Operator-Valued and Multivariate Free Berry-Esseen Theorems

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Dedicated to Professor Friedrich Götze on the occasion of his 60th birthday

Abstract We address the question of a Berry-Esseen type theorem for the speed of convergence in a multivariate free central limit theorem. For this, we estimate the difference between the operator-valued Cauchy transforms of the normalized partial sums in an operator-valued free central limit theorem and the Cauchy transform of the limiting operator-valued semicircular element. Since we have to deal with in general non-self-adjoint operators, we introduce the notion of matrix-valued resolvent sets and study the behavior of Cauchy transforms on them.

Keywords Free Berry-Esseen • operator valued • multivariate • linearization trick • matrix valued spectrum

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1 Introduction

In classical probability theory the famous Berry-Esseen theorem gives a quantitative statement about the order of convergence in the central limit theorem. It states in its simplest version: If $(X_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables with mean 0 and variance 1, then the distance between

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$S_n := \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ and a normal variable γ of mean 0 and variance 1 can be estimated in terms of the Kolmogorov distance Δ by

$$\Delta(S_n, \gamma) \leq C \frac{1}{\sqrt{n}} \rho,$$

where C is a constant and ρ is the absolute third moment of the variables X_i. The question for a free analogue of the Berry-Esseen estimate in the case of one random variable was answered by Chistyakov and Götze in [2] (and independently, under the more restrictive assumption of compact support of the X_i, by Kargin [10]): If $(X_i)_{i\in\mathbb{N}}$ is a sequence of free and identically distributed variables with mean 0 and variance 1, then the distance between $S_n := \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ and a semicircular variable s of mean 0 and variance 1 can be estimated as

$$\Delta(S_n, s) \leq c \frac{|m_3| + \sqrt{m_4}}{\sqrt{n}},$$

where $c > 0$ is an absolute constant and m_3 and m_4 are the third and fourth moment, respectively, of the X_i.

In this paper we want to present an approach to a multivariate version of a free Berry-Esseen theorem. The general idea is the following: Since there is up to now no suitable replacement of the Kolmorgorov metric in the multivariate case, we will, in order to describe the speed of convergence of a d-tuple $(S_n^{(1)}, \ldots, S_n^{(d)})$ of partial sums to the limiting semicircular family $(s_1, \ldots, s_d)$, consider the speed of convergence of $p(S_n^{(1)}, \ldots, S_n^{(d)})$ to $p(s_1, \ldots, s_d)$ for any self-adjoint polynomial p in d non-commuting variables. By using the linearization trick of Haagerup and Thorbjørnsen [5,6], we can reformulate this in an operator-valued setting, where we will state an operator-valued free Berry-Esseen theorem. Because estimates for the difference between scalar-valued Cauchy transforms translate by results of Bai [1] to estimates with respect to the Kolmogorov distance, it is convenient to describe the speed of convergence in terms of Cauchy transforms. On the level of deriving equations for the (operator-valued) Cauchy transforms we can follow ideas which are used for dealing with speed of convergence questions for random matrices; here we are inspired in particular by the work of Götze and Tikhomirov [4], but see also [1].

Since the transition from the multivariate to the operator-valued setting leads to operators which are, even if we start from self-adjoint polynomials p, in general not self-adjoint, we have to deal with (operator-valued) Cauchy transforms defined on domains different from the usual ones. Since most of the analytic tools fail in this generality, we have to develop them along the way.

As a first step in this direction, the present paper (which is based on the unpublished preprint [13]) leads finally to the proof of the following theorem:

Theorem 1.1. *Let* $(\mathcal{C}, \tau)$ *be a non-commutative* C^**-probability space with* τ *faithful and put* $\mathcal{A} := \mathrm{M}_m(\mathbb{C}) \otimes \mathcal{C}$ *and* $E := \mathrm{id} \otimes \tau$. *Let* $(X_i)_{i\in\mathbb{N}}$ *be a sequence of non-zero elements in the operator-valued probability space* $(\mathcal{A}, E)$. *We assume:*

- *All X_i's have the same $*$-distribution with respect to E and their first moments vanish, i.e. $E[X_i] = 0$.*
- *The X_i are $*$-free with amalgamation over $\mathrm{M}_m(\mathbb{C})$ (which means that the $*$-algebras $\mathcal{X}_i$, generated by $\mathrm{M}_m(\mathbb{C})$ and X_i, are free with respect to E).*
- *We have $\sup_{i\in\mathbb{N}} \|X_i\| < \infty$.*

Then the sequence $(S_n)_{n\in\mathbb{N}}$ defined by

$$S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i, \qquad n \in \mathbb{N}$$

converges to an operator-valued semicircular element s. Moreover, we can find $\kappa > 0$, $c > 1$, $C > 0$ and $N \in \mathbb{N}$ such that

$$\|G_s(b) - G_{S_n}(b)\| \leq C \frac{1}{\sqrt{n}} \|b\| \qquad \text{for all } b \in \Omega \text{ and } n \geq N,$$

where

$$\Omega := \Big\{ b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa,\ \|b\| \cdot \|b^{-1}\| < c \Big\}$$

and where G_s and G_{S_n} denote the operator-valued Cauchy transforms of s and of S_n, respectively.

Applying this operator-valued statement to our multivariate problem gives the following main result on a multivariate free Berry Esseen theorem.

Theorem 1.2. *Let $(x_i^{(k)})_{k=1}^{d}$, $i \in \mathbb{N}$, be free and identically distributed sets of d self-adjoint non-zero random variables in some non-commutative C^*-probability space $(\mathcal{C}, \tau)$, with τ faithful, such that the conditions*

$$\tau(x_i^{(k)}) = 0 \qquad \text{for } k = 1, \ldots, d \text{ and all } i \in \mathbb{N}$$

and

$$\sup_{i\in\mathbb{N}} \max_{k=1,\ldots,d} \|x_i^{(k)}\| < \infty$$

are fulfilled. We denote by $\Sigma = (\sigma_{k,l})_{k,l=1}^{d}$, where $\sigma_{k,l} := \tau(x_i^{(k)} x_i^{(l)})$, their joint covariance matrix. Moreover, we put

$$S_n^{(k)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i^{(k)} \qquad \text{for } k = 1, \ldots, d \text{ and all } n \in \mathbb{N}.$$

Then $(S_n^{(1)}, \ldots, S_n^{(d)})$ converges in distribution to a semicircular family $(s_1, \ldots, s_d)$ of covariance Σ. We can quantify the speed of convergence in the following way. Let p be a (not necessarily self-adjoint) polynomial in d non-commutating variables and put

$$P_n := p(S_n^{(1)}, \ldots, S_n^{(d)}) \qquad \text{and} \qquad P := p(s_1, \ldots, s_d).$$

Then, there are constants $C > 0$, $R > 0$ and $N \in \mathbb{N}$ (depending on the polynomial) such that

$$|G_P(z) - G_{P_n}(z)| \leq C \frac{1}{\sqrt{n}} \qquad \text{for all } |z| > R \text{ and } n \geq N,$$

where G_P and G_{P_n} denote the scalar-valued Cauchy transform of P and of P_n, respectively.

In the case of a self-adjoint polynomial p, we can consider the distribution measures μ_n and μ of the operators P_n and P from above, which are probability measures on $\mathbb{R}$. Moreover, let $\mathcal{F}_{\mu_n}$ and $\mathcal{F}_\mu$ be their cumulative distribution functions. In order to deduce estimates for the Kolmogorov distance

$$\Delta(\mu_n, \mu) = \sup_{x \in \mathbb{R}} |\mathcal{F}_{\mu_n}(x) - \mathcal{F}_\mu(x)|$$

one has to transfer the estimate for the difference of the scalar-valued Cauchy transforms of P_n and P from near infinity to a neighborhood of the real axis. A partial solution to this problem was given in the appendix of [14], which we will recall in Sect. 4. But this leads to the still unsolved question, whether $p(s_1, \ldots, s_d)$ has a continuous density. We conjecture that the latter is true for any self-adjoint polynomial in free semicirculars, but at present we are not aware of a proof of that statement.

The paper is organized as follows. In Sect. 2 we recall some basic facts about holomorphic functions on domains in Banach spaces. The tools to deal with matrix-valued Cauchy transform will be presented in Sect. 3. Section 4 is devoted to the proof of Theorems 1.1 and 1.2.

2 Holomorphic Functions on Domains in Banach Spaces

For reader's convenience, we briefly recall the definition of holomorphic functions on domains in Banach spaces and we state the theorem of Earle-Hamilton, which will play a major role in the subsequent sections.

