theta}; \quad \text{when} \quad |y| \to \infty \tag{5.85}$$

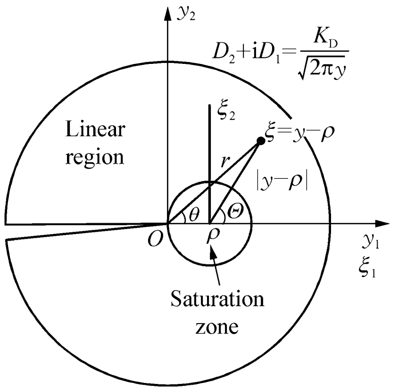


Fig. 5.5 Asymptotic analysis sketch of a crack with local saturation

where K_D is the electric displacement intensity factor. Now we discuss an infinite piezoelectric material with an impermeable crack subjected to electric loading as shown in Eq. (5.85). As shown in Eqs. (5.57) and (5.69), the approximate solutions of the electric displacement can be taken as

$$\begin{aligned} &D_2 + iD_1 = K_D\Big/\sqrt{2\pi y}, \quad w'(z) = -D_1 + iD_2 = iK_D\Big/\sqrt{2\pi y}; \quad \text{in} \quad \Omega_0 \\ &D_2 + iD_1 = D_c e^{-i\theta}, \quad \text{in} \quad \Omega_s \\ &\xi = \xi_1 + i\xi_2 = y - \rho = |y-\rho|e^{i\Theta} \end{aligned} \tag{5.86}$$

where $\rho = K_D^2\big/\left(2\pi D_c^2\right)$ is the radius of the saturation zone, $w(z)$ represents electric displacement complex potential, Ω_0 denotes the linear zone, and Ω_s denotes the saturation zone. Equation (5.86) satisfies the boundary condition on the crack surface and Eq. (5.85) at infinity. On the interface between Ω_0 and Ω_s, $\xi_0 = \rho e^{i\Theta}$. The constitutive equation is shown in Eq. (5.9), but here the slight different form is used:

$$\varepsilon_{\alpha\beta} = (1+\nu)\left(\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right)/Y + Q(1+q)D_\alpha D_\beta - Qq(1+\nu)D_\gamma D_\gamma \delta_{\alpha\beta} \tag{5.87}$$

where Y is elastic modulus and ν is Poisson ratio, Q and q are the electrostrictive coefficients. Apply the superposition method to solve this problem: Problem (1) is that a plate without crack is subjected to the above electric displacement fields. In this problem on the artificial cut corresponding to the original crack we can get the tractions $\sigma_{22}^c, \sigma_{21}^c$. Problem (2) is that the artificial cut is subjected tractions $-\sigma_{22}^c$, $-\sigma_{21}^c$. The solution of the original problem is the sum of solutions of these two problems.

According to Eqs. (5.86) and (5.87), the strains in the saturation zone induced by the saturation electric displacements are

$$\varepsilon_r^s = -QqD_c^2(1+\nu), \quad \varepsilon_\theta^s = QD_c^2(1-q\nu), \quad \varepsilon_{r\theta}^s = 0 \tag{5.88}$$

The strains in Eq. (5.88) satisfy the compatible equation automatically, so they do not produce stresses. Neglecting the rigid displacements the displacements corresponding to these strains are

$$\begin{aligned} &u_r^{\mathrm{s}} = -QqD_{\mathrm{c}}^2(1+\nu)r, \quad u_\theta^s = QD_{\mathrm{c}}^2(1+q)r\theta \\ &u_1^{\mathrm{s}} + \mathrm{i}u_2^{\mathrm{s}} = QD_{\mathrm{c}}^2[-(1+\nu)q + \mathrm{i}(1+q)\theta](\xi+\rho) \end{aligned} \tag{5.89}$$

Analogous to Eqs. (5.6), (5.16), (5.17), and (5.18) in the linear zone we have

$$\begin{aligned} &2G(u_1+\mathrm{i}u_2) = K\phi(\xi) - \xi\overline{\phi'(\xi)} - \overline{\psi(\xi)} + hw(\xi)\overline{w'(\xi)} + 4(1-\nu)m^{-1}h\int\left[\overline{w'(\xi)}\right]^2\mathrm{d}\bar{\xi} \\ &\sigma_{22}+\sigma_{11} = 2\left[\phi'(\xi)+\overline{\phi'(\xi)}\right] - 2hw'(\xi)\overline{w'(\xi)} \\ &\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} = 2[\bar{\xi}\phi''(\xi)+\psi'(\xi)] - 2hw''(\xi)\overline{w(\xi)} \\ &\mathrm{i}(P_1+\mathrm{i}P_2) = \left[\xi\overline{\phi'(\xi)}+\phi(\xi)+\overline{\psi(\xi)} - hw(\xi)\overline{w'(\xi)}\right]_A^B \end{aligned} \tag{5.90}$$

where

$$K = 3-4\nu, \quad h = \frac{1-(1+2\nu)q}{2}\frac{GQ}{1-\nu}, \quad m = 2\frac{1-(1+2\nu)q}{1+q} \tag{5.91}$$

In the saturation zone we have

$$\begin{aligned} &2G(u_1+\mathrm{i}u_2) = K\phi(\xi) - \xi\overline{\phi'(\xi)} - \overline{\psi(\xi)} + 2G\left(u_1^{\mathrm{s}} + iu_2^{\mathrm{s}}\right) \\ &\sigma_{22}+\sigma_{11} = 2\left[\phi'(\xi)+\overline{\phi'(\xi)}\right] \\ &\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} = 2[\bar{\xi}\phi''(\xi)+\psi'(\xi)] \\ &\mathrm{i}(P_1+\mathrm{i}P_2) = \left[\xi\overline{\phi'(\xi)}+\phi(\xi)+\overline{\psi(\xi)}\right]_A^B \end{aligned} \tag{5.92}$$

The two group solutions shown in Eqs. (5.90) and (5.92) should satisfy the continuity conditions of displacements and stresses on the interface between linear and saturation zones. The solution in the linear zone should also satisfy the boundary conditions at infinity.

Solution of problem (1) Assume the solutions are:

$$\begin{aligned} &\phi(\xi) = -\sigma_0\rho\ln(\xi/\rho) + \phi_2(\xi), \quad \psi(\xi) = -\sigma_0\rho\ln(\xi/\rho) + \psi_2(\xi); \quad \xi\in\Omega_0 \\ &\phi(\xi) = \phi_1(\xi), \quad \psi(\xi) = \psi_1(\xi), \quad \sigma_0 = \left[(1+q)/8\left(1-\nu^2\right)\right]YQD_{\mathrm{c}}^2; \quad \xi\in\Omega_{\mathrm{s}} \end{aligned} \tag{5.93}$$

Substitution of Eq. (5.93) into Eqs. (5.90) and (5.92) yields

$$\begin{aligned}&2G(u_1+\mathrm{i}u_2)+\mathrm{i}(P_1+\mathrm{i}P_2)=4(1-\nu)\left\{[-\sigma_0\rho\ln(\xi/\rho)+\phi_2(\xi)]+m^{-1}h\int\left[\overline{w'(\xi)}\right]^2d\bar{\xi}\right\};\ \xi\in\Omega_0\\&2G(u_1+\mathrm{i}u_2)+\mathrm{i}(P_1+\mathrm{i}P_2)=4(1-\nu)\phi_1(\xi)+2G\left(u_1^{\mathrm{s}}+\mathrm{i}u_2^{\mathrm{s}}\right);\quad \xi\in\Omega_{\mathrm{s}}\end{aligned}\tag{5.94}$$

From the continuity conditions of displacements and resultant forces on the interface we have

$$\phi_2(\xi_0)-\phi_1(\xi_0)=\sigma_0\rho[\ln(\xi_0/\rho)+m/2-1](1+\xi_0/\rho)\tag{5.95}$$

where $\xi_0=\rho\mathrm{e}^{\mathrm{i}\Theta}$ is the value of ξ on the interface. Assuming the displacements vanish at infinity, by the standard analytic continuation theory from Eq. (5.95) we find

$$\begin{aligned}\phi_2(\xi)&=\sigma_0\rho\left[\left(1+\frac{\xi}{\rho}\right)\ln\frac{\xi}{\xi+\rho}+1\right],\\ \phi_1(\xi)&=\sigma_0\rho\left[-\left(1+\frac{\xi}{\rho}\right)\left(\ln\frac{\xi+\rho}{\rho}+\frac{1}{2}m-1\right)+1\right]\end{aligned}\tag{5.96}$$

Analogously from the continuity conditions of resultant forces $i(P_1+iP_2)$ on the interface we have

$$\psi_2(\xi_0)-\psi_1(\xi_0)=-\sigma_0\rho[-(1+m)(\rho/\xi_0)+m/2-1-\ln(\xi_0/\rho)]\tag{5.97}$$

Assuming the displacements vanish at infinity, by the standard analytic continuation theory from Eq. (5.97)

$$\begin{aligned}\psi_2(\xi)&=\sigma_0\rho[(1+m)(\rho/\xi)+\ln[\xi/(\xi+\rho)]-m+4(1-\nu)]\\ \psi_1(\xi)&=\sigma_0\rho[-\ln[(\xi+\rho)/\rho]-m/2+3-4\nu]\end{aligned}\tag{5.98}$$

Finally we have

$$\begin{aligned}\phi(\xi)&=\sigma_0\rho\left[-\ln\frac{\xi}{\rho}+\left(1+\frac{\xi}{\rho}\right)\ln\frac{\xi}{\xi+\rho}+1\right]\\ \psi(\xi)&=\sigma_0\rho\left[(1+m)\frac{\rho}{\xi}-\ln\frac{\xi}{\rho}+\ln\frac{\xi}{\xi+\rho}-m+4(1-\nu)\right]\quad\text{in}\quad\Omega_0\end{aligned}\tag{5.99}$$

$$\begin{aligned}\varphi(\xi)&=\sigma_0\rho\left[-\left(1+\frac{\xi}{\rho}\right)\left(\ln\frac{\xi+\rho}{\rho}+\frac{1}{2}m-1\right)+1\right]\\ \psi(\xi)&=\sigma_0\rho\left[-\ln\frac{\xi+\rho}{\rho}-\frac{1}{2}m+3-4\nu\right];\quad\text{in}\quad\Omega_{\mathrm{s}}\end{aligned}\tag{5.100}$$

Solution of problem (2) Eqs. (5.90) and (5.99) yield

$$\sigma_{22}^{c} = \sigma_0 \left[2\ln\frac{y_1-\rho}{y_1} - (1+m)\left(\frac{\rho}{y_1-\rho}\right)^2 \right], \quad \sigma_{21}^{c} = 0 \tag{5.101}$$

When the crack surface is subjected to $-\sigma_{22}^{c}$, the solution is

$$\begin{aligned} \phi'(z) &= \sigma_0 \left\{ -\ln\frac{y-\rho}{y} + \frac{1}{2}(1+m)\left(\frac{\rho}{y-\rho}\right)^2 \right. \\ &\left. \quad -\frac{1}{4}(1+m)\rho^{3/2}\frac{y+\rho}{\sqrt{y}(y-\rho)^2} - 2\mathrm{arcoth}\sqrt{\frac{y}{\rho}} + 2\sqrt{\frac{\rho}{y}} \right\} \\ \psi'(y) &= -y\phi''(y) \end{aligned} \tag{5.102}$$

Solution of the original problem Superposing solutions of problems (1) and (2), finally we get the following. In the linear zone Ω_0,

$$\begin{aligned} \frac{\sigma_{22}+\sigma_{11}}{2} &= \sigma_0 \mathrm{Re}\left\{ (1+m)\left(\frac{\rho}{y-\rho}\right)^2 - \frac{1}{2}(1+m)\sqrt{\frac{\rho}{y}}\frac{\rho(y+\rho)}{(y-\rho)^2} \right. \\ &\left. \quad - 4\mathrm{arcoth}\sqrt{\frac{y}{\rho}} + 4\sqrt{\frac{\rho}{y}} \right\} - \sigma_0 m\frac{\rho}{|y-\rho|} \\ \frac{\sigma_{22}-\sigma_{11}}{2} + \mathrm{i}\sigma_{12} &= \sigma_0 \left\{ \frac{\bar{y}-\rho}{y-\rho} - \frac{\bar{y}}{y} - (1+m)\left(\frac{\rho}{y-\rho}\right)^2 + m\frac{\rho}{y-\rho}\sqrt{\frac{\bar{y}-\rho}{y-\rho}} + (\bar{y}-y) \right. \\ &\left. \times \left[\left(\sqrt{\frac{\rho}{y}}-1\right)\frac{\rho}{y(y-\rho)} - (1+m)\frac{\rho^2}{(y-\rho)^3} - \frac{1+m}{8}\sqrt{\frac{\rho}{y}}\frac{\rho}{y(y-\rho)^3}\left(\rho^2-6\rho y-3y^2\right) \right] \right\} \end{aligned} \tag{5.103}$$

In the saturation zone Ω_s,

$$\begin{aligned} \frac{\sigma_{22}+\sigma_{11}}{2} &= \sigma_0 \mathrm{Re}\left\{ -2\ln\frac{y-\rho}{\rho} + (1+m)\left(\frac{\rho}{y-\rho}\right)^2 \right. \\ &\left. \quad -\frac{1}{2}(1+m)\sqrt{\frac{\rho}{y}}\frac{\rho(y+\rho)}{(y-\rho)^2} - 4\mathrm{arcoth}\sqrt{\frac{y}{\rho}} + 4\sqrt{\frac{\rho}{y}} \right\} - \sigma_0 m \\ \frac{\sigma_{22}-\sigma_{11}}{2} + \mathrm{i}\sigma_{12} &= \sigma_0 \left\{ -\frac{\bar{y}}{y} + (\bar{y}-y)\left[\left(\sqrt{\frac{\rho}{y}}-1\right)\frac{\rho}{y(y-\rho)} \right. \right. \\ &\left. \left. \quad - (1+m)\frac{\rho^2}{(y-\rho)^3} - \frac{1+m}{8}\sqrt{\frac{\rho}{y}}\frac{\rho}{y(y-\rho)^3}\left(\rho^2-6\rho y-3y^2\right) \right] \right\} \end{aligned} \tag{5.104}$$

It is also found that in the saturation zone the stresses at the real crack tip has the singularity $1/\sqrt{r}$ and at the effective crack tip $(y=\rho)$ has the logarithmic singularity. Because accuracy of the electric field is of the order ρ/a, the accuracy of solutions of the mechanical stresses is still in the same order.

Following the elastoplastic fracture mechanics, Beom et al. (2006) also discussed the modified boundary layer theory, i.e., replaced Eq. (5.85) by

$$D_2 + \mathrm{i}D_1 = K_\mathrm{D} \Big/ \sqrt{2\pi z} + \mathrm{i}T; \quad \text{when} \quad |z| \to \infty$$

where T is a finite electric displacement parallel to the crack surface.

Beom (1999)discussed the singular behavior near a crack tip in an electrostrictive material with the elastic behavior shown in Eq. (5.87), and for the electric behavior, he took the Ramberg-Osgood type constitutive equation

$$\begin{aligned} &E_\alpha = -2Q(1+q)\sigma_{\alpha\beta}D_\beta + 2Qq(1+\nu)\sigma_{\beta\beta}D_\alpha + 2YQ^2q^2D_\beta D_\beta D_\alpha + D_\alpha f(D)/D \\ &f(D) = (E_\mathrm{c}/D_\mathrm{c})D + kE_\mathrm{c}(D/D_\mathrm{c})^n; \quad n > 3 \end{aligned} \tag{5.105}$$

where k and n are material constants; $E = f(D)$ is the uniaxial dielectric response in the absence of stress. In this case he got $\boldsymbol{\sigma} \propto r^{-1/2}, \boldsymbol{D} \propto r^{-1/(n+1)}$.

5.4 Pyroelectric Material

5.4.1 Generalized Two-Dimensional Linear Thermo-electro-elastic Problem

In engineering the extensive applied governing equation is Eq. (2.89) with independent variables $(\boldsymbol{\varepsilon}, \boldsymbol{E}, \vartheta), \quad \vartheta = T - T_0$ for the pyroelectric materials:

$$\begin{aligned} &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta, \quad D_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta \\ &s = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + C\vartheta/T_0, \quad \vartheta = T - T_0 \end{aligned} \tag{5.106}$$

The thermal conduction and the entropy equations are

$$q_i = -\lambda_{ij}T_{,j}, \quad T_{,j} = \vartheta_{,j} = -\lambda_{ji}^{-1}q_i; \quad -q_{i,i} = T\dot{s} - \dot{r} \tag{5.107}$$

The mechanical, electric, and thermal boundary conditions are

$$\begin{array}{lllll} \sigma_{ij}n_j = T_i^*, & \text{on} \quad a_\sigma; & \text{or} & u_i = u_i^*, & \text{on} \quad a_u \\ D_i n_i = -\sigma^*, & \text{on} \quad a_D; & \text{or} & \varphi = \varphi^*, & \text{on} \quad a_\varphi \\ q_i n_i = q_n = q_0^*, & \text{on} \quad a_q; & \text{or} & T = T^*, & \text{on} \quad a_T \end{array} \tag{5.108}$$

where T^*, σ^*, q_0^* are the traction, electric charge per area, and normal heat flow per area.

The continuity conditions on the interface are

$$\sigma_{ij}^{+}n_j=\sigma_{ij}^{-}n_j,\quad u_i^{+}=u_i^{-},\quad D_i^{+}=D_i^{-};\quad \varphi^{+}=\varphi^{-},T^{+}=T^{-},\quad q^{+}=q^{-}\quad \text{on}\quad L \tag{5.109}$$

The governing equations in $(\boldsymbol{u},\varphi,\vartheta)$ are

$$\begin{aligned}&\left(C_{ijlk}u_l+e_{kij}\varphi\right)_{,ik}-\alpha_{ij}\vartheta_{,i}+\left(f_j^{\mathrm{m}}+f_j^{\mathrm{e}}\right)=\rho\ddot{u}_j\\&\left(-\epsilon_{ik}\varphi+e_{ijk}u_j\right)_{,ik}+\tau_iT_{,i}=\rho_{\mathrm{e}}\\&\lambda_{ij}T_{,j}=-q_i\end{aligned} \tag{5.110}$$

For a multiply connected domain, the displacement and electric potential must satisfy the uniqueness condition Eq. (3.7).

The thermo-electro-elastic fundamental theory of the pyroelectric material was studied a long time (Tiersten 1971; Mindlin 1974). For a static problem with stationary temperature, from Eq. (5.110) we get

$$\left(C_{ijrs}u_r+e_{sij}\varphi\right)_{,si}=\alpha_{ij}\vartheta_{,i},\quad \left(-\epsilon_{is}\varphi+e_{irs}u_r\right)_{,si}=-\tau_i\vartheta_{,i},\quad -q_{i,i}=\left(\lambda_{ij}\vartheta_{,j}\right)_{,i}=0 \tag{5.111}$$

From Eq. (5.111) it is seen that the generalized displacements are dependent to the temperature, but the temperature is independent to the generalized displacements. So the temperature can be solved independently (Hwu 1992; Shen and Kuang 1998). Because ϑ is real, it is assumed that

$$\vartheta(x_1,x_2)=\mathrm{g}'(z_T)+\overline{\mathrm{g}'(z_T)}=2\mathrm{Re}\,\mathrm{g}'(z_T),\quad z_T=x_1+\mu_Tx_2 \tag{5.112}$$

Substitution of Eq. (5.112) into the third equation in Eq. (5.111) yields

$$\left(\lambda_{11}+2\mu_T\lambda_{12}+\mu_T^2\lambda_{22}\right)\mathrm{g}'''(z)=0 \tag{5.113}$$

As in Sect. 3.2.1, from Eq. (5.113) we get a pair of conjugate complex roots $\mu_T,\bar{\mu}_T$ with $\mathrm{Im}\mu_T>0$:

$$\begin{aligned}&\lambda_{11}+2\mu_T\lambda_{12}+\mu_T^2\lambda_{22}=0,\quad \lambda_{11}+\mu_T\lambda_{12}=-\mu_T(\lambda_{12}+\mu_T\lambda_{22})\\&\mu_T=(-\lambda_{12}+\mathrm{i}\alpha)/\lambda_{22},\quad \alpha=\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}=\lambda_{22}(\mu_T-\bar{\mu}_T)/2\mathrm{i}=-\mathrm{i}(\lambda_{21}+\mu_T\lambda_{22})\end{aligned} \tag{5.114}$$

where α is real. Using Eq. (5.114) the Fourier's law can be written as

$$\begin{aligned} q_i &= -2\mathrm{Re}[(\lambda_{i1}+\mu_T\lambda_{i2})\mathrm{g}''(z_T)], \quad q_n = -2\mathrm{Im}[\alpha(\mu_T n_1 - n_2)\mathrm{g}''(z_T)] \\ q_1 &= -2\mathrm{Re}[(\lambda_{11}+\mu_T\lambda_{12})\mathrm{g}''(z_T)] = 2\mathrm{Re}[\mathrm{i}\alpha\mu_T\mathrm{g}''(z_T)] = -2\mathrm{Im}[\alpha\mu_T\mathrm{g}''(z_T)] \\ q_2 &= -2\mathrm{Re}[(\lambda_{21}+\mu_T\lambda_{22})\mathrm{g}''(z_T)] = -2\mathrm{Re}[\mathrm{i}\alpha\mathrm{g}''(z_T)] = 2\mathrm{Im}[\alpha\mathrm{g}''(z_T)] \end{aligned} \tag{5.115}$$

By using Eq. (3.27) the total heat flow $\hat{q}$ through a line segment from z_0 to z is

$$\hat{q} = \int_{z_0}^{z_T} q_i n_i \mathrm{d}s = 2\mathrm{Re}\int_{z_0}^{z_T} \mathrm{i}\alpha(\mu_T\mathrm{d}x_2 + \mathrm{d}x_1)\mathrm{g}''(z_T) = -2\mathrm{Im}\{\alpha[\mathrm{g}'(z_T) - \mathrm{g}'(z_0)]\} \tag{5.116}$$

When ϑ is solved, the terms in the right side of the first and second equations in Eq. (5.111) become known. The special solution introduced by the temperature ϑ can be assumed as

$$\boldsymbol{U}_T = [U_{TP}]^{\mathrm{T}} = \boldsymbol{c}\mathrm{g}(z_T), \quad U_{Ti} = u_{Ti} = c_i\mathrm{g}(z_T), \quad U_{T4} = \varphi_T = c_4\mathrm{g}(z_T) \tag{5.117}$$

where a subscript in upper case P takes the value 1,2,3, or 4 and a subscript in lower case $i, j, \ldots$ takes the value 1, 2, or 3, as shown in Sect. 3.2.1. Substitution of Eq. (5.117) into the first and second equations in Eq. (5.111) yields the equations to determine $\boldsymbol{c} = [c_1, c_2, c_3, c_4]^{\mathrm{T}}$:

$$\begin{aligned} &\left[C_{j1k1} + \mu_T\left(C_{j1k2} + C_{j2k1}\right) + \mu_T^2 C_{j2k2}\right]c_k + \left[e_{1j1} + \mu_T\left(e_{2j1} + e_{1j2}\right) + \mu_T^2 e_{2j2}\right]c_4 \\ &\quad = \alpha_{1j} + \mu_T\alpha_{2j} \\ &\left[e_{1k1} + \mu_T(e_{2k1} + e_{1k2}) + \mu_T^2 e_{2k2}\right]c_k - \left[\epsilon_{11} + \mu_T(\epsilon_{12} + \epsilon_{21}) + \mu_T^2\epsilon_{22}\right]c_4 = -\tau_1 - \mu_T\tau_2; \quad \text{or} \\ &\left[\boldsymbol{Q} + \mu_T\left(\boldsymbol{R} + \boldsymbol{R}^T\right) + \mu_T^2\boldsymbol{T}\right]\boldsymbol{c} = \boldsymbol{D}(\mu_T)c = \boldsymbol{\chi}_1 + \mu_T\boldsymbol{\chi}_2, \quad \boldsymbol{\chi}_i = [\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, -\tau_i]^{\mathrm{T}} \end{aligned} \tag{5.118}$$

where $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}$ are expressed in Eq. (3.13). The generalized stress introduced by temperature is

$$\begin{aligned} \sigma_{Tij} &= 2\mathrm{Re}[(C_{ijkl}c_k + e_{lij}c_4)z_{T,l} - \alpha_{ij}]\mathrm{g}'(z_T) \\ D_{Ti} &= 2\mathrm{Re}[(e_{ikl}c_k + \epsilon_{il}c_4)z_{T,l} + \tau_i]\mathrm{g}'(z_T) \end{aligned} \tag{5.119}$$

The solution for the thermo-electro-elastic analysis in pyroelectric material is the sum of the special solution and the general solution of the corresponding homogeneous equations. For the stationary temperature the general solution is

$$\boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z_P) + \boldsymbol{c}\mathrm{g}(z_T)], \quad \text{or} \quad \boldsymbol{U} = 2\mathrm{Re}[\boldsymbol{A}\langle f(z_P)\rangle\boldsymbol{V} + \boldsymbol{c}\mathrm{g}(z_T)] \tag{5.120}$$

The stress can be expressed as

$$\begin{aligned}
&\boldsymbol{\Sigma}_1 = -2\mathrm{Re}[\boldsymbol{B}\mu_p\boldsymbol{F}(z_P) + \boldsymbol{d}\mu_T g'(z_T)], \quad \boldsymbol{\Sigma}_2 = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{F}(z_P) + \boldsymbol{d}g'(z_T)] \\
&\boldsymbol{d} = \left(\boldsymbol{R}^{\mathrm{T}} + \mu_T\boldsymbol{T}\right)\boldsymbol{c} - \boldsymbol{\chi}_2 = \{-(\boldsymbol{Q} + \mu_T\boldsymbol{R})\boldsymbol{c} + \boldsymbol{\chi}_1\}/\mu_T \\
&d_j = (C_{j2kl}c_k + e_{l2j}c_4)z_{T,l} - \alpha_{2j} = -[(C_{j1kl}c_k + e_{l1j}c_4)z_{T,l} - \alpha_{1j}]/\mu_T \\
&d_4 = (e_{2kl}c_k + \epsilon_{2l}c_4)z_{T,l} + \tau_2 = -[(e_{1kl}c_k + \epsilon_{1l}c_4)z_{T,l} + \tau_2]/\mu_T
\end{aligned} \tag{5.121}$$

where $\boldsymbol{F}(z_j) = \boldsymbol{f}'(z_j)$. Introduce the stress function $\boldsymbol{\Phi}$:

$$\begin{aligned}
&\boldsymbol{\Phi} = [\Phi_i, \Phi_4]^{\mathrm{T}} = 2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z) + \boldsymbol{d}g(z_T)], \quad \text{or} \quad \boldsymbol{\Phi} = 2\mathrm{Re}[\boldsymbol{B}\langle f(z_P)\rangle\boldsymbol{V} + \boldsymbol{d}g(z_T)] \\
&\Sigma_{2P} = \Phi_{P,1}, \quad \Sigma_{1P} = -\Phi_{P,2}; \quad T_i = -\mathrm{d}\Phi_i/\mathrm{d}s, \quad D_n = -\mathrm{d}\Phi_4/\mathrm{d}s; \quad P = 1,2,4, \quad i = 1,2
\end{aligned} \tag{5.122}$$

Equations (5.112), (5.115), (5.120), and (5.122) are the general solutions of the thermo-electro-elastic analysis in the pyroelectric material. Combining these equations and the appropriate boundary conditions, we can solve all the thermo-electro-elastic problems. For the multi-connected region the generalized displacement and temperature should satisfy the uniqueness condition.

5.4.2 A Thermal Impermeable Elliptic Hole in a Pyroelectric Material

As an example in this section, we discuss a generalized 2D problem of a pyroelectric material that occupied the region S with an elliptic hole that occupied the region S^c filled with air under uniform generalized stresses $(\boldsymbol{\sigma}^\infty, \boldsymbol{D}^\infty)$ and heat flow $\boldsymbol{q}^\infty$ (see Fig. 3.3). The interface L between the material and the hole is free of generalized forces and is thermal insulated (Lu et al. 1998; Gao et al. 2002). The boundary conditions are

$$\begin{aligned}
&\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{D} = \boldsymbol{D}^\infty, \quad \boldsymbol{q} = \boldsymbol{q}^\infty; \quad \text{at} \quad \text{infinity} \\
&\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0, \quad \varphi = \varphi^c, \quad \boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^c\cdot\boldsymbol{n} = -\epsilon_0(\nabla\cdot\varphi^c)\cdot\boldsymbol{n}; \quad \text{on} \quad L
\end{aligned} \tag{5.123}$$

Temperature field in the piezoelectric material with a thermal insulated hole As shown in Sect. 5.4.1, the temperature can be solved independently. As in Sect. 3.4 the transform method is used to solve this problem. The mapping function for z_T plane to ς_T plane is similar to z_j plane to ς_j plane in Eq. (3.86), but μ_j is replaced by μ_T, i.e.,

$$z_T = \omega_T(\varsigma_T) = R_T\left(\varsigma_T + m_T\varsigma_T^{-1}\right); \quad R_T = (a - \mathrm{i}\mu_T b)/2, \quad m_T = (a + \mathrm{i}\mu_T b)/(a - \mathrm{i}\mu_T b) \tag{5.124}$$

The interface L in z plane is mapped to Γ in ς plane. The temperature field can be chosen as

$$g'(z_T) = \beta_T z_T + \hat{g}'_0(z_T) = \beta_T \omega_T(\varsigma_T) + g'_0(\varsigma_T), \quad g'_0(\varsigma_T) = \hat{g}'_0[\omega_T(\varsigma_T)] \quad (5.125)$$

where β_T is a complex constant and $g'_0(\varsigma_T)$ is holomorphic outside the unit circle in ς_T plane. Equations (5.112), (5.115), and (5.125) yield

$$q_1^\infty = 2\mathrm{Re}[\mathrm{i}\alpha\mu_T\beta_T], \quad q_2^\infty = -2\mathrm{Re}[\mathrm{i}\alpha\beta_T]; \quad \beta_T = -\mathrm{i}\left(q_1^\infty + \bar{\mu}_T q_2^\infty\right)/\alpha(\mu_T - \bar{\mu}_T) \quad (5.126)$$

Because the interface is thermal insulated, Eqs. (5.116) and (5.126) yield

$$\begin{aligned} &\mathrm{Re}[\mathrm{i}\alpha g'(\sigma) - \mathrm{i}\alpha g'(\sigma_0)] = 0, \quad \text{or} \\ &\mathrm{i}\left[g'_0(\sigma) - \bar{g}'_0(\bar{\sigma})\right] = \mathrm{i}\left\{\beta_T R_T(\sigma + \bar{m}_T\bar{\sigma}) - \bar{\beta}_T \bar{R}_T(\bar{\sigma} + \bar{m}_T\sigma)\right\} \\ &\quad = (1/2\alpha)\left[a q_2^\infty(\sigma + \bar{\sigma}) + \mathrm{i}b q_1^\infty(\sigma - \bar{\sigma})\right] \end{aligned} \quad (5.127)$$

where σ is the value of ς_T on Γ. Multiplying Eq. (5.127) by $\int_L [\mathrm{d}\sigma/(\sigma - \varsigma)]$ and using the Cauchy integral formula we get

$$g'_0(\varsigma_T) = \delta_T\varsigma_T^{-1}, \quad g'(\varsigma_T) = \beta z_T + g'_0(\varsigma_T); \quad \delta_T = \left[(1/2\mathrm{i}\alpha)\left(a q_2^\infty - \mathrm{i}b q_1^\infty\right)\right] \quad (5.128)$$

From Eqs. (5.125) and (5.128) in z plane, we get

$$\begin{aligned} &g'(z_T) = \beta_T z_T + \delta_T\varsigma_T^{-1}(z_T) \\ &g(z_T) = (1/2)\beta_T z_T^2 + R_T\delta_T \ln\varsigma_T(z_T) + (1/2)R_T m_T \varsigma_T^{-2}(z_T) \end{aligned} \quad (5.129)$$

where $\beta_T z_T$ represents the complex potential of a uniform heat flow $\boldsymbol{q}^\infty$ in an infinite material without hole.

Superposition method By means of superposition, the solution of the original problem can be obtained as the sum of the following three problems:

(1) A pyroelectric material with an elliptic hole under boundary conditions

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\sigma}^\infty, \quad \boldsymbol{D} = \boldsymbol{D}^\infty, \quad \boldsymbol{q} = \boldsymbol{0}; \quad \text{at} \quad \text{infinity} \\ &\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0, \quad \varphi = \varphi^{\mathrm{c}}, \quad \boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^{\mathrm{c}}\cdot\boldsymbol{n} = -\epsilon_0(\boldsymbol{\nabla}\cdot\varphi^{\mathrm{c}})\cdot\boldsymbol{n}; \quad \text{on} \quad L \end{aligned} \quad (5.130a)$$

Problem (1) can be reduced to the following problem: a piezoelectric material, with an elliptic hole, subjected generalized stresses at infinity under constant temperature, which has been discussed in Sect. 3.4.

(2) A pyroelectric material without elliptic hole under boundary conditions

$$\boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{D} = \mathbf{0}, \quad \boldsymbol{q} = \boldsymbol{q}^{\infty}; \quad \text{at} \quad \text{infinity} \tag{5.130b}$$

The solution is

$$\begin{aligned}
&\mathrm{g}'(z_T) = \beta_T z_T, \quad q_1 = q_1^{\infty}, \quad q_2 = q_2^{\infty}; \quad \sigma_{ij} = 0; \quad D_i = 0 \\
&\vartheta(x_1, x_2) = 2\mathrm{Re}(\beta_T z_T) = -\left(\lambda_{11}\lambda_{22} - \lambda_{12}^2\right)^{-1}\left[\left(\lambda_{22}q_1^{\infty} - \lambda_{12}q_2^{\infty}\right)x_1 + \left(\lambda_{11}q_2^{\infty} - \lambda_{12}q_1^{\infty}\right)x_2\right]
\end{aligned} \tag{5.131}$$

This temperature field does not affect the generalized stress field, because a linear temperature field always satisfies the strain compatible equation.

(3) A pyroelectric material with an elliptic hole under boundary and single valued conditions

$$\begin{aligned}
&\boldsymbol{\sigma} = \mathbf{0}, \quad \boldsymbol{D} = \mathbf{0}, \quad \boldsymbol{q} = \mathbf{0}; \quad \text{at} \quad \text{infinity} \\
&\boldsymbol{\sigma}\cdot\boldsymbol{n} = \mathbf{0}, \quad \boldsymbol{q}\cdot\boldsymbol{n} = -\boldsymbol{q}^{\infty}\cdot\boldsymbol{n}, \quad \varphi = \varphi^{\mathrm{c}}, \quad \boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^{\mathrm{c}}\cdot\boldsymbol{n} = -\epsilon_0(\nabla\cdot\varphi^{\mathrm{c}})\cdot\boldsymbol{n}; \quad \text{on} \quad L \\
&\oint_L \mathrm{d}U = \oint_{\Gamma} \mathrm{d}U = 0
\end{aligned} \tag{5.130c}$$

Now we discuss the solution of the problem (3) Subtracting the solution of problem (2) from Eq. (5.129), the temperature potential in ς plane of problem (3) can be obtained:

$$\mathrm{g}(\varsigma_T) = R_T\left[\delta_T \ln\varsigma_T + (1/2)m_T\varsigma_T^{-2}\right] \tag{5.132}$$

The electric field inside the hole filled with air is fully the same as that in Sect. 3.4.2 and Eqs. (3.81), (3.82a), (3.82b), (3.83), (3.84), and (3.85) are still held. The complex potential $\phi(\varsigma)$ is still expressed by Eq. (3.85), i.e.,

$$\begin{aligned}
&\varphi^{\mathrm{c}}(\rho,\psi) = \phi(\varsigma) + \overline{\phi(\varsigma)} \\
&\phi(\varsigma) = \sum_{k=-\infty}^{\infty} h_k\varsigma^k, \quad h_{-k} = \rho_0^{2k}h_k = m^k h_k \quad (\text{not} \quad \text{summed} \quad \text{on} \quad k), \quad \rho_0 \le |\varsigma| \le 1
\end{aligned} \tag{5.133}$$

From Eq. (5.120) it is seen that $\boldsymbol{f}(z_P)$ and $\mathrm{g}(z_T)$ have the similar role in the generalized displacements, so $\boldsymbol{f}(\varsigma_P)$ in S can be assumed in the following form:

$$\boldsymbol{f}(\varsigma_P) = \langle\ln(\varsigma_P)\rangle\boldsymbol{p} + \boldsymbol{f}_0(\varsigma_P), \quad \boldsymbol{f}_0(\varsigma_P) = \sum_{k=1}^{\infty}\left(\varsigma_P^{-k}\right) \quad \boldsymbol{a}_k; \quad |\varsigma_P| \ge 1 \tag{5.134}$$

Substitution of Eqs. (5.132) and (5.133) into Eq. (5.120) yields

$$\boldsymbol{U} = 2\mathrm{Re}\left\{\boldsymbol{A}\left[\langle\ln\varsigma_P\rangle\boldsymbol{p} + \boldsymbol{f}_0(\varsigma_P)\right] + \boldsymbol{c}R_T\delta_T\left[\ln\varsigma_T + (1/2)m_T\varsigma_T^{-2}\right]\right\} \tag{5.135}$$

In Eqs. (5.132), (5.133), (5.134), and (5.135) functions $\mathrm{g}(\varsigma)$, $\boldsymbol{f}(\varsigma)$, $\phi(\varsigma)$ are all the functions of ς, but in Eq. (5.130c) we need their derivatives with s and n on the L in the z plane, so the following relations are needed. Eq. (3.82) yields

$$\begin{aligned}
&x_1 = a\cos\psi, \quad x_2 = b\sin\psi; \quad \mathrm{d}x_1/\mathrm{d}s = -a\sin\psi\mathrm{d}\psi/\mathrm{d}s; \quad \mathrm{d}x_2/\mathrm{d}s = b\cos\psi\mathrm{d}\psi/\mathrm{d}s\\
&\rho\sin\theta = a\sin\psi, \quad \rho\cos\theta = b\cos\psi; \quad \mathrm{d}s = \rho\mathrm{d}\psi, \quad \rho^2 = a^2\sin^2\psi + b^2\cos^2\psi\\
&\partial\varsigma_T/\partial\psi = \partial\varsigma_P/\partial\psi = \mathrm{i}e^{\mathrm{i}\psi} = \mathrm{i}\sigma, \quad \partial z/\partial\psi = -a\sin\psi + \mathrm{i}b\cos\psi = \rho(-\sin\theta + \mathrm{i}\cos\theta)\\
&\partial z_T/\partial\psi_T = \rho(-\sin\theta_T + \mu_T\cos\theta_T), \quad \partial z_j/\partial\psi_j = \rho\left(-\sin\theta_j + \mu_j\cos\theta_j\right); \quad \text{on} \quad \Gamma
\end{aligned} \tag{5.136}$$

Using Eq. (5.136) it is easy to get

$$\begin{aligned}
\mathrm{g}_{,s} &= \frac{\partial \mathrm{g}}{\partial\varsigma_T}\frac{\partial\varsigma_T}{\partial\psi}\frac{\partial\psi_T}{\partial z_T}\left(\frac{\partial z_T}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial z_T}{\partial x_2}\frac{\partial x_2}{\partial s}\right) = \mathrm{i}\frac{\sigma \mathrm{g}'(\sigma)}{\rho} = \mathrm{i}\frac{R_T\delta_T}{\rho}\left(1 - m_T\frac{1}{\sigma^2}\right)\\
f_{P,s} &= \frac{\partial f_P}{\partial\varsigma_P}\frac{\partial\varsigma_P}{\partial\psi}\frac{\partial\psi}{\partial z_P}\left(\frac{\partial z_P}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial z_P}{\partial x_2}\frac{\partial x_2}{\partial s}\right) = \mathrm{i}\frac{\sigma f'_{P,s}(\sigma)}{\rho} = \mathrm{i}\frac{p_P}{\rho} - \frac{\mathrm{i}}{\rho}\sum_{k=1}^{\infty} k a_{Pk}\sigma^{-k}\\
\phi_{,n} &= \frac{\partial\phi}{\partial\varsigma}\frac{\partial\varsigma}{\partial\psi}\frac{\partial\psi}{\partial z}\left(\frac{\partial z}{\partial x_1}n_1 + \frac{\partial z}{\partial x_2}n_2\right) = \frac{\sigma\phi'(\sigma)}{\rho} = \frac{1}{\rho}\sum_{k=1}^{\infty} k\left(h_k\sigma^k - h_{-k}\sigma^{-k}\right)
\end{aligned} \tag{5.137}$$

Substituting Eqs. (5.135), (5.137), and (5.122) into the connective conditions on the interface Γ and the single valued condition in Eq. (5.130c) and then comparing the coefficients of the corresponding terms on both sides in result equations, we get

$$\begin{aligned}
&A_{iP}p_P - \bar{A}_{iP}\bar{p}_P + c_iR_T\delta_T + \bar{c}_i\bar{R}_T\bar{\delta}_T = 0; \quad \text{(single valued condition)}, \quad P = 1,2,4\\
&B_{iP}p_P - \bar{B}_{iP}\bar{p}_P + d_iR_T\delta_T - \bar{d}_i\bar{R}_T\bar{\delta}_T = 0; \quad (-\mathrm{d}\Phi_i/\mathrm{d}s = T_i, \quad -\mathrm{d}\Phi_4/\mathrm{d}s = D_n)\\
&kB_{ik}a_{Pk} - \mathrm{i}\epsilon_0 k\left(h_k m^k + \bar{h}_k\right)\delta_{P4} = \begin{cases} -d_iR_T\delta_T m_T, & k = 2\\ 0, & k \neq 2\end{cases}; \quad (\boldsymbol{D}\cdot\boldsymbol{n} = \boldsymbol{D}^{\mathrm{c}}\cdot\boldsymbol{n})\\
&A_{4k}a_{4k} - \left(h_k m^k + \bar{h}_k\right) = \begin{cases} -(1/2)c_4R_T m_T\delta_T, & k = 2\\ 0 & k \neq 2\end{cases}; \quad (\varphi = \varphi^{\mathrm{c}}); \quad \text{on} \quad \Gamma
\end{aligned} \tag{5.138}$$

where δ_{P4} is Kronecker delta. Solving undetermined coefficients finally yields

$$\begin{aligned}
&\phi(\varsigma) = h_2\left(\varsigma^2 + m^2\varsigma^{-2}\right); \quad \varphi^{\mathrm{c}}(\varsigma) = 2\mathrm{Re}\phi(\varsigma) \quad \rho_0 \leq |\varsigma| \leq 1\\
&\boldsymbol{f}(\varsigma_P) = \langle\ln(\varsigma_P)\rangle\boldsymbol{p} + \boldsymbol{a}_2\langle\varsigma_P^{-2}\rangle; \quad |\varsigma_j| \geq 1\\
&\mathrm{g}(\varsigma_T) = R_T\delta_T\left[\ln\varsigma_T + (1/2)m_T\varsigma_T^{-2}\right]; \quad |\varsigma_T| \geq 1\\
&\boldsymbol{a}_k = 0, \quad h_k = 0; \quad \text{if} \quad k \neq 2
\end{aligned} \tag{5.139}$$

It is seen from Eq. (5.139) that $\mathrm{g}'(\varsigma_T)$, $f'_P(\varsigma_P) \to 0$ when $|\varsigma_P|$, $|\varsigma_T| \to \infty$. So the boundary conditions at infinity are satisfied also.

In Eq. (5.139) $\varphi^{c}(\varsigma)$ can also be rewritten as

$$\varphi^{c}(x_1, x_2) = -2m(d_2 + \bar{d}_2) + R^{-2}\left(d_2 z^2 + \bar{d}_2 \bar{z}^2\right) \tag{5.140}$$

Therefore, the electric field in the elliptic hole varies linearly with the coordinates.

5.5 Interface Crack in Dissimilar Pyroelectric Material

5.5.1 General Discussion

The fundamental theory of the pyroelectric material has been discussed in Sect. 5.4. Now the interface crack in dissimilar pyroelectric material (see Fig. 4.2) will be discussed (Shen and Kuang 1998; Gao and Wang 2001). The general solutions $\boldsymbol{U}\left(z_j, z_T\right)$, $\boldsymbol{\Phi}\left(z_j, z_T\right)$, and ϑ are shown in Eqs. (5.120), (5.122), and (5.112), respectively. The boundary conditions are assumed:

$$\begin{aligned}
&\boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = \boldsymbol{\Phi}_{\mathrm{II},1}(x_1) = \boldsymbol{\Sigma}_0(x_1), \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1) = q_0(x_1), \quad x \in L_{\mathrm{c}} \\
&\hat{\boldsymbol{d}}(x_1) = \boldsymbol{U}_{\mathrm{I}}(x_1) - \boldsymbol{U}_{\mathrm{II}}(x_1) = \boldsymbol{0}, \quad \boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = \boldsymbol{\Phi}_{\mathrm{II},1}(x_1) \\
&\vartheta_{\mathrm{I}}(x_1) = \vartheta_{\mathrm{II}}(x_1), \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1), \quad x \in L - L_{\mathrm{c}} \\
&\boldsymbol{\Sigma}_{\mathrm{I}}(x_1) = \boldsymbol{\Sigma}_{\mathrm{II}}(x_1) \to \boldsymbol{0}, \quad q_{\mathrm{I}n} = q_{\mathrm{II}n} \to 0; \quad |z| \to \infty
\end{aligned} \tag{5.141}$$

where $\hat{\boldsymbol{d}}$ is the displacement disconnected value between crack surfaces. Equation (5.141) shows that on whole axis x_1 we have

$$\boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = \boldsymbol{\Phi}_{\mathrm{II},1}(x_1), \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1); \quad -\infty < x_1 < \infty, \quad x_2 = 0 \tag{5.142}$$

From Equation (5.115) it is known that $q_2 = -\mathrm{i}\alpha \mathrm{g}''(z_T) + \mathrm{i}\alpha \bar{\mathrm{g}}''(\bar{z}_T)$, where z_T, α are shown in Eqs. (5.112) and (5.114), respectively. Equation (5.142) yields

$$\begin{aligned}
&-\mathrm{i}\alpha_{\mathrm{I}} \mathrm{g}_{\mathrm{I}}''(x_1) + \mathrm{i}\alpha_{\mathrm{I}} \bar{\mathrm{g}}_{\mathrm{I}}''(\bar{x}_1) = -\mathrm{i}\alpha_{\mathrm{II}} \mathrm{g}_{\mathrm{II}}''(x_1) + \mathrm{i}\alpha_{\mathrm{II}} \bar{\mathrm{g}}_{\mathrm{II}}''(\bar{x}_1) \quad \text{or} \\
&\mathrm{i}\alpha_{\mathrm{I}} \mathrm{g}_{\mathrm{I}}''^{+}(x_1) + \mathrm{i}\alpha_{\mathrm{II}} \bar{\mathrm{g}}_{\mathrm{II}}''^{+}(x_1) = \mathrm{i}\alpha_{\mathrm{II}} \mathrm{g}_{\mathrm{II}}''^{-}(x_1) + \mathrm{i}\alpha_{\mathrm{I}} \bar{\mathrm{g}}_{\mathrm{I}}''^{-}(x_1)
\end{aligned} \tag{5.143}$$

Analogous to Eq. (4.22) from Eq. (5.143) we have

$$\bar{\mathrm{g}}_{\mathrm{II}}''(z_T) = -(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}}) \mathrm{g}_{\mathrm{I}}''(z_T), \quad x_2 > 0; \quad \bar{\mathrm{g}}_{\mathrm{I}}''(z_T) = -(\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}}) \mathrm{g}_{\mathrm{II}}''(z_T), \quad x_2 < 0 \tag{5.144}$$

It is assumed that the temperature satisfies the same equation:

$$\bar{\mathrm{g}}_{\mathrm{II}}'(z_T) = -(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}}) \mathrm{g}_{\mathrm{I}}'(z_T), \quad x_2 > 0; \quad \bar{\mathrm{g}}_{\mathrm{I}}'(z_T) = -(\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}}) \mathrm{g}_{\mathrm{II}}'(z_T), \quad x_2 < 0 \tag{5.145}$$

Equations (5.112) and (5.145) yield

$$\begin{aligned}\vartheta_{\rm I}(x_1) &= {\rm g}'_{\rm I}(x_1) + \bar{\rm g}'_{\rm I}(\bar{x}_1) = {\rm g}'_{\rm I}(x_1) - (\alpha_{\rm II}/\alpha_{\rm I}){\rm g}'_{\rm II}(x_1) \\ \vartheta_{\rm II}(x_1) &= {\rm g}'_{\rm II}(x_1) + \bar{\rm g}'_{\rm II}(\bar{x}_1) = {\rm g}'_{\rm II}(x_1) - (\alpha_{\rm I}/\alpha_{\rm II}){\rm g}'_{\rm I}(x_1)\end{aligned} \tag{5.146}$$

Analogously from Eqs. (5.142), (5.145), and (5.122) we get

$$\begin{aligned}\boldsymbol{B}_{\rm I}\boldsymbol{F}_{\rm I}(z) + (\boldsymbol{d}_{\rm I} + \bar{\boldsymbol{d}}_{\rm II}\alpha_{\rm I}/\alpha_{\rm II}){\rm g}'_{\rm I}(z) &= \bar{\boldsymbol{B}}_{\rm II}\bar{\boldsymbol{F}}_{\rm II}(z), \quad x_2 > 0 \\ \boldsymbol{B}_{\rm II}\boldsymbol{F}_{\rm II}(z) + (\boldsymbol{d}_{\rm II} + \bar{\boldsymbol{d}}_{\rm I}\alpha_{\rm II}/\alpha_{\rm I}){\rm g}'_{\rm II}(z) &= \bar{\boldsymbol{B}}_{\rm I}\bar{\boldsymbol{F}}_{\rm I}(z), \quad x_2 < 0\end{aligned} \tag{5.147}$$

Equations (5.120) and (5.147) yield

$$\begin{aligned}\boldsymbol{U}'_{\rm I}(x_1) = {}& \boldsymbol{A}_{\rm I}\boldsymbol{F}_{\rm I}(x_1) + c_{\rm I}{\rm g}'_{\rm I}(x_1) + \bar{\boldsymbol{A}}_{\rm I}\bar{\boldsymbol{B}}_{\rm I}^{-1}[\boldsymbol{B}_{\rm II}\boldsymbol{F}_{\rm II}(x_1) \\ & + (\boldsymbol{d}_{\rm II} + \bar{\boldsymbol{d}}_{\rm I}\alpha_{\rm II}/\alpha_{\rm I}){\rm g}'_{\rm II}(x_1)] - (\alpha_{\rm II}/\alpha_{\rm I})\bar{c}_{\rm I}{\rm g}'_{\rm II}(x_1) \\ \boldsymbol{U}'_{\rm II}(x_1) = {}& \boldsymbol{A}_{\rm II}\boldsymbol{F}_{\rm II}(x_1) + \boldsymbol{c}_{\rm II}{\rm g}'_{\rm II}(x_1) + \bar{\boldsymbol{A}}_{\rm II}\bar{\boldsymbol{B}}_{\rm II}^{-1}[\boldsymbol{B}_{\rm I}\boldsymbol{F}_{\rm I}(x_1) \\ & + (\boldsymbol{d}_{\rm I} + \bar{\boldsymbol{d}}_{\rm II}\alpha_{\rm I}/\alpha_{\rm II}){\rm g}'_{\rm I}(x_1)] - (\alpha_{\rm I}/\alpha_{\rm II})\bar{\boldsymbol{c}}_{\rm II}{\rm g}'_{\rm I}(x_1)\end{aligned} \tag{5.148}$$

5.5.2 *The Solution of Temperature*

Using Eq. (5.146) and $\vartheta_{\rm I}(x_1) = \vartheta_{\rm II}(x_1)$ on the connective surface yields

$${\rm g}'_{\rm I}(x_1)[1 + (\alpha_{\rm I}/\alpha_{\rm II})] = {\rm g}'_{\rm II}(x_1)[1 + (\alpha_{\rm II}/\alpha_{\rm I})], \quad x \notin L_{\rm c} \tag{5.149}$$

So we can construct a function $\theta(z_T)$ analytic in whole z_T plane except $L_{\rm c}$:

$$\theta(z_T) = \begin{cases} [1 + (\alpha_{\rm I}/\alpha_{\rm II})]{\rm g}_{\rm I}(z_T), & x_2 > 0 \\ [1 + (\alpha_{\rm II}/\alpha_{\rm I})]{\rm g}_{\rm II}(z_T), & x_2 < 0 \end{cases}, \quad x \notin L_{\rm c} \tag{5.150}$$

The heat flow on the crack surface is

$$\begin{aligned}q_{\rm I2} &= -\lambda_{2j}\vartheta_{,j} = -{\rm i}\alpha_{\rm I}{\rm g}''_{\rm I}(x_1) + {\rm i}\alpha_{\rm I}\bar{\rm g}''_{\rm I}(\bar{x}_1) = -{\rm i}\alpha_{\rm I}{\rm g}''_{\rm I}(x_1) - {\rm i}\alpha_{\rm II}{\rm g}''_{\rm II}(x_1) \\ &= -{\rm i}[\alpha_{\rm I}\alpha_{\rm II}/(\alpha_{\rm I} + \alpha_{\rm II})]\left[\theta''^{+}(x_1) + \theta''^{-}(x_1)\right]\end{aligned} \tag{5.151}$$

So the boundary condition of the heat flow on the crack surface is reduced to

$$\theta''^{+}(x_1) + \theta''^{-}(x_1) = {\rm i}[(\alpha_{\rm I} + \alpha_{\rm II})/\alpha_{\rm I}\alpha_{\rm II}]q_0(x_1), \quad x \in L_{\rm c} \tag{5.152}$$

Its solution is

$$\begin{aligned} \theta''(z_T) &= \frac{\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}}}{2\pi\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}} Z_0(z_T)\int_{L_c}\frac{q_0(x_1)\mathrm{d}x_1}{Z_0^+(x_1)(x_1-z_T)} + Z_0(z_T)C(z_T) \\ Z_0(z_T) &= \prod_{j=1}^{n}\left(z_T-a_j\right)^{-1/2}\left(z_T-b_j\right)^{-1/2} \end{aligned} \tag{5.153}$$

where $C(z_T)$ is the polynomial degree n of z_T.

5.5.3 The Solution of Generalized Stress

Because on $L-L_c$ $i\boldsymbol{d}'(x_1)=0$, so

$$\begin{aligned} &\boldsymbol{H}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \left\{\mathrm{i}[\boldsymbol{c}_{\mathrm{I}} + (\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\bar{\boldsymbol{c}}_{\mathrm{II}}] + \bar{\boldsymbol{Y}}_{\mathrm{II}}[\boldsymbol{d}_{\mathrm{I}} + (\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\bar{\boldsymbol{d}}_{II}]\right\}\mathrm{g}'_{\mathrm{I}}(x_1) \\ &\quad = \bar{\boldsymbol{H}}\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1) + \left\{\mathrm{i}[\boldsymbol{c}_{\mathrm{II}} + (\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})\bar{\boldsymbol{c}}_{\mathrm{I}}] + \bar{\boldsymbol{Y}}_{\mathrm{I}}[\boldsymbol{d}_{\mathrm{II}} + (\alpha_{\mathrm{II}}/\alpha_{\mathrm{I}})\bar{\boldsymbol{d}}_{\mathrm{I}}]\right\}\mathrm{g}'_{\mathrm{II}}(x_1), \quad x \notin L_c \end{aligned} \tag{5.154}$$

where $\boldsymbol{H}=\boldsymbol{Y}_{\mathrm{I}}+\bar{\boldsymbol{Y}}_{\mathrm{II}},\ \boldsymbol{Y}_\alpha=\mathrm{i}\boldsymbol{A}_\alpha\boldsymbol{B}_\alpha^{-1}$ $(\alpha=\mathrm{I},\mathrm{II})$. So we can construct a function $\boldsymbol{h}(z)$ analytic in whole z plane except L_c:

$$\boldsymbol{h}(z) = \begin{cases} \boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z) + (\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})^{-1}\boldsymbol{H}^{-1}[i(\alpha_{\mathrm{II}}\boldsymbol{c}_{\mathrm{I}}+\alpha_{\mathrm{I}}\bar{\boldsymbol{c}}_{\mathrm{II}}) \\ \quad +\bar{\boldsymbol{Y}}_{\mathrm{II}}(\alpha_{\mathrm{II}}\boldsymbol{d}_{\mathrm{I}}+\alpha_{\mathrm{I}}\bar{\boldsymbol{d}}_{\mathrm{II}})]\theta'(z), \quad x_2>0 \\ \boldsymbol{H}^{-1}\bar{\boldsymbol{H}}\Big\{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z) + (\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})^{-1}\bar{\boldsymbol{H}}^{-1}[i(\alpha_{\mathrm{I}}\boldsymbol{c}_{\mathrm{II}}+\alpha_{\mathrm{II}}\bar{\boldsymbol{c}}_{\mathrm{I}}) \\ \quad +\bar{\boldsymbol{Y}}_{\mathrm{I}}(\alpha_{\mathrm{I}}\boldsymbol{d}_{\mathrm{II}}+\alpha_{\mathrm{II}}\bar{\boldsymbol{d}}_{\mathrm{I}})]\theta'(z)\Big\}, \quad x_2<0 \end{cases} \tag{5.155}$$

Using Eqs. (5.145), (5.147), and (5.155), Eq. (5.122) can be reduced to

$$\begin{aligned} \boldsymbol{\Phi}_{\mathrm{I},1}(x_1) &= \boldsymbol{h}^+(x_1) + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1) - \boldsymbol{\eta}_1\theta'^+(x_1) - \boldsymbol{\eta}_2\theta'^-(x_1) \\ \boldsymbol{\eta}_1 &= -\bar{\boldsymbol{\eta}}_2 = (\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})^{-1}\boldsymbol{H}^{-1}\left\{[\mathrm{i}(\alpha_{\mathrm{II}}\boldsymbol{c}_{\mathrm{I}}+\alpha_{\mathrm{I}}\bar{\boldsymbol{c}}_{\mathrm{II}}) + \bar{\boldsymbol{Y}}_{\mathrm{II}}(\alpha_{\mathrm{II}}\boldsymbol{d}_{\mathrm{I}}+\alpha_{\mathrm{I}}\bar{\boldsymbol{d}}_{\mathrm{II}})] - \alpha_{\mathrm{II}}\boldsymbol{d}_{\mathrm{I}}\right\} \\ \boldsymbol{\eta}_2 &= -\bar{\boldsymbol{\eta}}_1 = (\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})^{-1}\bar{\boldsymbol{H}}^{-1}\left\{[\mathrm{i}(\alpha_{\mathrm{I}}\boldsymbol{c}_{\mathrm{II}}+\alpha_{\mathrm{II}}\bar{\boldsymbol{c}}_{\mathrm{I}}) + \bar{\boldsymbol{Y}}_{\mathrm{I}}(\alpha_{\mathrm{I}}\boldsymbol{d}_{\mathrm{II}}+\alpha_{\mathrm{II}}\bar{\boldsymbol{d}}_{\mathrm{I}})] - \alpha_{\mathrm{I}}\boldsymbol{d}_{\mathrm{II}}\right\} \end{aligned} \tag{5.156}$$

Substituting Eq. (5.156) into the generalized stress boundary condition in (5.141) yields

$$\boldsymbol{h}^+(x_1) + \bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\boldsymbol{h}^-(x_1) = \tilde{\boldsymbol{\Sigma}}_0(x_1), \quad \tilde{\boldsymbol{\Sigma}}_0(x_1) = \boldsymbol{\Sigma}_0(x_1) + \boldsymbol{\eta}_1\theta'^+(x_1) + \boldsymbol{\eta}_2\theta'^-(x_1) \tag{5.157}$$

Equation (5.157) is identical with (4.28) except using $\tilde{\boldsymbol{\Sigma}}_0(x_1)$ instead of $\boldsymbol{\Sigma}_0(x_1)$, so its solution is still expressed by Eqs. (4.41a) and (4.9):

$$\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{h}(z)=\boldsymbol{Q}(z)\left[\boldsymbol{C}(z)+\frac{1}{2\pi\mathrm{i}}\int_L\frac{\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_0(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1-z)}\right],\quad \boldsymbol{Q}(z)=\left\langle Y_0^{(j)}(z)\right\rangle \tag{5.158}$$

From Eq. (5.157) it is seen that its homogeneous equation is fully identical with (4.29) and does not relate to the temperature, so the eigenvalues and eigenvectors of both equations are also the same. Therefore, $\boldsymbol{Q}(z)$ and $\boldsymbol{\Omega}$ in Eq. (5.158) are still expressed by Eq. (4.37).

On the connective surface $\boldsymbol{h}^+(x_1)=\boldsymbol{h}^-(x_1)=\boldsymbol{h}(x_1)$, $\theta'^+(x_1)=\theta'^-(x_1)=\theta'(x_1)$, so we have

$$\boldsymbol{\Sigma}_2(x_1)=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)=\left(\mathbf{I}+\bar{\boldsymbol{H}}^{-1}\boldsymbol{H}\right)\boldsymbol{h}(x_1)-(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\theta'(x_1),\quad x\in L-L_{\mathrm{c}} \tag{5.159}$$

The open displacement disconnected value $\hat{\boldsymbol{d}}$ behind the crack tip is

$$\hat{\boldsymbol{d}}'(x_1)=\boldsymbol{U}_{\mathrm{I}}'(x_1)-\boldsymbol{U}_{\mathrm{II}}'(x_1)=-\mathrm{i}\boldsymbol{H}[\boldsymbol{h}^+(x_1)-\boldsymbol{h}^-(x_1)],\quad x\notin L_{\mathrm{c}} \tag{5.160}$$

5.5.4 A Single Interface Crack

In the case of a crack of length $2a$, we have $Z_0(z_T)=\sqrt{z_T^2-a^2}$. If only the normal heat flow q_0 on the crack surface, Eq. (5.153) yields

$$\begin{aligned}\theta''(z_T)&=\mathrm{i}q_0^*\left[1-\left(z_T\Big/\sqrt{z_T^2-a^2}\right)\right]+C_1z_T+C_0\\ \theta'(z_T)&=\mathrm{i}q_0^*\left(z_T-\sqrt{z_T^2-a^2}\right),\quad q_0^*=q_0[(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})/2\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}]\end{aligned} \tag{5.161}$$

where $C_1=0$ due to $\boldsymbol{q}\cdot\boldsymbol{n}=\mathbf{0}$ at infinity, and $C_0=0$ due to the temperature single value condition $\int_{-a}^{a}\left[\theta''^+(x_1)-\theta''^-(x_1)\right]\mathrm{d}x_1=0$. Equation (5.150) yields

$$g_I''(z_T)=[\alpha_{\mathrm{II}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})]\theta''(z_T),\quad g_{\mathrm{II}}''(z_T)=[\alpha_{\mathrm{I}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})]\theta''(z_T) \tag{5.162}$$

Because Eq. (5.158) is decoupling, on the crack surface, for normalized Ω we have

$$\boldsymbol{h}(z)=\boldsymbol{\Omega}\boldsymbol{Q}(z)\left\{\boldsymbol{C}(z)+\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{\tilde{\boldsymbol{\Sigma}}_0(x_1)\mathrm{d}x_1}{\boldsymbol{Q}^+(x_1)(x_1-z)}\right\},\quad Y_0^{(i)}(z)=\frac{1}{\sqrt{z^2-a^2}}\left(\frac{z-a}{z+a}\right)^{\mathrm{i}\varepsilon_i} \tag{5.163}$$

In Eq. (5.163) the integrated function containing $\sqrt{z^2-a^2}$, so when use Eq. (4.18), $g^*=-1$ should be used due to $\lim_{z\to x^-}\sqrt{z^2-a^2}=-\lim_{z\to x^+}\sqrt{z^2-a^2}$. These integrals are

$$\frac{1}{2\pi\,\mathrm{i}}\int_{-a}^{a}\frac{\mathrm{d}x_1}{Y_0^{(j)+}(x_1)(x_1-z)}=\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\left\{\frac{1}{Y_0^{(j)}(z)}-\left(z+2\mathrm{i}\varepsilon_j a\right)\right\}$$
$$\frac{1}{2\pi\,\mathrm{i}}\int_{-a}^{a}\frac{x_1\mathrm{d}x_1}{Y_0^{(j)+}(x_1)(x_1-z)}=\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\left\{\frac{z}{Y_0^{(j)}(z)}-\left[z^2+2\mathrm{i}\varepsilon_j az-\frac{a^2}{2}\left(1+4\varepsilon_j^2\right)\right]\right\}$$
$$\frac{1}{2\pi\,\mathrm{i}}\int_{-a}^{a}\frac{\mathrm{i}\sqrt{a^2-x_1^2}\mathrm{d}x_1}{Y_0^{(j)+}(x_1)(x_1-z)}=\frac{1}{1-\mathrm{e}^{2\pi\varepsilon_j}}\left\{\frac{\sqrt{z^2-a^2}}{Y_0^{(j)}(z)}-\left[z^2+2\mathrm{i}\varepsilon_j az-a^2\left(1+2\varepsilon_j^2\right)\right]\right\} \tag{5.164}$$

Using Eq. (5.164), Eq. (5.163) is reduced to

$$\begin{aligned}\boldsymbol{h}(z)=&\boldsymbol{\Omega Q}(z)(\boldsymbol{C}_1z+\boldsymbol{C}_0)+\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle 1-(z+2\mathrm{i}\varepsilon_* a)Y_0^{(j)}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0\\&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle z-\left[z^2+2\mathrm{i}\varepsilon_j az-\frac{a^2}{2}\left(1+4\varepsilon_j^2\right)\right]Y_0^{(j)}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\\&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1-\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle\sqrt{z^2-a^2}-\left[z^2+2\mathrm{i}\varepsilon_j az-a^2(1+2\varepsilon j)\right]Y_0^{(j)}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)\end{aligned} \tag{5.165}$$

At infinity, $\boldsymbol{Q}(z)\to\boldsymbol{I}/z$, $\theta'(z_T)\to 0$, $\boldsymbol{\Sigma}_2(x_1)=\boldsymbol{0}$, from Eqs. (5.159) and (5.165) we get

$$\begin{aligned}&\boldsymbol{C}_1=\mathrm{i}q_0^*\left\langle\frac{2\mathrm{i}\varepsilon_j a}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)+\mathrm{i}q_0^*\left\langle\frac{2\mathrm{i}\varepsilon_j a}{1-\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)\langle 2\mathrm{i}\varepsilon_j a\rangle\\&C_1^{(j)}=\mathrm{i}q_0^*\left\{\frac{2\mathrm{i}\varepsilon_j a}{1+\mathrm{e}^{2\pi\varepsilon_j}}\bar{\Omega}_{jk}^{\mathrm{T}}(\eta_{1k}+\eta_{2k})+\frac{2\mathrm{i}\varepsilon_j a}{1-\mathrm{e}^{2\pi\varepsilon_j}}\bar{\Omega}_{jk}^{\mathrm{T}}(\eta_{2k}-\eta_{1k})\right\},\quad\bar{\Omega}_{jk}^{\mathrm{T}}=\Omega_{kj}\end{aligned} \tag{5.166}$$

Substitution of Eq. (5.166) into Eq. (5.165) yields

$$\begin{aligned}\boldsymbol{h}(z)=&\boldsymbol{\Omega Q}(z)\boldsymbol{C}_0+\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle 1-\left(z+2\mathrm{i}\varepsilon_j a\right)Y_0^{(j)}(z)\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0\\&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle z-\left[z^2-\frac{a^2}{2}\left(1+4\varepsilon_j^2\right)\right]Y_0^{(j)}(z)\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\\&+\mathrm{i}q_0^*\boldsymbol{\Omega}\left\langle\frac{1}{1-\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\left\langle\sqrt{z^2-a^2}-\left[z^2-a^2\left(1+2\varepsilon_j^2\right)\right]Y_0^{(j)}(z)\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)\end{aligned} \tag{5.167}$$

$\boldsymbol{C}_0$ is determined by the single value condition, and according to Eq. (5.160) it is equivalent to

$$\boldsymbol{H}\int_{-a}^{a}[\boldsymbol{h}^{+}(x_1)-\boldsymbol{h}^{-}(x_1)]\mathrm{d}x_1=0,\quad H_{ij}\int_{-a}^{a}\left[h_j^{+}(x_1)-h_j^{-}(x_1)\right]\mathrm{d}x_1=0 \tag{5.168}$$

On the crack surface there has $\left\langle Y_0^{(j)-}(x_1)\right\rangle=-\left\langle \mathrm{e}^{2\pi\varepsilon_j}Y_0^{(j)+}(x_1)\right\rangle$ or $\boldsymbol{Q}^{+}-\boldsymbol{Q}^{-}=\left\langle 1+\mathrm{e}^{2\pi\varepsilon_j}\right\rangle\boldsymbol{Q}^{+}$. Using the following equation (Shen and Kuang 1998)

$$\int_{-a}^{a}\frac{x^n}{\sqrt{a^2-x^2}}\left(\frac{a-x}{a+x}\right)^{\mathrm{i}\varepsilon}\mathrm{d}x=\begin{cases}\pi/\cosh\pi\varepsilon & \text{when } n=0\\ -2\mathrm{i}\pi a\varepsilon/\cosh\pi\varepsilon & \text{when } n=1\\ (1-4\varepsilon^2)\pi a^2/2\cosh\pi\varepsilon & \text{when } n=2\end{cases} \tag{5.169}$$

and noting $\int_{-a}^{a}\sqrt{x_1^2-a^2}\mathrm{d}x_1=\pm\,\mathrm{i}\pi a^2/2$, from the single valued condition we get

$$-\boldsymbol{C}_0=\mathrm{i}q_0^{*}a^2\left\langle\frac{4\varepsilon_j^2}{1+\mathrm{e}^{2\pi\varepsilon_j}}\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)+\mathrm{i}q_0^{*}a^2\left\langle\frac{1+8\varepsilon_j^2+2\mathrm{i}\cosh\pi\varepsilon_j}{2(1-\mathrm{e}^{2\pi\varepsilon_j})}\right\rangle\overline{\boldsymbol{\Omega}}^{\mathrm{T}}(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1) \tag{5.170}$$

The stress intensity is

$$\boldsymbol{K}=[K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_D]^{\mathrm{T}}=\lim_{x_1\to b_j}\sqrt{2\pi(x_1-b_j)}\boldsymbol{\Omega}\left\langle\left(x_1-b_j\right)^{-\mathrm{i}\varepsilon_j}\right\rangle\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}_2(x_1) \tag{5.171}$$

where $\boldsymbol{\Sigma}_2(x_1)$ is determined by Eq. (5.159).

For a homogeneous material $\boldsymbol{A}_{\mathrm{I}}=\boldsymbol{A}_{\mathrm{II}}=\boldsymbol{A}$, and $\boldsymbol{H}=\bar{\boldsymbol{H}}$, $\boldsymbol{C}_0=\boldsymbol{0}$, $Y_0^{(j)}=1\Big/\sqrt{z_j^2-a^2}$. So the solution is

$$\begin{aligned}&\theta''(z_T)=\mathrm{i}q_0^{*}\left\{1-\frac{z_T}{\sqrt{z_T^2-a^2}}\right\},\quad \theta'(z_T)=\mathrm{i}q_0^{*}\left(z_T-\sqrt{z_T^2-a^2}\right)\\ &\mathrm{g}_{\mathrm{I}}''(z_T)=\mathrm{g}_{\mathrm{II}}''(z_T)=\frac{1}{2}\theta''(z_T)=\frac{\mathrm{i}q_0^{*}}{2}\left\{1-\frac{z_T}{\sqrt{z_T^2-a^2}}\right\}\\ &\boldsymbol{h}(z)=\frac{1}{2}\boldsymbol{\Omega}\langle 1-z\boldsymbol{Q}(z)\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0+\mathrm{i}q_0^{*}\boldsymbol{\Omega}\left\langle z-\frac{2z^2-a^2}{2}\boldsymbol{Q}(z)\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\eta}\end{aligned} \tag{5.172}$$

And the asymptotic stress field near the crack tip $x_1=a$ is

$$\begin{aligned}\boldsymbol{\Sigma}_2(x_1)&=\boldsymbol{\Phi}_{\mathrm{I},1}(x_1)=2\boldsymbol{h}(x_1)-2\boldsymbol{\eta}\theta'(x_1)\\ &=-\boldsymbol{\Omega}\left\langle\sqrt{\frac{a}{2}}\frac{1}{\sqrt{x_1-a}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0-\mathrm{i}q_0^{*}a\boldsymbol{\Omega}\left\langle\sqrt{\frac{a}{2}}\frac{1}{\sqrt{x_1-a}}\right\rangle\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\eta}\end{aligned} \tag{5.173}$$

The stress intensity factor at $x_1 = a$ is

$$\begin{aligned} \boldsymbol{K} &= [K_{\mathrm{II}}, K_{\mathrm{I}}, K_{\mathrm{III}}, K_D]^{\mathrm{T}} = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)}\boldsymbol{\Sigma}_2(x_1) \\ &= -\sqrt{\pi a}\left(\boldsymbol{\Omega}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\Sigma}_0 + \mathrm{i}q_0^* a \boldsymbol{\Omega}\bar{\boldsymbol{\Omega}}^{\mathrm{T}}\boldsymbol{\eta}\right) = -\sqrt{\pi a}\left(\boldsymbol{\Sigma}_0 + \mathrm{i}q_0^* a\boldsymbol{\eta}\right) \end{aligned} \tag{5.174}$$

5.6 Point Heat Source and Interaction with Cracks

5.6.1 Point Heat Source in Piezoelectric and Bi-piezoelectric Material

Hwu (1990) discussed the thermal stress in an anisotropic elastic material. Shen et al. (1995) and Shen and Kuang (1998) discussed the thermal stress in a pyroelectric material, the point heat source, and their interactions.

1. *Heat source in a homogeneous material* For a point heat source, the temperature $\vartheta = T - T_0$ can be expressed as

$$\begin{aligned} &\vartheta(x_1, x_2) = 2\mathrm{Re}\, \mathrm{g}'(z_T), \quad z_T = x_1 + \mu_T x_2, \quad \mu_T = (-\lambda_{12} + i\alpha)/\lambda_{22} \\ &\mathrm{g}'(z_T) = \mathrm{g}_0'(z_T) = c\ln(z_T - z_{T0}), \quad \mathrm{g}_0''(z_T) = c/(z_T - z_{T0}) \end{aligned} \tag{5.175}$$

According to Eq. (5.116) for a point heat source with strength M located at $z_0(x_{10}, x_{20})$ in an infinite homogeneous pyroelectric material, c is determined by the following equation:

$$\begin{aligned} &M = \oint q_n \mathrm{d}s = -2\mathrm{Im}\{\alpha[\mathrm{g}'(z_T) - \mathrm{g}'(z_0)]\}_0^{2\pi} = -4\pi\alpha c; \quad c = -M/4\pi\alpha \\ &q_1 = 2\mathrm{Re}[\mathrm{i}\alpha\mu_T \mathrm{g}''(z_T)], \quad q_2 = -2\mathrm{Re}[\mathrm{i}\alpha \mathrm{g}''(z_T)]; \quad \alpha = \lambda_{22}(\mu_T - \bar{\mu}_T)/2\mathrm{i} \end{aligned} \tag{5.176}$$

So finally the solution of the temperature in an infinite homogeneous pyroelectric material is

$$\vartheta = 2\mathrm{Re}\, \mathrm{g}_0'(z_T) = -(M/2\pi\alpha)\mathrm{Re}\ln(z_T - z_{T0}), \quad z_{T0} = x_{10} + \mu_T x_{20} \tag{5.177}$$

2. *Heat source in a bimaterial* The solving method of a heat source in a bimaterial is analogous to that in Paragraph 3.6.2. Let the point heat source with strength M be located at $z_0(x_{10}, x_{20})$ in material II that occupied $S^-, x_2 < 0$. The solution can be assumed as

$$\begin{aligned} &\mathrm{g}'(z_T) = \begin{cases} \mathrm{g}_{\mathrm{I}}'(z_T), & z_T \in S^+ \\ \mathrm{g}_{\mathrm{II}}'(z_T) + \mathrm{g}_0'(z_T), & z_T \in S^- \end{cases} \\ &\mathrm{g}_0'(z_T) = c_{\mathrm{II}}\ln(z_T - z_{T0}), \quad \mathrm{g}_0''(z_T) = c_{\mathrm{II}}/(z_T - z_{T0}), \quad c_{\mathrm{II}} = -M/4\pi\alpha_{\mathrm{II}} \end{aligned} \tag{5.178}$$

Because heat flow and temperature are continuous in whole axis x_1, so according to Eqs. (5.115) and (5.112) it yields

$$\begin{aligned}&\alpha_{\mathrm{I}}g_{\mathrm{I}}''(x_1)-\alpha_{\mathrm{I}}\overline{g_{\mathrm{I}}''(x_1)}=\alpha_{\mathrm{II}}g_{\mathrm{II}}''(x_1)-\alpha_{\mathrm{II}}\overline{g_{\mathrm{II}}''(x_1)}+\alpha_{\mathrm{II}}g_0''(x_1)-\alpha_{\mathrm{II}}\overline{g_0''(x_1)}\\&g_{\mathrm{I}}'(x_1)+\overline{g_{\mathrm{I}}'(x_1)}=g_{\mathrm{II}}'(x_1)+g_0'(x_1)+\overline{g_{\mathrm{II}}'(x_1)}+\overline{g_0'(x_1)}\end{aligned} \tag{5.179}$$

If $\mathbf{q}\to\mathbf{0},T\to 0$ when $|z|\to\infty$, like Eqs. (3.161), (3.162), (3.163), (3.164), (3.165) or (4.22), (4.23) we have

$$\begin{aligned}&\alpha_{\mathrm{I}}g_{\mathrm{I}}''(z_T)+\alpha_{\mathrm{II}}\overline{g_{\mathrm{II}}''}(z_T)-\alpha_{\mathrm{II}}g_0''(z_T)=0,\quad \alpha_{\mathrm{II}}g_{\mathrm{II}}''(z_T)+\alpha_{\mathrm{I}}\overline{g_{\mathrm{I}}''}(z_T)-\alpha_{\mathrm{II}}\overline{g_0''}(z_T)=0\\&g_{\mathrm{I}}'(z_T)-\overline{g_{\mathrm{II}}'}(z_T)-g_0'(z_T)=0,\quad g_{\mathrm{II}}'(z_T)-\overline{g_{\mathrm{I}}'}(z_T)+\overline{g_0'}(z_T)=0\end{aligned} \tag{5.180}$$

Equations (5.178), (5.179), and (5.180) yield

$$\begin{aligned}&g'(z_T)=\begin{cases}g_{\mathrm{I}}'(z_T)=2\alpha_2 g_0'(z_T), & z\in S^+\\ g_{\mathrm{II}}'(z_T)+g_0'(z_T)=(\alpha_2-\alpha_1)\bar{g}_0'(z_T)+g_0'(z_T), & z\in S^-\end{cases}\\&\alpha_1=\alpha_{\mathrm{I}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}}),\quad \alpha_2=\alpha_{\mathrm{II}}/(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})\end{aligned} \tag{5.181}$$

On the interface $x_2=0$ we have

$$q_2=-\mathrm{i}\alpha_{\mathrm{I}}g_{\mathrm{I}}''(z_T)+\mathrm{i}\alpha_{\mathrm{I}}\overline{g_{\mathrm{I}}''(z_T)}=-2\mathrm{i}\alpha_{\mathrm{I}}\alpha_2\left[g_0''(x_1)-\bar{g}_0''(x_1)\right] \tag{5.182}$$

Because the generalized stress and displacement are continuous on whole axis x_1, according to Eqs. (3.161), (3.162), (3.163), (3.164), and (3.165), we can derive

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)-\bar{\boldsymbol{B}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z)-[(\alpha_2-\alpha_1)\bar{\boldsymbol{d}}_{\mathrm{II}}-2\alpha_2\boldsymbol{d}_{\mathrm{I}}+\boldsymbol{d}_{\mathrm{II}}]g_0'(z)=0\\&\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\bar{\boldsymbol{B}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z)+[(\alpha_2-\alpha_1)\boldsymbol{d}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{d}}_{\mathrm{I}}+\bar{\boldsymbol{d}}_{\mathrm{II}}]\bar{g}_0'(z)=0\end{aligned} \tag{5.183}$$

and

$$\begin{aligned}&\boldsymbol{A}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(z)-\bar{\boldsymbol{A}}_{\mathrm{II}}\bar{\boldsymbol{F}}_{\mathrm{II}}(z)-[(\alpha_2-\alpha_1)\bar{\boldsymbol{c}}_{\mathrm{II}}-2\alpha_2\boldsymbol{c}_{\mathrm{I}}+\boldsymbol{c}_{\mathrm{II}}]g_0'(z)=0\\&\boldsymbol{A}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(z)-\bar{\boldsymbol{A}}_{\mathrm{I}}\bar{\boldsymbol{F}}_{\mathrm{I}}(z)+[(\alpha_2-\alpha_1)\boldsymbol{c}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{c}}_{\mathrm{I}}+\bar{\boldsymbol{c}}_{\mathrm{II}}]\bar{g}_0'(z)=0\end{aligned} \tag{5.184}$$

From Eqs. (5.183) and (5.184), the stress functions are

$$\begin{aligned}\boldsymbol{F}_{\mathrm{I}}(z_j)&=\mathrm{i}\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}[(\alpha_2-\alpha_1)\bar{\boldsymbol{c}}_{\mathrm{II}}-2\alpha_2\boldsymbol{c}_{\mathrm{I}}+\boldsymbol{c}_{\mathrm{II}}]g_0'(z_j)\\&\quad+\boldsymbol{B}_{\mathrm{I}}^{-1}\boldsymbol{H}^{-1}\boldsymbol{Y}_{\mathrm{II}}[(\alpha_2-\alpha_1)\bar{\boldsymbol{d}}_{\mathrm{II}}-2\alpha_2\boldsymbol{d}_{\mathrm{I}}+\boldsymbol{d}_{\mathrm{II}}]g_0'(z_j)\\\boldsymbol{F}_{\mathrm{II}}(z_j)&=-\mathrm{i}\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}[(\alpha_2-\alpha_1)\boldsymbol{c}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{c}}_{\mathrm{I}}+\bar{\boldsymbol{c}}_{\mathrm{II}}]\bar{g}_0'(z_j)\\&\quad-\boldsymbol{B}_{\mathrm{II}}^{-1}\bar{\boldsymbol{H}}^{-1}\bar{\boldsymbol{Y}}_{\mathrm{I}}[(\alpha_2-\alpha_1)\boldsymbol{d}_{\mathrm{II}}-2\alpha_2\bar{\boldsymbol{d}}_{\mathrm{I}}+\bar{\boldsymbol{d}}_{\mathrm{II}}]\bar{g}_0'(z_j)\end{aligned} \tag{5.185}$$

According to Eq. (5.121) on $x_2=0$ we have

$$\boldsymbol{\Sigma}_2(x_1)=2\mathrm{Re}\left\{\boldsymbol{B}_{\mathrm{II}}\boldsymbol{F}_{\mathrm{II}}(x_1)+\boldsymbol{d}_{\mathrm{II}}g_{\mathrm{II}}'(x_1)\right\}=2\mathrm{Re}\left\{\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1)+\boldsymbol{d}_{\mathrm{I}}g_{\mathrm{I}}'(x_1)\right\} \tag{5.186}$$

5.6.2 The Point Heat Source Located at the External of an Elliptic Inclusion

Let an infinite piezoelectric material II occupied region Ω^- with an elliptic inclusion I, occupied region Ω^+ of major semiaxis a and minor axis b directed along the material principle axes x_1 and x_2, respectively. The interface of Ω^- and Ω^+ is denoted by L, its normal is denoted by $\boldsymbol{n}$ directed the inside of the inclusion or the outside of the piezoelectric material. At infinity $\boldsymbol{T}_{\text{II}} = \boldsymbol{0},\ q_n = 0$ and the connective conditions on the L are

$$\boldsymbol{T}_{\text{I}} = \boldsymbol{T}_{\text{II}}, \quad q_{\text{I}2}(x_1) = q_{\text{II}2}(x_1), \quad x \in L \tag{5.187}$$

Let a point heat source at z_0 with strength M be located in the piezoelectric material. We shall use the transform method to solve this problem (Qin 1998, 1999). The transform function from z plane to ς plane is shown in Eqs. (3.82) and (3.86). The point ς_0 in ς plane is corresponding to point z_0 in z plane. L is transformed to Γ. Let $\mathrm{g}_0(\varsigma_T)$ be the fundamental solution in the ς plane when the piezoelectric material occupies the whole space and as in Sect. 5.6.1 we take

$$\mathrm{g}_0'(\varsigma_T) = c\ln(\varsigma_T - \varsigma_{0T}), \quad c = -M/4\pi\alpha_{\text{II}}; \quad \vartheta_0(x_1, x_2) = 2\mathrm{Re}\,\mathrm{g}_0'(\varsigma_T) \tag{5.188}$$

Obviously $\mathrm{g}_0(\varsigma_T)$ is analytic in the inclusion Ω^+. Assume the solution of the problem is

$$\mathrm{g}'(\varsigma_T) = \begin{cases} \mathrm{g}_{\text{I}}'(\varsigma_T), & \varsigma_T \in \Omega^+ - \Omega_0 \\ \mathrm{g}_{\text{II}}'(\varsigma_T) + \mathrm{g}_0'(\varsigma_T), & \varsigma_T \in \Omega^- \end{cases} \tag{5.189}$$

where $\mathrm{g}_{\text{I}}'(\varsigma_T)$ and $\mathrm{g}_{\text{II}}'(\varsigma_T)$ are analytic functions in $\Omega^+ - \Omega_0$ and Ω^-, and Ω_0 is the region $\rho \le \rho_0 = \sqrt{m}, 0 \le \theta < 2\pi$ and on $\Omega_0\,\phi(\rho_0 \mathrm{e}^{\mathrm{i}\psi}) = \phi(\rho_0 \mathrm{e}^{-\mathrm{i}\psi})$ (see Sect. 3.4.2).

According to Eqs. (5.175) and (5.176), the continuity conditions of temperature ϑ and heat $q_n\mathrm{d}s$ through a differential arc on Γ can be reduced to

$$\begin{aligned} &\mathrm{g}_{\text{I}}'(\sigma) + \overline{\mathrm{g}_{\text{I}}'(\sigma)} = \mathrm{g}_{\text{II}}'(\sigma) + \overline{\mathrm{g}_{\text{II}}'(\sigma)} + \mathrm{g}_0'(\sigma) + \overline{\mathrm{g}_0'(\sigma)}, \\ &\alpha_{\text{I}}\mathrm{g}_{\text{I}}'(\sigma) - \alpha_{\text{I}}\overline{\mathrm{g}_{\text{I}}'(\sigma)} = \alpha_{\text{II}}\mathrm{g}_{\text{II}}'(\sigma) - \alpha_{\text{II}}\overline{\mathrm{g}_{\text{II}}'(\sigma)} + \alpha_{\text{II}}\mathrm{g}_0'(\sigma) - \alpha_{\text{II}}\overline{\mathrm{g}_0'(\sigma)} \end{aligned} \tag{5.190}$$

It is noted that $\mathrm{g}_{\text{I}}'(\sigma)$ is analytic only in an annular region $\Omega^+ - \Omega_0$. Similar to Eqs. (3.84) and (3.85), it yields

$$\begin{aligned} &\mathrm{g}_{\text{I}}'(\varsigma_T) = \sum_{k=1}^{\infty}\left(d_k\varsigma_T^k + d_{-k}\varsigma_T^{-k}\right) = \sum_{k=1}^{\infty} d_k\left(\varsigma_T^k + \upsilon_k\varsigma_T^{-k}\right) \\ &\upsilon_k = \rho_0^{2k} = m_{T\,\text{I}}^k = \left[(a + \mathrm{i}\mu_{T\,\text{I}}b)/(a - \mathrm{i}\mu_{T\,\text{I}}b)\right]^k, \quad \rho_0 \le |\varsigma| \le 1 \end{aligned} \tag{5.191}$$

So Eq. (5.190) can be reduced to

$$\sum_{k=1}^{\infty}(d_k+\bar{\upsilon}_k\bar{d}_k)\sigma^k-\bar{g}'_{\mathrm{II}}(1/\sigma)-g'_0(\sigma)=g'_{\mathrm{II}}(\sigma)+\bar{g}'_0(1/\sigma)-\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\sigma^{-k}$$

$$(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\sum_{k=o}^{\infty}(d_k-\bar{d}_k\bar{\varpi}_k)\sigma^k+\bar{g}'_{\mathrm{II}}(1/\sigma)-g'_0(\sigma)$$
$$=g'_{\mathrm{II}}(\sigma)-\bar{g}'_0(1/\sigma)+(\alpha_{\mathrm{I}}/\alpha_{\mathrm{II}})\sum_{k=o}^{\infty}(\bar{d}_k-\upsilon_k d_k)\sigma^{-k} \tag{5.192}$$

From Eq. (5.192) it is known that the functions at the left side in Eq. (5.192) are analytic in the region Ω^+, whereas those on the right side are analytic in the region Ω^-, and they are continuous on Γ. So these functions are analytic in whole plane and must be constants. So we have

$$\theta_1(\varsigma)=\begin{cases}\sum_{k=1}^{\infty}(d_k+\bar{\upsilon}_k\bar{d}_k)\sigma^k-\overline{g'_{\mathrm{II}}}(1/\varsigma)-g'_0(\varsigma), & \varsigma\in\Omega^+\\ g'_{\mathrm{II}}(\varsigma)-\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\sigma^{-k}+\overline{g'_0}(1/\varsigma), & \varsigma\in\Omega^-\end{cases}$$
$$\theta_2(\varsigma)=\begin{cases}\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(d_k-\bar{\upsilon}_k\bar{d}_k)\sigma^k+\alpha_{\mathrm{II}}\overline{g'_{\mathrm{II}}}(1/\varsigma)-\alpha_{\mathrm{II}}g'_0(\varsigma), & \varsigma\in\Omega^+\\ \alpha_{\mathrm{II}}g'_{\mathrm{II}}(\varsigma)+\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(\bar{d}_k-\upsilon_k d_k)\sigma^{-k}-\alpha_{\mathrm{II}}\overline{g'_0}(1/\varsigma), & \varsigma\in\Omega^-\end{cases} \tag{5.193}$$

If there are no generalized external forces acting at infinite, these constants must be zero, i.e., $\theta_1(\infty)=\theta_2(\infty)=0$, so $\theta_1(\varsigma)=\theta_2(\varsigma)=0$ and from Eq. (5.193) we get

$$\begin{aligned}&\sum_{k=1}^{\infty}(d_k+\bar{\upsilon}_k\bar{d}_k)\sigma^k=\bar{g}'_{\mathrm{II}}(1/\varsigma)+g'_0(\varsigma),\\ &\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(d_k-\bar{\upsilon}_k\bar{d}_k)\sigma^k=-\alpha_{\mathrm{II}}\bar{g}'_{\mathrm{II}}(1/\varsigma)+\alpha_{\mathrm{II}}g'_0(\varsigma), \quad \varsigma\in\Omega^+\\ &\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\sigma^{-k}=g'_{\mathrm{II}}(\varsigma)+\bar{g}'_0(1/\varsigma),\\ &\alpha_{\mathrm{I}}\sum_{k=o}^{\infty}(\bar{d}_k-\upsilon_k d_k)\sigma^{-k}=-\alpha_{\mathrm{II}}g'_{\mathrm{II}}(\varsigma)+\alpha_{\mathrm{II}}\bar{g}'_0(1/\varsigma), \quad \varsigma\in\Omega^-\end{aligned} \tag{5.194}$$

Solving Eq. (5.194) yields

$$\begin{aligned}&\sum_{k=1}^{\infty}[(\alpha_{\mathrm{I}}+\alpha_{\mathrm{II}})d_k+(\alpha_{\mathrm{II}}-\alpha_{\mathrm{I}})\bar{\upsilon}_k\bar{d}_k]\varsigma^k=2\alpha_{\mathrm{II}}g'_0(\varsigma)\\ &g'_{\mathrm{II}}(\varsigma)=-\overline{g'_0}(1/\varsigma)+\sum_{k=1}^{\infty}(\bar{d}_k+\upsilon_k d_k)\varsigma^{-k}\end{aligned} \tag{5.195}$$

Solving $d_k, \mathrm{g}'_{\mathrm{II}}(\varsigma)$ and using ς_T instead of ς, from Eq. (5.189), $\mathrm{g}'(\varsigma_T)$ is obtained:

$$\mathrm{g}'(\varsigma_T) = \begin{cases} \mathrm{g}'_{\mathrm{I}}(\varsigma_T) = \sum_{k=1}^{\infty} d_k \left[\varsigma_T^k + \upsilon_k \varsigma_T^{-k}\right], & \varsigma_T \in \Omega^+ - \Omega_0 \\ \tilde{\mathrm{g}}'_{\mathrm{II}}(\varsigma_T) = \mathrm{g}'_0(\varsigma_T) - \bar{\mathrm{g}}'_0(1/\varsigma) + \sum_{k=1}^{\infty} (\bar{d}_k + \upsilon_k d_k)\varsigma_T^{-k}, & \varsigma_T \in \Omega^- \end{cases}$$
$$\vartheta(x_1, x_2) = 2\mathrm{Re}\,\mathrm{g}'(\varsigma_T), \quad \mathrm{g}'_0(z_T) = -(M/4\pi\alpha_{II})\ln(\varsigma_T - \varsigma_{0T}) \tag{5.196}$$

5.6.3 *Interaction of an Impermeable Crack with a Singularity in a Piezoelectric Bimaterial*

Let a mechanical singular generalized load with strength $(\boldsymbol{b}, \boldsymbol{p})$ be located at z_0 in material II that occupied the lower half-plane $\Omega^-, x_2 < 0$. An insulated crack $(-a, a)$ is located on the interface $x_2 = 0$. According to Eqs. (3.165), (3.166), and (3.160), the generalized stress on the interface introduced by the singularity in a piezoelectric material is

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathrm{I}2} = \boldsymbol{\Sigma}_{\mathrm{II}2} = \boldsymbol{\Sigma}_2(x_1) = \boldsymbol{\Phi}_{\mathrm{I},1}(x_1) = 2\mathrm{Re}\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) = 2\mathrm{Re}\left[\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} + \boldsymbol{Y}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{II}}\mathbf{g}_{\mathrm{II}}(z)\right] \\ = 2\mathrm{Re}\left[\boldsymbol{H}^{-1}(\bar{\boldsymbol{Y}}_{\mathrm{II}} + \boldsymbol{Y}_{\mathrm{II}})\boldsymbol{B}_{\mathrm{II}}\langle 2\pi\mathrm{i}(x_1 - z_0)\rangle^{-1}\left(\bar{\boldsymbol{B}}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{A}_{\mathrm{II}}^{\mathrm{T}}\boldsymbol{V}\right)\right] \end{aligned} \tag{5.197}$$

The original problem can be simply solved by the superposition method: The singularity in a piezoelectric material without crack and an external force $-\boldsymbol{\Sigma}_2(x_1)$ applies on the crack surface. The last problem has been solved in Sect. 4.2.4.

