CLT for Stationary Normal Markov Chains via Generalized Coboundaries

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Dedicated to Friedrich Götze on the occasion of his sixtieth birthday

Abstract Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary Markov chain with a stationary probability distribution μ on the state space of X and the transition operator $Q : L_2(\mu) \to L_2(\mu)$. Let $f \in L_2(\mu)$ be a function on the state space of X. The solvability in $L_2(\mu)$ of the *Poisson equation* $f = g - Qg$ implies that the stationary sequence $(f(X_n))_{n\in\mathbb{Z}}$ can be represented in the form

$$f(X_n) = \big(g(X_{n+1}) - (Qg)(X_n)\big) + \big(g(X_n) - g(X_{n+1})\big) = \eta_n + \zeta_n \; (n \in \mathbb{Z}).$$

Here $\eta = (\eta_n)_{n\in\mathbb{Z}}$ is a stationary sequence of square integrable *martingale differences*, and $\zeta = (\zeta_n)_{n\in\mathbb{Z}}$ is an L_2-*coboundary* that is a difference of two consecutive elements of a stationary sequence of square integrable random variables. This representation reduces the Central Limit Theorem (CLT) question for $(f(X_n))_{n\in\mathbb{Z}}$ to the well-studied case of martingale differences. However, in many situations the martingale approximation as a tool in limit theorems works well, though the above martingale-coboundary representation does not hold. In particular, if the transition operator Q is *normal* in $L_2(\mu)$, 1 is a simple eigenvalue of Q, and the assumptions

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(1) $\sigma_f^2 = \int_D \frac{1-|z|^2}{|1-z|^2}\rho_f\, dz < \infty$,
(2) $\lim_{n\to\infty} n^{-\frac{1}{2}} | \sum_{k=0}^{n-1} Q^k f|_2 = 0$

hold true for a real-valued function $f \in L_2(\mu)$, the Central Limit Theorem for $(f(X_n))_{n\in\mathbb{Z}}$ was established via the martingale approximation.

In the present paper we show that under condition (1) $(f(X_n))_{n\in\mathbb{Z}}$ admits a generalized form of the martingale-coboundary representation as the sum of a square integrable stationary martingale difference and a *generalized coboundary*. The latter is a stationary sequence of random variables which are increments of a stationary sequence of *m-functions* introduced in the paper. Furthermore, it turns out that assumption (2) means exactly that the generalized coboundary can be neglected in the limit. Connection with generalized solutions to the Poisson equation is also studied.

Keywords Generalized coboundary • Limit theorems • Markov chain • Martingale approximation • Normal transition operator • Poisson equation

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1 Introduction

Let $\eta = (\eta_n)_{n\in\mathbb{Z}}$ be a stationary (in the strict sense) sequence of integrable random variables. Assume also that η is a sequence of *martingale differences* that is for every n

$$\mathbb{E}(\eta_n|\eta_{n-1}, \eta_{n-2}, \dots) = 0.$$

Let, moreover, η be ergodic, real-valued and $\mathbb{E}\eta_n^2 < \infty$. Then, according to the classical results of Billingsley [1] and Ibragimov [11], the sequence η satisfies the Central Limit Theorem (CLT). It is known that under the same conditions also the Functional Central Limit Theorem (FCLT) and the Law of the Iterated Logarithm (including its functional form due to Strassen) are valid. Under appropriate assumptions some of these results extend to not necessarily stationary sequences or arrays of martingale differences.

A natural idea is to use a certain approximation by martingales (that is the sums of martingale differences) to establish limit theorems of the above-mentioned type for the sums of dependent random variables more general than martingale differences. More precisely, one needs to construct a martingale difference approximation of the random sequence in question and represent the error of this approximation in a form which allows us, under the appropriate normalization, to neglect by this error in the limit. In the stationary setup an approach to this problem was proposed in [7] basing on the so-called martingale-coboundary representation. The latter means that a stationary sequence $\xi = (\xi_n)_{n\in\mathbb{Z}}$ admits the representation

$$\xi_n = \eta_n + \zeta_n, n \in \mathbb{Z}, \tag{1}$$

where $\eta = (\eta_n)_{n\in\mathbb{Z}}$ is a stationary sequence of *martingale differences*, and $\zeta = (\zeta_n)_{n\in\mathbb{Z}}$ is a so-called *coboundary* which can be written as

$$\zeta_n = \theta_n - \theta_{n-1}, n \in \mathbb{Z}, \tag{2}$$

with a certain stationary sequence $\theta = (\theta_n)_{n\in\mathbb{Z}}$. One says in this case that ζ *is a coboundary of* θ; we speak of a B*-coboundary* if each of θ_n in the above representation belongs to a certain Banach space B of random variables. It is assumed that the random sequences ξ, η, θ in this representation are defined on a common probability space so that they are jointly stationary. A convenient way to formulate this type of interrelation between random sequences (which does not lead to any loss of generality) is to assume that a probability preserving invertible map T acts on the basic probability space so that

$$\zeta_{n+1} = \zeta_n \circ T,\ \eta_{n+1} = \eta_n \circ T,\ \theta_{n+1} = \theta_n \circ T\ (n \in \mathbb{Z}).$$

As to the asymptotic distributions of the sums

$$\sum_{k=0}^{n-1} \xi_k, n \geq 1, \tag{3}$$

normalized by dividing by positive reals tending to ∞, it is clear that one can neglect by the contribution of the sequence ζ into these sums and extend to $\xi = \eta + \zeta$ certain limit theorems originally known to hold for the martingale difference η. To deduce the martingale-coboundary representation for a stationary sequence ξ, some conditions need to be imposed on ξ. These conditions are usually stated in terms of a *compatible filtration* $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ (that is a family of sub-σ-fields satisfying $\cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \cdots$ and $T^{-1}\mathcal{F}_n = \mathcal{F}_{n+1}$) on the basic probability space. Specifying such a filtration is a standard prerequisite to develop the martingale approximation for stationary sequences. Given such a filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$, we need to distinguish between a general *non-adapted* sequence and an *adapted* sequence $\xi = (\xi_n)$ where ξ_n is $\mathcal{F}_n$-measurable for every $n \in \mathbb{Z}$. The latter case can be treated easier and is equivalent to the study of functions of a stationary Markov chain with a general measurable state space. Though only very special Markov chains emerge in this context, and, on the other hand, both adapted and non-adapted cases can be studied, basing on the martingale-coboundary representation and without any reference to markovianity [7], it is the whole class of general Markov chains where the application of the martingal-coboundary decomposition can be done in a very natural, simple and elegant way in terms of a condition related to the so-called