Definition 2.1. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two complex Banach spaces and let $D \subseteq X$ be an open subset of X. A function $f : D \to Y$ is called

- **Strongly holomorphic**, if for each $x \in D$ there exists a bounded linear mapping $Df(x) : X \to Y$ such that

$$\lim_{y \to 0} \frac{\|f(x+y) - f(x) - Df(x)y\|_Y}{\|y\|_X} = 0.$$

- **Weakly holomorphic**, if it is locally bounded and the mapping

$$\lambda \mapsto \phi(f(x+\lambda y))$$

is holomorphic at $\lambda = 0$ for each $x \in D$, $y \in Y$ and all continuous linear functionals $\phi : Y \to \mathbb{C}$.

An important theorem due to Dunford says, that a function on a domain (i.e. an open and connected subset) in a Banach space is strongly holomorphic if and only if it is weakly holomorphic. Hence, we do not have to distinguish between both definitions.

Definition 2.2. Let D be a nonempty domain in a complex Banach space $(X, \|\cdot\|)$ and let $f : D \to D$ be a holomorphic function. We say, that $f(D)$ **lies strictly inside** D, if there is some $\epsilon > 0$ such that

$$B_\epsilon(f(x)) \subseteq D \qquad \text{for all } x \in D$$

holds, whereby we denote by $B_r(y)$ the open ball with radius r around y.

The remarkable fact, that strict holomorphic mappings are strict contractions in the so-called Carathéodory-Riffen-Finsler metric, leads to the following theorem of Earle-Hamilton (cf. [3]), which can be seen as a holomorphic version of Banach's contraction mapping theorem. For a proof of this theorem and variations of the statement we refer to [7].

Theorem 2.3 (Earle-Hamilton, 1970). *Let $\emptyset \neq D \subseteq X$ be a domain in a Banach space $(X, \|\cdot\|)$ and let $f : D \to D$ be a bounded holomorphic function. If $f(D)$ lies strictly inside D, then f has a unique fixed point in D.*

3 Matrix-Valued Spectra and Cauchy Transforms

The statement of the following lemma is well-known and quite simple. But since it turns out to be extremely helpful, it is convenient to recall it here.

Lemma 3.1. *Let $(A, \|\cdot\|)$ be a complex Banach-algebra with unit 1. If $x \in A$ is invertible and $y \in A$ satisfies $\|x - y\| < \sigma \frac{1}{\|x^{-1}\|}$ for some $0 < \sigma < 1$, then y is invertible as well and we have*

$$\|y^{-1}\| \leq \frac{1}{1-\sigma}\|x^{-1}\|.$$

Proof. We can easily check that

$$\sum_{n=0}^{\infty} \big(x^{-1}(x-y)\big)^n x^{-1}$$

is absolutely convergent in A and gives the inverse element of y. Moreover we get

$$\|y^{-1}\| \leq \sum_{n=0}^{\infty} \big(\|x^{-1}\| \|x-y\|\big)^n \|x^{-1}\| < \frac{1}{1-\sigma}\|x^{-1}\|,$$

which proves the stated estimate. □

Let $(\mathcal{C}, \tau)$ be a non-commutative C^*-probability space, i.e., $\mathcal{C}$ is a unital C^*-algebra and τ is a unital state (positive linear functional) on $\mathcal{C}$; we will always assume that τ is faithful. For fixed $m \in \mathbb{N}$ we define the operator-valued C^*-probability space $\mathcal{A} := \mathrm{M}_m(\mathbb{C}) \otimes \mathcal{C}$ with conditional expectation

$$E := \mathrm{id}_m \otimes \tau : \mathcal{A} \to \mathrm{M}_m(\mathbb{C}),\ b \otimes c \mapsto \tau(c)b,$$

where we denote by $\mathrm{M}_m(\mathbb{C})$ the C^*-algebra of all $m \times m$ matrices over the complex numbers $\mathbb{C}$. Under the canonical identification of $\mathrm{M}_m(\mathbb{C}) \otimes \mathcal{C}$ with $\mathrm{M}_m(\mathcal{C})$ (matrices with entries in $\mathcal{C}$), the expectation E corresponds to applying the state τ entrywise in a matrix. We will also identify $b \in \mathrm{M}_m(\mathbb{C})$ with $b \otimes 1 \in \mathcal{A}$.

Definition 3.2. For $a \in \mathcal{A} = \mathrm{M}_m(\mathcal{C})$ we define the **matrix-valued resolvent set**

$$\rho_m(a) := \{b \in \mathrm{M}_m(\mathbb{C}) \mid b - a \text{ is invertible in } \mathcal{A}\}$$

and the **matrix-valued spectrum**

$$\sigma_m(a) := \mathrm{M}_m(\mathbb{C}) \backslash \rho_m(a).$$

Since the set $\mathrm{GL}(\mathcal{A})$ of all invertible elements in $\mathcal{A}$ is an open subset of $\mathcal{A}$ (cf. Lemma 3.1), the continuity of the mapping

$$f_a : \mathrm{M}_m(\mathbb{C}) \to \mathcal{A},\ b \mapsto b - a$$

implies, that the matrix-valued resolvent set $\rho_m(a) = f_a^{-1}(\mathrm{GL}(\mathcal{A}))$ of an element $a \in \mathcal{A}$ is an open subset of $\mathrm{M}_m(\mathbb{C})$. Hence, the matrix-valued spectrum $\sigma_m(a)$ is always closed.

Although the behavior of this matrix-valued generalizations of the classical resolvent set and spectrum seems to be quite similar to the classical case (which is of course included in our definition for $m = 1$), the matrix valued spectrum is in general not bounded and hence not a compact subset of $\mathrm{M}_m(\mathbb{C})$. For example, we have for all $\lambda \in \mathbb{C}$, that

$$\sigma_m(\lambda 1) = \{b \in \mathrm{M}_m(\mathbb{C}) \mid \lambda \in \sigma_{\mathrm{M}_m(\mathbb{C})}(b)\},$$

i.e. $\sigma_m(\lambda 1)$ consists of all matrices $b \in \mathrm{M}_m(\mathbb{C})$ for which λ belongs to the spectrum $\sigma_{\mathrm{M}_m(\mathbb{C})}(b)$. Particularly, $\sigma_m(\lambda 1)$ is unbounded for $m \geq 2$.

In the following, we denote by $\mathrm{GL}_m(\mathbb{C}) := \mathrm{GL}(\mathrm{M}_m(\mathbb{C}))$ the set of all invertible matrices in $\mathrm{M}_m(\mathbb{C})$.

Lemma 3.3. *Let $a \in \mathcal{A}$ be given. Then for all $b \in \mathrm{GL}_m(\mathbb{C})$ the following inclusion holds:*

$$\{\lambda b \mid \lambda \in \rho_{\mathcal{A}}(b^{-1}a)\} \subseteq \rho_m(a)$$

Proof. Let $\lambda \in \rho_{\mathcal{A}}(b^{-1}a)$ be given. By definition of the usual resolvent set this means that $\lambda 1 - b^{-1}a$ is invertible in $\mathcal{A}$. It follows, that

$$\lambda b - a = b\big(\lambda 1 - b^{-1}a\big)$$

is invertible as well, and we get, as desired, $\lambda b \in \rho_m(a)$. □

Lemma 3.4. *For all $0 \neq a \in \mathcal{A}$ we have*

$$\left\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \frac{1}{\|a\|}\right\} \subseteq \rho_m(a)$$

and

$$\sigma_m(a) \cap \mathrm{GL}_m(\mathbb{C}) \subseteq \left\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| \geq \frac{1}{\|a\|}\right\}.$$

Proof. Obviously, the second inclusion is a direct consequence of the first. Hence, it suffices to show the first statement.

Let $b \in \mathrm{GL}_m(\mathbb{C})$ with $\|b^{-1}\| < \frac{1}{\|a\|}$ be given. It follows, that $h := 1 - b^{-1}a$ is invertible, because

$$\|1 - h\| = \|b^{-1}a\| \leq \|b^{-1}\| \cdot \|a\| < 1.$$

Therefore, we can deduce, that also

$$b - a = b\big(1 - b^{-1}a\big) \tag{1}$$

is invertible, i.e. $b \in \rho_m(a)$. This proves the assertion. □

The main reason to consider matrix-valued resolvent sets is, that they are the natural domains for matrix-valued Cauchy transforms, which we will define now.

Definition 3.5. For $a \in \mathcal{A}$ we call

$$G_a : \rho_m(a) \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto E\big[(b-a)^{-1}\big]$$

the **matrix-valued Cauchy transform** of a.

Note that G_a is a continuous function (and hence locally bounded) and induces for all $b_0 \in \rho_m(a)$, $b \in \mathrm{M}_m(\mathbb{C})$ and bounded linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ a function

$$\lambda \mapsto \phi\big(G_a(b_0 + \lambda b)\big),$$

which is holomorphic in a neighborhood of $\lambda = 0$. Hence, G_a is weakly holomorphic and therefore (as we have seen in the previous section) strongly holomorphic as well.