From Eq. (5.197) it is seen that the effect of a mechanical singularity is equivalent to adding an external force $-\boldsymbol{\Sigma}_2(x_1)$ on the crack surface, and it does not affect the heat flow.

A point heat source with strength M located at z_0 in material II, $\Omega^-, x_2 < 0$ will produce the heat flow, as shown in Eq. (5.82), and generalized surface traction, as shown in Eq. (5.186), i.e., the point heat source affects both the stress and temperature fields. A point heat source in a bimaterial is equivalent to a point heat source in an infinite homogeneous piezoelectric material II and on the crack surface superposed the following loads:

$$q_T = -q_2 = -\mathrm{i}\alpha_{\mathrm{I}}\alpha_2\left[\mathrm{g}''_0(x_1) - \bar{\mathrm{g}}''_0(x_1)\right], \quad \boldsymbol{t}_2 = -\boldsymbol{\Sigma}_2(x_1) = -2\mathrm{Re}\left[\boldsymbol{B}_{\mathrm{I}}\boldsymbol{F}_{\mathrm{I}}(x_1) + \boldsymbol{d}_{\mathrm{I}}\mathrm{g}'_{\mathrm{I}}(x_1)\right] \tag{5.198}$$

where $\mathrm{g}'_0(x_1), \mathrm{g}'_{\mathrm{I}}(x_1), \boldsymbol{F}_{\mathrm{I}}(x_1)$ are calculated from Eqs. (5.175), (5.178), and (5.185), respectively.

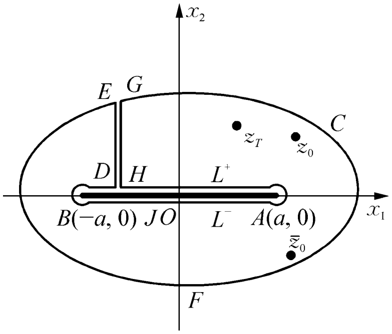


Fig. 5.6 Integral path Λ for the integral $\varPhi$

As an example we discuss the interaction of the above point heat source with a single crack located at $(-a, a)$ (Shen and Kuang 1998). It is assumed that the boundary conditions are

$$\begin{aligned} &\boldsymbol{\varPhi}_{\mathrm{I},i}(x_1) = \boldsymbol{\varPhi}_{\mathrm{II,i}}(x_1) = \mathbf{0}, \quad q_{\mathrm{I}2}(x_1) = q_{\mathrm{II}2}(x_1) = 0, \quad x \in L_{\mathrm{c}} \\ &\boldsymbol{\varPhi}_{\mathrm{I},i}(x_1) = \boldsymbol{\varPhi}_{\mathrm{II,i}}(x_1) = \mathbf{0}, \quad q_i = T = 0, \quad |z| \to \infty; \quad i = 1, 2 \end{aligned} \tag{5.199}$$

Substitution of Eqs. (5.198) and (5.178) into Eq. (5.153) yields

$$\begin{aligned} \theta''(z_T) &= \frac{\alpha_{\mathrm{I}} + \alpha_{\mathrm{II}}}{2\pi\alpha_{\mathrm{I}}\alpha_{\mathrm{II}}} Z_0(z_T) \int_{L_{\mathrm{c}}} \frac{q_T(x_1)\mathrm{d}x_1}{Z_0^+(x_1)(x_1 - z_T)} + Z_0(z_T)C(z_T) \\ &= -\frac{M\mathrm{i}}{4\pi^2\alpha_{\mathrm{II}}} Z_0(z_T) \int_{L_{\mathrm{c}}} \left(\frac{1}{x_1 - z_0} - \frac{1}{x_1 - \bar{z}_0} \right) \frac{1}{Z_0^+(x_1)(x_1 - z_T)} \mathrm{d}x_1 + Z_0(z_T)C(z_T) \end{aligned} \tag{5.200}$$

where $Z_0(z_T) = \left(z_T^2 - a^2\right)^{-1/2}$. The integral in Eq. (5.200) can be integrated. At first we discuss the contour integral

$$\varPhi = \int_{\Lambda} \frac{1}{Z_0^+(x_1)(x_1 - z_T)(x_1 - z_0)(x_1 - \bar{z}_0)} \mathrm{d}x_1$$

where Λ is shown in Fig. 5.6. Inside the contour there are three poles: $z_T, z_0, \bar{z}_0$. Using the residual theorem the $\varPhi$ is reduced to

$$\varPhi = 2\pi\mathrm{i} \left\{ \frac{\sqrt{z_T^2 - a^2}}{(z - z_0)(z - \bar{z}_0)} + \frac{\sqrt{z_0^2 - a^2}}{(z_0 - z)(z_0 - \bar{z}_0)} + \frac{\sqrt{\bar{z}_0^2 - a^2}}{(\bar{z}_0 - z)(\bar{z}_0 - z_0)} \right\}$$

On the other hand it is easy to prove that the integral $\varPhi$ on the path *DEFGH* vanishes whereas on the path *HJD* equals

$$2\int_{L_{\mathrm{c}}} \frac{z_0 - \bar{z}_0}{Z_0^+(x_1)(x_1 - z_T)(x_1 - z_0)(x_1 - \bar{z}_0)} \mathrm{d}x_1 = \int_{L_{\mathrm{c}}} \left(\frac{1}{x_1 - z_0} - \frac{1}{x_1 - \bar{z}_0} \right) \frac{1}{Z_0^+(x_1)(x_1 - z_T)} \mathrm{d}x_1$$

From the condition at infinity in Eq. (5.199) and the single valued condition of temperature we find $C(z_T) = 0$. So Eq. (5.200) is reduced to

$$\theta''(z_T) = \frac{M}{2\pi\alpha_{\rm II}} \frac{1}{\sqrt{z_T^2 - a^2}} \left\{ \frac{\sqrt{\bar{z}_0^2 - a^2}}{z_T - \bar{z}_0} - \frac{\sqrt{z_0^2 - a^2}}{z_T - z_0} \right\} + \frac{M}{2\pi\lambda_0} \left\{ \frac{1}{z_T - z_0} - \frac{1}{z_T - \bar{z}_0} \right\} \tag{5.201}$$

Using $T = 0$ at infinity finally we get

$$\theta'(z_T) = \frac{M}{4\pi\alpha_{\rm II}} \ln\left\{ \frac{\sqrt{\bar{z}_0^2 - a^2} + \bar{z}_0}{\sqrt{z_0^2 - a^2} + z_0} \frac{\sqrt{z_T^2 - a^2}\sqrt{z_0^2 - a^2} + z_T z_0 - a^2}{\sqrt{z_T^2 - a^2}\sqrt{\bar{z}_0^2 - a^2} + z_T \bar{z}_0 - a^2} \right\} \tag{5.202}$$

Substituting Eq. (5.202) into (5.158) $\bar{\boldsymbol{\Omega}}^{\rm T}\boldsymbol{h}(z)$ can be obtained:

$$\begin{aligned} &\bar{\boldsymbol{\Omega}}^{\rm T}\boldsymbol{h}(z) = \boldsymbol{Q}(z)\left[\boldsymbol{C}(z) + \frac{1}{2\pi {\rm i}} \int_L \frac{\bar{\boldsymbol{\Omega}}^{\rm T}\tilde{\boldsymbol{\Sigma}}_0(x_1){\rm d}x_1}{\boldsymbol{Q}^+(x_1)(x_1 - z)}\right], \quad \boldsymbol{Q}(z) = \left\langle Y_0^{(j)}(z) \right\rangle \\ &\tilde{\boldsymbol{\Sigma}}_0(x_1) = \boldsymbol{\eta}_1\theta'^+(x_1) + \boldsymbol{\eta}_2\theta'^-(x_1) \end{aligned} \tag{5.203}$$

From Eq. (5.155) $\boldsymbol{F}_{\rm I}(z_j)$ and $\boldsymbol{F}_{\rm II}(z_j)$ can be obtained.

Gao and Wang (2001) discussed the permeable crack problem, Herrmann and Loboda (2003) discussed the contact zone model in pyroelectric material, and Norris (1994) discussed the dynamic Green function in piezoelectric material.

5.7 Functionally Graded Piezoelectric Material

5.7.1 Fundamental Equations in Antiplane Shear Problem

Functionally graded piezoelectric material (FGPM) is a kind of material with continuously varying properties (Wu et al. 1996) which is very useful as a transit layer instead of the bonding agent in order to avoid the very large stresses near the interface. Li and Weng (2002) discussed the antiplane crack problem (Fig. 5.7) with varied material constants for a transversely material:

$$\begin{aligned} &C_{44}(x_2) = C_{44}^0(1 + \alpha|x_2|)^k, \quad e_{15} = e_{15}^0(1 + \alpha|x_2|)^k, \quad \epsilon_{11} = \epsilon_{11}^0(1 + \alpha|x_2|)^k \\ &\alpha = \left(\sqrt[k]{C_{44}^h/C_{44}^0} - 1\right)\Big/h = \left(\sqrt[k]{e_{15}^h/e_{15}^0} - 1\right)\Big/h = \left(\sqrt[k]{\epsilon_{11}^h/\epsilon_{11}^0} - 1\right)\Big/h \end{aligned} \tag{5.204}$$

where $C_{44}^0, e_{15}^0, \epsilon_{11}^0$ are the values at $x_2 = 0$ and $C_{44}^h, e_{15}^h, \epsilon_{11}^h$ are the values at $x_2 = \pm h$; k and α are material constants. It is assumed that the geometry, material behavior,

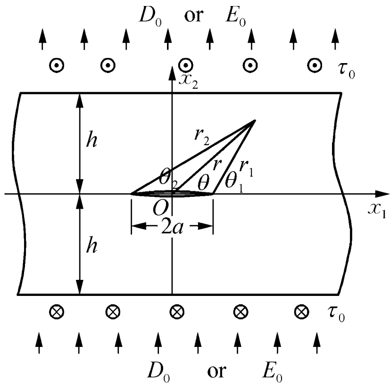


Fig. 5.7 An antiplane crack in FGPM

and applied loading are symmetric about the x_2-axis, so we only need to study the part of $x_1 \geq 0, x_2 \geq 0$ and $|x_2| = x_2$. The fundamental equations (4.238) and (4.239) of antiplane shear problem discussed in Sect. 4.8.1 are still held in a FGPM, but the material constants are functions of coordinates.

Substitution of Eq. (5.204) into Eq. (4.239) yields

$$\begin{aligned} &C_{44}^{0}\left[\nabla^{2}u_{3} + (k\alpha/\xi)u_{3,2}\right] + e_{15}^{0}\left[\nabla^{2}\varphi + (k\alpha/\xi)\varphi_{,2}\right] = 0 \\ &e_{15}^{0}\left[\nabla^{2}u_{3} + (k\alpha/\xi)u_{3,2}\right] - \epsilon_{11}^{0}\left[\nabla^{2}\varphi + (k\alpha/\xi)\varphi_{,2}\right] = 0, \quad \xi = 1 + \alpha x_{2} \end{aligned} \tag{5.205}$$

where $\nabla^2 = \partial/\partial x_1^2 + \partial/\partial x_2^2$. In general case $\left(e_{15}^0\right)^2 + C_{44}^0\epsilon_{11}^0 \neq 0$, so we also have

$$\nabla^{2}u_{3} + (k\alpha/\xi)u_{3,2} = 0, \quad \nabla^{2}\varphi + (k\alpha/\xi)\varphi_{,2} = 0 \tag{5.206}$$

The boundary and connective conditions on $x_2 = 0$ are

$$\begin{aligned} &\sigma_{32}(x_1,0) = 0, \quad E_1(x_1,0^+) = E_1^c(x_1,0^-), \quad D_2(x_1,0^+) = D_2^c(x_1,0^-), \quad 0 \leq x_1 < a \\ &u_3(x_1,0) = 0, \quad \varphi(x_1,0) = 0, \quad \sigma_{32}(x_1,0^+) = \sigma_{32}(x_1,0^-), \quad a \leq x_1 < \infty \end{aligned} \tag{5.207a}$$

where the right superscript "c" means that the related variable is in the air. The boundary conditions on $x_2 = h$ are divided into two forms dependent to giving D_2 or E_2:

$$\begin{aligned} &\text{Case 1:} \quad D_2(x_1,h) = D_0, \quad \sigma_{32}(x_1,h) = \tau_h = \left(C_{44}^{h*}/C_{44}^{h}\right)\tau_0 - \left(e_{15}^{h}/\epsilon_{11}^{h}\right)D_0 \\ &\text{Case 2:} \quad E_2(x_1,h) = E_0, \sigma_{32}(x_1,h) = \tau_h = \tau_0 - e_{15}^{h}E_0 \ ; \quad 0 \leq x_1 < \infty \end{aligned} \tag{5.207b}$$

where D_0 and E_0 are the external electric displacement and field, respectively; τ_0 is the stress at zero electric loading, $C_{44}^{h*} = C_{44}^{h} + \left(e_{15}^{h}\right)^2/\epsilon_{11}^{h}$.

5.7.2 Solution of the Antiplane Shear Problem

Considering the symmetry about x_2-axis, Li and Weng (2002) used the Fourier cosine transforms to solve this problem. Let

$$
\begin{aligned}
u_3(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty \xi^{-\beta}\{A_1(s)\mathrm{I}_\beta(\xi s/\alpha)+A_2(s)\mathrm{K}_\beta(\xi s/\alpha)\}\cos(sx_1)\mathrm{d}s + a_1x_2\\
\varphi(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty \xi^{-\beta}\{B_1(s)\mathrm{I}_\beta(\xi s/\alpha)+B_2(s)\mathrm{K}_\beta(\xi s/\alpha)\}\cos(sx_1)\mathrm{d}s - b_1x_2
\end{aligned}
\tag{5.208}
$$

where $\beta=(k-1)/2$; I_β and K_β are the first and second kind of modified Bessel's functions, respectively; $A_i(s)$ and $B_i(s)$ are undetermined functions; a_1, b_1 are real constants. Equation (5.208) yields

$$
\begin{aligned}
\sigma_{31}(x_1,x_2) &= -\frac{2}{\pi}\int_0^\infty s\xi^{-\beta}\big[(C_{44}A_1+e_{15}B_1)\mathrm{I}_\beta(\xi s/\alpha)\\
&\quad +(C_{44}A_2+e_{15}B_2)K_\beta(\xi s/\alpha)\big]\sin(sx_1)\mathrm{d}s\\
\sigma_{32}(x_1,x_2) &= -\frac{2}{\pi}\int_0^\infty \Big\{(C_{44}A_1+e_{15}B_1)\Big[\beta\alpha\xi^{-\beta-1}\mathrm{I}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{I}'_\beta(\xi s/\alpha)\Big]\\
&\quad +(C_{44}A_2+e_{15}B_2)\Big[\{\beta\alpha\xi^{-\beta-1}\mathrm{K}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{K}'_\beta(\xi s/\alpha)\Big]\Big\}\cos(sx_1)\mathrm{d}s\\
&\quad + C_{44}a_1 - e_{15}b_1
\end{aligned}
\tag{5.209}
$$

$$
\begin{aligned}
D_1(x_1,x_2) &= -\frac{2}{\pi}\int_0^\infty s\xi^{-\beta}\big[(e_{15}A_1-\epsilon_{11}B_1)\mathrm{I}_\beta(\xi s/\alpha)\\
&\quad +(e_{15}A_2-\epsilon_{11}B_2)\mathrm{K}_\beta(\xi s/\alpha)\big]\sin(sx_1)\mathrm{d}s\\
D_2(x_1,x_2) &= -\frac{2}{\pi}\int_0^\infty \Big\{\beta\alpha\xi^{-\beta-1}\mathrm{I}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{I}'_\beta(\xi s/\alpha)\Big\}\\
&\quad \times(e_{15}A_1-\epsilon_{11}B_1)+\Big[\beta\alpha\xi^{-\beta-1}\mathrm{K}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{K}'_\beta(\xi s/\alpha)\Big]\\
&\quad \times(e_{15}A_2-\epsilon_{11}B_2)\}\cos(sx_1)\mathrm{d}s + e_{15}a_1+\epsilon_{11}b_1
\end{aligned}
\tag{5.210}
$$

$$
\begin{aligned}
E_1(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty s\xi^{-\beta}\big[B_1\mathrm{I}_\beta(\xi s/\alpha)+B_2\mathrm{K}_\beta(\xi s/\alpha)\big]\sin(sx_1)\mathrm{d}s\\
E_2(x_1,x_2) &= \frac{2}{\pi}\int_0^\infty \Big\{B_1\Big[\beta\alpha\xi^{-\beta-1}\mathrm{I}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{I}'_\beta(\xi s/\alpha)\Big]\\
&\quad +B_2\Big[\beta\alpha\xi^{-\beta-1}\mathrm{K}_\beta(\xi s/\alpha)-s\xi^{-\beta}\mathrm{K}'_\beta(\xi s/\alpha)\Big]\Big\}\cos(sx_1)\mathrm{d}s + b_1
\end{aligned}
\tag{5.211}
$$

where $\mathrm{I}'_\beta, \mathrm{K}'_\beta$ are the derivatives of $\mathrm{I}_\beta, \mathrm{K}_\beta$.

In the air between the crack surfaces, we have

$$D_1^{\rm c} = \epsilon^{\rm c} E_1^{\rm c}, \quad D_2^{\rm c} = \epsilon^{\rm c} E_2^{\rm c}, \quad \nabla^2 \varphi^{\rm c} = 0 \tag{5.212}$$

Its solution can be assumed as

$$\begin{aligned}
&\varphi^{\rm c}(x_1, x_2) = \frac{2}{\pi}\int_0^\infty C(s)\sinh(sx_2)\cos(sx_1){\rm d}s, \quad 0 \le x_1 < a \\
&D_1^{\rm c}(x_1, 0) = 0, \quad D_2^{\rm c}(x_1, 0) = -\frac{2}{\pi}\int_0^\infty \epsilon^c sC(s)\cos(sx_1){\rm d}s \\
&E_1^{\rm c}(x_1, 0) = 0, \quad E_2^{\rm c}(x_1, 0) = -\frac{2}{\pi}\int_0^\infty sC(s)\cos(sx_1){\rm d}s
\end{aligned} \tag{5.213}$$

where $C(s)$ is an unknown function. Using the boundary conditions on $x_2 = h$ yields

$$\begin{aligned}
&A_2(s) = RA_1(s), \quad B_2(s) = RB_1(s) \\
&R = -\frac{\beta\alpha(1+\alpha h)^{-1}{\rm I}_\beta[(1+\alpha h)s/\alpha] - s{\rm I}'_\beta(1+\alpha h)s/\alpha}{\beta\alpha(1+\alpha h)^{-1}{\rm K}_\beta[(1+\alpha h)s/\alpha] - s{\rm K}'_\beta(1+\alpha h)s/\alpha} \\
&a_1 = \left(e_{15}^h D_0 + \epsilon_{11}^h \tau_0\right)/C_{44}^{h*}\epsilon_{11}^h, \quad b_1 = \left(C_{44}^h D_0 - e_{15}^h \tau_0\right)/C_{44}^{h*}\epsilon_{11}^h \quad \text{(case 1)} \\
&a_1 = \left(e_{15}^h E_0 + \tau_0\right)/C_{44}^h, \quad b_1 = E^\infty \quad \text{(case 2)}
\end{aligned} \tag{5.214}$$

Substituting $E_1(x_1,0), \varphi(x_1,0), E_1^{\rm c}(x_1,0)$ into the corresponding boundary conditions in Eq. (5.207) yields the following dual integral equation:

$$\begin{aligned}
&\int_0^\infty sB_1(s)\left\{I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right\}\sin(sx_1){\rm d}s = 0, \quad 0 \le x_1 < a \\
&\int_0^\infty B_1(s)\left\{I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right\}\cos(sx_1){\rm d}s = 0, \quad a \le x_1 < \infty
\end{aligned} \tag{5.215}$$

If let

$$B_1(s)\left\{I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right\} = \left(\pi a^2/2\right)\int_0^1 \sqrt{\eta}\Phi(\eta)J_0(sa\eta){\rm d}\eta \tag{5.216}$$

where J_0 is the zero-order Bessel function of the first kind, then the second equation in Eq. (5.215) is satisfied automatically and the first equation in Eq. (5.215) requires $\Phi(\eta) = 0$. So it is easy to obtain $B_1(s) = 0$ and then straightly $B_2(s) = 0$.

Substituting $\sigma_{32}(x_1,0), u_3(x_1,0)$ into the corresponding boundary conditions in Eq. (5.207) and noting $B_1(s) = B_2(s) = 0$, the following dual integral equation is obtained:

$$\begin{aligned}
&\int_0^\infty sF(s)A(s)\cos(sx_1){\rm d}s = (\pi/2)\left(C_{44}^0 a_1 - e_{15}^0 b_1\right)/C_{44}^0, \quad 0 \le x_1 < a \\
&\int_0^\infty A(s)\cos(sx_1){\rm d}s = 0, \quad a \le x_1 < \infty
\end{aligned} \tag{5.217}$$

where

$$A(s) = A_1(s)\left[I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right]$$
$$F(s) = \frac{\left[\beta\alpha I_\beta(s/\alpha) - sI'_\beta(s/\alpha)\right] + R\left[\beta\alpha K_\beta(s/\alpha) - sK'_\beta(s/\alpha)\right]}{s\left[I_\beta(s/\alpha) + RK_\beta(s/\alpha)\right]} \tag{5.218}$$

The solution of Eq. (5.217) can be written as

$$A(s) = \frac{\pi a^2}{2}\frac{\hat{C}^0_{44}}{C^0_{44}}\int_0^1 \sqrt{\eta}\Psi(\eta)J_0(sa\eta)\mathrm{d}\eta, \quad \hat{C}^0_{44} = C^0_{44}a_1 - e^0_{15}b_1 \tag{5.219}$$

Equation (5.219) satisfies the second equation in Eq. (5.217) automatically. In order to satisfy the first equation in Eq. (5.217), $\Psi(\eta)$ should be satisfied by the following Fredholm integral equation of the second kind:

$$\Psi(\eta) + \int_0^1 \psi(\eta)G(\eta,\eta')\mathrm{d}\eta' = \sqrt{\eta}$$
$$G(\eta,\eta') = \sqrt{\eta\eta'}\int_0^\infty s[F(s/a) - 1]J_0(s\eta)J_0(s\eta')\mathrm{d}s \tag{5.220}$$

5.7.3 The Generalized Stress Asymptotic Fields Near the Crack Tip

The singular generalized stress fields are determined by the behavior of the solution when $s \to \infty$. Using integration by parts to decompose Eq. (5.219) into singular and regular parts,

$$A(s) = \frac{\pi a}{2}\frac{\hat{C}^0_{44}}{C^0_{44}}\frac{1}{s}\left\{\Psi(1)J_1(sa) - \int_0^1 \eta J_1(sa\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{\Psi(\eta)}{\sqrt{\eta}}\right]\mathrm{d}\eta\right\} \tag{5.221}$$

where J_1 is the first-order Bessel function of the first kind. The integral in Eq. (5.221) is finite at the crack tip $x_1 = \pm a$, and the singular behavior is determined by the term containing $\Psi(1)$. It is noted that the modified Bessel functions have the following behaviors:

$$\lim_{s\to\infty} \mathrm{I}_\beta(s) = \left(1\middle/\sqrt{2\pi s}\right)\mathrm{e}^s, \quad \lim_{s\to\infty} \mathrm{I}'_\beta(s) = \left(1\middle/\sqrt{2\pi s}\right)\mathrm{e}^s$$
$$\lim_{s\to\infty} \mathrm{K}_\beta(s) = \left(\sqrt{\pi/2s}\right)\mathrm{e}^{-s}, \quad \lim_{s\to\infty} \mathrm{K}'_\beta(s) = -\left(\sqrt{\pi/2s}\right)\mathrm{e}^{-s} \tag{5.222}$$

After complex derivation we obtain

$$\begin{aligned}&\sigma_{31}=-\hat{C}_{44}^{0}a\Psi(1)\xi^{k/2}f_1(s),\quad \sigma_{32}=-\hat{C}_{44}^{0}a\Psi(1)\xi^{k/2}f_2(s)\\&D_1=-\frac{e_{15}^{0}\hat{C}_{44}^{0}}{C_{44}^{0}}a\Psi(1)\xi^{k/2}f_1(s),\quad D_2=-\frac{e_{15}^{0}\hat{C}_{44}^{0}}{C_{44}^{0}}a\Psi(1)\xi^{k/2}f_2(s)\\&E_1=0,\quad E_2=E_0\end{aligned}\tag{5.223}$$

where

$$\begin{aligned}f_1(s)&=\int_0^\infty J_1(sa)\mathrm{e}^{-sx_2}\sin(sx_1)\mathrm{d}s=-\frac{r}{a\sqrt{r_1r_2}}\sin\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\\f_2(s)&=\int_0^\infty J_1(sa)\mathrm{e}^{-sx_2}\cos(sx_1)\mathrm{d}s=\frac{1}{a}-\frac{r}{a\sqrt{r_1r_2}}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\end{aligned}\tag{5.224}$$

where the meanings of $r, r_1, r_2, \theta, \theta_1, \theta_2$ can be seen in Fig. 5.7. Let $\theta\to 0, \theta_2\to 0$, $r_2\to 2a, r\to a$ from Eq. (5.223) we get

$$\begin{aligned}&\sigma_{31}=-\left(K_{\mathrm{III}}\Big/\sqrt{2\pi r_1}\right)\sin(\theta_1/2),\quad \sigma_{32}=\left(K_{\mathrm{III}}\Big/\sqrt{2\pi r_1}\right)\cos(\theta_1/2),\\&D_1=-\left(K_{\mathrm{III}}^{D}\Big/\sqrt{2\pi r_1}\right)\sin(\theta_1/2),\quad D_2=\left(K_{\mathrm{III}}^{D}\Big/\sqrt{2\pi r_1}\right)\cos(\theta_1/2),\quad E_1=E_2=0\\&K_{\mathrm{III}}=\hat{C}_{44}^{0}\sqrt{\pi a}\Psi(1),\quad K_{\mathrm{III}}^{D}=\left(e_{15}^{0}/C_{44}^{0}\right)K_{\mathrm{III}}=\left(\hat{C}_{44}^{0}e_{15}^{0}/C_{44}^{0}\right)\sqrt{\pi a}\Psi(1)\end{aligned}\tag{5.225}$$

It is found that for the functional gradient piezoelectric material, the asymptotic fields of the generalized stress still have the singularity $1/\sqrt{r}$. Because a_1, b_1 is enclosed in $\hat{C}_{44}^{0}$ (see Eq. (5.219)), the generalized stress intensity factors are different for two different electric boundary conditions. It is also found that the electric field does not have singularity at the crack tip.

Yang et al. (2004) also discussed the electric field gradient effects in antiplane problems of polarized ceramics.

5.7.4 Plane Strain Problem

The constitutive equations of the in-plane problem are

$$\begin{aligned}&\sigma_{11}=C_{11}u_{1,1}+C_{13}u_{3,3}-e_{31}E_3,\quad \sigma_{33}=C_{13}u_{1,1}+C_{33}u_{3,3}-e_{33}E_3\\&\sigma_{13}=C_{44}\left(u_{1,3}+u_{3,1}\right)-e_{15}E_1,\quad D_1=e_{15}\left(u_{1,3}+u_{3,1}\right)+\epsilon_{11}E_1\\&D_3=e_{31}u_{1,1}+e_{33}u_{3,3}+\epsilon_{33}E_3\end{aligned}\tag{5.226}$$

It is assumed that the material properties are one dimensional dependent to x_3 as

$$(C_{11}, C_{13}, C_{33}, C_{44}, e_{31}, e_{33}, e_{15}, \epsilon_{11}, \epsilon_{33}) = \left(C_{11}^0, C_{13}^0, C_{33}^0, C_{44}^0, e_{31}^0, e_{33}^0, e_{15}^0, \epsilon_{11}^0, \epsilon_{33}^0\right) e^{\beta|x_3|} \tag{5.227}$$

where β is a material constant. The equilibrium equations in terms of generalized displacements are

$$\begin{aligned}
&C_{11}^0 u_{1,11} + C_{44}^0 u_{1,33} + \left(C_{13}^0 + C_{44}^0\right) u_{3,13} + \left(e_{31}^0 + e_{15}^0\right)\varphi_{,13} + \beta\left[C_{44}^0\left(u_{1,3} + u_{3,1}\right) + e_{15}^0 \varphi_{,1}\right] = 0 \\
&C_{44}^0 u_{3,11} + C_{33}^0 u_{3,33} + \left(C_{13}^0 + C_{44}^0\right) u_{1,13} + e_{15}^0 \varphi_{,11} + e_{33}^0 \varphi_{,33} + \beta\left(C_{13}^0 u_{1,1} + C_{33}^0 u_{3,3} + \epsilon_{33}^0 \varphi_{,3}\right) = 0 \\
&e_{15}^0 u_{3,11} + e_{33}^0 u_{3,33} + \left(e_{31}^0 + e_{15}^0\right) u_{1,13} - \epsilon_{11}^0 \varphi_{,11} - \epsilon_{33}^0 \varphi_{,33} + \beta\left(e_{31}^0 u_{1,1} + e_{33}^0 u_{3,3} - \epsilon_{33}^0 \varphi_{,3}\right) = 0
\end{aligned} \tag{5.228}$$

In the air between the crack surfaces, the governing equations are still shown in Eq. (5.212).

As in Sect. 5.7.1 it is assumed that the geometry, material behavior, and applied loading are all symmetric about the x_3-axis, so we only need to study the part of $x_1 \geq 0, x_3 \geq 0$ and $|x_2| = x_2$. The boundary conditions on the crack and connective surfaces are

$$\begin{aligned}
&\sigma_{33}(x_1, 0) = 0, \quad E_1(x_1, 0^+) = E_1^c(x_1, 0^-), \quad D_3(x_1, 0^+) = D_3^c(x_1, 0^-), \quad 0 \leq |x_1| < a \\
&u_3(x_1, 0) = 0, \quad \varphi(x_1, 0) = 0, \quad a \leq |x_1| < \infty; \quad \sigma_{31}(x_1, 0^+) = 0, \quad 0 \leq |x_1| < \infty
\end{aligned} \tag{5.229a}$$

The boundary conditions on the edge $x_3 = h$ are divided into two forms:

$$\begin{aligned}
&\text{case 1:} \quad \sigma_{33}(x_1, h) = \sigma_h = \frac{C_{33}^{0*}}{C_{33}^0}\sigma_0 - \frac{e_{33}^0}{\epsilon_{33}^0} D_0, \quad \sigma_{13}(x_1, h) = 0, \quad D_3(x_1, h) = D_0 \\
&\text{case 2:} \quad \sigma_{33}(x_1, h) = \sigma_h = \sigma_0 - e_{33}^0 E_0 e^{\beta h}, \quad \sigma_{13}(x_1, h) = 0, \; E_3(x_1, h) = E_0
\end{aligned} \tag{5.229b}$$

where D_0 and E_0 are the external electric displacement and field, respectively, and σ_0 is the stress at zero electric loading, $C_{33}^{0*} = C_{33}^0 + \left(e_{33}^0\right)^2 / \epsilon_{33}^0$.

The single valued condition of the generalized displacements is

$$\int_{-a}^{a} \psi(x_1) \mathrm{d}x_1 = 0, \quad \psi(x_1) = \mathrm{d}\left[U_3(x_1, 0^+) - U_3(x_1, 0^-)\right] / \mathrm{d}x_1, \quad 0 \leq |x_1| < a \tag{5.230}$$

where $\psi(x_1)$ is the generalized dislocation density and on the connective surface $\psi = 0$.

Ueda (2005) adopted the Fourier integral transform techniques to solve this problem. Let

$$\begin{aligned} u_1(x_1,x_3) &= \frac{2}{\pi}\sum_{j=1}^{6}\int_0^\infty a_j A_j(s)\mathrm{e}^{s\gamma_j x_3}\sin(sx_1)\mathrm{d}s \\ u_3(x_1,x_3) &= \frac{2}{\pi}\sum_{j=1}^{6}\int_0^\infty A_j(s)\mathrm{e}^{s\gamma_j x_3}\cos(sx_1)\mathrm{d}s + a_0\left(1-\mathrm{e}^{-\beta x_3}\right) \\ \varphi(x_1,x_3) &= -\frac{2}{\pi}\sum_{j=1}^{6}\int_0^\infty b_j A_j(s)\mathrm{e}^{s\gamma_j x_3}\cos(sx_1)\mathrm{d}s - b_0\left(1-\mathrm{e}^{-\beta x_3}\right) \end{aligned} \tag{5.231}$$

where $A_j(s)$ is undetermined function and a_0, b_0 are unknown constants; $\gamma_j(s), a_j(s)$, $b_j(s)$ are known functions. $\gamma_j(s)$ is the root of the following equation:

$$\begin{aligned} &(f_3q_3+g_2p_4)\gamma^6+(f_3q_2+g_2p_3+f_2q_3+g_1p_4)\gamma^5+(f_3q_1+g_2p_2+f_2q_2+g_1p_3 \\ &\quad +f_1q_3+g_0p_4)\gamma^4+(f_3q_0+g_2p_1+f_2q_1+g_1p_2+f_1q_2+g_0p_3+f_0q_3)\gamma^3 \\ &\quad +(f_2q_0+g_2p_0+f_1q_1+g_1p_1+f_0q_2+g_0p_2)\gamma^2+(f_1q_0+g_1p_0+f_0q_1+g_0p_1)\gamma \\ &\quad +(f_0q_0+g_0p_0)=0 \end{aligned} \tag{5.232}$$

For convenience let $\mathrm{Re}\gamma_j(s) < \mathrm{Re}\gamma_{j+1}(s), j = 1-5$. $a_j(s), b_j(s)$ are determined by

$$\begin{aligned} a_j(s) &= \frac{q_3\gamma_j^3+q_2\gamma_j^2+q_1\gamma_j+q_0}{g_2\gamma_j^2+g_1\gamma_j+g_0} \\ b_j(s) &= \frac{\left[\left(C_{13}^0+C_{44}^0\right)s\gamma_j+C_{13}^0\beta\right]a_j+C_{33}^0s\gamma_j^2+C_{33}^0\beta\gamma_j-C_{44}^0s}{e_{33}^0s\gamma_j^2+e_{33}^0\beta\gamma_j-e_{15}^0s} \end{aligned} \tag{5.233}$$

where

$$\begin{aligned} f_0 &= -\left(e_{31}^0e_{15}^0+C_{13}^0\epsilon_{11}^0\right)\beta s \\ f_1 &= \left(e_{31}^0e_{33}^0+C_{13}^0\epsilon_{33}^0\right)\beta^2-\left[e_{15}^0\left(e_{15}^0+e_{31}^0\right)+\epsilon_{11}^0\left(C_{13}^0+C_{44}^0\right)\right]s^2 \\ f_2 &= \left[\epsilon_{33}^0\left(2C_{13}^0+C_{44}^0\right)+e_{33}^0\left(2e_{31}^0+e_{15}^0\right)\right]\beta s \\ f_3 &= \left[\epsilon_{33}^0\left(C_{13}^0+C_{44}^0\right)+e_{33}^0\left(e_{31}^0+e_{15}^0\right)\right]s^2 \end{aligned} \tag{5.234a}$$

$$\begin{aligned} p_0 &= \left(C_{44}^0\epsilon_{11}^0+e_{15}^{02}\right)s^2, \quad p_1 = -\left(\epsilon_{33}^0C_{44}^0+2e_{33}^0e_{15}^0+\epsilon_{11}^0C_{33}^0\right)\beta s \\ p_2 &= \left(e_{33}^{02}+C_{33}^0\epsilon_{33}^0\right)\beta^2-\left(\epsilon_{33}^0C_{44}^0+2e_{33}^0e_{15}^0+\epsilon_{11}^0C_{33}^0\right)s^2 \\ p_3 &= 2\left(e_{33}^{02}+C_{33}^0\epsilon_{33}^0\right)\beta s, \quad p_4 = \left(e_{33}^{02}+C_{33}^0\epsilon_{33}^0\right)s^2 \end{aligned} \tag{5.234b}$$

$$\begin{aligned} g_0 &= e_{15}^0\left(e_{31}^0 e_{33}^0 + C_{13}^0 \epsilon_{33}^0\right)\beta^2 + C_{11}^0\left(e_{15}^0 \epsilon_{33}^0 - e_{33}^0 \epsilon_{11}^0\right)s^2 \\ g_1 &= \left\{e_{15}^0\left[\epsilon_{33}^0\left(C_{13}^0 + C_{44}^0\right) + e_{33}^0\left(e_{31}^0 + e_{15}^0\right)\right] + \left(e_{31}^0 + e_{15}^0\right)\left(e_{31}^0 e_{33}^0 + C_{13}^0 \epsilon_{33}^0\right)\right. \\ &\quad \left. - C_{44}^0\left(e_{15}^0 \epsilon_{33}^0 - e_{33}^0 \epsilon_{11}^0\right)\right\}\beta s \\ g_2 &= \left\{\left(e_{15}^0 + e_{31}^0\right)\left[\epsilon_{33}^0\left(C_{13}^0 + C_{44}^0\right) + e_{33}^0\left(e_{31}^0 + e_{15}^0\right)\right] - C_{44}^0\left(e_{15}^0 \epsilon_{33}^0 - e_{33}^0 \epsilon_{11}^0\right)\right\}s^2 \end{aligned} \tag{5.234c}$$

$$\begin{aligned} q_0 &= e_{33}^0\left(C_{44}^0 \epsilon_{11}^0 + e_{15}^{02}\right)\beta s \\ q_1 &= -e_{15}^0\left(e_{33}^{02} + C_{33}^0 \epsilon_{33}^0\right)\beta^2 + \left[\left(\epsilon_{33}^0 C_{44}^0 + e_{33}^0 e_{15}^0\right) - \left(e_{15}^0 \epsilon_{33}^0 - e_{31}^0 \epsilon_{11}^0\right)\left(C_{13}^0 + C_{44}^0\right)\right]s^2 \\ q_2 &= -\left(e_{31}^0 + 2e_{15}^0\right)\left(e_{33}^{02} + C_{33}^0 \epsilon_{33}^0\right)\beta s, \quad q_3 = -\left(e_{31}^0 + e_{15}^0\right)\left(e_{33}^{02} + C_{33}^0 \epsilon_{33}^0\right)s^2 \end{aligned} \tag{5.234d}$$

Let

$$\varphi^c(x_1, x_3) = \frac{2}{\pi}\int_0^\infty B(s)\sinh(sx_3)\cos(sx_1)\mathrm{d}s, \quad -a < x_1 < a \tag{5.235}$$

where $B(s)$ is undetermined function.

Using the dislocation density $\psi(x_1)$, $\sigma_{33}(x_1, 0) = 0,\ 0 \le x_1 < a$, other boundary conditions, and Eq. (5.235), finally we can get the following singular integral equation:

$$\frac{1}{\pi}\int_{-a}^{a}\psi(t)\left[\frac{1}{t - x_1} + M_1(t, x_1) + M_2(t, x_1)\right]\mathrm{d}t = \frac{\sigma_h}{Z_0^\infty} \tag{5.236}$$

The expressions of $M_1(t, x_1)$ and $M_2(t, x_1)$ are omitted here.

Equations (5.236) and (5.231) form a singular integral equation system. Let

$$\psi(t) = \left(\sigma_h / Z_0^\infty\right)\Phi(u) \Big/ \sqrt{1 - u^2}, \quad u = t/a \tag{5.237}$$

Substitute Eq. (5.237) into (5.236) and then use the Gauss-Jacobi numerical integral technique to solve the integral equation. The generalized stress intensity factors are

$$K_{\mathrm{I}} = \lim_{x \to a^+}\sqrt{2\pi(x_1 - a)}\sigma_{33}(x_1, 0) = \sigma_0\sqrt{\pi a}\Phi(1), \quad K_{\mathrm{D}} = \left(Z_3^\infty / Z_0^\infty\right)K_{\mathrm{I}} \tag{5.238}$$

A lot of literatures studied the functional graded piezoelectric materials, such as Zhou and Chen (2008), Chen et al. (2003), and Wang and Zhang (2004).

References

Beom HG (1999) Singular behavior near a crack tip in an electrostrictive material. J Mech Phys Solids 47:1027–1049

Beom HG, Kim YH, Cho C, Kim CB (2006) Asymptotic analysis of an impermeable crack in an electrostrictive material subjected to electric loading. Int J Solids Struct 43:6869–6886

Chen J, Liu Z-X, Zou Z-G (2003) Dynamic response of a crack in a functionally graded interface of two dissimilar piezoelectric half-planes. Arch Appl Mech 72:686–696
Cherepanov GP (1979) Mechanics of brittle fracture. McGraw-Hill, New York
Gao C-F, Wang M-Z (2001) Collinear permeable cracks in thermopiezoelectric materials. Mech Mater 33:1–9
Gao C-F, Zhao Y-T, Wang M-Z (2002) An exact and explicit treatment of an elliptic hole problem in thermopiezoelectric media. Int J Solids Struct 39:2665–2685
Gao C-F, Mai YW, Zhang N (2010) Solution of a crack in an electrostrictive solid. Int J Solids Struct 47:444–453
Hao T-H, Gong X, Suo Z (1996) Fracture mechanics for the design of ceramic multilayer actuators. J Mech Phys Solids 44:23–48
Herrmann KP, Loboda VV (2003) Fracture-mechanical assessment of interface cracks with contact zones in piezoelectric bimaterials under thermoelectromechanical loadings II. Electrically impermeable interface cracks. Int J Solids Struct 40:4219–4237
Hwu CB (1990) Thermal stresses in an anisotropic plate disturbed by an insulated elliptic hole or crack. Trans ASME J Appl Mech 57:916–922
Hwu CB (1992) Thermoelastic interface crack problems in dissimilar anisotropic media. Int J Solids Struct 29:2077–2090
Jiang Q, Kuang Z-B (2003) Stress analysis of electrostrictive material with an elliptic defect. Sci China Ser G 46:492–500
Jiang Q, Kuang Z-B (2004) Stress analysis in two dimensional electrostrictive material with an elliptic rigid conductor. Eur J Mech A Solids 23:945–956
Jiang Q, Kuang Z-B (2005) Stress analysis in an infinite electrostrictive plate with an elliptic inclusion. Acta Mech Solida Sin 26:55–61 (in Chinese)
Jiang Q, Kuang Z-B (2006) Stress analysis in cracked infinite electrostrictive plate with local saturation electric fields. Int J Eng Sci 44:1583–1600
Jiang Q, Kuang Z-B (2007) Stress and electric fields in an infinite electrostrictive plate with an elliptic inhomogeneity. Eur J Mech A Solids 26:225–241
Knops RJ (1963) Two-dimensional electrostriction. Q J Mech Appl Math 16:377–388
Kuang Z-B, Jiang Q (2006) Stress analysis in two dimensional electrostrictive material under general loading[C]. In: Yang W (ed) IUTAM symposium mechanics and reliability of actuating materials. Springer, Dordrecht, pp 68–84
Li C, Weng GJ (2002) Antiplane crack problem in functional graded piezoelectric materials. J Appl Mech 69:481–488
Liang J, Han J, Wang B, Du S (1995) Electroelastic modeling of anisotropic piezoelectric materials with an elliptic inclusion. Int J Solids Struct 32:2989–3000
Lu P, Tan MJ, Liew KM (1998) Piezothermoelastic analysis of a piezoelectric material with an elliptic cavity under uniform heat flow. Arch Appl Mech 68:719–733
McMeeking RM (1989) Electrostrictive stressed near crack-like flaws. J Appl Math Phys (ZAMP) 40(615–627)
Mindlin RD (1974) Equations of high frequency vibrations of thermopiezoelectric crystal plates. Int J Solids Struct 10:625–637
Muskhelishvili NI (1975) Some basic problems of mathematical theory of elasticity. Noordhoof, Leyden
Norris AN (1994) Dynamic Green's functions in anisotropic piezoelectric, thermolelastic and poroelastic solids. Proc R Soc Lond A447:175–188
Qin Q-H (1998) Thermoelectroelastic Green's function for a piezoelectric plate containing an elliptic hole. Mech Mater 30:21–29
Qin Q-H (1999) Thermoelectroelastic Green's function for thermal load inside or on the boundary of an elliptic inclusion. Mech Mater 31:611–626
Shen S-P, Kuang Z-B (1998) Interface crack in bi-piezothermoelastic media and the interaction with a point heat source. Int J Solids Struct 35:3899–3915

Shen S-P, Wang X, Kuang Z-B (1995) The general solution and the interface crack in a pyroelectric material. Acta Mech Solida Sin 16:283–293 (in Chinese)
Smith TE, Warren WE (1966) Electromagnetic force in deformable electrostriction. J Math Phys 45:45–51; Corrigenda: 47. 109–110 (1968)
Tiersten HE (1971) On the nonlinear equations of thermo-electroelasticity. Int J Eng Sci 9:587–604
Ueda S (2005) Electromechanical response of a center crack in a functional graded piezoelectric strip. Smart Mat Struct 14:1133–1138
Wang B-L, Zhang XH (2004) A mode III crack in a functionally graded piezoelectric material strip. J Appl Mech 71:327–333
Wu CM, Kahn M, Moy W (1996) Piezoelectric ceramics with functional gradients: a new application in material design. J Am Ceram Soc 79:809–812
Yang W, Suo Z (1994) Cracking in ceramic actuators caused by electrostriction. J Mech Phys Solids 42:649–663
Yang XM, Hu YT, Yang JS (2004) Electric field gradient effects in anti-plane problems of polarized ceramics. Int J Solids Struct 41:6801–6811
Zhou Z-G, Chen Z-T (2008) Basic solution of a mode-I limited-permeable crack in functional graded piezoelectric/piezomagnetic materials. Int J Solids Struct 45:2265–2296

Chapter 6
Electroelastic Wave

Abstract In this chapter the electroelastic wave in piezoelectric and pyroelectric materials are discussed. In the electrically quasi-static approximation in an infinite space, there are three independent elastic waves for the piezoelectric material, and there is no independent electric wave. In the pyroelectric material a temperature wave has happened. In the reflection and transmission of waves, the inhomogeneous wave theory is effective; a quasi-surface wave is revealed in the electrically quasi-static approximation. In some particular cases the coupling between elastic equation and Maxwell electrodynamics equation needs to be studied together, and in these cases there are three elastic waves and two electric waves in the piezoelectric material. Surface acoustic waves (SAW) are extensively used in engineering. In order to improve performance of SAW devices, SAW devices may work in a biasing state. In this chapter a small perturbation superposed on finite generalized displacements is discussed in detail, and some surface waves under the biasing state are studied. The inertial entropy theory is used to derive the governing equation of the temperature wave with finite propagation velocity. The general dynamic analyses of interface cracks are given shortly, and some wave scattering problems from a crack tip are also discussed.

Keywords Electroelastic and temperature plane waves • Surface wave • Biasing state • Wave scattering

6.1 Electroelastic Waves in Piezoelectric Materials

6.1.1 Fundamental Equations in Electroelastic Wave

The elastic waves in isotropic and anisotropic materials and electroelastic waves in piezoelectric materials have been discussed in many literatures, such as Fischer (1955), Auld (1973), Dieulesaint and Royer (1980), and Nayfen (1995). Except Sect. 6.8 in this book, we discuss some linear problems in piezoelectric and

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pyroelectric materials under the condition of quasi-static electric field. In the linear problem the Maxwell stress is not considered. The constitutive equations are shown in Eq. (3.2), and the generalized momentum equation without generalized body forces in terms of generalized displacements under electrically static condition is

$$C_{ijkl}u_{l,kj} + e_{kij}\varphi_{,kj} = \rho\ddot{u}_i, \quad e_{ikl}u_{l,ki} - \epsilon_{ij}\varphi_{,ji} = 0 \tag{6.1}$$

Equation (6.1) gives the elastic wave equation. The electric displacement does not have its own independent wave, but it propagates following the elastic waves through the constitutive equation.

In this book we only discuss the plane wave, which can be expressed in two forms. For the generalized displacements $\boldsymbol{U} = [u_1, u_2, u_3, u_4]^{\mathrm{T}}, \quad u_4 = \varphi$ we have

$$\begin{aligned} &U_i = U_{0i}F(k_m x_m - \omega t) = U_{0i}F[k(n_m x_m - ct)], \quad \boldsymbol{U}_0 = [u_{01}, u_{02}, u_{03}, u_{04} = \varphi_0]^{\mathrm{T}} \\ &U_i = U_{0i}F[\omega(L_m x_m - t)], \quad k_m = kn_m, \quad L_m = n_m/c, \quad \omega = kc \end{aligned} \tag{6.2}$$

where $\boldsymbol{U}_0$ is the wave polarization vector or the amplitude vector and the ratio of its components represents the particle displacement direction, ω is the circular frequency, c is the phase velocity, $\boldsymbol{k}$ is the wave vector, $\boldsymbol{L}$ is the slowness vector, and $F(y)$ is a certain function of y. For an ideal piezoelectric material, the energy is not dissipative, so wave vector $\mathbf{k} = k\mathbf{n}$, where $k = 2\pi/\lambda$ is the wave number, λ is the wave length, $\boldsymbol{n}$ is the wave propagation direction Eq. (6.2) yields

$$\ddot{U}_i = U_{0i}\omega^2 F'', \quad u_{l,jk} = u_{0l}k_j k_k F'', \quad \varphi_{,jk} = \varphi_0 k_j k_k F'', \quad E_j = -\varphi_0 k_j F' \tag{6.3}$$

where $F'(y) = \partial F/\partial y$. Substituting Eq. (6.3) into (6.1) yields the Christoffel equation:

$$\begin{aligned} &\rho c^2 u_{0i} = \Gamma_{il}u_{0l} + e_i^*\varphi_0, \quad e_i^* u_{0i} - \epsilon^*\varphi_0 = 0 \\ &\Gamma_{il} = \Gamma_{li} = C_{ijkl}n_j n_k, \quad e_i^* = e_{kij}n_j n_k, \quad \epsilon^* = \epsilon_{jk}n_j n_k \end{aligned} \tag{6.4}$$

or

$$\boldsymbol{\Lambda}(\boldsymbol{k}, c)\boldsymbol{U}_0 = \mathbf{0}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \Gamma_{il} - \rho c^2\delta_{il} & e_i^* \\ e_l^* & -\epsilon^* \end{bmatrix} \tag{6.5}$$

In order for $\boldsymbol{U}_0$ to have nontrivial solution, $\boldsymbol{\Lambda}$ must satisfy the following secular equation:

$$|\boldsymbol{\Lambda}| = \begin{vmatrix} \Gamma_{il} - \rho c^2\delta_{il} & e_i^* \\ e_l^* & -\epsilon^* \end{vmatrix} = \begin{vmatrix} \Gamma_{11} - \rho c^2 & \Gamma_{12} & \Gamma_{13} & e_1^* \\ \Gamma_{21} & \Gamma_{22} - \rho c^2 & \Gamma_{23} & e_2^* \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} - \rho c^2 & e_3^* \\ e_1^* & e_2^* & e_3^* & -\epsilon^* \end{vmatrix} = 0 \tag{6.6}$$

where $\boldsymbol{\Gamma}$ is symmetric and called Christoffel tensor. ρc^2 is the eigenvalue, and $\boldsymbol{U}_0$ is the corresponding eigenvector. Eliminating φ_0 from Eq. (6.4) yields

$$\rho c^2 u_{0i} = \bar{\Gamma}_{il} u_{0l}, \quad \left|\bar{\Gamma}_{il} - \rho c^2 \delta_{il}\right| = 0, \quad \bar{\Gamma}_{il} = \Gamma_{il} + e_i^* e_l^* / \epsilon^* \tag{6.7}$$

From Eq. (6.7) it is known that ρc^2 has three roots: $\rho c_1^2, \rho c_2^2, \rho c_3^2$. Corresponding to each ρc_i^2 there is an eigenvector $\boldsymbol{u}_0^{(i)}$ with one undetermined component. Sometimes for convenience we let the undetermined component equal to 1 or adopt the normalized eigenvector $\bar{\boldsymbol{u}}_0^{(i)} \boldsymbol{u}_0^{(j)} = \mathbf{I}$. Equation (6.7) yields

$$\rho c^2 = \frac{\bar{\Gamma}_{il} u_{0i} u_{0l}}{u_{0i} u_{0i}} = \frac{\bar{C}_{ijkl} n_j n_k u_{0i} u_{0l}}{u_{0i} u_{0i}}, \quad \bar{C}_{ijkl} = C_{ijkl} + \frac{(e_{pij} n_p)(e_{qkl} n_q)}{\epsilon_{jk} n_j n_k} \tag{6.8}$$

From Eq. (6.8) it is known that ρc^2 is real, because

$$\begin{aligned} \bar{C}_{ijkl} n_j n_k u_{0i} u_{0l} &= \bar{C}_{ijkl} \left[(u_{0i} n_j + u_{0j} n_i)(u_{0i} n_j - u_{0j} n_i)\right]\left[(u_{0l} n_k + u_{0k} n_l)(u_{0l} n_k - u_{0k} n_l)\right]/4 \\ &= \bar{C}_{ijkl} (u_{0i} n_j + u_{0j} n_i)(u_{0l} n_k + u_{0k} n_l)/4 \geq 0 \end{aligned}$$

Because ρc^2 is real, there are three orthogonal plane waves. In general $\boldsymbol{u}_0^{(i)}$ is not parallel or perpendicular to the wave propagation direction $\boldsymbol{n}$. The wave $\boldsymbol{u}_0^{(1)}$ closest to $\boldsymbol{n}$ is called the quasi-longitudinal wave, which has the largest phase velocity c_1, while the other two waves $\boldsymbol{u}_0^{(2)}, \boldsymbol{u}_0^{(3)}$ are located on the plane close to the plane perpendicular to $\boldsymbol{n}$ and called the quasi-shear waves with slower velocity c_2 and $c_3 < c_2$.

6.1.2 Energy Propagation

According to Eqs. (1.57) and (1.58), the energy equation can be reduced to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathfrak{A}_\mathrm{t} \, \mathrm{d}V = -\int_a P_j n_j \, \mathrm{d}a, \quad \mathfrak{A}_\mathrm{t} = \mathfrak{A} + K, \quad P_j = -\sigma_{ij} \dot{u}_i + \varphi \dot{D}_j \\ &\mathfrak{A} = (1/2) C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + (1/2) \beta_{ij} D_i D_j, \quad K = (1/2) \rho \dot{u}_i \dot{u}_i \\ &\int_a P_j n_j \, \mathrm{d}a = -\int_a (T_i \dot{u}_i + \varphi \sigma) \mathrm{d}a = \int_a (\sigma_{ij} \dot{u}_i - \varphi \dot{D}_j) n_j \, \mathrm{d}a \end{aligned} \tag{6.9}$$

where $\mathfrak{A}_\mathrm{t}$ is the total energy density, $\dot{\mathfrak{A}}_\mathrm{t}$ is the rate of the total energy, $\int_a P_j n_j \mathrm{d}a$ represents the rate of the traversing external energy through the boundary and $\boldsymbol{P}$ is the Poynting vector. In Eq. (6.9) the heat flow is not considered. If $\mathfrak{A}$ is expressed with ε and E, Eq. (6.9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathfrak{A}_\mathrm{t}\,\mathrm{d}V + \int_a P_j n_j\,\mathrm{d}a = 0, \quad P_j = -\sigma_{ij}\dot{u}_i - D_j\dot{\varphi}$$
$$\mathfrak{A} = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + (1/2)\epsilon_{ij}E_iE_j, \quad K = (1/2)\rho\dot{u}_i\dot{u}_i, \tag{6.10}$$

Define the energy transport velocity $\boldsymbol{V}^\mathrm{e}$ as

$$\boldsymbol{V}^\mathrm{e} = \boldsymbol{P}/g_\mathrm{t} \tag{6.11}$$

Equation (6.4) yields

$$\rho c^2 u_{0i}u_{0i} = \Gamma_{il}u_{0l}u_{0i} + e_i^*\varphi_0 u_{0i} = \Gamma_{il}u_{0l}u_{0i} + \epsilon^*\varphi_0\varphi_0, \quad e_i^* u_{0i}\varphi_0 - \epsilon^*\varphi_0\varphi_0 = 0 \tag{6.12}$$

Equations (6.4), (6.10), and (6.12) yield

$$\mathfrak{A} = (1/2)(C_{ijkl}u_{0i}u_{0l}n_jn_k + \epsilon_{ij}\varphi_0\varphi_0 n_in_j)\left(F'^2/c^2\right) = (1/2)\rho u_{0i}^2 F'^2$$
$$K = (1/2)\rho u_{0i}^2 F'^2, \quad K + \mathfrak{A} = \rho u_{0i}^2\, F'^2, \quad u_{0i}^2 = u_{0i}u_{0i}{}^2 \tag{6.13}$$
$$P_i = -\sigma_{ij}\dot{u}_j - D_i\dot{\varphi} = (C_{ijkl}u_{0j}u_{0l}n_k + \epsilon_{ij}\varphi_0\varphi_0 n_j)\left(F'^2/c\right)$$

Substitution of Eq. (6.13) into Eq. (6.11) yields

$$V_i^\mathrm{e} = (C_{ijkl}u_{0j}u_{0l}n_k + \epsilon_{ij}\varphi_0\varphi_0 n_j)/\rho u_{0m}^2 c, \quad V_i^\mathrm{e} n_i = c \tag{6.14a}$$

For the normalized displacement vector $(u_{0m}u_{0m} = 1)$, Eq. (6.14a) becomes

$$V_i^\mathrm{e} = (C_{ijkl}u_{0j}u_{0l}n_k + \epsilon_{ij}\varphi_0\varphi_0 n_j)/\rho c, \quad \boldsymbol{V}^\mathrm{e}\cdot\boldsymbol{n} = c \tag{6.14b}$$

where $\boldsymbol{V}^\mathrm{e}$ gives the energy transport direction, the direction of the acoustic ray. The projection of $\boldsymbol{V}^\mathrm{e}$ on $\boldsymbol{n}$ is equal to the phase velocity c, so $|\boldsymbol{V}^\mathrm{e}| \geq c$.

6.1.3 Group Velocity

Usually for a monochromatic wave, Eq. (6.2) is written in a complex number form:

$$u_i = u_{0i}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = u_{0i}\mathrm{e}^{\mathrm{i}k(\boldsymbol{n}\cdot\boldsymbol{x}-ct)}, \quad \varphi = \varphi_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = \varphi_0\mathrm{e}^{\mathrm{i}k(\boldsymbol{n}\cdot\boldsymbol{x}-ct)} \tag{6.15}$$

A general plane wave is dealt with the superposition method. For a chromatic dispersion wave, the wave velocity is dependent to the frequency, and the group velocity is defined as

$$V_j^\mathrm{g} = \partial\omega/\partial k_j, \quad k_j = kn_j \tag{6.16}$$

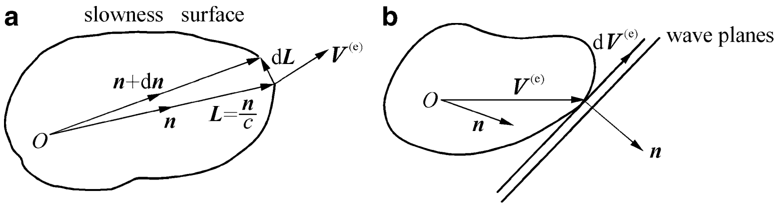


Fig. 6.1 (**a**) Energy velocity is orthogonal to slowness surface. (**b**) Wave surface and propagation direction

Multiplying Eq. (6.7) by k^2 yields

$$\left|\rho(kc)^2\delta_{il} - \left(C_{ijkl}kn_jkn_k + ke_i^*ke_l^*/\epsilon^*\right)\right| = 0 \tag{6.17}$$

It is found that the relation between c and $\left(n_j, e_i^*\right)$ is identical with the relation between $\omega = kc$ and $\left(kn_j, ke_i^*\right)$, so

$$V_j^{\mathrm{g}} = \partial\omega/\partial k_j = \partial(kc)/\partial(kn_j) = \partial c/\partial n_j = V_i^{\mathrm{e}} \tag{6.18}$$

where Eq. (6.12) has been used. Therefore, the energy transport velocity is identical with the group velocity for a non-dissipative plane wave, but for a dissipative plane wave, they may be different.

6.1.4 Characteristic Surfaces

In the illustration of the phenomena of an electroelastic wave propagation, the characteristic surfaces, including the velocity surface, slowness surface, and wave surface, are very useful.

1. *Velocity surface* When the propagation direction is varied, the locus of the ends of the phase velocity vector $\boldsymbol{c} = c\boldsymbol{n}$ forms a velocity surface. In a piezoelectric material there are three different velocities, so there are three velocity surfaces.

2. *Slowness surface* The end of the slowness vector $\boldsymbol{L} = \boldsymbol{n}/c$ draws a slowness surface. $\boldsymbol{L}$ is parallel to $\boldsymbol{c}$ and $Lc = 1, L = |\boldsymbol{L}|$. The slowness surface is important in dealing with reflection and transmission problem in crystals due to similarity with the index surface in optics. The energy transport velocity is always perpendicular to the slowness surface (Fig. 6.1a). In fact we have

$$\begin{aligned}
&\frac{\partial L_i}{\partial n_k} = \frac{\partial(n_i/c)}{\partial n_k} = \frac{\delta_{ik}}{c} - \frac{n_i}{c^2}\frac{\partial c}{\partial n_k} \\
&\boldsymbol{V}^{\mathrm{e}} \cdot \frac{\partial \boldsymbol{L}}{\partial n_k} = V_i^{\mathrm{e}}\frac{\partial L_i}{\partial n_k} = \frac{1}{c}\left(V_k^{\mathrm{e}} - \frac{V_i^{\mathrm{e}} n_i}{c}\frac{\partial c}{\partial n_k}\right) = \frac{1}{c}\left(V_k^{\mathrm{e}} - \frac{\partial c}{\partial n_k}\right) = 0
\end{aligned} \tag{6.19}$$

3. *Wave surface* The wave surface is the locus of the ends of the energy transport velocity. The propagation direction of a plane wave is perpendicular to the wave surface (Fig. 6.1b). In fact according to Eqs. (6.14b) and (6.19) from $\boldsymbol{V}^{\mathrm{e}} \cdot \boldsymbol{L} = 1$, $\boldsymbol{V}^{\mathrm{e}} \cdot \mathrm{d}\boldsymbol{L} = 0$, it can be derived as

$$\boldsymbol{L} \cdot \mathrm{d}\boldsymbol{V}^{\mathrm{e}} = 0, \quad \text{or} \quad \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{V}^{\mathrm{e}} = 0 \tag{6.20}$$

6.1.5 Reflection and Transmission of the Plane Wave in Piezoelectric Materials

In order to save the size of this chapter, the wave propagation in an infinite space and the reflection and transmission problem of the plane wave in piezoelectric materials will be discussed with the thermo-electro-elastic wave in pyroelectric materials together.

6.2 Surface Wave

6.2.1 Surface Waves in Structures

Surface waves have been studied a long time (Gulyaev 1969; Auld 1973; Dieulesaint and Royer 1980; Nayfen 1995). Surface acoustic waves (SAW) including Rayleigh wave, Love wave, Lamb wave, and B-G wave are extensively used in transducers, actuators, filters, delay lines, oscillators, signal processing, acoustic imaging, mobile communication, nondestructive evaluation, biomedical ultrasound, and flow noise. Surface acoustic wave device includes a piezoelectric substrate, at least one interdigital transducer (IDT) disposed on the piezoelectric substrate, an input end and an output end connected to the IDT. The energy of the surface acoustic wave is mainly concentrated near the surface.

1. *Semi-infinite media* In 1885 Rayleigh found a surface wave at the surface of a semi-infinite medium, which is called Rayleigh wave. Rayleigh wave is a complex wave and attenuates along the normal direction of the surface. Its penetration depth is 2λ. In the isotropic media it is constituted of a longitudinal wave (L-wave) and a shear wave (S-wave) with $\pi/2$ phase shift. The transverse surface wave with polarization parallel to the surface cannot happen in an elastic media, but it can happen in a semi-infinite piezoelectric material and called B-G wave (Bleustein 1968; Biryukov et al. 1995), whose penetration depth is about 100λ larger than that in Rayleigh wave.

2. *Bi-semi-infinite media* In a bi-semi-infinite medium, Rayleigh wave can propagate on both sides of the interface and this surface wave is called Stoneley wave.

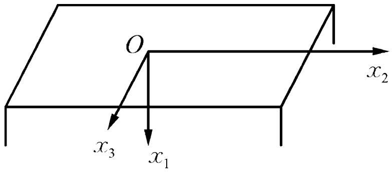


Fig. 6.2 Surface wave propagating in x_1x_2 plane

3. *Infinite plate* For a plate bounded by two parallel infinite planes, when the plate thickness is of the order of the wave length, one gets Lamb waves. Lamb wave possesses longitudinal and shear components, so it can be either symmetric or antisymmetric.

4. *Layer structures* Typically a layer structure is constituted of two or multiple layers of different materials, especially thin films deposited on a thick substrate. When the shear wave velocity of the film is larger than that in the substrate, the Love transverse shear surface wave will be happened.

6.2.2 General Procedure for Solving Surface Wave Problem

Let coordinates ox_i with its origin be at the free surface in a semi-infinite space. A surface wave propagates in (x_1, x_2) plane; x_1 is perpendicular to the free surface (Fig. 6.2). The generalized displacements decrease exponentially in direction x_1, i.e.,

$$u_i = u_{0i}e^{-kbx_1}e^{ik(x_2n_2+x_3n_3-ct)}, \quad \varphi = \varphi_0 e^{-kbx_1}e^{ik(x_2n_2+x_3n_3-ct)} \tag{6.21}$$

where $b > 0$ is the unknown attenuated coefficient. Equation (6.21) can also be written as

$$u_i = u_{0i}e^{ik(x_jn_j-ct)}, \quad \varphi = \varphi_0 e^{ik(x_jn_j-ct)}, \quad b = -in_1, \quad \mathrm{Im}\, n_1 > 0, \quad j = 1, 2, 3 \tag{6.22}$$

where n_1 is not the directional cosine, but an unknown related to the attenuated coefficient.

Substituting Eq. (6.22) into Eq. (6.1) yields Eq. (6.4) and the corresponding eigen-equation (6.6), but in the surface wave case, c and n_1 are unknown. Usually let c be a parameter and solve n_1 from the eigen-equation. Because the eigen-equation is an eight-order algebraic equation with real coefficients, n_1 has 4 pairs of complex roots. But there are 4 roots appropriate because Im $n_1 > 0$ is required. Corresponding 4 eigenvalues $n_1^{(r)}, r = 1, 2, 3, 4$, there are 4 group eigenvectors $u_{0i}^{(r)}, \varphi_0^{(r)}$. The general solution is

$$\begin{gathered}
u_i = \sum_{r=1}^{4} A_r u_{0i}^{(r)} e^{ik\left(x_jn_j-ct\right)}, \quad \varphi = \sum_{r=1}^{4} A_r \varphi_0^{(r)} e^{ik\left(x_jn_j-ct\right)}, \quad j = 1, 2, 3 \\
u_i = \sum_{r=1}^{4} A_r u_{0i}^{(r)} e^{-kb_rx_1} e^{ik(x_2n_2+x_3n_3-ct)}, \quad \varphi = \sum_{r=1}^{4} A_r \varphi_0^{(r)} e^{-kb_rx_1} e^{ik(x_2n_2+x_3n_3-ct)}
\end{gathered} \tag{6.23}$$

Unknowns c and A_r in Eq. (6.23) are determined by the boundary conditions on the free surface. The boundary conditions on a free surface is

$$\sigma_{i1} = C_{i1kl}u_{k,l} + e_{ki1}\varphi_{,k} = \sum_{r=1}^{4} A_r\sigma_{i1}^{(r)}\mathrm{e}^{\mathrm{i}k(x_2n_2+x_3n_3-ct)} = 0; \quad \text{when} \quad x_1 = 0$$
$$\sigma_{i1}^{(r)} = -\mathrm{i}k\left(C_{i1kl}n_k u_{0l}^{(r)} + e_{ki1}n_k\varphi_0^{(r)}\right) \tag{6.24}$$

Let the environment of the piezoelectric material be air. In air $\nabla^2\varphi^c = 0$, we can assume

$$\varphi^c = \varphi_0^c\mathrm{e}^{\mathrm{i}k(x_jn_j-ct)}, \quad D_1^c = -\mathrm{i}k\epsilon_0\varphi_0^c\mathrm{e}^{\mathrm{i}k(x_jn_j-ct)} \quad x_1 \geq 0, \quad j = 1,2,3$$
$$\varphi^c = \varphi_0^c\mathrm{e}^{\mathrm{i}k(x_2n_2+x_3n_3-ct)}, \quad D_1^c = -\mathrm{i}k\epsilon_0\varphi_0^c\mathrm{e}^{\mathrm{i}k(x_2n_2+x_3n_3-ct)}; \quad x_1 = 0\ (n_1 = 1) \tag{6.25}$$

There are two kinds of the electric boundary conditions:

$$\text{Electrically open case}: \quad \varphi = \varphi^c, \quad D_1 = D_1^c, \quad \text{when} \quad x_1 = 0 \tag{6.26a}$$

$$\text{Electrically shorted case}: \quad \varphi^c = 0, \quad \text{when} \quad x_1 = 0 \tag{6.26b}$$

Combining Eqs. (6.24) and (6.26a) or (6.26b), five homogeneous equations with five unknowns A_r, φ_0^c are obtained. In order for A_r, φ_0^c to have nontrivial solutions, the determinant of coefficients before them must be zero. From this condition the wave velocity c and A_r, φ_0^c are obtained. Usually only one c can satisfy condition $\mathrm{Im}n_1 > 0$.

The coupling coefficient k_e is defined as (Laurent et al. 2000)

$$k_\mathrm{e}^2 = U_\mathrm{me}^2/U_\mathrm{m}U_\mathrm{e} = 2(c_\mathrm{f} - c_\mathrm{s})/c_\mathrm{f}(1 + \epsilon^c/\epsilon) \approx 2(c_\mathrm{f} - c_\mathrm{s})/c_\mathrm{f} \tag{6.27}$$

where c_f is the wave velocity under electrically open case and c_s is the wave velocity under electrically shorted case. $U_\mathrm{me}, U_\mathrm{m}, U_\mathrm{e}$ are the mutual electromechanical coupling energy, mechanical energy, and electric energy, respectively.

6.2.3 Surface Waves in a Semi-infinite Piezoelectric Material

Equation (6.23) has three displacement waves and an electric potential. It is a general 3D piezoelectric Rayleigh wave denoted by $\bar{R}_3$. In the material principle coordinate system, the number of the independent material constants will be obviously reduced by crystal symmetry. So the following simpler surface waves will happen:

1. $\Gamma_{13} = \Gamma_{23} = e_1^* = e_2^* = 0$, and Eq. (6.4) or (6.6) splits to the following two equations:

$$\begin{bmatrix} \Gamma_{11}-\rho c^2 & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22}-\rho c^2 \end{bmatrix}\begin{Bmatrix} u_{01} \\ u_{02} \end{Bmatrix} = \mathbf{0}, \quad \begin{bmatrix} \Gamma_{33}-\rho c^2 & e_3^* \\ e_3^* & -\epsilon^* \end{bmatrix}\begin{Bmatrix} u_{03} \\ \varphi_0 \end{Bmatrix} = \mathbf{0} \tag{6.28}$$

where the first equation represents the pure 2D elastic Rayleigh wave denoted by R_2 and the second equation represents the transverse piezoelectric wave denoted by B-G wave. According to $\Gamma_{13}=\Gamma_{23}=e_1^*=e_2^*=0$, some relations between material constants can be derived.

2. $\Gamma_{13}=\Gamma_{23}=e_3^*=0$, and Eq. (6.4) or (6.6) splits to the following two equations:

$$(\Gamma_{33}-\rho c^2)u_{03}=0, \quad \begin{bmatrix} \Gamma_{11}-\rho c^2 & \Gamma_{12} & e_1^* \\ \Gamma_{21} & \Gamma_{22}-\rho c^2 & e_2^* \\ e_1^* & e_2^* & -\epsilon^* \end{bmatrix}\begin{Bmatrix} u_{01} \\ u_{02} \\ \varphi_0 \end{Bmatrix} = \mathbf{0} \tag{6.29}$$

where the first equation represents the pure elastic transverse shear wave and the second equation represents the 2D piezoelectric Rayleigh wave denoted by $\bar{R}_2$. According to $\Gamma_{13}=\Gamma_{23}=e_3^*=0$, some relations between material constants can be derived.

Li et al. (2005b) and Li et al. (2005a) adopted the modified Mindlin (1968) polarization gradient theory to discuss the surface wave and showed that the gradient effect can make the surface wave dispersive which is different with the classical linear theory. This phenomenon may be meaningful in high-frequency short surface wave. In the later sections we mainly discuss the surface wave with initial stress or biasing electric field and a few problems about wave scattering from a crack.

6.3 Fundamental Theory of Layered Structure with Generalized Biasing Stresses

6.3.1 A Small Perturbation Superposed on Finite Generalized Displacements

In order to improve performance or select the most suitable operating conditions of SAW devices, such as selectivity of filters, stability of oscillators, and temperature compensation, the generalized biasing displacements or stresses are applied to the SAW devices to establish a biasing state. At the same time because of the material behaviors between the layer and substrate are different, the initial stresses and initial strains in the layer are produced unavoidably during manufacture process. Sometimes the initial stress is great with the magnitude of 1 GPa. The presence of initial stress causes changes in the speed of surface acoustic wave, frequency shift, controlling the selectivity of a filter and temperature compensation of devices, etc. A middle layer in the multilayered structure can be used to adjust the range of phase velocity of SAW and to improve its property (Khuri-Yakub and Kino 1974; Assour et al. 2000).

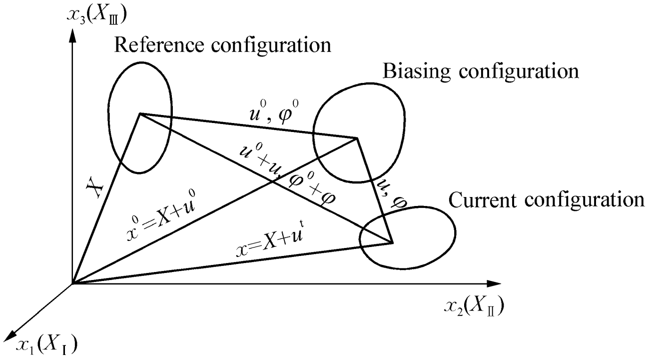


Fig. 6.3 Different configurations in finite deformation

The biasing stress and electric fields usually are large and assumed known, but the external signal or perturbation is small. So the problem is a small perturbation superposed on finite generalized displacements (Tiersten 1978; Sinha et al. 1985; Su et al. 2005). The fundamental theory of finite deformation can be seen in many books (Ogden 1984; Kuang 2002). Some fundamental formulas can be found in Sect. 1.3.4. Take the natural configuration without generalized stress as the reference configuration. In this theoretical part the notations shown in Sect. 1.3.4 are adopted. The same coordinate system in the current and initial configurations is taken, i.e., $x_I = x_i$, so that for the case without differential symbol, we have $\bar{\sigma}_{IJ} = \bar{\sigma}_{ij}$, but the differentiation with the capital or small letter subscript is different. Let $\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{\varepsilon}}^0, \bar{\boldsymbol{D}}^0, \bar{\boldsymbol{E}}^0, \boldsymbol{u}^0, \varphi^0, \bar{\boldsymbol{f}}^0, \bar{\boldsymbol{T}}^0, \bar{\rho}_e^0, \bar{\sigma}^{*0}$ be variables in the biasing state. The small perturbation variables in the reference configuration are denoted with $\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{E}}, \boldsymbol{u}, \varphi, \bar{\boldsymbol{f}}, \bar{\boldsymbol{T}}, \bar{\rho}_e, \bar{\sigma}^*$ (Fig. 6.3), where $\bar{\boldsymbol{\sigma}}$ is the Kirchhoff stress and $\bar{\boldsymbol{\varepsilon}}$ is the Green strain. The current total variables described in the reference configuration are

$$\text{Generalized geometric equation:} \quad \bar{\varepsilon}^t_{KL} = (1/2)\left(u^t_{K,L} + u^t_{L,K} + u^t_{M,K}u^t_{M,L}\right); \quad \bar{E}^t_I = -\varphi^t_{,I} \tag{6.30}$$

$$\text{Generalized motion equation:} \quad \left(\bar{\sigma}^t_{KM}\delta_{lM} + \bar{\sigma}^t_{KM}u^t_{l,M}\right)_{,K} + \bar{f}^t_l = \bar{\rho}\ddot{u}^t_l; \quad \bar{D}^t_{I,I} = \bar{\rho}^t_e \tag{6.31}$$

$$\text{Boundary conditions:} \quad x_{l,M}\bar{\sigma}^t_{KM}\bar{n}^t_K = \bar{T}^{*t}_l, \quad \bar{D}^t_K\bar{n}^t_K = -\bar{\sigma}^{*t}, \quad \text{or} \quad u^t_i = u^{t*}_i, \quad \varphi^t = \varphi^{t*} \tag{6.32}$$

where

$$\begin{aligned} &\bar{\boldsymbol{\sigma}}^t = \bar{\boldsymbol{\sigma}}^0 + \bar{\boldsymbol{\sigma}}, \quad \bar{\boldsymbol{D}}^t = \bar{\boldsymbol{D}}^0 + \bar{\boldsymbol{D}}, \quad \boldsymbol{u}^t = \boldsymbol{u}^0 + \boldsymbol{u}, \quad \boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}^t, \quad \boldsymbol{x}^0 = \boldsymbol{X} + \boldsymbol{u}^0 \\ &\bar{\boldsymbol{f}}^t = \bar{\boldsymbol{f}}^0 + \bar{\boldsymbol{f}}, \quad \bar{\boldsymbol{T}}^t = \bar{\boldsymbol{T}}^0 + \bar{\boldsymbol{T}}, \quad \bar{\rho}^t_e = \bar{\rho}^0_e + \bar{\rho}_e, \quad \bar{\sigma}^{*t} = \bar{\sigma}^{*0} + \bar{\sigma}^*, \quad \varphi^t = \varphi^0 + \varphi \end{aligned} \tag{6.33}$$

Because the biasing state is an equilibrium state, so

$$\begin{aligned} &(\bar{\sigma}^0_{KM}\delta_{lM} + \bar{\sigma}^0_{KM}u^0_{l,M})_{,K} + \bar{f}^0_l = 0, \quad \bar{D}^0_{I,I} = \bar{\rho}^0_e \\ &\left(\delta_{lM} + u^0_{l,M}\right)\bar{\sigma}^0_{KM}\bar{n}^0_K = \bar{T}^{*0}_l, \quad \bar{D}^0_K\bar{n}^0_K = -\bar{\sigma}^{*0} \end{aligned} \tag{6.34}$$

Subtracting the first equation in Eq. (6.34) from Eq. (6.31) and ignoring small terms in high order, such as $u_{m,K}u_{m,L}$, we find

$$\left(\bar{\sigma}_{KM}\delta_{lM} + \bar{\sigma}_{KM}u^0_{l,M} + \bar{\sigma}^0_{KM}u_{l,M}\right)_{,K} + \bar{f}_l = \rho_0\ddot{u}_l; \quad \bar{D}_{K,K} = \bar{\rho}_{\mathrm{e}} \tag{6.35}$$

Subtracting the second equation in Eq. (6.34) from Eq. (6.32) yields

$$\begin{aligned} &\left(\bar{\sigma}_{Kl} + \bar{\sigma}^0_{KM}u_{l,M} + u^0_{l,M}\bar{\sigma}_{KM}\right)\bar{n}^t_K + \left(\bar{n}^t_K - \bar{n}^0_K\right)\left(\bar{\sigma}^0_{Kl} + \bar{\sigma}^0_{KM}u^0_{l,M}\right) = \bar{T}_l \\ &\bar{D}_K\bar{n}^t_K + \bar{D}^0_K\left(\bar{n}^t_K - \bar{n}^0_K\right) = -\bar{\sigma}^* \end{aligned} \tag{6.36}$$

From Eq. (1.45) we can derive

$$\bar{n}^0_K = \left|\partial X_I \big/ \partial x^0_j\right|\left(\partial x^0_i/\partial X_K\right)n_i\mathrm{d}a\big/\mathrm{d}\bar{a}^0, \quad \bar{n}^t_K = \left|\partial X_I/\partial x_j\right|(\partial x_i/\partial X_K)n_i\mathrm{d}a/\mathrm{d}\bar{a}^t$$

The change of the normal of the boundary can only be obtained after solving the problem. But usually the difference between $\bar{n}^t_k$ and $\bar{n}^0_k$ is small and let $\bar{n}^t_k = \bar{n}^0_k$ to simplify the boundary conditions.

Sometimes the three-order coefficients in the constitutive equation are needed. Let

$$\begin{aligned} &\bar{\sigma}^t_{IJ} = C_{IJKL}\bar{\varepsilon}^t_{KL} + (1/2)C_{IJKLMN}\bar{\varepsilon}^t_{KL}\bar{\varepsilon}^t_{MN} - e_{MIJ}\bar{E}^t_M - e_{MIJKL}\bar{\varepsilon}^t_{KL}\bar{E}^t_M - (1/2)l_{MNIJ}\bar{E}^t_M\bar{E}^t_N \\ &\bar{D}^t_M = \epsilon_{MN}\bar{E}^t_N + (1/2)\epsilon_{MNJ}\bar{E}^t_N\bar{E}^t_J + e_{MIJ}\bar{\varepsilon}^t_{IJ} + (1/2)e_{MIJKL}\bar{\varepsilon}^t_{IJ}\bar{\varepsilon}^t_{KL} + l_{MNIJ}\bar{\varepsilon}^t_{IJ}\bar{E}^t_N \end{aligned} \tag{6.37}$$

Similarly for the biasing state, we have

$$\begin{aligned} &\bar{\sigma}^0_{IJ} = C_{IJKL}\bar{\varepsilon}^0_{KL} + (1/2)C_{IJKLMN}\bar{\varepsilon}^0_{KL}\bar{\varepsilon}^0_{MN} - e_{MIJ}\bar{E}^0_M - e_{MIJKL}\bar{\varepsilon}^0_{KL}\bar{E}^0_M - (1/2)l_{MNIJ}\bar{E}^0_M\bar{E}^0_N \\ &\bar{D}^0_M = \epsilon_{MN}\bar{E}^0_N + (1/2)\epsilon_{MNJ}\bar{E}^0_N\bar{E}^0_J + e_{MIJ}\bar{\varepsilon}^0_{IJ} + (1/2)e_{MIJKL}\bar{\varepsilon}^0_{IJ}\bar{\varepsilon}^0_{KL} + l_{MNIJ}\bar{\varepsilon}^0_{IJ}\bar{E}^0_N \end{aligned} \tag{6.38}$$

Subtracting Eq. (6.38) from (6.37) and neglecting the small terms in higher order, such as $u_{M,K}u_{M,L}$, $u^0_{P,N}u^0_{K,M}u_{K,L}$, then we get the results that are expressed in terms of generalized displacements:

$$\bar{\sigma}_{IJ} = \hat{C}_{IJKL}u_{K,L} + \hat{e}_{MIJ}\varphi_{,M}, \quad \bar{D}_K = e^*_{KIJ}u_{I,J} - \epsilon^*_{KN}\varphi_{,N} \tag{6.39}$$

where

$$
\begin{aligned}
\hat{C}_{IJKL} &= C_{IJKL} + C_{IJML}u^0_{K,M} + C_{IJKLMN}u^0_{K,L} + e_{MIJKL}\varphi^0_{,M} \\
\hat{e}_{MIJ} &= e_{MIJ} + e_{MIJKL}u^0_{K,L} - l_{MNIJ}\varphi^0_{,N} \\
e^*_{MIJ} &= e_{MIJ} + e_{MNJ}u^0_{I,N} + e_{MIJKL}u^0_{K,L} - l_{MNIJ}\varphi^0_{,N} \\
\epsilon^*_{MN} &= \epsilon_{MN} - \epsilon_{MNJ}\varphi^0_{,J} + l_{MNIJ}u^0_{I,J}
\end{aligned}
\tag{6.40}
$$

Substitution of Eq. (6.39) into Eqs. (6.35) and (6.36) finally yields

$$
\begin{aligned}
&\left(\bar{\sigma}^*_{Kl} + \bar{\sigma}^0_{KM}u_{l,M}\right)_{,K} + \bar{f}_l = \rho_0\ddot{u}_l, \quad \bar{D}_{K,K} = \bar{\rho}_{\mathrm{e}} \\
&\left(\sigma^*_{Kl} + \sigma^0_{KM}u_{l,M}\right)n_K = \bar{T}^*_l, \quad \bar{D}_K n_K = -\bar{\sigma}^*
\end{aligned}
\tag{6.41}
$$

where C^*_{KlPJ}, e^*_{KPJ}, etc. are effective material constants and

$$
\begin{aligned}
&\bar{\sigma}^*_{Kl} = \bar{\sigma}_{KM}\left(\delta_{lM} + u^0_{l,M}\right) = C^*_{KlPJ}u_{P,J}, \quad \bar{D}_K = e^*_{KPJ}u_{P,J}; \quad P = 1,2,3,4 \\
&C^*_{KlIJ} = \hat{C}_{KMIJ}\left(\delta_{lM} + u^0_{l,M}\right), \quad C^*_{Kl4J} = \hat{e}_{JKM}\left(\delta_{lM} + u^0_{l,M}\right), \quad u_{4,J} = \varphi_{,J} \\
&\bar{D}_K = e^*_{KPJ}u_{P,J}, \quad e^*_{KIJ} = e^*_{KIJ}, \quad e^*_{K4J} = -\epsilon^*_{KN}
\end{aligned}
\tag{6.42}
$$

where the capital letter subscript P takes the value 1, 2, 3, 4. Equation (6.41) can also be rewritten as

$$
\begin{aligned}
&\left(C^{**}_{KlIJ}u_{I,J} + e^{**}_{NKl}\varphi_{,N}\right)_{,K} + \bar{f}_l = \rho_0\ddot{u}_l, \quad \left(e^*_{KIJ}u_{I,J} - \epsilon^*_{KN}\varphi_{,N}\right)_{,K} = \rho_{\mathrm{e}} \\
&\left(C^{**}_{KlIJ}u_{I,J} + e^{**}_{NKl}\varphi_{,N}\right)\bar{n}^t_K = \bar{T}_j, \quad \left(e^*_{KIJ}u_{I,J} - \epsilon^*_{KN}\varphi_{,N}\right)\bar{n}^t_K = -\bar{\sigma}^* \\
&C^{**}_{KlIJ} = C^*_{KlIJ} + \bar{\sigma}^0_{KJ}\delta_{Il}, \quad e^{**}_{NKl} = \hat{e}_{NKM}\left(\delta_{lM} + u^0_{l,M}\right) \approx e^*_{NKl}
\end{aligned}
\tag{6.43}
$$

When $\boldsymbol{u}, \varphi$ are solved, $\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{D}}$ is obtained; then from Eq. (1.45) the generalized Cauchy stresses $\boldsymbol{\sigma}, \boldsymbol{D}$ can be obtained.

6.3.2 *Simplifications of the Governing Equations for Some Cases*

1. *Initial stress configuration taken as reference configuration* If we use the configuration with initial stress as the reference configuration, then $\boldsymbol{u}^0, \varphi^0$ are not needed or let $\boldsymbol{u}^0 = \boldsymbol{0}, \varphi^0 = 0$ and $\bar{\sigma}^*_{Kl} = \bar{\sigma}_{KM} = \sigma_{kl}, \bar{D}_K = D_k$, so formulas are much simpler. But the material coefficients must take the "tangent modulus," or the constitutive equations are measured at the state with given generalized biasing

displacements and stresses. The total generalized stress and displacements are the sum of the initial values and the perturbation values.

2. *Small initial generalized displacements* If the initial generalized displacements and stresses are also small, i.e., $\boldsymbol{u}^0 + \boldsymbol{u}, \varphi^0 + \varphi$ are small compared with 1, the terms containing them can be neglected, so all generalized stress $\bar{\boldsymbol{\sigma}}^* = \boldsymbol{\sigma}, \bar{\boldsymbol{D}} = \boldsymbol{D}$.

6.4 Love Wave in $ZnO/SiO_2/Si$ Structure with Initial Stresses

6.4.1 Transfer Matrix Method

Figure 6.4 shows a three-layer structure constituted of the substrate Si, the first layer SiO_2 of thickness h_1 and the second layer ZnO of thickness h_2 and $h_1 + h_2 = h$. The origin of the global coordinate system is located at interface of the substrate and first layer. The top surface of the layer $x_1 = -h$ is free of stress and the environment is air. Experiments show that the distribution of initial stresses along the depth x_1 in the layers is shown in Fig. 6.4. The thickness of the substrate is much larger than that of layers and can be treated as a half-space. The initial stresses in the substrate are negligible. In order to obtain more exact solution, the first layer is further divided into $1 \sim m$ sublayers, and the second layer is divided into $m + 1 \sim N$ sublayers. The substrate is denoted by layer 0 and the air is denoted by layer $N + 1$ (Fig. 6.5).