Poisson equation. More specifically, let $X = (X_n)_{n\in\mathbb{Z}}$, μ and $Q : L_2(\mu) \to L_2(\mu)$ be, respectively, a stationary Markov chain, its stationary probability distribution and its transition operator. Let $f \in L_2(\mu)$ be a function on the state space of X. Then the solvability in $g \in L_2$ of the *Poisson equation*

$$f = g - Qg \tag{4}$$

implies the applicability of the above mentioned martingale-coboundary representation to the stationary sequence $(f(X_n))_{n\in\mathbb{Z}}$ (see Abstract for an explanation; notice that the converse is also true). Moreover, in this case η turns out to be an L_2-*coboundary* (that is a coboundary of a square integrable sequence θ). In the context of limit theorems this was independently observed in [13] (for the particular case of Harris recurrent chains) and in [8] (for the general ergodic case).

Now we will explain the topic of the present paper. It was recognized during the last three decades that the martingale approximation as a tool in limit theorems is still effective in an area where the martingale-coboundary representation with an L_2-coboundary not always holds. This means that, under some assumptions, ζ in representation (1) needs not be an L_2-coboundary to make a contribution to (3) which is, having being divided by $\sqrt{n}$, negligible in the limit. The first CLT result of such kind was obtained for stationary Markov chains with *normal* transition operators [2,9] (recall that a bounded operator in a Hilbert space is said to be *normal* if it commutes with its adjoint). More specifically, let the chain $(X_n)_{n\in\mathbb{Z}}$ introduced above have a normal transition operator Q in $L_2(\mu)$ (we call such a chain *normal*, too). Assuming that 1 is a simple eigenvalue of Q and, for an $f \in L_2(\mu)$, the equation

$$f = (I - Q)^{1/2} g \tag{5}$$

(called the *fractional Poisson equation of order* $1/2$ [3]) has a solution $g \in L_2$, the CLT holds for $(f(X_n))$. Independently, under the same condition the CLT and the FCLT for stationary Markov chains with *selfadjoint* transition operators were established in [12]. Moreover, in the normal case the most general known condition for the CLT to hold was proposed in [10]. This compound condition consists of two assumptions which appear in Theorem 4.1 of the present paper as (1) and (2).

Later the CLT [14] and the FCLT [15] (see also [16] for an alternative proof) were established for stationary Markov chains with not necessarily normal transition operators under a ceratin hypothesis we call the *Maxwell-Woodroofe condition*. This condition (which we just mention without further discussion in the present paper) is stronger than the requirement that (5) is solvable in L_2, but is less restrictive than the assumption of the L_2-solvability of (4). These results were achieved by means of the martingale approximation based on relation (1). Obtaining bounds for the sequences ξ and ζ in (1) is somewhat tricky, especially in proofs of the FCLT. This impressive development, however, left open certain important questions, some part of which will be touched in the present paper under the assumption of normality. In our opinion, the key problem here is finding a suitable extension to a more general setup of the known relation between the Poisson equation and the martingale-coboundary representation. This could clarify the structure of the sequence ζ and should be helpful, in particular, when one needs to show that this sequence is negligible. One may expect that the fractional Poisson equation (5) plays an important role in this problem. However, some known facts show that the relation between the solvability of (5) and the applicability of limit theorems is not so simple. For example, even for selfadjoint transition operators a natural fractional modification of the martingale-coboundary representation in general does not hold under the assumption that (5) is solvable in L_2 (see [4] for a counterexample). Further, as we mentioned above, for a function f of a normal Markov chain to satisfy the CLT, a weaker condition than the L_2-solvability of (5) is known (see [10] and the present paper). Moreover, without the assumption of normality the solvability of (5) in L_2 does no longer imply that the variances of sums (3) grow linearly in n [19].

These and other facts stimulate attempts to find a more precise substitute for the Poisson condition in the context of the CLT and other limit theorems. In this paper we analyze further the compound condition used in [10] to deduce the CLT for normal Markov chains. To make shorter the discussion of our approach, we only deal in this Introduction with those functions on the state space which are *completely nondeterministic*. It is this class of functions to which the study of the general case will be reduced at the cost of a certain additional assumption. We generalize the known relation between the Poisson equation (of degree one) and the martingale-coboundary representation. This is achieved by extending the class of admissible solutions of the Poisson equation along with extending the class of possible ingredients of the martingale-coboundary representation. Notice that commonly in the first case we deal with functions defined on the state space of a Markov chain, while in the second one we deal with functions on its path space which forms our basic probability space. Correspondingly, we are led to two kinds of extensions of the related L_2-spaces. We call their elements *t-functions* and *m-functions*, respectively (in general, they are not functions at all). We use the martingale decomposition with respect to a given filtration to construct the space of *m-functions* as an extension of the L_2-space on the basic probability space. To construct the space of t-functions we use a system of operators which can be very loosely described as compressions of the system of conditional expectations defined by the filtration mentioned above. In fact, the definition of this system of operators involves, along with the powers of Q and Q^*, the so-called *defect operators*. This way we arrive at expressions which are well-known in the theory of non-selfadjoint operators, in particular, in connection with dilations, characteristic functions and functional models. In the context of limit theorems, we finally obtain two conditions parallel to (1) and (2) in the abstract. The first of them requires, for an L_2-function on the state space, the solvability of the Poisson equation (of degree one) in *t-functions* (under the assumption of normality of Q this is exactly (1)). This condition guarantees (and is equivalent to) the generalized martingale-coboundary representation involving m-function. The conditions for the applicability of the CLT to a function f can be expressed in terms of the t-function solving the Poisson equation with f in the right hand side. Finally we obtain conditions for the CLT to apply to a function f formulated entirely in terms of this function and the transition operator Q, without any reference to the path space of the Markov chain.

The main conclusion which can be done from the present paper is that in the normal case the (generalized) coboundary can be completely and explicitly restored in a very simple way from the martingale difference part of the martingale-coboundary representation. This martingale difference part (rather than the coboundary) seems to be the most natural functional parameter in the situation of the present paper.

A natural framework for some part of our considerations is given by the classical dilation theory of (not necessarily normal) contractions in Hilbert spaces [17, 18]. However, there are some probabilistic aspects which can not be treated in the framework of a purely Hilbert space theory (martingale difference nature of wandering subspaces, limit theorems). As to our approach to constructing extensions of Hilbert spaces in terms of filtrations, it can have some parallels in the analytic function theory in the unit disk. Clarifying these connections and considering the general non-normal case or other limit theorems require additional study and will not be discussed here.