Because the structure of $\rho_m(a)$ and therefore the behavior of G_a might in general be quite complicated, we restrict our attention to a suitable restriction of G_a. In this way, we will get some additional properties of G_a.

The first restriction enables us to control the norm of the matrix-valued Cauchy transform on a sufficiently nice subset of the matrix-valued resolvent set.

Lemma 3.6. *Let $0 \neq a \in \mathcal{A}$ be given. For $0 < \theta < 1$ the matrix valued Cauchy transform G_a induces a mapping*

$$G_a : \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \theta \cdot \frac{1}{\|a\|}\Big\} \to \Big\{b \in \mathrm{M}_m(\mathbb{C}) \mid \|b\| < \frac{\theta}{1-\theta} \cdot \frac{1}{\|a\|}\Big\}.$$

Proof. Lemma 3.4 (c) tells us, that the open set

$$U := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \theta \cdot \frac{1}{\|a\|}\Big\}$$

is contained in $\rho_m(a)$, i.e. G_a is well-defined on U. Moreover, we get from (1)

$$(b-a)^{-1} = \big(1 - b^{-1}a\big)^{-1} b^{-1} = \sum_{n=0}^{\infty} \big(b^{-1}a\big)^n b^{-1}$$

and hence

$$\|G_a(b)\| \leq \|(b-a)^{-1}\| \leq \|b^{-1}\| \sum_{n=0}^{\infty} \big(\|b^{-1}\| \|a\|\big)^n < \frac{\theta}{1-\theta} \cdot \frac{1}{\|a\|} \tag{2}$$

for all $b \in U$. This proves the claim. □

To ensure, that the range of G_a is contained in $\mathrm{GL}_m(\mathbb{C})$, we have to shrink the domain again.

Lemma 3.7. *Let $0 \neq a \in \mathcal{A}$ be given. For $0 < \theta < 1$ and $c > 1$ we define*

$$\Omega := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \theta \cdot \frac{1}{\|a\|},\ \|b\| \cdot \|b^{-1}\| < c\Big\}$$

and

$$\Omega' := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b\| < \frac{\theta}{1-\theta} \cdot \frac{1}{\|a\|}\Big\}.$$

If the condition

$$\frac{\theta}{1-\theta} < \frac{\sigma}{c}$$

is satisfied for some $0 < \sigma < 1$*, then the matrix-valued Cauchy transform* G_a *induces a mapping* $G_a : \Omega \to \Omega'$ *and we have the estimates*

$$\|G_a(b)\| \leq \|(b-a)^{-1}\| < \frac{\theta}{1-\theta} \cdot \frac{1}{\|a\|} \qquad \text{for all } b \in \Omega \tag{3}$$

and

$$\|G_a(b)^{-1}\| < \frac{1}{1-\sigma} \cdot \|b\| \qquad \text{for all } b \in \Omega. \tag{4}$$

Proof. For all $b \in \Omega$ we have

$$G_a(b) - b^{-1} = E\big[(b-a)^{-1} - b^{-1}\big] = E\Big[\sum_{n=1}^{\infty} \big(b^{-1}a\big)^n b^{-1}\Big],$$

which enables us to deduce

$$\|G_a(b) - b^{-1}\| \leq \|b^{-1}\| \sum_{n=1}^{\infty} \big(\|b^{-1}\| \|a\|\big)^n \leq \frac{\theta}{1-\theta} \cdot \|b^{-1}\| < \frac{\theta}{1-\theta} \cdot \frac{c}{\|b\|} < \sigma \cdot \frac{1}{\|b\|}.$$

Using Lemma 3.1, this implies $G_a(b) \in \mathrm{GL}_m(\mathbb{C})$ and (4). Since we already know from (2) in Lemma 3.6, that (3) holds, it follows $G_a(b) \in \Omega'$ and the proof is complete. □

Remark 3.8. Since domains of our holomorphic functions should be connected it is necessary to note, that for $\kappa > 0$ and $c > 1$

$$\Omega = \big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa,\ \|b\| \cdot \|b^{-1}\| < c\big\}$$

and for $r > 0$

$$\Omega' = \big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b\| < r\big\}$$

are pathwise connected subsets of $\mathrm{M}_m(\mathbb{C})$. Indeed, if $b_1, b_2 \in \mathrm{GL}_m(\mathbb{C})$ are given, we consider their polar decomposition $b_1 = U_1 P_1$ and $b_2 = U_2 P_2$ with unitary matrices $U_1, U_2 \in \mathrm{GL}_m(\mathbb{C})$ and positive-definite Hermitian matrices $P_1, P_2 \in \mathrm{GL}_m(\mathbb{C})$ and define (using functional calculus for normal elements in the C^*-algebra $\mathrm{M}_m(\mathbb{C})$)

$$\gamma : [0,1] \to \mathrm{GL}_m(\mathbb{C}),\ t \mapsto U_1^{1-t} P_1^{1-t} U_2^t P_2^t.$$

Then γ fulfills $\gamma(0) = b_1$ and $\gamma(1) = b_2$, and $\gamma([0,1])$ is contained in Ω and Ω' if b_1, b_2 are elements of Ω and Ω', respectively.

Since the matrix-valued Cauchy transform is a solution of a special equation (cf. [8, 12]), we will be interested in the following situation:

Corollary 3.9. *Let* $\eta : \mathrm{GL}_m(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C})$ *be a holomorphic function satisfying*

$$\|\eta(w)\| \leq M\|w\| \qquad \text{for all } w \in \mathrm{GL}_m(\mathbb{C})$$

for some $M > 0$*. Moreover, we assume that*

$$bG_a(b) = 1 + \eta(G_a(b))G_a(b) \qquad \text{for all } b \in \Omega$$

holds. Let $0 < \theta, \sigma < 1$ *and* $c > 1$ *be given with*

$$\frac{\theta}{1-\theta} < \sigma \min\left\{\frac{1}{c}, \frac{\|a\|^2}{M}\right\}$$

and let Ω *and* Ω' *be as in Lemma 3.7.*

Then, for fixed $b \in \Omega$*, the equation*

$$bw = 1 + \eta(w)w, \qquad w \in \Omega' \tag{5}$$

has a unique solution, which is given by $w = G_a(b)$*.*

Proof. Let $b \in \Omega$ be given. For all $w \in \Omega'$ we get

$$\|\eta(w)\| \leq M\|w\| \leq \frac{\theta}{1-\theta} \cdot \frac{M}{\|a\|}$$

and therefore

$$\|b^{-1}\eta(w)\| \leq \|b^{-1}\|\|\eta(w)\| \leq \frac{\theta}{1-\theta} \cdot \frac{M}{\|a\|^2} \cdot \theta < \theta\sigma < 1.$$

This means, that $1 - b^{-1}\eta(w)$ and hence $b - \eta(w)$ is invertible with

$$\begin{aligned}\|(b-\eta(w))^{-1}\| &\leq \|b^{-1}\|\|(1-b^{-1}\eta(w))^{-1}\| \\ &\leq \|b^{-1}\|\sum_{n=0}^{\infty}\|b^{-1}\eta(w)\|^n \\ &< \frac{\theta}{1-\theta\sigma} \cdot \frac{1}{\|a\|},\end{aligned}$$

and shows, that we have a well-defined and holomorphic mapping

$$\mathcal{F} : \Omega' \to \mathrm{M}_m(\mathbb{C}), \; w \mapsto (b - \eta(w))^{-1}$$

with

$$\|\mathcal{F}(w)\| = \|(b - \eta(w))^{-1}\| < \frac{\theta}{1 - \theta\sigma} \cdot \frac{1}{\|a\|} < \frac{\theta}{1 - \theta} \cdot \frac{1}{\|a\|}$$

and therefore $\mathcal{F}(w) \in \Omega'$.