For a multilayer structure the transfer matrix method is a useful technique (Thomson 1950; Stewart and Yong 1994; Liu et al. 2003b; Su et al. 2005). The basis of the transfer matrix method is that for any sublayer k to establish, a transfer matrix maps the generalized stresses and displacements from its lower surface to upper surface. Successive application of the transfer method through sublayer 0 to $N + 1$ and invoking corresponding interface continuity conditions leads to a set of equations relating to the boundary conditions on the free surface. Combining the conditions at infinity, we can get enough equations to solve

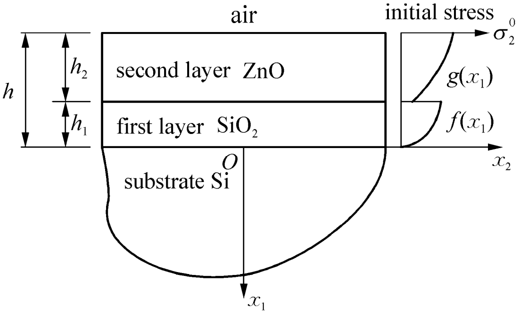


Fig. 6.4 Three-layer structure

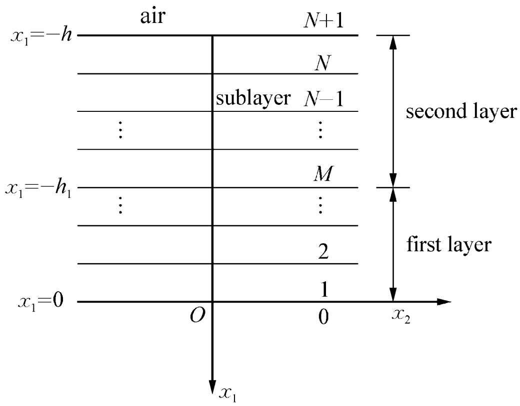


Fig. 6.5 Layered structure divided into N sublayers

the problem. The state space approach with appropriate selected variables is a convenient method to establish the transfer matrix. Here $u_i, \varphi, \sigma_{i1}, D_1$ are used as state variables. For convenience in the following part, the subscripts all adopt the small letters except the capital letter P which takes the value 1, 2, 3, 4. It is noted that in this part the differentiation with a small letter is still considered in the reference configuration. According to the first equation in Eq. (6.41), in each sublayer we have

$$\bar{\sigma}^*_{i1,1} = \rho_0 \ddot{u}_i - \bar{\sigma}^*_{i2,2} - \bar{\sigma}^*_{i3,3} - \bar{\sigma}^0_{jk} u_{i,jk} - \bar{\sigma}^0_{jk,j} u_{i,k} \tag{6.44}$$

Assuming the incident wave is located in the plane x_2x_3, the solution of the generalized displacement U_P $(P = 1, 2, 3, 4)$ is

$$U_P = A_P(x_1)\exp[\mathrm{i}(k_2x_2 + k_3x_3 - \omega t)] = A_P(x_1)\exp[\mathrm{i}(k_\alpha x_\alpha - \omega t)], \quad \alpha = 2, 3 \tag{6.45}$$

Substitution of Eqs. (6.45) and (6.42) into Eq. (6.44) yields

$$\begin{aligned}\bar{\sigma}^*_{i1,1} = \Big\{ &-\rho_0\omega^2 A_i - \mathrm{i}k_\beta\left(C^*_{i\beta P1}A_{P,1} + \mathrm{i}k_\gamma C^*_{i\beta P\gamma}A_P\right) - \bar{\sigma}^0_{11}A_{i,11} - 2\mathrm{i}k_\gamma\bar{\sigma}^0_{1\gamma}A_{i,1} \\ &+ k_\beta k_\gamma \bar{\sigma}^0_{\beta\gamma}A_i - \bar{\sigma}^0_{j1,j}A_{i,1} - \mathrm{i}k_\beta\bar{\sigma}^0_{j\beta,j}A_i\Big\}\exp[\mathrm{i}(k_\alpha x_\alpha - \omega t)]\end{aligned} \tag{6.46}$$

Usually $\bar{\sigma}^0_{11}, \bar{\sigma}^0_{12}, \bar{\sigma}^0_{13}$ are small and can be dropped. Let

$$\bar{\sigma}^*_{ij} = \hat{\sigma}_{ij}(x_1)\exp[\mathrm{i}(k_\alpha x_\alpha - \omega t)], \quad \bar{D}_i = T_{i+6}(x_1)\exp[\mathrm{i}(k_\alpha x_\alpha - \omega t)] \tag{6.47}$$

Equation (6.46) can be rewritten as

$$\hat{\sigma}_{i1,1} + \mathrm{i}k_\beta C^*_{i\beta P1} A_{P,1} + \bar{\sigma}^0_{j1,j} A_{i,1} = \left(-\rho_0\omega^2 + k_\beta k_\gamma \bar{\sigma}^0_{\beta\gamma} - \mathrm{i}k_\beta \bar{\sigma}^0_{j\beta,j}\right) A_i + k_\beta k_\gamma C^*_{i\beta P\gamma} A_P \tag{6.48a}$$

The second equation in Eq. (6.41) and Eq. (6.42) can also be expressed as

$$\begin{aligned} &T_{7,1} + \mathrm{i}k_\beta e^*_{\beta p1} A_{P,1} = k_\beta k_\gamma e^*_{\beta p\gamma} A_P \\ &C^*_{1jP1} A_{P,1} = \hat{\sigma}_{j1} - \mathrm{i}k_\beta C^*_{1jP\beta} A_P, \quad e^*_{1P1} A_{P,1} = T_7 - \mathrm{i}k_\beta e^*_{1P\beta} A_P \end{aligned} \tag{6.48b}$$

Introduce (in Voigt notation)

$$\sigma_n = T_n(x_1) \exp[\mathrm{i}(k_\alpha x_\alpha - \omega t)], \quad n = 1, 2, \ldots, 6 \tag{6.49}$$

where $\sigma_n (n = 1, 2, \ldots, 6)$ represents $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{32}, \sigma_{31}, \sigma_{12}$, respectively, and let $\hat{\sigma}_{i1} = T_1$, $\hat{\sigma}_{i2} = T_6$, and $\hat{\sigma}_{i3} = T_5$. Introduce an eight-dimensional vector $\boldsymbol{v}_m$:

$$\boldsymbol{v}_m(x_1) = [A_{1m}, A_{2m}, A_{3m}, A_{4m}, T_{1m}, T_{6m}, T_{5m}, T_{7m}]^{\mathrm{T}} \tag{6.50}$$

where A_{1m}, A_{2m}, A_{3m} are the amplitudes of u_1, u_2, u_3, respectively; A_{4m} is the amplitude of φ; T_{1m}, T_{6m}, T_{5m} are the amplitudes of $\sigma_{11}, \sigma_{21}, \sigma_{31}$, respectively; and T_{7m} is the amplitude of D_1. Using Eqs. (6.48a) and (6.48b) for any sublayer k, the eight-dimensional state equation with unknown vector $\boldsymbol{v}_m$ is

$$\left[\boldsymbol{B}_m(x_1)\frac{\mathrm{d}}{\mathrm{d}x_1} - \boldsymbol{F}_m(x_1)\right]\boldsymbol{v}_m(x_1) = \boldsymbol{0}, \quad \text{or} \quad \left[\frac{\mathrm{d}}{\mathrm{d}x_1} - \boldsymbol{B}_m^{-1}(x_1)\boldsymbol{F}_m(x_1)\right]\boldsymbol{v}_m(x_1) = \boldsymbol{0} \tag{6.51}$$

where $\boldsymbol{B}_m^{-1}(x_1)\boldsymbol{F}_m(x_1)$ is the state matrix of the sublayer k and

$$\boldsymbol{B}_m = \begin{bmatrix} B_m(11) & \mathrm{i}k_\beta C^*_{1\beta21} & \mathrm{i}k_\beta C^*_{1\beta31} & \mathrm{i}k_\beta C^*_{1\beta41} & 1 & 0 & 0 & 0 \\ \mathrm{i}k_\beta C^*_{2\beta11} & B_m(22) & \mathrm{i}k_\beta C^*_{2\beta31} & \mathrm{i}k_\beta C^*_{2\beta41} & 0 & 1 & 0 & 0 \\ \mathrm{i}k_\beta C^*_{3\beta11} & \mathrm{i}k_\beta C^*_{3\beta21} & B_m(33) & \mathrm{i}k_\beta C^*_{3\beta41} & 0 & 0 & 1 & 0 \\ \mathrm{i}k_\beta e^*_{\beta11} & \mathrm{i}k_\beta e^*_{\beta21} & \mathrm{i}k_\beta e^*_{\beta31} & \mathrm{i}k_\beta e^*_{\beta41} & 0 & 0 & 0 & 1 \\ C^*_{1111} & C^*_{1121} & C^*_{1131} & C^*_{1141} & 0 & 0 & 0 & 0 \\ C^*_{1211} & C^*_{1221} & C^*_{1231} & C^*_{1241} & 0 & 0 & 0 & 0 \\ C^*_{1311} & C^*_{1321} & C^*_{1331} & C^*_{1341} & 0 & 0 & 0 & 0 \\ e^*_{111} & e^*_{121} & e^*_{131} & e^*_{141} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_m(11) = \sigma^0_{j1,j} + \mathrm{i}k_\beta c^*_{1\beta11}, \quad B_m(22) = \sigma^0_{j1,j} + \mathrm{i}k_\beta c^*_{2\beta21}, \quad B_m(33) = \sigma^0_{j1,j} + \mathrm{i}k_\beta c^*_{3\beta31} \tag{6.52}$$

and

$$\boldsymbol{F}_m = \begin{bmatrix} F_m(11) & k_\beta k_\gamma C^*_{1\beta 2\gamma} & k_\beta k_\gamma C^*_{132\gamma} & k_\beta k_\gamma C^*_{1\beta 4\gamma} & 0 & 0 & 0 & 0 \\ k_\beta k_\gamma C^*_{2\beta 1\gamma} & F_m(22) & k_\beta k_\gamma C^*_{2\beta 3\gamma} & k_\beta k_\gamma C^*_{2\beta 4\gamma} & 0 & 0 & 0 & 0 \\ k_\beta k_\gamma C^*_{3\beta 1\gamma} & k_\beta k_\gamma C^*_{3\beta 2\gamma} & F_m(33) & k_\beta k_\gamma C^*_{3\beta 4\gamma} & 0 & 0 & 0 & 0 \\ k_\beta k_\gamma e^*_{\beta 1\gamma} & k_\beta k_\gamma e^*_{\beta 2\gamma} & k_\beta k_\gamma e^*_{\beta 3\gamma} & k_\beta k_\gamma e^*_{\beta 4\gamma} & 0 & 0 & 0 & 0 \\ -\mathrm{i}k_\beta C^*_{111\beta} & -\mathrm{i}k_\beta C^*_{112\beta} & -\mathrm{i}k_\beta C^*_{113\beta} & -\mathrm{i}k_\beta C^*_{114\beta} & 1 & 0 & 0 & 0 \\ -\mathrm{i}k_\beta C^*_{121\beta} & -\mathrm{i}k_\beta C^*_{122\beta} & -\mathrm{i}k_\beta C^*_{123\beta} & -\mathrm{i}k_\beta C^*_{124\beta} & 0 & 1 & 0 & 0 \\ -\mathrm{i}k_\beta C^*_{131\beta} & -\mathrm{i}k_\beta C^*_{132\beta} & -\mathrm{i}k_\beta C^*_{133\beta} & -\mathrm{i}k_\beta C^*_{134\beta} & 0 & 0 & 1 & 0 \\ -\mathrm{i}k_\beta e^*_{11\beta} & -\mathrm{i}k_\beta e^*_{12\beta} & -\mathrm{i}k_\beta e^*_{13\beta} & -\mathrm{i}k_\beta e^*_{14\beta} & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.53}$$

$$F_m(11) = -\rho_0\omega^2 + k_\beta k_\gamma \sigma^0_{\beta\gamma} - \mathrm{i}k_\beta \sigma^0_{j\beta,j} + k_\beta k_\gamma c^*_{1\beta 1\gamma}$$
$$F_m(22) = -\rho_0\omega^2 + k_\beta k_\gamma \sigma^0_{\beta\gamma} - \mathrm{i}k_\beta \sigma^0_{j\beta,j} + k_\beta k_\gamma c^*_{2\beta 2\gamma}$$
$$F_m(33) = -\rho_0\omega^2 + k_\beta k_\gamma \sigma^0_{\beta\gamma} - \mathrm{i}k_\beta \sigma^0_{j\beta,j} + k_\beta k_\gamma c^*_{3\beta 3\gamma}$$

The solution of Eq. (6.51) is

$$\begin{aligned} &\boldsymbol{v}_m(x_1) = \boldsymbol{Q}_m\boldsymbol{R}_m\boldsymbol{a}_m, \quad \boldsymbol{R}_m = \mathrm{diag}[\exp(b_{1m}x_1), \exp(b_{2m}x_1), \ldots, \exp(b_{8m}x_1)] \\ &\boldsymbol{Q}_m = [h_{1m}, h_{2m}, \ldots, h_{8m}], \quad \boldsymbol{a}_m = [a_{1m}, a_{2m}, \ldots, a_{8m}]^{\mathrm{T}} \end{aligned} \tag{6.54}$$

where b_{jm} and h_{jm} are the eigenvalue and eigenvector of the state matrix, respectively, and a_{jm} is an undetermined constant in the sublayer m. The generalized stresses and displacements at the bottom of the structure can be related to those at its top through the transfer matrix $\boldsymbol{P}_m(x_{1m} - d_m, x_{1m})$:

$$\boldsymbol{v}_m(x_{1m} - d_m) = \boldsymbol{P}_m(x_{1m} - d_m, x_{1m})\boldsymbol{v}_m(x_{1m}) \tag{6.55}$$

Equations (6.54) and (6.55) yield

$$\boldsymbol{P}_m(x_{1m} - d_m, x_{1m}) = \boldsymbol{Q}_m\boldsymbol{R}_m(-d_m)\boldsymbol{Q}_m^{-1} \tag{6.56}$$

where x_{1m} is the coordinate at the bottom surface of the sublayer m and d_m is its thickness. Using the basic relations of the transfer matrix, it is found

$$\boldsymbol{P}(x'_1, x_1) = \boldsymbol{P}\left(x'_1, x''_1\right)\boldsymbol{P}\left(x''_1, x_1\right) \tag{6.57}$$

This leads to

$$\boldsymbol{P}(-h, 0) = \prod_{m=1}^{N} \boldsymbol{P}_m(x_{1m} - d_m, x_{1m}), \quad \boldsymbol{v}_N(-h) = \boldsymbol{P}(-h, 0)\, \boldsymbol{v}_0(0) \tag{6.58}$$

where $\boldsymbol{v}_N(-h)$ and $\boldsymbol{v}_1(0)$ are the state vectors at the upper and lower surfaces of the structure, respectively.

6.4.2 Love Wave in Zno/SiO$_2$/Si Structure with Initial Stress

Figure 6.4 shows a ZnO/SiO$_2$/Si multilayer structure. SiO$_2$ and Si are isotropic elastic materials, and ZnO is a transverse isotropic piezoelectric material with poling direction along x_3. In a transversely isotropic piezoelectric material, the number of material constants is only ten: $C_{11} = C_{22}, C_{12}, C_{13} = C_{23}, C_{33}, C_{44} = C_{55}$, $C_{66} = (C_{11} - C_{12})/2, e_{31} = e_{32}, e_{15} = e_{24}, e_{33}, \epsilon_{11} = \epsilon_{22}, \epsilon_{33}$ Let the biasing stresses be $\sigma_{33}^0(x_1)$ and $\sigma_{22}^0(x_1)$. Other stress components and the biasing potential φ^0 is assumed to be zero. Love wave is a transverse shear wave, so only $u_3(x_1, x_2, t)$ and $\varphi(x_1, x_2, t)$ are not zero. Let Love wave propagate along the positive direction of x_2, so only $k_2 = k$ is not zero. In this case Eqs. (6.50), (6.52), and (6.53) are simplified to

$$\boldsymbol{v}_m(x_3) = [A_{3m}, A_{4m}, T_{5m}, T_{7m}]^{\mathrm{T}}$$
$$\boldsymbol{B}_m = \begin{bmatrix} \mathrm{i}kC_{45}^* & \mathrm{i}ke_{14}^* & 1 & 0 \\ \mathrm{i}ke_{25}^* & -\mathrm{i}k\,\epsilon_{21}^* & 0 & 1 \\ C_{55}^* & e_{15}^* & 0 & 0 \\ e_{15}^* & -\epsilon_{11}^* & 0 & 0 \end{bmatrix}, \quad \boldsymbol{F}_m = \begin{bmatrix} -\rho_0\omega^2 + \left(C_{44}^* + \sigma_{22}^0\right)k^2 & e_{24}^*k^2 & 0 & 0 \\ e_{24}^*k^2 & -\epsilon_{22}^*k^2 & 0 & 0 \\ -\mathrm{i}kC_{54}^* & -\mathrm{i}ke_{25}^* & 1 & 0 \\ -\mathrm{i}ke_{14}^* & \mathrm{i}k\,\epsilon_{12}^* & 0 & 1 \end{bmatrix} \tag{6.59}$$

where effective material constants can be calculated from Eq. (6.42). For ZnO and SiO$_2$, $C_{45} = C_{54} = e_{14} = e_{25} = \epsilon_2 = \epsilon_1 = 0$, so for small initial stresses it yields

$$\boldsymbol{B}_m = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ C_{55}^* & e_{15}^* & 0 & 0 \\ e_{15}^* & -\epsilon_{11}^* & 0 & 0 \end{bmatrix}, \quad \boldsymbol{F}_m = \begin{bmatrix} -\rho_0\omega^2 + \left(C_{44}^* + \sigma_{22}^0\right)k^2 & e_{24}^*k^2 & 0 & 0 \\ e_{24}^*k^2 & -\epsilon_{22}^*k^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.60}$$

For the $6mm$-type ceramics $C_{44} = C_{55}, e_{15} = e_{24}, \epsilon_{11} = \epsilon_{22}$, so the differences between c_{44}^* and c_{55}^*, e_{15}^* and e_{24}^*, and ϵ_{11}^* and ϵ_{22}^* can be neglected. In this case the eigenvalues of $\boldsymbol{B}_m^{-1}(x_1)\boldsymbol{F}_m(x_1)$ are obtained as

$$b_{1m,2m} = \pm k, \quad b_{3m,4m} = \pm kq_m, \quad q_m = \sqrt{1 - \left[\left(\rho c^2 - \sigma_{22}^0\right)/\bar{C}_{55}\right]}$$
$$\bar{C}_{55} = C_{55}^* + \left(e_{15}^*\right)^2/\epsilon_{11}^*, \quad c = \omega/k \tag{6.61}$$

where c is the phase velocity. Correspondingly the eigenvector matrix $\boldsymbol{Q}_m$ is

$$\boldsymbol{Q}_m = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & e_{15}^*/\epsilon_{11}^* & e_{15}^*/\epsilon_{11}^* \\ e_{15}^*k & -e_{15}^*k & \bar{C}_{55}q_mk & -\bar{C}_{55}q_mk \\ -\epsilon_{11}^*k & \epsilon_{11}^*k & 0 & 0 \end{bmatrix} \tag{6.62}$$

Substitution of Eqs. (6.61) and (6.62) into Eq. (6.54) yields

$$\boldsymbol{v}_m(x_1)=\begin{bmatrix}0&0&1&1\\1&1&e_{15}^*/\epsilon_{11}^*&e_{15}^*/\epsilon_{11}^*\\e_{15}^*k&-e_{15}^*k&\bar{C}_{55}q_mk&-\bar{C}_{55}q_mk\\-\epsilon_{11}^*k&\epsilon_{11}^*k&0&0\end{bmatrix}\times\begin{bmatrix}e^{kx_{1m}}&0&0&0\\0&e^{-kx_{1m}}&0&0\\0&0&e^{kq_mx_{1m}}&0\\0&0&0&e^{-kq_mx_{1m}}\end{bmatrix}\begin{Bmatrix}a_{1m}\\a_{2m}\\a_{3m}\\a_{4m}\end{Bmatrix}\tag{6.63}$$

According to Eq. (6.56) the transfer matrix of the sublayer m is

$$\begin{aligned}&\boldsymbol{P}_m(x_{1m}-d_m,x_{1m})\\&=\begin{bmatrix}\cosh(kq_md_m)&0&-\dfrac{\sinh(kq_md_m)}{\bar{C}_{55}q_mk}&-\dfrac{e_{15}^*\sinh(kq_md_m)}{\bar{C}_{55}\epsilon_{11}^*q_mk}\\P(21)&\cosh(kd_m)&-\dfrac{e_{15}^*\sinh(kq_md_m)}{\bar{C}_{55}\epsilon_{11}^*q_mk}&P(24)\\P(31)&-e_{15}^*k\sinh(kd_m)&\cosh(kq_md_m)&P(34)\\-e_{15}^*k\sinh(kd_m)&\epsilon_{11}^*k\sinh(kd_m)&0&\cosh(kd_m)\end{bmatrix}\\&P(21)=\frac{e_{15}^*}{\epsilon_{11}^*}[\cosh(kq_md_m)-\cosh(kd_m)],\quad P(24)=\frac{\sinh(kd_m)}{\epsilon_{11}^*k}-\frac{e_{15}^{*2}\sinh(kq_md_m)}{\bar{C}_{55}\epsilon_{11}^{*2}q_mk}\\&P(31)=-\bar{C}_{55}q_mk\sinh(kq_md_m)+e_{15}^{*2}\sinh(kd_m)/\epsilon_{11}^*\\&P(34)=-\left(e_{15}^*/\epsilon_{11}^*\right)\cosh(kd_m)+\left(e_{15}^*/\epsilon_{11}^*\right)\cosh(kq_md_m)\end{aligned}\tag{6.64}$$

Because the Love wave is confined to layers and near the substrate surface, the generalized displacements are attenuated in the substrate. In the substrate we have

$$\boldsymbol{v}_0(x_1)=\boldsymbol{Q}_0\left[0,a_{20}e^{b_{20}x_1},0,a_{40}e^{b_{40}x_1}\right]^{\mathrm{T}}\tag{6.65}$$

$\boldsymbol{Q}_0$ can be obtained by substituting material constants of the substrate into Eq. (6.62). At $x_1=0$ we have

$$\boldsymbol{v}_0(x_1)=\boldsymbol{Q}_0[0,a_{20},0,a_{40}]^{\mathrm{T}}\tag{6.66}$$

The electric potential φ_{N+1} and the electric displacement $D_{1(N+1)}$ in the air $x_1<-h$ can be expressed as

$$\varphi_{N+1}(x_1,t)=a_{N+1}\exp(kx_1)\exp[\mathrm{i}(k_\alpha x_\alpha-\omega t)],\quad D_{1(N+1)}=-\epsilon_0\varphi_{N+1,1}\tag{6.67}$$

where ϵ_0 is the permittivity of air and a_{N+1} is an undetermined constant.

The mechanical boundary conditions are

$$\begin{aligned} &\sigma_{13} = 0, \quad \text{at} \quad x_1 = -h \\ &\sigma_{13}^{+} = \sigma_{13}^{-}, \quad u_3^{+} = u_3^{-}, \quad \text{at} \quad x_1 = 0 \end{aligned} \tag{6.68}$$

The electric boundary conditions between air and ZnO are divided into two kinds: electrically open and electrically shorted, i.e.,

$$\begin{aligned} &\varphi_N = \varphi_{N+1}, \quad D_{1\,(N)} = D_{1\,(N+1)}, \quad \text{at} \quad x_1 = -h \quad \text{(electrically open)} \\ &\varphi_N = 0, \quad \text{at} \quad x_1 = -h \quad \text{(electrically shorted)} \end{aligned} \tag{6.69}$$

The electric boundary conditions between the substrate and SiO$_2$ are

$$D_1^{+} = D_1^{-}, \quad \varphi^{+} = \varphi^{-}, \quad \text{at} \quad x_1 = 0 \tag{6.70}$$

The continuity condition at $x_1 = 0$ expressed by the state vector $\boldsymbol{v}(x_1)$ is

$$\boldsymbol{v}_0(x_1) = \boldsymbol{v}_1(x_1), \quad \text{at} \quad x_1 = 0 \tag{6.71}$$

From Eqs. (6.58), (6.66), and (6.71), the continuity condition of $\boldsymbol{v}(x_1)$ at $x_1 = -h$ is

$$\boldsymbol{v}_N(-h) = \boldsymbol{P}(-h, 0)\boldsymbol{v}_0(0) = \boldsymbol{M}[0, a_{20}, 0, a_{40}]^{\mathrm{T}}, \quad \boldsymbol{M} = \prod_{m=1}^{N} \boldsymbol{P}_m(x_{1m} - d_m, x_{1m})\boldsymbol{Q}_0 \tag{6.72}$$

According to the boundary conditions at $x_1 = -h$, $u_{3(N+1)}$ or u_{3air} is not needed, from the electrically open case we get

$$\begin{aligned} \begin{Bmatrix} A_{4N} \\ T_{5N} \\ T_{7N} \end{Bmatrix} &= \begin{bmatrix} M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} & M_{34} \\ M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{Bmatrix} a_{20} \\ 0 \\ a_{40} \end{Bmatrix} = \begin{bmatrix} M_{22} & M_{24} \\ M_{32} & M_{34} \\ M_{42} & M_{44} \end{bmatrix} \begin{Bmatrix} a_{20} \\ a_{40} \end{Bmatrix} \\ &= \begin{Bmatrix} a_{N+1}\exp(-kh) \\ 0 \\ -\epsilon_0 a_{N+1} k \exp(-kh) \end{Bmatrix} \end{aligned} \tag{6.73}$$

or

$$\begin{bmatrix} M_{22} & M_{24} & -\exp(-kh) \\ M_{32} & M_{34} & 0 \\ M_{42} & M_{44} & \epsilon_0 k \exp(-kh) \end{bmatrix} \begin{Bmatrix} a_{20} \\ a_{40} \\ a_{N+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.74}$$

In order to obtain nontrivial solutions for a_{20}, a_{40}, a_{N+1} the coefficient determinant in Eq. (6.74) should be vanished. So the equation to determine the phase velocity c_f in electrically open case is

$$(M_{42} + \epsilon_0 kM_{22})M_{34} - (M_{44} + \epsilon_0 kM_{24})M_{32} = 0 \tag{6.75}$$

Similarly for the electrically shorted case, the equation to determine the phase velocity c_s is

$$\begin{aligned} &\begin{Bmatrix} A_{4N} \\ T_{5N} \end{Bmatrix} = \begin{bmatrix} M_{22} & M_{24} \\ M_{32} & M_{34} \end{bmatrix} \begin{Bmatrix} a_{20} \\ a_{40} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \text{or} \\ &M_{22}M_{34} - M_{24}M_{32} = 0 \end{aligned} \tag{6.76}$$

There are many papers to discuss on the problem of Love wave, such as Danoyan and Pilliposian (2007) and Liu et al. (2001). Du et al. (2008) discussed the propagation of Love waves in prestressed piezoelectric layered structures loaded with viscous liquid.

6.4.3 The Distribution of the Initial Stresses

Because layers are very thin, the mechanical stresses have only occurred in layers, i.e., at $x_1 = -h_1, \bar{\sigma}^0_{i1} = 0;\ x_1 = 0, \bar{\sigma}^0_{ij} = 0$. The continuity conditions of the initial stresses at $x_1 = -h_1$ are that u^0_2 and u^0_3 are continuous, or

$$\bar{\varepsilon}^0_{2\mathrm{ZnO}} = \bar{\varepsilon}^0_{2\mathrm{SiO_2}} = \bar{\varepsilon}^0_2, \quad \bar{\varepsilon}^0_{3\mathrm{ZnO}} = \bar{\varepsilon}^0_{3\mathrm{SiO_2}} = \bar{\varepsilon}^0_3, \quad \bar{\varepsilon}^0_{23\mathrm{ZnO}} = \bar{\varepsilon}^0_{23\mathrm{SiO_2}} = \bar{\varepsilon}^0_{23}, \quad \text{at} \quad x_1 = -h_1 \tag{6.77}$$

Using the constitutive equation from Eq. (6.77), we can derive the following relation:

$$\begin{aligned} &\bar{\sigma}^0_{1\mathrm{ZnO}} = C_{11}\bar{\varepsilon}^0_{1\mathrm{ZnO}} + C_{12}\bar{\varepsilon}^0_{2\mathrm{ZnO}} + C_{13}\bar{\varepsilon}^0_{3\mathrm{ZnO}}, \quad \sigma^0_{2\mathrm{ZnO}} = C_{12}\bar{\varepsilon}^0_{1\mathrm{ZnO}} + C_{11}\bar{\varepsilon}^0_{2\mathrm{ZnO}} + C_{13}\bar{\varepsilon}^0_{3\mathrm{ZnO}} \\ &\bar{\sigma}^0_{3\mathrm{ZnO}} = C_{13}\bar{\varepsilon}^0_{1\mathrm{ZnO}} + C_{13}\bar{\varepsilon}^0_{2\mathrm{ZnO}} + C_{33}\bar{\varepsilon}^0_{3\mathrm{ZnO}}, \\ &Y\bar{\varepsilon}^0_{2\mathrm{SiO_2}} = \bar{\sigma}^0_{2\mathrm{SiO_2}} - \nu\bar{\sigma}^0_{3\mathrm{SiO_2}}, \quad Y\bar{\varepsilon}^0_{3\mathrm{SiO_2}} = \bar{\sigma}^0_{3\mathrm{SiO_2}} - \nu\bar{\sigma}^0_{2\mathrm{SiO_2}} \end{aligned} \tag{6.78}$$

where C_{ij} is the elastic constant of ZnO and Y, ν is the elastic constant of SiO_2. So at $x_1 = -h_1$, the initial stresses in ZnO and SiO_2 must satisfy the following relation:

$$\begin{aligned} YC_{11}\bar{\sigma}^0_{2\mathrm{ZnO}} = &\left[C^2_{11} - C^2_{12} - \nu C_{13}(C_{11} - C_{12})\right]\bar{\sigma}^0_{2\mathrm{SiO_2}} \\ &+ \left[C_{13}(C_{11} - C_{12}) - \nu\left(C^2_{11} - C^2_{12}\right)\right]\bar{\sigma}^0_{3\mathrm{SiO_2}} \end{aligned} \tag{6.79}$$

Similarly we can get $\bar{\sigma}^0_{3\mathrm{ZnO}}$, but it is not needed. It can be assumed that $\sigma^0_{2\mathrm{SiO_2}}, \sigma^0_{2\mathrm{ZnO}}$ are varied exponentially (Fig. 6.4), i.e.,

$$\begin{aligned} &\bar{\sigma}^0_{2\mathrm{SiO_2}}(x_1) = f(x_1) = (\mathrm{e}^{x_1} - 1)\big/\big(\mathrm{e}^{-h_1} - 1\big)\bar{\sigma}^0_{2\mathrm{SiO_2}}(-h), \quad -h_1 \le x_1 \le 0 \\ &\bar{\sigma}^0_{2\mathrm{ZnO}}(x_1) = \mathrm{g}(x_1) = (\mathrm{e}^{x_1} - 1)\big/\big(\mathrm{e}^{-h} - 1\big)\bar{\sigma}^0_{2\mathrm{ZnO}}(-h), \quad -h \le x_1 \le -h_1 \end{aligned} \tag{6.80}$$

It is also noted that the generalized displacements and stresses at the initial state should be obtained directly by experiments or calculated by the updated Lagrange method which needs multiple steps from the natural state to initial state. The updated Lagrange method and other methods in plasticity can be utilized to the problems discussed here.

6.4.4 Numerical Example

In the paper of Su et al. (2005), they assumed $\sigma^0_{3\mathrm{SiO_2}} = L\sigma^0_{2\mathrm{SiO_2}}$ to simplify calculation, where L is a proportional coefficient. They adopted the following material constants:

$$\begin{aligned} &\mathrm{ZnO}: \ \rho = 5{,}665\,\mathrm{kg/m^3}, \ C_{11} = 209.6, \ C_{12} = 120.5, \ C_{13} = 104.6, \ C_{44} = 242.3\,(\mathrm{MPa}); \\ &\quad e_{15} = -0.48, \ e_{31} = -0.573, \ e_{33} = 1.32\,(\mathrm{C/m^2}); \ \epsilon_{11} = 0.67, \ \epsilon_{33} = 0.799\,(10^{-10}\mathrm{F/m}) \\ &\mathrm{SiO_2}: \ \rho = 2{,}200\,\mathrm{kg/m^3}, \ \lambda = 78.5, \ G = 31.2\,(\mathrm{MPa}); \ \epsilon_{11} = 0.33, \ \epsilon_{33} = 0.33\,(10^{-10}\mathrm{F/m}) \\ &\mathrm{Si}: \ \rho = 2{,}328\,\mathrm{kg/m^3}, \ \lambda = 165.75, \ G = 79.4\,(\mathrm{MPa}); \ \epsilon_{11} = 1.035, \ \epsilon_{33} = 1.035\,(10^{-10}\mathrm{F/m}) \end{aligned}$$

Figure 6.6 shows the change of the phase velocity c_{f0} of the Love wave with kh under the case that: electrically open, without initial stresses, $h_2 = 10^{-5}$ m and different h_1. It is seen that for all h_1 when $kh \to 0$, $c_{\mathrm{f0}} \to c_{\mathrm{Si}}$; c_{f0} decreases with increasing kh. When $kh \to \infty$ $c_{\mathrm{f0}} \to c_{\mathrm{ZnO}}$ if $h_1 < h_2$ and $h_1 \sim h_2$; or $c_{\mathrm{f0}} \to c_{\mathrm{SiO_2}}$ if $h_1 \gg h_2$. c_{f0} is in the following range:

$$\begin{aligned} &(c_{\mathrm{ZnO}}, c_{\mathrm{SiO_2}}) < c_{\mathrm{f0}} < c_{\mathrm{Si}} \\ &c_{\mathrm{ZnO}} = \sqrt{\frac{C_{55} + e_{15}^2}{\rho_{\mathrm{ZnO}}}} = 2{,}841.5\,(\mathrm{m/s}), \quad c_{\mathrm{SiO_2}} = \sqrt{\frac{\mu_{\mathrm{SiO_2}}}{\rho_{\mathrm{SiO_2}}}} = 3{,}765.9(\mathrm{m/s}), \\ &c_{Si} = \sqrt{\frac{\mu_{Si}}{\rho_{Si}}} = 5{,}840\,(\mathrm{m/s}) \end{aligned} \tag{6.81}$$

Figure 6.7 shows the change of $\Delta c/c_{\mathrm{f0}}$ with kh under the case that: electrically open case, $\bar{\sigma}^0_{2\mathrm{ZnO}} = 200$ MPa, $L = 1, h_2 = 10^{-5}$ m and different h_1. Here $\Delta c = c_{\mathrm{f}} - c_{\mathrm{f0}}$ and c_{f} is the Love wave velocity with initial stress. From Figs. 6.6 and 6.7, it is seen that the middle layer has significant role.

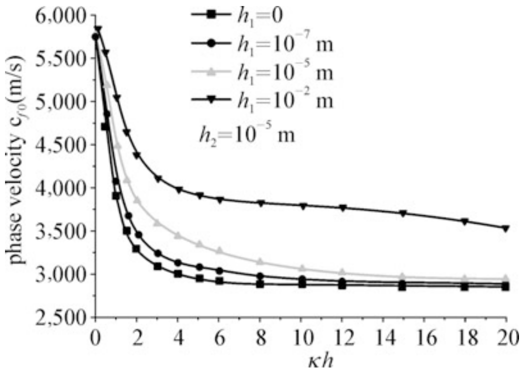


Fig. 6.6 Variation of phase velocity c_{f0} with kh under conditions: electrically open, without initial stress

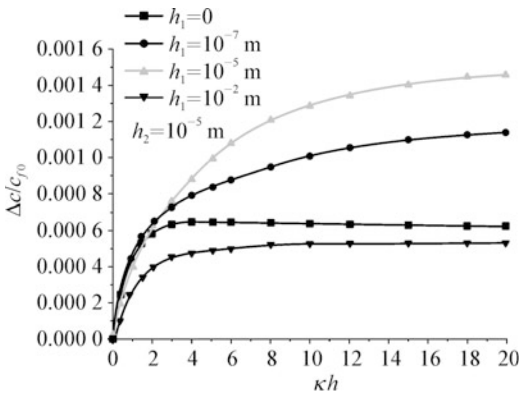


Fig. 6.7 Variation of $\Delta c/c_{f0}$ with kh under conditions: electrically open, $\sigma^0_{2\mathrm{ZnO}}(-h) = 200\,\mathrm{MPa}, \quad L = 1$

6.5 Other Surface Waves

6.5.1 B-G Wave in a Prestressed Piezoelectric Structure

Because the penetration depth of the B-G wave is about 10–100 λ, the application of B-G wave is limited in microwave techniques. However, the application of layered structures can significantly reduce the penetration depth. Jin et al. (2001) and Liu et al. (2003a) discussed the prestressed layered piezoelectric structures. The layer and substrate are all made by piezoelectric materials, and the poling directions of the layer and substrate are along the positive and negative axes x_3, respectively (see Fig. 6.2). The B-G wave in layered structure can also be considered as a kind of the Love wave. The basic equations have been discussed in Sect. 6.3 and governing equations can be seen in Eq. (6.43). In B-G wave only $u_3(x_1, x_2, t)$ and $\varphi(x_1, x_2, t)$ are not zero. Neglecting terms containing $u^0_{K,L}u^0_{M,N}$, Eq. (6.43) is reduced to

$$\begin{aligned}&\left(C^*_{1331}+\sigma^0_{11}\right)u_{3,11}+\left(C^*_{1332}+C^*_{2331}+2\sigma^0_{12}\right)u_{3,12}+\left(C^*_{2332}+\sigma^0_{22}\right)u_{3,22}\\&\quad+e^*_{131}\varphi_{,11}+\left(e^*_{132}+e^*_{231}\right)\varphi_{,12}+e^*_{232}\varphi_{,22}=\rho_0\ddot{u}_3\\&e^*_{131}u_{3,11}+\left(e^*_{132}+e^*_{231}\right)u_{3,12}+e^*_{232}u_{3,22}-\epsilon^*_{11}\varphi_{,11}-2\,\epsilon^*_{12}\varphi_{,12}-\epsilon^*_{22}\varphi_{,22}=0\end{aligned}\tag{6.82}$$

The variables in the substrate are denoted by a superscript "M." Because in the substrate $x_1>0$ there is no initial stress, the governing equations are

$$\begin{aligned}&C^{\mathrm{M}}_{1331}u^{\mathrm{M}}_{3,11}+2C^{\mathrm{M}}_{1332}u^{\mathrm{M}}_{3,12}+C^{\mathrm{M}}_{2332}u^{\mathrm{M}}_{3,22}+e^{\mathrm{M}}_{131}\varphi^{\mathrm{M}}_{,11}+2e^{\mathrm{M}}_{132}\varphi^{\mathrm{M}}_{,12}+e^{\mathrm{M}}_{232}\varphi^{\mathrm{M}}_{,22}=\rho_0u^{\mathrm{M}}_3\\&e^{\mathrm{M}}_{131}u^{\mathrm{M}}_{3,11}+2e^{\mathrm{M}}_{132}u^{\mathrm{M}}_{3,12}+e^{\mathrm{M}}_{232}u^{\mathrm{M}}_{3,22}-\epsilon^{\mathrm{M}}_{11}\varphi^{\mathrm{M}}_{,11}-2\,\epsilon^{\mathrm{M}}_{12}\varphi^{\mathrm{M}}_{,12}-\epsilon^{\mathrm{M}}_{22}\varphi^{\mathrm{M}}_{,22}=0\end{aligned}\tag{6.83}$$

The boundary and the interface continuity conditions are

$$\begin{aligned}&\bar{\sigma}^*_{13}+\sigma^0_{1k}u_{3,k}=0,\quad \text{at}\quad x_1=-h\\&\varphi=\varphi^c,\quad D_1=D^c_1,\ \text{(electrically open)};\quad \varphi=0,\ \text{(electrically shorted)}\quad \text{at}\quad x_1=-h\\&u_3=u^M_3,\quad \bar{\sigma}^*_{13}=\sigma^M_{13};\quad \varphi=\varphi^M,\quad \bar{D}_1=D^M_1,\quad \text{at}\quad x_1=0\\&u_3,\quad \varphi\to 0,\quad \text{when}\quad x_1\to+\infty;\quad \varphi^c\to 0,\quad \text{when}\quad x_1\to-\infty\end{aligned}\tag{6.84}$$

Let B-G wave propagate along positive x_2 direction. The generalized displacements in layer are assumed

$$u_3=\alpha_3\exp\left(\mathrm{i}kbx_1\right)\exp[\mathrm{i}k(x_2-ct)],\quad \varphi=\alpha_4\exp(\mathrm{i}kbx_1)\exp[\mathrm{i}k(x_2-ct)]\tag{6.85}$$

Substitution of Eq. (6.85) into Eq. (6.82) yields

$$\begin{bmatrix}\left(c^*_{1331}+\sigma^0_{11}\right)b^2+\left(c^*_{1332}+c^*_{2331}+2\sigma^0_{12}\right)b+\left(c^*_{2332}+\sigma^0_{22}\right)-\rho_0c^2 & e^*_{131}b^2+\left(e^*_{132}+e^*_{231}\right)b+e^*_{232}\\ e^*_{131}b^2+\left(e^*_{132}+e^*_{231}\right)b+e^*_{232} & -\epsilon^*_{11}b^2-2\epsilon^*_{12}b-\epsilon^*_{22}\end{bmatrix}\begin{Bmatrix}\alpha_3\\ \alpha_4\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}\tag{6.86}$$

In order for α_3,α_4 to have nontrivial solutions, their coefficient determinate must be zero, or

$$A_4b^4+A_3b^3+A_2b^2+A_1b+A_0=0\tag{6.87}$$

where A_i is determined by Eq. (6.86) and omitted here. Equation (6.87) has 4 eigenvalues $b_p(p=1,2,3,4)$ and a pair of eigenvectors α_3,α_4 corresponding to each b_p and

$$\beta_p = \frac{\alpha_4^{(p)}}{\alpha_3^{(p)}} = \frac{e_{131}^* b_p^2 + \left(e_{132}^* + e_{231}^*\right) b_p + e_{232}^*}{\epsilon_{11}^* b_p^2 + 2\epsilon_{12}^* b_p + \epsilon_{22}^*} \tag{6.88}$$

Substituting Eq. (6.88) into Eq. (6.85) yields the generalized displacements in layer

$$\begin{aligned} u_3 &= \sum_{p=1}^{4} \alpha_3^{(p)} \exp(\mathrm{i}kb_p x_1) \exp[\mathrm{i}k(x_2 - ct)] \\ \varphi &= \sum_{p=1}^{4} \beta_p \alpha_3^{(p)} \exp(\mathrm{i}kb_p x_1) \exp[\mathrm{i}k(x_2 - ct)] \end{aligned} \tag{6.89}$$

Similar to the layer and noting $u_3, \varphi \to 0$, when $x_1 \to \infty$, the generalized displacements in the substrate only have two eigenvalues $b_q^{\mathrm{M}}(q = 1, 2)$ with positive image parts, i.e.,

$$\begin{aligned} u_3^M &= \sum_{p=1}^{2} \alpha_3^{M(q)} \exp\left(\mathrm{i}kb_q^M x_1\right) \exp[\mathrm{i}k(x_2 - ct)] \\ \varphi^M &= \sum_{p=1}^{2} \beta_q^M \alpha_3^{M(q)} \exp\left(\mathrm{i}kb_q^M x_1\right) \exp[\mathrm{i}k(x_2 - ct)] \end{aligned} \tag{6.90}$$

From $\varphi^c_{,11} + \varphi^c_{,22} = 0$ and the connective conditions at $x_1 = -h$ in Eq. (6.84), it is assumed

$$\varphi^c = \sum_{p=1}^{4} \beta_p \alpha_3^{(p)} \exp(\mathrm{i}kb_p x_1) \exp[k(x_1 + h)] \exp[\mathrm{i}k(x_2 - ct)] \tag{6.91}$$

Substituting Eqs. (6.89), (6.90), and (6.91) into Eq. (6.84) yields six homogeneous equations for vector $\boldsymbol{\alpha}$:

$$\boldsymbol{P\alpha} = \mathbf{0}, \quad [P_{ij}]\{\alpha_j\} = \{0\}; \quad \boldsymbol{\alpha} = \left[\alpha_3^{(1)}, \alpha_3^{(2)}, \alpha_3^{(3)}, \alpha_3^{(4)}, \alpha_3^{M(1)}, \alpha_3^{M((1))}\right]^{\mathrm{T}} \tag{6.92}$$

In the electrically open case at $x_1 = -h$ for $j = 1, 2, 3, 4$, we have

$$\begin{aligned} &P_{1j} = \mathrm{i}k\left[\left(c_{1331}^* + \sigma_{11}^0 + e_{131}^* \beta_j\right) b_j + c_{1332}^* + \sigma_{12}^0 + e_{132}^* \beta_j\right] \exp(-\mathrm{i}kb_j h) \\ &P_{2j} = \mathrm{i}k\left[\left(e_{131}^* - \epsilon_{11}^* \beta_j\right) b_j + e_{132}^* - \left(\epsilon_{12}^* + i\,\epsilon^c\right) \beta_j\right] \exp(-\mathrm{i}kb_j h) \\ &P_{3j} = \mathrm{i}k\left[\left(c_{1331}^* + \sigma_{11}^0 + e_{131}^* \beta_j\right) b_j + c_{1332}^* + \sigma_{12}^0 + e_{132}^* \beta_j\right] \\ &P_{4j} = 1, \quad P_{5j} = \beta_j, \quad P_{6j} = \mathrm{i}k\left[\left(e_{131}^* - \epsilon_{11}^* \beta_j\right) b_j + e_{132}^* - \epsilon_{12}^* \beta_j\right] \end{aligned} \tag{6.93a}$$

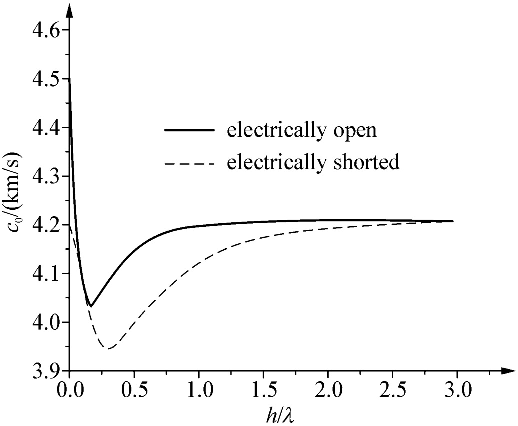


Fig. 6.8 Dispersion relations for the fundamental mode of B-G wave (without initial stress)

For $j = 5, 6$ we have

$$P_{1j} = 0, \quad P_{2j} = 0, \quad P_{3j} = -\mathrm{i}k\left[\left(\hat{c}^{M*}_{1331} + e^{M*}_{131}\beta^M_{j-4}\right)b^M_{j-4} + \hat{c}^{M*}_{1332} + e^*_{132}\beta^M_{j-4}\right]$$
$$P_{4j} = -1, \quad P_{5j} = -\beta^M_{j-4}, \quad P_{6j} = -\mathrm{i}k\left[\left(e^{M*}_{131} - \epsilon^{M*}_{11}\beta^M_{j-4}\right)b^M_{j-4} + e^{M*}_{132} - \epsilon^{M*}_{12}\beta^M_{j-4}\right] \tag{6.93b}$$

The equation to determine the phase velocity of B-G wave under electrically open case is

$$|\boldsymbol{P}| = 0 \tag{6.94}$$

For the electrically shorted case at $x_1 = -h, P_{2j}$ in Eq. (6.93a) should be replaced by

$$P_{2j} = \beta_j \exp\left(-\mathrm{i}kb_jh\right), \quad \text{for} \quad j = 1 - 4 \tag{6.95}$$

In the paper of Liu et al. (2003a), they found that the penetration depth is dramatically reduced in a $LiNbO_3$ layer piezoelectric material and the effect of the third-order piezoelectric coefficients (Cho and Yamanouchi 1987) is meaningful for the low-frequency case. Figure 6.8 shows the dispersion relations for the fundamental mode of B-G surface wave in the absence of initial stress, where $c_0 = c_{\mathrm{f}0}$ for electrically open case and $c_0 = c_{\mathrm{s}0}$ for electrically shorted case. It is found that for a given value of h/λ, the phase velocity of the electrically open case is greater than that of electrically shorted case, i.e., $c_{\mathrm{f}0} > c_{\mathrm{s}0}$. In the low-frequency limit, $h/\lambda \to 0$, the wave mode tends to the B-G surface wave in a piezoelectric half-space, i.e., $c_{\mathrm{f}0} \to 4.538\,\mathrm{km/s}, c_{\mathrm{s}0} \to 4.203\,\mathrm{km/s}$. Figure 6.9 shows the variations of $\Delta c_{\mathrm{f}}/c_{\mathrm{f}0}$ and $\Delta c_{\mathrm{s}}/c_{\mathrm{s}0}$ with h/λ under the initial stress $\bar{\sigma}^0_{22} = 40\,\mathrm{MPa}$, where $c_{\mathrm{f}}, c_{\mathrm{s}}$ are the phase velocity in layer with initial stress and $\Delta c_{\mathrm{f}} = c_{\mathrm{f}} - c_{\mathrm{f}0}$ or $\Delta c_{\mathrm{s}} = c_{\mathrm{s}} - c_{\mathrm{s}0}$.

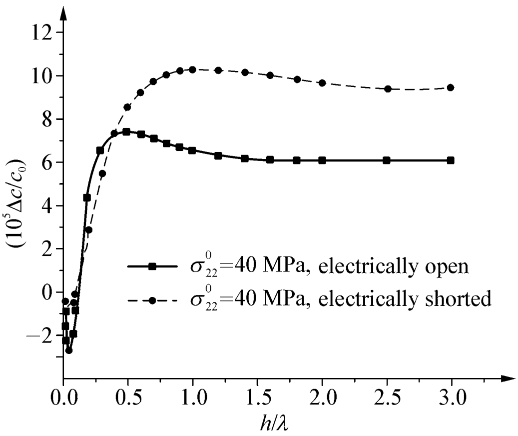


Fig. 6.9 Variations of $\Delta c/c_0$ with h/λ for different initial stresses

6.5.2 Rayleigh Wave in a Prestressed Structure

Discuss an approximately transversely isotropic $LiNbO_3$ piezoelectric film of thickness h polarized x_1-axis deposited on a sapphire substrate (see Fig. 6.2). Usually the thickness of the layer is some micrometers, so the substrate can be considered as semi-infinite. The basic equation Eq. (6.43) in the layer is reduced to

$$C^*_{KIlJ} u_{I,JK} + \bar{\sigma}^0_{KJ} u_{l,JK} + e^*_{NKl} \varphi_{,NK} = \rho_0 \ddot{u}_l, \quad e^*_{KIJ} u_{I,JK} - \epsilon^*_{KN} \varphi_{,NK} = 0 \tag{6.96}$$

In the following the superscript * on material coefficients will be omitted. In the substrate the basic equation is

$$C^{\mathrm{M}}_{ijkl} u^{\mathrm{M}}_{k,li} + e^{\mathrm{M}}_{kij} \varphi^{\mathrm{M}}_{,ki} = \rho \ddot{u}^{\mathrm{M}}_j, \quad e^{\mathrm{M}}_{ikl} u^{\mathrm{M}}_{k,li} - \epsilon^{\mathrm{M}}_{ik} \varphi^{\mathrm{M}}_{,ik} = 0 \tag{6.97}$$

The boundary and the interface continuity conditions are

$$\begin{aligned}
&\bar{\sigma}^*_{1j} + \bar{\sigma}^0_{1k} u_{j,k} = 0, \quad x_1 = -h \\
&\varphi = \varphi^c, \quad D_1 = D^c_1, \quad \text{(electrically open)}; \quad \varphi = 0, \quad \text{(electrically shorted)} \quad \text{at} \quad x_1 = -h \\
&\bar{\sigma}^*_{1j} + \bar{\sigma}^0_{1k} u_{j,k} = \sigma^{\mathrm{M}}_{1j}, \quad u_j = u^{\mathrm{M}}_j, \quad \varphi = \varphi^{\mathrm{M}}, \quad \bar{D}_1 = D^{\mathrm{M}}_1; \quad \text{at} \quad x_1 = 0 \\
&u_j, \varphi \to 0, \quad \text{when} \quad x_1 \to +\infty; \quad \varphi^c \to 0, \quad \text{when} \quad x_1 \to -\infty
\end{aligned} \tag{6.98}$$

Let the wave propagate along x_2 and take the form

$$u_i = B_i e^{ikbx_1} e^{ik(x_2 - ct)}, \quad \varphi = B_4 e^{ikbx_1} e^{ik(x_2 - ct)} \tag{6.99}$$

where B_i is the amplitude of ith component. Substitution of Eq. (6.99) into Eq. (6.96) yields

$$\begin{aligned}
&\boldsymbol{\Gamma B} = \boldsymbol{0}, \quad \text{or} \quad \left[\Gamma_{\alpha\beta}\right]\{B_\alpha\} = \{0\}; \quad \boldsymbol{B} = [B_1, B_2, B_3, B_4]^{\mathrm{T}}, \quad \alpha, \beta = 1-4 \\
&\Gamma_{jk} = C_{1jk1} + b(C_{3jk1} + C_{1jk3}) + b^2 C_{3jk3} + \delta_{jk}\left(\sigma_{11}^0 + 2b\sigma_{13}^0 + b^2\sigma_{33}^0 - \rho c^2\right) \\
&\Gamma_{j4} = e_{11j} + b(e_{13j} + e_{31j}) + b^2 e_{33j}, \quad \Gamma_{4j} = e_{1j1} + b(e_{1j3} + e_{3j1}) + b^2 e_{3j3} \\
&\Gamma_{44} = -(\epsilon_{11} + 2b\epsilon_{13} + b^2\epsilon_{33}); \quad i, j = 1, 2, 3
\end{aligned} \tag{6.100}$$

In order to get nontrivial solution of $\boldsymbol{B}$, its coefficient determinate need be zero, i.e.,

$$A_8 b^8 + A_7 b^7 + A_6 b^6 + A_5 b^5 + A_4 b^4 + A_3 b^3 + A_2 b^2 + A_1 b^1 + A_0 = 0 \tag{6.101}$$

where A_i is determined by Eq. (6.100). Equation (6.101) is the eighth-order equation of b with the Rayleigh wave velocity c as a parameter. From Eq. (6.101) eight b_q can be obtained and for each b_q an eigenvector with four components can be obtained. For convenience we let

$$B_{iq} = \beta_{iq} B_{1q}, \quad \beta_{1q} = 1, \quad i = 1-4, \quad q = 1-8 \tag{6.102}$$

After B_{iq} is obtained the generalized displacements in layer can be expressed as

$$u_i = \sum_{q=1}^{8} \beta_{iq} B_{1q} \mathrm{e}^{\mathrm{i}kb_q x_1} \mathrm{e}^{\mathrm{i}k(x_2 - ct)}, \quad \varphi = \sum_{q=1}^{8} \beta_{4q} B_{1q} \mathrm{e}^{\mathrm{i}kb_q x_1} \mathrm{e}^{\mathrm{i}k(x_2 - ct)} \tag{6.103}$$

Analogously the generalized displacements in substrate can be expressed as

$$u_i^{\mathrm{M}} = \sum_{q=1}^{4} \beta_{iq}^{\mathrm{M}} \beta_{1q}^{\mathrm{M}} \mathrm{e}^{\mathrm{i}kb_q^{\mathrm{M}} x_1} \mathrm{e}^{\mathrm{i}k(x_2 - ct)}, \quad \varphi^{\mathrm{M}} = \sum_{q=1}^{4} \beta_{4q}^{\mathrm{M}} \beta_{1q}^{\mathrm{M}} \mathrm{e}^{\mathrm{i}kb_q^{\mathrm{M}} x_1} \mathrm{e}^{\mathrm{i}k(x_2 - ct)} \tag{6.104}$$

Analogous to Eq. (6.91) for the electrically open case, the electric potential in the air is

$$\varphi^c = \sum_{q=1}^{8} \beta_{4q} B_{1q} \mathrm{e}^{-\mathrm{i}kb_q h} \mathrm{e}^{k(h + x_1)} \mathrm{e}^{\mathrm{i}k(x_2 - ct)} \tag{6.105}$$

Substitution of Eqs. (6.103), (6.104), and (6.105) into Eq. (6.98) yields

$$\boldsymbol{PB} = \boldsymbol{0}, \quad \boldsymbol{P} = [P_{mn}], \quad \boldsymbol{B} = \left\{B_{1q}, B_{1q}^M\right\}^{\mathrm{T}}; \quad m, n = 1-12 \tag{6.106}$$

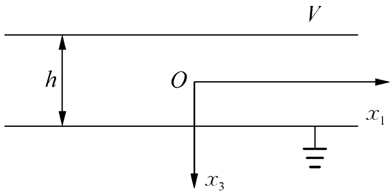


Fig. 6.10 Lamb wave in piezoelectric plate with biasing electric field

where for $j,k = 1,2,3;\ n = 1-8$ we have

$$\begin{aligned} &P_{jn} = \mathrm{i}k\left\{\left[C_{3jk1} + C_{3jk3}b_n + \delta_{jk}\left(\sigma^0_{13} + \sigma^0_{33}b_n\right)\right]\beta_{kn} + (e_{13j} + e_{33j}b_n)\beta_{4n}\right\}\mathrm{e}^{-\mathrm{i}kb_nh} \\ &P_{4n} = \mathrm{i}k\{(e_{3k1} + e_{3k3}b_n)\beta_{kn} - (\epsilon_{31} + \epsilon_{33}b_n + \mathrm{i}\epsilon_0)\beta_{4n}\}\mathrm{e}^{-\mathrm{i}kb_nh} \\ &P_{j+4,n} = \mathrm{i}k\left\{\left[C_{3jk1} + C_{3jk3}b_n + \delta_{jk}\left(\sigma^0_{13} + \sigma^0_{33}b_n\right)\right]\beta_{kn} + \left(e_{13j} + e_{33j}b_n\right)\beta_{4n}\right\} \\ &P_{k+7,n} = \beta_{kn}, \quad P_{11,n} = \beta_{4n}, \quad P_{12,n} = \mathrm{i}k\{(e_{3k1} + e_{3k3}b_n)\beta_{kn} - (\epsilon_{31} + \epsilon_{33}b_n)\beta_{4n}\} \end{aligned} \tag{6.107a}$$

and for $j,k = 1,2,3;\ n = 9-12$ we have

$$\begin{aligned} &P_{jn} = 0, \quad P_{4n} = 0, \quad P_{j+4,n} = -\mathrm{i}k\left\{\left[C^M_{3jk1} + C^M_{3jk3}b^M_n\right]\beta^M_{kn} + (e^M_{13j} + e^M_{33j}b^M_n)\beta^M_{4n}\right\} \\ &P_{k+7,n} = -\beta^M_{k,n-8}, \quad P_{11,n} = -\beta^M_{4,n-8}, \\ &P_{12,n} = -\mathrm{i}k\left\{\left(e^M_{3k1} + e^M_{3k3}b^M_{n-8}\right)\beta^M_{kn} - \left(\epsilon^M_{31} + \epsilon^M_{33}b^M_{n-8}\right)b^M_{4,n-8}\right\} \end{aligned} \tag{6.107b}$$

The phase velocity c of the Rayleigh wave $\bar{R}_3$ should satisfy Eqs. (6.100) and (6.106) simultaneously. From this condition c can be obtained by iteration method.

For the electrically shorted case, the fourth row in $[P_{mn}]$ should be changed to

$$P_{4n} = \beta_{4n}\mathrm{e}^{-\mathrm{i}kb_nh}, \quad n = 1-8; \quad P_{4n} = 0, \quad n = 9-12 \tag{6.108}$$

Liu et al. (2003d) discussed the phase velocity and the electromechanical coupling coefficient, $k^2 = 2(c_\mathrm{f} - c_\mathrm{s})/c_\mathrm{f}$. The results show that for a given value of h/λ, the phase velocity of the electrically open case is greater than that of the shorted case. The phase velocity approaches the Rayleigh wave velocity of the substrate when $h/\lambda \to 0$ and tends to the Rayleigh wave velocity of the layer for large $h/\lambda > 1.5$.

Babich and Lukyanov (1998) discussed the surface wave in a curved layered structure. Liu et al. (2003c, d) discussed the Love wave in a layered structure with functionally graded isotropic substrate.

6.5.3 Lamb Waves in Piezoelectric Plate with Biasing Electric Field

Lamb waves were researched a long time (Joshi and Jin 1991), it shows a large sensitivity to mass loading, and the zero-order antisymmetric mode can be applied in contact with a liquid with a small attenuation (Laurent et al. 2000). Figure 6.10 shows a thin infinite transversely isotropic piezoelectric plate of thickness h polarized x_3-axis. Liu et al. (2002a, b) assumed that a small biasing voltage V is applied to the electrode deposited on the upper surface and the electrode on the lower surface is grounded. The deference between the natural and initial configurations is neglected in their analysis. The electric field $\boldsymbol{E}^0 = (V/h)\mathbf{i}_3$, $\mathbf{i}_3$ is the unit vector on the axis x_3. According to the external loading, the generalized stresses can be assumed as constants. In this section the Voigt notations are used. The static electric force acted on the upper and lower surfaces of the piezoelectric material produced by the electric charges on the electrodes is neglected. The boundary conditions are assumed

$$\begin{aligned}&\boldsymbol{T}^0 = \mathbf{0}, \quad \text{on} \quad x_3 = \pm h/2; \quad \varepsilon^0_{1j} = 0, \quad x_1 = \pm\infty; \quad \varepsilon^0_{2j} = 0, \quad x_2 = \pm\infty \\ &\varphi^0 = V, \quad \text{on} \quad x_3 = -h/2; \quad \varphi^0 = 0, \quad \text{on} \quad x_3 = h/2, \quad x_1 = \pm\infty, \quad x_2 = \pm\infty\end{aligned} \tag{6.109}$$

A transversely isotropic material polarized x_3-axis only has ten independent constants. From Eq. (6.109) and the constitutive equation (3.2), it is obtained

$$\begin{aligned}&\sigma^0_1 = \eta_\sigma E^0_3, \quad \sigma^0_2 = \eta'_\sigma E^0_3, \quad D^0_3 = \eta_D E^0_3; \quad E^0_3 = (V/h); \quad \text{other} \quad \sigma^0_i = D^0_i = 0 \\ &\eta_\sigma = C_{13}e_{33}/C_{33} - e_{31}, \quad \eta'_\sigma = C_{23}e_{33}/C_{33} - e_{23}, \quad \eta_D = e^2_{33}/C_{33} + \epsilon_{33}\end{aligned} \tag{6.110}$$

According to Eq. (6.41) for the small perturbation in a 2D problem, we have

$$\begin{aligned}&\sigma_{1,1} + \sigma_{5,3} + \sigma^0_1 u_{1,11} = \rho_0 \ddot{u}_1, \quad \sigma_{5,1} + \sigma_{3,3} + \sigma^0_1 u_{3,11} = \rho_0 \ddot{u}_3 \\ &D_{1,1} + D_{3,3} = 0\end{aligned} \tag{6.111}$$

And the constitutive equation

$$\begin{aligned}&\sigma_1 = C_{11}\varepsilon_1 + C_{13}\varepsilon_3 - e_{31}E_3, \quad \sigma_3 = C_{13}\varepsilon_1 + C_{33}\varepsilon_3 - e_{33}E_3 \\ &\sigma_5 = C_{44}\varepsilon_5 - e_{15}E_1, \quad D_1 = e_{15}\varepsilon_5 + \epsilon_{11}E_1, \quad D_3 = e_{31}\varepsilon_1 + e_{33}\varepsilon_3 + \epsilon_{33}E_3\end{aligned} \tag{6.112}$$

Substitution of Eqs. (6.110) and (6.112) into Eq. (6.111) yields

$$\begin{aligned}&\left(C_{11} + \eta_\sigma E^0_3\right)u_{1,11} + C_{44}u_{1,33} + (C_{13} + C_{44})u_{3,13} + (e_{31} + e_{15})\varphi_{,13} = \rho_0 \ddot{u}_1 \\ &(C_{13} + C_{44})u_{1,13} + \left(C_{44} + \eta_\sigma E^0_3\right)u_{3,11} + C_{33}u_{3,33} + e_{15}\varphi_{,11} + e_{33}\varphi_{,33} = \rho_0 \ddot{u}_3 \\ &(e_{31} + e_{15} + \eta_D E^0_3)u_{1,13} + e_{15}u_{3,11} + (e_{33} + \eta_D E^0_3)u_{3,33} - \epsilon_{11}\varphi_{,11} - \epsilon_{33}\varphi_{,33} = 0\end{aligned} \tag{6.113}$$

It is assumed that the solutions of the antisymmetric Lamb waves are (Liu et al. 2002a)

$$\begin{aligned}&u_1 = B_1 \sin(kbx_3)\exp[\mathrm{i}k(x_1 - ct)], \quad u_3 = B_2 \cos(kbx_3)\exp[\mathrm{i}k(x_1 - ct)] \\ &\varphi = B_3 \cos(kbx_3)\exp[\mathrm{i}k(x_1 - ct)]\end{aligned} \tag{6.114}$$

and assumed that the solutions of the symmetric Lamb waves are (Liu et al. 2002b)

$$\begin{aligned} &u_1 = B_1 \cos(kbx_3) \exp[ik(x_1 - ct)], \quad u_3 = B_2 \sin(kbx_3) \exp[ik(x_1 - ct)] \\ &\varphi = B_3 \sin(kbx_3) \exp[ik(x_1 - ct)] \end{aligned} \tag{6.115}$$

where B_i is the undetermined constant. Substitution of Eqs. (6.114) and (6.115) into Eq. (6.113) yields

$$\begin{aligned} &\pm \left[C_{11} - \rho_0 c^2 + C_{44} b^2 + \eta_\sigma E_3^0\right] B_1 + (C_{13} + C_{44}) ibB_2 + (e_{31} + e_{15}) ibB_3 = 0 \\ &(C_{13} + C_{44}) ibB_1 \mp \left[C_{44} - \rho_0 c^2 + C_{33} b^2 + \eta_\sigma E_3^0\right] B_2 \mp (e_{15} + e_{33} b^2) B_3 = 0 \\ &(e_{31} + e_{15} + \eta_D E_3^0) ibB_1 \mp \left[e_{15} + (e_{33} + \eta_D E_3^0) b^2\right] B_2 \pm (\epsilon_{11} + \epsilon_{33} b^2) B_3 = 0 \end{aligned} \tag{6.116}$$

where the upper and lower symbols in "$\pm$" and "$\mp$" are used for the antisymmetric and symmetric solutions, respectively. In order to get nontrivial solutions of B_1, B_2, B_3, their coefficient determinant must be zero. So we get a third-order equation of b^2 containing phase velocity c as an unknown parameter:

$$A_1(c) b^6 + A_2(c) b^4 + A_3(c) b^2 + A_4(c) = 0 \tag{6.117}$$

Solving Eq. (6.117) we get three solutions of b^2 and select appreciate one $b_l (l = 1, 2, 3)$ in b^2. Substituting $b_l (l = 1, 2, 3)$ into Eq. (6.114) or (6.115) yields the amplitude ratios B_{1l}/B_{3l}, $B_{2l}/B_{3l}, l = 1, 2, 3$. Substituting $b_l, B_{1l}/B_{3l}, B_{2l}/B_{3l}$ $(l = 1, 2, 3)$ back into Eq. (6.114) or (6.115) and then into boundary conditions, we can finally get three homogeneous equations of B_{31}, B_{32}, B_{33}. Let the determinant of the coefficients of B_{31}, B_{32}, B_{33} equal to zero; the equation to determine c is obtained. The details can be seen in the original papers.

Sharma and Pal (2004) also discussed propagation of Lamb waves in a transversely isotropic piezo-thermo-elastic plate. Li et al. (2005b) discussed the spatial dispersion of short surface acoustic waves in piezoelectric ceramics.

6.6 Waves in Pyroelectrics

6.6.1 Generalized Thermodynamics of Temperature Wave in Thermoelasticity

The infinite wave speed problem (Banerjee and Bao 1974) and the Landau second sound speed in liquid helium and in some solids at low temperatures (Landau 1941; Jackson and Walker 1971) induced the development of the generalized heat, thermoelastic, and thermo-piezoelectric wave theories. The temperature wave from heat pulses at low temperature propagates with a finite phase velocity. The main simpler generalized theories with a finite velocity are Kaliski (1965)-Lord-Shulman (K-L-S) theory (1967), Green-Lindsay (G-L) theory (1972), and inertial entropy theory (Kuang 2009). The temperature wave equation can also be established on the extended irreversible thermodynamics and can be found in Joseph and Preziosi's papers (1989, 1990). In the K-L-S theory for an isotropic thermoelastic material, the following Cattaneo-Vernotte heat conduction formula (Vernotte 1958; Cattaneo 1958) was used to replace the Fourier's law, but the classical entropy equation and the Helmholtz free energy are kept, i.e.,

$$\begin{aligned} & q_i + \tau_0 \dot{q}_i = -\lambda \vartheta_{,i}, \quad T\dot{s} = \dot{r} - q_{i,i}, \quad g(\varepsilon_{kl}, \vartheta) = \mathfrak{A}(\varepsilon_{kl}, s) - \vartheta s \\ & \sigma_{ij} = \partial g / \partial \varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - \alpha_{ij}\vartheta, \quad s = -\partial g/\partial \vartheta = \alpha_{ij}\varepsilon_{ij} + C\vartheta/T_0 \end{aligned} \tag{6.118}$$

where τ_0 represents the relaxation time and is a material parameter. From Eq. (6.118) we find

$$\begin{aligned} & q_{i,i} = -T\dot{s} = T\left[(\partial^2 g/\partial \vartheta^2)\dot{\vartheta} + (\partial^2 g/\partial \vartheta \partial \varepsilon_{ij})\dot{\varepsilon}_{ij}\right] \\ & \lambda \vartheta_{,ii} = T\left[(\partial^2 g/\partial \vartheta^2)(\dot{\vartheta} + \tau_0 \ddot{\vartheta}) + (\partial^2 g/\partial \vartheta \partial \varepsilon_{ij})(\dot{\varepsilon}_{ij} + \tau_0 \ddot{\varepsilon}_{ij})\right] \end{aligned}$$

Then neglecting many small terms, finally, they got

$$\begin{aligned} & \lambda \vartheta_{,ii} = C(\dot{\vartheta} + \tau_0 \ddot{\vartheta}) + \alpha T_0 (\dot{\varepsilon}_{kk} + \tau_0 \ddot{\varepsilon}_{kk}) \\ & [G/(1-2\nu)]u_{j,ij} + Gu_{i,jj} - [2G(1+\nu)/(1-2\nu)]\alpha \vartheta_{,i} = \rho \ddot{u}_i \end{aligned} \tag{6.119}$$

where $\alpha_{ij} = \alpha \delta_{ij}$. The second equation in Eq. (6.119) is the momentum equation.

The G-L theory (1972) was based on modifying the Clausius-Duhemin inequality and the energy equation; They used a new temperature function $\phi(T, \dot{T})$ instead of the usual temperature T. i.e.,

$$\begin{aligned} & \int_V \dot{s}\,\mathrm{d}V - \int_V (r/\phi)\,\mathrm{d}V + \int_a (q_i/\phi) n_i\,\mathrm{d}a \geq 0, \quad \phi = \phi(T, \dot{T}), \quad T = \phi(T, 0) \\ & g = \mathfrak{A} - \phi s, \quad g = g(T, \dot{T}, \varepsilon_{ij}) \end{aligned} \tag{6.120}$$

Substituting Eq. (6.120) into the momentum and energy equations, after complex manipulation and linearization and neglecting small terms finally get (here we take the form in small deformation for an isotropic material)

$$\begin{aligned} & \lambda T_{,ii} = C(\dot{T} + \tau_0 \ddot{T}) + \gamma T_0 \dot{\varepsilon}_{jj}, \quad \sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i \\ & \sigma_{ij} = [2G\nu/(1-2\nu)]\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - \gamma(\theta + \tau_1 \dot{\theta}) \end{aligned} \tag{6.121}$$

where τ_0, τ_1, and γ are material constants.

The derivation of the governing equations is very complex when using K-L-S or G-L theory, but the derivation is very simple when the inertial entropy theory is used as shown at the next section.

6.6.2 The Inertial Entropy Theory of Temperature Wave

In Sect. 1.7.2 the inertial entropy theory (Kuang 2009) is introduced. Equation (1.84) gives

$$T\dot{s}+T\dot{s}^{(\mathrm{a})}=\dot{r}-q_{i,i};\quad \dot{s}^{(\mathrm{a})}=\rho_{\mathrm{s}}\ddot{T},\quad \rho_{\mathrm{s}}=\rho_{\mathrm{s}0}C/T \tag{6.122}$$

The Fourier's law given in Eqs. (1.71) or (5.107) is

$$q_i=-\lambda_{ij}T_{,j},\quad T_{,j}=\vartheta_{,j}=-\lambda_{ji}^{-1}q_i;\quad -q_{i,i}=T\dot{s}-\dot{r} \tag{6.123}$$

The constitutive (or state) and evolution equations given in (5.106) are

$$\begin{aligned}&\sigma_{ij}=C_{ijkl}\varepsilon_{kl}-e_{kij}E_k-\alpha_{ij}\vartheta,\quad D_i=\epsilon_{ij}E_j+e_{ikl}\varepsilon_{kl}+\tau_i\vartheta\\&s=\alpha_{ij}\varepsilon_{ij}+\tau_iE_i+C\vartheta/T_0,\quad \vartheta=T-T_0\end{aligned} \tag{6.124}$$

where α_{ij} is the stress-temperature coefficient. Equations (6.122), (6.123), and (6.124) yield

$$(\alpha_{ij}\varepsilon_{ij}+\tau_iE_i+C\vartheta/T_0)^{\cdot}+\rho_{\mathrm{s}0}(C/T)\ddot{\vartheta}=\dot{r}/T+(\lambda_{ij}T_{,j})_{,i}/T \tag{6.125}$$

When material coefficients are all constants and the variation of temperature is not large from (6.125) we have

$$(C/T_0)(\rho_{\mathrm{s}0}\ddot{\vartheta}+\dot{\vartheta})-\lambda_{ij}\vartheta_{,ji}/T_0=\dot{r}/T_0-\alpha_{ij}\dot{u}_{i,j}+\tau_i\dot{\varphi}_{,i} \tag{6.126}$$

Equation (6.126) is a temperature wave equation with finite phase velocity.

The generalized momentum equations are

$$\begin{aligned}&\sigma_{ij,j}+f_i=\rho\ddot{u}_i,\quad \text{or}\quad \rho\ddot{u}_i=C_{ijkl}u_{k,lj}+e_{kij}\varphi_{,kj}-\alpha_{ij}\vartheta_{,j}+f_i\\&D_{i,i}=\rho_{\mathrm{e}},\quad \text{or}\quad e_{ikj}u_{k,ji}-\epsilon_{ij}\varphi_{,ji}+\tau_i\vartheta_{,i}=\rho_{\mathrm{e}}\end{aligned} \tag{6.127}$$

It is obvious that the experimental studies for the inertial entropy theories are very important.

6.6.3 The Homogeneous Thermo-electro-elastic Wave in Pyroelectric Material

Under the quasi-static electric approximation, the governing equations of the waves in pyroelectric material in the inertial entropy theory are shown in Eqs. (6.126) and (6.125). Like Eq. (6.119), using Eq. (6.118) in a piezoelectric material, the extended K-L-S equation can also be obtained. When $f_j^{\mathrm{m}}, f_j^{\mathrm{e}}, \rho_{\mathrm{e}}, r$ are not considered and assuming the variation of temperature is small (i.e., let $T \approx T_0$), the inertial entropy theory, the K-L-S theory can be expressed in a unified equation system:

$$\begin{aligned} &C_{ijkl}u_{k,lj} + e_{kij}\varphi_{,kj} - \alpha_{ij}\vartheta_{,j} = \rho\ddot{u}_i, \quad e_{ikj}u_{k,ji} - \epsilon_{ij}\varphi_{,ji} + \tau_i\vartheta_{,i} = 0 \\ &T_0\alpha_{ij}(\dot{u}_{i,j} + \xi_1\ddot{u}_{i,j}) - T_0\tau_i(\dot{\varphi}_{,i} + \xi_2\ddot{\varphi}_{,i}) + C(\dot{\vartheta} + \xi_0\ddot{\vartheta}) = \lambda_{ij}\vartheta_{,ji} \end{aligned} \quad (6.128)$$

When $\xi_1 = \xi_2 = 0, \xi_0 = \rho_{s0}$, Eq. (6.128) represents the inertial entropy theory and $\xi_1 = \xi_2 = \xi_0 = \tau_0$ represents the K-L-S theory. In Sect. 1.6.2 we have pointed out that there are some questions in the K-L-S theory. Here we can also show that (1) from Eq. (6.118) we get $T\dot{s} - \tau_0 T\ddot{s} = \lambda_{ij}T_{,ji} + (\dot{r} + \tau_0\ddot{r})$, so it is difficult to consider that s is a state function. (2) It is difficult to physically explain why Eq. (6.128) also has the inertial terms $\tau_0 T_0(-\tau_i\ddot{\varphi}_{,i}, \alpha_{ij}\ddot{u}_{i,j})$. (3) The Fourier thermal conductive equation is substantially an irreversible phenomenon, which is in the same level with the mechanical viscous effect, as seen from the equation of the entropy production. So the viscous effect in elasticity produced by the Cattaneo-Vernotte heat conductive equation is a second effect.

For a plane wave Eq. (6.2) becomes

$$u_k = U_k\mathrm{e}^{\mathrm{i}(kn_mx_m - \omega t)}, \quad \varphi = \Phi\mathrm{e}^{\mathrm{i}(kn_mx_m - \omega t)}, \quad \vartheta = \Theta\mathrm{e}^{\mathrm{i}(kn_mx_m - \omega t)} \quad (6.129)$$

or

$$u_k = U_k\mathrm{e}^{i\omega(L_mx_m - t)}, \quad \varphi = \Phi\mathrm{e}^{i\omega(L_mx_m - t)}, \quad \vartheta = \Theta\mathrm{e}^{i\omega(L_mx_m - t)}; \quad L_m = kn_m/\omega = n_m/c \quad (6.130)$$

where U, Φ, Θ are the amplitudes of the displacement, electric potential, and temperature, respectively. In general k is a complex number:

$$k = \alpha + \mathrm{i}\beta, \quad \mathrm{e}^{\mathrm{i}(kn_mx_m - \omega t)} = \mathrm{e}^{-\beta n_mx_m}\mathrm{e}^{\mathrm{i}(\alpha n_mx_m - \omega t)}, \quad c = \omega/\alpha, \quad \boldsymbol{c} = (\omega/\alpha)\boldsymbol{n} \quad (6.131)$$

where c is the phase velocity and β is the attenuation coefficient. Substituting Eq. (6.129) into (6.128) and dropping the common factor $\exp[\mathrm{i}(kn_mx_m - \omega t)]$ we obtain the Christoffel equation:

$$
\begin{aligned}
&(\Gamma_{ik}^{*}k^2-\rho\omega^2\delta_{ik})U_k+e_i^{*}k^2\Phi+\mathrm{i}\alpha_i^{*}k\Theta=0\\
&e_k^{*}k^2U_k-\epsilon^{*}k^2\Phi-\mathrm{i}\tau^{*}k\Theta=0\\
&T_0\alpha_k^{*}k\omega(1-\mathrm{i}\xi_1\omega)U_k-T_0\tau^{*}k\omega(1-\mathrm{i}\xi_2\omega)\Phi+(\lambda^{*}k^2-C\xi_0\omega^2-\mathrm{i}C\omega)\Theta=0
\end{aligned}
\tag{6.132}
$$

or

$$
\boldsymbol{\Lambda}(k,\omega,\boldsymbol{n})\boldsymbol{U}=\mathbf{0},\quad \boldsymbol{U}=[U_1,U_2,U_3,\Phi,\Theta]^{\mathrm{T}}
\tag{6.133}
$$

where

$$
\boldsymbol{\Lambda}=\begin{bmatrix}
\Gamma_{11}^{*}k^2-\rho\omega^2 & \Gamma_{12}^{*}k^2 & \Gamma_{13}^{*}k^2 & e_1^{*}k^2 & \mathrm{i}\alpha_1^{*}k\\
\Gamma_{21}^{*}k^2 & \Gamma_{22}^{*}k^2-\rho\omega^2 & \Gamma_{23}^{*}k^2 & e_2^{*}k^2 & \mathrm{i}\alpha_2^{*}k\\
\Gamma_{31}^{*}k^2 & \Gamma_{32}^{*}k^2 & \Gamma_{33}^{*}k^2-\rho\omega^2 & e_3^{*}k^2 & \mathrm{i}\alpha_3^{*}k\\
e_1^{*}k^2 & e_2^{*}k^2 & e_3^{*}k^2 & -\epsilon^{*}k^2 & -\mathrm{i}\tau^{*}k\\
\alpha_1^{*}k\omega\eta_1 & \alpha_2^{*}k\omega\eta_1 & \alpha_3^{*}k\omega\eta_1 & -\tau^{*}k\omega\eta_2 & T_0^{-1}(\lambda^{*}k^2-C\eta_3)
\end{bmatrix}
\tag{6.134}
$$

with

$$
\begin{aligned}
&\Gamma_{ik}^{*}=C_{ijkl}n_jn_l,\quad e_i^{*}=e_{kij}n_kn_j,\quad \alpha_i^{*}=\alpha_{ij}n_j\\
&\tau^{*}=\tau_jn_j,\quad \epsilon^{*}=\epsilon_{jk}n_kn_j,\quad \lambda_j^{*}=\lambda_{ij}n_i,\quad \lambda^{*}=\lambda_j^{*}n_j\\
&\eta_1=1-\mathrm{i}\xi_1\omega,\quad \eta_2=1-\mathrm{i}\xi_2\omega,\quad \eta_3=\xi_0\omega^2+\mathrm{i}\omega
\end{aligned}
\tag{6.135}
$$

In order to get the nontrivial solution of $\boldsymbol{U}$, it is necessary that

$$
\det\Lambda(k,\omega,\boldsymbol{n})=0
\tag{6.136a}
$$

Equation (6.136) is called the secular equation and can be expanded to

$$
\left[F_8(\omega,\mathbf{n})k^8+F_6(\omega,\mathbf{n})k^6+F_4(\omega,\mathbf{n})k^4+F_2(\omega,\mathbf{n})k^2+F_0(\omega,\mathbf{n})\right]k^2=0
\tag{6.136b}
$$

where $F_i(\omega,\boldsymbol{n})$ is known functions of $(\omega,\boldsymbol{n})$. So one k^2 is zero in Eq. (6.136), i.e., the wave velocity of the electric potential is infinite or the electric potential does not have its own independent wave mode. From Eq. (6.136) we can solve four independent eigenvalues or wave velocity, and for each wave velocity an independent mode from Eq. (6.133) is obtained. There are total four independent modes: the quasi-longitudinal (QL) wave with highest wave velocity, fast quasi-transverse wave (FT), slow quasi-transverse wave (ST), and a temperature wave (T).

From Eq. (6.132) we can also eliminate $\boldsymbol{\Phi}$ to get equations with independent variables U_k,ϑ.

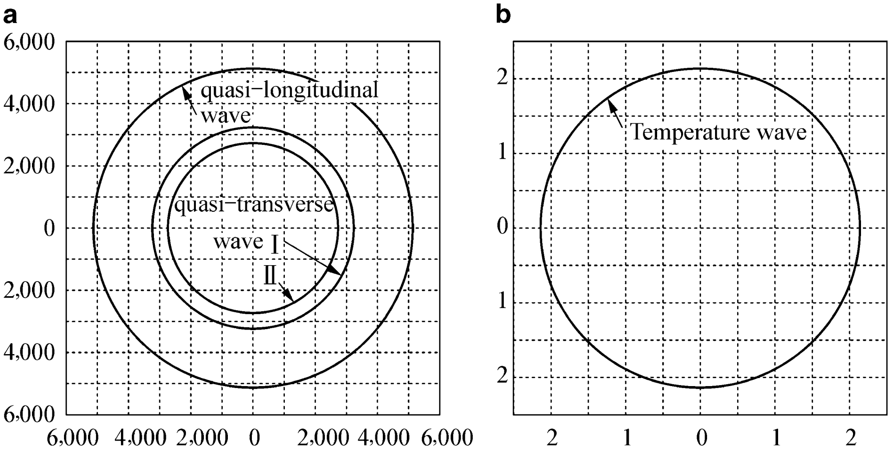


Fig. 6.11 Cross sections of the velocity surfaces in an isotropic plane (x_1, x_2): (**a**) elastic waves and (**b**) temperature wave

6.6.4 An Example

Now we discuss the character surfaces for material $BaTiO_3$ under $\omega = 2\pi \times 10^6\,s^{-1}$, $\gamma = 0$. Material constants of $BaTiO_3$ with poling axis x_3 are

$$C_{11} = 150, \quad C_{12} = 66, \quad C_{13} = 66, \quad C_{33} = 146, \quad C_{44} = 44, \quad C_{66} = 43(\text{MPa});$$
$$e_{13} = -4.35, \quad e_{33} = 17.5, \quad e_{15} = 11.4(\text{C/m}^2);$$
$$\epsilon_{11} = 9.87, \quad \epsilon_{33} = 11.15(10^{-9}\text{C/Vm}); \quad \lambda_{11} = 1.1, \quad \lambda_{33} = 3.5\text{J/mKs};$$
$$\alpha_{11}^{\varepsilon} = 8.53, \quad \alpha_{33}^{\varepsilon} = 1.99(10^{-6}/\text{K}); \quad \tau = 5.53(10^{-3}\text{C/m}^2\text{K})$$
$$\alpha_{11} = \alpha_{22} = (C_{11} + C_{12})\alpha_{11}^{\varepsilon} + (C_{13} + e_{31})\alpha_{33}^{\varepsilon}, \quad \alpha_{33} = 2C_{13}\alpha_{11}^{\varepsilon} + (C_{33} + e_{33})\alpha_{33}^{\varepsilon}$$

where $\alpha_{11}^{\varepsilon}$, $\alpha_{33}^{\varepsilon}$ are the usual thermal expansion coefficients. Figure 6.11 (a) and (b) gives the velocity surfaces of the elastic waves and temperature wave in the isotropic plane (x_1, x_2), respectively; Fig. 6.12 (a) and (b) gives the velocity surfaces of the elastic waves and temperature wave in the anisotropic plane (x_1, x_3), respectively; Fig. 6.13 (a) and (b) gives the slowness surfaces of the elastic waves and temperature wave in the anisotropic plane (x_1, x_3), respectively. The dotted lines in the Fig. 6.13 represent the velocity or slowness surfaces for purely elastic material. The numerical results show that the attenuation of the temperature wave is large, but for the elastic waves, they are small and may be negative for certain ρ_{s0}. The results of Ezzat et al. (2002) and Yuan and Kuang (2008) also showed that the temperature wave can enforce the elastic wave when the temperature is decreased. It means that the term containing ρ_{s0} enforces the elastic wave, or when the temperature decreased, the released inertial heat is partly transformed to the elastic wave. It may be a restriction of ρ_{s0}. This phenomenon has been discussed in Sect. 1.7.5.

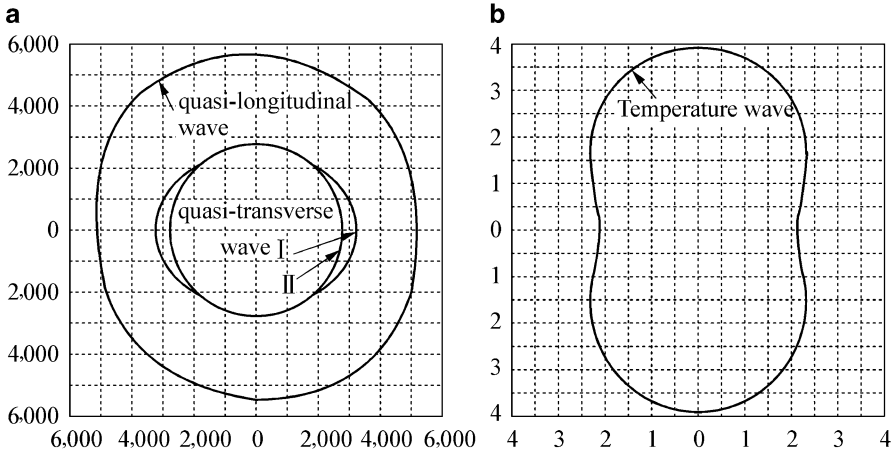


Fig. 6.12 Cross sections of the velocity surfaces in an anisotropic plane (x_1, x_3): (**a**) elastic waves and (**b**) temperature wave

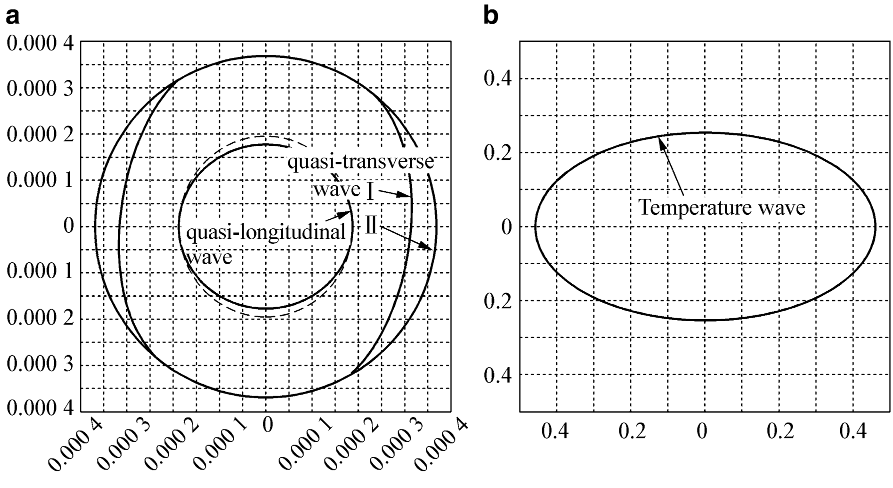


Fig. 6.13 Cross sections of the slowness surfaces in an anisotropic plane (x_1, x_3): (**a**) elastic waves and (**b**) temperature wave

6.6.5 Inhomogeneous Wave

In the framework of the inhomogeneous wave theory generally, the wave vector is $\boldsymbol{k} = \boldsymbol{P} + \mathrm{i}\boldsymbol{A}$, where $\boldsymbol{P}$ and $\boldsymbol{A}$ are two real vectors (Buchen 1971; Borcherdt 1973). The vector $\boldsymbol{n} = \boldsymbol{P}/|\boldsymbol{P}|$ represents the wave propagation direction which is perpendicular to the wave surface with equal phase, and $\boldsymbol{m} = \boldsymbol{A}/|\boldsymbol{A}|$ represents the maximum

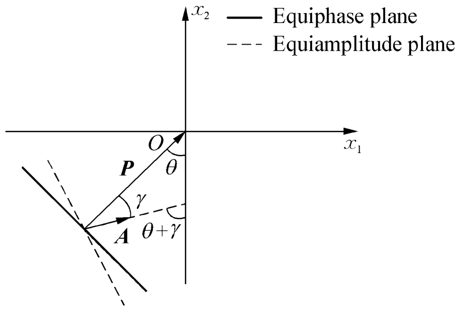


Fig. 6.14 Inhomogeneous wave

attenuation direction which is perpendicular to the equal-amplitude surface. The angle γ between $\boldsymbol{n}$ and $\boldsymbol{m}$ is called the attenuation angle (Fig. 6.14). The surface wave may be considered as an inhomogeneous wave with $\gamma = \pi/2$ and P is parallel to the surface. In general case how to determine γ is not very clear (Krebes 1983). For an inhomogeneous plane wave, we have (Yuan and Kuang 2010)

$$\begin{aligned} &f = f_0 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = f_0 \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)}, \quad \boldsymbol{k} = \boldsymbol{P} + \mathrm{i}\boldsymbol{A}, \quad \boldsymbol{P} = P\boldsymbol{n}, \quad \boldsymbol{A} = A\boldsymbol{m} \\ &k_j = P_j + \mathrm{i}A_j = Pn_j + \mathrm{i}Am_j, \quad P = \sqrt{P_1^2 + P_2^2}, \quad A = \sqrt{A_1^2 + A_2^2} \end{aligned} \tag{6.137}$$

Let θ denote the angle between $\boldsymbol{n}$ and the ordinate, so

$$\boldsymbol{n} = [\sin\theta, \cos\theta]^{\mathrm{T}}, \quad \boldsymbol{m} = [\sin(\theta+\gamma), \cos(\theta+\gamma)]^{\mathrm{T}}, \quad \boldsymbol{n}\cdot\boldsymbol{m} = \cos\gamma \tag{6.138}$$

For an inhomogeneous wave we need four variables (P, A, θ, γ) to describe it, but for a homogeneous wave we only need three variables (P, A, θ) due to $\boldsymbol{n} = \boldsymbol{m}, \gamma = 0$ and $k_1 = (P + \mathrm{i}A)\sin\theta,\ k_2 = (P + \mathrm{i}A)\cos\theta$. So $\boldsymbol{k}$ can be expressed by one complex number.

An inhomogeneous plane wave can be written as

$$u_k = U_k \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)}, \quad \varphi = \Phi \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)}, \quad \vartheta = \Theta \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)} \tag{6.139}$$

Substituting Eq. (6.139) into (6.129) and dropping the common factor we get the Christoffel equation:

$$\begin{gathered} \boldsymbol{\Lambda}(\boldsymbol{k},\omega)\boldsymbol{U} = \boldsymbol{0}, \quad \boldsymbol{U} = [U_1, U_2, U_3, \Phi, \Theta]^{\mathrm{T}} \\ \boldsymbol{\Lambda} = \begin{bmatrix} \Gamma_{11}^*(\boldsymbol{k}) - \rho\omega^2 & \Gamma_{12}^*(\boldsymbol{k}) & \Gamma_{13}^*(\boldsymbol{k}) & e_1^*(\boldsymbol{k}) & \mathrm{i}\alpha_1^*(\boldsymbol{k}) \\ \Gamma_{21}^*(\boldsymbol{k}) & \Gamma_{22}^*(\boldsymbol{k}) - \rho\omega^2 & \Gamma_{23}^*(\boldsymbol{k}) & e_2^*(\boldsymbol{k}) & \mathrm{i}\alpha_2^*(\boldsymbol{k}) \\ \Gamma_{31}^*(\boldsymbol{k}) & \Gamma_{32}^*(\boldsymbol{k}) & \Gamma_{33}^*(\boldsymbol{k}) - \rho\omega^2 & e_3^*(\boldsymbol{k}) & \mathrm{i}\alpha_3^*(\boldsymbol{k}) \\ e_1^*(\boldsymbol{k}) & e_2^*(\boldsymbol{k}) & e_3^*(\boldsymbol{k}) & -\epsilon^*(\boldsymbol{k}) & -\mathrm{i}\tau^*(\boldsymbol{k}) \\ T_0\alpha_1^*(\boldsymbol{k})\eta_1 & T_0\alpha_2^*(\boldsymbol{k})\eta_1 & T_0\alpha_3^*(\boldsymbol{k})\eta_1 & -T_0\tau^*(\boldsymbol{k})\eta_2 & \lambda^* - C\eta_3 \end{bmatrix} \end{gathered} \tag{6.140}$$

where

$$\begin{aligned} &\Gamma_{ik}^{*}(\boldsymbol{k}) = C_{ijkl}k_j k_l, \quad e_i^{*}(\boldsymbol{k}) = e_{kij}k_k k_j, \quad \alpha_i^{*}(\boldsymbol{k}) = \alpha_{ij}k_j \\ &\tau^{*}(\boldsymbol{k}) = \tau_j k_j, \quad \epsilon^{*}(\boldsymbol{k}) = \epsilon_{jk}k_k k_j, \quad \lambda_j^{*} = \lambda_{ij}n_i, \quad \lambda^{*}(\boldsymbol{k}) = \lambda_{ij}k_i k_j \\ &\eta_1 = 1 - \mathrm{i}\xi_1\omega, \quad \eta_2 = 1 - \mathrm{i}\xi_2\omega, \quad \eta_3 = \xi_0\omega^2 + \mathrm{i}\omega \end{aligned} \tag{6.141}$$

The secular equation corresponding to Eq. (6.140) is

$$|\boldsymbol{\Lambda}| = 0 \tag{6.142a}$$

Substituting $k_j = Pn_j + iAm_j$ and decomposing $|\boldsymbol{\Lambda}| = 0$ into real and imaginary parts we get two coupling real equations of (P, A, θ, γ):

$$\mathrm{Re}|\boldsymbol{\Lambda}| = 0, \quad \mathrm{Im}|\boldsymbol{\Lambda}| = 0 \tag{6.142b}$$

Giving (θ, γ), (P, A) can be obtained from Eq. (6.142), so (k_1, k_2). It means that k_1 and k_2 are obtained simultaneously. In order to (P, A) are not negative it needs $-\pi/2 < \gamma < \pi/2$.