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2 Contractions, Transition Operators and Markov Chains

2.1 Some Notation

Let $(\mathcal{S}, \mathcal{M})$ and $Q : \mathcal{S} \times \mathcal{M} \to [0, 1]$ be a measurable space and a transition probability (= Markov kernel) on it. Assume that for Q there exists a stationary probability μ on $(\mathcal{S}, \mathcal{M})$ so that $\int_{\mathcal{S}} Q(s, A)\mu(ds) = \mu(A), A \in \mathcal{M}$. By the same symbol Q we will denote the transition (or Markov) operator defined on bounded measurable functions f by the relation $(Qf)(\cdot) = \int_{\mathcal{S}} f(s)Q(\cdot, ds)$. For every $p \in [1, \infty]$ the same formula defines in $L_p(\mu)$ an operator Q of norm 1 preserving positivity and acting identically on constants.

For every $n \in \mathbb{Z}$ denote by $\mathcal{S}_n$ a copy of $\mathcal{S}$, and set $\Omega = \prod_{n\in\mathbb{Z}} \mathcal{S}_n$. Assume that $X = (X_n)_{n\in\mathbb{Z}}$ is a stationary homogeneous Markov chain which has μ as the one-dimensional distribution and Q as the transition operator. The latter means that

$$\mathbb{E}(f(X_n)|X_{n-1}, X_{n-2}, \dots) = (Qf)(X_{n-1})$$

for every $n \in \mathbb{Z}$ and every bounded measurable f. We assume that the chain X is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such a way that, for every $n \in \mathbb{Z}$, $X_n(\omega) = s_n$, the n-th entry of $\omega = (\dots, s_{-1}, s_0, s_1, \dots)$, and also $\mathcal{F} = \sigma(X_n, n \in \mathbb{Z})$, the σ-field generated by all $X_n, n \in \mathbb{Z}$. We denote by T the $\mathbb{P}$-preserving bi-measurable invertible self-map of Ω uniquely determined by the relations $X_n(T(\cdot)) = X_{n+1}(\cdot), n \in \mathbb{Z}$. Starting with $(\mathcal{S}, \mathcal{M})$, Q and μ, we can always construct such a chain and related objects whenever $(\mathcal{S}, \mathcal{M})$ is a standard Borel space. It is known that the transformation T of $(\Omega, \mathcal{F}, \mathbb{P})$ is *ergodic* (that is there is no $A \in \mathcal{F}$ with $T^{-1}A = A$ and $\mathbb{P}(A)(1-\mathbb{P}(A)) \neq 0$) if and only if 1 is a simple eigenvalue of the transition operator Q.

For sub-σ-fields $\mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{F}' \subseteq \mathcal{F}$ the standard notations $L_p(\mathcal{S}, \mathcal{M}', \mu)$ and $L_p(\Omega, \mathcal{F}', \mathbb{P})$ will be abbreviated to $L_p(\mathcal{M}', \mu)$ and $L_p(\mathcal{F}', \mathbb{P})$, or even to $L_p(\mu)$ or $L_p(\mathbb{P})$ if $\mathcal{M}' = \mathcal{M}$ or $\mathcal{F}' = \mathcal{F}$.

The Markov chain X generates on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ an increasing filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ and a decreasing filtration $(\mathcal{F}^n)_{n\in\mathbb{Z}}$ where $\mathcal{F}_n = \sigma(X_k, k \leq n)$ and $\mathcal{F}^n = \sigma(X_k, k \geq n)$, $n \in \mathbb{Z}$. These filtrations are compatible with T in the sense that $T^{-1}\mathcal{F}_n = \mathcal{F}_{n+1} \supseteq \mathcal{F}_n$ and $T^{-1}\mathcal{F}^n = \mathcal{F}^{n+1} \subseteq \mathcal{F}^n$ for $n \in \mathbb{Z}$. The increasing filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ can be completed to obtain $(\mathcal{F}_n)_{-\infty\leq n\leq\infty}$ by setting $\mathcal{F}_{-\infty} = \bigcap_{n\in\mathbb{Z}} \mathcal{F}_n$ and $\mathcal{F}_{\infty} = \bigvee_{n\in\mathbb{Z}} \mathcal{F}_n$. Analogously, $(\mathcal{F}^n)_{-\infty\leq n\leq\infty}$ is a completion of $(\mathcal{F}^n)_{n\in\mathbb{Z}}$ defined by setting $\mathcal{F}^{-\infty} = \bigvee_{n\in\mathbb{Z}} \mathcal{F}^n$ and $\mathcal{F}^{\infty} = \bigcap_{n\in\mathbb{Z}} \mathcal{F}^n$. The above filtrations give rise to the families $(\mathbb{E}_n)_{-\infty\leq n\leq\infty}$ and $(\mathbb{E}^n)_{-\infty\leq n\leq\infty}$ of conditional expectations. Let U and I be a unitary operator defined by $Uf = f \circ T$, $f \in L_2(\mu)$, and the identity operator, respectively. We denote by $|\cdot|_p$ and $||\cdot||_p$ the $L_p(\mu)$-norm and the $L_p(\mathbb{P})$-norm, respectively. The symbols $(\cdot,\cdot)$ and $||\cdot||$ denote the inner product in every Hilbert space and the norm in abstract Hilbert spaces.

Recall that a *contraction* is an operator in a Hilbert space whose norm is less than or equal to one. The transition operator Q in the situation described above defines a contraction in $L_2(\mu)$. Since the measurable map $X_0 : \Omega \to \mathcal{S}$ transforms the measure $\mathbb{P}$ to the measure μ, the mapping $L_2(\mu) \ni f \mapsto \tilde{f} \stackrel{\text{def}}{=} f \circ X_0 \in L_2(\mathbb{P})$ is an isometric embedding of $L_2(\mu)$ to $L_2(\mathbb{P})$. We have to emphasize that in many respects we just reproduce (or go in parallel to) well-known points from the dilation theory of contractions in Hilbert spaces [17, 18].