Now, we want to show that $\mathcal{F}(\Omega')$ lies strictly inside Ω'. We put

$$\epsilon := \min\left\{\frac{1}{2} \cdot \frac{1}{\|b\| + \sigma\|a\|}, \left(1 - \frac{1 - \theta}{1 - \theta\sigma}\right) \cdot \frac{\theta}{1 - \theta} \cdot \frac{1}{\|a\|}\right\} > 0$$

and consider $w \in \Omega'$ and $u \in \mathrm{M}_m(\mathbb{C})$ with $\|u - \mathcal{F}(w)\| < \epsilon$. At first, we get

$$\|b - \eta(w)\| \leq \|b\| + \|\eta(w)\| \leq \|b\| + \frac{M}{\|a\|} \cdot \frac{\theta}{1 - \theta} \leq \|b\| + \sigma\|a\|$$

and thus

$$\|u - (b - \eta(w))^{-1}\| = \|u - \mathcal{F}(w)\| < \epsilon \leq \frac{1}{2} \cdot \frac{1}{\|b\| + \sigma\|a\|} \leq \frac{1}{2} \cdot \frac{1}{\|b - \eta(w)\|},$$

which shows $u \in \mathrm{GL}_m(\mathbb{C})$, and secondly

$$\begin{aligned}\|u\| &= \|u - (b - \eta(w))^{-1}\| + \|\mathcal{F}(w)\| \\ &< \epsilon + \frac{1 - \theta}{1 - \theta\sigma} \cdot \frac{\theta}{1 - \theta} \cdot \frac{1}{\|a\|} \\ &< \frac{\theta}{1 - \theta} \cdot \frac{1}{\|a\|}\end{aligned}$$

which shows $u \in \Omega'$.

Let now $w \in \Omega'$ be a solution of (5). This implies that

$$w^{-1}\mathcal{F}(w) = w^{-1}(b - \eta(w))^{-1} = \big(bw - \eta(w)w\big)^{-1} = 1,$$

and hence $\mathcal{F}(w) = w$. Since $\mathcal{F} : \Omega' \to \Omega'$ is holomorphic on the domain Ω' and $\mathcal{F}(\Omega')$ lies strictly inside Ω', it follows by the Theorem of Earle-Hamilton, Theorem 2.3, that $\mathcal{F}$ has exactly one fixed point. Because $G_a(b)$ (which is an element of Ω' by Lemma 3.7) solves (5) by assumption and hence is already a fixed point of $\mathcal{F}$, it follows $w = G_a(b)$ and we are done. □

Remark 3.10. Let $(\mathcal{A}', E')$ be an arbitrary operator-valued C^*-probability space with conditional expectation $E' : \mathcal{A}' \to \mathrm{M}_m(\mathbb{C})$. This provides us with a unital (and continuous) $*$-embedding $\iota : \mathrm{M}_m(\mathbb{C}) \to \mathcal{A}'$. In this section, we only considered the special embedding

$$\iota : \mathrm{M}_m(\mathbb{C}) \to \mathcal{A}, \; b \mapsto b \otimes 1,$$

which is given by the special structure $\mathcal{A} = \mathrm{M}_m(\mathbb{C}) \otimes \mathcal{C}$. But we can define matrix-valued resolvent sets, spectra and Cauchy transforms also in this more general framework. To be more precise, we put for all $a \in \mathcal{A}'$

$$\rho_m(a) := \{b \in \mathrm{M}_m(\mathbb{C}) \mid \iota(b) - a \text{ is invertible in } \mathcal{A}'\}$$

and $\sigma_m(a) := \mathrm{M}_m(\mathbb{C}) \backslash \rho_m(a)$ and

$$G_a : \rho_m(a) \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto E'\big[(\iota(b) - a)^{-1}\big].$$

We note, that all the results of this section stay valid in this general situation.

4 Multivariate Free Central Limit Theorem

4.1 Setting and First Observations

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence in the operator-valued probability space $(\mathcal{A}, E)$ with $\mathcal{A} = \mathrm{M}_m(\mathcal{C}) = \mathrm{M}_m(\mathbb{C}) \otimes \mathcal{C}$ and $E = \mathrm{id} \otimes \tau$, as defined in the previous section. We assume:

- All X_i's have the same $*$-distribution with respect to E and their first moments vanish, i.e. $E[X_i] = 0$.
- The X_i are $*$-free with amalgamation over $\mathrm{M}_m(\mathbb{C})$ (which means that the $*$-algebras $\mathcal{X}_i$, generated by $\mathrm{M}_m(\mathbb{C})$ and X_i, are free with respect to E).
- We have $\sup_{i\in\mathbb{N}} \|X_i\| < \infty$.

If we define the linear (and hence holomorphic) mapping

$$\eta : \mathrm{M}_m(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto E[X_i b X_i],$$

we easily get from the continuity of E, that

$$\|\eta(b)\| \le \Big(\sup_{i\in\mathbb{N}} \|X_i\|\Big)^2 \|b\| \qquad \text{for all } b \in \mathrm{M}_m(\mathbb{C})$$

holds. Hence we can find $M > 0$ such that $\|\eta(b)\| < M\|b\|$ holds for all $b \in \mathrm{M}_m(\mathbb{C})$. Moreover, we have for all $k \in \mathbb{N}$ and all $b_1, \dots, b_k \in \mathrm{M}_m(\mathbb{C})$

$$\sup_{i\in\mathbb{N}} \|E[X_i b_1 X_i \dots b_k X_i]\| \le \Big(\sup_{i\in\mathbb{N}} \|X_i\|\Big)^{k+1} \|b_1\| \cdots \|b_k\|.$$

Since $(X_i)_{i\in\mathbb{N}}$ is a sequence of centered free non-commutative random variables, Theorem 8.4 in [15] tells us that the sequence $(S_n)_{n\in\mathbb{N}}$ defined by

$$S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i, \qquad n \in \mathbb{N}$$

converges to an operator-valued semicircular element s. Moreover, we know from Theorem 4.2.4 in [12] that the operator-valued Cauchy transform G_s satisfies

$$bG_s(b) = 1 + \eta(G_s(b))G_s(b) \qquad \text{for all } b \in U_r,$$

where we put $U_r := \{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < r\} \subseteq \rho_m(s)$ for all suitably small $r > 0$.

By Proposition 7.1 in [9], the boundedness of the sequence $(X_i)_{i\in\mathbb{N}}$ guarantees boundedness of $(S_n)_{n\in\mathbb{N}}$ as well. In order to get estimates for the difference between the Cauchy transforms G_s and G_{S_n} we will also need the fact, that $(S_n)_{n\in\mathbb{N}}$ is bounded away from 0. The precise statement is part of the following lemma, which also includes a similar statement for

$$S_n^{[i]} := S_n - \frac{1}{\sqrt{n}} X_i = \frac{1}{\sqrt{n}} \sum_{\substack{j=1 \\ j\neq i}}^{n} X_j \qquad \text{for all } n \in \mathbb{N} \text{ and } 1 \leq i \leq n.$$

Lemma 4.1. *In the situation described above, we have for all $n \in \mathbb{N}$ and all $1 \leq i \leq n$*

$$\|S_n\| \geq \|\alpha\|^{\frac{1}{2}} \qquad \text{and} \qquad \|S_n^{[i]}\| \geq \sqrt{1 - \frac{1}{n}}\|\alpha\|^{\frac{1}{2}},$$

where $\alpha := E[X_i^* X_i] \in \mathrm{M}_m(\mathbb{C})$.

Proof. By the $*$-freeness of $X_1, X_2, \ldots,$ we have

$$E[X_i^* X_j] = E[X_i^*] \cdot E[X_j] = 0, \qquad \text{for } i \neq j$$

and thus

$$\|S_n\|^2 = \|S_n^* S_n\| \geq \|E[S_n^* S_n]\| = \frac{1}{n}\left\| \sum_{i,j=1}^{n} E[X_i^* X_j] \right\| = \|\alpha\|.$$

Similarly

$$\begin{aligned} \|S_n^{[i]}\|^2 &= \|(S_n^{[i]})^* S_n^{[i]}\| \\ &\geq \|E[(S_n^{[i]})^* S_n^{[i]}]\| \\ &= \left\| E[S_n^* S_n] - \frac{1}{n} E[X_i^* X_i] \right\| \\ &= \frac{n-1}{n}\|\alpha\|, \end{aligned}$$

which proves the statement. □

We define for $n \in \mathbb{N}$

$$R_n : \rho_m(S_n) \to \mathcal{A},\ b \mapsto \big(b - S_n\big)^{-1}$$

and for $n \in \mathbb{N}$ and $1 \leq i \leq n$

$$R_n^{[i]} : \rho_m(S_n^{[i]}) \to \mathcal{A},\ b \mapsto \big(b - S_n^{[i]}\big)^{-1}.$$

Lemma 4.2. *For all $n \in \mathbb{N}$ and $1 \leq i \leq n$ we have*

$$R_n(b) = R_n^{[i]}(b) + \frac{1}{\sqrt{n}} R_n^{[i]}(b) X_i R_n^{[i]}(b) + \frac{1}{n} R_n(b) X_i R_n^{[i]}(b) X_i R_n^{[i]}(b) \tag{6}$$

and

$$R_n(b) = R_n^{[i]}(b) + \frac{1}{\sqrt{n}} R_n^{[i]}(b) X_i R_n(b) \tag{7}$$

for all $b \in \rho_m(S_n) \cap \rho_m(S_n^{[i]})$.