Similar to the homogeneous wave, Eq. (6.142) only has four independent eigenvalues $\boldsymbol{k}_i = P_i\boldsymbol{n} + \mathrm{i}A_i\boldsymbol{m}$ $(i = 1, 2, 3, 4)$ corresponding four phase velocities:

$$c_i = \omega / P_i, \quad P_i = \sqrt{(P_i n_1)^2 + (P_i n_2)^2} \tag{6.143}$$

Corresponding each complex $\boldsymbol{k}_i$, from Eq. (6.142) we can get the amplitude vectors or eigenvectors $\boldsymbol{U}_i$. In each $\boldsymbol{U}_i$, $U_{1i} : U_{2i} : U_{3i} : \Phi_i(= U_{4i}) : \Theta_i(= U_{5i})$ is determined, i.e., only one component, say, $U_{j1} = \beta_j$, is undetermined. So there are only four undetermined amplitude components, and the general solution of the wave propagation problem is

$$\begin{aligned} &u_k = \sum_{j=1}^{4} \beta_j U_k^{(j)} \mathrm{e}^{\mathrm{i}(k_m^{(j)}x_m - \omega t)}, \quad \varphi = \sum_{j=1}^{4} \beta_j \Phi^{(j)} \mathrm{e}^{\mathrm{i}(k_m^{(j)}x_m - \omega t)}, \quad \vartheta = \sum_{j=1}^{4} \beta_j \Theta^{(j)} \mathrm{e}^{\mathrm{i}(k_m^{(j)}x_m - \omega t)} \\ &\mathrm{e}^{\mathrm{i}(k_m^{(j)}x_m - \omega t)} = \mathrm{e}^{\mathrm{i}\left[(P^{(j)}\boldsymbol{n} + \mathrm{i}A^{(j)}\boldsymbol{m})\cdot\boldsymbol{x} - \omega t\right]} = \mathrm{e}^{-A^{(j)}\boldsymbol{m}\cdot\boldsymbol{x}} \mathrm{e}^{\mathrm{i}(P^{(j)}\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)} \end{aligned} \tag{6.144}$$

where $\beta_j (i = 1, 2, 3, 4)$ is an undetermined coefficient.

The numerical calculations for $BaTiO_3$ show that the effect of γ on the velocity surfaces of elastic waves is limited and there is a certain effect on the velocity surfaces of the temperature. There are certain effects on the attenuation coefficients of all waves.

6.7 Reflection and Transmission of Waves in Pyroelectric and Piezoelectric Materials

6.7.1 General Theory

Consider the problem of two semi-infinite pyroelectric materials I and II bounded on the interface $x_2 = 0$ subjected to an inhomogeneous harmonic incident wave of frequency ω with an incident angle θ from the lower semi-plane I, $x_2 < 0$, (Fig. 6.15) (Kuang and Yuan 2011; Zhou et al. 2012). In Fig. 6.15 only one reflection wave and one transmission wave are drawn for clarity. The mechanical, electrical, and thermal continuity conditions on the interface are (MCC), (ECC), and (TCC), respectively

$$\begin{aligned} &\text{MCC}: \quad u_i^I = u_i^{II}, \quad \sigma_{ij}^I n_i^I + \sigma_{ij}^{II} n_i^{II} = 0, \qquad \text{(6 conditions)} \\ &\text{ECC}: \quad \varphi^I = \varphi^{II}, \quad D_i^I n_i^I + D_i^{II} n_i^{II} = 0, \qquad \text{(2 conditions)} \\ &\text{TCC}: \quad \vartheta^{\mathrm{I}} = \vartheta^{\mathrm{II}}, \quad \lambda_{ij}^{\mathrm{I}} \vartheta_{,j}^{\mathrm{I}} n_i^{\mathrm{I}} + \lambda_{ij}^{\mathrm{II}} \vartheta_{,j}^{\mathrm{II}} n_i^{\mathrm{II}} = 0, \quad \text{(2 conditions)} \end{aligned} \tag{6.145}$$

where $n_i^{\mathrm{II}} = -n_i^{\mathrm{I}}$. There are totally ten continuity conditions on the interface.

Let an incident wave with a wave vector $\boldsymbol{k}^{(0)}$ be in the semi-infinite plane I, $x_2 \leq 0$, and corresponding displacement, electric potential, and relative temperature can be expressed by

$$u_k^{(0)} = U_k^{(0)} \mathrm{e}^{\mathrm{i}\left(k_m^{(0)} x_m - \omega t\right)}, \quad \varphi^{(0)} = \Phi^{(0)} \mathrm{e}^{\mathrm{i}\left(k_m^{(0)} x_m - \omega t\right)}, \quad \vartheta^{(0)} = \Theta^{(0)} \mathrm{e}^{\mathrm{i}\left(k_m^{(0)} x_m - \omega t\right)}, \tag{6.146}$$

where $U_k^{(0)}$, $\Phi^{(0)}$, $\Theta^{(0)}$ and $k_m^{(0)}$ are all known. The reflection wave in the semi-infinite plane I, $x_2 \leq 0$, can be expressed by

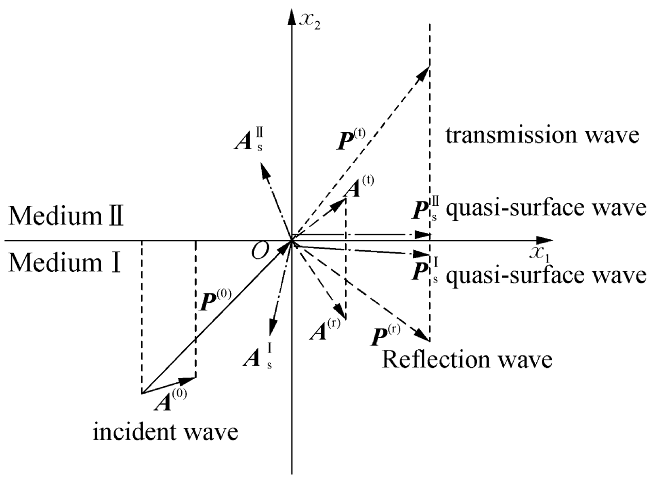


Fig. 6.15 A sketch of reflection and transmission of inhomogeneous waves

$$u_k^{(r)} = \sum_{j=1}^{N} \beta_j^{(r)} U_k^{(r,j)} \mathrm{e}^{\mathrm{i}\left(k_m^{(r,j)} x_m - \omega t\right)}, \quad \varphi^{(r)} = \sum_{j=1}^{N} \beta_j^{(r)} \Phi^{(r,j)} \mathrm{e}^{\mathrm{i}\left(k_m^{(r,j)} x_m - \omega t\right)}$$
$$\vartheta^{(r)} = \sum_{j=1}^{N} \beta_j^{(r)} \Theta^{(r,j)} \mathrm{e}^{\mathrm{i}\left(k_m^{(r,j)} x_m - \omega t\right)} \tag{6.147}$$

and the transmission wave in the semi-infinite plane II, $x_2 \geq 0$, can be expressed by

$$u_k^{(t)} = \sum_{j=1}^{N} \beta_j^{(t)} U_k^{(t,j)} \mathrm{e}^{\mathrm{i}\left(k_m^{(t,j)} x_m - \omega t\right)}, \quad \varphi^{(t)} = \sum_{j=1}^{N} \beta_j^{(t)} \Phi^{(t,j)} \mathrm{e}^{\mathrm{i}\left(k_m^{(t,j)} x_m - \omega t\right)}$$
$$\vartheta^{(t)} = \sum_{j=1}^{N} \beta_j^{(t)} \Theta^{(t,j)} \mathrm{e}^{\mathrm{i}\left(k_m^{(t,j)} x_m - \omega t\right)} \tag{6.148}$$

In Eqs. (6.147) and (6.148), N is the number of the independent waves. It is obvious that

$$u_k^{\mathrm{I}} = u_k^{(0)} + u_k^{(r)}, \quad u_k^{\mathrm{II}} = u_k^{(t)}, \quad \varphi^{\mathrm{I}} = \varphi^{(0)} + \varphi^{(r)}, \quad \varphi^{\mathrm{II}} = \varphi^{(t)}, \quad \vartheta^{\mathrm{I}} = \vartheta^{(0)} + \vartheta^{(r)}, \quad \vartheta^{\mathrm{II}} = \vartheta^{(t)}$$
$$\sigma_{ij}^{\mathrm{I}} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(r)}, \quad \sigma_{ij}^{\mathrm{II}} = \sigma_{ij}^{(t)}, \quad D_i^{\mathrm{I}} = D_i^{(0)} + D_i^{(r)}, \quad D_i^{\mathrm{II}} = D_i^{(t)} \tag{6.149}$$

When waves propagate in the x_1-x_2 plane, the following synchronism condition should be held:

$$k_1^{(0)} = k_1^{(r,j)} = k_1^{(t,j)}, \quad k_1^{(\alpha,j)} = k^{(\alpha,j)} n_1^{(\alpha,j)}, \quad (\alpha = r, t;\ j = 1 - N) \tag{6.150}$$

Decomposing Eq. (6.150) into real and imaginary parts yields

$$P^{(0)} \sin\theta^{(0)} = P^{(r,j)} \sin\theta^{(r,j)} = P^{(t,j)} \sin\theta^{(t,j)}$$
$$A^{(0)} \sin\left(\theta^{(0)} + \gamma^{(0)}\right) = A^{(r,j)} \sin\left(\theta^{(r,j)} + \gamma^{(r,j)}\right) = A^{(t,j)} \sin\left(\theta^{(t,j)} + \gamma^{(t,j)}\right) \tag{6.151}$$

From Eqs. (6.137) and (6.151), we can get the generalized Snell's law from the real part:

$$\frac{\sin\theta^{(0)}}{c^{(0)}} = \frac{\sin\theta^{(r,j)}}{c^{(r,j)}} = \frac{\sin\theta^{(t,j)}}{c^{(t,j)}}, \quad c^{(0)} = \frac{\omega}{P^{(0)}}, \quad c^{(r,j)} = \frac{\omega}{P^{(r,j)}},$$
$$c^{(t,j)} = \frac{\omega}{P^{(t,j)}};\ (j = 1 - N) \tag{6.152}$$

From Eq. (6.152), $\theta^{(r,j)},\theta^{(t,j)},\gamma^{(r,j)},\gamma^{(t,j)}$ can be solved when $\theta^{(0)},\gamma^{(0)},c^{(0)}$ and $c^{(r,j)}$, $c^{(t,j)}$ are known. In the reflection and transmission wave case, $k_1^{(0)}=k_1^{(r,j)}=k_1^{(t,j)}$ are known and unknowns are $k_2^{(r,j)},k_2^{(t,j)}$ in Eq. (6.142). In this case except four bulk waves as that in the infinite space, a new kind of wave will be revealed. The numerical examples show that this new wave propagates almost parallel to the interface, but the maximum attenuation direction is almost perpendicular to the interface. So we call it quasi-surface wave or QS wave. The similar waves called evanescent wave in the previous literatures for piezoelectric by Auld (1973) and Every and Neiman (1992) had been discussed. Sharma et al. (2008) discussed also the wave reflection and transmission in pyroelectric materials.

Substituting Eqs. (6.146), (6.147), (6.148), and (6.149) into Eq. (6.145), the ten boundary conditions on the interface can be expressed as

$$
\begin{aligned}
&U_k^{(0)}+\sum_{j=1}^{5}\beta_j^{(r)}U_k^{(r,j)}=\sum_{j=1}^{5}\beta_j^{(t)}U_k^{(t,j)},\, k=1-3,\\
&C_{2iml}^{(r)}k_l^{(0)}U_m^{(0)}+e_{m2i}^{(r)}k_m^{(0)}\Phi^{(0)}+\mathrm{i}\alpha_{i2}^{(r)}\Theta^{(0)}+\sum_{j=1}^{5}\beta_j^{(r)}(C_{2iml}^{(r)}k_l^{(r,j)}U_m^{(r,j)}+e_{m2i}^{(r)}k_m^{(r,j)}\Phi^{(r,j)}\\
&\quad+\mathrm{i}\alpha_{i2}^{(r)}\Theta^{(r,j)}=\sum_{j=1}^{5}\beta_j^{(t)}\left(C_{2iml}^{(t)}k_l^{(t,j)}U_m^{(t,j)}+e_{m2i}^{(t)}k_m^{(t,j)}\Phi^{(t,j)}+\mathrm{i}\alpha_{i2}^{(t)}\Theta^{(t,j)}\right),\quad i=1-3
\end{aligned}
\tag{6.153a}
$$

$$
\begin{aligned}
&\Phi^{(0)}+\sum_{j=1}^{5}\beta_j^{(r)}\Phi^{(r,j)}=\sum_{j=1}^{5}\beta_j^{(t)}\Phi^{(t,j)}\\
&\quad-\epsilon_{2m}^{(r)}k_m^{(0)}\Phi^{(0)}+e_{2pm}^{(r)}k_m^{(0)}U_p^{(0)}-\mathrm{i}\tau_i^{(r)}\Theta^{(0)}+\sum_{j=1}^{5}\beta_j^{(r)}\left(e_{2pm}^{(r)}k_m^{(r,j)}U_p^{(r,j)}-\epsilon_{2m}^{(r)}k_m^{(r,j)}\Phi^{(r,j)}\right.\\
&\left.-\mathrm{i}\tau_i^{(r)}\Theta^{(r,j)}\right)=\sum_{j=1}^{5}\beta_j^{(t)}\left(e_{2pm}^{(t)}k_m^{(t,j)}U_p^{(t,j)}-\epsilon_{2m}^{(t)}k_m^{(t,j)}\Phi^{(t,j)}-\mathrm{i}\tau_i^{(t)}\Theta^{(t,j)}\right)
\end{aligned}
\tag{6.153b}
$$

$$
\begin{aligned}
&\Theta^{(0)}+\sum_{j=1}^{5}\beta_j^{(r)}\Theta^{(r,j)}=\sum_{j=1}^{5}\beta_j^{(t)}\Theta^{(t,j)}\\
&\lambda_{2m}^{(r)}k_m^{(0)}\Theta^{(0)}+\lambda_{2m}^{(r)}\sum_{j=1}^{5}\beta_j^{(r)}k_m^{(r,j)}\Theta^{(r,j)}=\lambda_{2m}^{(t)}\sum_{j=1}^{5}\beta_j^{(t)}k_m^{(t,j)}\Theta^{(t,j)}
\end{aligned}
\tag{6.153c}
$$

Therefore, in the reflection and transmission waves, there are ten complex unknown amplitude coefficients $\beta_j^{(r)}$ and $\beta_j^{(t)}$ $(j=1-5)$ with total ten complex interface continuity conditions. This shows that the reflection and transmission waves are solvable.

The general expression of the wave energy flow and its ratio of the reflection and transmission are defined as

$$\dot{W}_i = -\sigma_{kl}\dot{u}_k + \varphi\dot{D}_i - \lambda_{ik}\vartheta_{,k}\vartheta/T_0, \quad e^{(j)} = <\dot{W}_2^{(j)}> / <\dot{W}_2^{(0)}>, \qquad (6.154)$$

where the symbol <> expresses the average value over one period of a physical variable and $\dot{W}_2^{(j)}$ is the energy flow component corresponding to $\beta^{(j)}$ along x_2 direction.

Omitting the terms related to temperature, the governing equations of the piezoelectric materials are obtained.

The above theory of acoustic wave in piezoelectric materials is based on the quasi-electrostatic description, because the sound speed c_a is several orders smaller than the electromagnetic wave speed c_e. The precision of this approximation is very high. The electromagnetic corrections to the surface acoustic speeds only have the order $(c_a/c_e)^2 \approx 10^{-8}$. The exceptional case is the incidence under small angle of the order of c_a/c_e to the normal of the interface, due to the generalized Snell's law or the synchronism condition (Darinskii et al. 2008). In this case the incident elastic wave can be converted into the electromagnetic waves. However, the magnitudes of the tangential components of the wave amplitudes are in order of c_a/c_e, so very small due to the small incident angle.

6.7.2 Numerical Example

As an example, we discuss 2D propagation waves in PZT-6B/BaTiO$_3$ material combination, which are transversely isotropic materials with poling axis x_3 (Zhou et al. 2012). In 2D case there is only one transverse wave QT.

The material data for BaTiO$_3$ are given in 6.6.4. The material data for PZT-6B are given as

$$\begin{aligned}
&C_{11}=168\times10^9,\ \ C_{13}=60\times10^9,\ \ C_{33}=163\times10^9,\ \ C_{44}=27.1\,(\mathrm{MPa}),\\
&e_{13}=-0.9,\ \ e_{33}=7.1,\ \ e_{15}=4.6\,(\mathrm{C/m^2}),\ \ \epsilon_{11}=3.6\times10^{-9},\ \ \epsilon_{33}=3.4\times10^{-9}(\mathrm{C/Vm}),\\
&\alpha_{11}^{\varepsilon}=7\times10^{-6},\ \ \alpha_{33}^{\varepsilon}=7(10^{-6}/\mathrm{K}),\ \ \lambda_{11}=1.2,\ \ \lambda_{33}=1.2(\mathrm{J/msK}),\ \ \tau=3.7(10^{-4}\mathrm{C/m^2K}),\\
&\rho=7{,}600\,(\mathrm{kg/m^3}),\ \ \omega=2\pi\times10^6\mathrm{s}^{-1},\ \ C=420\,(\mathrm{J/kgK}),\ \ \rho_{s0}=10^{-14}\mathrm{s}^{-1}.
\end{aligned}$$

Figure 6.16 shows the variations of the amplitude coefficients $|\beta_i|$ and the energy flow ratios $e^{(j)}$ of the reflection and transmission waves with the incident angle θ of the QL incident wave from PZT-6B to BaTiO$_3$. Figure 6.16a gives the amplitude coefficients for reflected waves Ref-QL and Ref-QT and transmitted wave Tran-QL and Tran-QT. It is found that when the incident angle θ exceeds the critical angle $\theta_{cr}\,(\theta_{cr} \approx 61.2°)$, the Tran-QL wave becomes evanescent propagating along the interface. Figure 6.16b shows the energy flow ratios normal to the interface for the Ref-QL, Ref-QT, Tran-QL, and Tran-QT. It is found that the sum of Ref-QL

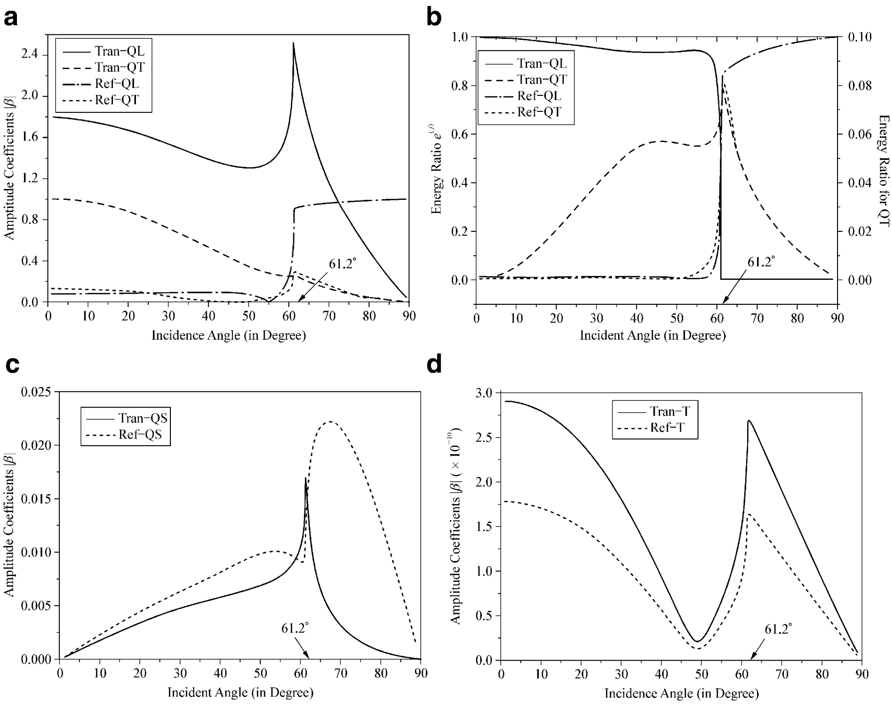


Fig. 6.16 Variations of $|\beta_i|$ and $e^{(j)}$ with the incident angle θ of QL incident wave from PZT-6B to $BaTiO_3$: (**a**) coefficients of QL and QT waves, (**b**) energy flow ratios of QL and QT waves, (**c**) coefficients of QS wave, and (**d**) coefficients of T wave

and Tran-QL waves is far larger than the sum of Ref-QT and Tran-QT waves. Figure 6.16c gives the amplitude coefficients for the quasi-surface (QS) waves. The amplitude coefficients of QS waves are much less than those of other elastic waves. Figure 6.16d shows the amplitude coefficients of the reflected and transmitted temperature T waves. The amplitude coefficients of temperature waves are far less than those of other waves discussed in the example. The energy flow normal to the interface for the temperature wave is also very little and dissipates quickly.

Kuang and Yuan (2011) discussed the 2D reflection problem from the interface of $BiTiO_3$/vaccum with the boundary conditions

$$\sigma_{2j}^{(o)} + \sigma_{2j}^{(r)} = 0, \quad D_2^{(o)} + D_2^{(r)} = 0, \quad \lambda_{2j}\left(\vartheta_j^{(o)} + \vartheta_j^{(r)}\right) = 0, \quad j = 1, 2$$

In this case there are no transmitted waves. The quasi-surface wave becomes surface wave. They found that the wave velocity of the quasi-surface wave is significantly dependent to the incident angle due to the generalized Snell's law. When the incident wave is the elastic wave, the reflected wave is mainly the elastic wave, the quasi-surface wave is weaker, and the reflected temperature wave is very limited. The effect of the attenuation angle γ is very limited.

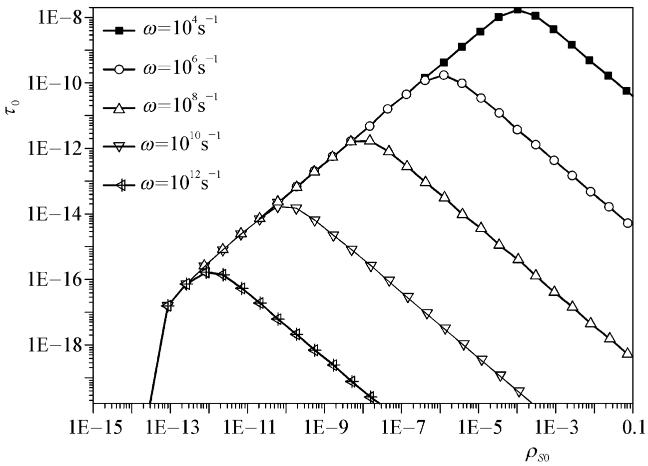


Fig. 6.17 Phase diagram of the attenuation coefficient

6.7.3 Viscous Effect

The experimental results showed that the viscous relaxation times are about $10^{-6} - 10^{-8}$s for various metals under shock-loading conditions (Mineev and Mineev 1997; Ma et al. 2011). Ezzat et al. (2002) discussed the generalized thermo-viscoelasticity with G-L theory. Lionetto et al. (2005) studied the boundary value problem of one-dimensional semi-infinite piezoelectric rod subjected to a sudden heat based on K-L-S theory. They found that the thermal relaxation and the viscous effects were evident in short time for the thermal shock in viscoelastic-piezoelectric material. Kuang (2011) and Kuang and Zhou (2012) introduced material constant β_{ijkl} to discuss tentatively the viscoelastic effect in the inertial entropy theory. The constitutive equation (6.124) is changed to

$$\begin{aligned} &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + \beta_{ijkl}\dot{\varepsilon}_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta, \quad D_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta, \\ &\quad s = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + C\vartheta/T_0 \end{aligned} \tag{6.155}$$

The governing equation (6.128) becomes

$$\begin{aligned} &C_{ijkl}u_{k,lj} + \beta_{ijkl}\dot{\varepsilon}_{kl} + e_{kij}\varphi_{,kj} - \alpha_{ij}\vartheta_{,j} = \rho\ddot{u}_i, \quad e_{ikj}u_{k,ji} - \epsilon_{ij}\varphi_{,ji} + \tau_i\vartheta_{,i} = 0 \\ &\alpha_{ij}\dot{u}_{i,j} - \tau_i\dot{\varphi}_{,i} + (C/T_0)(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}) = \lambda_{ij}\vartheta_{,ji}/T \end{aligned} \tag{6.156}$$

Figure 6.17 gives the phase diagram of attenuation coefficient of QL wave for various ρ_{s0}, τ_0 for a plane wave with $\gamma = 0$ for various ω, where $\beta_{ijkl} = \tau_0 C_{ijkl}$ is assumed. In Fig. 6.17 the attenuation coefficient is positive if the region is above the lines and negative if the region is below the lines. In the region with negative attenuation coefficient, there is an enlarged factor before the elastic wave amplitudes. However, it is not to say that on the propagation path the elastic waves are enhanced, because the elastic wave amplitudes are proportional to the temperature wave amplitude (see Sect. 1.7.6).

It is found that if $\tau_0 = 0$, negative damping occurred, but for $\rho_{s0} = 0$ there is no damping region. How to explain and use the negative damping it is also a meaningful problem.

In the shock problem it is better to take the integral-type viscoelastic constitutive equation (Kuang 2002; Ezzat et al. 2002).

6.7.4 Waves in Piezoelectric Materials

The governing equations in the piezoelectric materials can be obtained by omitting the terms containing temperature in the governing equations of the pyroelectric materials. For the plane wave from Eq. (6.132) the Christoffel equation is

$$(\Gamma^*_{ik}k^2 - \rho\omega^2\delta_{ik})U_k + e^*_i k^2 \Phi = 0, \quad e^*_k k^2 U_k - \epsilon^* k^2 \Phi = 0 \tag{6.157}$$

or

$$\begin{aligned}
&\boldsymbol{\Lambda}(k,\omega,\boldsymbol{n})\boldsymbol{U} = \boldsymbol{0}, \quad \boldsymbol{U} = [U_1, U_2, U_3, \Phi]^{\mathrm{T}} \\
&\boldsymbol{\Lambda}(k,\omega,\boldsymbol{n}) = \begin{bmatrix} \Gamma^*_{11}k^2 - \rho\omega^2 & \Gamma^*_{12}k^2 & \Gamma^*_{13}k^2 & e^*_1 k^2 \\ \Gamma^*_{21}k^2 & \Gamma^*_{22}k^2 - \rho\omega^2 & \Gamma^*_{23}k^2 & e^*_2 k^2 \\ \Gamma^*_{31}k^2 & \Gamma^*_{32}k^2 & \Gamma^*_{33}k^2 - \rho\omega^2 & e^*_3 k^2 \\ e^*_1 k^2 & e^*_2 k^2 & e^*_3 k^2 & -\epsilon^* k^2 \end{bmatrix}
\end{aligned} \tag{6.158}$$

where $\Gamma^*_{ij}, e^*_j, \epsilon^*$ are shown in Eq. (6.135). Other theories can be discussed similarly.

Pang et al. (2008) discussed the reflection of plane waves at the interface between piezoelectric piezomagnetic media.

6.8 Coupling Problem of Elastic and Electromagnetic Waves in Piezoelectric Material

6.8.1 Governing Equations in Pyroelectric Materials

In this section we shall discuss the coupling of elastic wave with electromagnetic wave shortly. It is assumed that there are no body force and body electric charge in the material. According to Eq. (1.4) the independent Maxwell equations for the case without electric current are

$$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla \times \boldsymbol{H} = \dot{\boldsymbol{D}} \tag{6.159}$$

It is assumed that the material is nonmagnetic, so the constitutive equations are

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} - \boldsymbol{e} \cdot \boldsymbol{E} - \boldsymbol{\alpha}\theta, \quad s = \boldsymbol{\alpha} : \boldsymbol{\varepsilon} + \boldsymbol{\tau} \cdot \boldsymbol{E} + C\vartheta/T_0, \\ &\boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E} + \boldsymbol{e} : \boldsymbol{\varepsilon} + \boldsymbol{\tau}\vartheta, \quad \boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H} \end{aligned} \tag{6.160}$$

Equations (6.159) and (6.160) yield the electromagnetic wave equation

$$\ddot{\boldsymbol{D}} = \nabla \times \dot{H} = -\nabla \times (\boldsymbol{\mu}^{-1} \cdot \nabla \times \boldsymbol{E}); \quad \text{or} \quad \epsilon_{ij}\ddot{E}_j + e_{ikl}\ddot{\varepsilon}_{kl} + \tau_i\ddot{\vartheta} = -\varpi_{lkj}\varpi_{pni}\mu_{nj}^{-1}E_{k,lp} \tag{6.161}$$

where ϖ_{ijk} is the permutation notation. The momentum and thermal equations are

$$C_{ijkl}u_{k,lj} - e_{kij}E_{k,j} - \alpha_{ji}\vartheta_{,j} = \rho\ddot{u}_i, \quad \alpha_{ij}\dot{u}_{i,j} + \tau_i\dot{E}_i + (C/T_0)(\dot{\vartheta} + \rho_{s0}\ddot{\vartheta}) = \lambda_{ij}\vartheta_{,ji}/T \tag{6.162}$$

The continuity conditions on an interface for a wave reflection and transmission problem are

$$\begin{aligned} &\boldsymbol{u}^{\mathrm{I}} = \boldsymbol{u}^{\mathrm{II}}, \boldsymbol{\sigma}^{\mathrm{I}} \cdot \boldsymbol{n} = \boldsymbol{\sigma}^{\mathrm{II}} \cdot \boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{E}^{\mathrm{I}} = \boldsymbol{n} \times \boldsymbol{E}^{\mathrm{II}}, \\ &\boldsymbol{n} \times \boldsymbol{H}^{\mathrm{I}} = \boldsymbol{n} \times \boldsymbol{H}^{\mathrm{II}}, \vartheta^{\mathrm{I}} = \vartheta^{\mathrm{II}}, \boldsymbol{q}^{\mathrm{I}} \cdot \boldsymbol{n} = \boldsymbol{q}^{\mathrm{II}} \cdot \boldsymbol{n} \end{aligned} \tag{6.163}$$

Equations (6.161), (6.162), and (6.163) are the electroelastic coupling governing equations in pyroelectric materials.

6.8.2 *Coupling Problem of Plane Wave in Piezoelectric Materials*

Kyame (1949), Auld (1973), and Every and Neiman (1992) discussed the electroelastic coupling waves in piezoelectric materials. From Eqs. (6.160), (6.161), and (6.162), the governing equations in piezoelectric materials with isotropic magnetic behavior are

$$\begin{aligned} &\nabla \cdot (\boldsymbol{C} : \boldsymbol{\varepsilon}) - \nabla \cdot (\boldsymbol{e} \cdot \boldsymbol{E}) = \rho\ddot{\boldsymbol{u}} \\ &\mu_0(\boldsymbol{\epsilon} \cdot \ddot{\boldsymbol{E}} + \boldsymbol{e} : \ddot{\boldsymbol{\varepsilon}}) = -\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla(\nabla \cdot \boldsymbol{E}) + \nabla^2\boldsymbol{E}; \quad \text{or} \\ &C_{ijkl}u_{k,lj} - e_{kij}E_{k,j} = \rho\ddot{u}_i, \quad \mu_0\left(e_{ikl}\ddot{u}_{k,l} + \epsilon_{ij}\ddot{E}_j\right) = E_{i,mm} - E_{m,mi} \end{aligned} \tag{6.164}$$

In the coupling problem it is convenient to use the velocity instead of displacement. For a plane wave it is assumed

$$v_k = \dot{u}_k = V_k \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)}, \quad E_i = E_{0i} \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)}; \quad u_k = U_k \mathrm{e}^{\mathrm{i}(k_m x_m - \omega t)}, \quad V_k = -\mathrm{i}\omega U_k \tag{6.165}$$

Substituting Eq. (6.165) into Eq. (6.164) yields the Christoffel equation

$$\begin{aligned} &\left(C_{ijkl} k_l k_j - \rho\omega^2 \delta_{ik}\right) V_k + e_{kij} \omega k_j E_{0k} \\ &e_{ikl} \mu_0 \omega k_l V_k + \left[(k_j k_j \delta_{im} - k_m k_i) - \omega^2 \mu_0 \epsilon_{im}\right] E_{0m} \end{aligned} \tag{6.166}$$

or

$$\begin{aligned} &\boldsymbol{\Lambda}(\boldsymbol{k})\boldsymbol{U} = \boldsymbol{0}, \quad \boldsymbol{U} = [V_1, V_2, V_3, E_{01}, E_{02}, E_{03}]^{\mathrm{T}} \\ &\boldsymbol{\Lambda} = \begin{bmatrix} \Gamma^*_{11} - \rho\omega^2 & \Gamma^*_{12} & \Gamma^*_{13} & e^*_{11} & e^*_{21} & e^*_{31} \\ \Gamma^*_{21} & \Gamma^*_{22} - \rho\omega^2 & \Gamma^*_{23} & e^*_{12} & e^*_{22} & e^*_{32} \\ \Gamma^*_{31} & \Gamma^*_{32} & \Gamma^*_{33} - \rho\omega^2 & e^*_{13} & e^*_{23} & e^*_{33} \\ e^{**}_{11} & e^{**}_{12} & e^{**}_{13} & \gamma^*_{11} & \gamma^*_{12} & \gamma^*_{13} \\ e^{**}_{21} & e^{**}_{22} & e^{**}_{23} & \gamma^*_{21} & \gamma^*_{22} & \gamma^*_{23} \\ e^{**}_{31} & e^{**}_{32} & e^{**}_{33} & \gamma^*_{31} & \gamma^*_{32} & \gamma^*_{33} \end{bmatrix} \end{aligned} \tag{6.167}$$

where

$$\Gamma^*_{ik} = C_{ijkl} k_l k_j, \quad e^*_{ki} = e_{kij} \omega k_j, \quad e^{**}_{ki} = \mu_0 e^*_{ki}, \quad \gamma^*_{ik} = \left[(k_j k_j \delta_{ik} - k_i k_k) - \omega^2 \mu_0 \epsilon_{ik}\right] \tag{6.168}$$

The corresponding secular equation $\det \boldsymbol{\Lambda} = 0$ is a 6×6 determinant of the coefficients including V_m and E_{0m}. Every and Neiman (1992) discussed the approximate solution.

Now discuss a plane wave propagating along x_1 axis (so $k_1 = k, k_2 = k_3 = 0$) in a transversely isotropic piezoelectric material with x_3 polarization. In a transversely isotropic piezoelectric material, the material constants in Voigt notations are $e_{31} = e_{32}, e_{15} = e_{24}, e_{33}, \epsilon_{11} = \epsilon_{22}, \epsilon_{33}$, $C_{11} = C_{22}, C_{12}, C_{13} = C_{23}, C_{33}, C_{44} = C_{55}, C_{66} = (C_{11} - C_{12})/2$. Therefore, the secular equation is

$$\begin{aligned} |\boldsymbol{\Lambda}| &= \begin{vmatrix} C_{11}k^2 - \rho\omega^2 & 0 & 0 & 0 & 0 & e_{31}\omega k \\ 0 & C_{66}k^2 - \rho\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{44}k^2 - \rho\omega^2 & e_{15}\omega k & 0 & 0 \\ 0 & 0 & \mu_0 e_{15}\omega k & -\omega^2 \mu_0 \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & k^2 - \omega^2 \mu_0 \epsilon_{11} & 0 \\ \mu_0 e_{31}\omega k & 0 & 0 & 0 & 0 & k^2 - \omega^2 \mu_0 \epsilon_{33} \end{vmatrix} \\ &= (C_{66}k^2 - \rho\omega^2)(k^2 - \omega^2 \mu_0 \epsilon_{11})\left[(C_{11}k^2 - \rho\omega^2)(k^2 - \omega^2 \mu_0 \epsilon_{33}) - \mu_0 e^2_{31} \omega^2 k^2\right] \\ &\quad \times \left[-\omega^2 \mu_0 \epsilon_{11}(C_{44}k^2 - \rho\omega^2) - \mu_0 e_{15}\omega k e_{15}\omega k\right] = 0 \end{aligned} \tag{6.169}$$

Equations (6.169) and (6.167) can be decomposed into four groups. The modes and the corresponding wave velocities $c_i = \omega/k_i$ can be given as follows:

Purely acoustic wave: mode, $(C_{66}k^2 - \rho\omega^2)V_2 = 0$; velocity, $c_{s6} = \sqrt{C_{66}/\rho}$

Purely electromagnetic wave: mode, $(k^2 - \omega^2\mu_0\epsilon_{11})E_{02} = 0$; velocity, $c_e = \sqrt{1/\mu_0\epsilon_{11}}$.

Stiffened acoustic wave (electrically quasi-static): modes, $(C_{44}k^2 - \rho\omega^2)V_3 + e_{15}\omega k E_{01} = 0$, $\mu_0 e_{15}\omega k V_3 - \omega^2\mu_0\epsilon_{11}E_{01} = 0$; velocity, $c_s^* = \sqrt{\left(C_{44} + e_{15}^2/\epsilon_{11}\right)/\rho}$

Quasi-acoustic and quasi-electromagnetic coupling wave:

$$\text{modes,}\ (C_{11}k^2 - \rho\omega^2)V_1 - e_{31}\omega k E_{03} = 0, \quad \mu_0 e_{31}\omega k V_1 + (k^2 - \omega^2\mu_0\epsilon_{33})E_{03} = 0$$

$$\text{velocities,} \left\{ \begin{array}{l} c_{q\,\text{elctr.}} \\ c_{q\,\text{acust.}} \end{array} \right. = \frac{1}{2}\left(\frac{1}{\mu_0\epsilon_{33}} + \frac{C_{11}}{\rho} + \frac{e_{31}^2}{\rho\epsilon_{33}}\right)\left[1 \pm \sqrt{1 - \frac{4\rho\mu_0\epsilon_{33}C_{11}}{\left(\rho + \mu_0\epsilon_{33}C_{11} + \mu_0 e_{31}^2\right)^2}}\right]$$

6.9 Transverse Wave Scattering from a Semi-infinite Conducting Crack

6.9.1 Fundamental Theory

Discuss a transversely isotropic piezoelectric material with isotropic magnetic behavior and isotropic plane $x_1 - x_2$. Assume the electromechanical coupling occurred between antiplane displacement $\boldsymbol{u}(0, 0, u_3)$ and in-plane electric field $\boldsymbol{E}(E_1, E_2, 0)$. For mode-III problem Eq. (6.164) becomes

$$\begin{aligned} &C_{44}\nabla^2 u_3 - e_{15}\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho\ddot{u}_3, \quad \boldsymbol{\nabla}(\,) = \mathbf{i}_1(\,)_{,1} + \mathbf{i}_2(\,)_{,2} \\ &\ddot{\boldsymbol{D}} = \epsilon_{11}\ddot{\boldsymbol{E}} + e_{15}\boldsymbol{\nabla}\ddot{u}_3 = \boldsymbol{\nabla}\times\dot{\boldsymbol{H}} = -\mu_0^{-1}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) \end{aligned} \tag{6.170}$$

Let

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \dot{\boldsymbol{A}}/c_e; \quad c_e = 1/\sqrt{\mu_0\epsilon_{11}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{A} + \dot{\varphi}/c_e = 0 \tag{6.171}$$

The last one in Eq. (6.171) is a gauge condition to make $\boldsymbol{A}$ unique.

For a general mode-III case from Eq. (6.170), we obtained the electromagneto-acoustic wave equations:

$$C_{44}\nabla^2 u_3 + e_{15}\left(\nabla^2\varphi - \ddot{\varphi}/c_e^2\right) = \rho\ddot{u}_3, \quad \boldsymbol{\nabla}^2\boldsymbol{A} - \ddot{\boldsymbol{A}}/c_e^2 = -\,\mu_0 e_{15}c_e\boldsymbol{\nabla}\dot{u}_3 \tag{6.172}$$

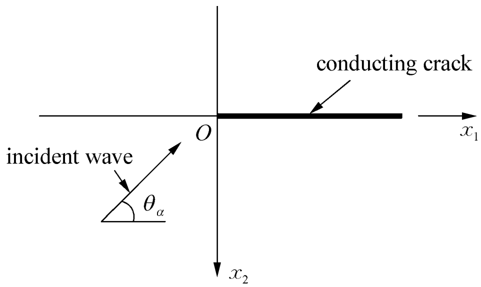


Fig. 6.18 Transverse wave scattering from a semi-infinite conducting crack

where $\nabla \cdot \boldsymbol{E} = -\nabla^2\varphi + \ddot{\varphi}/c_e^2$ has been used. For the electrically quasi-stationary (EQS) case, we have $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}} = \boldsymbol{0}$, so Eq. (6.170) yields

$$C_{44}\nabla^2 u_3 + e_{15}\left(\nabla^2\varphi - \ddot{\varphi}/c_e^2\right) = \rho\ddot{u}_3, \quad e_{15}\nabla^2 u_3 = \epsilon_{11}\left(\nabla^2\varphi - \ddot{\varphi}/c_e^2\right) \tag{6.173}$$

In EQS case the electroelastic wave does not couple with magnetic field. Let

$$\psi = \varphi - (e_{15}/\epsilon_{11})\hat{c}u_3, \quad \hat{c} = c_e^2/\left(c_e^2 - c_s^{*2}\right), \quad c_s^* = \sqrt{\left(C_{44} + e_{15}^2/\epsilon_{11}\right)/\rho} \tag{6.174}$$

Using Eq. (6.174), Eq. (6.173) can be reduced to

$$\nabla^2 u_3 - L_s^{*2}\ddot{u}_3 = 0, \quad \nabla^2\psi - L_e^2\ddot{\psi} = 0; \quad L_s^* = 1/c_s^*, \quad L_e = 1/c_e \ll L_s^* \tag{6.175}$$

If term $\ddot{\varphi}/c_e^2$ is neglected, Eq. (6.173) is reduced to Eq. (4.239) for the electrically static problem. So the difference between the electrically quasi-stationary and static problems is very small, but Eq. (6.175) forms two hyperbolic equations, which may sometimes solve the problem easier. The constitutive equations are

$$\begin{aligned} &\sigma_{13} = C_{44}^* u_{3,1} + e_{15}\psi_{,1}, \quad \sigma_{23} = C_{44}^* u_{3,2} + e_{15}\psi_{,2}; \quad C_{44}^* = C_{44} + \hat{c}e_{15}^2/\epsilon_{11} \\ &D_1 = e_{15}(1-\hat{c})u_{3,1} - \epsilon_{11}\psi_{,1}, \quad D_2 = e_{15}(1-\hat{c})u_{3,2} - \epsilon_{11}\psi_{,2} \end{aligned} \tag{6.176}$$

6.9.2 Transverse Wave Scattering from a Semi-infinite Conducting Crack

Figure 6.18 shows the diffraction of an incident shear wave through a semi-infinite conducting crack in a transversely isotropic piezoelectric material. Li (1996) used the governing equations Eq. (6.175) to solve this problem and called it "quasi-hyperbolic approximation" method. The generalized displacement in the material is

$$u_3(x_1,x_2,t)=u_3^{(i)}+u_3^{(s)},\quad \psi(x_1,x_2,t)=\psi^{(i)}+\psi^{(s)},\quad \left(\varphi(x_1,x_2,t)=\varphi^{(i)}+\varphi^{(s)}\right) \tag{6.177}$$

where the superscripts "(i)" and "(s)" denote the incident and scattering fields, respectively. The incident acoustic wave is assumed in the following form:

$$u_3^{(i)}(x_1,x_2,t)=u_{30}^{(i)}G\left(t-L_s^* n_m x_m\right),\quad G(t)=H(t)\int_0^t \mathrm{g}(\tau)\mathrm{d}\tau \tag{6.178}$$

where $\mathrm{g}(\tau)$ is a given real function, $H(t)$ is the Heaviside function, $U_0^{(i)}$ is the amplitudes of incident acoustic wave, and $n_1=\cos\theta_\alpha,\ n_2=-\sin\theta_\alpha$. For convenience the field variables in the upper half-space ($x_2<0$) and lower space ($x_2>0$) are labeled by supermarks " $-$ " and " $+$," respectively. In order to apply the Wiener-Hopf technique an artificial interface $x_2=0, x_1<0$ is introduced. Using Eq. (6.176) the boundary conditions on the crack and the artificial surfaces are

$$\begin{aligned}&\sigma_{23}^{\pm}(x_1,0,t)=\sigma_{23}^{(i)}+\sigma_{23}^{\pm(s)}=0,\quad \varphi^{\pm}(x_1,0,t)=\varphi^{(i)}+\varphi^{\pm(s)}=0;\quad 0\le x_1<\infty;\\&u_3^{+}(x_1,0,t)=u_3^{-}(x_1,0,t),\quad D_2^{+}(x_1,0,t)=D_2^{-}(x_1,0,t);\quad x_1<0\end{aligned} \tag{6.179}$$

The initial and radiation conditions are as follows:

$$\begin{aligned}&u_3^{(s)}(x_1,x_2,t)=\dot{u}_3^{(s)}(x_1,x_2,t)=0,\quad \varphi^{(s)}(x_1,x_2,t)=\dot{\varphi}^{(s)}(x_1,x_2,t)=0;\quad t<0\\&\lim_{|x|\to\infty}\left[u_3^{(s)}(x_1,x_2,t),\quad \dot{u}_3^{(s)}(x_1,x_2,t),\quad \varphi^{(s)}(x_1,x_2,t),\quad \dot{\varphi}^{(s)}(x_1,x_2,t)\right]=0;\quad t>0\end{aligned} \tag{6.180}$$

6.9.3 Derive the Wiener-Hopf Equations in Laplace Transform Region

Introduce the one-side Laplace transform $\bar{f}(x_1,x_2,p)$ with respect to time of $f(x_1,x_2,t)$ and its inverse transform:

$$\bar{f}(x_1,x_2,p)=\int_0^\infty f(x_1,x_2,t)\mathrm{e}^{-pt}\,\mathrm{d}t,\quad f(x_1,x_2,t)=\frac{1}{2\pi i}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\bar{f}(x_1,x_2,p)\mathrm{e}^{pt}\,\mathrm{d}p \tag{6.181}$$

where $f(x_1, x_2, t)$ is called original function, $\bar{f}(x_1, x_2, p)$ is image function, and $p = \alpha + \mathrm{i}\beta$ is the Laplace transform complex parameter. $\bar{f}(x_1, x_2, p)$ is an analytic function in the plane $\mathrm{Re}\, p > \alpha_0$, where α_0 is the growth exponent of $f(t)$. The integral path in Eq. (6.181) is called Bromwich path. The two-side Laplace transform $\bar{f}^*(\varsigma, x_2, p)$ with respect to x_1 of $\bar{f}(x_1, x_2, p)$ is defined as

$$\begin{aligned} \bar{f}^*(\varsigma, x_2, p) &= \int_{-\infty}^{\infty} \bar{f}(x_1, x_2, p)\, \mathrm{e}^{-p\varsigma x_1} \mathrm{d}x_1 \\ f(x_1, x_2, p) &= \frac{p}{2\pi i} \int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty} \bar{f}^*(\varsigma, x_2, p)\, \mathrm{e}^{p\varsigma x_1} \mathrm{d}\varsigma \end{aligned} \tag{6.182}$$

Applying Eqs. (6.181), and (6.175), using the integral by parts and the initial and radiation conditions we find

$$\bar{u}^*_{3,22} - p^2\alpha^2\bar{u}^*_3 = 0, \quad \bar{\psi}^*_{,22} - p^2\beta^2\bar{\psi}^* = 0; \quad \alpha(\varsigma) = \sqrt{L_s^{*2} - \varsigma^2}, \quad \beta(\varsigma) = \sqrt{L_e^2 - \varsigma^2} \tag{6.183}$$

To satisfy the boundary conditions at infinity, the solution is chosen in the following form:

$$\left.\begin{aligned} \bar{u}_3^{*+}(\varsigma, x_2, p) &= (1/p^2)A^+(\varsigma)\mathrm{e}^{-p\alpha x_2} \\ \bar{\psi}^{*+}(\varsigma, x_2, p) &= (1/p^2)B^+(\varsigma)\mathrm{e}^{-p\beta x_2} \end{aligned}\right\}, x_2 > 0; \quad \left.\begin{aligned} \bar{u}_3^{*-} &= -(1/p^2)A^-(\varsigma)\mathrm{e}^{p\alpha x_2} \\ \bar{\psi}^{*-} &= -(1/p^2)B^-(\varsigma)\mathrm{e}^{p\beta x_2} \end{aligned}\right\}, x_2 < 0 \tag{6.184}$$

where $\mathrm{Re}\alpha(\varsigma) \geq 0, \mathrm{Re}\beta(\varsigma) \geq 0$ in the ζ plane with branch cuts on the $\mathrm{Im}\varsigma = 0$:

$$\text{For } \alpha: \mathrm{Re}\varsigma < -L_s^* \quad \text{and} \quad \mathrm{Re}\varsigma > L_s^*; \quad \text{For } \beta: \quad \mathrm{Re}\varsigma < -L_e \quad \text{and} \quad \mathrm{Re}\varsigma > L_e \tag{6.185}$$

Li et al. (2005a) adopted the Wiener-Hopf technique (Noble 1958; Zhu and Kuang 1995) to solve above problem. Introduce unknown functions:

$$\begin{aligned} &\sigma_-(x_1, t) = \begin{cases} \sigma_{23}^{\pm}, & x_1 < 0 \\ 0, & x_1 \geq 0 \end{cases}, \qquad \varphi_-(x_1, t) = \begin{cases} \varphi^{\pm}, & x_1 < 0 \\ 0, & x_1 \geq 0 \end{cases}, \\ &\Delta w_+(x_1, t) = \begin{cases} 0, & x_1 < 0 \\ u_3^+ - u_3^-, & x_1 \geq 0 \end{cases}, \quad \Delta D_+(x_1, t) = \begin{cases} 0, & x_1 < 0 \\ D_2^+ - D_2^-, & x_1 \geq 0 \end{cases} \\ &\sigma_{23}^{\pm} = \sigma_{23}^{\pm}(x_1, 0, t), \quad \varphi^{\pm} = \varphi^{\pm}(x_1, 0, t), \quad D_2^{\pm} = D_2^{\pm}(x_1, 0, t) \end{aligned} \tag{6.186}$$

So the boundary conditions can be expanded to the full range of the x_1-axis:

$$\begin{aligned} &\sigma_{23}^{\pm}(x_1, 0, t) = \sigma_-(x_1, t) - \sigma_{23}^{(i)}(x_1, 0, t), \quad \varphi^{\pm}(x_1, 0, t) = \varphi_-(x_1, t) - \varphi^{(i)}(x_1, 0, t), \\ &\quad -\infty < x_1 < \infty \\ &u_3^+(x_1, 0, t) - u_3^-(x_1, 0, t) = \Delta w_+(x_1, t), \quad D_2^+(x_1, 0, t) - D_2^-(x_1, 0, t) = \Delta D_+(x_1, t), \\ &\quad -\infty < x_1 < \infty \end{aligned} \tag{6.187}$$

The double Laplace transform of Eq. (6.187) is

$$\begin{aligned}
&\bar{\sigma}_{23}^{*\pm}(\varsigma,0,p)=\Sigma_{-}(\varsigma)/p-\bar{\sigma}_{23}^{*(i)}(\varsigma,0,p),\ \ \bar{\varphi}^{*\pm}(\varsigma,0,p)=\Phi_{-}(\varsigma)/p^{2}-\bar{\varphi}^{*(i)}(\varsigma,0,p)\\
&\frac{1}{2}\left[\bar{u}_{3}^{*+}(\varsigma,0,p)-\bar{u}_{3}^{*-}(\varsigma,0,p)\right]=\frac{\Delta U_{+}(\varsigma)}{p^{2}},\ \ \frac{1}{2}\left[D_{2}^{*+}(\varsigma,0,p)-D_{2}^{*-}(\varsigma,0,p)\right]=\frac{\Delta D_{+}(\varsigma)}{p^{2}}\\
&\Sigma_{-}(\varsigma)=p\int_{-\infty}^{0}\sigma_{-}^{*}(x_{1},p)\mathrm{e}^{-p\varsigma x_{1}}\mathrm{d}x_{1},\ \ \Phi_{-}(\varsigma)=p^{2}\int_{-\infty}^{0}\Phi_{-}^{*}(x_{1},p)\mathrm{e}^{-p\varsigma x_{1}}\mathrm{d}x_{1}\\
&\Delta U_{+}(\varsigma)=(p^{2}/2)\int_{0}^{\infty}\Delta w_{+}^{*}(x_{1},p)\mathrm{e}^{-p\varsigma x_{1}}\mathrm{d}x_{1},\ \ \Delta D_{+}(\varsigma)=(p/2)\int_{0}^{\infty}\Delta D_{+}^{*}(x_{1},p)\mathrm{e}^{-p\varsigma x_{1}}\mathrm{d}x_{1}
\end{aligned}\tag{6.188}$$

Substituting Eq. (6.184) and the transformed constitutive equation obtained from Eq. (6.176) into Eq. (6.188) we get

$$\begin{aligned}
&\bar{\sigma}_{23}^{*+}+\bar{\sigma}_{23}^{*-}:\ \ -C_{44}^{*}\alpha(\varsigma)A_{s}(\varsigma)-e_{15}\beta(\varsigma)B_{s}(\varsigma)=\Sigma_{-}(\varsigma)-p\bar{\sigma}_{23}^{*(i)}\\
&\bar{\varphi}^{*+}-\bar{\varphi}^{*-}:\ \ (e_{15}/\epsilon_{11})\hat{c}A_{s}(\varsigma)+B_{s}(\varsigma)=0\\
&\bar{u}_{3}^{*+}-\bar{u}_{3}^{*-}:\ \ A_{s}(\varsigma)=\Delta U_{+}(\varsigma)\\
&\bar{\sigma}_{23}^{*+}-\bar{\sigma}_{23}^{*-}:\ \ -C_{44}^{*}\alpha(\varsigma)A_{as}(\varsigma)-e_{15}\beta(\varsigma)B_{as}(\varsigma)=0\\
&\bar{\varphi}^{*+}+\bar{\varphi}^{*-}:\ \ (e_{15}/\epsilon_{11})\hat{c}A_{as}(\varsigma)+B_{as}(\varsigma)=\Phi_{-}(\varsigma)-p^{2}\bar{\varphi}^{*(i)}\\
&D_{2}^{*+}-D_{2}^{*-}:\ \ -e_{15}(1-\hat{c})\alpha(\varsigma)A_{as}(\varsigma)+\epsilon_{11}\beta(\varsigma)B_{as}(\varsigma)=\Delta D_{+}(\varsigma)\\
&A_{s}=(A^{+}+A^{-})/2,\ \ A_{as}=(A^{+}-A^{-})/2;\ \ B_{s}=(B^{+}+B^{-})/2,\ \ B_{as}=(B^{+}-B^{-})/2
\end{aligned}\tag{6.189}$$

From Eq. (6.189) two decoupled Wiener-Hopf equations can be obtained:

$$-C_{44}^{*}\mathcal{K}(\varsigma)\Delta U_{+}(\varsigma)=\Sigma_{-}(\varsigma)-p\bar{\sigma}_{23}^{*(i)};\quad \mathcal{K}(\varsigma)=\alpha(\varsigma)-k_{e}^{2}\beta(\varsigma)\tag{6.190a}$$

$$\frac{C_{44}^{*}\mathcal{K}(\varsigma)\Delta D_{+}(\varsigma)}{\alpha(\varsigma)\beta(\varsigma)\left[e_{15}^{2}(1-\hat{c})+\epsilon_{11}C_{44}^{*}\right]}=\Phi_{-}(\varsigma)-p^{2}\bar{\varphi}^{*(i)};\quad k_{e}^{2}=\frac{e_{15}^{2}}{\epsilon_{11}C_{44}^{*}}\hat{c}\tag{6.190b}$$

The $\bar{\sigma}_{23}^{*(i)}$ and $\bar{\varphi}^{*(i)}$ in Eqs. (6.190a) and (6.190b) are the double Laplace transform of $\sigma_{23}^{(i)}$ and $\varphi^{(i)}$, respectively, and for an incident acoustic wave are equal to

$$\begin{aligned}
&\bar{\sigma}_{23}^{*(i)}(\varsigma,0,p)=-\frac{\sigma_{0}\bar{g}(p)}{p\left(\varsigma+L_{s}^{*}n_{1}\right)};\quad \bar{\varphi}^{*(i)}(\varsigma,0,p)=-\frac{\varphi_{0}\bar{g}(p)}{p^{2}\left(\varsigma+L_{s}^{*}n_{1}\right)}\\
&\sigma_{0}=C_{44}^{*}L_{s}^{*}n_{2}u_{30}^{(i)};\quad \varphi_{0}=-(e_{15}/\epsilon_{11})\hat{c}u_{30}^{(i)}
\end{aligned}\tag{6.191}$$

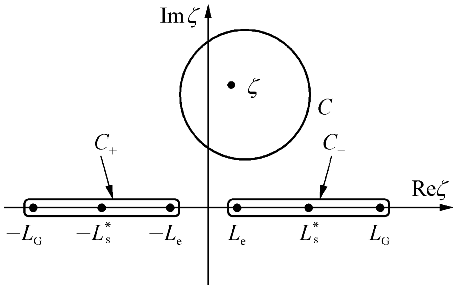


Fig. 6.19 Integration paths used for product decomposition of $H(\varsigma)$ in ζ plane

6.9.4 Decomposing of the Function $\mathcal{K}(\varsigma)$

In order to solve Eqs. (6.190) and (6.191), it is needed to factorize the function $\mathcal{K}(\xi)$ into sectionally analytic functions in the left and right half ς plane, respectively. Let (Li 2000)

$$\begin{aligned} &\mathcal{K}(\varsigma) = \alpha(\varsigma) - k_e^2\beta(\varsigma) = \left(1 - k_e^4\right)\frac{\left(L_G^2 - \varsigma^2\right)}{\alpha(\varsigma) + k_e^2\beta(\varsigma)}, \quad L_G = \sqrt{\frac{L_s^{*2} - k_e^4 L_e^2}{1 - k_e^4}} \\ &\alpha(\varsigma) + k_e^2\beta(\varsigma) = \left(1 + k_e^2\right)\sqrt{L_G^2 - \varsigma^2}\,\Omega(\varsigma), \quad \Omega(\varsigma) = \frac{\alpha(\varsigma) + k_e^2\beta(\varsigma)}{\left(1 + k_e^2\right)\sqrt{L_G^2 - \varsigma^2}} \end{aligned} \tag{6.192}$$

When $|\varsigma| \to \infty$, $\alpha \to \beta = \sqrt{-\varsigma^2}$ and $\Omega(\varsigma) \to 1$. Factorize $\Omega(\varsigma)$ into sectionally analytic functions $\Omega_+(\varsigma)$ and $\Omega_-(\varsigma)$ and $\Omega(\varsigma) = \Omega_+(\varsigma)\Omega_-(\varsigma)$. Let

$$\ln\,\Omega(\varsigma) = \ln\,\Omega_+(\varsigma) + \ln\,\Omega_-(\varsigma) = \frac{1}{2\pi \mathrm{i}}\oint_C \frac{\ln\,\Omega(z)}{z - \varsigma}\mathrm{d}z \tag{6.193}$$

where C is the integration path located in the ς plane with cuts C_+ and C_- (Fig. 6.19). There are three branch points inside C_+ or C_-.

Using the Cauchy principle value (PV) integration around C_+, it is obtained

$$\begin{aligned} &\Theta(\varsigma) = \arg[\Omega(\varsigma)] = \begin{cases} \pm\pi/2, & -L_G < \mathrm{Re}\varsigma < -L_s^*, \quad \mathrm{Im}\varsigma = \pm 0 \\ \pm\arctan\,\Xi(\varsigma), & -L_s^* < \mathrm{Re}\varsigma < -L_e, \quad \mathrm{Im}\varsigma = \pm 0 \end{cases} \\ &\Xi(\varsigma) = \frac{k_e^2\sqrt{(\varsigma - L_e)(\varsigma + L_e)}}{\sqrt{\left(L_s^* - \varsigma\right)\left(L_s^* + \varsigma\right)}} = \frac{k_e^2\sqrt{\left(\varsigma^2 - L_e^2\right)}}{\sqrt{\left(L_s^{*2} - \varsigma^2\right)}} \end{aligned} \tag{6.194}$$

By using the Cauchy's integral theorem, it is obtained

$$\Omega_\pm(\varsigma) = \sqrt{\frac{L_s^* \pm \varsigma}{L_G \pm \varsigma}}\exp\left\{-\frac{1}{\pi}\int_{L_e}^{L_s^*}\arctan[\Xi(\eta)]\frac{\mathrm{d}\eta}{\eta \pm \varsigma}\right\} \tag{6.195}$$

where $\Omega_{\pm}(\varsigma)$ is corresponding to the notations " $\pm$ " at right hand of the equality, respectively. Therefore, it yields

$$\begin{aligned}
&\alpha(\varsigma)+k_e^2\beta(\varsigma)=\left(1+k_e^2\right)\sqrt{L_s^{*2}-\varsigma^2}M_+(\varsigma)M_-(\varsigma)\\
&\mathcal{K}(\varsigma)=\left(1-k_e^4\right)\frac{\left(L_G^2-\varsigma^2\right)}{\alpha(\varsigma)+k_e^2\beta(\varsigma)}=\left(1-k_e^2\right)\frac{\left(L_G^2-\varsigma^2\right)}{\sqrt{L_s^{*2}-\varsigma^2}}S_+(\varsigma)S_-(\varsigma)\\
&M_{\pm}(\varsigma)=\exp\left[-\frac{1}{\pi}\int_{L_e}^{L_s^*}\arctan\Xi(\eta)\frac{\mathrm{d}\eta}{\eta\pm\varsigma}\right],\quad S_{\pm}(\varsigma)=\left[M_{\pm}(\varsigma)\right]^{-1}
\end{aligned}\tag{6.196}$$

6.9.5 Solutions of the Wiener-Hopf Equations

Substitution of Eqs. (6.196) and (6.191) into Eq. (6.190a) yields

$$-C_{44}^{**}\frac{\left(L_G^2-\varsigma^2\right)}{\sqrt{L_s^{*2}-\varsigma^2}}\Delta U_+(\varsigma)S_+(\varsigma)S_+(\varsigma)=\Sigma_-(\varsigma)+\frac{\sigma_0\bar{g}(p)}{\left(\varsigma+L_s^*n_1\right)},\quad C_{44}^{**}=C_{44}^*\left(1-k_e^2\right)\tag{6.197}$$

Introduce

$$R_-(\varsigma)=\frac{\sqrt{L_s^*-\varsigma}}{(L_G-\varsigma)S_-(\varsigma)},\quad \frac{R_-(\varsigma)}{\varsigma+L_s^*n_1}=\frac{R_-(\varsigma)-R_-\left(-L_s^*n_1\right)}{\varsigma+L_s^*n_1}+\frac{R_-\left(-L_s^*n_1\right)}{\varsigma+L_s^*n_1}\tag{6.198}$$

Equations (6.197) and (6.198) yield

$$\begin{aligned}
&-C_{44}^{**}\frac{(L_G+\varsigma)}{\sqrt{L_s^*+\varsigma}}\Delta U_+(\varsigma)S_+(\varsigma)-\frac{\sigma_0\bar{g}(p)R_-\left(-L_s^*n_1\right)}{\left(\varsigma+L_s^*n_1\right)}\\
&=\Sigma_-(\varsigma)R_-(\varsigma)+\frac{\sigma_0\bar{g}(p)\left[R_-(\varsigma)-R_-\left(-L_s^*n_1\right)\right]}{\left(\varsigma+L_s^*n_1\right)}
\end{aligned}\tag{6.199}$$

It is known that the functions at the left side in Eq. (6.199) are analytic in the right half-plane $\mathrm{Re}\varsigma>0$ and equal to zero at infinity, whereas those on the right side are analytic in the left half-plane $\mathrm{Re}\varsigma<0$ and they are continuous on $\mathrm{Im}\varsigma=0$. So according to Liouville theorem (Lavrenchive and Shabat 1951), these functions are analytic in whole plane and must be zero. So

$$\begin{aligned}
\Delta U_+(\varsigma)&=-\frac{\sigma_0\bar{g}(p)\sqrt{L_s^*+\varsigma}R_-\left(-L_s^*n_1\right)}{C_{44}^{**}(L_G+\varsigma)\left(\varsigma+L_s^*n_1\right)S_+(\varsigma)},\\
\Sigma_-(\varsigma)&=\frac{\sigma_0\bar{g}(p)}{\left(\varsigma+L_s^*n_1\right)}\left[\frac{R_-\left(-L_s^*n_1\right)}{R_-(\varsigma)}-1\right]
\end{aligned}\tag{6.200}$$

Analogously from Eqs. (6.196) and (6.190b), it can be obtained

$$\begin{aligned} \Delta D_+(\varsigma) &= -\frac{\varphi_0 \bar{g}(p)\left(L_s^* + \varsigma\right)\sqrt{L_e + \varsigma} R'_-\left(-L_s^* n_1\right)\left[e_{15}^2(1-\hat{c}) + \epsilon_{11} C_{44}^*\right]}{C_{44}^{**}(L_G + \varsigma)\left(\varsigma + L_s^* n_1\right) S_+(\varsigma)} \\ \Phi_-(\varsigma) &= \frac{\varphi_0 \bar{g}(p)}{\left(\varsigma + L_s^* n_1\right)}\left[\frac{R'_-\left(-L_s^* n_1\right)}{R'_-(\varsigma)} - 1\right]; \quad R'_-(\varsigma) = \frac{\left(L_s^* + \varsigma\right)\sqrt{L_e + \varsigma}}{(L_G + \varsigma) S_-(\varsigma)} \end{aligned} \tag{6.201}$$

Substituting Eqs. (6.200) and (6.201) into Eq. (6.189), one can obtain $A_s(\varsigma), B_s(\varsigma)$, $A_{as}, B_{as}(\varsigma)$ and $A^{\pm}(\varsigma), B^{\pm}(\varsigma)$:

$$\begin{aligned} A^{\pm}(\varsigma) &= -[\sigma_0 \pm \varphi_0 A_0(\varsigma)]\bar{g}(p)\Lambda(\varsigma) \\ B^{\pm}(\varsigma) &= [\sigma_0 \hat{c}(e_{15}/\epsilon_{11}) \pm \varphi_0 B_0(\varsigma)]\bar{g}(p)\Lambda(\varsigma) \\ A_0(\varsigma) &= \frac{e_{15}\sqrt{L_e + \varsigma}\sqrt{L_e + L_s^* n_1}\sqrt{L_s^* + L_s^* n_1}}{\sqrt{L_s^* - \varsigma}} \\ B_0(\varsigma) &= \frac{C_{44}^*\sqrt{L_s^* + \varsigma}\sqrt{L_s^* + L_t n_1}\sqrt{L_e + L_s^* n_1}}{\sqrt{L_e - \varsigma}} \\ \Lambda(\varsigma) &= \frac{(L_G - \varsigma)\sqrt{L_s^* + L_s^* n_1} S_-(\varsigma)}{\left(\varsigma + L_s^* n_1\right)\sqrt{L_s^* - \varsigma}(L_G + L_s^* n_1) S_-\left(-L_s^* n_1\right) C_{44}^* \mathcal{K}(\varsigma)} \end{aligned} \tag{6.202}$$

Substituting Eq. (6.202) into (6.184) and carrying out the inverse transform with ς we find

$$\begin{aligned} \bar{u}_3(x_1, x_2, p) &= -\frac{1}{2\pi \mathrm{i} p}\int_{\varsigma_\alpha - \mathrm{i}\infty}^{\varsigma_\alpha + \mathrm{i}\infty}\{[\sigma_0 + \varphi_0 A_0(\varsigma)\mathrm{sgn}(x_2)]\bar{g}(p)\Lambda(\varsigma)\} \\ &\quad \times \exp\left\{-p\left[\alpha(\varsigma)\mathrm{sgn}(x_2)x_2 - \varsigma x_{1,}\right]\right\}\mathrm{d}\varsigma \\ \bar{\psi}(x_1, x_2, p) &= \frac{1}{2\pi \mathrm{i} p}\int_{\varsigma_\beta - \mathrm{i}\infty}^{\varsigma_\beta + \mathrm{i}\infty}\left\{\left[\sigma_0 \hat{c}\frac{e_{15}}{\epsilon_{11}} + \varphi_0 B_0(\varsigma)\mathrm{sgn}(x_2)\right]\bar{g}(p)\Lambda(\varsigma)\right\} \\ &\quad \times \exp\{-p[\beta(\varsigma)\mathrm{sgn}(x_2)x_2 - \varsigma x_1]\}\mathrm{d}\varsigma \end{aligned} \tag{6.203}$$

6.9.6 *Scattering Fields in Front of the Crack Tip*

The one-side Laplace inversion with time can be obtained by replacing the original Bromwich path with deformed Cagniard-de Hoop inversion contours (Fig. 6.20) in the ς plane (Li et al. 2005a). The inversion procedure is given only for $x_2 > 0$; for $x_2 < 0$ the procedure is the same and is omitted. Along the Cagniard-de Hoop contours, the exponentials in Eq. (6.203) take the form e^{-pt}:

$$\alpha(\varsigma)x_2 - \varsigma x_1 = t, \quad \varsigma \epsilon \Gamma_\alpha, \Gamma_{\alpha\beta}; \quad \beta(\varsigma)x_2 - \varsigma x_1 = t, \quad \varsigma \epsilon \Gamma_\beta \tag{6.204}$$

A cut from $-L_s^*$ to $-L_e$ is needed due to two branch points $\varsigma = -L_s^*$ and $\varsigma = -L_e$. So a supplement path $\Gamma_{\alpha\beta}$ is needed to avoid the cut for Γ_α. The physical

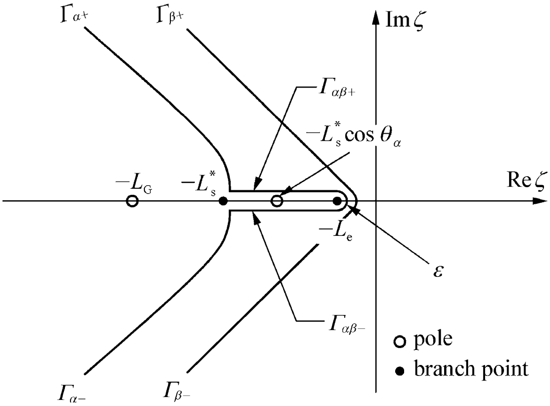


Fig. 6.20 The deformed Cagniard-de Hoop inversion paths $\Gamma_\alpha, \Gamma_\beta, \Gamma_{\alpha\beta}$

interpretation of $\Gamma_{\alpha\beta}$ is that the integral along $\Gamma_{\alpha\beta}\left(-L_s^* \cos\theta_\alpha \le \varsigma \le -L_e\right)$ represents an electroacoustic head wave, or a quasi-surface wave, which almost propagates in parallel with the boundary surface. The electroacoustic head wave in piezoelectric material was proved by experiments of Liu et al. (1989). The path Γ_β always avoids the cut. Let $x_1 = r\cos\theta,\ x_2 = r\sin\theta$ from Eq. (6.204) we get

$$\varsigma_{\alpha\pm} = -t\cos\theta \pm \mathrm{i}\sin\theta\sqrt{t^2 - L_s^{*2}r^2}/r;\quad L_s^* r \le t < \infty$$
$$\varsigma_{\beta\pm} = -t\cos\theta \pm \mathrm{i}\sin\theta\sqrt{t^2 - L_e^2 r^2}/r;\quad L_e r \le t < \infty$$
$$\varsigma_{\alpha\beta\pm} = -t\cos\theta \pm \sin\theta\sqrt{L_s^{*2}r^2 - t^2}/r \pm \mathrm{i}\varepsilon;\quad t_{\alpha 0} \le t < L_s^* r;\quad t_{\alpha 0} = \sqrt{L_s^{*2} - L_e^2}x_2 + L_e x_1$$

Finally, the exact inversions are found:

$$\begin{aligned} u_3^{(s)}(x_1, x_2, t) &= \int_0^t G(t-\tau) u_{3\delta}^{(s)}(x_1, x_2, \tau)\mathrm{d}\tau + u_{3r}^{(s)}(x_1, x_2, t) \\ \psi^{(s)}(x_1, x_2, t) &= \int_0^t G(t-\tau)\psi_\delta^{(s)}(x_1, x_2, \tau)\mathrm{d}\tau + \psi_r^{(s)}(x_1, x_2, t) \end{aligned} \tag{6.205}$$

where $u_{3\delta}^{(s)}, \psi_\delta^{(s)}$ denote the scattering fields due to the impulsive incident wave and $u_{3r}^{(s)}, \psi_r^{(s)}$ represent the reflective and transmission waves:

$$\begin{aligned} u_{3\delta}^{(s)} &= -\frac{1}{\pi}\mathrm{Re}\left\{(\sigma_0 + \varphi_0 A_0(\varsigma_{\alpha+})\mathrm{sgn}(x_2))\Lambda(\varsigma_{\alpha+})\frac{\alpha(\varsigma_{\alpha+})}{\sqrt{t^2 - L_s^{*2}r^2}}\right\} H\left(t - L_s^* r\right) \\ &\quad + \frac{1}{\pi}\mathrm{Im}\left\{\left(\sigma_0 + \varphi_0 A_0(\varsigma_{\alpha\beta+})\mathrm{sgn}(x_2)\right)\Lambda(\varsigma_{\alpha\beta+})\frac{\alpha(\varsigma_{\alpha\beta+})}{\sqrt{t^2 - L_s^{*2}r^2}}\right\}\left[H(t - t_0) - H\left(t - L_s^* r\right)\right] \\ \psi_\delta^{(s)} &= \frac{1}{\pi}\left\{\left(\sigma_0 \hat{c}\frac{e_{15}}{\epsilon_{11}} + \varphi_0 B_0(\varsigma_{\beta+})\mathrm{sgn}(x_2)\right)\Lambda(\varsigma_{\beta+})\frac{\beta(\varsigma_{\beta+})}{\sqrt{t^2 - L_e^2 r^2}}\right\} H(t - L_e r) \\ t_0 &= \sqrt{L_s^{*2} - L_e^2}x_2 + L_e x_1, \quad r = \sqrt{x_1^2 + x_2^2} \end{aligned} \tag{6.206}$$

$$u_{3r}^{(s)} = \begin{cases} [\mathrm{Re}\Pi(\theta_\alpha)]U_0^{(i)}G(t_*) - [\mathrm{Im}\Pi(\theta_\alpha)]U_0^{(i)}\left\{\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{G(t_*)}{\tau - t}\mathrm{d}\tau\right\}, & 0 \le \theta < \theta_\alpha \\ 0, & \theta_\alpha \le \theta < 2\pi - \theta_\alpha \\ -U_0^{(i)}G(t_*), & 2\pi - \theta_\alpha \le \theta < 2\pi \end{cases}$$
$$\varphi_r^{(s)}(x_1, x_2, t) = (e_5/\epsilon_{11})\hat{c}u_3^{(r)}(x_1, x_2, t), \qquad 0 \le \theta < 2\pi$$
$$\Pi(\theta_\alpha) = \frac{L_s^* n_2 - k_e\sqrt{L_e^2 - L_s^{*2}n_1^2}}{L_s^* n_2 + k_e\sqrt{L_e^2 - L_s^{*2}n_1^2}}, \quad t_* = t - L_s^*(n_1 x_1 - n_2 x_2) \tag{6.207}$$

For an incident shear wave with $\cos\theta_\alpha < L_e/L_s^*$ through a semi-infinite conducting crack in a transversely isotropic piezoelectric material, in front of the crack tip except the incident acoustic wave, there have been electroacoustic reflection, transmission, scattering and head waves, and electric scattering wave. Because in the "quasi-hyperbolic approximation" the Faraday's electric induction by a changing magnetic field is not considered, it cannot be used to solve the problem for $-L_s^*\cos\theta_\alpha > -L_e$ and the electric incident wave.

6.10 Transient Response of a Mode-I Crack

6.10.1 Fundamental Equations

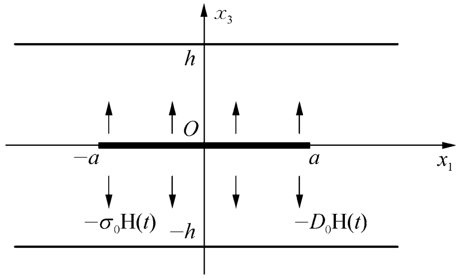
Figure 6.21 shows a transverse isotropic piezoelectric strip of width $2h$ with a central crack of length $2a$ subjected to loadings $-\sigma_0 H(t), -D_0 H(t)$ on the crack surface, where $H(t)$ is the Heaviside step function. Axis x_1 is along the crack direction and the polarized axis x_3 is perpendicular to the crack. In plane $x_1 x_3$ there are generalized displacements (u_1, u_3, φ) and stresses $(\sigma_1, \sigma_3, \sigma_5, D_1, D_3)$. Applying the Voigt notations the constitutive equations are

$$\sigma_1 = C_{11}\varepsilon_1 + C_{13}\varepsilon_3 - e_{31}E_3, \quad \sigma_3 = C_{13}\varepsilon_1 + C_{33}\varepsilon_3 - e_{33}E_3, \quad \sigma_5 = C_{44}\varepsilon_5 - e_{15}E_1$$
$$D_1 = \epsilon_{11}E_1 + e_{15}\varepsilon_5, \quad D_3 = \epsilon_{33}E_3 + e_{31}\varepsilon_1 + e_{33}\varepsilon_3 \tag{6.208}$$

The generalized momentum equations in displacements are

$$\begin{aligned} &C_{11}u_{1,11} + C_{44}u_{1,33} + (C_{13} + C_{44})u_{3,13} + (e_{31} + e_{15})\varphi_{,13} = \rho\ddot{u}_1 \\ &(C_{13} + C_{44})u_{1,13} + C_{44}u_{3,11} + C_{33}u_{3,33} + e_{15}\varphi_{,11} + e_{33}\varphi_{,33} = \rho\ddot{u}_3 \\ &(e_{31} + e_{15})u_{1,13} + e_{15}u_{3,11} + e_{33}u_{3,33} - \epsilon_{11}\varphi_{,11} - \epsilon_{33}\varphi_{,33} = 0 \end{aligned} \tag{6.209}$$

Fig. 6.21 A piezoelectric strip with a crack



The mechanical and electric impermeable conditions are

$$\begin{aligned}
&\sigma_3(x_1, 0, t) = -\sigma_0 H(t), \quad D_3(x_1, 0, t) = -D_0 H(t), \quad -a < x_1 < a \\
&u_3(x_1, 0, t) = \varphi(x_1, 0, t) = 0, \quad a < |x_1| < \infty, \quad \sigma_5(x_1, 0, t) = 0, \quad -\infty < x_1 < \infty \\
&\sigma_3(x_1 \pm h/2, t) = \sigma_5(x_1, \pm h/2, t) = D_3(x_1, \pm h/2, t) = 0, \quad -\infty < x_1 < \infty
\end{aligned} \tag{6.210}$$

When the derivatives of variables are zeros at the initial time, the Laplace transform (Eq. 6.181) of governing equations (6.208), (6.209), and (6.210) are

$$\begin{aligned}
&\bar{\sigma}_1 = C_{11}\bar{\varepsilon}_1 + C_{13}\bar{\varepsilon}_3 - e_{31}\bar{E}_3, \quad \bar{\sigma}_3 = C_{13}\bar{\varepsilon}_1 + C_{33}\bar{\varepsilon}_3 - e_{33}\bar{E}_3, \quad \bar{\sigma}_5 = C_{44}\bar{\varepsilon}_5 - e_{15}\bar{E}_1 \\
&\bar{D}_1 = \epsilon_{11}\bar{E}_1 + e_{15}\bar{\varepsilon}_5, \quad \bar{D}_3 = \epsilon_{33}\bar{E}_3 + e_{31}\bar{\varepsilon}_1 + e_{33}\bar{\varepsilon}_3
\end{aligned} \tag{6.211}$$

$$\begin{aligned}
&C_{11}\bar{u}_{1,11} + C_{44}\bar{u}_{1,33} + (C_{13} + C_{44})\bar{u}_{3,13} + (e_{31} + e_{15})\bar{\varphi}_{,13} = \rho p^2 \bar{u}_1 \\
&(C_{13} + C_{44})\bar{u}_{1,13} + C_{44}\bar{u}_{3,11} + C_{33}\bar{u}_{3,33} + e_{15}\bar{\varphi}_{,11} + e_{33}\bar{\varphi}_{,33} = \rho p^2 \bar{u}_3 \\
&(e_{31} + e_{15})\bar{u}_{1,13} + e_{15}\bar{u}_{3,11} + e_{33}\bar{u}_{3,33} - \epsilon_{11}\bar{\varphi}_{,11} - \epsilon_{33}\bar{\varphi}_{,33} = 0
\end{aligned} \tag{6.212}$$

$$\begin{aligned}
&\bar{\sigma}_3 = -\sigma_0/p, \quad \bar{D}_2 = -D_0/p, \quad -a < x_1 < a, \quad x_3 = 0 \\
&\bar{u}_3 = \bar{\varphi} = 0, \quad a < |x_1| < \infty, \quad x_3 = 0, \quad \bar{\sigma}_5 = 0, \quad -\infty < x_1 < \infty, \quad x_3 = 0 \\
&\bar{\sigma}_3 = \bar{\sigma}_5 = \bar{D}_2 = 0, \quad -\infty < x_1 < \infty, \quad x_3 = \pm h/2
\end{aligned} \tag{6.213}$$

6.10.2 Reduction to Singular Integration Equations

Because the problem is symmetric with respect to $x_3 = 0$, it is only needed to consider the upper part. In the Laplace transform region, Wang and Yu (2001) adopted the solutions in the following Fourier integrals:

$$\begin{aligned}
\bar{u}_1 &= (2/\pi) \sum_{j=1}^{6} \int_0^\infty q_j A_j(\xi) e^{-\gamma_j x_3} \sin(\xi x_1)\, \mathrm{d}\xi \\
\bar{u}_3 &= (2/\pi) \sum_{j=1}^{6} \int_0^\infty a_j A_j(\xi) e^{-\gamma_j x_3} \cos(\xi x_1)\, \mathrm{d}\xi \\
\bar{\varphi} &= (2/\pi) \sum_{j=1}^{6} \int_0^\infty b_j A_j(\xi) e^{-\gamma_j x_3} \cos(\xi x_1)\, \mathrm{d}\xi
\end{aligned} \tag{6.214}$$

where $A_j(\xi)(j=1-6)$ are unknown functions. Substitution of Eq. (6.214) into Eq. (6.212) yields

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{j=1}^{6}\int_0^\infty A_j(\xi)e^{-\gamma_j x_3}\sin(\xi x_1)\\
&\quad\times\left[\left(-C_{11}\xi^2+C_{44}\gamma_j^2-\rho p^2\right)q_j+(C_{13}+C_{44})\xi\gamma_j a_j+(e_{31}+e_{15})\gamma_j b_j\xi\right]\mathrm{d}\xi=0\\
&\frac{2}{\pi}\sum_{j=1}^{6}\int_0^\infty A_j(\xi)e^{-\gamma_j x_3}\cos(\xi x_1)\\
&\quad\times\left[-(C_{13}+C_{44})\xi\gamma_j q_j+\left(C_{33}\gamma_j^2-C_{44}\xi^2-\rho p^2\right)a_j+\left(e_{33}\gamma_j^2-e_{15}\xi^2\right)b_j\right]\mathrm{d}\xi=0\\
&\frac{2}{\pi}\sum_{j=1}^{6}\int_0^\infty A_j(\xi)e^{-\gamma_j x_3}\cos(\xi x_1)\\
&\quad\times\left[-(e_{31}+e_{15})\xi\gamma_j q_j+\left(e_{33}\gamma_j^2-\xi^2 e_{15}\right)a_j+\left(\epsilon_{11}\xi^2-\epsilon_{33}\gamma_j^2\right)b_j\right]\mathrm{d}\xi=0
\end{aligned}
\tag{6.215}
$$

From Eq. (6.215) it is obtained

$$
\begin{aligned}
&\left(-C_{11}\xi^2+C_{44}\gamma_j^2-\rho p^2\right)q_j+(C_{13}+C_{44})\xi\gamma_j a_j+(e_{31}+e_{15})\gamma_j b_j\xi=0\\
&\quad-(C_{13}+C_{44})\xi\gamma_j q_j+\left(C_{33}\gamma_j^2-C_{44}\xi^2-\rho p^2\right)a_j+\left(e_{33}\gamma_j^2-e_{15}\xi^2\right)b_j=0\\
&\quad-(e_{31}+e_{15})\xi\gamma_j q_j+\left(e_{33}\gamma_j^2-\xi^2 e_{15}\right)a_j+\left(\epsilon_{11}\xi^2-\epsilon_{33}\gamma_j^2\right)b_j=0
\end{aligned}
\tag{6.216}
$$

So the coefficient determinant of q_j, a_j, b_j must be zero, i.e.,

$$
\begin{vmatrix}
C_{44}\gamma^2-C_{11}\xi^2-\rho p^2 & (C_{13}+C_{44})\xi\gamma & (e_{31}+e_{15})\gamma\xi\\
-(C_{13}+C_{44})\xi\gamma & C_{33}\gamma^2-C_{44}\xi^2-\rho p^2 & e_{33}\gamma^2-e_{15}\xi^2\\
-(e_{31}+e_{15})\xi\gamma & e_{33}\gamma^2-\xi^2 e_{15} & \epsilon_{11}\xi^2-\epsilon_{33}\gamma^2
\end{vmatrix}=0
\tag{6.217}
$$

From Eq. (6.217) it can be obtained $\gamma_j(j=1-6)$. For convenience let $q_j=1$ in Eq. (6.214), which has not lost the generality, and a_j, b_j can be determined from Eq. (6.216):

$$
\begin{aligned}
&a_j=\frac{\Delta_a(\gamma_j)}{\Delta_0(\gamma_j)},\quad b_j=\frac{\Delta_b(\gamma_j)}{\Delta_0(\gamma_j)}\\
&\Delta_a(\gamma_j)=\left(C_{11}\xi^2-C_{44}\gamma_j^2+\rho p^2\right)\left(e_{33}\gamma_j^2-e_{15}\xi^2\right)-(C_{13}+C_{44})(e_{31}+e_{15})\xi^2\gamma_j^2\\
&\Delta_b(\gamma_j)=(C_{13}+C_{44})^2\xi^2\gamma_j^2-\left(C_{11}\xi^2-C_{44}\gamma_j^2+\rho p^2\right)\left(C_{33}\gamma_j^2-C_{44}\xi^2-\rho p^2\right)\\
&\Delta_0(\gamma_j)=\left[(C_{13}+C_{44})\left(e_{33}\gamma_j^2-e_{15}\xi^2\right)-(e_{31}+e_{15})\left(C_{33}\gamma_j^2-C_{44}\xi^2-\rho p^2\right)\right]\xi\gamma_j
\end{aligned}
\tag{6.218}
$$

$A_j(\xi)(j=1-6)$ is determined by the boundary conditions.

Introduce the half of the generalized dislocation density:

$$f(x_1,p)=\begin{cases}\bar{u}^{+}_{3,1}, & -a<x_1<a\\ 0, & a\le|x_1|<\infty\end{cases},\quad g(x_1,p)=\begin{cases}\bar{\varphi}^{+}_{,1}, & -a<x_1<a\\ 0, & a\le|x_1|<\infty\end{cases} \tag{6.219a}$$

Around the crack the single-valued conditions are

$$\int_a^b\left(\bar{u}^{+}_{3,1}-\bar{u}^{-}_{3,1}\right)\mathrm{d}u\Rightarrow\int_a^b f(u,p)\mathrm{d}u=0,\quad \int_a^b g(u,p)\mathrm{d}u=0 \tag{6.219b}$$

Substituting Eq. (6.214) into (6.211), then into the boundary conditions (6.213), and applying conditions (6.219), in the interval $0<x_1<a$, the following singular integral equations are obtained (Erdogan and Gupta 1972; Erdogan 1975; Lu 1984):

$$\begin{aligned}&\frac{\alpha_1}{\pi}\int_0^a\frac{f(u,p)}{u-x_1}\mathrm{d}u+\frac{\alpha_2}{\pi}\int_0^a\frac{g(u,p)}{u-x_1}\mathrm{d}u+\frac{1}{\pi}\int_0^a[Q_{11}(u,x_1)f(u,p)+Q_{12}(u,x_1)g(u,p)]\mathrm{d}u=-\frac{\sigma_0}{p}\\&\frac{\alpha_3}{\pi}\int_0^a\frac{f(u,p)}{u-x_1}\mathrm{d}u+\frac{\alpha_4}{\pi}\int_0^a\frac{g(u,p)}{u-x_1}\mathrm{d}u+\frac{1}{\pi}\int_0^a[Q_{21}(u,x_1)f(u,p)+Q_{22}(u,x_1)g(u,p)]\mathrm{d}u=-\frac{D_0}{p}\end{aligned} \tag{6.220}$$

where

$$\begin{aligned}&Q_{ij}(u,x_1)=\int_0^\infty 2\left[P_{ij}(\xi,p)-\alpha_{ij}\right]\cos(\xi x_1)\sin(\xi u)\,\mathrm{d}\xi+\frac{\alpha_{ij}}{u+x_1},\quad i,j=1,2\\&\alpha_{11}=\alpha_1,\quad \alpha_{12}=\alpha_2,\quad \alpha_{21}=\alpha_3,\quad \alpha_{22}=\alpha_4\end{aligned} \tag{6.221}$$

And

$$\begin{aligned}&P_{11}(\xi,p)=\sum_{j=1}^{6}\frac{C_{33}\gamma_j a_j+e_{33}\gamma_j b_j-C_{13}\xi}{\xi\Delta(\xi,p)}\Delta_{j1}(\xi,p)\\&P_{12}(\xi,p)=\sum_{j=1}^{6}\frac{C_{33}\gamma_j a_j+e_{33}\gamma_j b_j-C_{13}\xi}{\xi\Delta(\xi,p)}\Delta_{j2}(\xi,p)\\&P_{21}(\xi,p)=\sum_{j=1}^{6}\frac{e_{33}\gamma_j a_j+\epsilon_{33}\gamma_j b_j-e_{13}\xi}{\xi\Delta(\xi,p)}\Delta_{j1}(\xi,p)\\&P_{22}(\xi,p)=\sum_{j=1}^{6}\frac{e_{33}\gamma_j a_j+\epsilon_{33}\gamma_j b_j-e_{13}\xi}{\xi\Delta(\xi,p)}\Delta_{j2}(\xi,p)\\&\Delta(\xi,p)=\det[M_{ij}],\quad M_{1j}=a_j,\quad M_{2j}=b_j\\&M_{3j}=-\left[C_{44}(\gamma_j+a_j\xi)+e_{15}b_j\xi\right],\quad M_{4j}=-\left[C_{44}(\gamma_j+a_j\xi)+e_{15}b_j\xi\right]e^{-\gamma_j h}\\&M_{5j}=\left[C_{31}\xi-C_{33}\gamma_j a_j-e_{33}\gamma_j b_j\right]e^{-\gamma_j h},\quad M_{6j}=\left[e_{31}\xi-e_{33}\gamma_j a_j+\epsilon_{33}\gamma_j b_j\right]e^{-\gamma_j h}\end{aligned} \tag{6.222}$$

where $\Delta_{jn}(j=1-6,n=1,2)$ is the complementary minor of matrix $|M_{ij}|$ with respect to the component M_{nj}.