2.2 *Normal Contractions*

A bounded operator Q in a Hilbert space H satisfying the relation $QQ^* = Q^*Q$ is said to be *normal*. We are mostly interested in normal contractions. If Q is a normal contraction in H and $f \in H$, there exists such a unique measure ρ_f on the closed unit disk $D \subset \mathbb{C}$ that

$$(Q^m f, Q^n f) = \int_D z^m \bar{z}^n \rho_f(dz)$$

for every $m, n \geq 0$. In particular, if μ is a stationary probability measure for a transition probability Q, the transition operator $Q : L_2(\mu) \to L_2(\mu)$ has the norm 1, hence is a contraction. If, moreover, Q is normal, the above formula applies to Q and every $f \in L_2(\mu)$. The spectral theory of normal operators allows us to investigate the Poisson equation (4) for a normal Q in terms of ρ_f. In particular, (4) is solvable in $L_2(\mu)$ for an $f \in L_2(\mu)$ if and only if

$$\int_D \frac{1}{|1-z|^2}\,\rho_f(dz) < \infty. \tag{6}$$

Clearly, the latter condition implies that f is orthogonal to all fixed points of Q.

2.3 Unitary Part of a Transition Operator and Its Deterministic σ-Field

Let Q be a contraction in a Hilbert space H. It is known [17, 18] that the subspace $H_u \subseteq H$ defined by the relation

$$H_u = \{f \in H : \dots = |Q^{*2} f| = |Q^* f| = |f| = |Qf| = |Q^2 f| = \dots\} \tag{7}$$

reduces the operator Q so that $H = H_u \oplus H_{cnu}$, where H_u and H_{cnu} are completely Q-invariant (that is both Q- and Q^*-invariant), $Q|_{H_u}$ is a unitary operator, and H_u is the greatest subspace with such properties. The operators $Q_u = Q|_{H_u}$ and $Q_{cnu} = Q|_{H_{cnu}}$ are called the *unitary part* and the *completely non-unitary part* of Q, respectively. In this notation we have

$$Q = Q_u \oplus Q_{cnu}.$$

In the case of a *normal* contraction Q this decomposition can be immediately deduced by means of the projection-valued spectral measure P_Q of the operator Q. We say that a projection-valued measure *is concentrated* on a Borel set $A \subseteq \mathbb{C}$ if it vanishes on every Borel set disjoint with A; the *restriction* of such a measure P to a Borel set A is another such a (uniquely defined) measure which is concentrated on A and agrees with P on every Borel subset of A; we denote this measure by P^A. Using this terminology and notation, P_Q is concentrated on the closed unit disk D so that $P_Q = P_Q^D$. Let $K = \{|z| \in \mathbb{C} : z = 1\}$ and $D_0 = \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle and the open unit disk. Then $H_u = P_Q(K)H$ and $H_{cnu} = P_Q(D_0)H$. Let P_Q^K and $P_Q^{D_0}$ be the restrictions of P_Q to K and to D_0, respectively. Then we have $P_Q = P_Q^K + P_Q^{D_0}$. By abuse of notation, we also have $P_Q^K = P_{Q_u}$ and $P_Q^{D_0} = P_{Q_{cnu}}$ (here P_{Q_u} and $P_{Q_{cnu}}$ are considered, due to the canonical inclusions of H_u and H_{cnu} in H, as measures with values in orthoprojections of H rather than of H_u or H_{cnu}).

For a normal contraction there exist simple criteria for the relations $f \in H_u$ and $f \in H_{cnu}$.

Proposition 2.1. *Let $Q : H \to H$ be a normal contraction, $H = H_u \oplus H_{cnu}$ the orthogonal decomposition defined above, $||\cdot||$ the norm in H and $f \in H$.*

Then $f \in H_u$ if and only if at least one of the relations $\lim_{n\to\infty} ||Q^n f|| = ||f||$, $\lim_{n\to\infty} ||Q^{*n} f|| = ||f||$ *holds. In fact, in this case equalities in (7) take place.*

Further, $f \in H_{cnu}$ if and only if at least one of the relations $\lim_{n\to\infty} ||Q^n f|| = 0$, $\lim_{n\to\infty} ||Q^{*n} f|| = 0$ *holds. If so, both of these relations hold simultaneously.*

Proof. Let $f = f_u + f_{cnu}$, where $f_u \in H_u$, $f_{cnu} \in H_{cnu}$, and let ρ_f be the spectral measure of f. Then we have

$$||Q^n f_u||^2 = ||Q^{*n} f_u||^2 = \int_K |z|^{2n} \rho_f(dz) = \rho_f(K) = ||f_u||^2,$$

$$||Q^n f_{cnu}||^2 = ||Q^{*n} f_{cnu}||^2 = \int_{D_0} |z|^{2n} \rho_f(dz) \underset{n\to\infty}{\downarrow} 0,$$

and

$$||Q^n f||^2 = ||Q^{*n} f||^2 = ||Q^n f_u||^2 + ||Q^n f_{cnu}||^2 \underset{n\to\infty}{\downarrow} ||f_u||^2.$$

These relations, along with the relation $||f||^2 = ||f_u||^2 + ||f_{cnu}||^2$, imply the assertions of the proposition. □

Let now $H = L_2(\mu)$ and Q be a transition operator with the stationary probability μ. Then, according to S. Foguel's theorem [5, 6], the subspace H_u is of the form $L_2(\mathcal{M}_{det}, \mu)$, where $\mathcal{M}_{det}$ is a sub-σ-field of $\mathcal{M}$ which we will call *deterministic*. (Caution: sometimes this term is used for the σ-fields related to the one-sided analogues of the condition (7)). Moreover, $Q|_{L_2(\mathcal{M}_{det},\mu)}$ defines a μ-preserving automorphism of $\mathcal{M}_{det}$, and $\mathcal{M}_{det}$ is the largest sub-σ-field of $\mathcal{M}$ with this property. In the Markov chain context we will use denotations H_{det} and H_{ndet} (from *deterministic* and *nondeterministic*) instead of H_u and H_{cnu}, respectively. The orthogonal projection P_{det} to $H_{det} = L_2(\mathcal{M}_{det}, \mu)$ coincides with the corresponding conditional expectation $\mathbb{E}^{\mathcal{M}_{det}} : L_2(\mu) \to L_2(\mathcal{M}_{det}, \mu)$; the range of the complementary projection $P_{ndet} = I - \mathbb{E}^{\mathcal{M}_{det}}$ is H_{ndet}. In the normal case the projection $P_{det} : L_2(\mu) \to L_2(\mathcal{M}_{det}, \mu)$ is exactly the spectral projection $P_Q(K)$ of the operator Q while the complementary projection P_{ndet} agrees with $P_Q(D_0)$.