Proof. We have

$$\begin{aligned}\big(b - S_n\big) R_n(b) \big(b - S_n^{[i]}\big) &= b - S_n^{[i]} \\ &= \big(b - S_n\big) + \frac{1}{\sqrt{n}} \big(b - S_n^{[i]}\big) R_n^{[i]}(b) X_i \\ &= \big(b - S_n\big) + \frac{1}{\sqrt{n}} \big(b - S_n\big) R_n^{[i]}(b) X_i + \frac{1}{n} X_i R_n^{[i]}(b) X_i,\end{aligned}$$

which leads, by multiplication with $R_n(b) = (b - S_n)^{-1}$ from the left and with $R_n^{[i]}(b) = (b - S_n^{[i]})^{-1}$ from the right, to (6).

Moreover, we have

$$\big(b - S_n^{[i]}\big) R_n(b) \big(b - S_n\big) = b - S_n^{[i]} = \big(b - S_n\big) + \frac{1}{\sqrt{n}} X_i,$$

which leads, by multiplication with $R_n(b) = (b - S_n)^{-1}$ from the right and with $R_n^{[i]}(b) = (b - S_n^{[i]})^{-1}$ from the left, to equation (7). □

Obviously, we have

$$G_n := G_{S_n} = E \circ R_n \qquad \text{and} \qquad G_n^{[i]} := G_{S_n^{[i]}} = E \circ R_n^{[i]}.$$

4.2 Proof of the Main Theorem

During this subsection, let $0 < \theta, \sigma < 1$ and $c > 1$ be given, such that

$$\frac{\theta}{1-\theta} < \sigma \min\left\{\frac{1}{c}, \frac{\|\alpha\|}{M}\right\} \tag{8}$$

holds. For all $n \in \mathbb{N}$ we define

$$\kappa_n := \theta \min\left\{\frac{1}{\|s\|}, \frac{1}{\|S_n\|}, \frac{1}{\|S_n^{[1]}\|}, \dots, \frac{1}{\|S_n^{[n]}\|}\right\}$$

and

$$\Omega_n := \left\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa_n, \|b\| \cdot \|b^{-1}\| < c\right\}.$$

Lemma 3.4 shows, that Ω_n is a subset of $\rho_m(S_n)$.

Theorem 4.3. *For all $2 \leq n \in \mathbb{N}$ the function G_n satisfies the following equation*

$$\Lambda_n(b)G_n(b) = 1 + \eta(G_n(b))G_n(b), \qquad b \in \Omega_n,$$

where

$$\Lambda_n : \Omega_n \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto b - \Theta_n(b)G_n(b)^{-1},$$

with a holomorphic function

$$\Theta_n : \Omega_n \to \mathrm{M}_m(\mathbb{C})$$

satisfying

$$\sup_{b\in\Omega_n} \|\Theta_n(b)\| \leq \frac{C}{\sqrt{n}}$$

with a constant $C > 0$, independent of n.

Proof. (i) Let $n \in \mathbb{N}$ and $b \in \rho_m(S_n)$ be given. Then we have

$$S_n R_n(b) = bR_n(b) - (b - S_n)R_n(b) = bR_n(b) - 1$$

and hence

$$E[S_n R_n(b)] = E\left[bR_n(b) - 1\right] = bG_n(b) - 1.$$

(ii) Let $n \in \mathbb{N}$ be given. For all

$$b \in \rho_{m,n} := \rho_m(S_n) \cap \bigcap_{i=1}^{n} \rho_m(S_n^{[i]})$$

we deduce from the formula in (6), that

$$E[S_n R_n(b)] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[X_i R_n(b)]$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(E\big[X_i R_n^{[i]}(b)\big] + \frac{1}{\sqrt{n}} E\big[X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big]$$

$$+ \frac{1}{n} E\big[X_i R_n(b) X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big]\Big)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Big(E\big[X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big] + \frac{1}{\sqrt{n}} E\big[X_i R_n(b) X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big]\Big)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Big(E\big[X_i G_n^{[i]}(b) X_i\big] G_n^{[i]}(b) + \frac{1}{\sqrt{n}} E\big[X_i R_n(b) X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big]\Big)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Big(\eta(G_n^{[i]}(b)) G_n^{[i]}(b) + r_{n,1}^{[i]}(b)\Big),$$

where

$$r_{n,1}^{[i]} : \rho_m(S_n) \cap \rho_m(S_n^{[i]}) \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto \frac{1}{\sqrt{n}} E\big[X_i R_n(b) X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big].$$

There we used the fact, that, since the $(X_j)_{j\in\mathbb{N}}$ are free with respect to E, also X_i is free from $R_n^{[i]}$, and thus we have

$$E\big[X_i R_n^{[i]}(b)\big] = E[X_i] E\big[R_n^{[i]}(b)\big] = 0$$

and

$$E\big[X_i R_n^{[i]}(b) X_i R_n^{[i]}(b)\big] = E\big[X_i E\big[R_n^{[i]}(b)\big] X_i\big] E\big[R_n^{[i]}(b)\big].$$

(iii) Taking (7) into account, we get for all $n \in \mathbb{N}$ and $1 \leq i \leq n$

$$G_n(b) = E\big[R_n(b)\big] = E\big[R_n^{[i]}(b)\big] + \frac{1}{\sqrt{n}} E\big[R_n^{[i]}(b) X_i R_n(b)\big] = G_n^{[i]}(b) - r_{n,2}^{[i]}(b)$$

and therefore

$$G_n^{[i]}(b) = G_n(b) + r_{n,2}^{[i]}(b)$$

for all $b \in \rho_m(S_n) \cap \rho_m(S_n^{[i]})$, where we put

$$r_{n,2}^{[i]} :\ \rho_m(S_n) \cap \rho_m(S_n^{[i]}) \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto -\frac{1}{\sqrt{n}} E\big[R_n^{[i]}(b) X_i R_n(b)\big].$$

(iv) The formula in (iii) enables us to replace $G_n^{[i]}$ in (ii) by G_n. Indeed, we get

$$\begin{aligned} E[S_n R_n(b)] &= \frac{1}{n} \sum_{i=1}^{n} \Big(\eta(G_n^{[i]}(b)) G_n^{[i]}(b) + r_{n,1}^{[i]}(b) \Big) \\ &= \frac{1}{n} \sum_{i=1}^{n} \Big(\eta\big(G_n(b) + r_{n,2}^{[i]}(b)\big)\big(G_n(b) + r_{n,2}^{[i]}(b)\big) + r_{n,1}^{[i]}(b) \Big) \\ &= \eta(G_n(b)) G_n(b) + \frac{1}{n} \sum_{i=1}^{n} r_{n,3}^{[i]}(b) \end{aligned}$$

for all $b \in \rho_{m,n}$, where the function

$$r_{n,3}^{[i]} : \ \rho_m(S_n) \cap \rho_m(S_n^{[i]}) \to \mathrm{M}_m(\mathbb{C})$$

is defined by

$$r_{n,3}^{[i]}(b) := \eta(G_n(b)) r_{n,2}^{[i]}(b) + \eta(r_{n,2}^{[i]}(b)) G_n(b) + \eta(r_{n,2}^{[i]}(b)) r_{n,2}^{[i]}(b) + r_{n,1}^{[i]}(b).$$

(v) Combining the results from (i) and (iv), it follows

$$b G_n(b) - 1 = E[S_n R_n(b)] = \eta(G_n(b)) G_n(b) + \Theta_n(b),$$

where we define

$$\Theta_n : \ \rho_{m,n} \to \mathrm{M}_m(\mathbb{C}), \ b \mapsto \frac{1}{n} \sum_{i=1}^{n} r_{n,3}^{[i]}(b).$$

Due to (8), Lemmas 3.4 and 3.7 show that $\Omega_n \subseteq \rho_{m,n}$ and $G_n(b) \in \mathrm{GL}_m(\mathbb{C})$ for $b \in \Omega_n$. This gives

$$\big(b - \Theta_n(b) G_n(b)^{-1}\big) G_n(b) = 1 + \eta(G_n(b)) G_n(b)$$

and hence, as desired, for all $b \in \Omega_n$

$$\Lambda_n(b) G_n(b) = 1 + \eta(G_n(b)) G_n(b).$$

(v) The definition of Ω_n gives, by Lemma 3 and by Lemma 4.1, the following estimates

$$\|G_n(b)\| \le \|R_n(b)\| \le \frac{\theta}{1-\theta} \cdot \frac{1}{\|S_n\|} \le \frac{\theta}{1-\theta} \cdot \frac{1}{\|\alpha\|^{\frac{1}{2}}}, \qquad b \in \Omega_n$$

and

$$\|G_n^{[i]}(b)\| \leq \|R_n^{[i]}(b)\| \leq \frac{\theta}{1-\theta} \cdot \frac{1}{\|S_n^{[i]}\|} \leq \frac{\theta}{1-\theta} \cdot \frac{1}{\sqrt{1-\frac{1}{n}\|\alpha\|^{\frac{1}{2}}}}, \qquad b \in \Omega_n.$$