6.10.3 Solutions

In order to use the standard numerical method, introduce the dimensionless variables:

$$\frac{u}{a} = \frac{\rho + 1}{2}, \quad \frac{x_1}{a} = \frac{r + 1}{2} \tag{6.223}$$

Equation (6.220) is rewritten as

$$\begin{aligned}
&\frac{\alpha_1}{\pi}\int_{-1}^{1}\frac{F(\rho,p)}{\rho - r}\mathrm{d}\rho + \frac{\alpha_2}{\pi}\int_{-1}^{1}\frac{V(\rho,p)}{\rho - r}\mathrm{d}\rho + \frac{1}{\pi}\int_{0}^{a}\left[\hat{Q}_{11}(\rho,r)F(\rho,p) + \hat{Q}_{12}(\rho,r)V(\rho,p)\right]\mathrm{d}\rho = -\frac{\sigma_0}{p}\\
&\frac{\alpha_3}{\pi}\int_{-1}^{1}\frac{F(\rho,p)}{\rho - r}\mathrm{d}\rho + \frac{\alpha_4}{\pi}\int_{-1}^{1}\frac{V(\rho,p)}{\rho - r}\mathrm{d}\rho + \frac{1}{\pi}\int_{0}^{a}\left[\hat{Q}_{21}(\rho,r)F(\rho,p) + \hat{Q}_{22}(\rho,r)V(\rho,p)\right]\mathrm{d}\rho = -\frac{D_0}{p}\\
&-1 < r < 1
\end{aligned} \tag{6.224}$$

where

$$\begin{aligned}
&F(\rho,p) = f\left(\frac{\rho + 1}{2}a, p\right), \quad V(\rho,p) = \mathrm{g}\left(\frac{\rho + 1}{2}a, p\right),\\
&\hat{Q}_{ij}(\rho,r) = \frac{a}{2}Q_{ij}\left(\frac{\rho + 1}{2}a, \frac{r + 1}{2}a\right)
\end{aligned} \tag{6.225}$$

Let

$$\begin{aligned}
&F(\rho,p) = \frac{R(\rho,p)}{\sqrt{1 - \rho^2}}, \quad V(\rho,p) = \frac{T(\rho,p)}{\sqrt{1 - \rho^2}}\\
&R(\rho,p) = \sum_{i=0}^{\infty} C_i \mathrm{T}_i(\rho), \quad T(\rho,p) = \sum_{i=0}^{\infty} D_i \mathrm{T}_i(\rho)
\end{aligned} \tag{6.226}$$

where $T_k(\rho)$ $(U_k(\rho))$ is the first (second) kind of the Chebyshev polynomials. Using the Gauss-Chebyshev formula yields a linear algebraic equation system

$$\begin{aligned}
&\sum_{k=1}^{n}\left\{\left[\frac{\alpha_1}{\rho_k - r_m} + \hat{Q}_{11}(\rho_k, r_m)\right]\frac{R(\rho_1,p)}{n} + \left[\frac{\alpha_2}{\rho_k - r_m} + \hat{Q}_{12}(\rho_k, r_m)\right]\frac{T(\rho_1,p)}{n}\right\} = -\frac{\sigma_0}{p}\\
&\sum_{k=1}^{n}\left\{\left[\frac{\alpha_3}{\rho_k - r_m} + \hat{Q}_{21}(\rho_k, r_m)\right]\frac{R(\rho_1,p)}{n} + \left[\frac{\alpha_4}{\rho_k - r_m} + \hat{Q}_{22}(\rho_k, r_m)\right]\frac{T(\rho_1,p)}{n}\right\} = -\frac{D_0}{p}
\end{aligned} \tag{6.227}$$

where

$$\mathrm{T}_n(\rho_k)=0,\quad \rho_k=\cos\left(\frac{2k-1}{2n}\pi\right);\quad \mathrm{U}_n(r_k)=0,$$
$$r_k=\cos\left(\frac{k}{n+1}\pi\right);\quad k=1,2,\ldots,n \tag{6.228}$$

Because $f(x_1,p),\mathrm{g}(x_1,p)$ are the odd functions of x_1, $f(0,p)=\mathrm{g}(0,p)=0$, or $R(-1,p)=T(-1,p)=0$. So $R(\rho_n,p)=T(\rho_n,p)=0$, since ρ_n is the closest to -1 in all ρ_k in the limit sense $n\to\infty$ (Erdogan and Gupta 1972; Achenbach et al. 1980). So Eq. (6.227) is a $2(n-1)\times 2(n-1)$ linear algebraic equations with $2(n-1)\times 2(n-1)$ variables $R(\rho_k,p),T(\rho_k,p)$. It is solvable.

Applying the following behavior of the Chebyshev polynomials

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{T}_n(u)\,\mathrm{d}u}{(u-x_1)\sqrt{1-u^2}}=-\frac{|x_1|}{x_1\sqrt{x_1^2-1}}\left(x_1-\frac{|x_1|\sqrt{x_1^2-1}}{x_1}\right)^n,$$
$$|x_1|>1,\quad n=0,1,\ldots \tag{6.229}$$

in the Laplace transform region, the dynamic stress factors can be expressed as

$$\bar{K}_{\mathrm{I}}(p)=\lim_{x\to a^+}\sqrt{2\pi(x_1-a)}\bar{\sigma}_3(x_1,0,p)=-\sqrt{\frac{\pi a}{2}}[\alpha_1R(1,p)+\alpha_2T(1,p)]$$
$$\bar{K}_D(p)=\lim_{x\to a^+}\sqrt{2\pi(x_1-a)}\bar{D}_3(x_1,0,p)=-\sqrt{\frac{\pi a}{2}}[\alpha_3R(1,p)+\alpha_4T(1,p)] \tag{6.230}$$

The dynamic stress factors $K_{\mathrm{I}}(t),K_D(t)$ in the physical plane are obtained by the Laplace inverse transform using the numerical method. The asymptotic generalized stresses and displacements are

$$\sigma_3(x_1,0,t)=\frac{1}{\sqrt{2\pi(x_1-a)}}K_{\mathrm{I}}(t),\quad D_3(x_1,0,t)=\frac{1}{\sqrt{2\pi(x_1-a)}}K_D(t)$$
$$\begin{Bmatrix}u_3(x_1,0,t)\\ \varphi(x_1,0,t)\end{Bmatrix}=\frac{\sqrt{2\pi(a-x_1)}}{\alpha_2\alpha_3-\alpha_1\alpha_4}\begin{bmatrix}-\alpha_4 & \alpha_2\\ \alpha_3 & -\alpha_1\end{bmatrix}\begin{Bmatrix}K_{\mathrm{I}}(t)\\ K_D(t)\end{Bmatrix} \tag{6.231}$$

And the energy release rate with the electric enthalpy is

$$\tilde{G}\mathrm{d}a=2\int_a^{a+\mathrm{d}a}\frac{1}{2}[\sigma_3(x_1,0,t),\quad D_3(x_1,0,t)]\begin{Bmatrix}u_3(x_1-\mathrm{d}a,0,t)\\ \varphi(x_1-\mathrm{d}a,0,t)\end{Bmatrix}\mathrm{d}x_1$$
$$G=[\pi/2(\alpha_2\alpha_3-\alpha_1\alpha_4)]\left[-\alpha_4K_{\mathrm{I}}^2(t)+2\alpha_2\alpha_3K_{\mathrm{I}}(t)K_D(t)-\alpha_1K_D^2(t)\right] \tag{6.232}$$

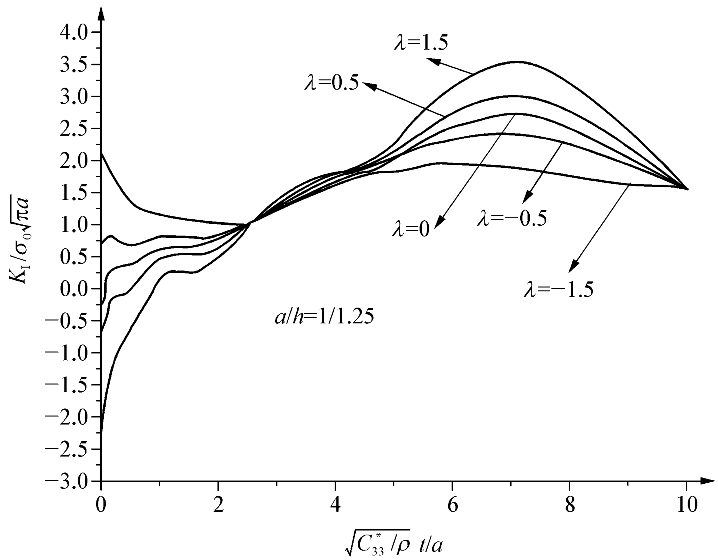


Fig. 6.22 Variation of $K_I/\sigma_0\sqrt{\pi a}$ with $\sqrt{C_{33}^*/\rho}\ t/a$ for various λ under $h/a = 1.25$ (Reprinted from Wang and Yu 2001, with permission from Mechanics of materials)

6.10.4 Numerical Example

In the numerical analysis the material is taken PZT-5H with material constants:

$$C_{11} = 12.6 \times 10^{10}, \quad C_{13} = 5.3 \times 10^{10}, \quad C_{33} = 11.7 \times 10^{10}, \quad C_{44} = 3.53 \times 10^{10}(\mathrm{N/m^2})$$
$$e_{31} = -6.5\,(\mathrm{C/m^2}), \quad e_{33} = 23.3\,(\mathrm{C/m^2}), \quad e_{21} = 17.0\,(\mathrm{C/m^2})$$
$$\epsilon_{11} = 15.1 \times 10^{-9}\,(\mathrm{C/Vm}), \quad \epsilon_{11} = 13.0 \times 10^{-9}\,(\mathrm{C/Vm}), \quad \rho = 7,500\,\mathrm{kg/m^3}$$

The solved values of α_k are

$$\alpha_1 = 5.094 \times 10^{10}, \quad \alpha_2 = 14.216, \quad \alpha_3 = 14.216, \quad \alpha_4 = -178.769 \times 10^{-10}$$

Figures 6.22 and 6.23 show the variations of the dimensionless generalized dynamic stress intensity factors $(K_I, K_D)/\sigma_0\sqrt{\pi a}$ with the dimensionless time $\sqrt{C_{33}^*/\rho}\ t/a$, $C_{33}^* = C_{33} + e_{33}^2/\epsilon_{33}$ under $h/a = 1.25$ and the different loading parameters $\lambda = -D_0\alpha_2/\sigma_0\alpha_4$. It can be seen that the dynamic intensity factors are increased at first and then decreased and after a long time they approach the static values.

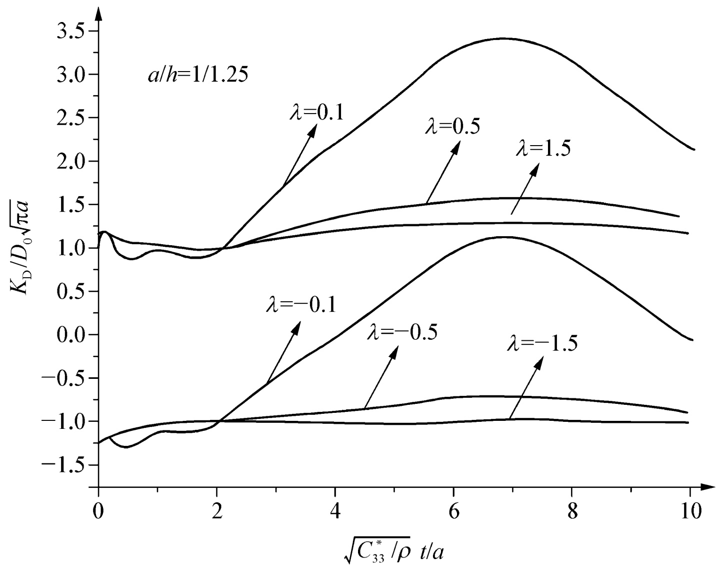


Fig. 6.23 Variation of $K_D/\sigma_0\sqrt{\pi a}$ with $\sqrt{C_{33}^*/\rho}\ t/a$ for various λ under $h/a = 1.25$ (Reprinted from Wang and Yu 2001, with permission from Mechanics of materials)

6.11 On the General Dynamic Analyses of Interface Cracks

6.11.1 *Governing Equations in Laplace-Fourier Transform Region*

In this section the electrically quasi-static assumption is adopted. The generalized momentum and constitutive equations are shown in Eqs. (3.2) and (6.1) or

$$\begin{aligned} &C_{ijkl}u_{l,kj} + e_{kij}\varphi_{,kj} = \rho\ddot{u}_i, \quad e_{ikl}u_{l,ki} - \epsilon_{ij}\varphi_{,ji} = 0 \\ &\sigma_{ij} = C_{ijkl}\epsilon_{kl} - e_{kij}E_k, \quad D_i = \epsilon_{ij}E_j + e_{ikl}\varepsilon_{kl} \end{aligned} \tag{6.233}$$

Shen et al. (1999) applied the Laplace-Fourier transform, i.e., at first adopted the Laplace transform (Eq. (6.181)) with respect to time and then used the Fourier transform (Eq. (4.242)) with respect to time x_1, to solve the problem. When the initial derivatives of variables are zero, the Laplace transform of the Eq. (6.233) is

$$\begin{aligned} &\bar{\sigma}_{ij} = C_{ijkl}\bar{u}_{k,l} + e_{kij}\bar{\varphi}_{,k}, \quad \bar{D}_i = -\epsilon_{ik}\bar{\varphi}_{,k} + e_{ikl}\bar{u}_{k,l} \\ &\left(C_{ijkl}\bar{u}_l + e_{kij}\bar{\varphi}\right)_{,ki} = \rho p^2\bar{u}_j, \quad \left(-\epsilon_{ik}\bar{\varphi} + e_{ikl}\bar{u}_l\right)_{,ki} = 0 \end{aligned} \tag{6.234}$$

Denote the Fourier transform of $\bar{u}_i(x_1, x_2, p)$ as $\bar{u}_i^*(s, x_2, p)$ and let

$$x = x_1. \quad y = \mathrm{i}sx_2 \tag{6.235}$$

By using $\bar{u}_{l,1}^* = \mathrm{i}s\bar{u}_l^*, \quad \bar{u}_{l,11}^* = (\mathrm{i}s)^2\bar{u}_l^*$, the Fourier transform of the second equation in Eq. (6.234) with respect to x_1 is

$$\begin{aligned}
&\left[C_{1jk1}^* + (C_{1jk2} + C_{2jk1})\partial/\partial y + C_{2jk2}\partial^2/\partial y^2\right]\bar{u}_k^* \\
&\quad + \left[e_{1j1} + (e_{1j2} + e_{2j1})\partial/\partial y + e_{2j2}\partial^2/\partial y^2\right]\bar{\varphi}^* = 0 \\
&\left[e_{1k1} + (e_{1k2} + e_{2k1})\partial/\partial y + e_{2k2}\partial^2/\partial y^2\right]\bar{u}_k^* \\
&\quad - \left[\epsilon_{11} + (\epsilon_{12} + \epsilon_{21})\partial/\partial y + \epsilon_{22}\partial^2/\partial y^2\right]\bar{\varphi}^* = 0 \\
&C_{1jk1}^* = C_{1jk1}^*(p, s) = C_{1jk1} + \rho(p^2/s^2)\delta_{jk}
\end{aligned} \tag{6.236}$$

Introduce notations

$$\boldsymbol{Q} = \begin{bmatrix} C_{1jk1}^* & e_{1j1} \\ e_{1k1} & -\epsilon_{11} \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} C_{1jk2} & e_{2j1} \\ e_{1k2} & -\epsilon_{12} \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} C_{2jk2} & e_{2j2} \\ e_{2k2} & -\epsilon_{22} \end{bmatrix}, \quad \bar{\boldsymbol{U}}^* = \begin{Bmatrix} \bar{u}_k^* \\ \bar{\varphi}^* \end{Bmatrix} \tag{6.237}$$

where $\boldsymbol{Q}$, $\boldsymbol{T}$ are positive definite. Applying Eq. (6.237), Eq. (6.236) becomes

$$\left[\boldsymbol{T}\partial^2/\partial y^2 + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}})\partial/\partial y + \boldsymbol{Q}\right]\bar{\boldsymbol{U}}^* = \boldsymbol{0} \tag{6.238}$$

Assume

$$\bar{\boldsymbol{U}}^*(s, y, p) = \boldsymbol{a}(s, p)\mathrm{e}^{\mu y(s,p)}, \quad \partial\bar{\boldsymbol{U}}^*/\partial y = \mu\bar{\boldsymbol{U}}^* \tag{6.239}$$

Substitution of Eq. (6.239) into Eq. (6.238) yields

$$\left[\boldsymbol{T}\mu^2 + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}})\mu + \boldsymbol{Q}\right]\boldsymbol{a} = \boldsymbol{0}, \quad |\boldsymbol{D}(\mu)| = \left|\boldsymbol{T}\mu^2 + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}})\mu + \boldsymbol{Q}\right| = 0 \tag{6.240}$$

Equations (6.237) and (6.240) are identical in the form with Eqs. (3.13) and (3.14), respectively, if use C_{1jk1}^* instead of C_{1jk1}. Analogous to Eq. (3.15), from $|\boldsymbol{D}(\mu)| = 0$ eight roots $\mu_j (j = 1 \sim 8)$ can be obtained. When $s \to \pm\infty$, Eq. (6.239) and μ_i approach the static solutions. The general solution of Eq. (6.239) is

$$\bar{\boldsymbol{U}}^*(s, y, p) = \sum_{k=1}^{4}\left[C_k\boldsymbol{a}_k(s, y, p)\mathrm{e}^{y\mu_k(s,p)} + C_{k+4}\boldsymbol{a}_{k+4}(s, y, p)\mathrm{e}^{y\mu_{k+4}(s,p)}\right] \tag{6.241}$$

The Fourier transform of the first equation in Eq. (6.234) is

$$\bar{\boldsymbol{\Sigma}}_2^* = \mathrm{i}s\left(\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{T}\partial/\partial y\right)\bar{\boldsymbol{U}}^* \tag{6.242}$$

6.11.2 Dynamical Interface Crack

The material I is located at the upper half-plane $S^+, x_2 > 0$; the material II is located at the lower half-plane S^-, $x_2 < 0$; $x_1 = 0$ is the interface and there is a crack of length $2a$ on it. The coordinate origin is selected at the center of the crack (Fig. 4.2b). For the materials I and II, Eqs. (6.238) and (6.242) are all held. The boundary conditions are

$$\begin{aligned}
&\boldsymbol{\Sigma}_2^{(\mathrm{I})} = \left[\sigma_{21}^{(\mathrm{I})}, \sigma_{22}^{(\mathrm{I})}, \sigma_{23}^{(\mathrm{I})}, D_2^{(\mathrm{I})}\right]^{\mathrm{T}} = \boldsymbol{\Sigma}_2^{(\mathrm{II})} = \boldsymbol{\tau}_0 H(t), \quad |x_1| < a \\
&\Sigma_{ij}^{(\mathrm{I})}(x_1, x_2, t) = \Sigma_{ij}^{(\mathrm{II})} = 0, \quad \text{when} \quad \left|\sqrt{x_1^2 + x_2^2}\right| \to \infty \\
&\boldsymbol{U}^{(\mathrm{I})}(u_1, u_2, u_3, \varphi) = \boldsymbol{U}^{(\mathrm{II})}, \quad \boldsymbol{\Sigma}_2^{(\mathrm{I})}(x_1) = \boldsymbol{\Sigma}_2^{(\mathrm{II})}(x_1) = \boldsymbol{\Sigma}_2(x_1, 0, t), \quad |x_1| > a, \quad x_2 = 0
\end{aligned} \tag{6.243}$$

where $\boldsymbol{\tau}_0$ is a constant vector. The initial conditions are

$$\boldsymbol{U}^{(\mathrm{I})}(x_1, x_2, 0) = \boldsymbol{U}^{(\mathrm{II})} = \mathbf{0}, \quad \dot{\boldsymbol{U}}^{(\mathrm{I})}(x_1, x_2, 0) = \dot{\boldsymbol{U}}^{(\mathrm{II})} = \mathbf{0} \tag{6.244}$$

The jump value of the generalized displacements on $x_2 = 0$ is defined as

$$\boldsymbol{\Delta U}(x_1) = \boldsymbol{U}^{(\mathrm{I})}(x_1, 0) - \boldsymbol{U}^{(\mathrm{II})}(x_1, 0), \quad \boldsymbol{\psi}(x_1) = \mathrm{d}\boldsymbol{\Delta U}(x_1)/\mathrm{d}x_1 \tag{6.245}$$

where $\boldsymbol{\psi}(x_1)$ is the dislocation density. The single-valued condition around the crack is

$$\int_{-a}^{a} \boldsymbol{\psi}(x_1, t)\mathrm{d}x_1 = 0, \quad \boldsymbol{\psi} = [\psi_1, \psi_2, \psi_3, \psi_4] \tag{6.246}$$

The Laplace-Fourier transform of Eqs. (6.245) and (6.246) is

$$\begin{aligned}
&\boldsymbol{\Delta}\bar{\boldsymbol{U}}^*(s, p) = \bar{\boldsymbol{U}}^{*(\mathrm{I})}(s, 0, p) - \bar{\boldsymbol{U}}^{*(\mathrm{II})}(s, 0, p) = -(\mathrm{i}/s)\bar{\boldsymbol{\psi}}^* = -(\mathrm{i}/s)\int_{-a}^{a} \bar{\boldsymbol{\psi}}(x_1, p)\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}x_1 \\
&\int_{-a}^{a} \bar{\boldsymbol{\psi}}^*(x_1, p)\mathrm{d}x_1 = 0; \quad \int_{-a}^{a} \bar{\boldsymbol{\psi}}(x_1, p)\mathrm{d}x_1 = 0
\end{aligned} \tag{6.247}$$

The Laplace transform of Eq. (6.243) is

$$\begin{aligned}
&\bar{\boldsymbol{\Sigma}}_2(x_1, 0) = \boldsymbol{\tau}_0/p, \quad |x_1| < a; \quad \bar{\boldsymbol{U}}^{(\mathrm{I})}(x_1, 0, t) = \bar{\boldsymbol{U}}^{(\mathrm{II})}(x_1, 0, t), \quad a < |x_1| < \infty \\
&\bar{\boldsymbol{\Sigma}}_2^{(\mathrm{I})}(x_1, 0) = \bar{\boldsymbol{\Sigma}}_2^{(\mathrm{II})}(x_1, 0) = \bar{\boldsymbol{\Sigma}}_2(x_1, 0), \quad |x_1| < \infty; \quad \bar{\boldsymbol{\Sigma}}_2^{(\mathrm{I})} = \bar{\boldsymbol{\Sigma}}_2^{(\mathrm{II})} = 0, \quad |\boldsymbol{x}| \to \infty
\end{aligned} \tag{6.248}$$

6.11.3 Reduced to the Singular Integral Equation

In order to make $\bar{\boldsymbol{U}}^*$ finite when $|s| \to \infty$, the solution of Eq. (6.241) is expressed as

$$\bar{\boldsymbol{U}}^{*(\mathrm{I})}(s,y,p) = \begin{cases} \boldsymbol{A}_1^{(\mathrm{I})}\boldsymbol{E}_1^{(\mathrm{I})}\boldsymbol{C}_1^{(\mathrm{I})}, & s<0 \\ \boldsymbol{A}_2^{(\mathrm{I})}\boldsymbol{E}_2^{(\mathrm{I})}\boldsymbol{C}_2^{(\mathrm{I})}, & s>0 \end{cases}; \quad \bar{\boldsymbol{U}}^{*(\mathrm{II})}(s,y,p) = \begin{cases} \boldsymbol{A}_1^{(\mathrm{II})}\boldsymbol{E}_1^{(\mathrm{II})}\boldsymbol{C}_1^{(\mathrm{II})}, & s>0 \\ \boldsymbol{A}_2^{(\mathrm{II})}\boldsymbol{E}_2^{(\mathrm{II})}\boldsymbol{C}_2^{(\mathrm{II})}, & s<0 \end{cases}$$
$$\boldsymbol{A}_1^{(\mathrm{N})} = \left[\boldsymbol{a}_1^{(\mathrm{N})},\boldsymbol{a}_2^{(\mathrm{N})},\boldsymbol{a}_3^{(\mathrm{N})},\boldsymbol{a}_4^{(\mathrm{N})}\right], \quad \boldsymbol{A}_2^{(\mathrm{N})} = \left[\boldsymbol{a}_5^{(\mathrm{N})},\boldsymbol{a}_6^{(\mathrm{N})},\boldsymbol{a}_7^{(\mathrm{N})},\boldsymbol{a}_8^{(\mathrm{N})}\right]; \quad \boldsymbol{E}_1^{(\mathrm{I})} = \left\langle \mathrm{e}^{-\mathrm{i}s\mu_j x_2}\right\rangle$$
$$\boldsymbol{E}_2^{(\mathrm{N})} = \left\langle \mathrm{e}^{-\mathrm{i}s\mu_{j+4} x_2}\right\rangle; \quad \boldsymbol{C}_1^{(N)} = \left[C_1^{(N)},C_2^{(N)},C_3^{(N)},C_4^{(N)}\right], \quad \boldsymbol{C}_2^{(\mathrm{N})} = \left[C_5^{(\mathrm{N})},C_6^{(\mathrm{N})},C_7^{(\mathrm{N})},C_8^{(\mathrm{N})}\right] \tag{6.249}$$

where $\mathrm{N} = \mathrm{I},\mathrm{II}$. Equation (6.249) can be rewritten as

$$\bar{\boldsymbol{U}}^{*(\mathrm{I})} = \boldsymbol{A}_1^{(\mathrm{I})}\boldsymbol{E}_1^{(\mathrm{I})}\boldsymbol{C}_1^{(\mathrm{I})} + \boldsymbol{A}_2^{(\mathrm{I})}\boldsymbol{E}_2^{(\mathrm{I})}\boldsymbol{C}_2^{(\mathrm{I})}; \quad \bar{\boldsymbol{U}}^{*(\mathrm{II})} = \boldsymbol{A}_1^{(\mathrm{II})}\boldsymbol{E}_1^{(\mathrm{II})}\boldsymbol{C}_1^{(\mathrm{II})} + \boldsymbol{A}_2^{(\mathrm{II})}\boldsymbol{E}_2^{(\mathrm{II})}\boldsymbol{C}_2^{(\mathrm{II})}$$
$$\boldsymbol{C}_2^{(\mathrm{I})} = \boldsymbol{C}_1^{(\mathrm{II})} = \boldsymbol{0}, \quad \text{when} \quad s<0; \quad \boldsymbol{C}_1^{(\mathrm{I})} = \boldsymbol{C}_2^{(\mathrm{II})} = \boldsymbol{0}, \quad \text{when} \quad s>0 \tag{6.250}$$

From Eq. (6.242) it is obtained

$$\bar{\boldsymbol{\Sigma}}_2^{*(\mathrm{I})}(s,y,p) = \begin{cases} \mathrm{i}s\boldsymbol{B}_1^{(\mathrm{I})}\mathbf{E}_1^{(\mathrm{I})}\boldsymbol{C}_1^{(\mathrm{I})} = \mathrm{i}s\boldsymbol{B}_1^{(\mathrm{I})}\boldsymbol{A}_1^{(\mathrm{I})^{-1}}\bar{\boldsymbol{U}}^{*(\mathrm{I})} = -s\boldsymbol{Y}_1^{(\mathrm{I})-1}\bar{\boldsymbol{U}}^{*(\mathrm{I})}, & s<0 \\ \mathrm{i}s\boldsymbol{B}_2^{(\mathrm{I})}\mathbf{E}_2^{(\mathrm{I})}\boldsymbol{C}_2^{(\mathrm{I})} = \mathrm{i}s\boldsymbol{B}_2^{(\mathrm{I})}\boldsymbol{A}_2^{(\mathrm{I})^{-1}}\bar{\boldsymbol{U}}^{*(\mathrm{I})} = -s\boldsymbol{Y}_2^{(\mathrm{I})^{-1}}\bar{\boldsymbol{U}}^{*(\mathrm{I})}, & s>0 \end{cases}$$
$$\bar{\boldsymbol{\Sigma}}_2^{*(\mathrm{II})}(s,y,p) = \begin{cases} \mathrm{i}s\boldsymbol{B}_1^{(\mathrm{II})}\mathbf{E}_1^{(\mathrm{II})}\boldsymbol{C}_1^{(\mathrm{II})} = \mathrm{i}s\boldsymbol{B}_1^{(\mathrm{II})}\boldsymbol{A}_1^{(\mathrm{II})^{-1}}\bar{\boldsymbol{U}}^{*(\mathrm{II})} = -s\boldsymbol{Y}_1^{(\mathrm{II})-1}\bar{\boldsymbol{U}}^{*(\mathrm{II})}, & s>0 \\ \mathrm{i}s\boldsymbol{B}_2^{(\mathrm{II})}\mathbf{E}_2^{(\mathrm{II})}\boldsymbol{C}_2^{(\mathrm{II})} = \mathrm{i}s\boldsymbol{B}_2^{(\mathrm{II})}\boldsymbol{A}_2^{(\mathrm{II})^{-1}}\bar{\boldsymbol{U}}^{*(\mathrm{II})} = -s\boldsymbol{Y}_2^{(\mathrm{II})^{-1}}\bar{\boldsymbol{U}}^{*(\mathrm{II})}, & s<0 \end{cases}$$
$$\boldsymbol{B}_1^{(N)} = \left[\boldsymbol{b}_1^{(\mathrm{N})},\boldsymbol{b}_2^{(\mathrm{N})},\boldsymbol{b}_3^{(\mathrm{N})},\boldsymbol{b}_4^{(\mathrm{N})}\right], \quad \boldsymbol{B}_2^{(\mathrm{N})} = \left[\boldsymbol{b}_5^{(\mathrm{N})},\boldsymbol{b}_6^{(\mathrm{N})},\boldsymbol{b}_7^{(\mathrm{N})},\boldsymbol{b}_8^{(\mathrm{N})}\right],$$
$$\boldsymbol{b}^{(\mathrm{N})} = \left(\boldsymbol{R}^{(\mathrm{N})\mathrm{T}} + \boldsymbol{T}^{(\mathrm{N})}\partial/\partial y\right)\boldsymbol{a}^{(\mathrm{N})}; \quad \mathrm{N} = \mathrm{I},\mathrm{II} \tag{6.251}$$

On the crack surface Eq. (6.251) becomes

$$\bar{\boldsymbol{\Sigma}}_2^{*(\mathrm{I})}(s,0,p) = \boldsymbol{R}^{(\mathrm{I})}\bar{\boldsymbol{U}}^{*(\mathrm{I})}; \quad \bar{\boldsymbol{\Sigma}}_2^{*(\mathrm{II})}(s,0,p) = \boldsymbol{R}^{(\mathrm{II})}\bar{\boldsymbol{U}}^{*(\mathrm{II})}$$
$$\boldsymbol{R}^{(\mathrm{I})} = \begin{cases} \mathrm{i}s\boldsymbol{B}_1^{(\mathrm{I})}\boldsymbol{A}_1^{(\mathrm{I})-1} = -s\boldsymbol{Y}_1^{(I)-1}, & s<0 \\ \mathrm{i}s\boldsymbol{B}_2^{(\mathrm{I})}\boldsymbol{A}_2^{(\mathrm{I})-1} = -s\boldsymbol{Y}_2^{(I)^{-1}}, & s>0 \end{cases}; \quad \boldsymbol{R}^{(\mathrm{II})} = \begin{cases} \mathrm{i}s\boldsymbol{B}_1^{(\mathrm{II})}\boldsymbol{A}_1^{(\mathrm{II})-1} = -s\boldsymbol{Y}_1^{(\mathrm{II})-1}, & s>0 \\ \mathrm{i}s\boldsymbol{B}_2^{(\mathrm{II})}\boldsymbol{A}_2^{(\mathrm{II})-1} = -s\boldsymbol{Y}_2^{(\mathrm{II})^{-1}}, & s<0 \end{cases} \tag{6.252}$$

where $\boldsymbol{Y}^{(\mathrm{N})} = \mathrm{i}\boldsymbol{A}^{(\mathrm{N})}\boldsymbol{B}^{(\mathrm{N})^{-1}}$. Because on the whole interface $\boldsymbol{\Sigma}_2^{(\mathrm{I})}(x_1) = \boldsymbol{\Sigma}_2^{(\mathrm{II})}(x_1)$, so

$$\boldsymbol{R}^{(\mathrm{I})}\bar{\boldsymbol{U}}^{*(\mathrm{I})} = \boldsymbol{R}^{(\mathrm{II})}\bar{\boldsymbol{U}}^{*(\mathrm{II})} \tag{6.253}$$

From Eqs. (6.247) and (6.253) it is obtained

$$\bar{\boldsymbol{U}}^{*(\mathrm{I})} = \boldsymbol{R}^{(\mathrm{II})}\boldsymbol{R}^{-1}\Delta\bar{\boldsymbol{U}}^{*}, \quad \bar{\boldsymbol{U}}^{*(\mathrm{II})} = \boldsymbol{R}^{(\mathrm{I})}\boldsymbol{R}^{-1}\Delta\bar{\boldsymbol{U}}^{*}; \quad \boldsymbol{R} = \boldsymbol{R}^{(\mathrm{II})} - \boldsymbol{R}^{(\mathrm{I})} \tag{6.254}$$

Combining Eqs. (6.242), (6.247), (6.252), (6.253), and (6.254) and performing the Fourier inverse transform, it is obtained

$$\begin{aligned}&\bar{\boldsymbol{\Sigma}}_2(x_1,0,p) = -(\mathrm{i}/2\pi)\int_{-a}^{a}\bar{\boldsymbol{\psi}}(\xi,p)\mathrm{d}\xi\int_{-\infty}^{\infty}(1/s)\boldsymbol{M}\mathrm{e}^{-\mathrm{i}s(\xi-x_1)}\mathrm{d}s, \quad |x_1|<\infty\\&\boldsymbol{M} = \boldsymbol{R}^{(\mathrm{II})}\boldsymbol{R}^{(\mathrm{I})}\boldsymbol{R}^{-1} = \boldsymbol{R}^{(\mathrm{I})}\boldsymbol{R}^{(\mathrm{II})}\boldsymbol{R}^{-1}\end{aligned} \tag{6.255}$$

Equation (6.255) is a singular integral equation, and its singular behavior is determined by the asymptotic behavior at infinity of the kernel function $s^{-1}\boldsymbol{M}(s,p)$. When $s\to\infty, \boldsymbol{Y}_j^{(\mathrm{N})}, \boldsymbol{Y}_j^{(\mathrm{N})-1}$ approach the static values, so are finite. At the static case $\boldsymbol{A}_2^{(\mathrm{N})} = \bar{\boldsymbol{A}}_1^{(\mathrm{N})}, \quad \boldsymbol{B}_2^{(\mathrm{N})} = \bar{\boldsymbol{B}}_1^{(\mathrm{N})}$. It is noted that for a constant, $\bar{\boldsymbol{A}}$ is not the Laplace transform of $\boldsymbol{A}$, but is the conjugate value of $\boldsymbol{A}$. From Eq. (6.252) it is obtained

$$\begin{aligned}&\lim_{s\to\infty}(1/s)\boldsymbol{R}^{(\mathrm{I})} = -\boldsymbol{Y}_{2\ \mathrm{static}}^{(\mathrm{I})-1} = \bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{I})-1}; \quad \lim_{s\to\infty}(1/s)\boldsymbol{R}^{(\mathrm{II})} = -\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{II})-1}\\&\lim_{s\to-\infty}(1/s)\boldsymbol{R}^{(\mathrm{I})} = -\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{I})-1}; \quad \lim_{s\to-\infty}(1/s)\boldsymbol{R}^{(\mathrm{II})} = -\boldsymbol{Y}_{2\ \mathrm{static}}^{(\mathrm{II})-1} = \bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{II})-1}\end{aligned} \tag{6.256}$$

So

$$\begin{aligned}&\lim_{s\to\infty}(1/s)\boldsymbol{M} = \bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{I})-1}\left(-s\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{II})-1}\right)\left[s\left(-\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{II})-1} - \bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{I})-1}\right)\right]^{-1} = \boldsymbol{M}_\infty\\&\lim_{s\to-\infty}(1/s)\boldsymbol{M} = -\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{I})-1}s\bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{II})-1}\left[s\left(\bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{II})-1} + \boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{I})-1}\right)\right]^{-1} = -\bar{\boldsymbol{M}}_\infty\\&\boldsymbol{M}_\infty = \bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{I})-1}\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{II})-1}\left(\boldsymbol{Y}_{1\ \mathrm{static}}^{(\mathrm{II})-1} + \bar{\boldsymbol{Y}}_{1\ \mathrm{static}}^{(\mathrm{I})-1}\right)\end{aligned} \tag{6.257}$$

or

$$\lim_{s\to\pm\infty}(1/s)\boldsymbol{M} = (s/|s|)\mathrm{Re}\,\boldsymbol{M}_\infty + \mathrm{i}\mathrm{Im}\,\boldsymbol{M}_\infty. \tag{6.258}$$

By separating the singular part in Eq. (6.255) and then substituting the result into the boundary condition Eq. (6.248), the following singular integral equation can be obtained:

$$\mathrm{Im}\,\boldsymbol{M}_\infty\bar{\boldsymbol{\psi}} + \frac{\mathrm{Re}\,\boldsymbol{M}_\infty}{\pi}\int_{-a}^{a}\xi\frac{\bar{\boldsymbol{\psi}}(\xi,p)}{t-x_1}\mathrm{d}\xi - \frac{\mathrm{i}}{2\pi}\int_{-a}^{a}\bar{\boldsymbol{\psi}}(t,p)\mathrm{d}t\int_{-\infty}^{\infty}\left(\frac{1}{s}\boldsymbol{M} + \boldsymbol{M}_\infty\right)\mathrm{e}^{-\mathrm{i}s(\xi-x_1)}\mathrm{d}s = \boldsymbol{\tau}_0 \tag{6.259}$$

Let λ_i be the eigenvector of $(\mathrm{Re}\,\boldsymbol{M}_\infty)^{-1}\mathrm{Im}\,\boldsymbol{M}_\infty$ and $\boldsymbol{\Lambda}$ be the matrix constituted of λ_i, then we get

$$\boldsymbol{\Lambda}(\mathrm{Re}\,\boldsymbol{M}_\infty)^{-1}(\mathrm{Im}\,\boldsymbol{M}_\infty)\boldsymbol{\Lambda}^{-1} = \mathrm{diag}(\boldsymbol{\Lambda}) = \langle \lambda_i \rangle \tag{6.260}$$

Multiplying both sides of Eq. (6.259) by $(\mathrm{Re}\,\boldsymbol{M}_\infty)^{-1}$, introducing the dimensionless variable $x = x_1/a, \eta = \xi/a$ and using Eq. (6.260), Eq. (6.259) can be reduced to

$$\begin{aligned} &\lambda_i \bar{\Psi}_{\lambda i}(x,p) + \frac{1}{\pi}\int_{-1}^{1} \frac{\bar{\Psi}_{\lambda i}(\eta,p)}{\eta - x}\mathrm{d}\eta + \int_{-1}^{1}\sum_{k=1}^{4} F_{ik}\bar{\Psi}_{\lambda i}(\eta,p)\mathrm{d}\eta = \bar{T}_{0i}(x,p) \\ &[F_{ik}] = -\frac{\mathrm{i}a}{2\pi}\boldsymbol{\Lambda}(\mathrm{Re}\,\boldsymbol{M}_\infty)^{-1}\left\{\int_{-\infty}^{\infty}\left(\frac{1}{s}\boldsymbol{M} + \boldsymbol{M}_\infty\right)e^{-isa(\eta - x)}\mathrm{d}s\right\}\boldsymbol{\Lambda}^{-1}, \boldsymbol{T}_0 = \boldsymbol{\Lambda}(\mathrm{Re}\,\boldsymbol{M}_\infty)^{-1}\bar{\boldsymbol{\tau}}_0 \end{aligned} \tag{6.261}$$

The solution of Eq. (6.261) can be expressed by the series of the Jacobi polynomials. Let

$$\begin{aligned} &\bar{\Psi}_{\lambda i}(x,p) = \sum_{n=0}^{\infty} C_{ni}(p)P_n^{(\alpha,\beta)}(x)w_i(x), \quad |x| < 1 \\ &w_i(x) = (1-x)^{\alpha_i}(1+x)^{\beta_i}, \quad \alpha_k = \frac{\mathrm{i}}{2\pi\mathrm{i}}\ln\frac{1-\lambda_k\mathrm{i}}{1+\lambda_k\mathrm{i}} - \frac{1}{2}, \quad \beta_k = -\frac{\mathrm{i}}{2\pi\mathrm{i}}\ln\frac{1-\lambda_k\mathrm{i}}{1+\lambda_k\mathrm{i}} - \frac{1}{2} \end{aligned} \tag{6.262}$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial, C_{ni} is unknown constant, and α_k, β_k determined by Eq. (6.262) are the singular indexes of the dynamical problem and usually are complex numbers. As in usual elastic problem, in the front of the crack tip, there is a small region in which the displacements of the upper and the lower surfaces may be imbedded to each other. Substituting Eq. (6.262) into Eqs. (6.261) and (6.247) in terms of the dimensionless length x, using the orthogonal relation of the Jacobi polynomials and $P_0^{(\alpha,\beta)}(t) = 1$ and the following relations

$$\begin{aligned} &\lambda_k P_n^{(\alpha_k,\beta_k)}(x)w_k(x) + \frac{1}{\pi}\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)\frac{w_k(t)}{t-x}\mathrm{d}t \\ &= \begin{cases} \dfrac{1}{2}\sqrt{1+\lambda_k^2}P_n^{(\alpha_k,\beta_k)}(x), & |x| < 1 \\ \dfrac{1}{2}\sqrt{1+\lambda_k^2}\left[(x-1)^{\alpha_k}(x-1)^{\beta_k}P_n^{(\alpha_k,\beta_k)}(x) + G_{kn}^{\infty}(x)\right], & |x| > 1 \end{cases}, \end{aligned} \tag{6.263}$$

the linear algebraic equations of C_n^k ($C_0^k = 0$) can be obtained. Where $G_{kn}^{\infty}(x)$ is the principle part of $P_n^{(\alpha_k,\beta_k)}(x)w_k(x)$ at infinity and is finite at $x = 1$, it is no contribution on the stress intensity factors. Take the first $N+1$ terms. The following $4N$ equations determined by C_n^k can be obtained:

$$
\begin{aligned}
&\frac{1}{2}\sqrt{1+\lambda_k^2}\theta_{j-1}^{(-\alpha_k,-\beta_k)}C_j^k+\sum_{n=1}^{\infty}\sum_{m=1}^{4}Y_{jn}^{km}C_n^m=q_{jk}, \quad k=1-4, \quad j=1-N\\
&q_{jk}=\int_{-1}^{1}\bar{T}_{0k}P_{j-1}^{(-\alpha_k,-\beta_k)}(x)w_k(x)\mathrm{d}x, \quad Y_{jn}^{km}=\int_{-1}^{1}H_n^{km}P_{j-1}^{(-\alpha_k,-\beta_k)}(x)w_k(x)\mathrm{d}x\\
&H_n^{km}(x,p)=\int_{-1}^{1}F_{km}(x,t,p)P_n^{(\alpha_k,\beta_k)}(t)w_k(t)\mathrm{d}t\\
&\theta_k^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)(k+\beta+1)}{(2k+\alpha+\beta+1)(k+\alpha+\beta+1)k!}, \quad \theta_0^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)(\beta+1)}{\Gamma(\alpha+\beta+2)}
\end{aligned}
\tag{6.264}
$$

The singular part of the generalized traction in front of the crack tip $|x|>a$ can be obtained from Eq. (6.255):

$$
\bar{\boldsymbol{\Sigma}}_2(x,0,p)=\mathrm{Re}\,\boldsymbol{M}_\infty\boldsymbol{\Lambda}^{-1}\sum_{n=1}^{N}\begin{bmatrix}(1/2)\sqrt{1+\lambda_k^2}(x-1)^{\alpha_k}(x+1)^{\beta_k}P_n^{(\alpha_k,\beta_k)}(x)C_n^k\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (1/2)\sqrt{1+\lambda_4^2}(x-1)^{\alpha_4}(x+1)^{\beta_4}P_n^{(\alpha_4,\beta_4)}(x)C_n^4\end{bmatrix}
\tag{6.265}
$$

The generalized stress intensity factors in the Laplace transform region are

$$
\bar{\boldsymbol{K}}=[\bar{K}_{\mathrm{II}},\bar{K}_{\mathrm{I}},\bar{K}_{\mathrm{III}},\bar{K}_D]^{\mathrm{T}}=\lim_{x\to1^+}\sqrt{2\pi}\langle(x-1)^{\alpha}\rangle\bar{\boldsymbol{\Sigma}}_2(x,0,p)
\tag{6.266}
$$

After solving $\bar{\boldsymbol{K}}$ in the Laplace transform region, the stress intensity factors in the physical region are obtained by the numerical Laplace inverse transform.

There are many papers to discuss the wave propagation in a piezoelectric material with defects, e.g., Li and Mataga (1996) discussed the semi-infinite crack propagation; Chen et al. (1998) discussed a Griffith moving crack along the interface of two dissimilar piezoelectric materials; Li and Weng (2002) discussed the Yoffe-type moving crack in a functionally graded piezoelectric material; and Ing and Wang (2004a, b) discussed the transient response of a semi-infinite propagating crack subjected to dynamic antiplane concentrated loading on the crack faces. Chen and Liu (2005) discussed the dynamic behavior of a functionally graded piezoelectric strip with periodic cracks vertical to the boundary. Shen et al. (2000) discussed the dynamics mode-III interfacial crack in nonlinear piezoelectric materials. Meikumyan (2007) discussed the diffraction of acoustic and electric waves in piezoelectric medium by an absorbent half-plane electrode.

References

Achenbach J, Keer L, Mendelsohn D (1980) Elastodynamic analysis of an edge crack. J Appl Mech 47:551–556

Assour MB, Elmazria O, Rioboo RJ, Sarry PA (2000) Modelling of SAW filter based on ZnO/diamond/Si layered structure including velocity dispersion. Appl Surf Sci 164:200–204

Auld BA (1973) Acoustic fields and waves in solids. Wiley, New York

Babich VM, Lukyanov VV (1998) Wave propagation along a curved piezoelectric layer. Wave Motion 28:1–11

Banerjee DK, Bao YH (1974) Thermoelastic waves in anisotropic solids. J Acoust Soc Am 56:1444–1454

Biryukov SV, Gulyaev YV, Kryiov VV, Plessky VP (1995) Surface acoustic waves in inhomogeneous media. Springer, New York

Bleustein JL (1968) A new surface wave in piezoelectric materials. Appl Phys Lett 13:412–413

Borcherdt RD (1973) Energy and plane waves in linear viscoelastic media. J Geophys Res 78:2442–2453

Buchen PW (1971) Plane waves in linear viscoelastic media. Geophys J R Astron Soc 23:531–542

Cattaneo C (1958) Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanee. Comptes Rendus de l'Académie des Sciences Paris 247:431–433

Chen J, Liu Z-X (2005) On the dynamic behavior of a functionally graded piezoelectric strip with periodic cracks vertical to the boundary. Int J Solids Struct 42:3133–3146

Chen Z-T, Karihaloo BL, Yu S-W (1998) A Griffith moving crack along the interface of two dissimilar piezoelectric materials. Int J Fract 91:197–203

Cho Y, Yamanouchi K (1987) Nonlinear, elastic, piezoelectric, electrostrictive and dielectric constants of Lithium niobate. J Appl Phys 61:875–886

Danoyan ZN, Pilliposian GT (2007) Surface electro-elastic Love waves in a layered structure with a piezoelectric substrate and a dielectric layer. Int J Solids Struct 44:5829–5847

Darinskii AN, Le Clezio E, Feuillard G (2008) The role of electromagnetic waves in the reflection of acoustic waves in piezoelectric crystals. Wave Motion 45:428–444

Dieulesaint E, Royer D (1980) Elastic waves in solids. Translated by Bastin A, Motz M. Wiley, New York

Du J, Xian K, Wang J, Young Y-K (2008) Propagation of Love waves in prestressed piezoelectric layered structures loaded with viscous liquid. Acta Mech Solida Sin 21:542–548

Erdogan F (1975) Complex function technique. In: Continuum physics, vol II. Academic, New York

Erdogan F, Gupta GD (1972) On the numerical solution of singular integral equations. Q Appl Math 32:525–553

Every AG, Neiman VI (1992) Reflection of electroacoustic waves in piezoelectric solids: mode conversion into four bulk waves. J Appl Phys 71:6018–6024

Ezzat MA, Otman MI, Ei-Karamany AS (2002) State space approach to generalized thermo-viscoelasticity with two relaxation times. Int J Eng Sci 40:283–302

Fischer FA (1955) Fundamentals of electroacoustics. Interscience, New York

Green AE, Lindsay KA (1972) Thermoelasticity. J Elast 2:1–7

Gulyaev YV (1969) Electroacoustic surface waves in solids. Soviet Phys-JETP (A translation of the "J Exp Theor Phys Acad Sci USSR") 9:37–38

Ing Y-S, Wang M-J (2004a) Explicit transient solutions for a mode-III crack subjected to dynamic concentrated loading in a piezoelectric medium. Int J Solids Struct 41:3849–3864

Ing Y-S, Wang M-J (2004b) Transient analysis of a mode-III crack propagating in a piezoelectric medium. Int J Solids Struct 41:6197–6214

Jackson HE, Walker CT (1971) Thermal conductivity, second sound and phonon-phonon interactions in NaF. Phys Rev B 3:1428–1439

Jin F, Wang Z-K, Wang T-J (2001) The Bleustein-Gulyaev (B-G) Wave in a piezoelectric layered half-space. Int J Eng Sci 39:1271–1285

Joseph DD, Preziosi L (1989) Heat waves. Rev Mod Phys 61:41–73
Joseph DD, Preziosi L (1990) Addendum to the paper: heat waves. Rev Mod Phys 62:375–391
Joshi SG, Jin Y (1991) Propagation of ultrasonic lamb waves in piezoelectric plates. J Appl Phys 70:4113–4120
Kaliski S (1965) Wave equation of thermoelasticity. Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Techniques 13:211–219
Khuri-Yakub BT, Kino GS (1974) A monolithic zinc-oxide-silicon convolver. Appl Phys Lett 25:188–193
Krebes ES (1983) The viscoelastic reflection/transmission problem: two special cases. Bull Seismol Soc Am 73:1673–1683
Kuang Z-B (2002) Nonlinear continuum mechanics. Shanghai Jiaotong University Press, Shanghai (in Chinese)
Kuang Z-B (2009) Variational principles for generalized dynamical theory of thermopiezoelectricity. Acta Mech 203:1–11
Kuang Z-B (2011) Some thermodynamic problems in continuum mechanics. In: Piraján JCM (ed) Thermodynamics – kinetics of dynamic systems. Intech Open Access Publisher, Croatia
Kuang Z-B, Yuan X-G (2011) Reflection and transmission theories of waves in pyroelectric and piezoelectric materials. J Sound Vib 330:1111–1120
Kuang Z-B, Zhou Z-D (2012) Waves in pyroelectric materials. ICTAM, Beijing
Kyame JJ (1949) Wave propagation in piezoelectric crystals. J Acoust Soc Am 21:159–167
Landau L (1941) The theory of superfluidity of helium II. J Phys 5:71–90
Laurent T, Bastien FO, Pommier J, Cachard A, Remiens D, Cattan E (2000) Lamb wave and plate mode in ZnO/Silicon and AIN/Silicon membrane application to sensors able to operate in contact with liquid. Sens Actuators 87:26–37
Lavrenchive MA, Shabat VA (1951) Method and theory of complex functions. National Technical and Theoretical Literature Press, Moscow; Лаврентьев МА, Шабат ВА (1951) Метод теории фукций коплексснго переменого. Государственное издадельство. технико – теоретический литературы. Москва
Li S (1996) The electromagneto-acoustic surface wave in a piezoelectric medium: the Bleustein-Gulyaev mode. J Appl Phys 80:5264–5269
Li S (2000) Transient wave propagation in a transversely isotropic piezoelectric half space. Z Angew Math Phys 51:236–266
Li S, Mataga PA (1996) Dynamic crack propagation in piezoelectric materials-part I. electrode solution, J Mech Phys Solids 44:1799–1830
Li C, Weng GJ (2002) Yoffe-type moving crack in a functionally graded piezoelectric material. Proc R Soc Lond A 458:381–399
Li S, To AC, Glaser SD (2005a) On scattering in a piezoelectric medium by a conducting crack. J Appl Mech 72:943–954
Li X-F, Yang JS, Jiang Q (2005b) Spatial dispersion of short surface acoustic waves in piezoelectric ceramics. Acta Mech 180:11–20
Lionetto F, Montagna F, Maffezzoli A (2005) Ultrasonic dynamic mechanical analysis of polymers. Appl Rheol 15:326–335
Liu Y, Wang C-h, Ying CF (1989) Electromagnetic acoustic head waves in piezoelectric media. Appl Phys Lett 55:434–436
Liu H, Wang Z-K, Wang T-J (2001) Effect of initial stress on the propagation behavior of Love waves in a layered piezoelectric structure. Int J Solids Struct 38:37–51
Liu H, Wang T-J, Wang Z-K, Kuang Z-B (2002a) Effect of a biasing electric field on the propagation of antisymmetric Lamb waves in piezoelectric plates. Int J Solids Struct 39:1777–1790
Liu H, Wang T-J, Wang Z-K, Kuang Z-B (2002b) Effect of a biasing electric field on the propagation of symmetric Lamb waves in piezoelectric plates. Int J Solids Struct 39:2031–2049
Liu H, Kuang Z-B, Cai Z-M (2003a) Propagation of Bleustein-Gulyaev waves in a prestressed layered piezoelectric structure. Ultrasonics 41:397–405
Liu H, Kuang Z-B, Cai Z-M (2003b) Application of transfer matrix method in analyzing the inhomogeneous initial stress problem in prestressed layered piezoelectric media. In: Watanabe K,

Ziegler F (eds) IUTAM symposium on dynamics of advanced materials and smart structures. Kluwer Academic Publishers, Dordrecht/Boston/London
Liu H, Kuang Z-B, Cai Z-M (2003c) Love wave in non-homogeneous layered structures. Acta Mech Sin 35:485–488
Liu H, Kuang Z-B, Cai Z-M (2003d) Propagation of surface acoustic waves in prestressed anisotropic layered piezoelectric structures. Acta Mech Solida Sin 16:16–23
Lord HW, Shulman Y (1967) A generalized dynamical theory of thermoelasticity. J Mech Phys Solids 15:299–309
Lu C-K (1984) A class of quadrature formulas of Chebyshev type for singular integrals. J Math Anal Appl 100:416–435
Ma XJ, Liu FS, Zhang MJ, Sun YY (2011) Viscosity of aluminum under shock-loading conditions. Chin Phys B 20:068301–068304
Meikumyan A (2007) On diffraction of acoustic and electric waves in piezoelectric medium by an absorbent half-plane electrode. Int J Solids Struct 44:3811–3827
Mindlin RD (1968) Polarization gradient in elastic dielectrics. Int J Solids Struct 4:637–642
Mineev VN, Mineev AV (1997) Viscosity of metals under shock-loading conditions. J De Phys IV 7(C3):583–585
Nayfen A (1995) Wave propagation in layered anisotropic media. Elsevier, Amsterdam
Noble B (1958) Methods based on the wiener-Hopf technique. Pergamon Press, New York
Ogden RW (1984) Nonlinear elastic deformations. Ellis Horwood/Halsted Press, Chichester/New York
Pang Y, Wang Y-S, Liu J-X, Fang D-N (2008) Reflection of plane waves at the interface between piezoelectric piezomagnetic media. Int J Eng Sci 46:1098–1110
Sharma JN, Pal M (2004) Propagation of Lamb waves in a transversely isotropic piezothermoelastic plate. J Sound Vib 270:587–610
Sharma JN, Walia V, Gupta SK (2008) Reflection of piezothermoelastic waves from the charge and stress free boundary of a transversely isotropic half space. Int J Eng Sci 46:131–146
Shen S-P, Nishioka T, Kuang Z-B (1999) Impact interfacial fracture for piezoelectric ceramic. Mech Res Commun 26:347–352
Shen S-P, Kuang Z-B, Nishioka T (2000) Dynamics mode-III interfacial crack in nonlinear piezoelectric materials. Int J Appl Electromagn Mech 11:211–222
Sinha AK, Tanski WJ, Lukaszek T, Ballato A (1985) Influence of biasing stress on the propagation of surface waves. J Appl Phys 57:767–776
Stewart JT, Yong YK (1994) Exact analysis of the propagation of acoustic waves in multilayered anisotropic piezoelectric plate. IEEE Trans Ultrason Ferroelectr Freq Control 41:375–390
Su J, Liu H, Kuang Z-B (2005) Love wave in ZnO/SiO_2/Si structure with initial stresses. J Sound Vib 286:981–999
Thomson WT (1950) Transmission of elastic waves through a stratified solid medium. J Appl Phys 21:89–93
Tiersten HF (1978) Perturbation theory for linear electroelastic equations for small fields superposed on a bias. J Acoust Soc Am 64:832–837
Vernotte P (1958) Les paradoxes de la théorie continue de l'equation de la chaleeur. Comptes Rendus de l'Académie des Sciences Paris 246:3154–3155
Wang X-Y, Yu S-W (2001) Transient response of a crack in piezoelectric strip subjected to the mechanical and electrical impacts: mode-I problem. Mech Mater 33:11–20
Yuan X-G, Kuang Z-B (2008) Waves in pyroelectrics. J Therm Stress 31:1190–1211
Yuan X-G, Kuang Z-B (2010) The inhomogeneous waves in pyroelectrics. J Therm Stress 33:172–186
Zhou Z-D, Yang F-P, Kuang Z-B (2012) Reflection and transmission of plane waves at the interface of pyroelectric bi-materials. J Sound Vib 331:3558–3566
Zhu J-J, Kuang Z-B (1995) On interface crack propagation with variable velocity for longitudinal shear problems. Int J Fract 72:121–144

Chapter 7
Three-Dimensional and Applied Electroelastic Problems

Abstract In this chapter, there are mainly two kinds of problems discussed. The first kind of problems is the 3D electroelastic problems: the potential function method, the solutions of the penny-shaped crack and elliptic inclusions. The second kind of problems is the applied electroelastic problems which are used in engineering: simple electroelastic problems, laminated piezoelectric plates containing classical and higher-order theories and piezoelectric composite shells. A unified first-order approximate theory of an electro-magneto-elastic thin plate derived by the physical variational principle is given when the electromagnetic induction effect can be neglected.

Keywords Penny-shaped crack • Laminated piezoelectric plate • Piezoelectric composite shell

7.1 Potential Function Methods in Transversely Isotropic Piezoelectric Materials

7.1.1 Governing Equations

The governing equations of transversely isotropic piezoelectric materials have been discussed in previous chapters. In the material principle coordinates, the constitutive equations are

$$\begin{aligned}
&\sigma_1 = C_{11}u_{1,1} + C_{12}u_{2,2} + C_{13}u_{3,3} + e_{31}\varphi_{,3}, && \sigma_2 = C_{12}u_{1,1} + C_{11}u_{2,2} + C_{13}u_{3,3} + e_{31}\varphi_{,3}\\
&\sigma_3 = C_{13}u_{1,1} + C_{13}u_{2,2} + C_{33}u_{3,3} + e_{33}\varphi_{,3}, && \sigma_4 = \sigma_{23} = C_{44}(u_{2,3} + u_{3,2}) + e_{15}\varphi_{,2}\\
&\sigma_5 = \sigma_{31} = C_{44}(u_{1,3} + u_{3,1}) + e_{15}\varphi_{,1}, && \sigma_6 = \sigma_{12} = C_{66}(u_{2,1} + u_{1,2})\\
&D_1 = e_{15}(u_{1,3} + u_{3,1}) - \epsilon_{11}\varphi_{,1}, && D_2 = e_{24}(u_{2,3} + u_{3,2}) - \epsilon_{11}\varphi_{,1}\\
&D_3 = e_{31}u_{1,1} + e_{31}u_{2,2} + e_{33}u_{3,3} - \epsilon_{33}\varphi_{,3}; && C_{66} = (C_{11} - C_{12})/2
\end{aligned} \tag{7.1}$$

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The generalized momentum equations are

$$
\begin{aligned}
&\sigma_{1,1}+\sigma_{6,2}+\sigma_{5,3}=\rho\ddot{u}_1, \quad \sigma_{6,1}+\sigma_{2,2}+\sigma_{4,3}=\rho\ddot{u}_2, \quad \sigma_{5,1}+\sigma_{4,2}+\sigma_{3,3}=\rho\ddot{u}_3\\
&D_{1,1}+D_{2,2}+D_{3,3}=0
\end{aligned}
\tag{7.2}
$$

where $i=1,2,3$. Substitution of Eq. (7.1) into Eq. (7.2) yields

$$
\begin{aligned}
&C_{11}u_{1,11}+C_{66}u_{1,22}+C_{44}u_{1,33}+(C_{12}+C_{66})u_{2,12}+(C_{13}+C_{44})u_{3,13}\\
&\quad+(e_{15}+e_{31})\varphi_{,13}=\rho u_{1,tt}\\
&(C_{12}+C_{66})u_{1,12}+C_{66}u_{2,11}+C_{11}u_{2,22}+C_{44}u_{2,33}+(C_{13}+C_{44})u_{3,23}\\
&\quad+(e_{15}+e_{31})\varphi_{,23}=\rho u_{2,tt}\\
&(C_{13}+C_{44})(u_{1,13}+u_{2,23})+C_{44}\nabla^2u_3+C_{33}u_{3,33}+e_{15}\nabla^2\varphi+e_{33}\varphi_{,33}=\rho u_{3,tt}\\
&(e_{15}+e_{31})(u_{1,13}+u_{2,23})+e_{15}\nabla^2u_3+e_{33}u_{3,33}-\epsilon_{11}\nabla^2\varphi-\epsilon_{33}\varphi_{,33}=0\\
&\nabla^2u=u_{,11}+u_{,22}
\end{aligned}
\tag{7.3}
$$

7.1.2 General Solution of the Static Problem (I)

Wang and Zheng (1995) discussed the general solution of (7.3) for the static problem by introducing potential functions. They assume

$$
u_1=\psi_{,1}-\chi_{,2}, \quad u_2=\psi_{,2}+\chi_{,1}, \quad u_3=k_1\psi_{,3}, \quad \varphi=k_2\psi_{,3} \tag{7.4}
$$

where k_1 and k_2 are undetermined constants and ψ and χ are potential functions. Substitution of Eq. (7.4) into Eq. (7.3) yields

$$
\begin{aligned}
&C_{66}\nabla^2\chi+C_{44}\chi_{,33}=0; \quad \text{or}\\
&\nabla^2\chi+\partial^2\chi/\partial z_0^2=0, \quad z_0=s_0x_3, \quad s_0^2=C_{66}/C_{44}=1/\lambda_0
\end{aligned}
\tag{7.5}
$$

$$
\begin{aligned}
&C_{11}\nabla^2\psi+[C_{44}+k_1(C_{13}+C_{44})+k_2(e_{15}+e_{31})]\psi_{,33}=0\\
&[(C_{13}+C_{44})+k_1C_{44}+k_2e_{15}]\nabla^2\psi+(k_1C_{33}+k_2e_{33})\psi_{,33}=0\\
&[(e_{15}+e_{31})+k_1e_{33}-k_2\,\epsilon_{11}]\nabla^2\psi+(k_1e_{33}-k_2\,\epsilon_{33})\psi_{,33}=0
\end{aligned}
\tag{7.6}
$$

In order to have nontrivial solution of Eq. (7.6), the following relations must be held:

$$
\begin{aligned}
&\frac{C_{44}+k_1(C_{13}+C_{44})+k_2(e_{15}+e_{31})}{C_{11}}=\frac{k_1C_{33}+k_2e_{33}}{(C_{13}+C_{44})+k_1C_{44}+k_2e_{15}}\\
&\quad=\frac{k_1e_{33}-k_2\,\epsilon_{33}}{(e_{15}+e_{31})+k_1e_{33}-k_2\,\epsilon_{11}}=\lambda
\end{aligned}
\tag{7.7}
$$

Eliminating k_1 and k_2, a cubic algebra equation of λ is obtained:

$$\begin{aligned}
&A\lambda^3 + B\lambda^2 + C\lambda + D = 0 \\
&A = e_{15}^2 + C_{44}\epsilon_{11}, \quad D = -C_{11}^{-1}\left(e_{33}^2 C_{44} + \epsilon_{33}C_{11}C_{33}\right) \\
&B = C_{11}^{-1}\{2e_{15}^2 C_{13} - e_{31}^2 C_{44} + 2e_{15}(e_{31}C_{13} - e_{33}C_{11}) + \epsilon_{11}\left(C_{13}^2 + 2C_{13}C_{44}\right) \\
&\qquad - \epsilon_{11}C_{11}C_{33} - \epsilon_{33}C_{11}C_{44}\} \\
&C = C_{11}^{-1}\Big\{(e_{15}+e_{31})^2 C_{33} - 2e_{33}(e_{15}+e_{31})(C_{13}+C_{44}) - (C_{13}+C_{44})^{-1}[(e_{15}+e_{31})C_{33} \\
&\qquad -(e_{15}+e_{11})C_{11}]C_{44}e_{15} + \epsilon_{11}C_{44}C_{33} + e_{33}^2 C_{11} - \epsilon_{33}(C_{13}+C_{44})^2 + \epsilon_{33}\left(C_{44}^2 + C_{11}C_{33}\right)\Big\}
\end{aligned} \tag{7.8}$$

Assume root λ_1 is positive real and λ_2 and λ_3 are either a pair of conjugate complex roots with positive real parts or positive real roots. Corresponding to each λ_i, a potential function ψ_j in Eq. (7.6) can be obtained:

$$\nabla^2\psi_j + \lambda_j \frac{\partial^2\psi_j}{\partial x_3^2} = \nabla^2\psi_j + \frac{\partial^2\psi_j}{\partial z_j^2} = 0, \quad z_j = s_j x_3, \quad s_j^2 = 1/\lambda_j; \quad j = 1,2,3 \tag{7.9}$$

Substituting λ_j into Eq. (7.7), k_{1j} and k_{2j} can be obtained. So the general solution of Eq. (7.3) can be expressed in potential functions:

$$\begin{aligned}
&u_1 = (\psi_1+\psi_2+\psi_3)_{,1} - \chi_{,2}, \quad u_2 = (\psi_1+\psi_2+\psi_3)_{,2} + \chi_{,1}, \\
&u_3 = k_{11}\psi_{1,3} + k_{12}\psi_{2,3} + k_{13}\psi_{3,3}, \quad \varphi = k_{21}\psi_{1,3} + k_{22}\psi_{2,3} + k_{23}\psi_{3,3}
\end{aligned} \tag{7.10}$$

Usually the numerical method is used to solve λ_j in Eq. (7.8) due to its complex roots. As an example for material PZT-6B with material constants,

$$\begin{aligned}
&C_{11} = 168(\mathrm{MPa}), \quad C_{33} = 163, \quad C_{44} = 27.1, \quad C_{12} = 60, \quad C_{13} = 60 \\
&e_{31} = -0.9\left(\mathrm{C/m^2}\right), \quad e_{33} = 7.1, \quad e_{15} = 4.6, \quad \epsilon_{11} = 36 \times 10^{-10}(\mathrm{F/m}), \\
&\epsilon_{33} = 34 \times 10^{-10}
\end{aligned}$$

The solved $\boldsymbol{\lambda}$ is $\lambda_1 = 3.92, \quad \lambda_2 = 0.73 + 0.87\mathrm{i}, \quad \lambda_3 = 0.73 - 0.87\mathrm{i}$.

7.1.3 General Solution of the Dynamic Problem

Ding et al. (1996) discussed the dynamic problem. Let

$$u_1 = \psi_{,2} - \chi_{,1}, \quad u_2 = -\psi_{,1} - \chi_{,2} \tag{7.11}$$

where ψ and χ are potential functions, but their meanings are different with that in Sect. 7.1.2. Substituting Eq. (7.11) into the first two equations in Eq. (7.3) yields

$$\begin{aligned} &B_{,2}-A_{,1}=0, \quad B_{,1}+A_{,2}=0; \quad B=C_{66}\nabla^2\psi+C_{44}\psi_{,33}-\rho\psi_{,tt} \\ &A=C_{11}\nabla^2\chi+C_{44}\chi_{,33}-\rho\chi_{,tt}-(C_{13}+C_{44})u_{3,3}-(e_{15}+e_{31})\varphi_{,3} \end{aligned} \tag{7.12}$$

Let $A=H_{,2}, \quad B=H_{,1}$,and Eq. (7.12) is reduced to $\nabla^2 H=0$. One particular solution is $H=$ constant. Adopt a particular solution

$$A=0,\ B=0 \tag{7.13}$$

Using this result, substituting Eq. (7.11) into the last two equations in Eq. (7.3) and listing the results with Eq. (7.13) together we get

$$C_{66}\nabla^2\psi+C_{44}\psi_{,33}-\rho\psi_{,tt}=0 \tag{7.14}$$

$$\begin{aligned} &\boldsymbol{D}\boldsymbol{G}=\boldsymbol{0}, \quad \boldsymbol{G}=[\chi,u_3,\varphi]^T \\ &\boldsymbol{D}=\begin{pmatrix} C_{11}\nabla^2+C_{44}\dfrac{\partial^2}{\partial x_3^2}-\rho\dfrac{\partial^2}{\partial t^2} & -(C_{13}+C_{44})\dfrac{\partial}{\partial x_3} & -(e_{15}+e_{31})\dfrac{\partial}{\partial x_3} \\ -(C_{13}+C_{44})\nabla^2\dfrac{\partial}{\partial x_3} & C_{44}\nabla^2+C_{33}\dfrac{\partial^2}{\partial x_3^2}-\rho\dfrac{\partial^2}{\partial t^2} & e_{15}\nabla^2+e_{33}\dfrac{\partial^2}{\partial x_3^2} \\ (e_{15}+e_{31})\nabla^2\dfrac{\partial}{\partial x_3} & -\left(e_{15}\nabla^2+e_{33}\dfrac{\partial^2}{\partial x_3^2}\right) & \epsilon_{11}\nabla^2+\epsilon_{33}\dfrac{\partial^2}{\partial x_3^2} \end{pmatrix} \end{aligned} \tag{7.15}$$

Introduce a new function F and let $|\boldsymbol{D}|F=0$ or

$$\begin{aligned} |\boldsymbol{D}|F=\Big\{&a\frac{\partial^6}{\partial x_3^6}+b\nabla^2\frac{\partial^4}{\partial x_3^4}+c\nabla^4\frac{\partial^2}{\partial x_3^2}+d\nabla^6+g\nabla^2\frac{\partial^4}{\partial t^4}+h\nabla^4\frac{\partial^2}{\partial t^2} \\ &+k\nabla^2\frac{\partial^2}{\partial x_3^2}\frac{\partial^2}{\partial t^2}+l\frac{\partial^4}{\partial x_3^4}\frac{\partial^2}{\partial t^2}+m\frac{\partial^2}{\partial x_3^2}\frac{\partial^4}{\partial t^4}\Big\}F=0 \end{aligned} \tag{7.16}$$

where

$$\begin{aligned} a&=C_{44}\left(e_{33}^2+C_{33}\epsilon_{33}\right) \\ b&=C_{33}\left[C_{44}\epsilon_{11}+(e_{15}+e_{31})^2\right]+\epsilon_{33}\left[C_{11}C_{33}+C_{44}^2-(C_{13}+C_{44})^2\right] \\ &\quad+e_{33}[2C_{44}e_{15}+C_{11}e_{33}-2(C_{13}+C_{44})(e_{15}+e_{31})] \\ c&=C_{44}\left[C_{11}\,\epsilon_{33}+(e_{15}+e_{31})^2\right]+\epsilon_{11}\left[C_{11}C_{33}+C_{44}^2-(C_{13}+C_{44})^2\right] \\ &\quad+e_{15}[2C_{11}e_{33}+C_{44}e_{15}-2(C_{13}+C_{44})(e_{15}+e_{31})] \\ d&=C_{11}\left(e_{15}^2+C_{44}\epsilon_{11}\right), \quad g=\rho^2\epsilon_{11}, \quad h=-\rho\left[e_{15}^2+(C_{11}+C_{44})\epsilon_{11}\right] \\ k&=-\rho\left[2e_{15}e_{33}+(C_{33}+C_{44})\,\epsilon_{11}+(C_{11}+C_{44})\epsilon_{33}+(e_{15}+e_{31})^2\right] \\ l&=-\rho\left[e_{33}^2+(C_{33}+C_{44})\epsilon_{33}\right], \quad m=\rho^2\epsilon_{33} \end{aligned} \tag{7.17}$$

After solving F, it can be proved that the three group solutions of χ, u_3, φ are

$$\chi = A_{i1}F, \quad u_3 = A_{i2}F, \quad \varphi = A_{i3}F; \qquad i = 1,2,3 \tag{7.18}$$

where A_{ij} in Eq. (7.18) is the algebraic complement of $|\boldsymbol{D}|$, i.e.,

$$\begin{aligned}
A_{11} &= \left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right) + \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)^2 \\
A_{12} &= \left[\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13} + C_{44}) + \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(e_{15} + e_{31})\right]\nabla^2\frac{\partial}{\partial x_3} \\
A_{13} &= \left[\left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13} + C_{44}) - \left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)(e_{15} + e_{31})\right]\nabla^2\frac{\partial}{\partial x_3}
\end{aligned} \tag{7.19}$$

$$\begin{aligned}
A_{21} &= \left[\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13} + C_{44}) + \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(e_{15} + e_{31})\right]\frac{\partial}{\partial x_3} \\
A_{22} &= \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(\epsilon_{11}\nabla^2 + \epsilon_{33}\frac{\partial^2}{\partial x_3^2}\right) + (e_{15} + e_{31})^2\nabla^2\frac{\partial^2}{\partial x_3^2} \\
A_{23} &= \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right) - (C_{13} + C_{44})(e_{15} + e_{31})\nabla^2\frac{\partial^2}{\partial x_3^2}
\end{aligned} \tag{7.20}$$

$$\begin{aligned}
A_{31} &= \left[\left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)(e_{15} + e_{31}) - \left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right)(C_{13} + C_{44})\right]\frac{\partial}{\partial x_3} \\
A_{32} &= (e_{15} + e_{31})(C_{13} + C_{44})\nabla^2\frac{\partial^2}{\partial x_3^2} - \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(e_{15}\nabla^2 + e_{33}\frac{\partial^2}{\partial x_3^2}\right) \\
A_{33} &= \left(C_{11}\nabla^2 + C_{44}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(C_{44}\nabla^2 + C_{33}\frac{\partial^2}{\partial x_3^2} - \rho\frac{\partial^2}{\partial t^2}\right) - (C_{13} + C_{44})^2\nabla^2\frac{\partial^2}{\partial x_3^2}
\end{aligned} \tag{7.21}$$

By substitution of Eq. (7.18) into Eq. (7.11), the general solution is

$$u_1 = \psi_{,2} - A_{i1}F_{,1}, \quad u_2 = -\psi_{,1} - A_{i1}F_{,2}, \quad u_3 = A_{i2}F, \quad \varphi = A_{i3}F; \quad i = 1,2,3 \tag{7.22}$$

The general solution of an axial-symmetric problem can be obtained from Eq. (7.22) if let $\psi = 0$ and F is independent of θ.

7.1.4 General Solution of the Static Problem (II)

Let all the potential functions be independent of the time the general solution of the static problem can be obtained from the results in Sect. 7.1.3. Equation (7.14) yields

$$(\nabla^2 + \partial^2/\partial z_0^2)\psi_0 = 0 \quad z_0^2 = s_0^2 x_3^2, \quad s_0^2 = C_{66}/C_{44} \tag{7.23}$$

Equation (7.16) can be reduced to

$$\left(\nabla^2+\frac{\partial^2}{\partial z_1^2}\right)\left(\nabla^2+\frac{\partial^2}{\partial z_2^2}\right)\left(\nabla^2+\frac{\partial^2}{\partial z_3^2}\right)F=0 \quad z_i^2=s_i^2x_3^2, \quad i=1,2,3 \tag{7.24}$$

where s_i^2 is the root of the following equation:

$$as^6-bs^4+cs^2-d=0 \tag{7.25}$$

No loss of generality let s_1 be real and assume $\mathrm{Re}(s_i)>0$. It is easy to prove that F_i satisfying the following equation is the solution of Eq. (7.24):

$$\left(\nabla^2+\frac{\partial^2}{\partial z_i^2}\right)F_i=0, \quad z_i^2=s_i^2x_3^2, \quad i=1,2,3 \tag{7.26}$$

The general solutions of Eq. (7.24) are

1. $s_1^2\neq s_2^2\neq s_3^2;\quad F=F_1+F_2+F_3$ (7.27)
2. $s_1^2\neq s_2^2=s_3^2\quad F=F_1+F_2+x_3F_3$ (7.28a)
3. $s_1^2=s_2^2=s_3^2;\quad F=F_1+x_3F_2+x_3^2F_3;$ (7.28b)

From Eq. (7.26), it is obtained that $\nabla^2=-\partial^2/\partial z_i^2,\ \partial/\partial x_3=s_i\partial/\partial z_i$. Substituting these results into Eq. (7.20) yields

$$\begin{aligned}
&A_{21}=\left(\beta_1\nabla^2+\beta_2\frac{\partial^2}{\partial x_3^2}\right)\frac{\partial}{\partial x_3}=(\beta_2s_i^2-\beta_1)s_i\frac{\partial^3}{\partial z_i^3}\\
&A_{22}=C_{11}\,\epsilon_{11}\nabla^4+\beta_3\nabla^2\frac{\partial^2}{\partial x_3^2}+C_{44}\,\epsilon_{33}\frac{\partial^4}{\partial x_3^4}=\left(C_{44}\,\epsilon_{33}s_i^4-\beta_3s_i^2+C_{11}\,\epsilon_{11}\right)\frac{\partial^4}{\partial z_i^4}\\
&A_{23}=C_{11}e_{15}\nabla^4+\beta_4\nabla^2\frac{\partial^2}{\partial x_3^2}+C_{44}e_{33}\frac{\partial^4}{\partial z_3^4}=\left(C_{44}e_{33}s_i^4-\beta_4s_i^2+C_{11}e_{15}\right)\frac{\partial^4}{\partial z_i^4}\\
&\beta_1=\epsilon_{11}(C_{13}+C_{44})+e_{15}(e_{15}+e_{31}),\quad \beta_2=\epsilon_{33}(C_{13}+C_{44})+e_{33}(e_{15}+e_{31})\\
&\beta_3=C_{11}\epsilon_{33}+C_{44}\epsilon_{11}+(e_{15}+e_{31})^2,\quad \beta_4=C_{11}e_{33}+C_{44}e_{15}-(C_{13}+C_{44})(e_{15}+e_{31})
\end{aligned} \tag{7.29}$$

The general solution Eq. (7.22) can be rewritten as

$$\begin{aligned}
&u_1=\frac{\partial\psi}{\partial x_2}+\sum_{i=1}^{3}\alpha_{i1}s_i\frac{\partial^4F_i}{\partial x_1\partial z_i^3},\quad u_2=-\frac{\partial\psi}{\partial x_1}+\sum_{i=1}^{3}\alpha_{i1}s_i\frac{\partial^4F_i}{\partial x_2\partial z_1^3},\\
&u_3=\sum_{i=1}^{3}\alpha_{i2}\frac{\partial^4F_i}{\partial z_i^4},\quad \varphi=\sum_{i=1}^{3}\alpha_{i3}\frac{\partial^4F_i}{\partial z_i^4}\\
&\alpha_{i1}=\beta_1-\beta_2s_i^2,\quad \alpha_{i2}=C_{11}\,\epsilon_{11}-\beta_3s_i^2+C_{44}\,\epsilon_{33}s_i^4,\quad \alpha_{i3}=C_{11}e_{15}-\beta_4s_i^2+C_{44}e_{33}s_i^4
\end{aligned} \tag{7.30}$$

If let $\alpha_{1i}s_i\partial^3 F_i/\partial z_i^3 = \psi_i,\ \psi_0 = -\psi$, Eq. (7.30) can be reduced to

$$u_1 = -\frac{\partial \psi_0}{\partial x_2} + \sum_{i=1}^{3}\frac{\partial \psi_i}{\partial x_1}, \quad u_2 = \frac{\partial \psi_0}{\partial x_1} + \sum_{i=1}^{3}\frac{\partial \psi_i}{\partial x_2}, \quad u_3 = \sum_{i=1}^{3} k_{i1}\frac{\partial \psi_i}{\partial z_i}$$
$$\varphi = \sum_{i=1}^{3} k_{i2}\frac{\partial \psi_i}{\partial z_i}; \quad k_{i1} = \alpha_{i2}/\alpha_{i1}s_i, \quad k_{i2} = \alpha_{i3}/\alpha_{i1}s_i \tag{7.31}$$

Equations (7.31) and (7.10) are formally the same.

7.2 A Penny-Shaped Crack in Transversely Isotropic Material

7.2.1 Governing Equations

Consider a transversely isotropic piezoelectric material weakened by a flat impermeable crack of radius a occupied region S in the plane $x_3 = 0$, subjected to distributed pressure $-p(x_1, x_2)$ and surface electric charge $q(x_1, x_2)$ (Fig. 7.1). In Fig. 7.1 only a pair of concentrated force and electric charge is shown. The Cartesian coordinates (x_1, x_2, x_3) and cylindrical coordinates (r, θ, z) are adopted simultaneously. Using the symmetry with respect to the crack surface, this problem can be reduced to a mixed boundary value problem for a half space subjected to the following boundary conditions:

$$\begin{aligned} &\sigma_{33} = -p(x_1, x_2), \quad D_3 = q(x_1, x_2), \quad \text{when} \quad (x_1, x_2) \in S \\ &u_3 = \varphi = 0, \quad \text{when} \quad (x_1, x_2) \notin S; \quad \sigma_{31} = \sigma_{32} = 0, \quad -\infty < (x_1, x_2) < \infty \end{aligned} \tag{7.32}$$

Chen and Shioya (1999) extended the method proposed by Fabrikant (1989) in the elasticity, to solve above problem. Introduce notation $\Lambda = \partial/\partial x_1 + \mathrm{i}\partial/\partial x_2$ and complex displacement $U = u_1 + \mathrm{i}u_2$. Let $w = x_3$, the generalized momentum equations in complex displacement is

$$\begin{aligned} &(1/2)(C_{11} + C_{66})\nabla^2 U + C_{44}U_{,33} + (1/2)(C_{11} - C_{66})\Lambda^2\bar{U} + (C_{13} + C_{44})\Lambda w_{,3} \\ &\quad + (e_{15} + e_{31})\Lambda\varphi_{,3} = 0 \\ &(1/2)(C_{13} + C_{44})(\bar{\Lambda}U + \Lambda\bar{U})_{,3} + C_{44}\nabla^2 w + C_{33}w_{,33} + e_{15}\nabla^2\varphi + e_{33}\varphi_{,33} = 0 \\ &(1/2)(e_{15} + e_{31})(\bar{\Lambda}U + \Lambda\bar{U})_{,3} + e_{15}\nabla^2 w + e_{33}w_{,33} - \epsilon_{11}\nabla^2\varphi - \epsilon_{33}\varphi_{,33} = 0 \end{aligned} \tag{7.33}$$

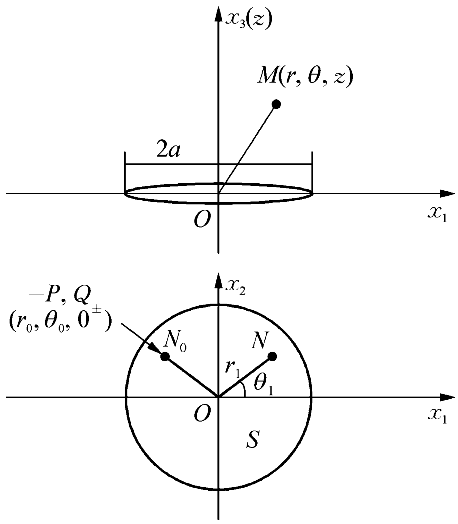


Fig. 7.1 A penny-shaped crack in a transversely isotropic piezoelectric material

where $\bar{U}, \bar{\Lambda}$ mean the conjugate value of U, Λ. By using the complex displacement, the general solution in potential functions, Eq. (7.31), become

$$U = \Lambda\left(\sum_{i=1}^{3}\psi_i + \mathrm{i}\psi_0\right), \quad w = \sum_{i=1}^{3} k_{i1}\frac{\partial\psi_i}{\partial z_i}, \quad \varphi = \sum_{i=1}^{3} k_{i2}\frac{\partial\psi_i}{\partial z_i} \tag{7.34}$$

The generalized stresses become

$$\begin{aligned}
&\sigma_{11} + \sigma_{22} = 2\nabla^2\sum_{i=1}^{3}(C_{11} - C_{66} - C_{13}s_i k_{i1} - e_{31}s_i k_{i2})\psi_i \\
&\sigma_{11} - \sigma_{22} + 2\mathrm{i}\sigma_{12} = 2C_{66}\Lambda^2(\psi_1 + \psi_2 + \psi_3 + \mathrm{i}\psi_0) \\
&\sigma_{31} + \mathrm{i}\sigma_{32} = \Lambda\left\{\sum_{i=1}^{3}[C_{44}(k_{i1} + s_i) + e_{15}k_{i2}]\frac{\partial\psi_i}{\partial z_i} + \mathrm{i}s_4 C_{44}\frac{\partial\psi_0}{\partial z_0}\right\} \\
&D_1 + \mathrm{i}D_2 = \Lambda\left\{\sum_{i=1}^{3}[e_{15}(k_{i1} + s_i) - \epsilon_{11}k_{i2}]\frac{\partial\psi_i}{\partial z_i} + \mathrm{i}s_4 e_{15}\frac{\partial\psi_0}{\partial z_0}\right\} \\
&\sigma_{33} = -\nabla^2\sum_{i=1}^{3}\gamma_{1i}\psi_i, \quad D_3 = -\nabla^2\sum_{i=1}^{3}\gamma_{2i}\psi_i, \quad \nabla^2 = -\partial^2/\partial z_i^2 \\
&\gamma_{1i} = -C_{13} + C_{33}s_i k_{i1} + e_{33}s_i k_{i2}, \quad \gamma_{2i} = -e_{31} + e_{33}s_i k_{i1} - \epsilon_{33}s_i k_{i2}
\end{aligned} \tag{7.35}$$

where s_i is the root of the Eq. (7.25).

7.2.2 Potential Theory Method of Crack Problem

The solution satisfying the boundary conditions in Eq. (7.32) can be expressed by two harmonic functions G and H:

$$\psi_i(z_i) = c_i G(z_i) + d_i H(z_i), \quad i = 1, 2, 3; \quad \psi_0(z_0) = 0 \tag{7.36}$$

where c_i, d_i are undetermined constants. G and H can be expressed by two potentials of a simple layer:

$$G(r, \theta, z) = \int_S \frac{\hat{u}(N)}{\rho(M, N)} \mathrm{d}S, \quad H(r, \theta, z) = \int_S \frac{\hat{\varphi}(N)}{\rho(M, N)} \mathrm{d}S \tag{7.37}$$

where $\hat{u}(N) = w(x_1, x_2, 0)$ and $\hat{\varphi}(N) = \hat{\varphi}(x_1, x_2, 0)$ are undetermined displacement and electric potential on the crack surface, respectively. $N(r_1, \theta_1, 0)$ is a point on S, $M(r, \theta, z)$ is a certain point in the material, and $\rho(M, N)$ is the distance between N and M. According to the property of the potential of a simple layer, the boundary conditions $w = \varphi = 0$ outside the crack in Eq. (7.37) are satisfied automatically. Inside the crack we have

$$(\partial G/\partial z)_{z=0} = -2\pi\hat{u}, \quad (\partial H/\partial z)_{z=0} = -2\pi\hat{\varphi} \tag{7.38}$$

Equations (7.34), (7.36), and (7.38) yield

$$\sum_{i=1}^{3} c_i k_{i1} = -\frac{1}{2\pi}, \quad \sum_{i=1}^{3} d_i k_{i1} = 0, \quad \sum_{i=1}^{3} c_i k_{i2} = 0, \quad \sum_{i=1}^{3} d_i k_{i2} = -\frac{1}{2\pi} \tag{7.39}$$

The boundary conditions of σ_{31}, σ_{32} in Eq. (7.32) demand

$$\sum_{i=1}^{3} c_i [C_{44}(k_{i1} + s_i) + e_{15} k_{i2}] = 0, \quad \sum_{i=1}^{3} d_i [C_{44}(k_{i1} + s_i) + e_{15} k_{i2}] = 0 \tag{7.40}$$

Combining Eqs. (7.39) and (7.40) yields

$$\begin{gathered} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{pmatrix} s_1 & s_2 & s_3 \\ k_{11} & k_{21} & k_{31} \\ k_{12} & k_{22} & k_{32} \end{pmatrix}^{-1} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}, \\ \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{pmatrix} s_1 & s_2 & s_3 \\ k_{11} & k_{21} & k_{31} \\ k_{12} & k_{22} & k_{32} \end{pmatrix}^{-1} \begin{Bmatrix} e_{15}/C_{44} \\ 0 \\ -1 \end{Bmatrix} \end{gathered} \tag{7.41}$$

$\hat{u}(N)$ and $\hat{\varphi}(N)$ can be determined from the first two boundary conditions in Eq. (7.32):

$$\begin{aligned}
p(N_0) &= -\eta_1 \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0,N)} \mathrm{d}S - \eta_2 \nabla^2 \iint_S \frac{\hat{\varphi}(N)}{\rho(N_0,N)} \mathrm{d}S, \\
q(N_0) &= -\eta_3 \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0,N)} \mathrm{d}S - \eta_4 \nabla^2 \iint_S \frac{\hat{\varphi}(N)}{\rho(N_0,N)} \mathrm{d}S \\
\eta_1 &= -\sum_{i=1}^{3} c_i \gamma_{1i}, \quad \eta_2 = -\sum_{i=1}^{3} d_i \gamma_{1i}, \quad \eta_3 = \sum_{i=1}^{3} c_i \gamma_{2i}, \quad \eta_4 = \sum_{i=1}^{3} d_i \gamma_{2i}
\end{aligned} \tag{7.42}$$

where $N_0(r_0, \theta_0, 0), N(r_1, \theta_1, 0) \in S$ and the integral is over all points on S. Equation (7.42) yields

$$\begin{aligned}
\eta_4 p(N_0) - \eta_2 q(N_0) &= -\frac{1}{4\pi^2 A} \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0,N)} \mathrm{d}S \\
\eta_1 q(N_0) - \eta_3 p(N_0) &= -\frac{1}{4\pi^2 A} \nabla^2 \iint_S \frac{\hat{\varphi}(N)}{\rho(N_0,N)} \mathrm{d}S, \quad A = \frac{1}{4\pi^2}(\eta_1\eta_4 - \eta_2\eta_3)
\end{aligned} \tag{7.43}$$

Equation (7.43) can be applied for a crack with any shape.

7.2.3 The Solution of a Circular Penny-Shaped Crack

For a circular penny-shaped crack of diameter $2a$, the solution of Eq. (7.43) is

$$\begin{aligned}
\hat{u} &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a \frac{1}{\rho} \arctan\left(\frac{\xi}{\rho}\right) [\eta_4 p(r_0, \theta_0) - \eta_2 q(r_0, \theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \\
\hat{\varphi} &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a \frac{1}{\rho} \arctan\left(\frac{\xi}{\rho}\right) [\eta_1 q(r_0, \theta_0) - \eta_3 p(r_0, \theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \\
\rho &= \sqrt{r_1^2 + r_0^2 - 2r_1 r_0 \cos(\theta - \theta_0)}, \quad \xi = \sqrt{(a^2 - r_1^2)(a^2 - r_0^2)} \Big/ a
\end{aligned} \tag{7.44}$$

Substituting Eq. (7.44) into (7.37) yields

$$\begin{aligned}
G(r, \theta, z) &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a K(r, \theta, z; r_0, \theta_0) [\eta_4 p(r_0, \theta_0) - \eta_2 q(r_0, \theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \\
H(r, \theta, z) &= \frac{2A}{\pi} \int_0^{2\pi} \int_0^a K(r, \theta, z; r_0, \theta_0) [\eta_1 q(r_0, \theta_0) - \eta_3 p(r_0, \theta_0)] r_0 \mathrm{d}r_0 \mathrm{d}\theta_0
\end{aligned} \tag{7.45}$$

The kernel function K in Eq. (7.45) is

$$K(M,N_0)=\int_0^{2\pi}\int_0^{a}\frac{1}{\rho(N,N_0)}\arctan\left[\frac{\sqrt{(a^2-r^2)(a^2-r_0^2)}}{a\rho(N,N_0)}\right]\frac{r_1\mathrm{d}r_1\mathrm{d}\theta_1}{\rho(M,N)} \tag{7.46}$$
$$K(M,N_0)=K(r,\theta,z;r_0,\theta_0)$$

Using the relation $\partial K/\partial z=-[2\pi/\rho(M,N_0)]\,\mathrm{arctan}[h/\rho(M,N_0)]$ (Fabrikant 1989), the derivatives of G and H in Eq. (7.45) are

$$\frac{\partial G}{\partial z}=-4A\int_0^{2\pi}\int_0^{a}\frac{1}{\rho(M,N_0)}\arctan\left[\frac{h}{\rho(M,N_0)}\right][\eta_4 p(r_0,\theta_0)-\eta_2 q(r_0,\theta_0)]r_0\mathrm{d}r_0\mathrm{d}\theta_0$$
$$\frac{\partial H}{\partial z}=-4A\int_0^{2\pi}\int_0^{a}\frac{1}{\rho(M,N_0)}\arctan\left[\frac{h}{\rho(M,N_0)}\right][\eta_1 q(r_0,\theta_0)-\eta_3 p(r_0,\theta_0)]r_0\mathrm{d}r_0\mathrm{d}\theta_0$$
$$h=\sqrt{(a^2-l^2)(a^2-r_0^2)}/a,\quad l=\left\{\sqrt{(r+a)^2+z^2}-\sqrt{(r-a)^2+z^2}\right\}\Big/2 \tag{7.47}$$

So for arbitrary polynomial distributed loadings p and q, all the generalized stresses can be expressed by elementary functions.