Remark 2.2. For an $f \in L_2(\mu)$ the orthogonal decomposition

$$f = f_{det} + f_{ndet}$$

with $f_{det} = P_{det} f$ and $f_{ndet} = P_{ndet} f$ leads to the decomposition of the stationary random sequence $(f(X_n))_{n\in\mathbb{Z}}$ into the sum of the sequences $(f_{det}(X_n))_{n\in\mathbb{Z}}$ and $(f_{ndet}(X_n))_{n\in\mathbb{Z}}$, the second of them having zero conditional expectation given the first one. Without additional assumptions the sequence $(f_{det}(X_n))_{n\in\mathbb{Z}}$ may be an arbitrary stationary sequence of square-integrable variables whose influence to the behavior of $(f(X_n))_{n\in\mathbb{Z}}$ is out of our control. The sequence $(f_{ndet}(X_n))_{n\in\mathbb{Z}}$, unlike $(f_{det}(X_n))_{n\in\mathbb{Z}}$, admits some further analysis. Under the assumption of normality of Q some problems (such as the Central Limit Theorem) concerning $(f(X_n))_{n\in\mathbb{Z}}$ can be treated in terms of the spectral measures of the functions f, f_{det} and f_{ndet}. Notice that $\rho_f = \rho_{f_{det}} + \rho_{f_{ndet}}$, where $\rho_{f_{det}}$ and $\rho_{f_{ndet}}$ are concentrated on K and D_0, respectively.

2.4 Defect Operators and Defect Spaces of a Contraction

We present here the definition and some properties of the *defect operators* and the *defect spaces* of a contraction $Q : H \to H$ (see [17] and [18] for proofs and more details). The operators

$$D_Q = (I - Q^* Q)^{\frac{1}{2}}, D_{Q^*} = (I - QQ^*)^{\frac{1}{2}}$$

are called the *defect operators* of Q. These operators are self-adjoint non-negative (in the spectral sense) contractions, satisfying

$$QD_Q = D_{Q^*} Q, \; D_Q Q^* = Q^* D_{Q^*}.$$

The spaces

$$\mathcal{D}_Q = \overline{D_Q H}, \; \mathcal{D}_{Q^*} = \overline{D_{Q^*} H}$$

are called *defect spaces* of Q. It follows from the above relations that

$$Q\mathcal{D}_Q \subseteq \mathcal{D}_{Q^*}, Q^*\mathcal{D}_{Q^*} \subseteq \mathcal{D}_Q.$$

In the case of a normal contraction Q the corresponding defect operators agree, and so are the defect subspaces. In this case the defect subspace is invariant with respect to both Q and Q^*, and the restriction $D_Q|_{H_{cnu}}$ of the defect operator to the completely non-unitary subspace is injective. Indeed, if $f \in H_{cnu}$ and $D_Q f = 0$ the spectral measure ρ_f is concentrated on D_0 by the first of these two relations, while by the second relation $(Qf, Qf) = (f, f)$; the latter means that ρ_f is concentrated on K, implying $\rho_f = 0$. Furthermore, it is easy to see from the consideration of spectral measures that $\mathcal{D}_Q = H_{cnu}$ if Q is a normal contraction.

Remark 2.3. When Q is a transition operator, its defect subspaces are in a natural unitary correspondence with the spaces of the forward and the backward martingale differences of the Markov chain X (see the next section of the paper; compare with [18], Sect. 3.2). □

3 Quasi-functions

3.1 Quasi-functions: m-Functions and t-Functions

Looking for a generalization of the martingale-coboundary representation and the Poisson equation, we need some more general objects than the $L_2(\mathbb{P})$-functions of the form $f(X_0)$ in the first case and $L_2(\mu)$-functions in the second one. The first problem is solved in terms of the filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ determined by the Markov chain, while the second one is treated in terms of the transition operator and some other auxiliary operators acting on $L_2(\mu)$-functions. In both cases we obtain a decomposition of an L_2-function into a series. Removing the requirement that the decomposition belongs to an L_2-function, we arrive at a class of objects which are given by their decompositions but, in general, are no longer functions. These objects called *quasi-functions* will be considered as elements of certain Banach spaces. These Banach spaces contain conventional L_2-spaces as dense subspaces such a way that every quasi-function can be represented in a canonical way as a limit of L_2-functions. Moreover, every operator we are interested in admits a canonical extension from L_2 to the appropriate Banach space. We will consider quasi-functions of two kinds. Quasi-functions of the first kind generalize conventional functions defined on the path space of the Markov chain under consideration and will be called *m-functions*; quasi-functions of the second kind, generalizing conventional functions defined on the state space of the Markov chain, will be called *t-functions*. It turns out that some conventional functions on the path space can have a martingale-coboundary representation in terms of *m-functions*; some of them are conventional L_2-functions, but some other are not. Also the Poisson equation for an L_2-function on the state space with no L_2-solution may be sometimes solved in *t-functions*.

As an introductory step, we start with considering the decompositions of functions from L_2-spaces.

3.2 Functions and m-Functions

For every $g \in L_2(\mu)$ we have the following martingale decomposition

$$\tilde{g} = \sum_{n=0}^{\infty}(\mathbb{E}_{-n} - \mathbb{E}_{-n-1})\tilde{g} + \mathbb{E}_{-\infty}\tilde{g}, \tag{8}$$

converging in the norm of $L_2(\mathbb{P})$. Rewriting the summands of (8) in terms of the operators Q, U and the embedding $g \mapsto \tilde{g}$, we have

$$\begin{aligned} &(\mathbb{E}_{-n} - \mathbb{E}_{-n-1})\tilde{g} \\ &= U^{-n}\mathbb{E}_0 U^n \tilde{g} - U^{-n-1}\mathbb{E}_0 U^{n+1}\tilde{g} = U^{-n}\mathbb{E}_0\mathbb{E}^n U^n \tilde{g} - U^{-n-1}\mathbb{E}_0\mathbb{E}^{n+1}U^{n+1}\tilde{g} \\ &= U^{-n}\widetilde{Q^n g} - U^{-n-1}\widetilde{Q^{n+1}g} \end{aligned} \tag{9}$$

and, with the limits in the norm of $L_2(\mathbb{P})$,

$$
\begin{aligned}
&\mathbb{E}_{-\infty}\tilde{g} \\
&= \lim_{n\to\infty} \mathbb{E}_{-n}\tilde{g} = \lim_{n\to\infty} U^{-n}\mathbb{E}_0 U^n \tilde{g} = \lim_{n\to\infty} U^{-n}\mathbb{E}_0\mathbb{E}^n U^n \tilde{g} \\
&= \lim_{n\to\infty} U^{-n}\widetilde{Q^n g} = \lim_{n\to\infty} U^{-n}\widetilde{Q^n g_{det}} \\
&= \tilde{g}_{det}.
\end{aligned} \tag{10}
$$