Therefore, we have for all $b \in \Omega_n$ by (ii)

$$\|r_{n,1}^{[i]}(b)\| \leq \frac{1}{\sqrt{n}}\|X_i\|^3\|R_n(b)\|\|R_n^{[i]}(b)\|^2 \leq \frac{1}{\sqrt{n}}\frac{n}{n-1}\Big(\frac{\theta}{1-\theta}\frac{1}{\|\alpha\|^{\frac{1}{2}}}\Big)^3\|X_i\|^3$$

and by (iii)

$$\|r_{n,2}^{[i]}(b)\| \leq \frac{1}{\sqrt{n}}\|X_i\|\|R_n(b)\|\|R_n^{[i]}(b)\| \leq \frac{1}{\sqrt{n-1}}\Big(\frac{\theta}{1-\theta}\frac{1}{\|\alpha\|^{\frac{1}{2}}}\Big)^2\|X_i\|$$

and finally by (iv)

$$\begin{aligned}
\|r_{n,3}^{[i]}(b)\| &\leq 2M\|G_n(b)\|\|r_{n,2}^{[i]}(b)\| + M\|r_{n,2}^{[i]}(b)\|^2 + \|r_{n,1}^{[i]}(b)\| \\
&\leq \frac{1}{\sqrt{n-1}}\Big(\frac{\theta}{1-\theta}\frac{1}{\|\alpha\|^{\frac{1}{2}}}\Big)^3\|X_i\| \cdot \\
&\quad \Big(2M + \frac{1}{\sqrt{n-1}}M\Big(\frac{\theta}{1-\theta}\frac{1}{\|\alpha\|^{\frac{1}{2}}}\Big)\|X_i\| + \sqrt{\frac{n}{n-1}}\|X_i\|^2\Big) \\
&\leq \frac{C}{\sqrt{n}}
\end{aligned}$$

for all $b \in \Omega_n$, where $C > 0$ is a constant, which is independent of n. Hence, it follows from (v) that

$$\sup_{b \in \Omega_n} \|\Theta_n(b)\| \leq \frac{C}{\sqrt{n}}.$$

□

The definition of Ω_n ensures, that

$$G := G_s : \ \rho_m(s) \to \mathrm{M}_m(\mathbb{C})$$

satisfies

$$bG(b) = 1 + \eta(G(b))G(b) \qquad \text{for all } b \in \Omega,$$

where

$$\Omega := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \theta \cdot \frac{1}{\|s\|},\ \|b\| \cdot \|b^{-1}\| < c\Big\} \supseteq \Omega_n.$$

We choose

$$0 < \gamma < \frac{c-1}{c+1} \qquad \text{and} \qquad 0 < \theta^* < (1-\gamma)\theta \tag{9}$$

(note, that $0 < \gamma < 1$) and we put $c^* := c - (1+c)\gamma$, which fulfills clearly $1 < c^* < c$. Since we have $\theta^* < \theta$ and $c^* < c$, we see

$$\frac{\theta^*}{1-\theta^*}c^* < \frac{\theta}{1-\theta}c < \sigma$$

and hence

$$\frac{\theta^*}{1-\theta^*} < \frac{\sigma}{c^*}. \tag{10}$$

Finally, we define

$$\kappa_n^* := \theta^* \min\left\{\frac{1}{\|s\|}, \frac{1}{\|S_n\|}, \frac{1}{\|S_n^{[1]}\|}, \dots, \frac{1}{\|S_n^{[n]}\|}\right\}$$

and

$$\Omega_n^* := \left\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa_n^*,\ \|b\| \cdot \|b^{-1}\| < c^*\right\} \subseteq \Omega_n.$$

Corollary 4.4. *There exists $N \in \mathbb{N}$ such that*

$$\Lambda_n(\Omega_n^*) \subseteq \Omega_n \qquad \text{for all } n \geq N.$$

Proof. Since we have by Theorem 4.3

$$\sup_{b\in\Omega_n} \|\Theta_n(b)\| \leq \frac{C}{\sqrt{n}}$$

for all $2 \leq n \in \mathbb{N}$, we can choose an $N \in \mathbb{N}$ such that

$$\sup_{b\in\Omega_n} \|\Theta_n(b)\| \leq \frac{\gamma}{c^*}(1-\sigma)$$

holds for all $n \geq N$. Now, we get for all $b \in \Omega_n^*$:

(i) $\Lambda_n(b)$ is invertible: Since (4) gives

$$\|G_n(b)^{-1}\| \leq \frac{1}{1-\sigma}\|b\| \qquad \text{for all } b \in \Omega_n,$$

we immediately get

$$\|\Lambda_n(b) - b\| \leq \|\Theta_n(b)\|\|G_n(b)^{-1}\| < \gamma\frac{\|b\|}{c^*} < \gamma\frac{1}{\|b^{-1}\|} < \frac{1}{\|b^{-1}\|}$$

(ii) We have $\|\Lambda_n(b)^{-1}\| < \kappa_n$: Using Lemma 3.1, we get from (i) that

$$\|\Lambda_n(b)^{-1}\| \leq \frac{1}{1-\gamma}\|b^{-1}\| < \frac{\kappa_n^*}{1-\gamma} < \kappa_n.$$

iii) We have $\|\Lambda_n(b)\|\|\Lambda_n(b)^{-1}\| < c$: Using

$$\|\Lambda_n(b) - b\| < \gamma\frac{\|b\|}{c^*}$$

from (i) and

$$\|\Lambda_n(b)^{-1}\| < \frac{1}{1-\gamma}\|b^{-1}\|$$

from (ii), we get

$$\begin{aligned}\|\Lambda_n(b)\|\|\Lambda_n(b)^{-1}\| &\leq \big(\|b\| + \|\Lambda_n(b) - b\|\big)\|\Lambda_n(b)^{-1}\| \\ &< \Big(1 + \frac{\gamma}{c^*}\Big)\frac{1}{1-\gamma} \cdot \|b\|\|b^{-1}\| \\ &< \frac{c^* + \gamma}{1-\gamma} < c.\end{aligned}$$

Finally, this shows $\Lambda_n(b) \in \Omega_n$. □

Corollary 4.5. *For all $n \geq N$ we have*

$$G_n(b) = G(\Lambda_n(b)) \qquad \textit{for all } b \in \Omega_n^*.$$

Proof. For all $n \in \mathbb{N}$ we define

$$\Omega_n' := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b\| < \frac{\kappa_n}{1-\theta}\Big\}.$$

Let $n \geq N$ and $b \in \Omega_n^*$ be given. We know, that

$$\Lambda_n(b)G(\Lambda_n(b)) = 1 + \eta(G(\Lambda_n(b)))G(\Lambda_n(b))$$

holds, i.e. $w = G(\Lambda_n(b)) \in \Omega_n'$ is a solution of the equation

$$\Lambda_n(b)w = 1 + \eta(w)w, \qquad w \in \Omega_n'.$$

Combining (8) with Lemma 4.1, we get

$$\frac{\theta}{1-\theta} < \sigma \min\Big\{\frac{1}{c}, \frac{\|\alpha\|}{M}\Big\} \leq \sigma \min\Big\{\frac{1}{c}, \frac{\|S_n\|^2}{M};\ n \in \mathbb{N}\Big\}.$$

Hence, the equation above has, by Theorem 3.9, the unique solution $w = G_n(b) \in \Omega_n'$. This implies, as desired, $G_n(b) = G(\Lambda_n(b))$. □

Corollary 4.6. *For all $n \geq N$ we have*

$$\|G(b) - G_n(b)\| \leq C' \frac{1}{\sqrt{n}} \|b\| \qquad \textit{for all } b \in \Omega_n^*,$$

where $C' > 0$ is a constant independent of n.