7.2.4 A Circular Penny-Shaped Crack Subjected to Generalized Concentrate Loading

Assume the penny-shaped crack is subjected to a pair of normal concentrated generalized loading $(-P,Q)$ at point $(r_0,\theta_0,0^{\pm}),r_0<a$ (Fig. 7.1). By using the general solution in the above section, after some manipulation, the generalized displacements and stresses can be obtained as

$$\begin{aligned}
u&=4A\sum_{i=1}^{3}[\tau_{i1}f_1(z_i)P+\tau_{i2}f_1(z_i)Q]\\
u_3&=-4A\sum_{i=1}^{3}k_{i1}[\tau_{i1}f_2(z_i)P+\tau_{i2}f_2(z_i)Q]\\
\varphi&=-4A\sum_{i=1}^{3}k_{i2}[\tau_{i1}f_2(z_i)P+\tau_{i2}f_2(z_i)Q]\\
\sigma_{33}&=4A\sum_{i=1}^{3}\gamma_{1i}[\tau_{i1}f_3(z_i)P+\tau_{i2}f_3(z_i)Q]\\
D_3&=4A\sum_{i=1}^{3}\gamma_{2i}[\tau_{i1}f_3(z_i)P+\tau_{i2}f_3(z_i)Q]
\end{aligned} \tag{7.48a}$$

where

$$f_1(z_i) = \frac{1}{t_0}\left\{\frac{z_i}{R_0}\arctan\frac{h_0}{R_0} - \frac{\sqrt{a^2-r_0^2}}{\alpha_0}\arctan\left(\frac{\bar{\alpha}_0}{\sqrt{m^2-a^2}}\right)\right\}, \quad f_2(z_i) = \frac{1}{R_0}\arctan\frac{h_0}{R_0}$$
$$f_3(z_i) = \frac{1}{R_0}\arctan\frac{h_0}{R_0} - \frac{h_0}{z_i\left(R_0^2+h_0^2\right)}\left(\frac{r^2-l^2}{m^2-l^2} - \frac{z_j^2}{R_0^2}\right), \quad \bar{\alpha}_0 = \sqrt{a^2 - rr_0 e^{-\mathrm{i}(\theta-\theta_0)}}$$
$$R_0 = \sqrt{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+z^2}, \quad m = \frac{1}{2}\left\{\sqrt{(r+a)^2+z^2} + \sqrt{(r-a)^2+z^2}\right\}$$
$$\tau_{i1} = c_i\eta_4 - d_i\eta_3, \;\; \tau_{i2} = d_i\eta_1 - c_i\eta_2, \;\; h_0 = \sqrt{(a^2-l^2)\left(a^2-r_0^2\right)}\Big/a, \;\; t_0 = re^{-\mathrm{i}\theta} - r_0 e^{-\mathrm{i}\theta_0} \tag{7.48b}$$

The generalized stress intensity factors are

$$K_{\mathrm{I}} = \frac{P}{\pi^{3/2}\sqrt{a}}\frac{\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}, \qquad K_D = \frac{Q}{\pi^{3/2}\sqrt{a}}\frac{\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)} \tag{7.49}$$

For the distributed loadings $(-p, q)$, the generalized stress intensity factors are

$$K_{\mathrm{I}} = \frac{\sqrt{2\pi}}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a \frac{p(r_0,\theta_0)\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)} r_0 \mathrm{d}r_0 \mathrm{d}\theta_0$$
$$K_{\mathrm{D}} = \frac{\sqrt{2\pi}}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a \frac{q(r_0,\theta_0)\sqrt{a^2-r_0^2}}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)} r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \tag{7.50}$$

For homogeneous distributed loadings $(-p_0, D_{30})$, the generalized stress intensity factors are

$$K_{\mathrm{I}} = 2p_0\sqrt{a/\pi}, \quad K_{\mathrm{D}} = 2D_{30}\sqrt{a/\pi} \tag{7.51}$$

From Eq. (7.51), it is seen that the stress intensity factor is determined independently by the mechanical loading and the electric displacement intensity factor is determined independently by the electric loading. Equation (7.51) obviously can be used to the case where homogeneous generalized stresses σ_{33} and $-D_3$ are applied at infinity.

There are many papers to discuss the penny-shaped crack, such as Huang (1997) and Wang (1992).

7.2.5 A Conducting Penny-Shaped Crack

Chen and Lim (2005) discussed the conducting penny-shaped crack. For a conducting crack, it adopts $\varphi = 0$ instead of $D_3 = q(x_1, x_2)$ in Eq. (7.32). The boundary conditions are

$$\begin{aligned}&\sigma_{33} = -p(x_1, x_2), \quad \text{when} \quad (x_1, x_2) \in S; \qquad w = 0, \quad \text{when} \quad (x_1, x_2) \notin S \\ &\varphi = 0, \qquad \sigma_{31} = \sigma_{32} = 0, \qquad -\infty < (x_1, x_2) < \infty\end{aligned} \tag{7.52}$$

According to $\varphi = 0, -\infty < (x_1, x_2) < \infty$ in Eq. (7.52), it is concluded that $H = 0$ in Eqs. (7.36) and (7.37). Equation (7.42) becomes

$$p(N_0) = -\eta_1 \nabla^2 \iint_S \frac{\hat{u}(N)}{\rho(N_0, N)} \mathrm{d}S, \quad \eta_1 = -\sum_{i=1}^{3} c_i \gamma_{1i} \tag{7.53}$$

Equation (7.45) is reduced to

$$G(r, \theta, z) = \frac{1}{2\pi^3 \eta_1} \int_0^{2\pi} \int_0^a K(r, \theta, z; r_0, \theta_0) p(r_0, \theta_0) r_0 \mathrm{d}r_0 \mathrm{d}\theta_0 \tag{7.54}$$

where $K(r, \theta, z; r_0, \theta_0)$ is still expressed by Eq. (7.46). The generalized displacements and stresses for a circular penny-shaped crack subjected to a concentrated force $-P$ are

$$\begin{aligned}&U = \frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} c_i f_1(z_i), \quad u_3 = -\frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} k_{i1} c_i f_2(z_i), \quad \varphi = -\frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} k_{i2} c_i f_2(z_i) \\ &\sigma_{33} = \frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} \gamma_{1i} c_i f_3(z_i), \quad D_3 = \frac{P}{\pi^2 \eta_1} \sum_{i=1}^{3} \gamma_{2i} c_i f_3(z_i)\end{aligned} \tag{7.55}$$

The functions in Eq. (7.55) are still expressed by Eq. (7.48b). The generalized stress intensity factors are

$$\begin{aligned}K_{\mathrm{I}} &= \frac{P}{\pi^{3/2}\sqrt{a}} \frac{\sqrt{a^2 - r_0^2}}{a^2 + r_0^2 - 2ar_0 \cos(\theta - \theta_0)}, \\ K_{\mathrm{D}} &= \frac{\beta}{\pi^{3/2}\sqrt{a}} \frac{\sqrt{a^2 - r_0^2}}{a^2 + r_0^2 - 2ar_0 \cos(\theta - \theta_0)}\end{aligned} \tag{7.56}$$

where $\beta = -\sum_{i=1}^{3} \gamma_{i1} c_i / \eta_1$. Above results are assumed that Eq. (7.27) or Eq. (7.24) has three different roots, $s_i, i = 1, 2, 3$. Chen and Lim (2005) also discussed other cases.

7.2.6 Solve an Impermeable Penny-Shaped Crack by Hankel Transform

In order to discuss the results of the Vickers indentation cracking of experiments, Jiang and Sun (2001) gave a solution of an impermeable penny-shaped crack with boundary conditions as shown in Eq. (7.32). They discussed an axisymmetric piezoelectric body under axisymmetric loading by using the Hankel transform method in cylindrical coordinates. In cylindrical coordinates, the constitutive equations are

$$\begin{Bmatrix} \sigma_r \\ \sigma_\vartheta \\ \sigma_z \\ \tau_{rz} \\ D_r \\ D_z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & -e_{31} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -e_{31} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & 0 & C_{44} & -e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & \epsilon_{11} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\vartheta \\ \varepsilon_z \\ \gamma_{rz} \\ E_r \\ E_z \end{Bmatrix} \tag{7.57}$$

The generalized geometric equations are

$$\varepsilon_r = u_{,r}, \quad \varepsilon_\vartheta = u/r, \quad \varepsilon_z = w_{,z}, \quad \gamma_{rz} = u_{,z} + w_{,r}; \quad E_r = -\varphi_{,r}, \quad E_z = -\varphi_{,z} \tag{7.58}$$

where u, w are the displacements along r, z directions, respectively.

The generalized equilibrium equations are

$$\begin{aligned} &\sigma_{r,r} + \tau_{rz,z} + (\sigma_r - \sigma_\vartheta)/r = 0, \quad \tau_{rz,r} + \sigma_{z,z} + \tau_{rz}/r = 0, \\ &\partial(rD_r)/\partial r + r\,\partial(D_z)/\partial z = 0 \end{aligned} \tag{7.59}$$

or

$$\begin{aligned} &C_{11}\left(u_{,rr} + \frac{1}{r}u_{,r} - \frac{u}{r^2}\right) + C_{44}u_{,zz} + (C_{13}+C_{44})w_{,rz} + (e_{15}+e_{31})\varphi_{,rz} = 0 \\ &(C_{13}+C_{44})\frac{1}{r}\left(ru_{,z}\right)_{,r} + C_{44}\left(w_{,rr} + \frac{1}{r}w_{,r}\right) + C_{33}w_{,zz} + e_{15}\left(\varphi_{,rr} + \frac{1}{r}\varphi_{,r}\right) + e_{33}\varphi_{,zz} = 0 \\ &(e_{15}+e_{31})\frac{1}{r}\left(ru_{,z}\right)_{,r} + e_{15}\left(w_{,rr} + \frac{1}{r}w_{,r}\right) + e_{33}w_{,zz} - \epsilon_{11}\left(\varphi_{,rr} + \frac{1}{r}\varphi_{,r}\right) - \epsilon_{33}\varphi_{,zz} = 0 \end{aligned} \tag{7.60}$$

Equation (7.32) becomes

$$\begin{aligned} &\sigma_z(r,0) = -p(r), \quad D_z(r,0) = q(r), \quad \text{when} \quad r < a \\ &w(r,0) = \varphi(r,0) = 0, \quad \text{when} \quad r > a; \quad \tau_{rz} = 0, \quad -\infty < (x_1, x_2) < \infty \end{aligned} \tag{7.61}$$

Introduce the Hankel transform pair with J_k; J_k is the Bessel's function of the first kind of order k:

$$\bar{F}(\xi, z) = \int_0^\infty F(r,z) r J_k(\xi r)\mathrm{d}r, \quad F(r,z) = \int_0^\infty \bar{F}(\xi,z)\xi J_k(\xi r)\mathrm{d}\xi \tag{7.62}$$

Applying the Hankel transform, Eq. (7,62) to Eq. (7,60) with $k=1$ for u and $k=0$ for w and φ yields

$$\begin{aligned}
&C_{44}\bar{u}'' - C_{11}\xi^2\bar{u} - (C_{13}+C_{44})\xi\bar{w}' - (e_{15}+e_{31})\xi\bar{\varphi}' = 0\\
&(C_{13}+C_{44})\xi\bar{u}' + C_{33}\bar{w}'' - C_{44}\xi^2\bar{w} + e_{33}\bar{\varphi}'' - e_{15}\xi^2\bar{\varphi} = 0\\
&(e_{15}+e_{31})\xi\bar{u}' + e_{33}\bar{w}'' - e_{15}\xi^2\bar{w} - \epsilon_{33}\bar{\varphi}'' + \epsilon_{11}\xi^2\bar{\varphi} = 0
\end{aligned} \tag{7.63}$$

where a prime indicates the derivative with respect to z. The solutions are assumed in the forms

$$\bar{u}(\xi,z) = \hat{u}(\xi)\mathrm{e}^{-\eta\,\xi\,z}, \quad \bar{w}(\xi,z) = \hat{w}(\xi)\mathrm{e}^{-\eta\,\xi\,z}, \quad \bar{\varphi} = \hat{\varphi}\mathrm{e}^{-\eta\,\xi\,z} \tag{7.64}$$

Substituting Eq. (7,64) into Eq. (7,63) yields

$$\begin{bmatrix} C_{44}\eta^2 - C_{11} & (C_{13}+C_{44})\eta & (e_{15}+e_{31})\eta \\ -(C_{13}+C_{44})\eta & C_{33}\eta^2 - C_{44} & e_{33}\eta^2 - e_{15} \\ -(e_{15}+e_{31})\eta & e_{33}\eta^2 - e_{15} & -\epsilon_{33}\eta^2 + \epsilon_{11} \end{bmatrix} \begin{Bmatrix} \hat{u}(\xi) \\ \hat{w}(\xi) \\ \hat{\varphi}(\xi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7.65}$$

In order to have nontrivial solutions for $\hat{u}$, $\hat{w}$ and $\hat{\varphi}$, it should obtain the characteristic equation

$$\eta^6 + B_1\eta^4 + B_2\eta^2 + B_3 = 0 \tag{7.66}$$

where B_1, B_2 and B_3 are coefficients constituted of material constants. Since coefficients in Eq. (7.66) are real, in general it has six roots. Because the upper half space ($z \geq 0$) is discussed, without loss of generality, we take eigenvalue η_1 is a real positive number and η_2, η_3 are, in general, a pair of complex conjugates with positive real part. One form of the corresponding eigenvectors is

$$\begin{aligned}
&\hat{u}_i(\xi) = \alpha_i\eta_i\hat{w}_i(\xi), \quad \alpha_i = \frac{(e_{15}+e_{31})\left(C_{33}\eta_i^2 - C_{44}\right) - \left(e_{33}\eta_i^2 - e_{15}\right)(C_{13}+C_{44})}{\left(e_{33}\eta_i^2 - e_{15}\right)\left(C_{44}\eta_i^2 - C_{11}\right) + (e_{15}+e_{31})(C_{13}+C_{44})\eta_i^2}\\
&\hat{\varphi}_i(\xi) = \gamma_i\eta_i\hat{w}_i(\xi), \quad \gamma_i = -\frac{\left(C_{44}\eta_i^2 - C_{11}\right)\alpha + (C_{13}+C_{44})}{(e_{15}+e_{31})\eta_i}, \quad i = 1,2,3
\end{aligned} \tag{7.67}$$

Let $\hat{w}_i(\xi) = (1/\eta_i\xi)f_i(\xi)$. Through the inverse transform, the general solutions are

$$\begin{aligned}
u(r,z) &= \sum_{i=1}^{3}\alpha_i\int_0^\infty f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_1(\xi r)\mathrm{d}\xi\\
\varphi(r,z) &= \sum_{i=1}^{3}\gamma_i\int_0^\infty f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi\\
w(r,z) &= \sum_{i=1}^{3}(1/\eta_i)\int_0^\infty f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi
\end{aligned} \tag{7.68}$$

Using relations $[J_1(\xi r)]_{,r} = \xi[J_0(\xi r) - (1/2)J_1(\xi r)],\ [J_0(\xi r)]_{,r} = -\xi J_1(\xi r)$, the generalized stresses are

$$\begin{aligned}
\sigma_z(r,z) &= \sum_{i=1}^{3}(C_{13}\alpha_i - C_{33} - e_{33}\eta_i\gamma_i)\int_0^\infty \xi f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi \\
\tau_{rz}(r,z) &= \sum_{i=1}^{3}(-C_{44}/\eta_i - C_{44}\eta_i\alpha_i - e_{15}\gamma_i)\int_0^\infty \xi f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_1(\xi r)\mathrm{d}\xi \\
D_z(r,z) &= \sum_{i=1}^{3}(e_{31}\alpha_i - e_{33} + \epsilon_{33}\eta_i\gamma_i)\int_0^\infty \xi f_i(\xi)\mathrm{e}^{-\eta_i\,\xi\,z}J_0(\xi r)\mathrm{d}\xi
\end{aligned} \tag{7.69}$$

For discussed penny-shaped crack (Fig. 7.1), since $\tau_{rz}(r,z)0 = 0$ due to symmetry, Eq. (7.69) yields

$$\sum_{i=1}^{3}(-C_{44}/\eta_i - C_{44}\eta_i\alpha_i - e_{15}\gamma_i)f_i = 0 \tag{7.70}$$

Eliminating f_3 from Eq. (7.68), (7.69), and (7.70) and then substituting the results into Eq. (7.61), two pairs of integral equations are obtained:

$$\begin{aligned}
&\int_0^\infty f_i(\xi)\xi J_0(\rho\xi)\mathrm{d}\xi = t_{i1}p + t_{i2}q, \quad \text{for} \quad \rho < 1; \quad \rho = r/a \\
&\int_0^\infty f_i(\xi)\xi J_0(\rho\xi)\mathrm{d}\xi = 0, \quad \text{for} \quad \rho > 1; \quad i = 1,2
\end{aligned} \tag{7.71}$$

where

$$\begin{aligned}
&t_{11} = k_{22}a^2/H, \quad t_{12} = -k_{12}a^2/H, \quad t_{21} = -k_{21}a^2/H, \quad t_{22} = k_{11}a^2/H \\
&k_{11} = C_{13}\alpha_1 - C_{33} - \gamma_1\eta_1 e_{33} + \Delta_1(C_{13}\alpha_3 - C_{33} - \gamma_3\eta_3 e_{33}) \\
&k_{12} = C_{13}\alpha_2 - C_{33} - \gamma_2\eta_2 e_{33} + \Delta_2(C_{13}\alpha_3 - C_{33} - \gamma_3\eta_3 e_{33}) \\
&k_{21} = e_{31}\alpha_1 - e_{33} + \gamma_1\eta_1\epsilon_{33} + \Delta_1(e_{13}\alpha_3 - e_{33} + \gamma_3\eta_3 e_{33}) \\
&k_{22} = e_{31}\alpha_2 - e_{33} + \gamma_2\eta_2\epsilon_{33} + \Delta_2(e_{13}\alpha_3 - e_{33} + \gamma_3\eta_3 e_{33}) \\
&H = k_{11}k_{22} - k_{12}k_{21}, \quad \Delta_1 = -\frac{C_{44}(1/\eta_1 + \eta_1\alpha_1) + e_{15}\gamma_1}{C_{44}(1/\eta_3 + \eta_3\alpha_3) + e_{15}\gamma_3}, \\
&\Delta_2 = -\frac{C_{44}(1/\eta_2 + \eta_2\alpha_2) + e_{15}\gamma_2}{C_{44}(1/\eta_3 + \eta_3\alpha_3) + e_{15}\gamma_3}
\end{aligned} \tag{7.72}$$

The solution of the dual integral equations in Eq. (7.71) is

$$f_i(\xi) = \frac{2}{\pi}\int_0^1 \mu\sin(\mu\xi)\mathrm{d}\mu\int_0^1 \frac{\rho[t_{i1}p(\rho) + t_{i2}q(\rho)]}{\sqrt{1-\rho^2}}\mathrm{d}\rho \tag{7.73}$$

When uniform pressure p_0 is applied over the area of radius $r = c < a$ and uniform charge q_0 applied over the area of $r = b < a$, we get

$$\sigma_z(\rho,0) = -\frac{2p_0}{\pi a}\left(1-\sqrt{1-(c/a)^2}\right)\left[\arcsin(1/\rho) - \frac{1}{\sqrt{\rho^2-1}}\right], \quad \rho \geq 1$$
$$D_z(\rho,0) = -\frac{2q_0}{\pi a}\left(1-\sqrt{1-(b/a)^2}\right)\left[\arcsin(1/\rho) - \frac{1}{\sqrt{\rho^2-1}}\right], \quad \rho \geq 1 \tag{7.74}$$

The solution for a point force $-P_0$ and point charge Q_0 acted at the crack center $(r = 0)$ can be obtained by using the limiting procedure $\lim_{c\to 0} \pi c^2 p_0 = P_0$ and $\lim_{b\to 0} \pi b^2 q_0 = Q_0$.

The generalized stress intensity factors for uniform loadings p_0 and q_0 applied over $r = a$ are the same as shown in Eq. (7.51). For the point loadings are

$$K_{\mathrm{I}} = P_0/(\pi a)^{3/2}, \quad K_{\mathrm{D}} = Q_0/(\pi a)^{3/2} \tag{7.75}$$

A modified stress intensity factor K_{I}^* for a semicircular surface crack in a homogeneous isotropic elastic material given by Cherepanov (1979) is

$$K_{\mathrm{I}}^* = \kappa(\theta)K_{\mathrm{I}}, \quad \kappa(\theta) = 1 + 0.2[(\pi - 2\theta)/\pi]^2 \tag{7.76}$$

In the Vicker's indentation cracking of experiments, a semicircular surface crack is located in an isotropic plane (x_1, x_2), the stress intensity factor can approximately adopt Eq. (7.76), but it should take $2P_0$ instead of P_0 in K_{I}.

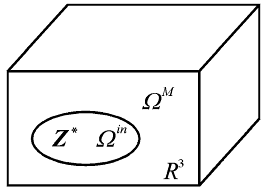
There were many literatures discussing the penny-shaped crack, such as Ueda (2007) which discussed a penny-shaped crack in a functionally graded piezoelectric strip under thermal loading.

7.3 Ellipsoidal Inclusion and Inhomogeneity

7.3.1 Basic Concept of Electroelastic Green Functions

A subdomain with prescribed eigenstrain in a matrix is usually called the inclusion, and material properties of the inclusion are the same with the matrix. A subdomain with different material properties from the matrix is usually called the inhomogeneity. The eigenstrain may be introduced by many physical phenomena, such as phase transformation strains, thermal strains, and plastic strains. The eigenstrain will produce self-equilibrium stresses in a constrained matrix. Eshelby's theory (1957)

Fig. 7.2 An elliptic inclusion



of the inclusion and inhomogeneity problems is important in the analyses of piezoelectric composite materials.

For convenience the notations given in Eq. (3.8) in Sect. 3.1 are adopted. A subscript in upper case takes the value 1,2,3, and 4 and a subscript in lower case takes the value 1,2, and 3. Figure 7.2 shows an ellipsoid inclusion occupied region Ω^{in} in a 3D space R^3. In Ω^{in} there is generalized eigenstrain $\boldsymbol{Z}^*\left(Z_{ij} = \varepsilon_{ij}^*, Z_{4j} = -E_j^*;\ i,j = 1,2,3;\ \text{without } Z_{44}\right)$. The constitutive equations with eigenstrain are

$$\Sigma_{iJ} = E_{iJKl}(Z_{Kl} - Z_{Kl}^*); \quad Z_{Kl}^*(\boldsymbol{x}) = Z_{Kl}^*, \quad \boldsymbol{x} \in \Omega; \quad Z_{Kl}^*(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \notin \Omega \quad (7.77)$$

where $Z_{Kl} = U_{K,l}, \boldsymbol{U} = [u_k, \varphi]^{\mathrm{T}}$. The generalized equilibrium equations are

$$\Sigma_{iJ,i} = -f_J; \qquad E_{iJKl}U_{K,li} = E_{iJKl}Z_{Kl,i}^*(\boldsymbol{x}) - f_J \quad (7.78)$$

where f_1, f_2, f_3 are components of the body force and $f_4 = -\rho_{\mathrm{e}}$ is the body electric charge density. From Eq. (7.78), it is seen that the role of $E_{iJKl}Z_{Kl,i}^*(\boldsymbol{x})$ is analogous to the body force.

The Green function $G_{KR,il}(\boldsymbol{x} - \boldsymbol{x}')$ in an infinite body is defined as

$$E_{iJKl}G_{KR,il}(\boldsymbol{x} - \boldsymbol{x}') + \delta_{JR}\delta(\boldsymbol{x} - \boldsymbol{x}') = \boldsymbol{0} \quad (7.79)$$

where δ_{JR} is the generalized Kronecker delta and $\delta(\boldsymbol{x} - \boldsymbol{x}')$ is the three-dimensional Dirac delta function. Except $\boldsymbol{x} = \boldsymbol{x}'$, $\delta(\boldsymbol{x} - \boldsymbol{x}') = 0$, and for a regular function $f(\boldsymbol{x})$,

$$\int_{-\infty}^{\infty} f(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')\mathrm{d}\boldsymbol{x}' = f(\boldsymbol{x}) \quad (7.80)$$

The Green function defined in Eq. (7.79) satisfies the generalized equilibrium equation. $G_{IJ}(\boldsymbol{x} - \boldsymbol{x}')$ and its derivative approach zero when $|\boldsymbol{x} - \boldsymbol{x}'| \to \infty$.

The electroelastic Green function $G_{IJ}(\boldsymbol{x} - \boldsymbol{x}')$ is extensively applied to study the inclusion and inhomogeneity problems in piezoelectric materials. $G_{ij}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the elastic displacement at $\boldsymbol{x}$ in the x_i direction due to a unit point force at $\boldsymbol{x}'$ in the x_j direction; $G_{i4}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the elastic displacement at $\boldsymbol{x}$ in the x_i direction due to a unit point charge at $\boldsymbol{x}'$; $G_{4j}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the electric potential at $\boldsymbol{x}$ due to a unit point force at $\boldsymbol{x}'$ in the x_j direction; and $G_{44}(\boldsymbol{x} - \boldsymbol{x}')$ denotes the electric potential at x due to a unit point charge at $\boldsymbol{x}'$.

7.3.2 Fourier Transform Method

Assume that the generalized eigenstrains and displacements can be expressed as (Mura 1987)

$$Z_{Kl}^*(\boldsymbol{x}) = \bar{Z}_{Kl}^*(\boldsymbol{\xi})e^{i\xi \cdot x}, \quad U_K(\boldsymbol{x}) = \bar{U}_K(\boldsymbol{\xi})e^{i\xi \cdot x} \tag{7.81}$$

Analogous to elasticity, substituting Eq. (7.81) into (7.78) and neglecting the body force yield

$$\begin{aligned} &\Pi_{JK}(\boldsymbol{\xi})\bar{U}_K = \Xi_J(\boldsymbol{\xi});, \qquad \Pi_{JK}(\boldsymbol{\xi}) = E_{iJKl}\xi_i\xi_l, \quad \Xi_J(\boldsymbol{\xi}) = -iE_{iJKl}\bar{Z}_{Kl}^*\xi_i \\ &\bar{U}_K = \Xi_J N_{JK}/D; \quad N_{MJ}(\boldsymbol{\xi}) = \frac{1}{2}\varpi_{IKL}\varpi_{JMN}\Pi_{KM}\Pi_{LN}, \quad D = |\Pi_{KM}| \end{aligned} \tag{7.82}$$

where ϖ_{JMN} is the permutation tensor and N_{MJ} is the algebraic complement of Π_{MJ} in the matrix $\boldsymbol{\Pi}$. For the general case, the Fourier integral transform is used (Mura 1987; Wang 1992):

$$\begin{aligned} Z_{Kl}^*(\boldsymbol{x}) &= \int_{-\infty}^{\infty} \bar{Z}_{Kl}^*(\boldsymbol{\xi})\, e^{i\xi \cdot x} d\xi, \quad \bar{Z}_{Kl}^*(\boldsymbol{\xi}) = \frac{1}{2\pi^3}\int_{-\infty}^{\infty} Z_{Kl}^*(\boldsymbol{x})\, e^{-i\xi \cdot x} d\boldsymbol{x} \\ U_K(\boldsymbol{x}) &= \int_{-\infty}^{\infty} \bar{U}_K(\boldsymbol{\xi})\, e^{i\xi \cdot x} d\xi, \quad \bar{U}_K(\boldsymbol{\xi}) = \frac{1}{2\pi^3}\int_{-\infty}^{\infty} U_K(\boldsymbol{x})\, e^{-i\xi \cdot x} d\boldsymbol{x} \end{aligned} \tag{7.83}$$

Analogous to elasticity, substituting Eq. (7.83) into (7.78) and using (7.82) yield

$$\begin{aligned} U_M(\boldsymbol{x}) &= -i\int_{-\infty}^{\infty} E_{iJKl}\bar{Z}_{Kl}^*(\boldsymbol{\xi})\xi_i N_{MJ}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\, e^{i\boldsymbol{\xi} \cdot x} d\boldsymbol{\xi} \quad \text{or} \\ U_M(\boldsymbol{x}) &= -\int_{-\infty}^{\infty} E_{iJKl}Z_{Kl}^*(\boldsymbol{x}')G_{MJ,i}(\boldsymbol{x}-\boldsymbol{x}')d\boldsymbol{x}' \\ G_{MJ}(\boldsymbol{x}-\boldsymbol{x}') &= \frac{1}{2\pi^3}\int_{-\infty}^{\infty} N_{MJ}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\, e^{i\boldsymbol{\xi} \cdot (x-x')} d\boldsymbol{\xi} \end{aligned} \tag{7.84}$$

where $G_{MJ,i}(\boldsymbol{x}-\boldsymbol{x}') = \partial G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')/\partial x_i = -\partial G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')/\partial x_i'$. If $G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')$ is a Green function in a finite region, then

$$U_M(\boldsymbol{x}) = \int_a E_{iJKl}Z_{Kl}^*(\boldsymbol{x}')n_i G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')da(x') - \int_V E_{iJKl}Z_{Kl,i}^*(\boldsymbol{x}')G_{MJ}(\boldsymbol{x}-\boldsymbol{x}')dV(\boldsymbol{x}') \tag{7.85}$$

7.3.3 Radon Transform Method

The Radon integral transform plays a fundamental role in the tomography, such as CT scanning in medicine. At first the basic concepts are introduced as follows (Deans 1983):

An arbitrary function $f(x_1, x_2)$ is defined on some domain Ω of a 2D plane $\boldsymbol{R}^2$. Let L be any line in the plane, then the mapping defined by the projection or line integral of f along all possible lines L is the 2D Radon transform of f, i.e.,

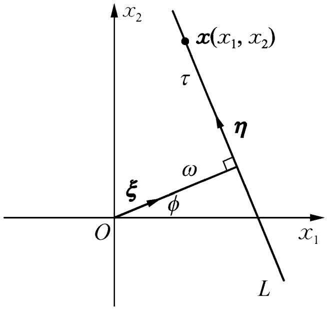


Fig. 7.3 A sketch of Radon transform

$$\bar{f}(\omega,\phi) = Rf = \int_L f(x_1,x_2)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(\omega\boldsymbol{\xi}+\tau\boldsymbol{\eta})\mathrm{d}\tau, \quad \text{or}$$
$$\bar{f}(\omega,\boldsymbol{\xi}) = \int_{R^2} f(\boldsymbol{x})\delta(\omega-\boldsymbol{\xi}\cdot\boldsymbol{x})\,\mathrm{d}a(\boldsymbol{x}) \tag{7.86}$$

where $\omega = \boldsymbol{\xi}\cdot\boldsymbol{x} = x_1\cos\phi + x_2\sin\phi$ is the perpendicular distance from the origin to L, $\boldsymbol{\xi} = (\cos\phi, \sin\phi)$ is a unit vector along $\boldsymbol{\omega}$ which defines the orientation of L, $\boldsymbol{\eta} = (-\sin\phi, \cos\phi)\perp\boldsymbol{\xi}$ is along L, τ is determined by $\boldsymbol{x} = \omega\boldsymbol{\xi}+\tau\boldsymbol{\eta}$, and ϕ is the angle between $\boldsymbol{\xi}$ and positive x_1-axis (Fig. 7.3). The last equation in Eq. (7.86) is easy extended to three- and higher-dimensional space. If $\bar{f}(\omega,\boldsymbol{\xi})$ is known for all ω and ϕ, then $\bar{f}(\omega,\boldsymbol{\xi})$ is a 2D Radon transform of $f(x_1,x_2)$. In an n-dimensional space, L represents $(n-1)$-dimensional hyperplane. Especially for a 3D space, L represents a plane. For n-dimensional space, the inversion Radon transform is

$$f(\boldsymbol{x}) = \frac{1}{2(2\pi\mathrm{i})^{n-1}}\boldsymbol{\Delta}_x^{(n-1)/2}\int_{|\boldsymbol{\xi}|=1} Rf(\boldsymbol{\xi},\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}a(\boldsymbol{\xi}) \tag{7.87}$$

where $\boldsymbol{\Delta}_x$ is the Laplacian operator. Especially for 3D space, Eq. (7.87) is reduced to

$$f(\boldsymbol{x}) = -\frac{1}{8\pi^2}\boldsymbol{\Delta}_3\int_{|\boldsymbol{\xi}|=1} Rf(\boldsymbol{\xi},\boldsymbol{\xi}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{\xi} = -\frac{1}{8\pi^2}\int_{|\boldsymbol{\xi}|=1}\left[\partial^2\bar{f}(\boldsymbol{\xi},\boldsymbol{\xi}\cdot\boldsymbol{x})/\partial\omega^2\right]_{\omega=\boldsymbol{\xi}\cdot x}\mathrm{d}a(\boldsymbol{\xi}) \tag{7.88}$$

Deeg (1980), Dunn and Taya (1993), and Dunn (1994) pointed out that the Radon transform can also be used to the electroelastic Green function $G_{IJ}(\boldsymbol{x}-\boldsymbol{x}')$, i.e.,

$$\bar{G}_{IJ}(\boldsymbol{\xi},\omega-\boldsymbol{\xi}\cdot\boldsymbol{x}') = \iint_{\boldsymbol{\xi}\cdot\boldsymbol{x}=\omega} G_{IJ}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}a(\boldsymbol{x})$$
$$G_{IJ}(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{8\pi^2}\iint_{|\boldsymbol{\xi}|=1}\left[\partial^2\bar{G}_{IJ}(\boldsymbol{\xi},\omega-\boldsymbol{\xi}\cdot\boldsymbol{x}')/\partial\omega^2\right]_{\boldsymbol{\xi}\cdot\boldsymbol{x}=\omega}\mathrm{d}a(\boldsymbol{\xi}) \tag{7.89}$$

where $\boldsymbol{\xi}$, ω are variables in the transform space and the integral domain is a 2D plane $\boldsymbol{\xi} \cdot \boldsymbol{x} = \omega$. In the inverse Radon transform, the integral domain is the surface of a unit sphere $|\boldsymbol{\xi}| = 1$. Using the Radon transform to Eq. (7.79), after some manipulation, yields

$$K_{JM}(\boldsymbol{\xi})(\partial^2/\partial x_k \partial x_k)\bar{G}_{MR}(\boldsymbol{\xi}, \omega - \boldsymbol{\xi} \cdot \boldsymbol{x}') + \delta_{JR}\delta(\omega - \boldsymbol{\xi} \cdot \boldsymbol{x}') = \mathbf{0} \tag{7.90}$$

where $K_{JM}(\boldsymbol{\xi}) = E_{iJMn}\xi_i\xi_n$. Using the inversion Radon transform from Eq. (7.90) yields

$$G_{MR}(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x} - \boldsymbol{x}'|} \iint_{|\boldsymbol{\xi}|=1} K_{MR}^{-1}(\boldsymbol{\xi})\delta(\boldsymbol{\xi} \cdot \boldsymbol{t})\mathrm{d}a(\boldsymbol{\xi}) \tag{7.91}$$

where $K_{JM}(\boldsymbol{\xi})K_{MR}^{-1}(\boldsymbol{\xi}) = \delta_{JR}$ and $\boldsymbol{t}$ is the unit vector along $\boldsymbol{x} - \boldsymbol{x}'$. Using the property of the Dirac delta, in an orthogonal coordinate system $\boldsymbol{t}$-$\boldsymbol{m}$-$\boldsymbol{n}$, Deeg (1980) reduced the integral in Eq. (7.91) to the following contour integral:

$$G_{MR}(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x} - \boldsymbol{x}'|} \int_C K_{MR}^{-1}(\boldsymbol{\xi})\delta(\boldsymbol{\xi} \cdot \boldsymbol{t})\mathrm{d}a(\boldsymbol{\xi}) \tag{7.92}$$

where C is the contour produced by $|\boldsymbol{\xi}| = 1$ in $\boldsymbol{m}$-$\boldsymbol{n}$ plane normal to $\boldsymbol{x} - \boldsymbol{x}'$. Compared to Fourier transform method, Eq. (7.92) is a simpler effective method to seek the Green function. The second partial derivatives of the electroelastic Green function is

$$G_{MR,kl}(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x} - \boldsymbol{x}'|} \frac{\partial^2}{\partial x_n \partial x_n} \int_{|\xi|=1} \xi_k \xi_l K_{MR}^{-1}(\boldsymbol{\xi})\delta[\boldsymbol{\xi} \cdot (\boldsymbol{x} - \boldsymbol{x}')]\mathrm{d}a(\boldsymbol{\xi}) \tag{7.93}$$

7.3.4 A Single Ellipsoidal Inclusion with Uniform Eigenstrains

Let the coordinates coincide with the principle axes of the material. For an ellipsoidal inclusion with uniform eigenstrains, the Eshelby's method is used. At first the inclusion is cut off from the matrix, so the inclusion and matrix are all free. The eigenstrains of the inclusion in the free state is denoted by $\boldsymbol{Z}^*$. The following generalized stresses are applied on the boundary of the inclusion:

$$T_J = -\Sigma_{iJ}^* \cdot n_i, \quad \Sigma_{iJ}^* = E_{iJMl}Z_{Ml}^* \tag{7.94}$$

Then put the inclusion subjected to above generalized stresses into the matrix. After this procedure, the original problem is transformed to a homogeneous material with a force $-T_J = E_{iJMl}Z^*_{Ml}$ acting on the counter which originally is the boundary of the inclusion. If Z^* is uniform, Eq. (7.94) yields (Deeg 1980)

$$U_{M,n}(\boldsymbol{\xi}) = -E_{iJKl}Z^*_{Kl}\int\int\int_{\Omega} G_{MJ,in}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x}') \tag{7.95}$$

Using Eq. (7.93) from Eq. (7.95) yields (Deeg 1980; Dunn and Taya 1993)

$$U_{M,n}(\boldsymbol{\xi}) = \frac{a_1a_2a_3}{4\pi}E_{iJKl}Z^*_{Kl}\int_{|\boldsymbol{\xi}|=1}\frac{1}{\alpha^3}\xi_k\xi_l K^{-1}_{MR}(\boldsymbol{\xi})\mathrm{d}a(\boldsymbol{\xi}), \quad \alpha = \sqrt{a_1^2\xi_1^2 + a_2^2\xi_2^2 + a_3^2\xi_3^2} \tag{7.96}$$

Analogous to the elastic inclusion problem, the generalized strains inside the ellipsoid induced by the uniform eigenstrains are also uniform and

$$\begin{aligned}
&Z_{Mn}(\boldsymbol{\xi}) = S_{MnKl}Z^*_{Kl}\\
&S_{MnKl} = \begin{cases}(1/8\pi)E_{iJKl}(I_{MJin}+I_{nJiM}), & M=1,2,3\\ (1/4\pi)E_{iJKl}I_{4Jin}, & M=4\end{cases}\\
&I_{MJin} = a_1a_2a_3\int_{|\boldsymbol{\xi}|=1}\left(1/\alpha^3\right)G_{MJin}(\boldsymbol{\xi})\mathrm{d}a(\boldsymbol{\xi}), \quad G_{MJin}(\boldsymbol{\xi}) = \xi_i\xi_n K^{-1}_{MJ}(\boldsymbol{\xi})
\end{aligned} \tag{7.97}$$

where S_{MnKl} is called the “electroelastic Eshelby tensor,” but it is not a tensor, i.e., it does not obey the tensor transform rule under a coordinate transformation. The key point to solve S_{MnKl} is to calculate $I_{MJin}(\boldsymbol{\xi})$. $I_{MJin}(\boldsymbol{\xi})$ can be transformed to the following form (Mikata 2000):

$$\begin{aligned}
I_{MJin}(\boldsymbol{\xi}) &= \int_{|\xi|=1} G_{MJin}(y_1/a_1, y_2/a_{12}, y_3/a_3)\mathrm{d}a(\boldsymbol{\xi})\\
&= \int_{-1}^{1}\mathrm{d}t\int_0^{2\pi} G_{MJin}(y_1/a_1, y_2/a_{12}, y_3/a_3)\mathrm{d}\phi\\
y_1 = \sqrt{1-t^2}\cos\phi,& \quad y_2 = \sqrt{1-t^2}\sin\phi, \quad y_3 = t
\end{aligned} \tag{7.98}$$

For transversely isotropic piezoelectric materials, Mikata (2000) pointed out that in $I_{MJin}(\boldsymbol{\xi})$ only $I_{1212}, I_{1313}, I_{1314}, I_{2323}, I_{2324}$ and $I_{11MJ}, I_{22MJ}, I_{33MJ}, MJ = 11, 22, 33, 44, 34$ are not zero, and the Eshelby tensor only has 36 components:

$S_{1111}, S_{1122}, S_{1133}, S_{1143}, S_{1212} = S_{1221} = S_{2112} = S_{2121}, S_{1313} = S_{1331} = S_{3113} = S_{3131}, S_{1341} = S_{3141}, S_{2211}, S_{2222}, S_{2233}, S_{2243}, S_{2323} = S_{2332} = S_{3223} = S_{3232}, S_{2342} = S_{3242}, S_{3311}, S_{3322}, S_{3333}, S_{3343}, S_{4113}, S_{4141}, S_{4223}, S_{4242}, S_{4311}, S_{4322}, S_{4333}, S_{4343}.$

For an elliptic cylindrical inclusion along the x_3-axis, the Eshelby tensor with 22 components is

$$\begin{aligned}
&S_{1111}=\frac{\alpha}{2(1+\alpha)^2}\left(3+\frac{C_{12}}{C_{11}}+\frac{2}{\alpha}\right),\quad S_{1122}=\frac{\alpha}{2(1+\alpha)^2}\left[\frac{(2+\alpha)C_{12}}{\alpha C_{11}}+\frac{2}{\alpha}\right]-1,\\
&S_{1133}=\frac{C_{13}}{(1+\alpha)C_{11}},\quad S_{1143}=\frac{e_{31}}{(1+\alpha)C_{11}},\quad S_{1212}=\frac{\alpha}{2(1+\alpha)^2}\left(\frac{1+\alpha+\alpha^2}{\alpha}-\frac{C_{12}}{C_{11}}\right),\\
&S_{1313}=\frac{1}{2(1+\alpha)},\quad S_{2233}=\frac{\alpha C_{13}}{(1+\alpha)C_{11}},\quad S_{2243}=\frac{\alpha e_{13}}{(1+\alpha)C_{11}},\quad S_{2323}=\frac{\alpha}{2(1+\alpha)},\\
&S_{2211}=\frac{\alpha}{2(1+\alpha)^2}\left[(1+2\alpha)\frac{C_{12}}{C_{11}}-1\right],\quad S_{2222}=\frac{\alpha}{2(1+\alpha)^2}\left(3+\frac{C_{12}}{C_{11}}+2\alpha\right),\\
&S_{4141}=\frac{1}{(1+\alpha)},\quad S_{4242}=\frac{\alpha}{(1+\alpha)},\quad \text{other}\ S_{MnKl}=0\\
&S_{1212}=S_{1221}=S_{2112}=S_{2121},S_{1313}=S_{1331}=S_{3113}=S_{3131},S_{2323}=S_{2332}=S_{3223}=S_{3232}
\end{aligned} \tag{7.99a}$$

For a penny-shaped crack perpendicular to x_3-axis, the Eshelby tensor with 18 components is

$$\begin{aligned}
&S_{1313}=S_{1331}=S_{3113}=S_{3131}=S_{2323}=S_{2332}=S_{3223}=S_{3232}=1/2\\
&S_{1341}=S_{3141}=S_{2342}=S_{3242}=\frac{e_{15}}{2C_{44}},\quad S_{3311}=S_{3322}=\frac{C_{13}\,\epsilon_{33}+e_{31}e_{33}}{C_{33}\,\epsilon_{33}+e_{33}^2}\\
&S_{3333}=S_{4343}=1,\quad S_{4311}=S_{4322}=\frac{C_{13}e_{33}-C_{33}e_{31}}{C_{33}\,\epsilon_{33}+e_{33}^2},\quad \text{other}\ S_{MJin}=0
\end{aligned} \tag{7.99b}$$

7.3.5 Ellipsoid Inhomogeneity

Analogous to elastic inhomogeneity problem, the electroelastic inhomogeneity problem can be handled by the equivalent inclusion method. Let E^{M}_{iJKl} and E^{in}_{iJKl} denote the material constants in the matrix and inhomogeneity, respectively. The generalized stress Σ^0_{Mn} is applied at infinity. Obviously for a homogeneous material, the generalized stress in material is also Σ^0_{Mn} and the corresponding strain is $Z^0_{Mn}=\left(E^M_{iJMn}\right)^{-1}\Sigma^0_{Mn}$. Assume the strain due to the inhomogeneity is Z_{Mn} and then the stress in the matrix and inhomogeneity are, respectively,

$$\Sigma^{\mathrm{M}}_{iJ}(\boldsymbol{x})=E^{\mathrm{M}}_{iJMn}\left[Z^0_{Mn}+Z_{Mn}(\boldsymbol{x})\right],\quad \Sigma^{\mathrm{in}}_{iJ}(\boldsymbol{x})=E^{\mathrm{in}}_{iJMn}\left[Z^0_{Mn}+Z_{Mn}(\boldsymbol{x})\right] \tag{7.100}$$

The key point of the equivalent inclusion method is that let the inhomogeneity possess the same material constants with the matrix, but the artificial eigenstrain Z^*_{Mn} is added. Then let

$$\Sigma^{\mathrm{in}}_{iJ}(\boldsymbol{x})=E^{\mathrm{in}}_{iJMn}\left[Z^0_{Mn}+Z_{Mn}(\boldsymbol{x})\right]=E^{\mathrm{M}}_{iJMn}\left[Z^0_{Mn}+Z_{Mn}(\boldsymbol{x})+Z^*_{Mn}\right] \tag{7.101}$$

Using $Z_{Mn}(\boldsymbol{x}) = S_{MnKl} Z^*_{Kl}$ in the inhomogeneity, so Eq. (7.101) yields

$$Z^*_{Kl} = \left[\left(E^{\mathrm{M}}_{iJMn} - E^{\mathrm{in}}_{iJMn}\right) S_{MnKl} + E^{\mathrm{M}}_{iJKl}\right]^{-1} \left(E^{\mathrm{in}}_{iJMn} - E^{\mathrm{M}}_{iJMn}\right) Z^0_{Mn} \tag{7.102}$$

Solving Z^*_{Kl}, the problem can be solved.

As an example an ellipsoidal piezoelectric sensor embedded in an elastic material is discussed (Fan and Qin 1995). The constitutive equations of the sensor (as an elliptic inclusion) and matrix are, respectively,

$$\sigma_{ij} = C^{\mathrm{in}}_{ijkl} \varepsilon_{kl} - e^{\mathrm{in}}_{kij} E_k, \quad D_i = e^{\mathrm{in}}_{ikl} \varepsilon_{kl} + \epsilon^{\mathrm{in}}_{ik} E_k; \quad \text{in } \Omega^{\mathrm{in}} \tag{7.103}$$

$$\sigma_{ij} = C^{\mathrm{M}}_{ijkl} \varepsilon_{kl}, \quad D_i = \epsilon^{\mathrm{M}}_{ik} E_k; \quad \text{in } \Omega^{\mathrm{M}} \tag{7.104}$$

Comparing Eqs. (7.103) and (7.104), one can consider the terms $e^{\mathrm{in}}_{kij} E_k, e^{\mathrm{in}}_{ikl} \varepsilon_{kl}$ in Eq. (7.103) produced by some kind of eigenstrains. When the matrix is subjected to uniform generalized stresses $\left(\sigma^0_{ij}, E^0_i\right)$ at infinity, the generalized stresses in the sensor are changed to

$$\sigma^{\mathrm{in}}_{ij} = \sigma^0_{ij} + \sigma'^{\mathrm{in}}_{ij} = C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + \varepsilon_{kl} - \varepsilon^{\mathrm{E}}_{kl}\right), \quad C^{\mathrm{in}}_{ijkl} \varepsilon^{\mathrm{E}}_{kl} = e^{\mathrm{in}}_{kij}\left(E_k + E^0_k\right) \tag{7.105}$$

$$D^{\mathrm{in}}_i = D^0_i + D'^{\mathrm{in}}_i = \epsilon^{\mathrm{in}}_{ik}\left(E^0_k + E_k - E^{\varepsilon}_k\right), \quad -\epsilon^{\mathrm{in}}_{ik} E^{\varepsilon}_k = e^{\mathrm{in}}_{ikl}\left(\varepsilon_{kl} + \varepsilon^0_{kl}\right) \tag{7.106}$$

By using the equivalent inclusion method, the original inhomogeneity problem subjected to uniform generalized stresses at infinity can be decoupled into two equivalent inclusion problems:

1. The elastic equivalent inclusion problem. Equation (7.105) can also be written as

$$\sigma^{\mathrm{in}}_{ij} = \sigma^0_{ij} + \sigma'^{\mathrm{in}}_{ij} = C^{\mathrm{M}}_{ijkl}\left(\varepsilon^0_{kl} + \varepsilon_{kl} - \varepsilon^{\mathrm{E}}_{kl} - \varepsilon^*_{kl}\right) \tag{7.107}$$

where ε^*_{kl} is the virtual eigenstrain. Using the Eshelby inclusion theory yields

$$\varepsilon_{kl} = S_{klmn} \varepsilon^{**}_{kl}, \quad \varepsilon^{**}_{kl} = \varepsilon^{\mathrm{E}}_{kl} + \varepsilon^*_{kl} \tag{7.108}$$

2. The dielectric equivalent inclusion problem. Equation (7.106) can also be written as

$$D^{\mathrm{in}}_i = D^0_i + D'^{\mathrm{in}}_i = \epsilon^{\mathrm{M}}_{ik}\left(E^0_k + E_k - E^{\varepsilon}_k - E^*_k\right) \tag{7.109}$$

where E^*_k is the virtual eigen electric field. Using the Eshelby inclusion theory yields

$$E_k = s_{kl} E^{**}_l, \quad E^{**}_k = E^{\varepsilon}_k + E^*_k \tag{7.110}$$

where $s_{kl} = S_{4k4l}$. Comparing Eqs. (7.105) and (7.107), it is concluded that

$$\begin{aligned} C^{\mathrm{M}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl} - \varepsilon^{**}_{kl}\right) &= C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl} - \varepsilon^{\mathrm{E}}_{kl}\right) \\ &= C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl}\right) - e^{\mathrm{in}}_{kij}\left(s_{kl}E^{**}_l + E^0_k\right) \end{aligned} \tag{7.111}$$

Comparing Eqs. (7.106) and (7.110), it is concluded that

$$\begin{aligned} &\epsilon^{\mathrm{in}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{\varepsilon}_k\right) = \epsilon^{\mathrm{M}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{**}_k\right), \qquad \text{or} \\ &E^{**}_l = \left[s_{ml}\left(\epsilon^{\mathrm{M}}_{im} - \epsilon^{\mathrm{in}}_{im}\right) - \epsilon^{\mathrm{M}}_{ik}\right]^{-1}\left[\left(\epsilon^{\mathrm{in}}_{im} - \epsilon^{\mathrm{M}}_{im}\right)E^0_k + e^{\mathrm{in}}_{imn}S_{mnpq}\varepsilon^{**}_{pq} + e^{\mathrm{in}}_{ikl}\varepsilon^0_{kl}\right] \end{aligned} \tag{7.112}$$

Solving ε^{**}_{kl} and E^{**}_k, the stress and electric fields are obtained in the sensor, i.e.,

$$\begin{aligned} \sigma^{\mathrm{in}}_{ij} &= C^{\mathrm{M}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl} - \varepsilon^{**}_{kl}\right) = C^{\mathrm{in}}_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^{**}_{kl}\right) - e^{\mathrm{in}}_{kij}\left(s_{kl}E^{**}_l + E^0_k\right) \\ D^{\mathrm{in}}_i &= \epsilon^{\mathrm{M}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{**}_k\right) = \epsilon^{\mathrm{in}}_{ik}\left(E^0_k + s_{kl}E^{**}_l - E^{\varepsilon}_k\right) \end{aligned} \tag{7.113}$$

The stress $\sigma^{\mathrm{out}}_{ij}$ and electric fields E^{out} in the matrix can be solved as follows:

$$\begin{aligned} &\sigma^{\mathrm{out}}_{ij} = \sigma^0_{ij} + \sigma'^{\mathrm{out}}_{ij} = \sigma^0_{ij} + [\![\sigma_{ij}]\!] + \sigma'^{\mathrm{in}}_{ij}, \quad [\![\sigma_{ij}]\!] = \sigma'^{\mathrm{out}}_{ij} - \sigma'^{\mathrm{in}}_{ij} = \sigma^{\mathrm{out}}_{ij} - \sigma^{\mathrm{in}}_{ij} \\ &E^{\mathrm{out}}_i = E^0_i + E'^{\mathrm{out}}_i = E^0_i + [\![E_i]\!] + E'^{\mathrm{in}}_i, \quad [\![E_i]\!] = E'^{\mathrm{out}}_i - E'^{\mathrm{in}}_i = E^{\mathrm{out}}_i - E^{\mathrm{in}}_i \end{aligned} \tag{7.114}$$

The displacement $\boldsymbol{u}$ and the surface traction $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0}$ across the interface of inclusion and matrix must be continuous, and jump of the displacement gradient $\nabla \otimes \boldsymbol{u}$ must be normal to the interface. The continuous conditions of the electric field and electric displacement $\boldsymbol{E} \times \boldsymbol{n}$ and $\boldsymbol{D} \cdot \boldsymbol{n}$ across the interface demand that the jump of $\boldsymbol{E}$ must be normal to the interface. So

$$\begin{aligned} &[\![u_i]\!] = u^{\mathrm{out}}_i - u^{\mathrm{in}}_i = 0, \quad [\![u_{i,j}]\!] = u^{\mathrm{out}}_{i,j} - u^{\mathrm{in}}_{i,j} = \lambda_i n_j; \quad [\![\sigma_{ij}]\!]n_j = 0 \\ &[\![E_i]\!] = \eta n_i, \quad [\![D_i]\!]n_i = 0 \end{aligned} \tag{7.115}$$

where $\boldsymbol{\lambda}, \eta$ are proportional constants. Substituting the constitutive equations into Eq. (7.115), it is obtained:

$$C_{ijkl}\lambda_k n_l n_j = -C_{ijkl}\varepsilon^{**}_{kl}n_j, \quad \eta\epsilon_{ik}n_i n_k = -\epsilon_{ik}n_i E^{**}_k \tag{7.116}$$

Therefore, the stress $\sigma^{\mathrm{out}}_{ij}$ and electric fields E^{out} in the matrix are

$$\sigma^{\mathrm{out}}_{ij} = \sigma^0_{ij} + C_{ijkl}\left(\lambda_k n_l + \varepsilon^{**}_{kl}\right) + \sigma'^{\mathrm{in}}_{ij}, \quad E^{\mathrm{out}}_i = E^0_i + \epsilon_{ik}\left(\eta n_k + E^{**}_k\right) + E'^{\mathrm{in}}_i \tag{7.117}$$

7.4 Some Simpler Practical Problems

7.4.1 Extension of a Rod

Figure 7.4 shows a transversely isotropic piezoelectric long cylindrical rod with polarized x_3-axis. The two silver-coated end faces are used as electrodes and subjected to uniform normal traction p. Using the first kind of constitutive equation, the solutions of this problem are as follows:

1. For shorted electrodes,

$$\begin{aligned}&\sigma_{33}=p,\quad \text{all other } \sigma_{ij}=0;\quad \varepsilon_{33}=s_{33}p,\quad \varepsilon_{11}=\varepsilon_{22}=s_{13}p;\\&E_3=E_1=E_2=0,\quad D_3=d_{33}p,\quad D_1=D_2=0;\quad \mathfrak{A}_s=\sigma_{33}\varepsilon_{33}/2=s_{33}p^2/2\end{aligned} \tag{7.118}$$

2. For open electrodes,

$$\begin{aligned}&\sigma_{33}=p,\quad \text{all other } \sigma_{ij}=0;\quad \varepsilon_{33}=s_{33}\left(1-d_{33}^2/\epsilon_{33}s_{33}\right)p,\quad \varepsilon_{11}=\varepsilon_{22}=s_{13}p+d_{31}E_3;\\&D_3=D_1=D_2=0,\ \ E_3=-(d_{33}/\epsilon_{33})p,\ \ E_1=E_2=0;\ \ U_o=(s_{33}/2)\left(1-d_{33}^2/\epsilon_{33}s_{33}\right)p^2\end{aligned} \tag{7.119}$$

   The rod appears to be stiffer for the open electrodes than sorted electrodes due to $d_{33}^2/\epsilon_{33}s_{33}>0$.

3. The longitudinal electromechanical coupling factor k_{33}:

$$k_{33}^2=(U_{\mathrm{s}}-U_{\mathrm{o}})/U_{\mathrm{s}}=d_{33}^2/\epsilon_{33}s_{33} \tag{7.120}$$

7.4.2 Torsion of a Piezoelectric Circular Cylinder

Figure 7.5 shows a transversely isotropic piezoelectric circular cylinder of length L, inner radius a, and outer radius b with polarized θ-axis. The two silver-coated end faces are used as electrodes and subjected to a torque M and charge Q_{e}. Using the second kind of constitutive equation, the general solution of this problem is

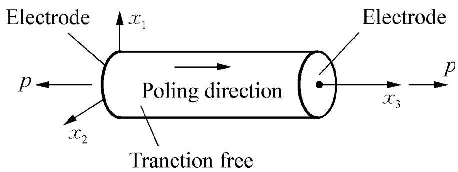


Fig. 7.4 An axial poled rod

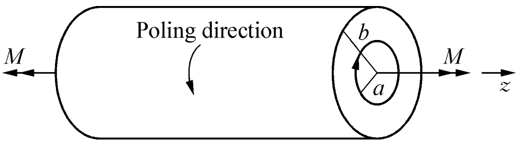


Fig. 7.5 A circular cylinder in torsion

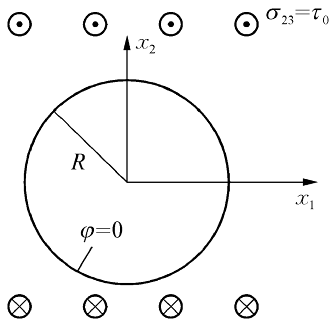


Fig. 7.6 An infinite plate with a *circular cylindrical hole* under longitudinal shear

$$
\begin{aligned}
&u_\theta = Arz, \quad u_r = u_z = 0; \quad \gamma_{\theta z} = Ar, \quad \sigma_{\theta z} = C_{44}Ar - e_{15}B; \\
&\varphi = -Bz, \quad E_z = B; \quad D_z = e_{15}Ar + \epsilon_{11}B \\
&M = \int_a^b \sigma_{\theta z}(2\pi r \mathrm{d}r)r = C_{44}AI_p - e_{15}2\pi B\left(b^3 - a^3\right)/3, \quad I_\mathrm{p} = \pi\left(b^4 - a^4\right)/2 \\
&Q_\mathrm{e} = \int_a^b D_z(2\pi r \mathrm{d}r) = e_{15}2\pi A\left(b^3 - a^3\right)/3 + \epsilon_{11}B\pi\left(b^2 - a^2\right)
\end{aligned}
\tag{7.121}
$$

1. Shorted electrodes : $B = 0, \quad A = M/C_{44}I_\mathrm{p}$ (7.122)
2. Open electrodes : $Q_\mathrm{e} = 0,$

$$
A = M\left[C_{44}I_\mathrm{p} + \frac{e_{15}^2}{\epsilon_{11}}\left(\frac{2\pi}{3}\right)^2 \frac{\left(b^3 - a^3\right)}{\pi\left(b^2 - a^2\right)}\right]^{-1} \tag{7.123}
$$

7.4.3 A Circular Hole Under Longitudinal Shear in an Infinite Piezoelectric Plate

Figure 7.6 shows a circular cylindrical hole of radius R in an unbounded transversely isotropic piezoelectric material with polarized x_3-axis under a uniform longitudinal shear stress $\sigma_{23} = \tau_0$ at $x_2 = \pm\infty$. The hole surface is a grounded electrode. The governing equations and boundary conditions for an electrically open case are

$$
\begin{aligned}
&\nabla^2 u_3 = 0, \quad \nabla^2\varphi = 0; \quad r > R \\
&\sigma_{rz} = 0, \quad \varphi = 0, \quad r = R; \quad \sigma_{23} = \tau_0, \quad x_2 = \pm\infty \\
&D_2 = 0, \quad x_2 = \pm\infty \text{ (electrically open)}; \quad E_2 = 0, \quad x_2 = \pm\infty \text{ (electrically shorted)}
\end{aligned}
\tag{7.124}
$$

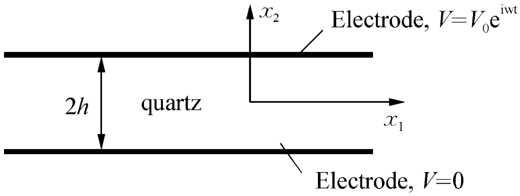


Fig. 7.7 An electrode quartz plate

For the electrically open case, the solution is

$$u_3 = \frac{\tau_0}{C_{44}(1+k^2)}\left[r + \left(1+2k^2\right)\frac{R^2}{r}\right]\sin\theta, \quad \varphi = \frac{\tau_0}{C_{44}\epsilon_{11}(1+k^2)}\left(r - \frac{R^2}{r}\right)\sin\theta \tag{7.125}$$

where $k^2 = e_{15}^2/C_{44}\epsilon_{11}$. For the electrically shorted case, the solution can also be obtained.

7.4.4 Thickness-Shear Vibration of a Quartz Plate

Figure 7.7 shows the sketch of a widely used piezoelectric resonator manufactured by rotated Y-cut quartz plate. Surfaces at $x_2 = \pm h$ are traction-free and electroded, with a driving voltage $V_0 e^{i\omega t}$. Let $u_3 = 0$ and $\partial u_2/\partial x_2 \approx 0$ due to the plate is thin enough. So the displacement and potential fields can be assumed in the following forms:

$$u_1 = U_1(x_2)e^{i\omega t}, \quad u_2(x_2) \approx 0, \quad u_3 = 0; \quad \varphi = \Phi(x_2)e^{i\omega t} \tag{7.126}$$

Under above assumptions, the constitutive equations are

$$\sigma_6 = C_{66}u_{1,2} + e_{26}\varphi_{,2}, \quad D_2 = e_{26}u_{1,2} - \epsilon_{22}\varphi_{,2} \tag{7.127}$$

The stresses $\sigma_5 = C_{56}u_{1,2} + e_{25}\varphi_{,2}$, $D_3 = e_{36}u_{1,2} - \epsilon_{23}\varphi_{,2}$ are omitted because they are not used. The generalized momentum equation and boundary condition are

$$\begin{aligned} &\sigma_{6,2} = -\rho\omega^2 u_1, \quad D_{2,2} = 0 \\ &\sigma_6 = 0, \quad \text{at } x_2 = \pm h; \quad \Phi(h) - \Phi(-h) = V_0 \end{aligned} \tag{7.128}$$

The general solutions are

$$\begin{aligned} &U_1 = A_1 \sin\lambda x_2 + A_2 \cos\lambda x_2; \quad \lambda = \sqrt{\rho/C_{66}^*}\,\omega, \;\; C_{66}^* = C_{66}\left(1+k^2\right), \;\; k = e_{26}\Big/\sqrt{C_{66}\epsilon_{22}} \\ &\Phi = (e_{26}/\epsilon_{22})(A_1 \sin\lambda x_2 + A_2 \cos\lambda x_2) + Bx_2 \end{aligned} \tag{7.129}$$

From the boundary conditions, we can get two group equations:

$$C_{66}^{*}A_{1}\lambda\cos\lambda h+e_{26}B=0,\quad 2(e_{26}/\epsilon_{22})A_{1}\sin\lambda h+2Bh=V_{0} \tag{7.130}$$

$$C_{66}^{*}A_{2}\lambda\sin\lambda h=0 \tag{7.131}$$

1. *Free vibration* $V_0 = 0$, *symmetric modes* From Eq. (7.131) it is obtained

$$\sin\lambda h=0;\quad \text{or}\quad \lambda_{n}h=n\pi/2,\quad \omega_{n}=(n\pi/2h)\sqrt{C_{66}^{*}/\rho},\quad n=0,2,4,6,\ldots, \tag{7.132}$$

where ω_n is the nth order resonance frequency. In the same time $A_2 \neq 0,\ A_1 = B = 0$. The corresponding symmetric modes are

$$U_{1}=\cos\lambda_{n}x_{2};\quad \Phi=(e_{26}/\epsilon_{22})\cos\lambda_{n}x_{2} \tag{7.133}$$

2. *Free vibration* $V_0 = 0$, *antisymmetric modes* In this case $A_1 \neq 0, B \neq 0, A_2 = 0$. Nontrivial solutions may exist in Eq. (7.130) if

$$C_{66}^{*}A_{1}\lambda h\cos\lambda h-\left(e_{26}^{2}/\epsilon_{22}\right)\sin\lambda h=0,\quad \text{or}\tan\lambda_{\nu}h=\lambda_{\nu}h(1+k^{2})/k^{2},\quad \omega_{\nu}=\lambda_{\nu}\sqrt{C_{66}^{*}/\rho} \tag{7.134}$$

From Eq. (7.130), it is obtained $B_{\nu} = -\left(C_{66}^{*}/e_{26}\right)A_1\lambda_{\nu}\cos\lambda_{\nu}h$. In this case $\sin \lambda h \neq 0$, so $A_2 = 0$. The corresponding antisymmetric modes are

$$U_{1}=\sin\lambda x_{2};\quad \Phi=(e_{26}/\epsilon_{22})\sin\lambda x_{2}-\left[\left(C_{66}^{*}/e_{26}\right)\lambda_{\nu}\cos\lambda_{\nu}h\right]x_{2} \tag{7.135}$$

3. *Forced vibration* From Eq. (7.131) we have $A_2 = 0$. From Eq. (7.130) we get

$$A_{1}=-\frac{V_{0}}{2}\frac{e_{26}V_{0}}{C_{66}^{*}\lambda h\cos\lambda h-\left(e_{26}^{2}/\epsilon_{22}\right)\sin\lambda h},\quad B=\frac{V_{0}}{2}\frac{C_{66}^{*}\lambda\cos\lambda h}{C_{66}^{*}\lambda h\cos\lambda h-\left(e_{26}^{2}/\epsilon_{22}\right)\sin\lambda h} \tag{7.136}$$

Yang (2005) gave also some other interest problems except the above examples in this section.

7.5 Laminated Piezoelectric Plates

7.5.1 Basic Concepts and Governing Equations

In the earlier work, the piezoelectric actuator structure constituted of an elastic substrate (beam or bar), electroded piezoelectric elements, and finite-thickness bonding layers. For a pair actuators fixed on the upper and lower surfaces, if the

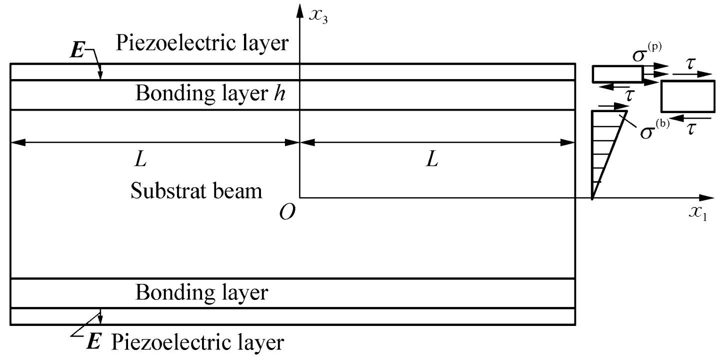


Fig. 7.8 A sketch of a simple intelligent beam

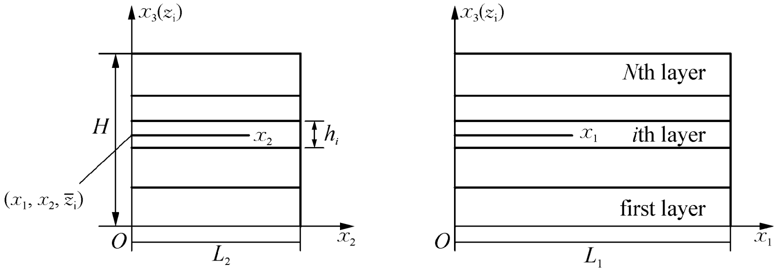


Fig. 7.9 A multiply laminated piezoelectric plate

same voltage is applied to both actuators, it results in pure extension, and if the opposite voltage is applied to both actuators, it results in bending (Fig. 7.8). In the present time, the "intelligent structure" may be a laminated piezoelectric beam, plate, shell and distributed actuator, sensor, and processor networks. In engineering the classical beam, plate and shell theory are commonly used, and sometimes will use the higher-order theories.

Consider an N-layer laminated piezoelectric plate of dimensions L_1 and L_2 in x_1 and x_2 directions and total thickness H in x_3-direction, and the ith layer has thickness h_i. The plate is polarized along x_3-axis. Except the global coordinate system, a local coordinate system (x_1, x_2, z_i) in the middle plane of ith layer is also adopted (Fig. 7.9). Layer 1 is the bottom layer and the layer N is the top layer. At each interface with perfect bonding between layers, continuity conditions of generalize displacements and tractions must be satisfied. For the ith interface between ith and $(i+1)$th layers in the local coordinate system, the continuity conditions are

$$\begin{aligned} &\boldsymbol{U}^{(i)}(x_1, x_2, h_i/2) = \boldsymbol{U}^{(i+1)}(x_1, x_2, -h_{i+1}/2), \\ &\boldsymbol{\Sigma}^{(i)}\boldsymbol{n}^{(i)}(x_1, x_2, h_i/2) = \boldsymbol{\Sigma}^{(i+1)}\boldsymbol{n}^{(i)}(x_1, x_2, -h_{i+1}/2), \quad i = 1 - (N-1) \end{aligned} \tag{7.137}$$

It is also noted that sometimes a bond-line may be simulated by a layer of small thickness. For each interface, there are eight continuity conditions, so for a laminate plate with N layers, there are $8(N-1)$ continuity conditions; on the lower surface $(z=-h_1/2)$ of the first layer and on the upper surface $(z=h_N/2)$ of the Nth layer, there are four boundary conditions, respectively. Therefore, there are total $8N$ boundary conditions to determine $8N$ unknowns. For an orthotropic material layer, the constitutive equation is shown in Eq. (3.69). Analogous to Eq. (7.3), the motion equations in terms of generalized displacements are

$$\begin{aligned}
&C_{11}u_{1,11}+C_{66}u_{1,22}+C_{55}u_{1,33}+(C_{12}+C_{66})u_{2,12}+(C_{13}+C_{55})u_{3,13}+(e_{15}+e_{31})\varphi_{,13}=\rho u_{1,tt}\\
&(C_{12}+C_{66})u_{1,12}+C_{66}u_{2,11}+C_{22}u_{2,22}+C_{44}u_{2,33}+(C_{23}+C_{44})u_{3,23}+(e_{24}+e_{32})\varphi_{,23}=\rho u_{2,tt}\\
&(C_{13}+C_{55})u_{1,13}+(C_{23}+C_{44})u_{2,23}+C_{55}u_{3,11}+C_{44}u_{3,22}+C_{33}u_{3,33}+e_{24}\varphi_{,22}+e_{33}\varphi_{,33}=\rho u_{3,tt}\\
&(e_{31}+e_{15})u_{1,13}+(e_{32}+e_{24})u_{2,23}+e_{15}u_{3,11}+e_{24}u_{3,22}+e_{33}u_{3,33}\\
&\quad-\epsilon_{11}\varphi_{,11}-\epsilon_{22}\varphi_{,22}-\epsilon_{33}\varphi_{,33}=0
\end{aligned} \tag{7.138}$$

7.5.2 *Bending in Simply Supported Orthotropic Laminated Rectangular Plate*

It is assumed that the bottom and lateral surfaces are free, and known normal traction and potential are imposed on the top surface:

$$\begin{aligned}
&q(x_1,x_2)=q_0\sin p_1x_1\sin p_2x_2,\quad \varphi(x_1,x_2)=\Phi_0\sin p_1x_1\sin p_2x_2\\
&p_1=(n\pi/L_1),\quad p_2=(m\pi/L_2);\quad \text{when}\quad x_3=H
\end{aligned} \tag{7.139}$$

For the simply supported orthotropic laminated rectangular plate, the solution in each layer is assumed (Heyliger 1997):

$$\begin{aligned}
&u_1=u_{10}e^{sx_3}\cos p_1x_1\sin p_2x_2,\quad u_2=u_{20}e^{sx_3}\sin p_1x_1\cos p_2x_2\\
&u_3=u_{30}e^{sx_3}\sin p_1x_1\sin p_2x_2,\quad \varphi=\varphi_0e^{sx_3}\sin p_1x_1\sin p_2x_2
\end{aligned} \tag{7.140}$$

where u_{i0},φ_0,s are undetermined constants and the superscript (i) is omitted. Substituting Eq. (7.140) into (7.138) yields

$$\begin{aligned}
&\boldsymbol{\Lambda}\boldsymbol{U}_0=\boldsymbol{0},\quad \boldsymbol{U}_0=[u_{10},u_{20},u_{30},\varphi_0]^{\mathrm{T}},\quad \boldsymbol{\Lambda}=\\
&\begin{bmatrix}
C_{11}p_1^2+C_{66}p_2^2-C_{55}s^2 & (C_{12}+C_{66})p_1p_2 & -(C_{13}+C_{55})p_1s & -(e_{15}+e_{31})p_1s\\
(C_{12}+C_{66})p_1p_2 & C_{66}p_1^2+C_{22}p_2^2-C_{44}s^2 & -(C_{23}+C_{44})p_2s & -(e_{24}+e_{32})p_2s\\
(C_{13}+C_{55})p_1s & (C_{23}+C_{44})p_2s & C_{55}p_1^2+C_{44}p_2^2-C_{33}s^2 & e_{15}p_1^2+e_{24}p_2^2-e_{33}s^2\\
(e_{15}+e_{31})p_1s & (e_{24}+e_{32})p_2s & e_{15}p_1^2+e_{24}p_2^2-e_{33}s^2 & -\epsilon_{11}p_1^2-\epsilon_{22}p_2^2+\epsilon_{33}s^2
\end{bmatrix}
\end{aligned} \tag{7.141}$$

Equation (7.141) will have nontrivial solution if $|\boldsymbol{\Lambda}|=0$. From which we can obtained eight eigenvector s_i, $s_i, i=1-8$. Corresponding each s_i, an eigenvector

$\boldsymbol{U}_{0j}$ $(u_{10i}, u_{20i}, u_{30i}, \varphi_{0i})$ with one unknown u_{10j} is obtained. As shown in Sect. 7.5.1, the problem can be solved uniquely.

7.5.3 Free Vibration of Laminates in Cylindrical Bending

Let $L_1 \to \infty$ in x_1 direction and all variables be independent with x_1. Assuming $u_1 = 0$, from Eq. (7.138), the generalized motion equations are

$$\begin{aligned} &C_{22}u_{2,22} + C_{44}u_{2,33} + (C_{23} + C_{44})u_{3,23} + (e_{24} + e_{32})\varphi_{,23} = \rho u_{2,tt} \\ &(C_{23} + C_{44})u_{2,23} + C_{44}u_{3,22} + C_{33}u_{3,33} + e_{24}\varphi_{,22} + e_{33}\varphi_{,33} = \rho u_{3,tt} \\ &(e_{32} + e_{24})u_{2,23} + e_{24}u_{3,22} + e_{33}u_{3,33} - \epsilon_{22}\varphi_{,22} - \epsilon_{33}\varphi_{,33} = 0 \end{aligned} \tag{7.142}$$

The continuity conditions on interface are shown in Eq. (7.137). For the free vibration, the mechanical boundary conditions on the top and bottom surfaces are

$$\sigma_{33}(x_2, h_N/2) = \sigma_{33}(x_2, -h_1/2) = \sigma_{23}(x_2, h_N/2) = \sigma_{23}(x_2, -h_1/2) = 0 \tag{7.143}$$

The electrical boundary conditions have two different kinds:

$$(1)\ \ \varphi(x_2, h_N/2) = \varphi(x_2, -h_1/2) = 0; \quad \text{or} \ \ (2)\ \ D_3(x_2, h_N/2) = D_3(x_2, -h_1/2) = 0 \tag{7.144}$$

For the cylindrical bending vibration in (x_2, x_3) plane, the boundary conditions on the lateral surfaces are

$$\sigma_{22}(0, x_3) = \sigma_{22}(0, L_1) = 0, \quad u_3(0, x_3) = u_3(0, L_1) = 0, \quad \varphi(0, x_3) = \varphi(0, L_1) = 0 \tag{7.145}$$

For each layer, there are six unknowns and six continuity conditions due to $u_1 = 0$. There are also total six boundary conditions on the top and bottom surfaces. In order to satisfy Eq. (7.145) automatically, Heyliger and Brooks (1995) took the solution in the following form:

$$(u_2, u_3, \varphi) = (u_{20}\cos px_2, u_{30}\sin px_2, \varphi_0 \sin px_2)\mathrm{e}^{sx_3}\mathrm{e}^{\mathrm{i}\omega t}, \quad p = n\pi/L_2 \tag{7.146}$$

Substituting Eq. (7.146) into (7.142) yields

$$\begin{bmatrix} -C_{22}p^2 + C_{44}s^2 + \rho\omega^2 & (C_{23} + C_{44})ps & (e_{24} + e_{32})ps \\ -(C_{23} + C_{44})ps & -C_{44}p^2 + C_{33}s^2 + \rho\omega^2 & -e_{24}p^2 + e_{33}s^2 \\ -(e_{32} + e_{24})ps & -e_{24}p^2 + e_{33}s^2 & \epsilon_{22}p^2 - \epsilon_{33}s^2 \end{bmatrix} \begin{Bmatrix} u_{20} \\ u_{30} \\ \varphi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7.147}$$

Setting the determinant of above matrix to zero for a nontrivial solution yields an eigen-equation. From the eigen-equation, we get six eigenvalues $s_i, i = 1 - 6$ for s. Corresponding each s_i, an eigenvector $\boldsymbol{U}_{0j}$ $(u_{10i}, u_{20i}, u_{30i}, \varphi_{0i})$ with one unknown u_{10j} can be obtained. As shown in Sect. 7.5.1, the problem can be solved uniquely.

7.5.4 A Mindlin-Type Plate Bending Theory

Consider an orthotropic piezoelectric plate of moderate thickness. Let (x_1, x_2) be located on the middle surface. The basic assumptions of the Mindlin bending theory are:

1. Straight lines normal to the $x_1 - x_2$ plane before deformation remain straight with unchanged length after deformation, but not compulsory normal to the mid-surface, i.e.,

$$\begin{aligned} &\varepsilon_{11} = u^0_{1,1} + x_3\psi_{1,1}, \quad \varepsilon_{22} = u^0_{2,2} + x_3\psi_{2,2}, \quad \gamma_{23} = u^0_{3,2} + \psi_2 \\ &\gamma_{13} = u^0_{3,1} + \psi_1, \quad \gamma_{12} = u^0_{1,2} + u^0_{2,1} + x_3(\psi_{1,2} + \psi_{2,1}) \end{aligned} \tag{7.148}$$

where $\boldsymbol{u}^0$ is the displacement in the mid-surface and ψ_1 and ψ_2 are the absolute cross-sectional rotations.

2. Stress σ_{33} can be neglected. So the constitutive equation can be written as

$$\begin{aligned} &\sigma_1 = \sigma_{11} = \bar{C}_{11}u_{1,1} + \bar{C}_{12}u_{2,2} + \bar{e}_{31}\varphi_{,3}, \quad \sigma_2 = \sigma_{22} = \bar{C}_{12}u_{1,1} + \bar{C}_{22}u_{2,2} + \bar{e}_{32}\varphi_{,3} \\ &\sigma_4 = \sigma_{23} = \bar{C}_{44}(u_{2,3} + u_{3,2}) + \bar{e}_{24}\varphi_{,2}, \quad \sigma_5 = \sigma_{31} = \bar{C}_{55}(u_{1,3} + u_{3,1}) + \bar{e}_{15}\varphi_{,1} \\ &\sigma_6 = \sigma_{12} = \bar{C}_{66}(u_{2,1} + u_{1,2}), \quad D_1 = \bar{e}_{15}(u_{1,3} + u_{3,1}) - \bar{\epsilon}_{11}\varphi_{,1} \\ &D_2 = \bar{e}_{24}(u_{2,3} + u_{3,2}) - \bar{\epsilon}_{22}\varphi_{,1}, \quad D_3 = \bar{e}_{31}u_{1,1} + \bar{e}_{32}u_{2,2} - \bar{\epsilon}_{33}\varphi_{,3} \end{aligned} \tag{7.149}$$

where $u_{3,3}$ has been eliminated by using $\sigma_3 = 0$, and

$$\bar{C}_{ij} = C_{ij} - C_{i3}C_{j3}/C_{33}, \quad \bar{e}_{ij} = e_{ij} - e_{33}C_{ji}/C_{33}, \quad \bar{\epsilon}_{ij} = \bar{\epsilon}_{ij} + e^2_{33}\delta_{i3}\delta_{j3}/C_{33} \tag{7.150}$$

are the reduced material coefficients.

Wang and Yang (2000) reviewed the higher-order theories of the piezoelectric plates. The equivalent single-layer models for the multiply layer plate (Krommer and Irschik 2000) are adopted here. Substituting Eq. (7.148) into (7.149) and integrating through the thickness yield the constitutive equation in the stress resultants and moments:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D \end{bmatrix} \begin{Bmatrix} u^0_{1,1} \\ u^0_{2,2} \\ u^0_{1,2} + u^0_{2,1} \\ \psi_{1,1} \\ \psi_{2,2} \\ \psi_{1,2} + \psi_{2,1} \end{Bmatrix} - \begin{Bmatrix} N_{1e} \\ N_{2e} \\ N_{12e} \\ M_{1e} \\ M_{2e} \\ M_{12e} \end{Bmatrix}$$

$$\begin{Bmatrix} q_2 \\ q_1 \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} - \begin{Bmatrix} q_{2e} \\ q_{1e} \end{Bmatrix} \tag{7.151}$$

where $N_1, N_2, N_{12} = N_6$ are the membrane forces, $M_1, M_2, M_{12} = M_6$ are the bending moments, and q_1, q_2 are the shear forces per unit length. The generalized stiffness in Eq. (7.151) are

$$(N_i, M_i) = \sum_{k=1}^{N} \int_{h_k} \sigma_i^{(k)}(1, x_3,)\mathrm{d}x_3, \ (i = 1, 2, 6); \quad (q_2, q_1) = \sum_{k=1}^{N} \int_{h_k} (\sigma_4^{(k)}, \sigma_5^{(k)})\mathrm{d}x_3$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_k} \bar{C}_{ij}^{(k)}(1, x_3, x_3^2)\mathrm{d}x_3, \quad S_{ij} = \sum_{k=1}^{N} \int_{h_k} \Phi_i \Phi_j \bar{C}_{ij}^{(k)} \mathrm{d}x_3$$

$$\begin{Bmatrix} N_{1e} & M_{1e} \\ N_{2e} & M_{2e} \\ N_{12e} & M_{12e} \end{Bmatrix} = \sum_{k=1}^{N} \int_{h_k} \begin{Bmatrix} \bar{e}_{31}^{(k)} \\ \bar{e}_{32}^{(k)} \\ 0 \end{Bmatrix} E_3^{(k)}(1, x_3)\mathrm{d}x_3, \quad \begin{bmatrix} q_{1e} \\ q_{2e} \end{bmatrix} = \sum_{k=1}^{N} \int_{h_k} \begin{Bmatrix} \Phi_5^2 \bar{e}_{24}^{(k)} E_1^{(k)} \\ \Phi_4^2 \bar{e}_{15}^{(k)} E_2^{(k)} \end{Bmatrix} \mathrm{d}x_3 \tag{7.152}$$

where Φ_i, Φ_j are shear factors which are determined by the shear stress distribution on the cross section and $N_{1e}, M_{1e}, N_{2e}, M_{2e}, N_{12e}, M_{12e}, q_{1e}, q_{2e}$ are introduced by piezoelectric effect. The electric variables will be studied layer by layer, and for each layer, there are two-type boundary conditions:

1. Electrically open Given the electric charge density $\sigma^{(i)}$ on the upper and lower surfaces of the layer. Usually $D_1^{(i)}$ and $D_2^{(i)}$ are neglected and $D_3^{(i)}$ is reserved. So $D_3^{(i)}$ is constant along the thickness direction due to Gauss equation $D_{3,3}^{(i)} = 0$:

$$D_j^{(i)} n_j = D_3^{(i)} = -\sigma^{(i)}, \quad E_3^{(i)} = -\sigma^{(i)} \Big/ \bar{\epsilon}_{33}^{(i)} \tag{7.153}$$

2. Electrically shorted Given the potential $V^{(i)}$ on the upper and lower surfaces of the layer. For convenience, assume the variation of the potential is linear along the thickness direction, so

$$E_3^{(i)} = V^{(i)}/h_i, \quad D_3^{(i)} = \bar{\epsilon}_{33}^{(i)} V^{(i)}/h_i; \quad E_1^{(i)} = E_2^{(i)} = 0 \tag{7.154}$$

In Mindlin theory, the motion equations are

$$\begin{aligned}&\rho_0\ddot{u}_1^0+\rho_1\ddot{\psi}_1=N_{1,1}+N_{12,2},\quad \rho_0\ddot{u}_2^0+\rho_1\ddot{\psi}_2=N_{2,2}+N_{12,1},\quad \rho_0\ddot{u}_3^0=q_{1,1}+q_{2,2}+p\\&\rho_1\ddot{u}_1^0+\rho_2\ddot{\psi}_1=M_{1,1}+M_{12,2}-q_1,\quad \rho_1\ddot{u}_2^0+\rho_2\ddot{\psi}_2=M_{2,2}+M_{12,1}-q_2\end{aligned}\tag{7.155}$$

where p is the transverse loading and

$$(\rho_0,\rho_1,\rho_2)=\sum_{k=1}^{N}\int_{h_k}\rho_0^{(k)}\left(1,x_3,x_3^2\right)\mathrm{d}x_3\tag{7.156}$$

Equations (7.148), (7.149), (7.150), (7.151), (7.152), (7.153), (7.154), (7.155), and (7.156) are the complete governing equations.

For a symmetrically laminated, transversely isotropic and simply supported plate, bending and extension are decouple, and the following relations are held:

$$\begin{aligned}&\bar{C}_{11}=\bar{C}_{22},\quad \bar{C}_{44}=\bar{C}_{55},\quad \bar{C}_{66}=\tfrac{1}{2}(\bar{C}_{11}-\bar{C}_{12});\quad \bar{e}_{31}=\bar{e}_{32},\quad \bar{e}_{15}=\bar{e}_{24};\quad \bar{\epsilon}_{11}=\bar{\epsilon}_{22}\\&D_{11}=D_{22},\quad D_{66}=(D_{11}-D_{12})/2,\quad S_{44}=S_{55},\quad M_{1e}=M_{2e}\end{aligned}\tag{7.157}$$

Eliminating the cross-sectional rotations ψ from Eqs. (7.151) and (7.155) for the bending vibration, a fourth-order partial differential equation for u_3^0 is obtained (Krommer and Irschik, 2000):

$$\begin{aligned}&D_{11}\nabla^2\nabla^2u_3^0-[(D_{11}/S_{44})\rho_0+\rho_2]\nabla^2\ddot{u}_3^0+\rho_0\ddot{u}_3^0+(\rho_0\rho_2/S_{44})\,\ddot{\ddot{u}}_3{}^0\\&\quad=p-(D_{11}/S_{44})\nabla^2p+(\rho_2/S_{44})\ddot{p}-\nabla^2M_{1e}\end{aligned}\tag{7.158}$$

where $\nabla^2u_3^0=u_{3,11}^0+u_{3,22}^0$ and $\ddot{\ddot{u}}_3{}^0=\partial^4u_3^0/\partial t^4$. When the external loadings are $p=p_0\mathrm{e}^{\mathrm{i}\omega t}$ and $M_{1e}=M_{10e}\mathrm{e}^{\mathrm{i}\omega t}$, the frequency equation of the bending vibration is

$$\begin{aligned}&D_{11}\nabla^2\nabla^2U_3^0+[(D_{11}/S_{44})\rho_0+\rho_2]\omega^2\nabla^2U_3^0-\rho_0\omega^2\left[1-(\rho_2/S_{44})\omega^2\right]U_3^0\\&\quad=p_0\left[1-(\rho_2/S_{44})\omega^2\right]-\nabla^2[(D_{11}/S_{44})p_0+M_{10e}]\end{aligned}\tag{7.159}$$

where the common factor $\mathrm{e}^{\mathrm{i}\omega t}$ is omitted and $u_3^0=U_3^0\mathrm{e}^{\mathrm{i}\omega t}$.

7.5.5 Third-Order Shear Deformation Theory of Laminate Plate

Consider the linear piezoelectric material, so the Maxwell stress and the environment need not be considered. The variational principle $\delta\Pi=0$ in Eq. (2.7) is becomes

$$\delta \Pi = \int_V \sigma_{ik} \delta u_{i,k} \mathrm{d}V + \int_V D_k \delta \varphi_{,k} \mathrm{d}V + \int_V \rho \ddot{u}_k \delta u_k \mathrm{d}V - \int_{a_\sigma} T_k^* \delta u_k \mathrm{d}a + \int_{a_\rho} \sigma^* \delta \varphi \mathrm{d}a = 0 \tag{7.160}$$

where the body force and body electric charge are neglected. For the orthotropic material of moderate thickness, Mitchell and Reddy (1995) adopted the displacements of the equivalent single-layer plate as

$$\begin{aligned} u_1 &= u_1^0(x_1, x_2, t) + \eta_1(x_3)\psi_1(x_1, x_2, t) - \eta_2(x_3)u_{3,1}^0(x_1, x_2, t) \\ u_2 &= u_2^0(x_1, x_2, t) + \eta_1(x_3)\psi_2(x_1, x_2, t) - \eta_2(x_3)u_{3,2}^0 \\ u_3 &= u_3^0(x_1, x_2, t) \\ \eta_1(x_3) &= x_3 - cx_3^3, \quad \eta_2(x_3) = cx_3^3, \quad c = 4/3h^2 \end{aligned} \tag{7.161}$$

where $\left(u_1^0, u_2^0, u_3^0\right)$ are the displacements of a point on the midplane and (ψ_1, ψ_2) are the rotations of a transverse normal at $x_3 = 0$ on the midplane about the x_2 and $-x_1$ axes, respectively. h is the total thickness of the plate. The strains corresponding to Eq. (7.161) are

$$\begin{aligned} &\varepsilon_{11} = u_{1,1}^0 + \eta_1\psi_{1,1} - \eta_2 u_{3,11}^0; \quad \varepsilon_{22} = u_{2,2}^0 + \eta_1\psi_{2,2} - \eta_2 u_{3,22}^0; \quad \varepsilon_{33} = 0 \\ &\gamma_{23} = u_{2,3} + u_{3,2} = \eta_{1,3}\psi_2 - \eta_{2,3}u_{3,2}^0 + u_{3,2}^0; \quad \gamma_{31} = u_{1,3} + u_{3,1} = \eta_{1,3}\psi_1 - \eta_{2,3}u_{3,1}^0 + u_{3,1}^0 \\ &\gamma_{12} = u_{1,2} + u_{2,1} = \left(u_{1,2}^0 + u_{2,1}^0\right) + \eta_1(\psi_{1,2} + \psi_{2,1}) - \eta_2\left(u_{3,12}^0 + u_{3,21}^0\right) \end{aligned} \tag{7.162}$$

From Eqs. (7.161) and (7.162), it is known that for $\eta_1(x_3), \eta_2(x_3)$ given in Eq. (7.161), the transverse shear strains γ_{13}, γ_{23} are zeros on the upper and lower surfaces and vary quadratically through the thickness. It is also without the normal strain. The potential is modeled on a discrete layer approximation as

$$\varphi(x_1, x_2, x_3, t) = \sum_{k=1}^{N} \sum_{j=1}^{m} f_k(x_3) \varphi^{(k,j)}(x_1, x_2, t) \tag{7.163}$$

where N is the layer number of the laminate plate, m is the number of interpolation points in a layer, and $\varphi^{(k,j)}$ is the potential at jth interpolation point of kth layer. $f_k(x_3)$ is the Lagrange interpolation function. It is noted that the potential on the upper surface of the $k-1$ layer must be equal to that on the lower surface of the k layer.