Deriving (9) and (10), we used the fact that $\tilde{g}$ is X_0-measurable, along with some standard properties of Markov chains. In (10) we also used the decomposition $g = g_{det} + g_{ndet}$, the relation $Q^n g_{ndet} \underset{n\to\infty}{\to} 0$ and the identity $U^{-1}\widetilde{Qg_{det}} = \widetilde{g_{det}}$ which can be explained as follows. Since the map $g \to \tilde{g}$ isometrically embeds $L_2(\mu)$ to $L_2(\mathbb{P})$, the subspace $H_{det} \subseteq L_2(\mu)$ is also embedded to $L_2(\mathbb{P})$. Furthermore, it can be easily verified that H_{det} is a completely invariant subspace of the unitary U, and that $U|_{H_{det}} = Q|_{H_{det}}$. Another way to express this is the relation $U^{-1}\widetilde{Qg_{det}} = \widetilde{g_{det}}$ used in (10). Moreover, this relation allows us to substitute $\tilde{g}$ by $\tilde{g}_{ndet}$ in the right-hand side of (9). With (10) and properly modified (9), the identity (8) can be rewritten as

$$
\tilde{g} = \sum_{n=0}^{\infty}(U^{-n}\widetilde{Q^n g_{ndet}} - U^{-n-1}\widetilde{Q^{n+1} g_{ndet}}) + \tilde{g}_{det}. \tag{11}
$$

Assuming now $g \in H_{ndet}$, we have the following martingale decomposition:

$$
\tilde{g} = \sum_{n=0}^{\infty} U^{-n}(\widetilde{Q^n g} - U^{-1}\widetilde{Q^{n+1} g}). \tag{12}
$$

Analyzing the right-hand side of (12), observe that all terms in this series are of the form $U^{-n}(\widetilde{r_n} - U^{-1}\widetilde{Qr_n})$, $r_n \in H_{ndet}$ $(n \in \mathbb{Z})$. Terms of such form are mutually orthogonal for different $n \in \mathbb{Z}$. Set

$$
L_n = \overline{\{U^n(\tilde{r} - U^{-1}\widetilde{Qr}) : r \in H_{ndet}\}} \ (n \in \mathbb{Z})
$$

and denote by M the closed subspace of $L_2(\mathbb{P})$ generated by all $L_n, n \in \mathbb{Z}$. In view of the mutual orthogonality of L_n we have

$$
M = \oplus_{n\in\mathbb{Z}} L_n \tag{13}
$$

(we use $\oplus$ both as a symbol of an exterior operation and also for the closed span of some orthogonal subspaces of a certain Hilbert space). The space M is a completely invariant subspace of the operator U. The operator $U|_M$ is unitarily equivalent to the two-sided shift operator, and every L_n is a wandering subspace for $U|_M$. From now on we will write U instead of $U|_M$. Denoting by $\bigvee$ the linear span of some set of liner subspaces, we also have

$$M = \overline{\bigvee_{n\in\mathbb{Z}} U^n H_{ndet}}. \tag{14}$$

Indeed, the left-hand side is contained in the right-hand one because of the obvious relation $L_n \subseteq \overline{U^n H_{ndet} \bigvee U^{n-1} H_{ndet}}$ ($n \in \mathbb{Z}$); the opposite inclusion is a consequence of (12) and the complete invariance of M with respect to U. We will also need the U^{-1}-invariant spaces

$$M_n = \oplus_{k\leq n} L_k \left(= \overline{\bigvee_{k\leq n} U^k H_{ndet}} \right), n \in \mathbb{Z}.$$

Since

$$||\tilde{h} - U^{-1}\widetilde{Qh}||^2 = ((I - Q^*Q)h, h),$$

setting for every $h \in H_{ndet}$

$$l(\tilde{h} - U^{-1}\widetilde{Qh}) = (I - Q^*Q)^{\frac{1}{2}} h$$

defines a unitary map $l : L_0 \to \mathcal{D}_Q$. As was observed in Sect. 2.4, for a normal transition operator Q we have $\mathcal{D}_Q = H_{ndet}$, and therefore l maps L_0 to H_{ndet}. Then the space M_0 is unitarily equivalent to the space of one-sided sequences of the elements of H_{ndet} via the correspondence

$$M_0 \ni \sum_{n\leq 0} U^n p_n \leftrightarrow (\ldots, l(p_{-1}), l(p_0)) \in H_{ndet} \otimes l_2(\mathbb{Z}_-), \tag{15}$$

where $(p_n)_{n\leq 0}$ is a sequence of elements of L_0 with $\sum_{n\leq 0} ||p_n||_2^2 < \infty$ and $H_{ndet} \otimes l_2(\mathbb{Z}_-)$ denotes the Hilbert space tensor product of Hilbert spaces. The elements of $H_{ndet} \otimes l_2(\mathbb{Z}_-)$ are sequences $(\ldots, a_{-1}, a_0)$ with $a_n \in H_{ndet}$ ($n \leq 0$) and $\sum_{n\leq 0} |a_n|_2^2 < \infty$. By this unitary equivalence the one-sided shift σ : $(\ldots, a_{-1}, a_0) \mapsto (\ldots, a_{-1}, a_0, 0)$ in the space $H_{ndet} \otimes l_2(\mathbb{Z}_-)$ corresponds to the isometric operator $U^{-1}|_{M_0}$, while the co-isometric inverse shift σ^*: $(\ldots, a_{-1}, a_0) \mapsto (\ldots, a_{-2}, a_{-1})$ corresponds to $(U^{-1}|_{M_0})^*$. Furthermore, since Q acts on the space H_{ndet}, we can define its coordinatewise action on $H_{ndet} \otimes l_2(\mathbb{Z}_-)$ by

$$Q(\ldots, a_{-1}, a_0) = (\ldots, Qa_{-1}, Qa_0).$$

We set $\hat{Q} = l^{-1}Ql : L_0 \to L_0$, and extend it (with the same notation and in agreement with (15)) to $\hat{Q} : M_0 \to M_0$ by setting $\hat{Q}(\sum_{n\leq 0} U^n p_n) = \sum_{n\leq 0} U^n \hat{Q} p_n$.