Proof. For all $b \in \Omega_n^* \subseteq \Omega_n \subseteq \Omega$ we have

$$\begin{aligned} G(b) - G_n(b) &= G(b) - G(\Lambda_n(b)) \\ &= E\big[(b-s)^{-1} - (\Lambda_n(b) - s)^{-1}\big] \\ &= E\big[(b-s)^{-1}(\Lambda_n(b) - b)(\Lambda_n(b) - s)^{-1}\big] \end{aligned}$$

and therefore by (4), which gives

$$\|G_n(b)^{-1}\| \leq \frac{1}{1-\sigma} \|b\| \qquad \text{for all } b \in \Omega_n^*,$$

and (since $\Lambda_n(b) \in \Omega_n \subseteq \Omega$) by (3)

$$\begin{aligned} \|G(b) - G_n(b)\| &\leq \|(b-s)^{-1}\| \cdot \|\Lambda_n(b) - b\| \cdot \|(\Lambda_n(b) - s)^{-1}\| \\ &\leq \Big(\frac{\theta}{1-\theta} \cdot \frac{1}{\|s\|}\Big)^2 \cdot \|\Theta_n(b)\| \cdot \|G_n(b)^{-1}\| \\ &\leq C' \frac{1}{\sqrt{n}} \|b\|, \end{aligned}$$

where

$$C' := \frac{C}{1-\sigma}\Big(\frac{\theta}{1-\theta} \cdot \frac{1}{\|s\|}\Big)^2 > 0.$$

This proves the corollary. □

We recall, that the sequence $(X_i)_{i\in\mathbb{N}}$ is bounded, which implies boundedness of the sequence $(S_n)_{n\in\mathbb{N}}$ as well. This has the important consequence, that

$$\kappa_n^* = \theta^* \min\Big\{\frac{1}{\|s\|}, \frac{1}{\|S_n\|}, \frac{1}{\|S_n^{[1]}\|}, \ldots, \frac{1}{\|S_n^{[n]}\|}\Big\} \geq \kappa^*$$

for some $\kappa^* > 0$. If we define

$$\Omega^* := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa^*,\ \|b\| \cdot \|b^{-1}\| < c^*\Big\},$$

we easily see $\Omega^* \subseteq \Omega_n^*$ for all $n \in \mathbb{N}$. Hence, by renaming Ω^* to Ω etc., we have shown our main Theorem 1.1.

We conclude this section with the following remark about the geometric structure of subsets of $\mathrm{M}_m(\mathbb{C})$ like Ω.

Lemma 4.7. *For $\kappa > 0$ and $c > 1$ we consider*

$$\Omega := \Big\{b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa,\ \|b\| \cdot \|b^{-1}\| < c\Big\}.$$

For $\lambda, \mu \in \mathbb{C}\backslash\{0\}$ we define

$$\Lambda(\lambda, \mu) := \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \mu & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu \end{pmatrix} \in \mathrm{GL}_m(\mathbb{C}).$$

If $\frac{1}{\kappa} < |\mu|$ holds, we have $\Lambda(\lambda, \mu) \in \Omega$ for all

$$\max\left\{\frac{1}{\kappa}, \frac{|\mu|}{c}\right\} < |\lambda| < c|\mu|. \tag{11}$$

Particularly, we have for all $|\lambda| > \frac{1}{\kappa}$, that $\lambda 1 \in \Omega$.

Proof. Let $\mu \in \mathbb{C}\backslash\{0\}$ with $\frac{1}{\kappa} < |\mu|$ be given. For all $\lambda \in \mathbb{C}\backslash\{0\}$, which satisfy (11), we get

$$\|\Lambda(\lambda, \mu)^{-1}\| = \|\Lambda(\lambda^{-1}, \mu^{-1})\| = \max\left\{|\lambda|^{-1}, |\mu|^{-1}\right\} < \kappa.$$

and

$$\begin{aligned} \|\Lambda(\lambda, \mu)\| \cdot \|\Lambda(\lambda, \mu)^{-1}\| &= \max\left\{|\lambda|, |\mu|\right\} \cdot \max\left\{|\lambda|^{-1}, |\mu|^{-1}\right\} \\ &= \begin{cases} |\mu||\lambda|^{-1}, & \text{if } |\lambda| < |\mu| \\ |\lambda||\mu|^{-1}, & \text{if } |\lambda| \geq |\mu| \end{cases} \\ &< c, \end{aligned}$$

which implies $\Lambda(\lambda, \mu) \in \Omega$. In particular, for $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\lambda| > \frac{1}{\kappa}$ we see that $\mu = \lambda$ fulfills (11) and it follows $\lambda 1 = \Lambda(\lambda, \lambda) \in \Omega$. □

4.3 *Application to Multivariate Situation*

4.3.1 Multivariate Free Central Limit Theorem

Let $(x_i^{(k)})_{k=1}^d$, $i \in \mathbb{N}$, be free and identically distributed sets of d self-adjoint non-zero random variables in some non-commutative C^*-probability space $(\mathcal{C}, \tau)$, with τ faithful, such that

$$\tau(x_i^{(k)}) = 0 \qquad \text{for } k = 1, \dots, d \text{ and all } i \in \mathbb{N}$$

and

$$\sup_{i \in \mathbb{N}} \max_{k=1,\dots,d} \|x_i^{(k)}\| < \infty. \tag{12}$$

We denote by $\Sigma = (\sigma_{k,l})_{k,l=1}^d$, where $\sigma_{k,l} := \tau(x_i^{(k)} x_i^{(l)})$, their joint covariance matrix. Moreover, we put

$$S_n^{(k)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i^{(k)} \qquad \text{for } k = 1, \dots, d \text{ and all } n \in \mathbb{N}.$$

We know (cf. [11]), that $(S_n^{(1)}, \dots, S_n^{(d)})$ converges in distribution as $n \to \infty$ to a semicircular family $(s_1, \dots, s_d)$ of covariance Σ. For notational convenience we will assume that $s_1, \dots, s_d$ live also in $(\mathcal{C}, \tau)$; this can always be achieved by enlarging $(\mathcal{C}, \tau)$.

Using Proposition 2.1 and Proposition 2.3 in [6], for each polynomial p of degree g in d non-commuting variables vanishing in 0, we can find $m \in \mathbb{N}$ and $a_1, \dots, a_d \in \mathrm{M}_m(\mathbb{C})$ such that

$$\lambda 1 - p(S_n^{(1)}, \dots, S_n^{(d)}) \quad \text{and} \quad \lambda 1 - p(s_1, \dots, s_d)$$

are invertible in $\mathcal{C}$ if and only if

$$\Lambda(\lambda, 1) - S_n \quad \text{and} \quad \Lambda(\lambda, 1) - s,$$

respectively, are invertible in $\mathcal{A} = \mathrm{M}_m(\mathcal{C})$. The matrices $\Lambda(\lambda, 1) \in \mathrm{M}_m(\mathbb{C})$ were defined in Lemma 4.7, and S_n and s are defined as follows:

$$S_n := \sum_{k=1}^d a_k \otimes S_n^{(k)} \in \mathcal{A} \qquad \text{for all } n \in \mathbb{N}$$

and

$$s := \sum_{k=1}^d a_k \otimes s_k \in \mathcal{A}.$$

If we also put

$$X_i := \sum_{k=1}^{d} a_k \otimes x_i^{(k)} \in \mathcal{A} \qquad \text{for all } i \in \mathbb{N},$$

then we have

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i.$$

We note, that the sequence $(X_i)_{i\in\mathbb{N}}$ is $*$-free with respect to the conditional expectation $E : \mathcal{A} = \mathrm{M}_m(\mathcal{C}) \to \mathrm{M}_m(\mathbb{C})$ and that all the X_i's have the same $*$-distribution with respect to E and that they satisfy $E[X_i] = 0$. In addition, (12) implies $\sup_{i\in\mathbb{N}} \|X_i\| < \infty$. Hence, the conditions of Theorem 1.1 are fulfilled. But before we apply it, we note that $(S_n)_{n\in\mathbb{N}}$ converges in distribution (with respect to E) to s, which is an $\mathrm{M}_m(\mathbb{C})$-valued semicircular element with covariance mapping

$$\eta :\ \mathrm{M}_m(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C}),\ b \mapsto E[sbs],$$

which is given by

$$\eta(b) = E[sbs] = \sum_{k,l=1}^{d} \mathrm{id} \otimes \tau[(a_k \otimes s_k)(b \otimes 1)(a_l \otimes s_l)] = \sum_{k,l=1}^{d} a_k b a_l \sigma_{k,l}.$$

Now, we get from Theorem 1.1 constants $\kappa^* > 0$, $c^* > 0$ and $C' > 0$ and $N \in \mathbb{N}$ such that we have for the difference of the operator-valued Cauchy transforms

$$G_s(b) := E[(b-s)^{-1}] \quad \text{and} \quad G_{S_n}(b) := E[(b-S_n)^{-1}]$$

the estimate

$$\|G_s(b) - G_{S_n}(b)\| \le C' \frac{1}{\sqrt{n}} \|b\| \qquad \text{for all } b \in \Omega^* \text{ and } n \ge N,$$

where we put

$$\Omega^* := \Big\{ b \in \mathrm{GL}_m(\mathbb{C}) \mid \|b^{-1}\| < \kappa^*,\ \|b\| \cdot \|b^{-1}\| < c^* \Big\}.$$