The middle plane is denoted by A and its boundary is L. Substituting Eqs. (7.161) and (7.163) into (7.160) and neglecting σ_{33} yield the following equations:

The variation of the mechanical energy:

$$
\begin{aligned}
\int_V \sigma_{ik}\delta u_{i,k}\mathrm{d}V &= \int_A \Big\{\Big(N_1\delta u^0_{1,1} + M_1\delta\psi_{1,1} - P_1\delta u^0_{3,11}\Big) + \Big(N_2\delta u^0_{2,2} + M_2\delta\psi_{2,2} - P_2\delta u^0_{3,22}\Big)\\
&+ \Big[N_6\Big(\delta u^0_{1,2} + \delta u^0_{2,1}\Big) + M_6\big(\delta\psi_{1,2} + \delta\psi_{2,1}\big) - 2P_6\delta u^0_{3,12}\Big]\\
&+ Q_4\Big(\delta\psi_2 + \delta u^0_{3,2}\Big) + Q_5\Big(\delta\psi_1 + \delta u^0_{3,1}\Big)\Big\}\mathrm{d}A\\
&= -\int_A \big\{\big(N_{1,1}\delta u^0_1 + M_{1,1}\delta\psi_1 - P_{1,11}\delta u^0_3\big) + \big(N_{2,2}\delta u^0_2 + M_{2,2}\delta\psi_2 - P_{2,22}\delta u^0_3\big)\\
&+ \big(N_{6,2}\delta u^0_1 + N_{6,1}\delta u^0_2 + M_{6,2}\delta\psi_1 + M_{6,1}\delta\psi_2 - 2P_{6,12}\delta u^0_3\big) + \big(Q_4\delta\psi_2 - Q_{4,2}\delta u^0_3\big)\\
&+ \big(Q_5\delta\psi_1 - Q_{5,1}\delta u^0_3\big)\big\}\mathrm{d}A + \int_L \Big\{\Big[N_1\delta u^0_1 + M_1\delta\psi_1 - \Big(P_1\delta u^0_{3,1} - P_{1,1}\delta u^0_3\Big)\Big]n_1\\
&+ \Big[N_2\delta u^0_2 + M_2\delta\psi_2 - \Big(P_2\delta u^0_{3,2} - P_{2,2}\delta u^0_3\Big)\Big]n_2 + \big(N_6\delta u^0_2 + M_6\delta\psi_2\big)n_1\\
&+ \big(N_6\delta u^0_1 + M_6\delta\psi_1\big)n_2 - \Big[P_6\delta u^0_{3,2}n_1 + P_6\delta u^0_{3,1}n_2 - \big(P_{6,1}n_2 + P_{6,2}n_1\big)\delta u^0_3\Big]\\
&+ (Q_4 n_2 + Q_5 n_1)\delta u^0_3\Big\}\mathrm{d}L
\end{aligned}
\tag{7.164}
$$

where the Voigt notation has been used and

$$
\begin{aligned}
&N_i = \int_{-h/2}^{h/2}\sigma_i\mathrm{d}x_3, \quad M_i = \int_{-h/2}^{h/2}\sigma_i\eta_1\mathrm{d}x_3, \quad P_i = \int_{-h/2}^{h/2}\sigma_i\eta_2\mathrm{d}x_3, \quad i = 1,2,6\\
&Q_i = \int_{-h/2}^{h/2}\sigma_i\Big[1 - 4(z/h)^2\Big]\mathrm{d}x_3, \quad i = 4,5
\end{aligned}
\tag{7.165}
$$

The variation of the electric energy:

$$
\begin{aligned}
\int_V D_i\delta\varphi_{,i}\mathrm{d}V &= \sum_{k=1}^{N}\left\{\int_V\sum_{j=1}^{m}\Big(D^{(k,j)}_\alpha\delta\varphi^{(k,j)}_{,\alpha} + D^{(k,j)}_3\delta\varphi^{(k,j)}_{,3}\Big)\mathrm{d}V\right\}\\
&= \sum_{k=1}^{N}\left\{\int_A -\sum_{j=1}^{m}\Big(P^{(k,j)}_{\alpha,\alpha} - G^{(k,j)}_3\Big)\delta\varphi^{(k,j)}\mathrm{d}A + \int_L\sum_{j=1}^{m}P^{(k,j)}_\alpha n_\alpha\delta\varphi^{(k,j)}\mathrm{d}L\right\}\\
P^{(k,j)}_\alpha &= \int_{h_{j-1}}^{h_j}D^{(k,j)}_\alpha f_k(x_3)\mathrm{d}x_3, \quad G^{(k,j)}_3 = \int_{h_{j-1}}^{h_j}D^{(k,j)}_3 f_{k,3}(x_3)\mathrm{d}x_3
\end{aligned}
\tag{7.166}
$$

The variation of the kinetic energy or inertial energy:

$$
\begin{aligned}
\int_V \rho\ddot{u}_i\delta u_i\mathrm{d}V &= \int_A\Big\{\Big(I_1\ddot{u}^0_1 + I_2\ddot{\psi}_1 - I_3\ddot{u}^0_{3,1}\Big)\delta u^0_1 + \Big(I_2\ddot{u}^0_1 + I_4\ddot{\psi}_1 - I_5\ddot{u}^0_{3,1}\Big)\delta\psi_1\\
&+ \Big[\Big(I_3\ddot{u}^0_1 + I_5\ddot{\psi}_1 - I_6\ddot{u}^0_{3,1}\Big)_{,1} + \Big(I_3\ddot{u}^0_2 + I_5\ddot{\psi}_2 - I_6\ddot{u}^0_{3,2}\Big)_{,2} + I_1\ddot{u}^0_3\Big]\delta u^0_3\\
&+ \Big[\Big(I_1\ddot{u}^0_2 + I_2\ddot{\psi}_2 - I_3\ddot{u}^0_{3,2}\Big)\delta u^0_2 + \Big(I_2\ddot{u}^0_2 + I_4\ddot{\psi}_2 - I_5\ddot{u}^0_{3,2}\Big)\delta\psi_2\Big]\Big\}\mathrm{d}A
\end{aligned}
\tag{7.167}
$$

where

$$
\begin{aligned}
I_1 &= \int_{-h/2}^{h/2} \rho \mathrm{d}x_3, \quad I_3 = \int_{-h/2}^{h/2} \rho\eta_2(x_3)\mathrm{d}x_3, \quad I_5 = \int_{-h/2}^{h/2} \rho\eta_1(x_3)\eta_2(x_3)\mathrm{d}x_3 \\
I_2 &= \int_{-h/2}^{h/2} \rho\eta_1(x_3)\mathrm{d}x_3, \quad I_4 = \int_{-h/2}^{h/2} \rho\eta_1^2(x_3)\mathrm{d}x_3, \quad I_6 = \int_{-h/2}^{h/2} \rho\eta_2^2(x_3)\mathrm{d}x_3
\end{aligned} \tag{7.168}
$$

In Eq. (7.167) term $-\int_L \left[\left(I_3\ddot{u}_1^0 + I_5\ddot{\psi}_1 - I_6\ddot{u}_{3,1}^0\right)n_1 + \left(I_3\ddot{u}_2^0 + I_5\ddot{\psi}_2 - I_6\ddot{u}_{3,2}^0\right)n_2\right]$ δu_3^0 has been neglected. This term is difficult to explanation.

The variation of the work of the generalized external force is as follows: The mechanical force acted on the equivalent single-layer plate should be distinguished two parts—T_i^* is the equivalent traction on the midplane A, and t_i^* is the equivalent resultant force on the lateral surface a. The electric charge acted on the kth layer should also be distinguished two parts: $\sigma^{(k)*}$ is the surface electric charge density of the kth layer, and $q^{(k)*}$ is the surface electric charge on the lateral surface. Neglecting some secondary terms we can get

$$
\begin{aligned}
-\int_{a_\sigma} T_k^*\delta u_k \mathrm{d}a + \int_{a_\rho} \sigma^*\delta\varphi \mathrm{d}a &= -\int_A \left\{T_1^*\delta u_1^0 + T_2^*\delta u_2^0 + T_3^*\delta u_3^0\right\}\mathrm{d}A \\
&- \int_L \left\{t_1^*\delta u_1^0 + t_2^*\delta u_2^0 + t_3^*\delta u_3^0\right\}\mathrm{d}L + \sum_{k=1}^{N}\left[\int_A \sigma^{(k)*} f_k \delta\varphi_j \mathrm{d}A + \int_L q^{(k)*} f_j \delta\varphi_j \mathrm{d}L\right]
\end{aligned} \tag{7.169}
$$

Substitution of Eqs. (7.163), (7.164), (7.165), (7.166), (7.167), (7.168), and (7.169) into Eq. (7.160) yields

$$
\begin{aligned}
&\delta u_{10}: N_{1,1} + N_{6,2} + T_1^* = I_1\ddot{u}_1^0 + I_2\ddot{\psi}_1 - I_3\ddot{u}_{3,1}^0 \\
&\delta u_{20}: N_{2,2} + N_{6,1} + T_2^* = I_1\ddot{u}_2^0 + I_2\ddot{\psi}_2 - I_3\ddot{u}_{3,2}^0 0 \\
&\delta u_{30}: P_{1,11} + P_{2,22} + 2P_{6,12} + Q_{4,2} + Q_{5,1} - T_3^* \\
&\qquad = \left(I_3\ddot{u}_1^0 + I_5\ddot{\psi}_1 - I_6\ddot{u}_{3,1}^0\right)_{,1} + \left(I_3\ddot{u}_2^0 + I_5\ddot{\psi}_2 - I_6\ddot{u}_{3,2}^0\right)_{,2} + I_1\ddot{u}_3^0 \\
&\delta\psi_1: M_{1,1} + M_{6,2} - Q_5 = I_2\ddot{u}_1^0 + I_4\ddot{\psi}_1 - I_5\ddot{u}_{3,1}^0 \\
&\delta\psi_2: M_{2,2} + M_{6,1} - Q_4 = I_2\ddot{u}_2^0 + I_4\ddot{\psi}_2 - I_5\ddot{u}_{3,2}^0 \\
&\delta\varphi^{(k,j)}: \sum_{j=1}^{m}\left(P_{\alpha,\alpha}^{(k,j)} - G_3^{(k,j)}\right) = \sum_{j=1}^{m}\sigma^{(j)*} f_j
\end{aligned} \tag{7.170}
$$

and the natural boundary conditions are

$$\begin{aligned}
&\delta u_1^0 : N_1 n_1 + N_6 n_2 + t_1^* = 0, \quad \delta u_2^0 : N_2 n_2 + N_6 n_1 + t_2^* = 0 \\
&\delta u_3^0 : P_{1,1} n_1 + P_{2,2} n_2 + P_{6,1} n_2 + P_{6,2} n + Q_4 n_2 + Q_5 n_1 + t_3^* = 0 \\
&\delta \psi_1 : M_1 n_1 + M_6 n_2 = 0, \qquad \delta \psi_2 : M_2 n_2 + M_6 n_1 = 0 \\
&\delta u_{3,1}^0 : P_1 n_1 + P_6 n_2 = 0, \qquad \delta u_{3,2}^0 : P_2 n_2 + P_6 n_1 = 0 \\
&\delta \varphi_j^k : \sum_{j=1}^{m} P_\alpha^{(j,k)} n_\alpha - q^{(k)*} f_j(x_3) = 0
\end{aligned} \tag{7.171}$$

It is also noted that on the boundaries given generalized displacements, we have

$$\delta u_1^0 = \delta u_2^0 = \delta u_3^0 = \delta u_{3,1}^0 = \delta u_{3,2}^0 = \delta \psi_1 = \delta \psi_2 = 0 \tag{7.172}$$

Mitchell and Reddy (1995) adopted Hamilton principle, $\delta \Pi_{\mathrm{H}} = \delta \Pi_{\mathrm{H1}}$ in Eq. (2.32); their results are slightly different. It can be seen that this complex approximate theory is difficult to exactly discuss and give some new simplified postulations are needed.

The generalized forces can be obtained from Eqs. (7.165) and (7.149). Let

$$N_i = \bar{N}_i + N_i^{\mathrm{p}}, \quad M_i = \bar{M}_i + M_i^{\mathrm{p}}, \quad P_i = \bar{P}_i + P_i^{\mathrm{p}}, \quad Q_i = \bar{Q}_i + Q_i^{\mathrm{p}} \tag{7.173a}$$

where the elastic variables are denoted by an over-bar and variables related to piezoelectric effect are denoted by a superscript "p," and

$$\begin{Bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_3 \\ \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{11}^* & A_{12}^* & 0 & A_{11}^{**} & A_{12}^{**} & 0 \\ A_{12} & A_{22} & 0 & A_{12}^* & A_{22}^* & 0 & A_{12}^{**} & A_{22}^{**} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & A_{66}^* & 0 & 0 & A_6^{**} \\ A_{11}^* & A_{12}^* & 0 & B_{11} & B_{12} & 0 & B_{11}^* & B_{12}^* & 0 \\ A_{12}^* & A_{22}^* & 0 & B_{12} & B_{22} & 0 & B_{12}^* & B_{22}^* & 0 \\ 0 & 0 & A_{66}^* & 0 & 0 & B_{66} & 0 & 0 & B_{66}^* \\ A_{11}^{**} & A_{12}^{**} & 0 & B_{11}^* & B_{12}^* & 0 & D_{11} & D_{12} & 0 \\ A_{12}^{**} & A_{22}^{**} & 0 & B_{12}^* & B_{22}^* & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & A_{66}^{**} & 0 & 0 & B_{66}^* & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1}^0 \\ u_{2,2}^0 \\ u_{1,2}^0 + u_{2,1}^0 \\ \psi_{1,1} \\ \psi_{2,2} \\ \psi_{1,2} + \psi_{2,1} \\ -u_{3,11}^0 \\ -u_{3,22}^0 \\ -2u_{3,12}^0 \end{Bmatrix} \tag{7.173b}$$

$$\begin{Bmatrix} \bar{Q}_4 \\ \bar{Q}_5 \end{Bmatrix} = \begin{pmatrix} F_{44} & 0 \\ 0 & F_{55} \end{pmatrix} \begin{Bmatrix} \psi_2 + u_{30,2} \\ \psi_1 + u_{30,1} \end{Bmatrix} \tag{7.173c}$$

$$\begin{aligned}
N_1^{\mathrm{p}} &= \sum_{k=1}^{N} e_{31}^{(k)} \sum_{j=1}^{m} \beta_1^{(k,j)} \varphi^{(k,j)}, \quad N_2^{\mathrm{p}} = \sum_{k=1}^{N} e_{32}^{(k)} \sum_{j=1}^{m} \beta_1^{(k,j)} \varphi^{(k,j)}, \quad N_6^{\mathrm{p}} = 0 \\
M_1^{\mathrm{p}} &= \sum_{k=1}^{N} e_{31}^{(k)} \sum_{j=1}^{m} \beta_2^{(k,j)} \varphi^{(k,j)}, \quad M_2^{\mathrm{p}} = \sum_{k=1}^{N} e_{32}^{(k)} \sum_{j=1}^{m} \beta_2^{(k,j)} \varphi^{(k,j)}, \quad M_6^{\mathrm{p}} = 0 \\
P_1^{\mathrm{p}} &= \sum_{k=1}^{N} e_{31}^{(k)} \sum_{j=1}^{m} \beta_3^{(k,j)} \varphi^{(k,j)}, \quad P_2^{\mathrm{p}} = \sum_{k=1}^{N} e_{32}^{(k)} \sum_{j=1}^{m} \beta_3^{(k,j)} \varphi^{(k,j)}, \quad P_6^{\mathrm{p}} = 0 \\
Q_4^{\mathrm{p}} &= \sum_{k=1}^{N} e_{24}^{(k)} \sum_{j=1}^{m} \beta_4^{(k,j)} \partial\varphi^{(k,j)}/\partial x_2, \quad Q_5^{p} = \sum_{k=1}^{N} e_{15}^{(k)} \sum_{j=1}^{m} \beta_4^{(k,j)} \partial\varphi^{(k,j)}/\partial x_1
\end{aligned} \tag{7.173d}$$

where

$$\begin{aligned}
&\mathbf{A}^{(k)} = \int_{-h_k/2}^{h_k/2} \mathbf{C}\mathrm{d}x_3, \quad \mathbf{A}^{(k)*} = \int_{-h_k/2}^{h_k/2} \mathbf{C}\mathrm{g}_1\mathrm{d}x_3, \quad \mathbf{A}^{(k)**} = \int_{-h_k/2}^{h_k/2} \mathbf{C}\mathrm{g}_2\mathrm{d}x_3, \quad \mathbf{B}^{(k)} = \int_{-h_k/2}^{h_k/2} \mathbf{C}\mathrm{g}_1^2\mathrm{d}x_3 \\
&\mathbf{B}^{(k)*} = \int_{-h_k/2}^{h_k/2} \mathbf{C}\mathrm{g}_1\mathrm{g}_2\mathrm{d}x_3, \quad \mathbf{D} = \int_{-h_k/2}^{h_k/2} \mathbf{C}\mathrm{g}_2^2\mathrm{d}x_3, \quad F_{44} = \int_{-h_k/2}^{h_k/2} C_{44}\mathrm{g}_{1,3}^2\mathrm{d}x_3 \\
&F_{55} = \int_{-h_k/2}^{h_k/2} C_{55}\mathrm{g}_{1,3}^2\mathrm{d}x_3 \beta_1^{(k,j)} = \int_{-h_k/2}^{h_k/2} f_{j,3}\mathrm{d}x_3, \quad \beta_2^{(k,j)} = \int_{-h_k/2}^{h_k/2} \mathrm{g}_1 f_{j,3}\mathrm{d}x_3, \\
&\beta_3^{(k,j)} = \int_{-h_k/2}^{h_k/2} \mathrm{g}_2 f_{j,3}\mathrm{d}x_3, \quad \beta_4^{(k,j)} = \int_{-h_k/2}^{h_k/2} \mathrm{g}_{1,3} f_j\mathrm{d}x_3; \quad \mathbf{C} = \begin{pmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{21} & \bar{C}_{22} & 0 \\ 0 & 0 & \bar{C}_{66} \end{pmatrix} \\
&(\mathbf{A}, \mathbf{A}^*, \mathbf{A}^{**}, \mathbf{B}, \mathbf{B}^*) = \sum_{k=1}^{N} \left(\mathbf{A}^{(k)}, \mathbf{A}^{(k)*}, \mathbf{A}^{(k)**}, \mathbf{B}^{(k)}, \mathbf{B}^{(k)*} \right)
\end{aligned} \tag{7.174}$$

Equations (7.170), (7.171), (7.172), (7.173a), (7.173b), (7.173c), (7.173d), and (7.174) are the complete governing equations.

7.5.6 Bending Theory of Timoshenko Beam

A narrow plate can be considered as a beam. The Timoshenko theory (Timoshenko and Woinowsky-Krieger 1959) considering the shear deformation of a beam of a moderate thickness can be obtained from the Mindlin theory. Let all variables be independent to x_2 in Mindlin theory, Eq. (7.149) is reduced to

$$\begin{aligned}
&\sigma_1 = \sigma_{11} = Y u_{1,1} + e_{31}\varphi_{,3}, \quad \sigma_5 = \sigma_{13} = G(u_{1,3} + u_{3,1}) + e_{15}\varphi_{,2} \\
&D_1 = e_{15}(u_{1,3} + u_{3,1}) - \epsilon_{11}\varphi_{,1}, \quad D_3 = e_{31}u_{1,1} - \epsilon_{33}\varphi_{,3}
\end{aligned} \tag{7.175}$$

where Y, G are elastic coefficients. Corresponding to Eq. (7.148), the deformation in Timoshenko beam is assumed as

$$\varepsilon_{11} = u^0_{1,1} + x_3\psi_{,1}, \quad \gamma_{13} = u^0_{3,1} + \psi \tag{7.176}$$

If $\psi = -u^0_{3,1}$, then $\gamma_{13} = 0$,Timoshenko beam is reduced to Bernoulli-Euler beam. Corresponding to Eqs. (7.173) and (7.174), we have

$$\begin{gathered}
\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{Bmatrix} u^0_{1,1} \\ \psi_{,1} \end{Bmatrix} - \begin{Bmatrix} N_e \\ M_e \end{Bmatrix}, \quad q = G\gamma_{13} - q_e \\
(A, B, D) = \sum_{k=1}^{N} \int_{h_k} Y^{(k)}\left(1, x_3.x_3^2\right)\mathrm{d}x_3, \quad G = \sum_{k=1}^{N} \int_{h_k} \Phi^2 G^{(k)}\mathrm{d}x_3 \\
(N_e, M_e) = \sum_{k=1}^{N} \int_{h_k} e^{(k)}_{31} E^{(k)}_3 (1, x_3.)\mathrm{d}x_3, \quad q_e = \sum_{k=1}^{N} \int_{h_k} \Phi^2 e^{(k)}_{15} E^{(k)}_1 \mathrm{d}x_3
\end{gathered} \tag{7.177}$$

Corresponding to Eq. (7.155), we have

$$\rho_0 \ddot{u}^0_1 + \rho_1 \ddot{\psi}_1 = N_{,1}, \quad \rho_0 \ddot{u}^0_3 = q_{,1} + p, \quad \rho_1 \ddot{u}^0_1 + \rho_2 \ddot{\psi}_1 = M_{,1} - q \tag{7.178}$$

where ρ_0, ρ_1, ρ_2 are shown in Eq. (7.156) and electric displacements and electric fields are shown in Eq. (7.153) or (7.154).

7.5.7 Bending Model of Beams of Crawley

Crawley and De Luis (1987) proposed an extension-bending model to study the simple intelligent beam structure as shown in Fig. 7.8. They assumed that the strain is uniform through the actuator thickness, the beam obeys Bernoulli-Euler rule, and the adhesive layer transfers loads only through shear. The formulas obtained by this model are well for extension, but not bending, especially for a thin plate (Crawley and Aanderson 1989).

7.6 The First-Order Approximate Theory of an Electro-magneto-elastic Thin Plate

7.6.1 Basic Postulations

The nonlinear theory of an electroelastic thin plate is not well established, and different authors proposed different theories. In this section, a first-order approximate theory of the quasi-static electro-magneto-elastic thin plate is recommended for small deformation, when the electromagnetic induction effect can be neglected. Let the origin of the coordinate system axes x_1, x_2 be located on the midplane and x_3 be upward normal to the midplane. The plate is bending upward. The role of x_3 is not the same as that of (x_1, x_2), so it will be discussed alone. For the present plate theory, three following basic postulations are assumed (Kuang 2011):

(1) $\sigma_{33} \ll \sigma_{\alpha 3} \ll \sigma_{\alpha\beta}, (\alpha = 1, 2)$, so σ_{33} is fully neglected and the effect of $\sigma_{\alpha 3}$ is considered partly.

(2) The Kirchhoff assumption is adopted, i.e.,

$$u_k = u_k^0 - x_3 u_{3,k}^0, \quad u_k^0 = u_k^0(x_1, x_2, t), \quad u_{k,3}^0 = 0, \quad (k = 1, 2, 3) \tag{7.179}$$

where $\mathbf{u}^0$ is the displacement on the middle surface and $\mathbf{u}$ is the displacement at a certain point in the plate. According to Eq. (7.179), we have $u_{3,\alpha} + u_{\alpha,3} = 0$, but it is not appropriate for the free boundary and should be modified approximately as shown later.

(3) The electromagnetic field obeys the 3D theory, but in order to consistent with the classical plate theory, the resultant electromagnetic force is reduced to the middle plane S $(\mathrm{d}S = \mathrm{d}x_1 \mathrm{d}x_2)$ or to the contour L of the middle plane. Usually when we solve the electromagnetic field, the electric field due to the direct piezoelectric effect can be approximately neglected compared to the lager applied electric field.

7.6.2 Governing Equations Derived from the First Method

In engineering the piezoelectric plate is surrounded by air, so the plate has only the interface with the air and does not have its own independent boundary. In air the mechanical stresses can be neglected, so only the electromagnetic field should be considered. It is assumed that there is no body force and body electric charge. According to Eqs. (2.19) and (2.21), there are two methods to establish the thin plate theory. Similar to Eq. (2.36), the first alternative form of the PVP, Eq. (2.19), for the static electromagnetic problem is modified as

$$\begin{aligned}
\delta\hat{\Pi} &= \delta\hat{\Pi}_1 + \delta\hat{\Pi}_2 - \delta W^{\mathrm{int}} \\
&= \int_V S_{kl}\delta u_{k,l}\mathrm{d}V + \int_V \rho\ddot{u}_k\delta u_k\mathrm{d}V - \int_{a_\sigma^{\mathrm{int}}} T_k^{*\,\mathrm{int}}\delta u_k\mathrm{d}a + \int_V D_k\delta\varphi_{,k}\mathrm{d}V + \int_V B_k\delta\psi_{,k}\mathrm{d}V \\
&\quad + \int_{V^{\mathrm{air}}} D_k^{\mathrm{air}}\delta\varphi_{,k}^{\mathrm{air}}\mathrm{d}V + \int_{V^{\mathrm{air}}} \sigma_{kl}^{\mathrm{M\,air}}\delta u_{k,l}^{\mathrm{air}}\mathrm{d}V + \int_{V^{\mathrm{air}}} B_k^{\mathrm{air}}\delta\psi_{,k}^{\mathrm{air}}\mathrm{d}V - \int_{a_\mu^{\mathrm{air}}} B_i^{*\mathrm{air}} n_i^{\mathrm{air}}\delta\psi^{\mathrm{air}}\mathrm{d}a \\
S_{kl} &= \sigma_{kl} + \sigma_{kl}^{\mathrm{M}}, \quad \sigma_{ik}^{\mathrm{M}} = D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n)\delta_{ik}
\end{aligned} \tag{7.180}$$

S_{kl}^{air} has the similar expression. The mechanical part related to the plate of the PVP is

$$\int_{a_\sigma^{\rm int}} \left[\left(S_{kl} - \sigma_{kl}^{\rm M\ air}\right)n_l - T_k^{*\ \rm int}\right]\delta u_k {\rm d}a - \int_V \left(S_{kl,l} - \rho \ddot{u}_k\right)\delta u_k {\rm d}V = 0 \tag{7.181}$$

Therefore, the thin plate theory can apply the usual elastic plate theory, but $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{S}$ and on the interface $\boldsymbol{T}^{*\ \rm int}$ is replaced by $\boldsymbol{T}^{*\ \rm int} + \boldsymbol{\sigma}^{\rm M\ air} \cdot \boldsymbol{n}$. According to the textbook of elastic plate theory, the bending theory of elastic electromagnetic thin plate can be expressed as

$$\text{Field equation:} \quad \mathrm{M}^{(\mathrm{S})}_{\alpha\beta,\beta\alpha} + \mathrm{q}^* = 0; \quad \text{in S} \tag{7.182}$$

Boundary conditions:

$$\textit{Clamped side} \quad u_3^0 = u_3^{0*}, \quad u_{3,n}^0 = u_{3,n}^{0*} \tag{7.183a}$$

$$\textit{Hinged side} \quad u_3^0 = u_3^{0*}, \quad M_n^{(S)} = M_n^* + M_n^{\rm M\ air}, \tag{7.183b}$$

$$\textit{Free side} \quad M_{nt,t}^{(S)} + Q_n^{(S)} = Q_n^* + Q_n^{M\ \rm air}, \quad M_n^{(S)} = M_n^* + M_n^{\rm M\ air} \tag{7.183c}$$

The notations in Eqs. (7.182) and (7.183) are

$$\begin{aligned} &M_{\alpha\beta}^{(S)} = \int_{-h}^{h} S_{\alpha\beta} x_3 {\rm d}x_3, \quad M_n^{\rm M\ air} = M_\alpha^{\rm M\ air} n_\alpha, \quad M_\alpha^{\rm M\ air} = \int_{-h}^{h} \sigma_{\alpha\beta}^{\rm M\ air} x_3 {\rm d}x_3 \\ &Q_\alpha^{(S)} = \int_{-h}^{h} S_{\alpha 3} {\rm d}x_3, \quad Q_n^{(S)} = Q_\alpha^{(S)} n_\alpha, \quad Q_\alpha^{\rm M\ air} = \int_{-h}^{h} \sigma_{\alpha 3}^{\rm M\ air} {\rm d}x_3 \end{aligned} \tag{7.184}$$

where $2h$ is the thickness of the plate, q^* is the distributed loading on the plate surface, and Q_n^* is the distributed loading on the lateral boundary of the midplane. At first the electromagnetic fields in plate and air are solved under the assumption that the elastic effect can be neglected. After solving the electromagnetic fields, the entire problem is reduced to a linear problem.

7.6.3 Governing Equations Derived from the Second Method

Similar to Eq. (2.36), the second alternative form of the PVP, Eq. (2.21), for the static electromagnetic problem is modified as

$$\begin{aligned} \delta\Pi' &= \delta\Pi_1' + \delta\Pi_2' - \delta W'^{\rm int} \\ &= \int_V \sigma_{kl}\delta u_{k,l} {\rm d}V - \int_V \left(\sigma_{jk,j}^{\rm M} - \rho\ddot{u}_k\right)\delta u_k {\rm d}V - \int_{a_\sigma^{\rm int}} \left(T_k^{\rm int\ *} + \sigma_{jk}^{\rm M\ env} n_j^{\rm env} - \sigma_{jk}^{\rm M} n_j\right)\delta u_k {\rm d}a \\ &\quad + \int_V D_k \delta\varphi_{,k} {\rm d}V + \int_V B_k \delta\psi_{,k} {\rm d}V + \int_{V^{\rm env}} D_k^{\rm env} \delta\varphi_{,k}^{\rm env} {\rm d}V + \int_{a_D^{\rm env}} \sigma^{\rm env\ *} \delta\varphi^{\rm env} {\rm d}a \\ &\quad + \int_{V^{\rm env}} B_k^{\rm env} \delta\psi_{,k}^{\rm env} {\rm d}V - \int_{V^{\rm env}} \sigma_{jk,j}^{\rm M\ env} \delta u_k^{\rm env} {\rm d}V + \int_{a_D} \sigma^* \delta\varphi {\rm d}a - \int_{a_\mu^{\rm env}} B_i^{*\rm env} n_i^{\rm env} \delta\psi^{\rm env} {\rm d}a = 0 \\ \sigma_{ik}^{\rm M} &= D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n)\delta_{ik}, \quad \text{Similar expression for } \sigma_{ik}^{\rm M\ env} \end{aligned} \tag{7.185}$$

According to postulation (3), the electromagnetic field obeys the 3D theory, which has been discussed in Chap. 2. Here only the mechanical part related to the plate of the variational principle is discussed in detail, which is

$$\int_V \sigma_{kl}\delta u_{k,l}\mathrm{d}V - \int_V \left(\sigma^{\mathrm{M}}_{jk,j} - \rho\ddot{u}_k\right)\delta u_k\mathrm{d}V - \int_{a^{\mathrm{int}}_\sigma}\left(T^{\mathrm{int}\,*}_k + \sigma^{\mathrm{M\,env}}_{jk}n^{\mathrm{env}}_j - \sigma^{\mathrm{M}}_{jk}n_j\right)\delta u_k\mathrm{d}a \tag{7.186}$$

Applying postulations (1) and (2) and noting $n_3 = 1, n_\alpha = 0$ on the midplane, $n_3 = 0$ on the lateral surface, it is obtained:

$$\begin{aligned}
\int_V \sigma_{kl}\delta u_{l,k}\mathrm{d}V &= \int_L N_{\alpha\beta}n_\beta\delta u^0_\alpha\mathrm{d}L - \int_s N_{\alpha\beta,\beta}\delta u^0_\alpha\mathrm{d}s - \int_L M_{\alpha\beta}n_\beta\delta u^0_{3,\alpha}\mathrm{d}L \\
&\quad + \int_L M_{\alpha\beta,\alpha}n_\beta\delta u^0_3\mathrm{d}L - \int_s M_{\alpha\beta,\beta\alpha}\delta u^0_3\mathrm{d}s \\
\int_V \rho\ddot{u}_k\delta u_k\mathrm{d}V &= \int_s\left(\rho_0\ddot{u}^0_3 + \rho_1\ddot{u}^0_{\alpha,\alpha} - \rho_2 u^0_{3,\alpha\alpha}\right)\delta u^0_3\mathrm{d}s \\
&\quad + \int_s\left(\rho_0\ddot{u}^0_\alpha - \rho_1\ddot{u}^0_{3,\alpha}\right)\delta u^0_\alpha\mathrm{d}s - \int_L\left(\rho_1\ddot{u}^0_k - \rho_2 u^0_{3,k}\right)n_k\delta u^0_3\mathrm{d}L \\
\int_{a^{\mathrm{int}}}\sigma^{\mathrm{M}}_{ip}n_i\delta u_p\mathrm{d}a &= \int_L N^{\mathrm{M}}_{\alpha p}n_\alpha\delta u^0_p\mathrm{d}L - \int_L M^{\mathrm{M}}_{\alpha\beta}n_\alpha\delta u^0_{3,\beta}\mathrm{d}L + \int_S p^{\mathrm{M}}_i\delta u^0_i\mathrm{d}S \\
\int_V \sigma^{\mathrm{M}}_{ip,i}\delta u_p\mathrm{d}V &= \int_s N^{\mathrm{M}}_{i\alpha,i}\delta u^0_\alpha\mathrm{d}S - \int_L M^{\mathrm{M}}_{i\alpha,i}n_\alpha\delta u^0_3\mathrm{d}L + \int_s M^{\mathrm{M}}_{i\alpha,i\alpha}\delta u^0_3\mathrm{d}S + \int_s N^{\mathrm{M}}_{i3,i}\delta u^0_3\mathrm{d}S \\
\int_{a^{\mathrm{int}}_\sigma} T^{*\mathrm{int}}_l\delta u_l\mathrm{d}a &= \int_S p^{*\mathrm{int}}_l\delta u^0_l\mathrm{d}S + \int_{L_\sigma} P^{*\mathrm{int}}_l\delta u^0_l\mathrm{d}L - \int_{L_\sigma} M^{*\mathrm{int}}_l\delta u^0_{3,l}\mathrm{d}L
\end{aligned} \tag{7.187}$$

where $2h$ is the thickness of the plate. The expression of $\sigma^{\mathrm{M\,air}}_{ip}$ is similar to σ^{M}_{ip}. The notations in Eq. (7.187) are

$$\begin{aligned}
&N_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}\mathrm{d}x_3, \quad M_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}x_3\mathrm{d}x_3, \quad N^{\mathrm{M}}_{\alpha p} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{\alpha p}\mathrm{d}x_3, \quad M^{\mathrm{M}}_{\alpha p} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{\alpha p}x_3\mathrm{d}x_3 \\
&p^{\mathrm{M}}_i = N^{\mathrm{M}}_{3i,3} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{3i,3}\mathrm{d}x_3 = \sigma^{\mathrm{M}}_{3i}(h) - \sigma^{\mathrm{M}}_{3i}(-h), \quad M^{\mathrm{M}}_{3i,3} = \int_{-h}^{h}\sigma^{\mathrm{M}}_{3i,3}x_3\mathrm{d}x_3 \\
&\rho_0 = \int_{-h}^{h}\rho\mathrm{d}x_3, \quad \rho_1 = \int_{-h}^{h}\rho x_3\mathrm{d}x_3, \quad \rho_2 = \int_{-h}^{h}\rho x_3^2\mathrm{d}x_3, \\
&p^{*\mathrm{int}}_l = T^{*\mathrm{int}}_l\big|_{-h}^{h}, \quad \text{on} \quad S; \quad P^{*\mathrm{int}}_l = \int_{-h}^{h}T^{*\mathrm{int}}_l\mathrm{d}x_3, \quad M^{*\mathrm{int}}_l = \int_{-h}^{h}T^{*\mathrm{int}}_l x_3\mathrm{d}x_3, \quad \text{on} \quad L
\end{aligned} \tag{7.188}$$

It is noted that when the Maxwell stresses on the upper and lower surfaces are reduced to the midplane, a distributed couple $\int_S\left(m^{\mathrm{M}}_{3\alpha} - m^{\mathrm{M\,env}}_{3\alpha}\right)\delta u^0_{3,\alpha}\mathrm{d}S$, $\quad m^{\mathrm{M}}_{3\alpha} = \left[\sigma^{\mathrm{M}}_{3\alpha}(h) + \sigma^{\mathrm{M}}_{3\alpha}(-h)\right](h/2)$ may be produced, but this effect is neglected in

Eq. (7.187) due to small h. It is also noted that Eq. (7.179) is not fully appropriate for the free lateral boundary. In fact from the variational formula, $\sigma_{\alpha 3}$ on the middle plane is approximately considered by $P_3^{\text{int}\,*}$, but $\sigma_{\alpha 3}$ on the free boundary should not be considered. So on the boundary L, a term $N_{3\beta} = \int_{-h}^{h} \sigma_{\beta 3} \mathrm{d}x_3$ should be added to the variational formula.

Substituting Eqs. (7.187) into Eq. (7.186), adding a term $\int_L N_{3\beta} n_\beta \delta u_3^0 \mathrm{d}L$, and finishing the variational calculation, we finally get:

The mechanical governing equations of the plane problem are

$$\begin{aligned} & N_{\alpha\beta,\beta} + N_{\alpha\beta,\beta}^{\mathrm{M}} + p_\alpha^{\mathrm{M\,env}} - p_\alpha^{\mathrm{M}} + p_\alpha^{*\mathrm{int}} = \rho_0 \ddot{u}_\alpha^0 \left(-\rho_1 \ddot{u}_{3,\alpha}^0\right); \quad \text{in} \quad S \\ & \left(N_{\alpha\beta} + N_{\alpha\beta}^{\mathrm{M}} - N_{\alpha\beta}^{\mathrm{M\,env}}\right) n_\beta = P_\alpha^{*\mathrm{int}}; \quad \text{on} \quad L_\sigma \end{aligned} \tag{7.189}$$

The mechanical governing field equation for the bending problem is

$$\begin{aligned} & M_{\alpha\beta,\beta\alpha} + M_{\alpha i,\alpha i}^{\mathrm{M}} + N_{\alpha 3,\alpha}^{\mathrm{M}} + p_3^{\mathrm{M\,env}} - p_3^{\mathrm{M}} + p_3^{*\mathrm{int}} = \rho_0 \ddot{u}_3^0 + \left(\rho_1 \ddot{u}_{\alpha,\alpha}^0 - \rho_2 u_{3,\alpha\alpha}^0\right); \quad \text{in} \quad S \\ & \int_L \left[\left(M_{\alpha\beta,\alpha} + M_{\alpha\beta,\alpha}^{\mathrm{M}}\right) n_\beta - P_3^{*\mathrm{int}} + \left(N_{3\beta} + N_{\beta 3}^{\mathrm{M}} - N_{\beta 3}^{\mathrm{M\,env}}\right) n_\beta - \left(\rho_1 \ddot{u}_k^0 - \rho_2 u_{3,k}^0\right)\right] \delta u_3^0 \mathrm{d}L \\ & - \int_L \left[M_{\alpha\beta} n_\beta + \left(M_{\beta\alpha}^{\mathrm{M}} - M_{\beta\alpha}^{\mathrm{M\,env}}\right) n_\beta - M_\alpha^{*\mathrm{int}}\right] \delta u_{3,\alpha}^0 \mathrm{d}L = 0 \end{aligned} \tag{7.190}$$

In Eq. (7.190) terms $\left(\rho_1 \ddot{u}_{\alpha,\alpha}^0 - \rho_2 u_{3,\alpha\alpha}^0\right)$ and $\left(\rho_1 \ddot{u}_k^0 - \rho_2 u_{3,k}^0\right)$ can be neglected. The usual three boundary conditions for the plate bending can easily be derived from Eq. (7.190).

According to the assumption (3), the electromagnetic field is reduced to

$$\begin{aligned} & D_{i,i} = \rho_{\mathrm{e}}, \quad B_{i,i} = 0, \quad \text{in} \quad V; \quad D_i n_i = -\sigma^*, \quad \text{on} \quad a_D; \quad B_i = B_i^*, \quad \text{on} \quad a_\mu \\ & \left(D_i - D_i^{\mathrm{env}}\right) n_i = -\sigma^{*\mathrm{int}}, \quad \left(B_i - B_i^{\mathrm{env}}\right) = B_i^{*\mathrm{int}}, \quad \text{on} \quad a^{\mathrm{int}} \end{aligned} \tag{7.191}$$

In deriving the above equations, the constitutive equations were not used, so the governing equations can be used for all materials satisfying the basic postulations (1) to (3).

7.6.4 Some Discussions

The soft electromagnetic plate under a uniform transverse magnetic field can be bending or buckling when the magnetic field exceeds a critical value (Moon and Pao 1968; Pao and Yeh 1973). The natural frequency of a soft electromagnetic plate can be changed under a longitudinal magnetic field (Zhou and Miya 1998). Zhou and Zheng (1997) pointed out that there was not a unified theory to discuss the above two problems, and they proposed a variational method attempting to unified deal with these problems. The key problem is to get the electromagnetic force acting on the plate. Though for the electromagnetically static problem and the problem without magnetic field the theory discussed above is appropriate, but for a MQS system ($\partial \boldsymbol{D}/\partial t = 0,\ \partial \boldsymbol{B}/\partial t \neq 0$), such as vibration problem in a magnetic field, it should be modified, the motional electric force should be considered. As an example, the transverse vibration of an elastic electroconductive plate is subjected to the external uniform magnetic field $\boldsymbol{H}_0 = H_{01}\mathbf{i}_1 + H_{02}\mathbf{i}_2$ parallel to the (x_1, x_2) plane only. The induced motional electric field $\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{j}$ in the plate due to the plate motion is (Librescu et al. 2004; Belubekyan et al. 2007)

$$\boldsymbol{e} = -\boldsymbol{v} \times \boldsymbol{B}_0, \quad \boldsymbol{v} = v\,\mathbf{i}_3; \quad \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0); \quad \boldsymbol{j} = \nabla \times \boldsymbol{h} \tag{7.192}$$

The corresponding Maxwell stress is

$$\begin{aligned} &\sigma_{ij}^{\mathrm{M}} = \sigma_{ij}^{\mathrm{M0}} + \sigma_{ij}^{\mathrm{M1}} \\ &\sigma_{ij}^{\mathrm{M0}} = B_{0i}H_{0j} - (1/2)B_{0m}H_{0m}\delta_{ij}, \quad \sigma_{ij}^{\mathrm{M1}} = b_i H_{0j} + B_{0i}h_j - B_{0m}h_m\delta_{ij} \end{aligned} \tag{7.193}$$

Belubekyan et al. (2007) adopted the electromagnetic body force $\boldsymbol{f} = \nabla \cdot \boldsymbol{\sigma}^{\mathrm{M}}$, but Librescu et al. (2004) adopted the Lorentz formula $\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{B}_0$. The MQS problems should be further studied.

7.7 Piezoelectric Composite Shells

7.7.1 First-Order Shear Deformation Theory

Consider a finite, simply supported, N-layered laminated circular cylindrical shell of mean radius R, length L, and thickness H. The shell is constituted of elastic orthotropic or radially polarized piezoelectric materials. The cylindrical coordinates (x, θ, z) are adopted with (x, θ) spanning the mid-surface and z along the normal (radial) direction. Analogous to the Mindlin plate theory, Kapuria et al. (1998) assumed that the displacements can be approximated as

$$u = u^0(x,\theta) + z\psi_1(x,\theta), \quad v = v^0(x,\theta) + z\psi_2(x,\theta), \quad w = w^0(x,\theta) \tag{7.194}$$

where u^0, v^0, w^0 are the displacement components on the mid-surface and ψ_1, ψ_2 are the rotations of its normal. The corresponding strains are

$$\begin{aligned} &\varepsilon_x = u^0_{,x} + z\psi_{1,x}, \quad \varepsilon_\theta = \left(v^0_{,\theta} + +z\psi_{2,\theta} + w^0\right)\Big/(R+z), \quad \varepsilon_z = 0, \quad \gamma_{zx} = \psi_1 + w^0_{,x} \\ &\gamma_{\theta z} = \psi_2 + \left(w^0_{,\theta} - v^0 - z\psi_2\right)\Big/(R+z), \quad \gamma_{\theta x} = v^0_{,x} + \left(u^0_\theta + z\psi_{1,\theta}\right)/(R+z) + z\psi_{2,x} \end{aligned} \tag{7.195}$$

On the mid-surface, the resultant membrane forces $N_x,\ N_\theta,\ N_{x\theta},\ N_{\theta x}$, transverse forces $Q_x,\ Q_\theta$, and resultant moments $M_x,\ M_\theta,\ M_{x\theta},\ M_{\theta x}$ are

$$\begin{bmatrix} N_x,\ N_\theta,\ N_{x\theta},\ N_{\theta x} \\ M_x,\ M_\theta,\ M_{x\theta},\ M_{\theta x} \end{bmatrix} = \int_{-H/2}^{H/2} \begin{bmatrix} 1 \\ z \end{bmatrix} \left[\sigma_x\left(1+\frac{z}{R}\right), \sigma_\theta, \sigma_{x\theta}\left(1+\frac{z}{R}\right), \sigma_{\theta x}\right] \mathrm{d}z$$
$$[Q_x,\ Q_\theta] = \int_{-H/2}^{H/2} [\sigma_{xz}(1+z/R), \sigma_{\theta z}] \mathrm{d}z \tag{7.196}$$

The constitutive equation is shown in Eq. (7.149). The equilibrium equations are

$$\begin{aligned} &N_{x,x} + N_{\theta x,\theta}/R + p_x = 0, \quad (Q_\theta + N_{\theta,\theta})/R + N_{x\theta,x} + p_\theta = 0, \\ &Q_{x,x} + (Q_{\theta,\theta} - N_\theta)/R + p_z = 0 \\ &M_{x,x} + M_{\theta x,\theta}/R - Q_x + m_x = 0, \quad M_{\theta,\theta}/R + M_{x\theta,x} - Q_\theta + m_\theta = 0 \\ &(p_x, p_\theta, p_z) = [(1+z/R)(\sigma_{zx}, \sigma_{z\theta}, \sigma_z)]_{-H/2}^{H/2}, \quad (m_x, m_\theta) = [(1+z/R)z(\sigma_{zx}, \sigma_{z\theta})]_{-H/2}^{H/2} \end{aligned} \tag{7.197}$$

where $p_x, p_\theta, p_z, m_x m_\theta$ are the external forces and moments, respectively. The boundary conditions are defined:

$$\begin{aligned} &N_x \quad \text{or} \quad u^0, \quad N_{x\theta} \quad \text{or} \quad v^0, \quad Q_x \quad \text{or} \quad w^0, \quad M_x \quad \text{or} \quad \psi_1, \quad M_{x\theta} \quad \text{or} \quad \psi_2; \\ &\quad \text{at} \quad x = 0 \quad \text{or} \quad L \\ &N_{\theta x} \quad \text{or} \quad u^0, \quad N_\theta \quad \text{or} \quad v^0, \quad Q_\theta \quad \text{or} \quad w^0, \quad M_{\theta x} \quad \text{or} \quad \psi_1, \quad M_\theta \quad \text{or} \quad \psi_2; \\ &\quad \text{at} \quad \theta = 0 \quad \text{or} \quad \theta_0 \end{aligned} \tag{7.198}$$

where θ_0 is the span of the cylindrical panel. Usually σ_z can be neglected. Using Eqs. (7.149) and (7.150), the equilibrium equations in terms of displacements can be obtained. Here it is omitted. The above theory is easily extended to the combined multiply layer shell.

The electric potential φ is assumed to vary linearly across the actuated layer, and the electric field is computed as $E_x = -\varphi_{,x},\ E_\theta = -\varphi_{,\theta}/(R+z),\ E_z = -\varphi_{,z}$.

For the classical shell theory, the transverse shear strains $\gamma_{zx}, \gamma_{\theta z}$ are neglected. Hence,

$$\begin{aligned} &u = u^0 - zw^0_{,x}, \quad v = v^0 - z\left(w^0_{,\theta} - v^0\right)\Big/R, \quad w = w^0(x,\theta); \\ &\varepsilon_x = u^0_{,x} - zw^0_{,xx}, \quad \varepsilon_\theta = v^0_{,\theta}\Big/R + \left(w^0 - zw^0_{,\theta\theta}\right)\Big/(R+z), \quad \varepsilon_z = 0, \\ &\gamma_{\theta x} = v^0_{,x} + \left(u^0_\theta + z\psi_{1,\theta}\right)/(R+z) + z\psi_{2,x}; \quad \psi_1 = -w^0_{,x}, \quad \psi_2 = \left(v^0 - w^0_{,\theta}\right)\Big/R \end{aligned} \tag{7.199}$$

The above thin shell theories yield poor predictions of the transverse stress components $\sigma_{xz}, \sigma_{\theta z}, \sigma_z$, so sometimes a post-processing technique is needed. The transverse stress can approximately be obtained from the 3D equilibrium equations (Kapuria et al. 1998):

$$\begin{aligned}(R+z)^2\sigma_{\theta z} &= -\int_{-h/2}^{z}\left[(R+z)\sigma_{\theta,\theta} + (R+z)^2\sigma_{\theta x,x}\right]\mathrm{d}z + c_1\\(R+z)^2\sigma_{xz} &= -\int_{-h/2}^{z}\left[\sigma_{\theta x,\theta} + (R+z)\sigma_{x,x}\right]\mathrm{d}z + c_2\\(R+z)^2\sigma_z &= -\int_{-h/2}^{z}\left[\sigma_\theta - \sigma_{z\theta,\theta} - (R+z)\sigma_{zx,x}\right]\mathrm{d}z + c_3\end{aligned} \tag{7.200}$$

where c_i is determined by the boundary conditions at the outer shell surface. Kapuria et al. (1998) compared the numerical results of the shell theory with that of the exact 3D theory and gave some comments. Saviz et al. (2007) proposed a layerwise model which is formulated by introducing piecewise continuous approximations through the thickness for each state variables. They showed that the results calculated by this model more consist with that from the 3D theory.

7.7.2 *The Cylindrical Bending of a Laminated Infinitely Long Shell*

The exact analytical solution of a cylindrical shell by 3D theory is difficult, but for some simpler cases, it is possible. Now discuss an infinitely long laminated orthotropic cylindrical shell with simple supported edges under purely cylindrical bending. The top and bottom layers are piezoelectric actuators, and the middle layer is an elastic orthotropic substrate. The cylindrical coordinates r, θ, z are used, where r, θ and z refer to the radial, circumferential, and axial directions, respectively, and u_r, u_θ and u_z are the corresponding displacements (Fig. 7.10).

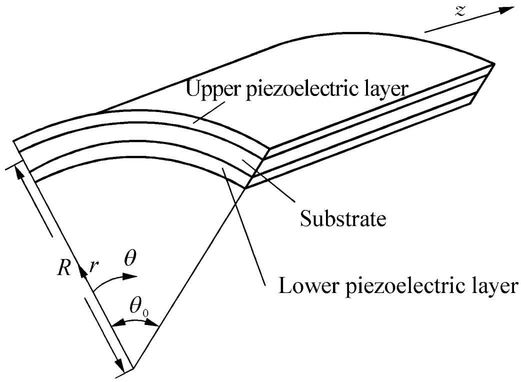
Because the shell is infinitely long, variables can be considered as the functions of (r, θ) only. The equilibrium, geometric, and constitutive equations are, respectively,

$$\begin{aligned}&\sigma_{r,r} + \sigma_{r\theta,\theta}/r + (\sigma_r - \sigma_\theta)/r = 0, \quad \sigma_{r\theta,r} + \sigma_{\theta,\theta}/r + 2\sigma_{r\theta}/r = 0\\&D_r + rD_{r,r} + D_{\theta,\theta} = 0\end{aligned} \tag{7.201}$$

$$\begin{aligned}&\varepsilon_r = u_{r,r}, \quad \varepsilon_\theta = (u_{\theta,\theta} + u_r)/r, \quad \gamma_{r\theta} = (u_{r,\theta} - u_\theta)/r + u_{\theta,r}\\&E_r = -\varphi_{,r}, \quad E_\theta = -\varphi_{,\theta}/r\end{aligned} \tag{7.202}$$

$$\begin{aligned}&\sigma_r = C_{11}\varepsilon_r + C_{12}\varepsilon_\theta - e_{33}E_r, \quad \sigma_\theta = C_{12}\varepsilon_r + C_{22}\varepsilon_\theta - e_{31}E_r\\&\sigma_{r\theta} = C_{66}\gamma_{r\theta} - e_{15}E_\theta, \quad D_r = e_{33}\varepsilon_r + e_{31}\varepsilon_\theta + \epsilon_r E_r, \quad D_\theta = e_{15}\gamma_{r\theta} + \epsilon_\theta E_\theta\end{aligned} \tag{7.203}$$

Fig. 7.10 Cylindrical bending of an infinitely long shell



where C_{ij} is the reduced material coefficients. The boundary conditions are

$$\begin{aligned}&1.\ \text{Simply supported } u_r=\sigma_\theta=\sigma_{\theta z}=0;\quad \varphi=0,\quad \text{when}\quad \theta=0,\quad \theta_0\\&2.\ \text{On the interfaces}\quad u_r,\ u_\theta,\ \sigma_r,\ \sigma_{r\theta}\quad \text{continuous and}\quad \varphi=0\\&3.\ \text{Upper surface of the outer actuator } \sigma_r=q_0\sin p\theta,\quad \sigma_{r\theta}=0,\quad \varphi=V\sin p\theta\\&4.\ \text{Lower surface of the inner actuator } \sigma_r=\sigma_{r\theta}=0;\quad D_r=0\end{aligned}\tag{7.204}$$

where $p=m\pi/\theta_0$,m is an integer, and V and q_0 are given values.

In order to satisfy the boundary conditions $u_r=\sigma_\theta=\varphi=0$ on the edges, Chen et al. (1996) adopted the following generalized displacements for actuators:

$$u_r=u_r^0(r)\sin p\theta,\quad u_\theta=u_\theta^0(r)\cos p\theta,\quad \varphi=\varphi^0(r)\sin p\theta \tag{7.205}$$

Substitution of Eq. (7.205) into Eq. (7.201) for actuators yields

$$\begin{aligned}&C_{11}\left(u''_{r0}+\frac{u'_{r0}}{r}\right)-\left(C_{22}+p^2C_{66}\right)\frac{u_{r0}}{r^2}-p(C_{66}+C_{12})\frac{u'_{\theta0}}{r}+p(C_{22}+C_{66})\frac{u_{\theta0}}{r^2}-e_{31}\frac{\varphi'_0}{r}=0\\&p(C_{12}+C_{66})\frac{u'_{r0}}{r}+p(C_{22}+C_{66})\frac{u_{r0}}{r^2}+C_{66}\left(u''_{\theta0}+\frac{u'_{\theta0}}{r}\right)-\left(p^2C_{22}+C_{66}\right)\frac{u_{\theta0}}{r^2}+pe_{31}\frac{\varphi'_0}{r}=0\\&e_{31}\frac{u'_{r0}}{r}-pe_{31}\frac{u'_{\theta0}}{r}-\epsilon_r\left(\varphi''_0+\frac{\varphi'_0}{r}\right)+p^2\epsilon_\theta\frac{\varphi_0}{r^2}=0\end{aligned}\tag{7.206}$$

where $f'=f_{,r},\quad f''=f_{,rr}$ for any f. Let

$$u_{r0}(r)=A_r r^s,\quad u_{\theta0}(r)=A_\theta r^s,\quad \varphi_0(r)=A_\varphi r^s \tag{7.207}$$

Substitution of Eq. (7.207) into Eq. (7.206) for actuators yields the homogeneous equation of A_r, A_θ, A_φ. In order to have nontrivial solutions for A_r, A_θ, A_φ, the coefficient determinate of them must be zero, so the following character equation is obtained:

$$
\begin{aligned}
&As^6 + Bs^4 + Cs^2 + D = 0 \\
&A = -C_{11}C_{66}\,\epsilon_r \\
&B = \left[C_{11}\left(p^2C_{22} + C_{66}\right) + C_{66}\left(C_{22} + p^2C_{66}\right) - p^2(C_{12} + C_{66})^2\right]\epsilon_r \\
&\qquad + p^2C_{11}C_{66}\,\epsilon_\theta + \left(C_{66} + p^2C_{11}\right)e_{31}^2 \\
&C = \left[-\left(C_{22} + p^2C_{66}\right)\left(p^2C_{22} + C_{66}\right) + p^2(C_{22} + C_{66})^2\right]\epsilon_r + \left[p^4(C_{12} + C_{66})^2\right. \\
&\qquad \left. -p^2C_{11}\left(p^2C_{22} + C_{66}\right) - p^2C_{66}\left(C_{22} + p^2C_{66}\right)\right]\epsilon_\theta + \left[2p^2(C_{22} + C_{66})\right. \\
&\qquad \left. -\left(p^2C_{22} + C_{66}\right)e_{31}^2 - p^2\left(C_{22} + p^2C_{66}\right)\right]e_{31}^2 \\
&D = \left[p^2\left(p^2C_{22} + C_{66}\right)\left(C_{22} + p^2C_{66}\right) - p^4(C_{22} + C_{66})^2\right]\epsilon_\theta
\end{aligned}
\tag{7.208}
$$

From Eq. (7.208) s has 6 real roots $s_j,\ j = 1 - 6$ for piezoelectric material. For each s_j, a group $(A_{rj}, A_{\theta j}, A_{\varphi j})$ with one unknown is obtained, so the general solution of Eq. (7.206) for each actuator is

$$
\begin{aligned}
&u_{r0} = \sum_{j=1}^{6} A_j r^{s_j}, \quad u_{\theta 0} = \sum_{j=1}^{6} A_j H_{\theta j} r^{s_j}, \quad \varphi_0 = \sum_{j=1}^{6} A_j H_{\varphi j} r^{s_j} \\
&H_{\theta j} = -\left\{p\left[(C_{12} + C_{66})s_j + (C_{22} + C_{66})\right]\left(-\epsilon_r s_j^2 + p^2\epsilon_\theta\right) - pe_{31}^2 s_j^2\right\}\Big/\Delta \\
&H_{\varphi j} = -\left\{\left(C_{66}s_j^2 - p^2C_{22} - C_{66}\right) + p^2\left[(C_{12} + C_{66})s_j + C_{22} + C_{66}\right]\right\}e_{31}s_j\Big/\Delta \\
&\Delta = \left(C_{66}s_j^2 - p^2C_{22} - C_{66}\right)\left(-\epsilon_r s_j^2 + p^2\epsilon_\theta\right) + pe_{31}^2 s_j^2
\end{aligned}
\tag{7.209}
$$

The governing equations of the middle orthotropic composite matrix can be obtained if the electric variables in Eq. (7.206) are omitted. Let the generalized displacements in matrix be

$$
u_r^{(m)} = a_r r^s \sin p\theta, \quad u_\theta^{(m)} = a_\theta r^s(r)\cos p\theta \tag{7.210}
$$

The character equation of a_r, a_θ is

$$
\begin{aligned}
&B's^4 + C's^2 + D' = 0 \\
&B' = C_{11}C_{66} \\
&C' = -C_{66}(C_{22} + p^2C_{66}) - C_{11}\left(p^2C_{22} + C_{66}\right) + p^2(C_{12} + C_{66})^2 \\
&D' = \left(C_{22} + p^2C_{66}\right)\left(p^2C_{22} + C_{66}\right) - p^2(C_{12} + C_{66})^2
\end{aligned}
\tag{7.211}
$$

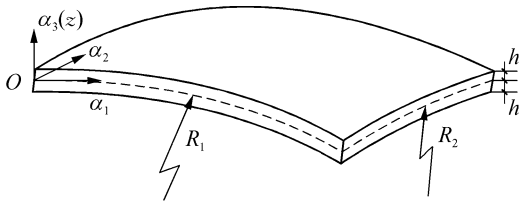


Fig. 7.11 Shallow piezoelectric shell

where the superscript M of C_{ij}^M is omitted. So the solution of the substrate is

$$\begin{aligned} u_{r0} &= \sum_{j=1}^{4} a_j H_{rj} r^{s_j}, \quad u_{\theta 0} = \sum_{j=1}^{4} a_j r^{s_j} \\ H_{rj} &= p\left[(C_{12}+C_{66})s_j - (C_{22}+C_{66})\right] \Big/ \left[C_{11}s_j^2 - (C_{22}+p^2 C_{66})\right] \end{aligned} \tag{7.212}$$

There are 16 unknowns: 6 A_j of the outer actuator, 6 A_j of the inner actuator, and 4 a_j of the middle substrate. There are also 16 boundary conditions: 4 conditions on each interface, 3 conditions on upper surface, 3 conditions on lower surface, and $\sigma_{\theta z} = 0$ at $\theta = 0, \theta_0$. Therefore, the problem is solved.

7.7.3 Approximate Theory of a Functionally Graded Shallow Piezoelectric Shell

Figure 7.11 shows a functionally graded shallow piezoelectric shell of thickness $2h$; (α_1, α_2) are the orthogonal curvilinear coordinates on the mid-surface; and its corresponding Lamé parameters are H_1, H_2 and radii of curvatures are R_1, R_2. α_3 is a linear coordinate and normal to the mid-surface. For a thin shell, $h \ll R_i$, $(i = 1, 2)$, R_i is approximately independent of α_3. Let (u_1, u_2, u_3, φ) be the generalized displacements in the orthogonal curvilinear coordinates, then

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{H_1 \partial \alpha_1} + \frac{u_2 \partial H_1}{H_1 H_2 \partial \alpha_2} + \frac{u_3}{R_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{H_2 \partial \alpha_2} + \frac{u_1 \partial H_2}{H_1 H_2 \partial \alpha_1} + \frac{u_3}{R_2} \\ \varepsilon_{33} &= \frac{\partial u_3}{\partial \alpha_3}, \quad \gamma_{23} = \frac{\partial u_2}{\partial \alpha_3} + \frac{\partial u_3}{H_2 \partial \alpha_2} - \frac{u_2}{R_2}, \quad \gamma_{13} = \frac{\partial u_1}{\partial \alpha_3} + \frac{\partial u_3}{H_1 \partial \alpha_1} - \frac{u_1}{R_1} \\ \gamma_{12} &= \frac{\partial u_1}{H_2 \partial \alpha_2} + \frac{\partial u_2}{H_1 \partial \alpha_1} - \frac{u_2 \partial H_2}{H_1 H_2 \partial \alpha_1} - \frac{u_1 \partial H_1}{H_1 H_2 \partial \alpha_2} \\ E_1 &= -\frac{\partial \varphi}{H_1 \partial \alpha_1}, \quad E_2 = -\frac{\partial \varphi}{H_2 \partial \alpha_2}, \quad E_3 = -\frac{\partial \varphi}{\partial \alpha_3} \end{aligned} \tag{7.213}$$

Let $C_{ijkl}(z), e_{kij}(z), \epsilon_{ij}(z)$, $z \equiv \alpha_3$. Wu et al. (2002) assumed

$$
\begin{aligned}
u_1 &= u_1^0(\alpha_1,\alpha_2) + zu_1^1(\alpha_1,\alpha_2), \quad u_2 = u_2^0(\alpha_1,\alpha_2) + zu_2^1(\alpha_1,\alpha_2) \\
u_3 &= u_3^0(\alpha_1,\alpha_2) + zu_3^1(\alpha_1,\alpha_2), \quad \varphi = \varphi^0(\alpha_1,\alpha_2) + z\varphi^1(\alpha_1,\alpha_2) + z^2\varphi^{(2)}(\alpha_1,\alpha_2)
\end{aligned}
\tag{7.214}
$$

Substitution of Eq. (7.214) into Eq. (7.213) yields the generalized strains, then substitutes the strains into the variational formula Eq. (7.160). Approximately take $1 + z/R_1 \approx 1,\ 1 + z/R_2 \approx 1$. Noting $\mathrm{d}V = H_1H_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}z$ and finishing the variational calculation, the approximate equations of the thin shell can be obtained. The generalized momentum equation is

$$
\begin{aligned}
\delta u_1^0 &: \frac{\partial(H_2N_{11})}{\partial\alpha_1} + \frac{\partial(H_1N_{12})}{\partial\alpha_2} + N_{12}\frac{\partial H_1}{\partial\alpha_2} - N_{22}\frac{\partial H_2}{\partial\alpha_1} + N_{31}\frac{H_1H_2}{R_1} \\
&\quad + H_1H_2\left(T_1^{*+} + T_1^{*-}\right) = H_1H_2\left(\rho_0\ddot{u}_1^0 + \rho_1\ddot{u}_1^1\right) \\
\delta u_2^0 &: (H_2N_{12})_{,1} + (H_1N_{22})_{,2} + N_{12}H_{2,1} - N_{11}H_{1,2} + N_{32}H_1H_2/R_2 \\
&\quad + H_1H_2\left(T_2^{*+} + T_2^{*-}\right) = H_1H_2\left(\rho_0\ddot{u}_2^0 + \rho_1\ddot{u}_2^1\right) \\
\delta u_3^0 &: (H_2N_{13})_{,1} + (H_1N_{32})_{,2} - N_{11}H_1H_2/R_1 - N_{22}H_1H_2/R_2 \\
&\quad + H_1H_2\left(T_3^{*+} + T_3^{*-}\right) = H_1H_2\left(\rho_0\ddot{u}_3^0 + \rho_1\ddot{u}_3^1\right)_3 \\
\delta u_1^1 &: (H_2M_{11})_{,1} + (H_1M_{12})_{,2} + M_{12}H_{1,2} - M_{22}H_{2,1} + M_{31}H_1H_2/R_1 \\
&\quad - N_{13}H_1H_2 + H_1H_2\left(hT_1^{*+} - hT_1^{*-}\right) = H_1H_2\left(\rho_1\ddot{u}_1^0 + \rho_2\ddot{u}_1^1\right) \\
\delta u_2^1 &: (H_2M_{12})_{,1} + (H_1M_{22})_{,2} + M_{12}H_{2,1} - M_{11}H_{1,2} + M_{32}H_1H_2/R_2 \\
&\quad - N_{23}H_1H_2 + H_1H_2\left(hT_2^{*+} - hT_2^{*-}\right) = H_1H_2\left(\rho_1\ddot{u}_2^0 + \rho_2\ddot{u}_2^1\right) \\
\delta u_3^1 &: (H_2M_{13})_{,1} + (H_1M_{23})_{,2} - M_{11}H_1H_2/R_1 - M_{22}H_1H_2/R_2 \\
&\quad - N_{33}H_1H_2 + H_1H_2\left(hT_3^{*+} - hT_3^{*-}\right) = H_1H_2\left(\rho_1\ddot{u}_3^0 + \rho_2\ddot{u}_3^1\right) \\
\delta\varphi^0 &: \left(H_2D_1^0\right)_{,1} + \left(H_1D_2^0\right)_{,2} + H_1H_2\left(\sigma_1^{*+} + \sigma_1^{*-}\right) = 0 \\
\delta\varphi^1 &: \left(H_2D_1^1\right)_{,1} + \left(H_1D_2^1\right)_{,2} - H_1H_2D_3^0 + H_1H_2\left(h\sigma_1^{*+} - h\sigma_1^{*-}\right) = 0 \\
\delta\varphi^{(2)} &: \left(H_2D_1^{(2)}\right)_{,1} + \left(H_1D_2^{(2)}\right)_{,2} - H_1H_2D_3^1 + H_1H_2\left(h^2\sigma_1^{*+} - h^2\sigma_1^{*-}\right) = 0
\end{aligned}
\tag{7.215}
$$

The natural boundary conditions are

$$
\begin{aligned}
&\delta u_1^0 : N_{11}n_1 + N_{12}n_2 = \bar{T}_1, && \delta u_2^0 : N_{12}n_1 + N_{22}n_2 = \bar{T}_2 \\
&\delta u_3^0 : N_{13}n_1 + N_{32}n_2 = \bar{T}_3, && \delta u_1^1 : M_{11}n_1 + M_{12}n_2 = \bar{M}_1 \\
&\delta u_2^1 : M_{12}n_1 + M_{22}n_2 = \bar{M}_2, && \delta u_3^1 : M_{13}n_1 + M_{23}n_2 = \bar{M}_3 \\
&\delta\varphi^0 : D_1^0n_1 + D_2^0n_2 = -\sigma^{0*}, && \delta\varphi^1 : D_1^1n_1 + D_2^1n_2 = -\sigma^{1*} \\
&\delta\varphi^{(2)} : D_1^{(2)}n_1 + D_2^{(2)}n_2 = -\sigma^{2*}
\end{aligned}
\tag{7.216}
$$

Let T_i^{*+} and T_i^{*-} denote the traction on the upper and lower surfaces, respectively, and

$$(\bar{T}_i, \bar{M}_i) = \int_{-h}^{h} T_i^*(1, z)\mathrm{d}z, \quad \left(D_i^0, D_i^1, D_i^{(2)}\right) = \int_{-h}^{h} D_i\left(1, z, z^2\right)\mathrm{d}z,$$

$$\left(\sigma^{0*}, \sigma^{1*}, \sigma^{(2)*}\right) = \int_{-h}^{h} \sigma^*\left(1, z, z^2\right)\mathrm{d}z \tag{7.217}$$

$$(\rho_0, \rho_1, \rho_2) = \sum_{k=1}^{N} \int_{h_k} \rho^{(k)}\left(1, x_3, x_3^2\right)\mathrm{d}x_3, \quad \left(N_{ij}, M_{ij}\right) = \int_{-h}^{h} (1, z)\sigma_{ij}\mathrm{d}z$$

It is noted that the generalized displacements must satisfy the boundary conditions when the variational formula Eq. (7.160) is used. Using the constitutive equation, the governing equations in terms of the generalized displacements are easily obtained.

7.7.4 Free Vibration of a Functionally Graded Piezoelectric Hollow Cylinder Filled with Compressible Fluid

Consider an orthotropic piezoelectric hollow cylinder of inner radius R, thickness h, and length L. Chen et al. (2004) adopted the cylindrical coordinates r, θ, z and adopted the state space method to analyze the free vibration of a functionally graded piezoelectric hollow cylinder filled with compressible fluid. Assumed all material constants and mass are the functions of r. From the constitutive equations, geometric equations, and motion equations, the state equation can be obtained as

$$\boldsymbol{Y}_{,r} = \boldsymbol{MY}, \quad \boldsymbol{Y} = [u_z, u_\theta, \sigma_r, D_r, \sigma_{rz}, \sigma_{r\theta}, u_r, \varphi]^{\mathrm{T}} \tag{7.218}$$

where $\boldsymbol{Y}$ is the state vector and $\boldsymbol{M}$ is 8×8 matrix. Chen et al. (2004) discussed the simply supported case with the boundary conditions:

$$u_r = u_\theta = \sigma_z = D_z = 0, \quad \text{at} \quad z = 0, L \tag{7.219}$$

In order to satisfy Eq. (7.219), they assumed that the solution of state vector can be expanded in double trigonometric series:

$$\begin{Bmatrix} u_z \\ u_\theta \\ \sigma_r \\ D_r \\ \sigma_{rz} \\ \sigma_{r\theta} \\ u_r \\ \varphi \end{Bmatrix} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \begin{Bmatrix} R_0\bar{u}_z(\eta)\cos m\pi\varsigma \cos n\theta \\ R_0\bar{u}_\theta(\eta)\sin m\pi\varsigma \sin n\theta \\ C_{44}^{\mathrm{out}}\bar{\sigma}_r(\eta)\sin m\pi\varsigma \cos n\theta \\ \sqrt{C_{44}^{\mathrm{out}}\epsilon_{33}^{\mathrm{out}}}\bar{D}_r(\eta)\cos m\pi\varsigma \cos n\theta \\ C_{44}^{\mathrm{out}}\bar{\sigma}_{rz}(\eta)\cos m\pi\varsigma \cos n\theta \\ C_{44}^{\mathrm{out}}\bar{\sigma}_{r\theta}(\eta)\sin m\pi\varsigma \sin n\theta \\ R_0\bar{u}_r(\eta)\sin m\pi\varsigma \cos n\theta \\ R_0\sqrt{C_{44}^{\mathrm{out}}/\epsilon_{33}^{\mathrm{out}}}\bar{\varphi}(\eta)\cos m\pi\varsigma \cos n\theta \end{Bmatrix} \mathrm{e}^{\mathrm{i}\omega t} \tag{7.220}$$

where $R_0 = R + h/2$, $\eta = r/R_0$, and $\varsigma = z/L$; m and n are integers; and ω is the angular frequency. Variables at the outer cylindrical surface $r = R + h$ are denoted by right superscript "out," and at the inner surface $r = R$ will be denoted by right superscript "inn." Substitution of Eq. (7.220) into Eq. (7.218) yields

$$\bar{\boldsymbol{Y}}_{,\eta} = \boldsymbol{N}\bar{\boldsymbol{Y}}, \quad \bar{\boldsymbol{Y}} = [\bar{u}_z, \bar{u}_\theta, \bar{\sigma}_r, \bar{D}_r, \bar{\sigma}_{rz}, \bar{\sigma}_{r\theta}, \bar{u}_r, \bar{\varphi}]^{\mathrm{T}} \tag{7.221}$$

where $\bar{\boldsymbol{Y}}$ is a constant vector and $\boldsymbol{N}$ is a 8×8 matrix and is not constant, so the solution cannot be obtained directly from Eq. (7.221). The approximate laminated model, for which the cylinder is divided into N thin layers, is adopted. For ith layer, $\boldsymbol{N}_i$ can be assumed constant and takes its value at midplane. Using the transfer matrix method as shown in Section 6.4.1, $\boldsymbol{Y}^{\mathrm{out}} = \boldsymbol{T}\boldsymbol{Y}^{\mathrm{inn}}$ can be obtained, through the transfer matrix $\boldsymbol{T}$, and the variables on the outer surface are expressed by the inner variables. The boundary condition on the inner cylindrical surface is solved by the fluid-solid coupling theory. For a nonviscous fluid, the connected conditions on the inner surface are

$$v_r = v_{\mathrm{f}r}, \quad p_{\mathrm{f}} + \sigma_r = 0, \quad \sigma_{rz} = \sigma_{r\theta} = 0, \quad \text{at} \quad r = R \tag{7.222}$$

where $v_{\mathrm{f}r}, v_r$ are the radial components of the velocity of the fluid and solid, respectively, and p_{f} is the fluid pressure. Finally, the boundary conditions on the inner and outer surfaces are

$$\begin{aligned} &\sigma_r^{\mathrm{inn}} = -\Omega^2 Q(\beta) u_r \rho_{\mathrm{f}}/\rho^{\mathrm{out}}, \quad \sigma_{rz}^{\mathrm{inn}} = \sigma_{r\theta}^{\mathrm{inn}} = 0, \quad \sigma_r^{\mathrm{out}} = \sigma_{rz}^{\mathrm{out}} = \sigma_{r\theta}^{\mathrm{out}} = 0 \\ &\Omega^2 = R^2\omega^2\rho^{\mathrm{out}}/C_{44}^{\mathrm{out}}, \quad \beta = \omega R/c_{\mathrm{f}}^2 - m\pi R/L \end{aligned} \tag{7.223}$$

where, $\rho_{\mathrm{f}}, \rho^{\mathrm{out}}, c_{\mathrm{f}}$ are the fluid density, solid density at $r = R + h$, and the sound velocity in fluid, respectively.

The electrically boundary conditions on $r = R,\ R + h$ are as follows:

Electrically open, $D_r = 0$, or electrically shorted, $\varphi = 0$, at

$$r = R,\ R + h \tag{7.224}$$

Substituting the boundary conditions at the inner and outer surfaces into Eq. (7.221), the frequency equation can be obtained. The frequency equations for the electrically shorted case and the electrically open case are different.

References

Belubekyan M, Ghazaryan K, Marzocca P, Cormier C (2007) Localized magnetoelastic bending vibration of an electroconductive elastic plate. J Appl Mech 74:1071–1077

Chen W-Q, Lim CW (2005) 3Dpoint force solution for a permeable penny-shaped crack embedded in an infinite transversely isotropic piezoelectric medium. Int J Solids Struct 131:231–246

Chen W-Q, Shioya T (1999) Fundamental solution for a penny-shaped crack in a piezoelectric media. Int J Solids Struct 47:1459–1475
Chen C-Q, Shen Y-P, Wang XM (1996) Exact solution of orthotropic cylindrical shell with piezoelectric layers under cylindrical bending. Int J Solids Struct 33:4481–4494
Chen W-Q, Bian ZG, Lv C-F, Ding HJ (2004) 3Dfree vibration analysis of a functionally graded piezoelectric hollow cylinder filled with compressible fluid. Int J Solids Struct 41:947–964
Cherepanov GP (1979) Mechanics of brittle fracture. McGraw-Hill, New York
Crawley EF, Aanderson E (1989) Detailed models of piezoceramic actuation of beams, In: Proceedings of the 30th AIAA/ASMI/ASCH/AHS/AHC structures, structural dynamics and materials conference, Washington, DC
Crawley EF, De Luis J (1987) Use of piezoelectric actuators as elements of intelligent structures. AIAA J 25:1373–1385
Deans SR (1983) The radon transform and some of its applications. Wiley, New York
Deeg WF (1980) The analysis of dislocation, crack and inclusion problems in piezoelectric solids. Ph.D. dissertation, Stanford University
Ding H-J, Chen B, Liang J (1996) General solutions for coupled equations for piezoelectric media. Int J Solids Struct 33:2283–2298
Dunn ML (1994) Electroelastic Green's functions for transversely isotropic piezoelectric media and their application to the solution of inclusion and inhomogeneity problems. Int J Eng Sci 32:119–131
Dunn ML, Taya M (1993) An analysis of piezoelectric composite materials containing ellipsoidal inhomogeneities. Proc R Soc Lond A 443:265–287
Eshelby JD (1957) The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc R Soc Lond A 241:376–396
Fabrikant VI (1989) Applications of potential theory in mechanics, a selection of new results. Kluwer Academic Publishers, Dordrecht/Boston/London
Fan H, Qin S (1995) A piezoelectric sensor embedded in a non- piezoelectric matrix. Int J Eng Sci 33:379–388
Heyliger P (1997) Exact solutions for simply supported laminated piezoelectric plates. J Appl Mech 64:299–306
Heyliger P, Brooks S (1995) Free vibration of piezoelectric laminates in cylindrical bending. Int J Solids Struct 32:2945–2960
Huang JH (1997) A fracture criterion of a penny-shaped crack in transversely isotropic piezoelectric media. Int J Solids Struct 34:2631–2644
Jiang LZ, Sun CT (2001) Analysis of indentation cracking in piezoceramics. Int J Solids Struct 38:1903–11918
Kapuria S, Sengupta S, Dumir C (1998) Assessment of shell theories for hybrid piezoelectric cylindrical shell under electromechanical load. Int J Mech Sci 40:461–477
Krommer M, Irschik H (2000) A Reissner-Mindlin-type plate theory including the direct piezoelectric and the pyroelectric effect. Acta Mech 141:51–69
Kuang Z-B (2011) Physical variational principle and thin plate theory in electro-magneto-elastic analysis. Int J Solids Struct 48:317–325
Librescu L, Hasanyan D, Ambur DR (2004) Electromagnetically conducting elastic plates in a magnetic field: modeling and dynamic implications. Int J Nonlinear Mech 39:723–729
Mikata Y (2000) Determination of piezoelectric Eshelby tensor in transversely isotropic piezoelectric solids. Int J Eng Sci 38:605–641
Mitchell JA, Reddy JN (1995) A refined hybrid plate theory for composite laminates with piezoelectric laminae. Int J Eng Sci 32:2345–2367
Moon EC, Pao YH (1968) Magnetoelastic buckling of a thin plate. J Appl Mech 35:53–68
Mura T (1987) Micromechanics of defects in solids, 2nd rev. edn. Martinus-Nijhoff, Deventer
Pao YH, Yeh CS (1973) A linear theory for soft ferromagnetic elastic bodies. Int J Eng Sci 11:415–436

Saviz MR, Shakeri M, Yas MH (2007) Electroelastic fields in a layered piezoelectric cylindrical shell under dynamic load. Smart Mater Struct 16:1683–1695

Timoshenko S, Woinowsky-Krieger S (1959) Theory of plates and shells. McGraw-Hill Book Company, New York

Ueda S (2007) A penny-shaped crack in a functionally graded piezoelectric strip under thermal loading. Eng Fract Mech 74:1255–1273

Wang B (1992) Three-dimensional analysis of an ellipsoidal inclusion in a piezoelectric material. Int J Solids Struct 29:293–308

Wang J, Yang J (2000) Higher-order theories of piezoelectric plates and applications. Appl Mech Rev 53:87–99

Wang Z-K, Zheng B-L (1995) The general solution of three dimensional problems in piezoelectric media. Int J Solids Struct 32:105–115

Wu X, Chen C, Shen Y, Tian X (2002) A high order theory for functionally graded piezoelectric shells. Int J Solids Struct 39:5325–5344

Yang J (2005) An introduction to the theory of piezoelectricity. Springer, New York

Zhou Y-H, Miya K (1998) A theoretical prediction of natural frequency of a ferromagnetic beam plate with low susceptibility in an in-plane magnetic field. J Appl Mech 65:121–126

Zhou Y-H, Zheng X-J (1997) A general expression of magnetic force for soft ferromagnetic plates in complex magnetic field. Int J Eng Sci 35:1405–1417

Chapter 8
Failure Theories of Piezoelectric Materials

Abstract In this chapter failure experiments and theories in piezoelectric materials are discussed. In present time the precision of experiments should still be improved. The failure theory in solids is very complicated and there is no unified critical criterion. It is clear that the critical energy for different failure version is different. Especially the version of brittle tension failure is significantly different with other versions. In piezoelectric ceramics the failure energy density of an electric field is much higher than that in mechanical loading. In this chapter the generalized stress intensity factor criterions; total, mechanical, and local energy release rate criterions; strain energy density factor criterion; modal strain energy density factor theory; small-scale domain switching theory; failure criterion of conductive cracks with charge-free zone model are studied. Some simple electric breakdown theories of solid dielectrics are also discussed.

Keywords Failure theories • Generalized stress intensity factor • Energy release rate • Modal strain energy • Charge-free zone model • Electric breakdown

8.1 Experimental Studies

The change of the microstructure, including plastic yielding, phase transformation, and failure theory, in solids is very complicated, and in present time there is no unified critical criterion to show these changes exactly. In general the change of the internal microstructure in the materials is caused by deformation, electromagnetic field and temperature. Experiments show that except the failure under tension, before failure the continuous deformation is revealed and then follows the change of the microstructure, so the failure and plastic yielding, etc. often posses the similar criterion. However, the failure of a brittle material under tension is less connected to the continuous deformation, so it should often be discussed in different theory. The experiments are necessary to get the practical failure criterions in engineering. Many experiments for failure have been carried out. Because the piezoelectric specimens are thin and small, manufacturing an ideal crack is very difficult. Usually the crack

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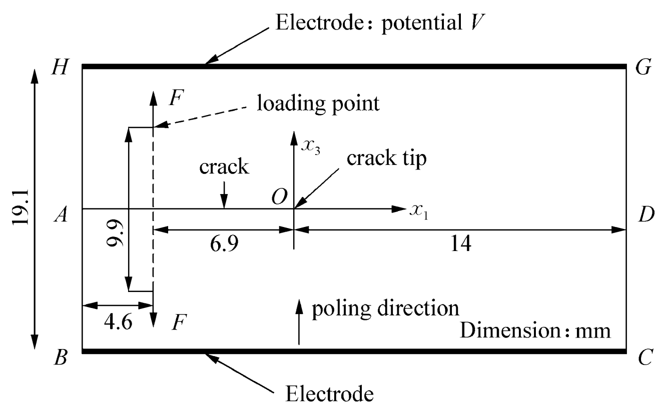


Fig. 8.1 Compact tension (*CT*) specimen

is a narrow slit with a circular end. The precision of the experimental results is not very well, and the experiments are needed in the future. The fatigue failure is not discussed in this book.

8.1.1 Test on Compact Tension Specimens

Park and Sun (1995) carried out the mode-I fracture tests of the compact tension (CT) specimen for PZT-4 ceramic. Two silver-coated faces at the upper and lower surfaces of the sample were used as electrodes. The sizes of the specimen are $25.5 \times 19.1 \times 5.1\,\text{mm}^3$ with crack length 11.5 mm as shown in Fig. 8.1. The polarization x_3-axis is perpendicular to the crack. The crack was created by cutting with a 0.46 mm thick diamond wheel and further cut by a sharp razor blade with diamond abrasive. Because the electric field exceeded $5\,\text{Kv/m}$ the electric discharging between electrodes through the air was observed, the specimen was immersed in a tube filled with silicone oil. The tests were to increase the tensile load until failure occurred under a certain electric field. Some experimental results can be found in Fig. 8.13. The results showed that the positive electric fields enforce the crack propagation or decrease the apparent fracture toughness K_{Ic}, while negative electric field impede crack propagation or increase the apparent fracture toughness.

Fang et al. (2004) carried out the tensile tests of the plate specimen with a central crack for PZT-5 ceramic. The sizes of the specimen are $40 \times 20 \times 3\,\text{mm}^3$ with the polarization x_3-axis. Their results are similar to that of Park and Sun.

8.1.2 Three-Point Bending Test with Asymmetric Crack

Park and Sun (1995) carried out the mixed-mode fracture tests on the three-point bend specimen with unsymmetrical crack for PZT-4 ceramic. The sizes of the specimen are $19.1 \times 9 \times 5.1\,\text{mm}^3$, and the poling direction was perpendicular to

Fig. 8.2 Specimen for mixed-mode fracture test

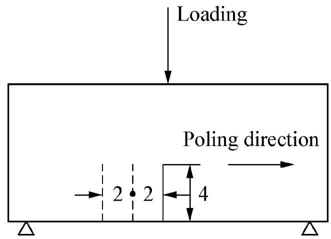
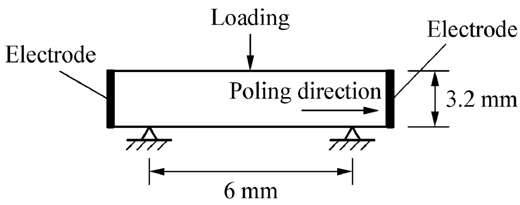


Fig. 8.3 Three-point smooth bending test specimen



the crack as shown in Fig. 8.2. The length of the crack is 4.0 mm and located at the midspan, 2 mm, 4 mm from the midspan, respectively. The center-cracked specimens produced mode-I fracture, and other two kinds of specimens exhibited mixed-mode fracture. The experimental results showed that the fracture in this test exhibits the same behaviors as that in CT specimens. It is also shown that the crack deviated from the midspan will increase the fracture load.

8.1.3 Three-Point Bending Test of Smooth Specimens

Fu and Zhang (2000) carried out three-point bending tests of smooth specimens to study the effect of an electric field on bending strength for PZT-841 ceramic (Fig. 8.3). The sizes of the specimen are $10 \times 4 \times 3.2\,\text{mm}^3$ and the span distance was 6.0 mm. The poling direction was perpendicular to the load. Two silver-coated faces at the ends of the sample were used as electrodes. The specimens were thermally depoled at 400 °C for 30 min. Both the loading jig and supports were insulated from the loading system. The generalized stresses will be calculated by the finite element method. Under a mechanical load of 340 N, the electric field was monotonically increased until fracture occurred. The results are shown in Fig. 8.4. Results show that for the depoled specimens, the positive and negative fields all decrease the bending strength.

8.1.4 Test on Conducting Crack

Heyer et al. (1998) carried out the mode-I fracture tests of the four-point bending specimen for PZT-PIC 151 ceramic. Two silver-coated faces at the upper and lower

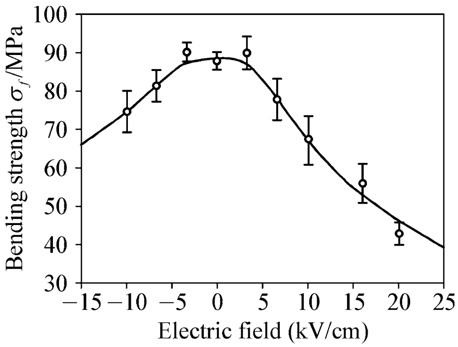


Fig. 8.4 Variations of bending strength with electric field (Reprinted from Fu and Zhang 2000, with permission from *Acta Materialia*)

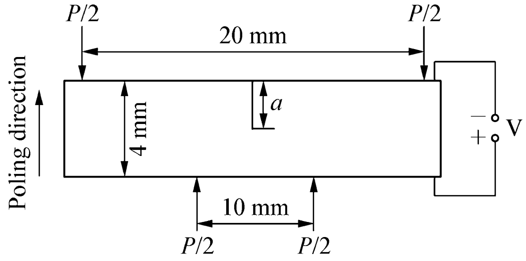


Fig. 8.5 Four-point bending specimen

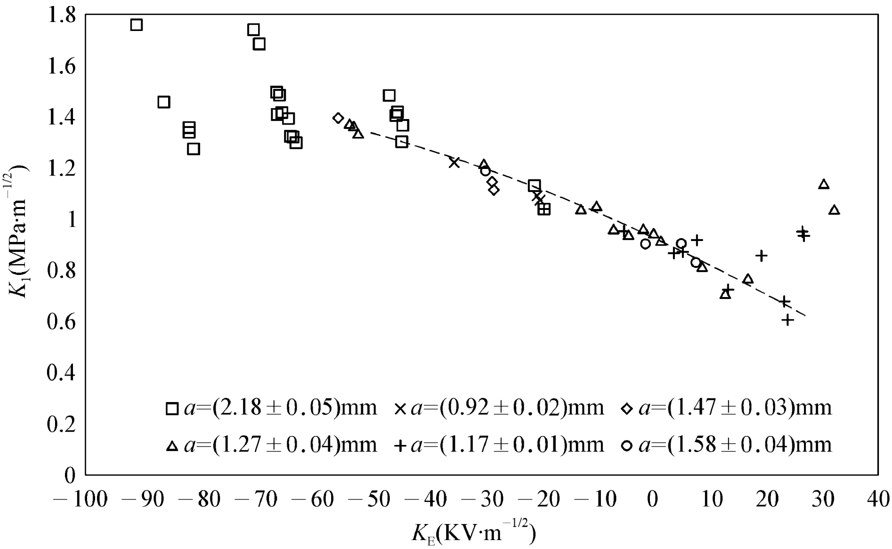


Fig. 8.6 Plot of the stress and electric field intensity factors (Reprinted from Heyer et al. 1998, with permission from *Acta Materialia*)

surfaces of the sample were used as electrodes. The thickness is 3 mm. The crack is filled with NaCl solution and its depth $a = 0.9 \sim 2.2$ mm as shown in Fig. 8.5. The generalized stress intensity factors are calculated by numerical method. The results are shown in Fig. 8.6.

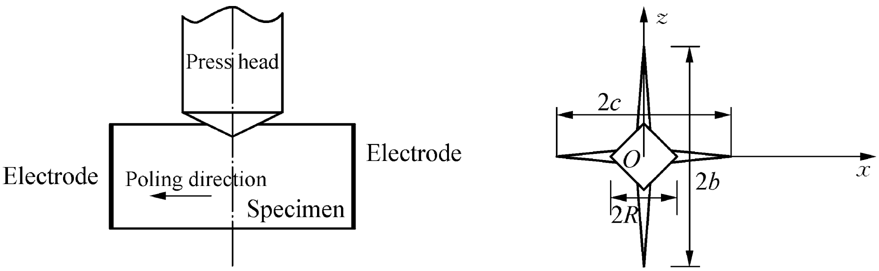


Fig. 8.7 Vicker indentation test

8.1.5 Vicker Indentation Test

The Vicker indentation test technique is pressing a square diamond pyramid into a specimen with a given external force. The resulting wedge force drives radial half penny-shaped cracks. The crack growth version in a Vicker indentation test is quite different with that in the CT test. In the direction perpendicular to the poling direction, the crack length is longer than that in the direction parallel to poling direction (Fig. 8.7). Experiment results (Jiang and Sun 2001) pointed out that for small loading (9 N), increasing the positive electric field increased the crack and increasing the negative electric field (absolute value) decreased the crack. For a larger loading (49 N), increasing the positive or negative electric field increased the crack, but in a negative electric field, the crack growth is smaller than that in the positive electric field. In the test of Wang and Singh (1997), the results are somewhat different.

8.2 Some Practical Failure Criterions

8.2.1 Generalized Stress Intensity Factor Criterion

In the linear- and small-scale yielding fracture mechanics for mode-I problem, the stress intensity factor K_{I} or the energy release rate G or J_{I} integral is used as the fracture criterion. For mixed fracture problem, the combination of $K_{\mathrm{I}}, K_{\mathrm{II}}$ and K_{III} is used. These theories are successful in engineering, but what combination should be used is still a problem. It is natural to extend these theories to electroelastic fracture problem, i.e., for the mixed fracture problem in piezoelectric material, the fracture criterion takes the following form:

$$f(K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}}, K_{\mathrm{D}}) = K_{\mathrm{c}} \tag{8.1}$$

However, it should be noted that the role of K_{D} is not fully the same as that of $K_{\mathrm{I}}, K_{\mathrm{II}}$ and K_{III}. Many experiments show that the electric energy at failure of a piezoelectric material is much larger than that in the mechanical fracture. Otherwise the large electric field can produce electric sparking to breakdown the dielectric. As examples Heyer et al. (1998) gave a fracture criterion fitting their measured data in the range $-50 < K_{\mathrm{E}} < 25\,(\mathrm{KVm}^{-1/2})$ for material PET-PIC 151 is

$$K_{\mathrm{I}}(\mathrm{MPam}^{1/2}) = 0.90 - 0.01\,K_{\mathrm{E}}(\mathrm{KVm}^{1/2}) - 0.00002\,K_{\mathrm{E}}^2 \tag{8.2}$$

Fang et al. (2004) gave a fracture criterion fitting their measured data in the range $K_{\mathrm{I}} < 1.5\mathrm{Pam}^{1/2}$ and $-2.5 \times 10^4\mathrm{Vm}^{-1/2} < K_{\mathrm{E}} < 4 \times 10^4\mathrm{Vm}^{-1/2}$ for material PET-5 is

$$K_{\mathrm{I}}(\mathrm{Pam}^{1/2}) + 18.193K_{\mathrm{E}}(\mathrm{Vm}^{-1/2}) - 2.641 \times 10^{-4}K_{\mathrm{E}}^2 = 803505.949 \tag{8.3}$$

8.2.2 Energy Release Rate Criterion

1. *Total Energy Release Rate Criterion*

For linear electroelastic theory, the original energy criterion of the crack extension is

$$G = R, \quad G = -\partial(U - W)/\partial a, \quad R = \mathrm{d}(\gamma_{\mathrm{s}} + \gamma_{\mathrm{i}})/\mathrm{d}a \tag{8.4}$$

where G is the energy release rate, R is crack extension resistance, U is the internal energy, W is the work done by the external force, γ_{s} is the energy to formed a new surface, and γ_{i} is the irreversible work at the crack tip. But we often use the electric enthalpy release rate $\tilde{G}$ as the energy release rate. If the crack tip is selected at the coordinate origin, $\tilde{G}$ is (Suo et al. 1992)

$$\tilde{G} = \lim_{\Delta \to 0} \frac{1}{2\Delta} \times \int_0^{\Delta} \left\{\sigma_{2j}(r)[u_j^+(\Delta - r) - u_j^-(\Delta - r)] + D_2(r)[\varphi^+(\Delta - r) - \varphi^-(\Delta - r)]\right\}\mathrm{d}r \tag{8.5}$$

where Δ is the crack virtual extension, r is the distance in front of the crack tip on axis x_1 and $\Delta - r$ is the distance behind the crack tip.