We are in position now to give a description of those elements of M_0 which are martingale decompositions (12) of certain $\tilde{g}$ with $g \in H_{ndet}$.

Proposition 3.1. *The following conditions on the series $\sum_{n\leq 0} U^n p_n \in M_0$ are equivalent:*

(1) the series $\sum_{n\le 0} U^n p_n$ *represents a decomposition* (12) *of certain* $\tilde{g}$ *with* $g \in H_{ndet}$*;*
(2) there exists such $p \in L_0$ *that* $\sum_{n\ge 0} ||\hat{Q}^n p||_2^2 < \infty$ *and* $p_{-n} = \hat{Q}^n p$ *for every* $n \ge 0$;
(3) there exists such $r \in H_{ndet}$ *that* $\sum_{n\ge 0} |Q^n r|_2^2 < \infty$ *and for every* $n \ge 0$ $l(p_{-n}) = Q^n r$.

Proof. Conditions (2) and (3) are equivalent because l is a unitary operator and $\hat{Q} = l^{-1} Q l$. Let us show that (1) implies (3). According to (12), for the martingale decomposition of an $\tilde{g}$ with $g \in H_{ndet}$ we have for $n \ge 0$ $p_{-n} = \widetilde{Q^n} g - U^{-1} \widetilde{Q^{n+1}} g$, so that $l(p_{-n}) = Q^n (I - Q^* Q)^{\frac{1}{2}} g = Q^n r$, where $r = (I - Q^* Q)^{\frac{1}{2}} g$. This follows that $\sum_{n\ge 0} |Q^n r|_2^2 = \sum_{n\ge 0} ((Q^* Q)^n (I - Q^* Q) g, g) = (g, g) < \infty$. Conversely, assuming (3), let $h \in H_{ndet}$ is such that $\sum_{n\ge 0} |Q^n r|_2^2 < \infty$. This is equivalent to

$$\int_D \frac{1}{1 - |z|^2} \rho_r(dz) < \infty,$$

which follows that there exists such $g \in H_{ndet}$ that $r = (I - Q^* Q)^{\frac{1}{2}} g$. Then in the martingale decomposition $\tilde{g} = \sum_{n\le 0} U^n p'_n$ we have for $n \le 0$ $p'_n = \widetilde{Q^{-n}} g - U^{-1} \widetilde{Q^{-n+1}} g$ or $l(p'_n) = Q^{-n} (I - Q^* Q)^{\frac{1}{2}} g = Q^{-n} r = l(p_n)$, and we conclude $p'_n = p_n$. □

Let $c_0(\mathbb{Z}_-)$ be the space of all complex sequences indexed by the elements of $\mathbb{Z}_-$ and tending to zero, $c_0(\mathbb{Z}_-)$ being supplied with the sup-norm. Then the injective tensor product $H_{ndet} \otimes_\epsilon c_0(\mathbb{Z}_-)$ is the space of all sequences $\overline{a} = (\ldots, a_{-1}, a_0)$ with $a_n \in H_{ndet}$ for $n \le 0$, $|a_n|_2 \underset{n\to -\infty}{\to} 0$ and with the norm of $\overline{a} = (\ldots, a_{-1}, a_0)$ defined as $\sup_{n\le 0} |a_n|_2$. The space $H_{ndet} \otimes l_2(\mathbb{Z}_-)$, represented as a space of sequences of elements of H_{ndet}, can be in a natural way continuously and injectively mapped into $H_{ndet} \otimes_\epsilon c_0(\mathbb{Z}_-)$. Notice that the shift operators σ_n and σ_n^* can be extended to $H_{ndet} \otimes_\epsilon c_0(\mathbb{Z}_-)$. Observe that $\sigma^{*n} \overline{a} \underset{n\to\infty}{\to} 0$ for every $\overline{a} \in H_{ndet} \otimes_\epsilon c_0(\mathbb{Z}_-)$. We can transfer this extension, via correspondence (15), to a space containing M_0. Elements of M_0 are sums $\sum_{n\le 0} U^n p_n$, where $p_n \in L_0, n \le 0$, and $\sum_{n\le 0} ||p_n||_2^2 < \infty$. Then the extended space denoted by M_0^{ext} and consisting of the formal sums $\sum_{n\le 0} U^n p_n$ where $p_n \in L_0, n \le 0$, $||p_n||_2 \underset{n\to -\infty}{\to} 0$; the norm of $\sum_{n\le 0} U^n p_n$ is defined as $\sup_{n\le 0} ||p_n||_2$. The operators $U^{-1}|_{M_0}$ and $(U^{-1}|_{M_0})^*$ admit obvious extensions to M_0^{ext} which we denote $U^{-1}|_{M_0^{ext}}$ and $(U^{-1}|_{M_0^{ext}})^*$. Analogously, every space M_n $(n \in \mathbb{Z})$ can be extended to the space M_n^{ext}. If we write the elements of M_n^{ext} as $\sum_{k\le n} U^k p_k$ with $p_k \in L_0 (k \le n)$, we obtain a growing sequence of subspaces of the space

$$M^{ext} = \{\sum_{n\in\mathbb{Z}} U^n p_n : p_n \in L_0 (n \in \mathbb{Z}), ||p_n||_2 \underset{|n|\to\infty}{\to} 0\}.$$

The space M^{ext} is an extension of M, and we will hold the notations U and U^{-1} for natural extensions of these operators from M to M^{ext}.

Definition 3.2. Elements of the Banach space M^{ext} are called *m-functions*.

Remark 3.3. There are also other operators which can be naturally extended from M to M^{ext}. For example, so are the projections $\mathbb{E}_n : \sum_{k\in\mathbb{Z}} U^k p_k \mapsto \sum_{k\leq n} U^k p_k$, $n \in \mathbb{Z}$. □

3.3 Functions and t-Functions

The space of t-functions which we are going to define extends the space $H_{ndet} \subseteq L_2(\mu)$. Functions from H_{ndet} admit some decomposition; t-functions will be defined in terms of a similar decomposition. The next problem to solve will be how to embed the space of t-functions to the space of m-functions generalizing the embedding $g \mapsto g \circ X_0$ of the space H_{ndet} to $L_2(\mathbb{P})$. This also will be done in terms of the corresponding decompositions.