Moreover, Proposition 2.3 in [6] tells us

$$\big(\lambda 1 - p(S_n^{(1)}, \dots, S_n^{(d)})\big)^{-1} = (\pi \otimes \mathrm{id}_{\mathcal{C}})\big((\Lambda(\lambda,1) - S_n)^{-1}\big)$$

and

$$\big(\lambda 1 - p(s_1, \dots, s_d)\big)^{-1} = (\pi \otimes \mathrm{id}_{\mathcal{C}'})\big((\Lambda(\lambda,1) - s)^{-1}\big),$$

where $\pi : \mathrm{M}_m(\mathbb{C}) \to \mathbb{C}$ is the mapping given by $\pi((a_{i,j})_{i,j=1,\dots,m}) := a_{1,1}$. Since $\tau \circ (\pi \otimes \mathrm{id}_{\mathcal{C}}) = \pi \circ E$, this implies a direct connection between the operator-valued Cauchy transforms of S_n and s and the scalar-valued Cauchy transforms of

$P_n := p(S_n^{(1)}, \dots, S_n^{(d)})$ and $P := p(s_1, \dots, s_d)$, respectively. To be more precise, we get

$$G_{P_n}(\lambda) := \tau[(\lambda - P_n)^{-1}] = \pi\big(G_{S_n}(\Lambda(\lambda, 1))\big)$$

and

$$G_P(\lambda) := \tau[(\lambda - P)^{-1}] = \pi\big(G_s(\Lambda(\lambda, 1))\big)$$

for all $\lambda \in \rho_{\mathcal{C}}(P_n)$ and $\lambda \in \rho_{\mathcal{C}}(P)$, respectively.

If we choose $\mu \in \mathbb{C}$ such that $|\mu| > \frac{1}{\kappa^*}$ holds, it follows from Lemma 4.7, that $\Lambda(\lambda, \mu) \in \Omega^*$ is fulfilled for all $\lambda \in A(\mu)$, where $A(\mu) \subseteq \mathbb{C}$ denotes the open set of all $\lambda \in \mathbb{C}$ satisfying (11), i.e.

$$A(\mu) := \Big\{\lambda \in \mathbb{C} \mid \max\Big\{\frac{1}{\kappa^*}, \frac{|\mu|}{c^*}\Big\} < |\lambda| < c^*|\mu|\Big\}.$$

If we apply Propositions 2.1 and 2.2 in [6] to the polynomial $\frac{1}{\mu^g}p$ (which corresponds to the operators $\frac{1}{\mu}S_n$ and $\frac{1}{\mu}S$), we easily deduce that

$$\lambda 1 - \frac{1}{\mu^{g-1}} p(S_n^{(1)}, \dots, S_n^{(d)}) \quad \text{and} \quad \lambda 1 - \frac{1}{\mu^{g-1}} p(s_1, \dots, s_d)$$

are invertible in $\mathcal{C}$ if and only if

$$\Lambda(\lambda, \mu) - S_n \quad \text{and} \quad \Lambda(\lambda, \mu) - S_n,$$

respectively, are invertible in $\mathcal{A}$. Moreover, we have

$$\mu^{g-1} G_{P_n}(\lambda \mu^{g-1}) = \pi\big(G_{S_n}(\Lambda(\lambda, \mu))\big)$$

and

$$\mu^{g-1} G_P(\lambda \mu^{g-1}) = \pi\big(G_s(\Lambda(\lambda, \mu))\big)$$

for all $\lambda \in \rho_{\mathcal{C}}(\frac{1}{\mu^{g-1}} P_n)$ and $\lambda \in \rho_{\mathcal{C}}(\frac{1}{\mu^{g-1}} P)$, respectively.

Particularly, for all $\lambda \in A(\mu)$ we get $\Lambda(\lambda, \mu) \in \Omega^*$ and hence $\lambda \in \rho_{\mathcal{C}}(\frac{1}{\mu^{g-1}} P_n) \cap \rho_{\mathcal{C}}(\frac{1}{\mu^{g-1}} P)$ for all $n \geq N$. Therefore, Theorem 1.1 implies

$$\begin{aligned}
|\mu|^{g-1} |G_P(\lambda \mu^{g-1}) - G_{P_n}(\lambda \mu^{g-1})| &= \big|\pi\big(G_s(\Lambda(\lambda, \mu)) - G_{S_n}(\Lambda(\lambda, \mu))\big)\big| \\
&\leq \big\|G_s(\Lambda(\lambda, \mu)) - G_{S_n}(\Lambda(\lambda, \mu))\big\| \\
&\leq C' \frac{1}{\sqrt{n}} \|\Lambda(\lambda, \mu)\| \\
&\leq C' \frac{1}{\sqrt{n}} \max\{|\lambda|, |\mu|\} \\
&\leq C' c^* |\lambda| \frac{1}{\sqrt{n}}
\end{aligned}$$

and hence

$$|G_P(\lambda\mu^{g-1}) - G_{P_n}(\lambda\mu^{g-1})| \leq C'c^* \frac{1}{\sqrt{n}} |\lambda\mu^{g-1}|.$$

This means, that

$$|G_P(z) - G_{P_n}(z)| \leq C'c^* \frac{1}{\sqrt{n}} |z|$$

holds for all $z \in \mathbb{C}$ with $\frac{z}{\mu^{g-1}} \in A(\mu)$ and all $n \geq N$. By definition of $A(\mu)$, we particularly get

$$|G_P(z) - G_{P_n}(z)| \leq C \frac{1}{\sqrt{n}} \qquad \text{for all } \frac{1}{c^*}|\mu|^g < |z| < c^*|\mu|^g \text{ and } n \geq N,$$

where we put $C := C'(c^*)^2|\mu|^g > 0$. Since $z \mapsto G_P(z) - G_{P_n}(z)$ is holomorphic on $\{z \in \mathbb{C} \mid |z| > R\}$ for $R := \frac{1}{c^*}|\mu|^g > 0$ and extends holomorphically to ∞, the maximum modulus principle gives

$$|G_P(z) - G_{P_n}(z)| \leq C \frac{1}{\sqrt{n}} \qquad \text{for all } |z| > R \text{ and } n \geq N.$$

This shows Theorem 1.2 in the case of a polynomial p vanishing in 0. For a general polynomial p, we consider the polynomial $\tilde{p} = p - p_0$ with $p_0 := p(0, \ldots, 0)$, which leads to the operators $\tilde{P} = P - p_0 1$ and $\tilde{P}_n = P_n - p_0 1$. Since we can apply the result above to $\tilde{p}$ and since the Cauchy transforms G_P and G_{P_n} are just translations of $G_{\tilde{P}}$ and $G_{\tilde{P}_n}$, respectively, the general statement follows easily.

4.3.2 Estimates in Terms of the Kolmogorov Distance

In the classical case, estimates between scalar-valued Cauchy transforms can be established (for self-adjoint operators) in all of the upper complex plane and lead then to estimates in terms of the Kolmogorov distance. In the case treated above, we have a statement about the behavior of the difference between two Cauchy transforms only near infinity. Even in the case, where our operators are self-adjoint, we still have to transport estimates from infinity to the real line, and hence we can not apply the results of Bai [1] directly. A partial solution to this problem was given in the appendix of [14] with the following theorem, formulated in terms of probability measures instead of operators. There we use the notation G_μ for the Cauchy transform of the measure μ, and put

$$D_R^+ := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0,\ |z| > R\}.$$

Theorem 4.8. *Let μ be a probability measure with compact support contained in an interval $[-A, A]$ such that the cumulative distribution function $\mathcal{F}_\mu$ satisfies*

$$|\mathcal{F}_\mu(x+t) - \mathcal{F}_\mu(x)| \leq \rho|t| \qquad \text{for all } x, t \in \mathbb{R}$$

for some constant $\rho > 0$. Then for all $R > 0$ and $\beta \in (0, 1)$ we can find $\Theta > 0$ and $m_0 > 0$ such that for any probability measure ν with compact support contained in $[-A, A]$, which satisfies

$$\sup_{z \in D_R^+} |G_\mu(z) - G_\nu(z)| \leq e^{-m}$$

for some $m > m_0$, the Kolmogorov distance $\Delta(\mu, \nu) := \sup_{x \in \mathbb{R}} |\mathcal{F}_\mu(x) - \mathcal{F}_\nu(x)|$ fulfills

$$\Delta(\mu, \nu) \leq \Theta \frac{1}{m^\beta}.$$

Obviously, this leads to the following questions: First, the stated estimate for the speed of convergence in terms of the Kolmogorov distance is far from the expected one. We hope to improve this result in a future work. Furthermore, in order to apply this theorem, we have to ensure that $p(s_1, \ldots, s_d)$ has a continuous density. As mentioned in the introduction, it is a still unsolved problem, whether this is always true for any self-adjoint polynomials p.

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