The J integral expressed with the electric enthalpy is defined as

$$J = \int_{\Gamma} (g n_1 - \sigma_{ij} n_j u_{i,1} - n_i D_i \varphi_{,1})\,\mathrm{d}l, \quad g = (1/2)(\sigma_{ij} u_{i,j} + D_i \varphi_{,i}) \tag{8.6}$$

where Γ is the integration contour around the crack tip and g is the electric enthalpy density. In the linear- or small-scale yielding case, it can be proved that $\tilde{G} = J$.

For the mode-I fracture tests of the compact tension (CT) specimen, as shown in Sect. 8.1.1, Park and Sun (1995) gave

$$\tilde{G}_{\mathrm{I}} == (\pi a/2)\left[2.29 \times 10^{-11}\left(\sigma_{33}^{\infty}\right)^{2} + 2.35 \times 10^{-11}\sigma_{33}^{\infty}E_{3}^{\infty} - 8.78 \times 10^{-9}\left(E_{3}^{\infty}\right)^{2}\right]\mathrm{N/m} \tag{8.7}$$

Equation (8.7) shows that when the contribution of an electric field is larger than that of the mechanical stress, the total energy release rate becomes negative and the crack cannot be extended. For the small stress and large electric field, the electric field always impedes crack propagation. It is contrary to the experimental results.

2. *Mechanical Strain Energy Release Rate Criterion*

Park and Sun (1995) proposed a mechanical strain energy release rate $G_{\mathrm{I}}^{\mathrm{M}}$ criterion for the piezoelectric materials. For the CT test, they got

$$\begin{aligned} G_{\mathrm{I}}^{\mathrm{M}} &= \lim_{\Delta \to 0} \frac{1}{2\Delta} \int_{0}^{\Delta} \left\{\sigma_{33}(r)\left[u_{3}^{+}(\Delta - r) - u_{3}^{-}(\Delta - r)\right]\right\} \\ &= (\pi a/2)\left[2.28 \times 10^{-11}\left(\sigma_{33}^{\infty}\right)^{2} + 1.21 \times 10^{-11}\sigma_{33}^{\infty}E_{3}^{\infty}\right]\mathrm{N/m} \end{aligned} \tag{8.8}$$

Equation (8.8) shows that the positive electric field (the direction of the external electric field is consistent with the poling electric field) increases the mechanical strain energy rate and decreases the fracture toughness. The results calculated from this criterion are consistent with that in test, as shown in Fig. 8.13. However, if the stresses are all zeros, Eq. (8.8) shows that the crack cannot be extended; it is also contrary with the experiment. It can be considered that for larger mechanical stress this criterion is well.

3. *Strain Energy Density Factor Criterion*

In elastic fracture mechanics, Sih (1973) proposed the strain energy density factor as the fracture criterion. Zuo and Sih (2000) and Shen and Nishioka (2000) extended this theory to the piezoelectric materials. The strain energy density factor S is defined as

$$S = \lim_{r \to 0} r\mathrm{d}U/\mathrm{d}V = \lim_{r \to 0} r\mathfrak{A}, \quad \mathfrak{A} = \mathrm{d}U/\mathrm{d}V = \sigma_{ij}\varepsilon_{ij}/2 + D_i E_i/2 \tag{8.9}$$

where $\mathfrak{A}$ is the strain energy density. The strain energy density factor criterion is assumed:

(a) At the crack tip, the minimum strain energy density $S_{\min}$ happened at $\theta = \theta_0$, $(\partial S/\partial \theta)_{\theta=\theta_0} = 0$, and $(\partial^2 S/\partial \theta^2)_{\theta=\theta_0} > 0$, where θ is the polar angle. Crack initiation will start at the direction of the max $S_{\min}$.

(b) When max $.S_{\min}$ reaches the critical value S_{c}, the crack begins propagation.

This theory is based on the stress state before crack extension. This theory is not related to the crack virtual extension which is demanded by the energy release rate theory.

According to Eq. (3.222) in general case, the stress asymptotic field near the crack tip for a piezoelectric material with polarized x_3-axis is

$$\boldsymbol{\Sigma}_1 = -\left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\,\boldsymbol{B}\left\langle \mu_k\Big/\sqrt{\Theta_k}\right\rangle\boldsymbol{B}^{-1}\boldsymbol{K}, \quad \boldsymbol{\Sigma}_2 = \left(1\Big/\sqrt{2\pi r}\right)\mathrm{Re}\,\boldsymbol{B}\left\langle 1\Big/\sqrt{\Theta_k}\right\rangle\boldsymbol{B}^{-1}\boldsymbol{K}$$
$$\Theta_k = \sqrt{\cos\theta + \mu_k\sin\theta}, \quad K_{\mathrm{I}} = \sigma_{33}^{\infty}\sqrt{\pi a},\ K_{\mathrm{II}} = \sigma_{31}^{\infty}\sqrt{\pi a},\ K_{\mathrm{III}} = \sigma_{32}^{\infty}\sqrt{\pi a},\ K_D = D_3^{\infty}\sqrt{\pi a} \tag{8.10}$$

Discuss the plane strain problem in the (x_1, x_3) plane. In this case $E_2 = 0, \varepsilon_{22} = 0$, so $\sigma_{22}\varepsilon_{22} = 0, D_2E_2 = 0$. The strain energy density factor becomes

$$S = \lim_{r\to 0} r\mathfrak{A} = (1/2)\lim_{r\to 0} r\left[\left(\boldsymbol{\Sigma}_1^{\mathrm{T}}\boldsymbol{\varepsilon}_1 + \boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{\varepsilon}_2\right) + (\sigma_{31}\varepsilon_{31} + \sigma_{23}\varepsilon_{23})\right]$$
$$\boldsymbol{\varepsilon}_1 = (\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, E_1)^{\mathrm{T}}, \quad \boldsymbol{\varepsilon}_2 = (\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}, E_2)^{\mathrm{T}} \tag{8.11}$$

where generalized strains can be obtained from the constitutive equations, so S can be expressed by the generalized stress intensity factors. Their numerical calculation results show that for the material PZT-4 in the range $-0.6\,(\mathrm{kV/cm}) < E_3 < 0.8\,(\mathrm{kV/cm})$, the theoretical results are consistent with previous experimental results.

8.2.3 Small-Scale Domain Switching Theory

Experiments show that under mechanical and electrical loadings, the intensified generalized stresses near a crack-like flaw lead to domain reorientation. An electric field can rotate the polar direction of a domain by either 180° or 90°, but a stress field rotates it only by 90°. Let the initial poling direction of a domain form an angle ϕ with x_1-axis and the variation of the polarization vector $\boldsymbol{\Delta P}_{\mathrm{s}}$ and the transformation strain tensor $\boldsymbol{\Delta\varepsilon}_0$ after a 90° rotation of a domain (Fig. 8.8) be, respectively,

$$\boldsymbol{\Delta P}_{\mathrm{s}} = \sqrt{2}P_{\mathrm{s}}\begin{bmatrix}\cos(\phi \pm 3\pi/4)\\ \sin(\phi \pm 3\pi/4)\end{bmatrix}, \quad \boldsymbol{\Delta\varepsilon}_0 = \varepsilon_0\begin{bmatrix}-\cos 2\phi & \sin 2\phi\\ \sin 2\phi & \cos 2\phi\end{bmatrix} \tag{8.12}$$

where $\boldsymbol{P}_{\mathrm{s}}$ is the spontaneous polarization, ε_0 is the spontaneous strain, and $\pm 3\pi/4$ are corresponding to the anticlockwise and clockwise, respectively.

The domain switching plays an important role in the crack extension theory. Zhu and Yang (1997) and Yang and Zhu (1998) proposed a small-scale domain switching theory to qualitatively discuss the fracture toughness variation which is related to the fracture criterion. In their discussion, they assumed outside the switching zone, the interaction between the stress and electric field is neglected and the material is assumed isotropic. In Sect. 5.1.3, the electric field of a permeable elliptic cavity in an electrostrictive material was studied under the assumption that the effect of the stress on the electric field was neglected, so the results in Sect. 5.1.3

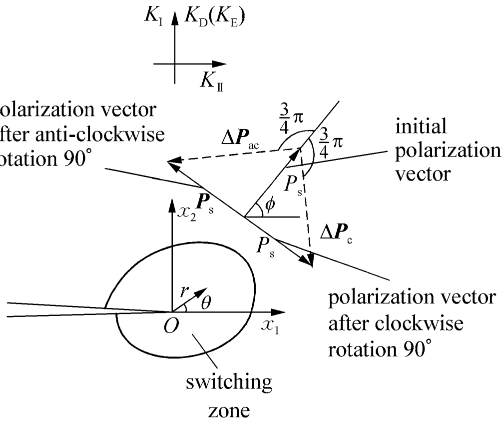


Fig. 8.8 90° switching domain near the crack tip

can be used here. When the external electric field is only E_2^{∞}, the electric asymptotic field near the right end of a narrow ellipse in a local coordinate system with the origin at the focus of the ellipse is (see Eq. (5.34))

$$E_2 \approx E_2^{\infty}\left\{\frac{1}{1+\bar{\delta}}\sqrt{\frac{a}{2r}}\mathrm{e}^{-\mathrm{i}\theta/2}+\frac{1+2\bar{\delta}}{2(1+\bar{\delta})}\right\} \tag{8.13}$$

In Eq. (8.13) the first term is a singular electric field, while the second term is a homogeneous electric field. When $\bar{\delta}$ is small, the singular electric field is dominant, while $\bar{\delta}$ is large, the homogeneous field is dominant. Outside the switching zone, the stress asymptotic field is

$$\sigma_{ij} = \left(K_{\mathrm{app}}\Big/\sqrt{2\pi r}\right)f_{ij}(\theta) \tag{8.14}$$

where K_{app} is the apparent stress intensity factor. The domain switching criterion can approximately expressed as (Hwang et al. 1995)

$$\boldsymbol{\sigma} : \Delta\boldsymbol{\varepsilon} + \boldsymbol{E} : \Delta\boldsymbol{P} \geq 2P_{\mathrm{s}}E_{\mathrm{c}} \tag{8.15}$$

where E_{c} is the coercive electric field. Substitution of Eqs. (8.12), (8.13), and (8.14) into Eq. (8.15) roughly yields the boundary of the switching zone R_0 as:

$$\sqrt{R_0} = \sqrt{\rho}R(\theta,\beta,\bar{\delta}) > 0; \quad \rho = \frac{1}{8\pi}\left(\frac{K_{\mathrm{app}}\varepsilon_0}{P_{\mathrm{s}}E_{\mathrm{c}}}\right); \quad \beta = \frac{K_{\mathrm{E}}P_{\mathrm{s}}}{K_{\mathrm{app}}\varepsilon_0}, \quad K_{\mathrm{E}} = E^{\infty}\sqrt{\pi a}$$
$$R = \left[\frac{\sqrt{2}}{1+\kappa}\beta\sin\left(\phi \pm \frac{3\pi}{4}-\frac{\theta}{2}\right)+\sin\theta\,\sin\left(2\phi+\frac{3\theta}{4}\right)\right]\left[1-\frac{E_2^{\infty}}{\sqrt{2}E_{\mathrm{c}}}\frac{\bar{\delta}}{1+\bar{\delta}}\sin\left(\phi\pm\frac{3\pi}{4}\right)\right]^{-1} \tag{8.16}$$

In mono-domain switching case, numerical results showed that when $\bar{\delta} = 10^3$, the uniform electric field is dominant and a positive electric field reduces the size of the switching zone, while the negative electric field enlarges the size. When $\bar{\delta} = 10^{-3}$, the singular electric field is dominant, and both the positive and negative electric fields enlarge the size of the switching zone.

At the crack tip region, the domain switching occurred. Solved the shape and the size of the switching zone, the toughness increment can be obtained by the transformation theory (Eshelby 1957; McCmeeking and Evans 1982; Budiansky et al. 1983). Zeng and Rajapakse (2001) and Rajapakse and Zeng (2001) discussed the toughness increment by domain switching also. Huang and Kuang (2003) discussed the influence of the switching wake on the facture toughness of ferroelectric materials. The domain switch theory need be improved.

8.3 The Local Energy Release Rate Theory

Usually a piezoelectric material has high mechanical strength, brittleness, and small deformation, but the electric saturation can be occurred under large electric field. Analogous to the Dugdale model in elastoplastic fracture mechanics, Gao et al. (1997) proposed the strip electric saturation model and local energy release rate theory to explain the effect of the electric field on the failure of a crack specimen in piezoelectric material. This model considers that near the crack tip the mechanical deformation is elastic, but the electric field is saturated on a line in front of the crack tip or treats dielectric ceramic as mechanically brittle and electrically ductile (Fig. 8.9). The fracture behavior is determined by the local J integral around the crack tip $x_1 = a$ only and does not enclosed the electric saturation end $x_1 = c$.

1. *Poling Axis Perpendicular to the Crack*

Discuss an infinite plate with a central crack of length $2a$ located on the axis x_1. The polarization x_3-axis is perpendicular to the crack (Fig. 8.9). In order to clearly explain the physical phenomenon, Gao et al. (1997) adopted the following simplified constitutive equation:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{12} \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} M & * & * & 0 & 0 & 0 & 0 & 0 & e \\ * & M & * & 0 & 0 & 0 & 0 & 0 & e \\ * & * & M & 0 & 0 & 0 & 0 & 0 & -e \\ 0 & 0 & 0 & M & 0 & 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 & M & 0 & -e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & \epsilon & 0 \\ -e & -e & e & 0 & 0 & 0 & 0 & 0 & \epsilon \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{32} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \\ E_1 \\ E_2 \\ E_3 \end{Bmatrix} \tag{8.17}$$

where $*$ means that the corresponding material constant does not appear in this model and is omitted. Here the plane strain problem is discussed.

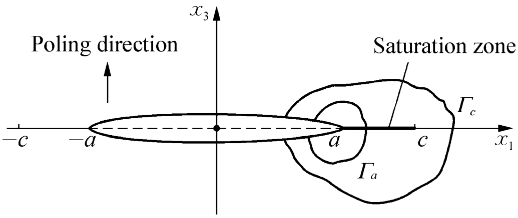


Fig. 8.9 The local *J* integral model

In Eq. (8.17) there are only three independent material constants M, e, ϵ. Assume the displacement only along x_3 direction, i.e.,

$$u_1 = 0, \quad u_3 = u_3(x_1, x_3), \quad E_i = -\varphi_{,i} \tag{8.18}$$

The equilibrium equation along direction x_1 is satisfied automatically due to $u_1 = 0$, so σ_{11} is not needed. Substitution of Eq. (8.18) into Eq. (8.17) yields

$$\begin{aligned} &\sigma_{13} = Mu_{3,1} + e\varphi_{,1}, \quad \sigma_{33} = Mu_{3,3} + e\varphi_{,3} \\ &D_1 = eu_{3,1} - \epsilon\varphi_{,1}, \quad D_3 = eu_{3,3} - \epsilon\varphi_{,3} \end{aligned} \tag{8.19}$$

Inserting Eq. (8.19) into generalized equilibrium equation $\sigma_{ij,j} = 0$ and $D_{i,i} = 0$ yields

$$\nabla^2 u_3 = 0, \quad \nabla^2 \varphi = 0 \tag{8.20}$$

Introduce complex potentials $U(z)$ and $\Phi(z)$, and let

$$\begin{aligned} &u_3 = \mathrm{Im}[U(z)]; \quad \varphi = \mathrm{Im}[\Phi(z)]; \quad \sigma_{33} + \mathrm{i}\sigma_{31} = MU'(z) + e\Phi'(z) \\ &D_3 + \mathrm{i}D_1 = eU'(z) - \epsilon\Phi'(z); \quad E_3 + \mathrm{i}E_1 = -\Phi'(z); \quad z = x_1 + \mathrm{i}x_3 \end{aligned} \tag{8.21}$$

Now discuss the strip electric saturation model. Assume in front of the crack that electric field reaches saturation on $a < x_1 \le c, x_3 = 0$. The boundary conditions are

$$\begin{aligned} &\sigma_{33} + \mathrm{i}\sigma_{31} = \sigma^\infty, \quad E_3 + \mathrm{i}E_1 = E^\infty; \quad \text{when} \quad |z| \to \infty \\ &\sigma_{33} = 0, \quad D_3 = 0; \quad \text{when} \quad |x_1| < a, \quad x_3 = 0 \\ &{u_3}^+ = {u_3}^-, \quad D_3 = D_s; \quad \text{when} \quad a < |x_1| \le c, \quad x_3 = 0 \end{aligned} \tag{8.22}$$

where σ^∞ and E^∞ are real constants. It is noted that σ_{31} cannot be zero in $|x_1| < a$ due to $u_1 = 0$ is assumed. The solution satisfying Eq. (8.22) is

$$\begin{aligned} &U'(z) = \frac{c_1 z}{\sqrt{z^2 - a^2}}, \quad \Phi'(z) = \frac{c_3 z}{\sqrt{z^2 - a^2}} + \frac{c_4 z}{\sqrt{z^2 - c^2}} - \frac{D_s}{\epsilon}\omega(z) \\ &c_1 = \frac{eE^\infty + \sigma^\infty}{M}, \quad c_3 = \frac{e(eE^\infty + \sigma^\infty)}{\epsilon M}, \quad c_4 = -\frac{(e^2 + \epsilon M)E^\infty + e\sigma^\infty}{\epsilon M} \\ &\omega(z) = \frac{2}{\pi}\left[\operatorname{arccot}\left(\frac{a}{z}\sqrt{\frac{z^2 - c^2}{c^2 - a^2}}\right) - \frac{z}{\sqrt{z^2 - c^2}} \arccos\left(\frac{a}{c}\right)\right] \end{aligned} \tag{8.23}$$

where $\omega(z)$ is similar to that in the Dugdale model and has the following behaviors:

$$\begin{aligned} &\omega(z) \to 0, \quad \text{when} \quad z \to \infty; \quad \operatorname{Im}\omega(z) = 0, \quad \text{when} \quad |x_1| > c \\ &\operatorname{Re}\omega(z) = 0, \quad \text{when} \quad |x_1| < a; \quad \operatorname{Re}\omega(z) = 1, \quad \text{when} \quad a < |x_1| < c \end{aligned} \tag{8.24}$$

The condition that the stresses are finite at $|x_1| = c$ yields the size of the saturation zone

$$\rho = c - a = a\sec\left(\frac{\pi}{2}\frac{(e^2 + \epsilon M)E^\infty + e\sigma^\infty}{MD_\mathrm{s}}\right) - a = a\sec\left(\frac{\pi}{2}\frac{D^\infty}{D_\mathrm{s}}\right) - a \tag{8.25}$$

Near the crack tip the stress field is singular, but the electric field is finite. The singular parts of the stresses are

$$\begin{aligned} &\sigma_{33} = \operatorname{Re}\{MU'(z) + e\Phi'(z)\} = M\frac{c_1(a+r)}{\sqrt{r^2 + 2ra}} + e\left[\frac{c_3(a+r)}{\sqrt{r^2+2ra}} - \frac{D_\mathrm{s}}{\epsilon}\omega(z)\right] \\ &u_3 = \operatorname{Im}\{U(z)\} = c_1\sqrt{a^2 - x^2}, \quad -a < x < a \end{aligned} \tag{8.26}$$

The local J integral J_a at the crack tip (Fig. 8.9) is:

$$J_\mathrm{a} = \int_{\Gamma_\mathrm{a}} (gn_1 - \sigma_{ij}n_j u_{j,1} - D_j n_j \varphi_{,1})\,\mathrm{d}s = \frac{\pi a}{2M}\left(1 + \frac{e^2}{\epsilon M}\right)(eE^\infty + \sigma^\infty)^2 \tag{8.27}$$

where $g = (1/2)(\sigma_{ij}\varepsilon_{ij} - D_i E_i)$ is the electric Gibbs free energy. Equation (8.26) can also be obtained by substituting Eq. (8.23) into Eq. (8.5) (Fang et al. 1999). The apparent J integral J_c whose integral path encloses the crack tip and the end of the strip electric saturation is

$$\begin{aligned} J_\mathrm{c} &= \int_{\Gamma_\mathrm{c}} (gn_1 - \sigma_{ij}n_j u_{j,1} - D_j n_j \varphi_{,1})\mathrm{d}s = J_\mathrm{c} + D_\mathrm{s}(\varphi^+ - \varphi^-)|_{x_1 = a} \\ &= \frac{\pi a}{2M}\left(1 + \frac{e^2}{\epsilon M}\right)(eE^\infty + \sigma^\infty)^2 - \frac{4D_\mathrm{s}a}{\pi\epsilon}\ln\left[\sec\left(\frac{\pi}{2}\frac{(e^2 + \epsilon M)E^\infty + e\sigma^\infty}{MD_\mathrm{s}}\right)\right] \\ &\approx \frac{\pi a}{2M}\left[(\sigma^\infty)^2 - (e^2 + \epsilon M)(E^\infty)^2\right] \end{aligned} \tag{8.28}$$

When $\rho \ll a$, the approximate equality is held in Eq. (8.28), which is just the solution for the linear problem. It is obvious that $J_\mathrm{c} \neq J_\mathrm{a}$. If using $J_\mathrm{c} = J_\mathrm{cr}$ as the fracture criterion, where J_cr is the critical value at fracture of J integral, both the positive and negative electric fields decrease J_c, so increases the fracture toughness. If using $J_\mathrm{a} = J_\mathrm{cr}$ as the fracture criterion, the positive electric fields decrease the apparent fracture toughness, while negative electric field increases the apparent fracture toughness. This is consistent with the experiment facts.

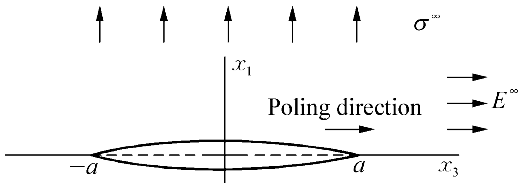


Fig. 8.10 A crack parallel to the poling direction

2. *Poling Axis Parallel to the Crack*

Let the crack be located on the axis x_3 (polarized axis) (Fig. 8.10). Take

$$u_3 = 0, \quad u_1 = u_1(x_1, x_3); \quad D_1 = -\psi_{,3}, \quad D_3 = \psi_{,1} \tag{8.29}$$

In the coordinate system shown in Fig. 8.10, the constitutive equations are

$$\begin{aligned} &\sigma_{13} = \bar{M}u_{1,3} + \bar{e}\psi_{,3}, \quad \sigma_{11} = \bar{M}u_{1,1} + \bar{e}\psi_{,1} \\ &E_3 = \bar{e}u_{1,1} - \bar{\epsilon}\psi_{,1}, \quad E_1 = -\bar{e}u_{1,3} + \bar{\epsilon}\psi_{,3} \\ &\bar{M} = M + e^2/\epsilon, \quad \bar{e} = e/\epsilon, \quad \bar{\epsilon} = -1/\epsilon \end{aligned} \tag{8.30}$$

According to $E_{1,3} = E_{3,1}$ and $\sigma_{11,1} + \sigma_{13,3} = 0$, it can still be derived that u_3 and ψ are all the harmonic functions. Introduce complex potentials $U(z)$ and $\Psi(z)$ and let

$$\begin{aligned} &u_1 = \mathrm{Im}[U(z)], \quad \psi = \mathrm{Im}[\Psi(z)]; \quad \sigma_{11} + \mathrm{i}\sigma_{13} = \bar{M}U'(z) + \bar{e}\Psi'(z) \\ &E_3 - \mathrm{i}E_1 = \bar{e}U'(z) - \bar{\epsilon}\Psi'(z); \quad D_3 - \mathrm{i}D_1 = \Psi'(z); \quad z = x_3 + \mathrm{i}x_1 \end{aligned} \tag{8.31}$$

The solution for a central crack is

$$U'(z) = \frac{\sigma^\infty z}{\bar{M}\sqrt{z^2 - a^2}} + \frac{\bar{e}(\bar{M}E^\infty - \bar{e}\sigma^\infty)}{\bar{M}(\bar{M}\bar{\epsilon} + \bar{e}^2)}, \quad \Psi'(z) = -\frac{\bar{M}E^\infty - \bar{e}\sigma^\infty}{\bar{M}\bar{\epsilon} + \bar{e}^2} \tag{8.32}$$

and

$$J_\mathrm{c} = J_\mathrm{a} = \pi a \sigma^{\infty 2}/2\bar{M} = \pi a\, \epsilon\, \sigma^{\infty 2}\big/\left[2M(\epsilon M + e^2)\right] \tag{8.33}$$

Equation (8.33) shows that the parallel electric field does not influence the fracture when the poling direction is parallel to the crack. It is also consistent with the experiment facts. If let $x_3 = x, x_1 = y$, the above formulas are identical with that in the paper of Gao et al. (1997).

Wang (2000) and Fulton and Gao (1997) discussed the fracture problem for the strip electric saturation model in a more general situation and pointed out that the local J integral criterion is consistent with the experimental results obtained by Park and Sun in a certain electric field range.

8.4 Failure Criterion of Conductive Cracks with Charge-Free Zone Model

8.4.1 Basic Concept of the Charge-Free Zone

In electronic and electromechanical devices made of piezoelectric ceramics, the embedded soft electrodes are widely used. These soft electrodes may be considered as conducting cracks. When an external electric field is parallel to the conducting crack, the induced charge will be produced on the crack surface in order to make the electric field inside the conducting crack remains zero. The same sign charges will be on the upper and lower surfaces near the crack tip, so the repellent force will open the crack. Zeller and Schneider (1984) proposed a model; they assumed that the charge mobility has a finite value when $E > E_{\mathrm{c}}$, while the charge mobility is zero when $E < E_{\mathrm{c}}$, where E_{c} is a critical value. Zhang et al. (2003, 2004) based on the above model proposed a charge-free zone (CFZ) model to discuss the failure behavior of conducting crack: When the electric intensity factor at the crack tip reaches a critical value, charges could be emitted from the tip. The emitted charges may form a charge cloud around the tip and shield the external electric field, so form a charge-free zone in front of the tip. Therefore the generalized stress field is singular at the crack tip because there is no electric charge in charge-free zone and the failure criterion can be expressed by the generalized stress intensity factors. The CFZ model is the extension of the dislocation-free model in elastic–plastic fracture mechanics (Ohr 1985; Majumdar and Burns 1983; Kuang et al. 1998).

8.4.2 Interaction of the Crack and a Point Charges in Front of It

Discuss an infinite piezoelectric material polarized along positive x_1-axis with a semi-infinite crack located on the minus x_1-axis subjected to a point electric charge q at z_0 (Fig. 8.11). The crack tip is selected as the origin of the coordinate system. At first discuss the interaction of a single point electric charge in front of the crack tip. Zhang et al. (2004) adopted the simplified constitutive equations shown in Eq. (8.17) with appropriate rearrangement, because the polarized axes are different. In order to discuss the problem qualitatively, it is assumed that

$$u_1 = 0, \quad u_3 = u_3(x_1, x_3) \tag{8.34}$$

$$\begin{aligned} \sigma_{13} &= M u_{3,1} + e\varphi_{,3}, \quad \sigma_{33} = M u_{3,3} - e\varphi_{,1} \\ D_1 &= -e u_{3,3} - \epsilon\varphi_{,1}, \quad D_3 = e u_{3,1} - \epsilon\varphi_{,3} \end{aligned} \tag{8.35}$$

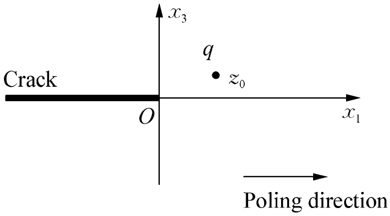


Fig. 8.11 A semi-infinite crack parallel to the poling direction

where u_3 and φ are all harmonic functions, so they can be expressed by complex functions $U(z)$ and $\Phi(z)$:

$$\begin{aligned} & u_3 = \mathrm{Im}[U(z)], \quad \varphi = \mathrm{Im}[\Phi(z)]; \quad z = x_1 + \mathrm{i}x_3 \\ & \sigma_{33} + \mathrm{i}\sigma_{31} = MU'(z) + \mathrm{i}e\Phi'(z); \quad \varepsilon_{33} + \mathrm{i}2\varepsilon_{31} = U'(z) \\ & D_1 - \mathrm{i}D_3 = -eU'(z) + \mathrm{i}\epsilon\Phi'(z); \quad E_3 + \mathrm{i}E_1 = -\Phi'(z) \end{aligned} \tag{8.36}$$

The boundary conditions on a conducting crack surface are

$$\sigma_{33} = 0, \quad E_1 = 0; \quad \text{when} \quad x_1 < 0, \quad x_2 = 0 \tag{8.37}$$

The solution satisfying the boundary conditions is

$$U = 0, \quad \Phi = -\frac{\mathrm{i}q}{2\pi\epsilon}\ln\left(\sqrt{z} - \sqrt{z_0}\right) + \frac{\mathrm{i}q}{2\pi\epsilon}\ln\left(\sqrt{z} - \sqrt{\bar{z}_0}\right) \tag{8.38}$$

Substitution of Eq. (8.38) into Eq. (8.36) yields

$$\begin{aligned} & E_1 - \mathrm{i}E_3 = \mathrm{i}\Phi'(z) = \frac{q}{4\pi\epsilon}\frac{\sqrt{z_0} + \sqrt{\bar{z}_0}}{\sqrt{z}\left[z + \sqrt{z}\left(\sqrt{\bar{z}_0} - \sqrt{z_0}\right) - \sqrt{z_0\bar{z}_0}\right]} \\ & \sigma_{33} + \mathrm{i}\sigma_{31} = e(E_1 - \mathrm{i}E_3); \quad D_1 - \mathrm{i}D_3 = \epsilon(E_1 - \mathrm{i}E_3); \quad \varepsilon_{33} + \mathrm{i}2\varepsilon_{31} = 0 \end{aligned} \tag{8.39}$$

When $z_0 = x_{01}$ is on the x_1-axis, Eq. (8.39) yields

$$\begin{aligned} & E_1 - \mathrm{i}E_3 = \frac{q}{2\pi\epsilon}\frac{\sqrt{x_{01}}}{\sqrt{z}[z - x_{01}]}, \quad K_{\mathrm{E}} = \lim_{z\to 0}\sqrt{2\pi z}(E_1 - \mathrm{i}E_3) = -\frac{q}{\epsilon\sqrt{2\pi x_{01}}} \\ & K_\sigma = \lim_{z\to 0}\sqrt{2\pi z}(\sigma_{33} + \mathrm{i}\sigma_{31}) = eK_{\mathrm{E}}, \quad K_{\mathrm{D}} = \lim_{z\to 0}\sqrt{2\pi z}(D_1 - \mathrm{i}D_3) = \epsilon K_{\mathrm{E}} \end{aligned} \tag{8.40}$$

8.4.3 The Condition to Form a Charge-Free Zone

Neglecting the effect of the deformation on the electric field the solution of an infinite material subjected to a point electric charge q is

$$U = 0, \quad \Phi = -\frac{\mathrm{i}q}{2\pi\epsilon}\ln(z - z_0), \quad E_1 - \mathrm{i}E_3 = \frac{q}{2\pi\epsilon}\frac{1}{z - z_0} \tag{8.41}$$

The image field $E_1^{(i)}, E_3^{(i)}$ introduced by the crack is Eq. (8.39) minus (8.41), i.e.,

$$E_1^{(i)} - \mathrm{i}E_3^{(i)} = \frac{q}{2\pi\epsilon}\left[\frac{\sqrt{z_0}+\sqrt{\bar{z}_0}}{2\sqrt{z}\left[z+\sqrt{z}\left(\sqrt{\bar{z}_0}-\sqrt{z_0}\right)-\sqrt{z_0\bar{z}_0}\right]} - \frac{1}{z-z_0}\right] \tag{8.42}$$

When $z_0 = x_{01}$, the image field of a point $z_1 = x_1$ near the crack tip is

$$E_1^{(i)} = \frac{q}{2\pi\epsilon}\left[\frac{\sqrt{x_{01}}}{\sqrt{x_1}[x_1 - x_{01}]} - \frac{1}{x_1 - x_{01}}\right] = -\frac{q}{2\pi\epsilon}\frac{1}{\sqrt{x_1}\left[\sqrt{x_1}+\sqrt{x_{01}}\right]} \tag{8.43}$$

and image force acted on q is

$$F_{\mathrm{i}} = qE_1^{(i)} = -\frac{q^2}{2\pi\epsilon}\frac{1}{2x_{01}} \tag{8.44}$$

So the image force is always to push the electric charge towards the crack. On the other hand, the external field exerts a driving force F_{a} on the charge. For simplicity discuss a charge, which is on the axis x_{01}. The driving force is given by

$$F_{\mathrm{a}} = K_{\mathrm{E}}q\Big/\sqrt{2\pi x_{01}} \tag{8.45}$$

where K_{E} is electric field intensity factor produced by the external field. According to Zeller and Schneider's model (1984), when the algebraic sum of the driving force and image force is larger than qE_{c}, the charge will be emitted from the crack tip, or the condition to form a charge-free zone is

$$F_{\mathrm{a}} + F_{\mathrm{i}} \geq qE_{\mathrm{c}}, \quad \frac{K_{\mathrm{E}}q}{\sqrt{2\pi x_{01}}} - \frac{q^2}{4\pi\epsilon}\frac{1}{x_{01}} \geq qE_{\mathrm{c}} \tag{8.46}$$

There are two points $x_1 = a$ and $x_2 = b$ (Fig. 8.12) satisfying Eq. (8.46):

$$\sqrt{x_{1,2}} = \frac{1}{2\sqrt{2\pi}E_{\mathrm{c}}}\left[K_{\mathrm{E}} \pm \left(K_{\mathrm{E}}^2 - \frac{2qE_{\mathrm{c}}}{\epsilon}\right)^{1/2}\right] \tag{8.47}$$

According to Zeller and Schneider's model (1984), a charge moves forward in the region (x_1, x_2) due to $F_{\mathrm{a}} + F_{\mathrm{i}} > qE_{\mathrm{c}}$ and stops at point x_2 due to $F_{\mathrm{a}} + F_{\mathrm{i}} = qE_{\mathrm{c}}$. When more and more charges are emitted from the crack tip, charges will pile up and form a charge trap zone (b, c) followed by the charge-free zone (o, b). It is assumed that b and c are constants.

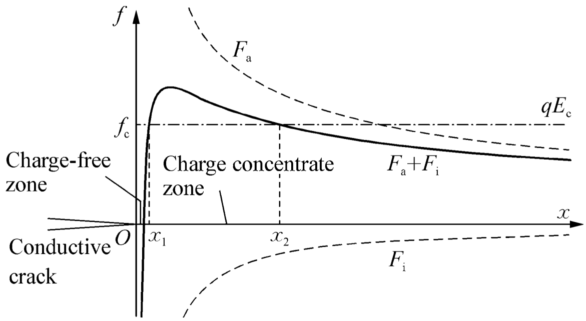


Fig. 8.12 A charge-free zone model

8.4.4 Failure Criterion of Charge-Free Zone Model

Assume the charge density in the charge trap zone is $f(x)$ with $f(b) = f(c) = 0$. The critical electric field is E_c, and the external applied stress intensity factor is $K_E^{(a)}$. Using Eqs. (8.40) and (8.45) the equilibrium condition in the charge trap zone is

$$\frac{K_E^{(a)}}{\sqrt{2\pi x_1}} + \frac{q}{2\pi\epsilon}\int_b^c \frac{f(x_1')\sqrt{x_1'}}{\sqrt{x_1}(x_1 - x_1')}dx_1' = E_c, \quad b \le x_1 \le c \tag{8.48}$$

In order to guarantee the existence and uniqueness of the solution of $f(x)$ in Eq. (8.48), it must be $f(b) = f(c) = 0$ or (Majumdar and Burns 1983)

$$\int_b^c \frac{E_c\sqrt{x_1'} - K_E^{(a)}/\sqrt{2\pi}}{\sqrt{(c - x_1')(x_1' - b)}}dx_1' = 0 \tag{8.49}$$

Equation (8.49) yields

$$K_E^{(a)} = 2\sqrt{2\pi c}E_c E(\pi/2, k)/\pi \tag{8.50}$$

where $E(\pi/2, k)$ is the complete elliptic integral of the second kind and $k = \sqrt{1 - b/c}$. The solution of Eq. (8.48) is

$$f(x_1') = -\frac{4\epsilon E_c b\sqrt{x_1'}}{\pi q\sqrt{c}}\frac{\sqrt{c - x_1'}}{x_1'(x_1' - b)}\Pi\left[\frac{\pi}{2}, \frac{x_1'(c - b)}{c(x_1' - b)}, k\right] \tag{8.51}$$

where $\Pi[\pi/2, n^2, k]$ is the complete elliptic integral of the third kind. Using Eq. (8.40) the electric intensity factor produced by the electric charges is

$$K_E^{(i)} = -\sqrt{2\pi}\frac{q}{2\pi\epsilon}\int_b^c \frac{f(x_1')}{\sqrt{x_1'}}dx_1' = -2\sqrt{\frac{2}{\pi}}E_c\left[\sqrt{c}E\left(\frac{\pi}{2}, k\right) - \sqrt{b}F\left(\frac{\pi}{2}, k\right)\right] = (\Omega - 1)K_E^{(a)}$$

$$K_\sigma^{(i)} = e(\Omega - 1)K_E^{(a)}, \quad K_D^{(i)} = \epsilon(\Omega - 1)K_E^{(a)}, \quad \Omega = \sqrt{\frac{b}{c}}F\left(\frac{\pi}{2}, k\right)\Big/E\left(\frac{\pi}{2}, k\right) \tag{8.52}$$

where $F(\pi/2, k)$ is the complete elliptic integral of the first kind. The local electric field intensity factor at the crack tip is the sum of the applied intensity factor and the intensity factor produced by the charges in the charge trap zone. So we can get

$$K_{\mathrm{E}} = K_{\mathrm{E}}^{(\mathrm{a})} + K_{\mathrm{E}}^{(\mathrm{i})} = \Omega K_{\mathrm{E}}^{(\mathrm{a})}, \quad K_{\mathrm{D}} = K_{\mathrm{D}}^{(\mathrm{a})} + \epsilon(\Omega - 1)K_{\mathrm{E}}^{(\mathrm{a})} = \epsilon\left(1 + \frac{e^2}{\epsilon M}\right)K_{\mathrm{E}}^{(\mathrm{a})} - \frac{e}{M}K_{\sigma}^{(\mathrm{a})}$$
$$K_{\sigma} = K_{\sigma}^{(\mathrm{a})} + e(\Omega - 1)K_{\mathrm{E}}^{(\mathrm{a})}, \quad K_{\varepsilon} = K_{\varepsilon}^{(\mathrm{a})} = \left[K_{\sigma}^{(\mathrm{a})} - eK_{\mathrm{E}}^{(\mathrm{a})}\right]/M \tag{8.53}$$

Using Eq. (8.53) the local J integral J_{a} is obtained:

$$J_{\mathrm{a}} = (1/2)(K_{\sigma}K_{\varepsilon} + K_{\mathrm{D}}K_{\mathrm{E}}) = (1/2M)\left(K_{\sigma}^{(\mathrm{a})} - eK_{\mathrm{E}}^{(\mathrm{a})}\right)^2 + (\epsilon/2)\left(\Omega K_{\mathrm{E}}^{(\mathrm{a})}\right)^2 \tag{8.54}$$

Assuming J_{cr} is the critical value of J_{a}, Eq. (8.54) yields the fracture criterion:

$$2MJ_{\mathrm{a}} = \left(K_{\sigma}^{(\mathrm{a})} - eK_{\mathrm{E}}^{(\mathrm{a})}\right)^2 + \epsilon M\left(\Omega K_{\mathrm{E}}^{(\mathrm{a})}\right)^2 = 2MJ_{\mathrm{cr}} \tag{8.55}$$

Under purely mechanical loading we can get the critical stress intensity factor $K_{\sigma\mathrm{cr}}$ and under purely electrical loading we can get the critical electric field intensity factor $K_{\mathrm{E\,cr}}$, or

$$K_{\sigma\,\mathrm{cr}}^2 = 2MJ_{\mathrm{cr}}, \quad \left(\epsilon M\Omega^2 + e^2\right)K_{\mathrm{E\,cr}}^2 = 2MJ_{\mathrm{cr}}; \quad \text{or}$$
$$K_{\sigma\,\mathrm{cr}} = \sqrt{2MJ_{\mathrm{cr}}}, \quad K_{\mathrm{E\,cr}} = \pm\sqrt{2MJ_{\mathrm{cr}}/\left(\epsilon M\Omega^2 + e^2\right)} \tag{8.56}$$

where $K_{\mathrm{E\,cr}}$ can be taken as positive or negative value. Using Eq. (8.56), Eq. (8.55) can be rewritten in dimensionless form:

$$\left(\frac{K_{\sigma}^{(\mathrm{a})}}{K_{\sigma\,\mathrm{cr}}}\right)^2 \mp \frac{2e}{\left(\epsilon M\Omega^2 + e^2\right)^{1/2}}\left(\frac{K_{\sigma}^{(\mathrm{a})}}{K_{\sigma\,\mathrm{cr}}}\right)\left(\frac{K_{\mathrm{E}}^{(\mathrm{a})}}{K_{\mathrm{E\,cr}}}\right) + \left(\frac{K_{\mathrm{E}}^{(\mathrm{a})}}{K_{\mathrm{E\,cr}}}\right)^2 = 1 \tag{8.57}$$

where the negative sign is for positive electric loading, while the positive sign for negative electric loading. From the derivation process, it is known that the following relation should be held:

$$K_{\sigma\,\mathrm{cr}}^2 = \left(\epsilon M\Omega^2 + e^2\right)K_{\mathrm{E\,cr}}^2 \tag{8.58}$$

The above relation may not be held for a real material. So from the engineering view, this constraint condition may be abandoned. It may be considered that $K_{\sigma\,\mathrm{cr}}$ and $K_{\mathrm{E\,cr}}$ are two independent experimental parameters and introduce weight coefficients in Eq. (8.57). If so, the criterion is more like the generalized stress intensity factor criterion Eq. (8.1), but it has certain theoretical foundation. For the problem with general constitutive equations, the results are also consistent with that in experiments (Zhang et al. 2003, 2004).

8.5 Modal Strain Energy Density Factor Theory

8.5.1 Normalized Generalized Stress and Strain Vectors in Piezoelectric Materials

As in the elastic case, the first kind of constitutive equations in Eq. (2.83) can be expressed in terms of Voigt vector, i.e.,

$$\begin{aligned}&\boldsymbol{\Gamma} = \boldsymbol{s}\cdot\boldsymbol{\Sigma}\\&\boldsymbol{\Gamma} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy}, D_x, D_y, D_z], \quad \boldsymbol{\Sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}, E_x, E_y, E_z]\end{aligned} \tag{8.59}$$

where $\gamma_{yz}, \gamma_{zx}, \gamma_{xy}$ are the engineering shear strain. Analogous to Eq. (1.40) the normalized generalized stress vector $\bar{\boldsymbol{\Sigma}}$ and strain vector $\bar{\boldsymbol{\Gamma}}$ in piezoelectric materials are defined as

$$\begin{aligned}&\bar{\boldsymbol{\Gamma}} = \boldsymbol{P}^{-1}\cdot\boldsymbol{\Gamma}, \quad \bar{\boldsymbol{\Sigma}} = \boldsymbol{P}\cdot\boldsymbol{\Sigma}, \quad \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} = \operatorname{diag}\left[1\ 1\ 1\ \sqrt{2}\ \sqrt{2}\ \sqrt{2}\ 1\ 1\ 1\right]\\&\bar{\boldsymbol{\Gamma}} = \bar{\boldsymbol{s}}\cdot\bar{\boldsymbol{\Sigma}}, \quad \bar{\boldsymbol{s}} = \boldsymbol{P}^{-1}\cdot\boldsymbol{s}\cdot\boldsymbol{P}^{-\mathrm{T}}\end{aligned} \tag{8.60}$$

where $\bar{\boldsymbol{s}}$ is the normalized generalized compliance matrix. Let the transform matrix of the coordinate systems ϕ' and ϕ be $\mathbf{Q} = [Q_{kl}]$, $Q_{kl} = \cos\left(\mathbf{i}_k, \mathbf{i}'_k\right)$, then

$$\boldsymbol{\Sigma}' = \boldsymbol{A}\cdot\boldsymbol{\Sigma}, \quad \boldsymbol{\Gamma}' = \boldsymbol{B}\cdot\boldsymbol{\Gamma}, \quad \bar{\boldsymbol{\Sigma}}' = \boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{P}^{-1}\cdot\bar{\boldsymbol{\Sigma}}, \quad \bar{\boldsymbol{\Gamma}}' = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{P}\cdot\bar{\boldsymbol{\Gamma}} \tag{8.61}$$

where

$$A = \begin{pmatrix}\boldsymbol{A}_{11} & 2\boldsymbol{A}_{12} & 0\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & 0\\ 0 & 0 & \boldsymbol{Q}\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12} & 0\\ 2\boldsymbol{A}_{21} & \boldsymbol{A}_{22} & 0\\ 0 & 0 & \boldsymbol{Q}\end{pmatrix}, \quad \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{B}^{-1} \tag{8.62}$$

where $\boldsymbol{A}_{11}, \boldsymbol{A}_{12}, \boldsymbol{A}_{21}, \boldsymbol{A}_{22}$ are shown in Eq. (1.39). It is easy to show that

$$\begin{aligned}&\boldsymbol{H} = \boldsymbol{P}\cdot\boldsymbol{A}\cdot\boldsymbol{P}^{-1} = \boldsymbol{P}^{-1}\cdot\boldsymbol{B}\cdot\boldsymbol{P}, \quad \boldsymbol{H}^{\mathrm{T}} = \boldsymbol{H}^{-1}\\&\bar{\boldsymbol{\Sigma}}' = \boldsymbol{H}\cdot\bar{\boldsymbol{\Sigma}}, \quad \bar{\boldsymbol{\Gamma}}' = \boldsymbol{H}\cdot\bar{\boldsymbol{\Gamma}}, \quad \bar{\boldsymbol{\Gamma}}' = \boldsymbol{H}\cdot\bar{\boldsymbol{\Gamma}} = \bar{\boldsymbol{s}}'\cdot\bar{\boldsymbol{\Sigma}}', \quad \bar{\boldsymbol{s}}' = \boldsymbol{H}\cdot\bar{\boldsymbol{s}}\cdot\boldsymbol{H}^{-1}\end{aligned} \tag{8.63}$$

Equation (8.63) shows that $\bar{\boldsymbol{\Sigma}}$ and $\bar{\boldsymbol{\Gamma}}$ are vectors in a nine-dimensional space with the orthogonal coordinate transform tensor $\boldsymbol{H}$.

8.5.2 Eigen Material Constants and Material Modes

Equation (8.60) shows that each component of $\bar{\boldsymbol{\Gamma}}$ are related to all nine components of $\bar{\boldsymbol{\Sigma}}$. Kuang et al. (2003) extended the Kelvin theory (Chen 1984; Ruhlevskii 1984; Arramon et al. 2000) to piezoelectric materials. In material there is a direction $\boldsymbol{M}$, along which $\bar{\boldsymbol{\Gamma}}$ and $\bar{\boldsymbol{\Sigma}}$ are parallel in the nine-dimensional space. The coordinates paralleling to $\boldsymbol{M}$ are called the material principle coordinates. In the material principle coordinates, $\bar{\boldsymbol{\Gamma}}$ is denoted by $\hat{\boldsymbol{\Gamma}}$, $\bar{\boldsymbol{\Sigma}}$ by $\hat{\boldsymbol{\Sigma}}$, and

$$
\begin{aligned}
&(\bar{\boldsymbol{s}} - \Lambda \mathbf{I}) \cdot \boldsymbol{M} = \mathbf{0}, \quad \hat{\boldsymbol{\Gamma}} = \boldsymbol{\Lambda} \cdot \hat{\boldsymbol{\Sigma}}, \quad \boldsymbol{\Lambda} = \mathrm{diag}[\Lambda_i] = \langle \Lambda_i \rangle \\
&|\bar{\boldsymbol{s}} - \Lambda \mathbf{I}| = \mathbf{0}; \quad \text{or} \quad \left| \boldsymbol{P}^{-1} \cdot \boldsymbol{s} \cdot \boldsymbol{P}^{-\mathrm{T}} - \Lambda \mathbf{I} \right| = \mathbf{0}
\end{aligned}
\tag{8.64}
$$

For the nondegenerate case, Eq. (8.64) has nine different Λ_i, where Λ_i is called ith eigen-compliance and $\boldsymbol{\Lambda}$ is called the eigen-compliance matrix. Usually $\hat{\boldsymbol{s}}$ is real symmetric, so Λ takes real value. For each Λ_i there is an eigenvector or material mode $\boldsymbol{M}_i$ with one arbitrary component. $\boldsymbol{M}_i$ and $\boldsymbol{M}_j$ are orthogonal to each other when $i \neq j$. The normalized orthogonal eigenvectors $\hat{\boldsymbol{M}}$ can be established by

$$
\hat{\boldsymbol{M}} = \left[\hat{\boldsymbol{M}}_i\right] = \left[\hat{\boldsymbol{M}}_1, \hat{\boldsymbol{M}}_2, \ldots, \hat{\boldsymbol{M}}_9\right], \quad \hat{\boldsymbol{M}}_i = \boldsymbol{M}_i / |\boldsymbol{M}_i|, \quad \hat{\boldsymbol{M}}\hat{\boldsymbol{M}}^{\mathrm{T}} = \mathbf{I}
\tag{8.65}
$$

$\hat{\boldsymbol{M}}$ is called the material mode matrix. The space spanning by basis vectors along $\hat{\boldsymbol{M}}$ is called the mode space. Usually the eigen-equation in Eq. (8.64) is degenerate due to the certain symmetry in the real materials, so the number of independent eigenvalues is less than nine, i.e., there are repeated roots in $\boldsymbol{\Lambda}$. However, the eigen-matrix in Eq. (8.64) is semisimple for the real material, so for a multiple root Λ_i, the number of the independent eigenvectors is the same as its multiplicity. Under the coordinate transformation ϕ to ϕ' we have $\hat{\boldsymbol{s}}' = \boldsymbol{H}\hat{\boldsymbol{s}}\boldsymbol{H}^{-1}$, i.e., $\hat{\boldsymbol{s}}'$ and $\hat{\boldsymbol{s}}$ are the similar matrix, so in coordinate systems ϕ and ϕ', the eigen-compliance matrix $\boldsymbol{\Lambda}$ is the same, but the eigenvector changes to $\boldsymbol{M}' = \boldsymbol{H}^{-1}\boldsymbol{M}$.

Analogous to the above discussion, we can also discuss the eigen elastic coefficient matrix $\boldsymbol{\lambda}$:

$$
\hat{\boldsymbol{\Sigma}} = \boldsymbol{\lambda} \cdot \hat{\boldsymbol{\Gamma}}, \quad \boldsymbol{\lambda} = \boldsymbol{\Lambda}^{-1}
\tag{8.66}
$$

8.5.3 Modal Stress, Modal Strain, and Modal Energy Density

Any normalized generalized stress vector $\bar{\boldsymbol{\Sigma}}$ and strain vector $\bar{\boldsymbol{\Gamma}}$ can be decomposed in a modal space:

$$
\bar{\boldsymbol{\Sigma}} = \sum_{j=1}^{m} \hat{\boldsymbol{\Sigma}}_j = \sum_{j=1}^{m} \hat{\Sigma}_j \boldsymbol{M}_j, \quad \bar{\boldsymbol{\Gamma}} = \sum_{j=1}^{m} \hat{\boldsymbol{\Gamma}}_j = \sum_{j=1}^{m} \hat{\Gamma}_j \boldsymbol{M}_j
\tag{8.67}
$$

where $\hat{\boldsymbol{\Sigma}}_j, \hat{\boldsymbol{\Gamma}}_j$ are the j-th modal stress and strain vectors, respectively and $\hat{\Sigma}_j, \hat{\Gamma}_j$ are their norms, respectively. Obviously $\hat{\boldsymbol{\Gamma}}_j = \boldsymbol{\Lambda}_j \hat{\boldsymbol{\Sigma}}_j$. The modal strain energy density $\mathfrak{A}_i$ of ith mode is

$$\mathfrak{A}_i = \hat{\boldsymbol{\Sigma}}_i^{\mathrm{T}} \hat{\boldsymbol{\Sigma}}_i / 2 = \Lambda_i \hat{\boldsymbol{\Sigma}}_i^2 / 2, \quad \text{no sum on } i \tag{8.68}$$

8.5.4 Modal Energy Density Factor (MEDF) Theory

It is believed that the energy possesses the central role in the change of the microstructure and failure. Because the resistance against the change of the microstructure is different in different deformation direction and mechanism, the role of the energy produced in different deformation version and mechanism is different. This fact shows that in the change of the microstructure and failure process, the energy possesses material structure anisotropic behavior. In the small-scale electric saturation case for the self-similarity extended crack, the failure criterion can be determined by the generalized stresses near the tip, so the modal energy density theory can be used. The MEDF failure theory can be expressed as follows:

Assume Λ_{p} is an r-repeated root and its corresponding independent modes are $\boldsymbol{M}_{\mathrm{p}i}, i = 1, 2, \ldots, r$. The subspace spanning by the basic vectors consisted of $\boldsymbol{M}_{\mathrm{p}i}$ is an isotropic subspace for Λ_{p}. Experiences show that the contribution to the failure of each deformation version in this subspace can be considered the same, so $\boldsymbol{M}_{\mathrm{p}1} + \boldsymbol{M}_{\mathrm{p}2} + \cdots + \boldsymbol{M}_{\mathrm{p}r}$ can be considered as one independent mode. Therefore, in the modal strain energy density, the modified number of the independent mode is $N \leq 9$. For the failure problem, the direction (tension or compression) of the generalized stress is also important. Experiments also show that the mechanism of the tension failure is somewhat different with other failure version, so the tension failure criterion should be given alone. Considering these factors, the modal strain energy density theory can be given as

$$\sum_{i=1}^{N} \left(a_i^+ \mathfrak{A}_i^+ + \beta_i a_i^- \mathfrak{A}_i^- \right) = \mathfrak{A}_{\mathrm{cr}}^+ + \beta \mathfrak{A}_{\mathrm{cr}}^- \tag{8.69}$$

where N is the modified number of the independent modes, a_i^+ and a_i^- are the weight coefficients considering the different modal energy, and the superscripts " $+$ " and " $-$ " express the different direction, β and β_i are the weight coefficients considering deformation direction. For the plastic deformation $a_i^+ = a_i^-$, $\beta = \beta_i = 1$. If all coefficients $a_i^{\pm} = 1, \beta = \beta_i = 1$, Eq. (8.69) is the total energy density criterion. If the generalized stress field is singular with singularity $1/\sqrt{r}$, Eq. (8.69) needs multiply r.

8.5.5 Eigen-Compliances and Material Modes of Some Materials

In practical calculation, Eq. (8.64), $\left(\boldsymbol{P}^{-1}\cdot\boldsymbol{s}\cdot\boldsymbol{P}^{-\mathrm{T}}-\Lambda\mathbf{I}\right)\boldsymbol{M}=\mathbf{0}$, is often used.

1. *Transverse Isotropic Material with Polarized x_3-Axis*

$$\boldsymbol{\Lambda}=\begin{bmatrix} s_{11}-\Lambda & s_{12} & s_{13} & 0 & 0 & 0 & 0 & 0 & d_{31}\\ s_{12} & s_{11}-\Lambda & s_{23} & 0 & 0 & 0 & 0 & 0 & d_{31}\\ s_{13} & s_{23} & s_{33}-\Lambda & 0 & 0 & 0 & 0 & 0 & d_{33}\\ 0 & 0 & 0 & \frac{1}{2}s_{44}-\Lambda & 0 & 0 & 0 & \frac{1}{\sqrt{2}}d_{15} & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}s_{44}-\Lambda & 0 & \frac{1}{\sqrt{2}}d_{15} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & (s_{11}-s_{12})-\Lambda & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}d_{15} & 0 & \epsilon_{11}-\Lambda & 0 & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}d_{15} & 0 & 0 & 0 & \epsilon_{11}-\Lambda & 0\\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 & 0 & 0 & \epsilon_{33}-\Lambda \end{bmatrix} \tag{8.70}$$

The first and second eigen-compliances are repeated roots $\boldsymbol{\Lambda}_1=\boldsymbol{\Lambda}_2$ and associated with two material modes

$$\begin{aligned} \Lambda_{1,2}&=\left(2\epsilon_{11}+s_{44}+\sqrt{(2\epsilon_{11}-s_{44})^2+8d_{15}^2}\right)\Big/4\\ \boldsymbol{M}_{11}^{\mathrm{T}}&=\left[0,0,0,0,\left(-2\epsilon_{11}+s_{44}+\sqrt{(2\epsilon_{11}-s_{44})^2+8d_{15}^2}\right)\Big/2\sqrt{2}d_{15},0,1,0,0\right]\\ \boldsymbol{M}_{12}^{\mathrm{T}}&=\left[0,0,0,\left(-2\epsilon_{11}+s_{44}+\sqrt{(2\epsilon_{11}-s_{44})^2+8d_{15}^2}\right)\Big/2\sqrt{2}d_{15},0,00,1,0\right] \end{aligned} \tag{8.71}$$

These two material modes represent the combined version of the shear strains out of the plane (x_1,x_2) and the electric field in the plane (x_1,x_2). The third and fourth eigen-compliances are repeated roots $\boldsymbol{\Lambda}_3=\boldsymbol{\Lambda}_4$ and associated with two material modes

$$\begin{aligned} &\Lambda_{3,4}=s_{11}-s_{12}\\ &\boldsymbol{M}_{31}^{\mathrm{T}}=[-1,1,0,0,0,0,0,0,0],\quad \boldsymbol{M}_{32}^{\mathrm{T}}=[0,0,0,0,0,1,0,0,0] \end{aligned} \tag{8.72}$$

These two material modes represent the 2D plane strain in (x_1,x_2) and uncoupled with the electric field. The fifth and sixth eigen-compliances are repeated roots $\boldsymbol{\Lambda}_5=\boldsymbol{\Lambda}_6$ and associated with two material modes

$$\begin{aligned} \Lambda_{5,6}&=\left(2\epsilon_{11}+s_{44}-\sqrt{(2\epsilon_{11}-s_{44})^2+8d_{15}^2}\right)\Big/4\\ \boldsymbol{M}_{51}^{\mathrm{T}}&=\left[0,0,0,0,\left(-2\epsilon_{11}+s_{44}-\sqrt{(2\epsilon_{11}-s_{44})^2+8d_{15}^2}\right)\Big/2\sqrt{2}d_{15},0,1,0,0\right]\\ \boldsymbol{M}_{52}^{\mathrm{T}}&=\left[0,0,0,\left(-2\epsilon_{11}+s_{44}-\sqrt{(2\epsilon_{11}-s_{44})^2+8d_{15}^2}\right)\Big/2\sqrt{2}d_{15},0,0,0,1,0\right] \end{aligned} \tag{8.73}$$

These two material modes are the counterparts of the first two material modes. The last three eigen-compliances are single roots

$$\begin{aligned}\Lambda_7 &= \sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}-\frac{a}{3}\\ \Lambda_8 &= \omega\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}-\frac{a}{3}\\ \Lambda_9 &= \omega^2\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}+\omega\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}-\frac{a}{3}\end{aligned} \tag{8.74}$$

where

$$\begin{aligned}&p=b-a^2/3,\quad q=2a^3/27-ab/3+c,\quad a=-(s_{33}+s_{11}+s_{12}+\epsilon_{33})\\ &b=-2s_{13}^2+s_{33}s_{11}+s_{33}s_{12}+s_{33}\epsilon_{33}+s_{11}\epsilon_{33}+s_{12}\epsilon_{33}-d_{33}^2-2d_{31}^2\\ &c=2\epsilon_{33}s_{13}^2+(s_{11}+s_{12})\left(d_{33}^2-s_{33}\epsilon_{33}\right)-4s_{13}d_{31}d_{33}+2d_{31}^2s_{33},\quad \omega=\left(-1+\mathrm{i}\sqrt{3}\right)\Big/2\end{aligned} \tag{8.75a}$$

and the corresponding material modes are

$$\begin{aligned}\boldsymbol{M}_7^{\mathrm{T}} &= \left[1,1,\frac{(s_{11}+s_{12}-\Lambda_7)d_{33}-2s_{13}d_{31}}{(s_{33}-\Lambda_7)d_{31}-s_{13}d_{33}},0,0,0,0,0,\frac{(s_{11}+s_{12}-\Lambda_7)d_{33}-2s_{13}d_{31}}{(\epsilon_{33}-\Lambda_7)s_{13}-d_{31}d_{33}}\right]\\ \boldsymbol{M}_8^{\mathrm{T}} &= \left[1,1,\frac{(s_{11}+s_{12}-\Lambda_8)d_{33}-2s_{13}d_{31}}{(s_{33}-\Lambda_8)d_{31}-s_{13}d_{33}},0,0,0,0,0,\frac{(s_{11}+s_{12}-\Lambda_8)d_{33}-2s_{13}d_{31}}{(\epsilon_{33}-\Lambda_8)s_{13}-d_{31}d_{33}}\right]\\ \boldsymbol{M}_9^{\mathrm{T}} &= \left[1,1,\frac{(s_{11}+s_{12}-\Lambda_9)d_{33}-2s_{13}d_{31}}{(s_{33}-\Lambda_9)d_{31}-s_{13}d_{33}},0,0,0,0,0,\frac{(s_{11}+s_{12}-\Lambda_9)d_{33}-2s_{13}d_{31}}{(\epsilon_{33}-\Lambda_9)s_{13}-d_{31}d_{33}}\right]\end{aligned} \tag{8.75b}$$

These three material modes represent the axial symmetric strains and the electric field out of the plane (x_1, x_2).

2. *Eigen-compliances of a cubic crystal*

$$\begin{aligned}&\Lambda_1=s_{11}+2s_{12},\quad \Lambda_{2,3}=s_{11}-s_{12},\\ &\Lambda_{4,5,6}=\tfrac{1}{2}\epsilon_{11}+\tfrac{1}{4}s_{44}+\tfrac{1}{4}\sqrt{4\,\epsilon_{11}^2-4s_{44}\,\epsilon_{11}+s_{44}^2+8d_{14}^2},\\ &\Lambda_{7,8,9}=\tfrac{1}{2}\epsilon_{11}+\tfrac{1}{4}s_{44}-\tfrac{1}{4}\sqrt{4\epsilon_{11}^2-4s_{44}\,\epsilon_{11}+s_{44}^2+8d_{14}^2}\end{aligned} \tag{8.76}$$

3. *Eigen-compliances of a hexagonal crystal*

$$\begin{aligned}
&\Lambda_1 = \epsilon_{33}, \quad \Lambda_{2,3} = s_{11} - s_{12}, \\
&\Lambda_4 = \tfrac{1}{2}s_{11} + \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{33} + \tfrac{1}{2}\sqrt{(-s_{11} - s_{12} - s_{33})^2 - 4\left(-2\epsilon_{13}^2 + s_{11}s_{33} + s_{12}s_{33}\right)}, \\
&\Lambda_5 = \tfrac{1}{2}s_{11} + \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{33} - \tfrac{1}{2}\sqrt{(-s_{11} - s_{12} - s_{33})^2 - 4\left(-2\epsilon_{13}^2 + s_{11}s_{33} + s_{12}s_{33}\right)}, \\
&\Lambda_{6,7} = \tfrac{1}{2}\epsilon_{11} + \tfrac{1}{4}s_{44} + \tfrac{1}{4}\sqrt{4\epsilon_{11}^2 - 4s_{44}\,\epsilon_{11} + s_{44}^2 + 8d_{14}^2}, \\
&\Lambda_{8,9} = \tfrac{1}{2}\epsilon_{11} + \tfrac{1}{4}s_{44} - \tfrac{1}{4}\sqrt{4\epsilon_{11}^2 - 4s_{44}\,\epsilon_{11} + s_{44}^2 + 8d_{14}^2}
\end{aligned} \tag{8.77}$$

4. *Eigen-compliances of a tetragonal crystal*

$$\begin{aligned}
&\Lambda_1 = \tfrac{1}{2}s_{66}, \quad \Lambda_2 = \epsilon_{33}, \quad \Lambda_3 = s_{11} - s_{12}, \\
&\Lambda_4 = \tfrac{1}{2}s_{11} + \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{33} + \tfrac{1}{2}\sqrt{(-s_{11} - s_{12} - s_{33})^2 - 4\left(-2\epsilon_{13}^2 + s_{11}s_{33} + s_{12}s_{33}\right)}, \\
&\Lambda_5 = \tfrac{1}{2}s_{11} + \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{33} - \tfrac{1}{2}\sqrt{(-s_{11} - s_{12} - s_{33})^2 - 4\left(-2\epsilon_{13}^2 + s_{11}s_{33} + s_{12}s_{33}\right)} \\
&\Lambda_{6,7} = \tfrac{1}{2}\epsilon_{11} + \tfrac{1}{4}s_{44} + \tfrac{1}{4}\sqrt{4\epsilon_{11}^2 - 4s_{44}\epsilon_{11} + s_{44}^2 + 8d_{14}^2}, \\
&\Lambda_{8,9} = \tfrac{1}{2}\epsilon_{11} + \tfrac{1}{4}s_{44} - \tfrac{1}{4}\sqrt{4\epsilon_{11}^2 - 4s_{44}\epsilon_{11} + s_{44}^2 + 8d_{14}^2}
\end{aligned} \tag{8.78}$$

5. *Eigen-compliances and material modes of an isotropic elastic material*

$$\begin{aligned}
&\Lambda_1 = s_{11} + 2s_{12} = 1/K, \quad \Lambda_{2,3,4,5,6} = s_{11} - s_{12} = 1/2G, \\
&\boldsymbol{M}_1^{\mathrm{T}} = \left[1\big/\sqrt{3}, 1\big/\sqrt{3}, 1\big/\sqrt{3}, 0, 0, 0\right], \quad \boldsymbol{M}_{21}^{\mathrm{T}} = \left[0, 1\big/\sqrt{2}, -1\big/\sqrt{2}, 0, 0, 0\right], \\
&\boldsymbol{M}_{22}^{\mathrm{T}} = \left[\sqrt{2/3}, -\sqrt{1/6}, -\sqrt{1/6}, 0, 0, 0\right], \quad \boldsymbol{M}_{23}^{\mathrm{T}} = [0, 0, 0, 1, 0, 0], \\
&\boldsymbol{M}_{24}^{\mathrm{T}} = [0, 0, 0, 0, 1, 0], \quad \boldsymbol{M}_{25}^{\mathrm{T}} = [0, 0, 0, 0, 0, 1] \\
&\boldsymbol{M}_0^{\mathrm{T}} = \boldsymbol{M}_{21}^{\mathrm{T}} + \boldsymbol{M}_{22}^{\mathrm{T}} + \boldsymbol{M}_{23}^{\mathrm{T}} + \boldsymbol{M}_{24}^{\mathrm{T}} + \boldsymbol{M}_{25}^{\mathrm{T}} \\
&\qquad = \left[\sqrt{2/3}, \sqrt{1/2} - \sqrt{1/6}, -\sqrt{1/2} - \sqrt{1/6}, 1, 1, 1\right]
\end{aligned} \tag{8.79}$$

where K and G are the volume compression modulus and shear modulus. In many cases $\boldsymbol{M}_{2i}^{\mathrm{T}}, i = 1 \sim 5$ can be replaced by $\boldsymbol{M}_0^{\mathrm{T}}$. Therefore, for an isotropic elastic material, there are only two different deformation versions: $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ corresponding to shape and volume changes, respectively. This is the theoretical foundation of the plastic yielding and the failure theory of the elastoplastic materials. The modal energy density theory is more complex, but more rational.

8.5.6 Example

The CT failure test of PZT-4 (Park and Sun 1995) is used as a numerical example to demonstrate the suitability of the MEDF theory. Material constants can be obtained from Sect. 4.4.1 by conversion. The eigen-compliances are

$$\Lambda_{1,2} = 1.3025 \times 10^{-8}\left(\mathrm{m}^2/\mathrm{N}\right), \quad \Lambda_3 = 1.1494 \times 10^{-8}, \quad \Lambda_{4,5} = 1.634 \times 10^{-11},$$
$$\Lambda_6 = 9.9855 \times 10^{-12}, \quad \Lambda_{7,8} = 9.0006 \times 10^{-12}, \quad \Lambda_9 = 3.5413 \times 10^{-12}$$

and the corresponding material modes are

$$\begin{aligned}
\boldsymbol{M}_{11} + \boldsymbol{M}_{12} &= [0, 0, 0, -0.02011, 0.02011, 0, 0.7068, -0.7068, 0]^{\mathrm{T}} \\
\boldsymbol{M}_3 &= [-0.01174, -0.01174, 0.02068, 0, 0, 0, 0, 0, 0.9995]^{\mathrm{T}} \\
\boldsymbol{M}_{41} + \boldsymbol{M}_{42} &= [-0.5, 0.5, 0, 0, 0, 0.70711, 0, 0, 0]^{\mathrm{T}} \\
\boldsymbol{M}_6 &= [0.3617, 0.3617, -0.8587, 0, 0, 0, 0, 0, 0.03091]^{\mathrm{T}} \\
\boldsymbol{M}_{71} + \boldsymbol{M}_{72} &= [0, 0, 0, -0.7068, -0.7068, 0, 0.02011, 0.02011, 0]^{\mathrm{T}} \\
\boldsymbol{M}_9 &= [0.6075, 0.6075, 0.5118, 0, 0, 0, 0, 0, 0.000918]^{\mathrm{T}}
\end{aligned}$$

It is seen that the deformation versions are three kinds: $\boldsymbol{M}_{11} + \boldsymbol{M}_{12}$ and $\boldsymbol{M}_{71} + \boldsymbol{M}_{72}$ represent the shear strain out of the plane and the in-plane electric field. In $\boldsymbol{M}_{11} + \boldsymbol{M}_{12}$ the role of the electric field is larger, but in $\boldsymbol{M}_{71} + \boldsymbol{M}_{72}$ the shear strain is larger; $\boldsymbol{M}_{41} + \boldsymbol{M}_{42}$ represents the in-plane stress; $\boldsymbol{M}_3, \boldsymbol{M}_6$ and $\boldsymbol{M}_9$ represent the axial symmetric strain and the electric fields out of the plane.

For the CT specimen in Park and Sun's test (1995), the generalized stress intensity factors are

$$K_{\mathrm{I}} = \sigma_3^{\infty}\sqrt{\pi a}, \quad \sigma_3^{\infty} = 4.4F/tc = 6.16 \times 10^4 F(\mathrm{MPa}); \quad K_{\mathrm{D}} = D_3^{\infty}\sqrt{\pi a}$$

The normalized generalized stress $\bar{\boldsymbol{\Sigma}}$ is

$$\sqrt{r}\bar{\boldsymbol{\Sigma}} = \sqrt{a/2}\left[8093F - 1.298E_3^{\infty}, 6922F + 10.2E_3^{\infty}, 6163F, 0, 0, 0, 0, 0, 1440F + 0.974E_3^{\infty}\right]^{\mathrm{T}}$$

where $E_3^{\infty} = D_3^{\infty} \times 10^8 - 1479F$ is obtained from the constitutive equations. Under the above loading the modal energy densities of $\boldsymbol{M}_3, \boldsymbol{M}_{41}, \boldsymbol{M}_{42}$ and $\boldsymbol{M}_6$ are not zero, and they are

$$\begin{aligned}
(r/a)\mathfrak{A}_3 &= \left(5 \times 10^{-12}\sigma_3^{\infty 2} + 0.417\sigma_3^{\infty}E_3^{\infty} + 8.69 \times 10^{-9}E_3^{\infty 2}\right)/4 \\
(r/a)(\mathfrak{A}_{41} + \mathfrak{A}_{42}) &= \left(1.47 \times 10^{-13}\sigma_3^{\infty 2} - 1.79 \times 10^{-11}\sigma_3^{\infty}E_3^{\infty} + 5.4 \times 10^{-10}E_3^{\infty 2}\right)/2 \\
(r/a)\mathfrak{A}_9 &= \left(1.4 \times 10^{-11}\sigma_3^{\infty 2} + 7.64 \times 10^{-11}\sigma_3^{\infty}E_3^{\infty} + 1.039 \times 10^{-10}E_3^{\infty 2}\right)/4 \\
(r/a)\mathfrak{A}_6 &= \left(5.44 \times 10^{-15}\sigma_3^{\infty 2} + 1.51 \times 10^{-12}\sigma_3^{\infty}E_3^{\infty} + 1.055 \times 10^{-10}E_3^{\infty 2}\right)/4 \approx 0
\end{aligned}$$

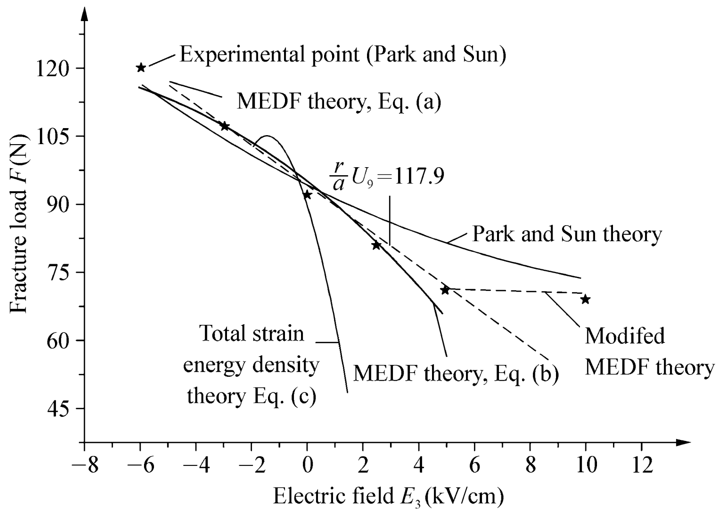


Fig. 8.13 Variations of the fracture load with the electric field

where $\mathfrak{A}_6$ can be neglected. The following criterions (by fitting the test data) are used:

$$(r/a)\mathfrak{A}_9 = \mathfrak{A}_{cr} \quad \text{or}$$
$$\left(1.4\times10^{-11}\sigma_3^{\infty2} + 7.64\times10^{-11}\sigma_3^{\infty}E_3^{\infty} + 1.039\times10^{-10}E_3^{\infty2}\right)/4 = 117.9 \tag{a}$$

$$(r/a)(0.02\mathfrak{A}_3 + 0.05\mathfrak{A}_4 + 0.93\mathfrak{A}_9) = \mathfrak{A}_{cr}, \quad \text{or}$$
$$\left(1.29\times10^{-11}\sigma_3^{\infty2} + 8.78\times10^{-11}\sigma_3^{\infty}E_3^{\infty} + 5.82\times10^{-10}E_3^{\infty2}\right)/4 = 112.9 \tag{b}$$

$$(r/a)(\mathfrak{A}_3 + \mathfrak{A}_4 + \mathfrak{A}_6 + \mathfrak{A}_9) == \mathfrak{A}_{cr}, \quad \text{or}$$
$$\left(1.95\times10^{-11}\sigma_3^{\infty2} + 4.59\times10^{-10}\sigma_3^{\infty}E_3^{\infty} + 9.98\times10^{-9}E_3^{\infty2}\right)/4 = 152.1 \tag{c}$$

Figure 8.13 shows that the theoretical results calculated from Eqs. (a) and (b) are consistent with the results in experiments when $-6\text{kV/cm} < E_3^{\infty} < 6\text{kV/cm}$, but after $E_3 > 6\text{kV/cm}$, the difference is obvious. It can be modified that after $E_3 > 6\text{kV/cm}$, we let $E_3 = 6\text{kV/cm}$ due to saturation. After this modification, the results calculated from Eqs. (a) and (b) are consistent with the results of experiments (Park and Sun 1995) in entire applied loading range. The formula (c) is the same as the total strain energy density factor theory, and the results calculated from it may be appropriated in a narrow loading region only.

8.6 Electric Breakdown of Solid Dielectrics

8.6.1 Energy Criterion

In electric apparatus electric breakdown is often happened. The breakdown is very complicated, here we only qualitatively discuss this problem from the view of the

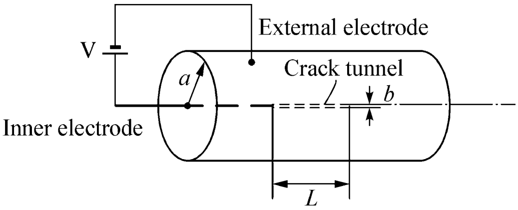


Fig. 8.14 An extending electric tubular channel model

fracture mechanics. The breakdown strength is sensitive to defects, electrodes, and environment. An insulating crack can intensify the field applied perpendicular to the crack, while a conducting crack intensifies the field applied parallel to the crack. Usually dielectric breakdown causes damage along a fine tubular channel. The tubular channel extends forward under external loading. Extending the Griffith theory (1921), Suo (1993) proposed an energy criterion to discuss the electric breakdown in dielectrics. Suo (1993) pointed out that the applied work is partly reversibly stored in the body and partly irreversibly spent to form the thin channel, i.e.,

$$\boldsymbol{f} \cdot \mathrm{d}\boldsymbol{u} + \varphi \mathrm{d}\rho_{\mathrm{e}} = \mathrm{d}\mathfrak{A} + \gamma \mathrm{d}l, \quad \mathrm{d}\mathfrak{A} = \sigma_{ij}\mathrm{d}\varepsilon_{ij} + E_i \mathrm{d}D_i \tag{8.80}$$

where $\boldsymbol{f}, \varphi, \rho_{\mathrm{e}}, \mathfrak{A}$ are the body force, electric potential, electric charge density, and internal energy density; γ is the work to create a unit length of channel; and $\mathrm{d}l$ is the increment of the channel. On the other hand, the driving force of the channel can be obtained by solving the electroelastic boundary problem, i.e.,

$$\mathrm{d}\Pi = -G\,\mathrm{d}l, \quad G = -\partial\Pi/\partial l, \quad \mathrm{d}\Pi = \mathrm{d}\mathfrak{A} - \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{u} - \varphi \mathrm{d}\rho_{\mathrm{e}} \tag{8.81}$$

where Π is the total potential energy. The energy criterion demands

$$G \geq \gamma \tag{8.82}$$

As an example, Fig. 8.14 shows a slender dielectric cylinder of radius a, inserted a needle-shaped inner electrode and on the cylindrical surface coated metal as the external electrode. The voltage between two electrodes is V. When the voltage reaches a critical value, a conductive channel, radius b and length L, emanates from the needle tip. When $L \gg a$, this problem can be considered as a coaxial transmission line, so the electric potential at a distance r from the center of the channel is

$$\varphi = \varphi_1 - \frac{q}{2\pi\epsilon} \ln\frac{r}{b}, \quad q = \frac{2\pi\epsilon(\varphi_1 - \varphi_2)}{\ln(a/b)} \tag{8.83}$$

where φ_1 and φ_2 are the potentials on the inner channel and external cylindrical surface and q is the electric charge on the channel of a unit length and is constant duo to constant $\varphi_1 - \varphi_2$. The work done by the external electric field is $q(\varphi_1 - \varphi_2)L$, so we obtain

$$\Pi = -(\varphi_1 - \varphi_2)\frac{qL}{2} = -\frac{\pi\epsilon L(\varphi_1 - \varphi_2)^2}{\ln(a/b)} \tag{8.84}$$

The energy release rate is

$$\widetilde{G} = -\frac{\partial \Pi}{\partial L} = \frac{\pi \epsilon (\varphi_1 - \varphi_2)^2}{\ln(a/b)} = \frac{\pi \epsilon V^2}{\ln(a/b)} \tag{8.85}$$

According to the energy criterion Eq. (8.82) when $\widetilde{G} \geq \gamma$, the channel will be extended.

8.6.2 J Integral Method

Beom and Kim (2008) discussed the application of J integral to breakdown analysis. From Eq. (2.143), it is known that the following conservation integral is held if the closed integral surface S is not enclosed singular point:

$$\int_V P_{ij,j} \mathrm{d}V = \oint_a \left(g\delta_{ij} - \Sigma_{\alpha j} U_{\alpha,i} \right) n_j \mathrm{d}a = 0 \tag{8.86}$$

where $\boldsymbol{P}$ is the energy-momentum tensor, g is the Gibbs free energy density, and n_i is the outward normal of the surface S. From this theory, it is easy to derive the three-dimensional J integral. For the pure electric loading case we have

$$J_i = \int_S \left(g\delta_{ij} - D_j \varphi_{,i} \right) n_j \mathrm{d}a \tag{8.87}$$

where the surface S with outward normal $\boldsymbol{n}$ is initiated from and stopped on a curve located on the surface defect. Analogous to 2D problem, J integral is equal to the energy release rate. When the channel is fixed, according to the virtual work principle it yields

$$\int_V \delta g \, \mathrm{d}V - \int_{S+S_c} T_i \delta \mathrm{u}_i \, \mathrm{d}a - \int_{S+S_c} D_i n_i \delta\varphi \, \mathrm{d}a = 0 \tag{8.88}$$

When the channel extends with velocity $\boldsymbol{v}$ in a similar version, the variation of the total potential energy Π is

$$\begin{aligned}
\delta \Pi &= \delta \int_V g \, \mathrm{d}V - \int_{a_\sigma} T_i \delta u_i \, \mathrm{d}a - \int_{a_D} D_i n_i \delta\varphi \, \mathrm{d}a \\
&= \int_V \delta g \, \mathrm{d}V + \int_{S_c} g v_i \delta t m_i \, \mathrm{d}a + \int_{S_u} T_i \delta u_i \, \mathrm{d}a + \int_{S_\varphi} D_i n_i \delta\varphi \, \mathrm{d}a \\
&= \int_{S_c} g v_i \delta t m_i \, \mathrm{d}a + \int_{S_u} T_i \delta u_i \, \mathrm{d}a + \int_{S_\varphi} D_i m_i \delta\varphi \, \mathrm{d}a
\end{aligned} \tag{8.89}$$

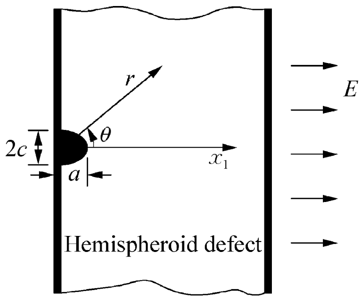


Fig. 8.15 Conducting hemispheroid defect

where $\boldsymbol{m}$ is the external normal of the defect head, so comparing with $\boldsymbol{n}$ in Eq. (8.87) we have $\boldsymbol{m} = -\boldsymbol{n}$. On S_u, u_i is given; on S_φ, φ is given. Using the relation

$$\delta\varphi = -\varphi_{,i} v_i \delta t, \quad \text{on} \quad S_c; \quad v_i = 0, \quad \delta\varphi = 0, \quad \text{on} \quad S - S_c \tag{8.90}$$

So Eq. (8.89) can be written as

$$\delta\Pi/\delta t = \int_{S_c} g v_j m_j \, \mathrm{d}a + \int_{S_c} D_i m_i v_j E_j \, \mathrm{d}a \tag{8.91}$$

Let the channel of length l is located on axis x_1 and extends along x_1, so $v_i = \delta_{i1} \mathrm{d}l/\mathrm{d}t$. Noting the outward normal $\boldsymbol{n} = -\boldsymbol{m}$ of the channel head, so the energy release rate $\tilde{G}$ is

$$\tilde{G} = -\frac{\delta\Pi}{\delta l} = -\frac{1}{\dot{l}} \frac{\delta\Pi}{\delta t} = \int_{S_c} (g n_1 + n_i D_i E_1) \mathrm{d}S \tag{8.92}$$

Equation (8.92) is identical with Eq. (8.87), i.e., $J = \tilde{G}$. Using Eq. (8.87) or (8.92), the effect of the defect shape can be considered.

As an example we discuss a semi-infinite medium with a conductive hemispherical defect of radius $a = c = R$ subjected to a remote uniform electric field E_0 as shown in Fig. 8.15. The solution of a conductive sphere embedded in an infinite dielectric under a remote uniform electric field can be seen in many textbooks. Using the symmetry, the solution of the hemisphere is

$$\varphi = -E_0\left(r - \frac{R^3}{r^3}\right)\cos\theta, \quad E_r = E_0\left[1 + 2\left(\frac{R}{r}\right)^3\right]\cos\theta, \quad E_\theta = -E_0\left[1 - \left(\frac{R}{r}\right)^3\right] \tag{8.93}$$

where r, θ are the sphere coordinates, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, and θ is the polar angle measured from the positive x_1-axis. In this case,

$$g = -(1/2)\,\epsilon\left(E_r^2 + E_\theta^2\right), \quad D_r = \epsilon E_r, \quad D_\theta = \epsilon E_\theta \tag{8.94}$$

and

$$J = (9/4)\pi\,\epsilon\, c^2 E_0^2 \tag{8.95}$$

Using the J integral, the electric breakdown can be further discussed.

For a semi-infinite medium with a conductive hemispheroid defect of major semiaxis a and minor semiaxis c subjected to a remote uniform electric field E_0 parallel to x_1-axis. Beom and Kim (2008) adopted the ellipsoid coordinate to solve this problem (Stratton 1941). Finally they got

$$\begin{aligned} &J = \pi \epsilon c^2 E_0^2 H(\lambda) \\ &H(\lambda) = \frac{(1-\lambda^2)\left[(1-\lambda^2) + 2\lambda^2 \ln \lambda\right]}{2\left[\sqrt{1-\lambda^2} - \lambda\left(\dfrac{\pi}{2} - \tan^{-1}\dfrac{\lambda}{\sqrt{1-\lambda^2}}\right)\right]}, \quad \text{when} \quad 0 < \lambda = \frac{a}{c} < 1 \\ &H(\lambda) = \frac{2\left(\lambda^2-1\right)\left[\left(\lambda^2-1\right) + 2\lambda^2 \ln \lambda\right]}{2\left[\sqrt{\lambda^2-1} + \lambda\left(\ln \dfrac{\lambda - \sqrt{\lambda^2-1}}{\lambda + \sqrt{\lambda^2-1}}\right)\right]}, \quad \text{when} \quad \lambda = \frac{a}{c} > 1 \end{aligned} \tag{8.96}$$

when $\lambda = a/c \to \infty$, the semi-ellipsoid reduces to a semi-penny-shaped crack and

$$J = \pi \epsilon c^2 E_0^2 (\lambda / \ln \lambda), \quad \lambda = a/c \tag{8.97}$$

For more complex cases, the finite element method is an appropriate method.

References

Arramon YP, Mchrabadi MM, Martin DW, Cowin SC (2000) A multi-dimensional anisotropic strength criterion based on Kelvin modes. Int J Solids Struct 37:2915–2935

Beom HG, Kim YH (2008) Application of J integral to breakdown analysis of a dielectric material. Int J Solids Struct 45:6045–6055

Budiansky B, Hutchinson JW, Lambropoulos JC (1983) Continuum theory of dilatant transformation toughening in ceramics. Int J Solids Struct 19:337–355

Chen ST (1984) New concepts of elasticity theory and an application. Acta Mech Sin 16:259–274 (in Chinese)

Eshelby JD (1957) The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc R Soc Lond A 241:376–396

Fang D, Liu B, Hwang K (1999) Energy analysis on fracture of ferroelectric ceramics. Int J Fract 100:401–408

Fang D, Zhang Z, Soh AK, Lee KL (2004) Fracture criteria of piezoelectric ceramics with defects. Mech Mater 36:917–928

Fu R, Zhang TY (2000) Influences of temperature and electric field on the bending strength of lead zirconate titanate ceramics. Acta Mater 48:1729–1740

Fulton CC, Gao H (1997) Electrical nonlinearity in fracture of piezoelectric ceramics. Appl Mech Rev 50:556–563

Gao HJ, Zhang TY, Tong P (1997) Local and global energy release rates for an electrically yielded crack in a piezoelectric ceramic. J Mech Phys Solids 45:491–510

Griffith AA (1921) The phenomena of rupture and flow in solids. Phil Trans R Soc Lond A 221:163–197

Heyer V, Schneider GA, Balke H, Drescher J, Bahr HA (1998) A fracture criterion for conducting cracks in homogeneously poled PZT-PIC151 ceramics. Acta Mater 46:6615–6622
Huang Z-Y, Kuang Z-B (2003) On the influence of the switching wake on the facture toughness of ferroelectric materials. Smart Mater Struct 12:1017–1022
Hwang SC, Lynch CS, McMeeking RM (1995) Ferroelectric/ferroelastic interactions and a polarization switching model. Acta Metall Mater 43:2073–2084
Jiang LZ, Sun CT (2001) Analysis of indentation cracking in piezoceramics. Int J Solids Struct 38:1903–11918
Kuang Z-B (2011) Theory of electroelasticity. Shanghai Jiaotong University Press, Shanghai (in Chinese)
Kuang Z-B, Gu H-C, Li Z-H (1998) The mechanical behavior of materials. Higher Education Publishing House, Beijing (in Chinese)
Kuang Z-B, Zhou Z-D, Chen Y, Zhao S-X (2003) Eigen-material constants, mode and failure criterion for piezoelectric media. Int J Appl Electromagn Mech 18:235–250
Majumdar BS, Burns SJ (1983) A Griffith crack shielded by a dislocation pile-up. Int J Fract 21:229–240
McCmeeking RM, Evans AG (1982) Mechanics of transformation-toughening in brittle materials. J Am Ceram Soc 65:242–246
Ohr SM (1985) An electron microscope study of crack tip deformation and its impact on the dislocation theory of fracture. Mater Sci Eng 72:1–35
Park S, Sun CT (1995) Fracture criteria for piezoelectric ceramics. J Am Ceram Soc 78:1475–1480
Rajapakse RKND, Zeng X (2001) Toughening of conducting cracks due to domain switching. Acta Mater 49:877–885
Ruhlevskii Y (1984) On Hooke's law. Appl Math Mech 48:420–435 (in Russian); РыхлевскийЯ. ОзаконеГука, прикладнаяматемат и каимехника 48:420–435 (1984)
Shen S-P, Nishioka T (2000) Fracture of piezoelectric materials: energy density criterion. Theor Appl Fract Mech 33:57–65
Sih GC (1973) Some basic problems in fracture mechanics and new concepts. J Eng Fract Mech 5:365–377
Stratton JA (1941) Electromagnetic theory. McGraw-Hill, New York
Suo Z (1993) Models for breakdown-resistant dielectric and ferroelectric ceramics. J Mech Phys Solids 41:1155–1176
Suo Z, Kuo CM, Barnett DM, Willis JR (1992) Fracture mechanics for piezoelectric ceramics. J Mech Phys Solids 40:739–765
Wang T-C (2000) Analysis of strip electric saturation model of crack problem in piezoelectric materials. Int J Solids Struct 37:6031–6049
Wang H, Singh RN (1997) Crack propagation in piezoelectric ceramics: effects of applied electric fields. J Appl Phys 81:7471–7479
Yang W, Zhu T (1998) Switch-toughening of ferroelectrics subjected to electric fields. J Mech Phys Solids 46:291–311
Zeller HR, Schneider WR (1984) Electrofracture mechanics of dielectric aging. J Appl Phys 56:455–459
Zeng X, Rajapakse RKND (2001) Domain switching induced fracture toughness variation in ferroelectrics. Smart Mater Struct 10:203–211
Zhang TY, Wang T, Zhao M (2003) Failure behavior and failure criterion of conductive cracks (deep notches) in thermally depoled PZT-4 ceramics. Acta Mater 51:4881–4895
Zhang TY, Zhao M, Liu G (2004) Failure behavior and failure criterion of conductive cracks (deep notches) in piezoelectric ceramics, I-the charge-free zone model. Acta Mater 52:2013–2024; II-experimental verification. Acta Mater 52:2025–2035 (2004)
Zhu T, Yang W (1997) Toughness variation of ferroelectrics by polarization switch under nonuniform electric field. Acta Mater 45:4695–4702
Zuo J-Z, Sih GC (2000) Energy density theory formulation and interpretation of cracking behavior for piezoelectric ceramics. Theor Appl Fract Mech 34:17–33

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