Taking in (11) the conditional expectation relative to X_0 (which is, in particular, the left inverse for the embedding $g \mapsto g \circ X_0$), we obtain

$$g = \sum_{n=0}^{\infty}(Q^{*n}Q^n - Q^{*(n+1)}Q^{n+1})g_{ndet} + g_{det}. \tag{16}$$

Now we again assume that $g \in H_{ndet}$. Then we have

$$g = \sum_{n=0}^{\infty}(Q^{*n}Q^n - Q^{*(n+1)}Q^{n+1})g \tag{17}$$

or

$$g = \sum_{n=0}^{\infty} Q^{*n}(I - Q^*Q)Q^n g, \tag{18}$$

where the series' converge in the norm $|\cdot|_2$.

Since Q is normal, we have

$$\begin{aligned}(g, g) &= \sum_{n=0}^{\infty}(Q^{*n}(I - Q^*Q)Q^n g, g)\\ &= \sum_{n=0}^{\infty}(Q^n(I - Q^*Q)^{\frac{1}{2}}g, Q^n(I - Q^*Q)^{\frac{1}{2}}g).\end{aligned} \tag{19}$$

Then, we have an isometric correspondence

$$H_{ndet} \ni g \leftrightarrow (\ldots, Q(I - Q^*Q)^{\frac{1}{2}} g, (I - Q^*Q)^{\frac{1}{2}} g) \in H_{ndet} \otimes l_2(\mathbb{Z}_-) \tag{20}$$

(the set $\mathbb{Z}_-$ rather than $\mathbb{Z}_+$ was chosen here for the future denotational convenience). According to Proposition 3.1, another description of the image of H_{ndet} in $H_{ndet} \otimes l_2(\mathbb{Z}_-)$ by the above correspondence is as follows:

$$\{(\ldots, Qr, r) : r \in H_{ndet}, \sum_{n \geq 0} |Q^n r|_2^2 < \infty\}.$$

We will identify this image with H_{ndet}. The extension $H_{ndet}^{ext} \supseteq H_{ndet}$ is then defined by

$$H_{ndet}^{ext} = \{(\ldots, Qr, r) : r \in H_{ndet}\}$$

(we consider H_{ndet}^{ext} as a subspace of $H_{ndet} \otimes_\epsilon c_0(\mathbb{Z}_-)$).

Definition 3.4. Elements of the Banach spaces H_{ndet}^{ext} are called *t-functions*.

It is clear from this definition that every t-function is a sequence of functions from H_{ndet} with some additional properties; in case the corresponding series converges in $L_2(\mu)$ its sum gives an H_{ndet}-representative of the corresponding t-function; otherwise a t-function is a proper generalized function; anyway, a t-function is a limit (in the sense of $H_{ndet} \otimes_\epsilon c_0(\mathbb{Z}_-)$) of functions from H_{ndet}.

Let us turn now to the embedding of t-functions to m-functions. Reformulating the mapping $f \mapsto \tilde{f} = f \circ X_0$, $f \in H_{ndet}$, in terms of decompositions, we obtain

$$\begin{aligned} &H_{ndet} \ni g \leftrightarrow (\ldots, (I - Q^*Q)^{\frac{1}{2}} g, Q(I - Q^*Q)^{\frac{1}{2}} g, (I - Q^*Q)^{\frac{1}{2}} g) \\ &\mapsto \sum_{n \leq 0} U^n l^{-1}\big(Q^{-n}(I - Q^*Q)^{\frac{1}{2}} g\big) = \tilde{g} \in M_0. \end{aligned} \tag{21}$$

This embedding can be described differently as

$$(\ldots, Qr, r) \mapsto \widetilde{(\ldots, Qr, r)} = \sum_{n \leq 0} U^n l^{-1}\big(Q^{-n} r\big), \tag{22}$$

which makes sense both for $H_{ndet} \to \widetilde{H_{ndet}} \subseteq M_0$ and for $H_{ndet}^{ext} \to \widetilde{H_{ndet}^{ext}} \subseteq M_0^{ext}$.

Proposition 3.5. *Let $Q : L_2(\mu) \to L_2(\mu)$ be a normal transition operator for a stationary Markov chain X and $f \in H_{ndet}$ have the spectral measure ρ_f. Then the following conditions on the function f are equivalent:*

(1) $f = g - Qg$ *with some* $g \in H_{ndet}^{ext}$*;*
(2) $\tilde{f} = h + \tilde{g} - U\tilde{g}$ *with some* $g \in H_{ndet}^{ext}$ *and* $h \in M_1$ *such that* $\mathbb{E}_0 h = 0$*;*
(3) $r \stackrel{def}{=} (I - Q)^{-1}(I - Q^*Q)^{\frac{1}{2}} f \in H_{ndet}$*;*
(4) $\sigma_f^2 \stackrel{def}{=} |r|_f^2 = \int_D \frac{1-|z|^2}{|1-z|^2} \rho_f \, dz < \infty$.

Moreover, deducing (2) from (1) or (3) we can always set $h = U\tilde{g} - \widetilde{Qg}$; also, we have $||h||_2^2 = |r|_2^2 = \sigma_f^2$.

Proof. Let (1) holds true. Then $\tilde{g} = \sum_{n\le 0} U^n l^{-1}\big(Q^{-n} r\big)$ for some $r \in H_{ndet}$, and

$$\tilde{f} = \tilde{g} - \widetilde{Qg} = U\tilde{g} - \widetilde{Qg} + \tilde{g} - U\tilde{g} = h + \tilde{g} - U\tilde{g}, \tag{23}$$

where

$$h = U\tilde{g} - \widetilde{Qg} = \sum_{n\le 0} U^{n+1} l^{-1}\big(Q^{-n} r\big) - \sum_{n\le 0} U^n l^{-1}\big(Q^{-n+1} r\big) = U^1 l^{-1}\big(r\big) \in L_1, \tag{24}$$

and (2) follows. To establish (2) $\to$ (1), apply $\mathbb{E}_0$ to the relation (2), obtaining $\tilde{f} = \tilde{g} - \mathbb{E}_0 U\tilde{g}$; then check $\mathbb{E}_0 U\tilde{g} = \widetilde{Qg}$ by means of the represention $\tilde{g} = \sum_{n\le 0} U^n l^{-1}\big(Q^{-n} r\big)$ with some $r \in H_{ndet}$.

Let us show now that (1) and (3) are equivalent. The relation (1) holds if and only if for some $r \in H_{ndet}$ and every $n \ge 0$ $Q^n Q(I - Q^*Q)^{\frac{1}{2}} f = Q^n(I - Q)r$. But this is equivalent to $Q(I - Q^*Q)^{\frac{1}{2}} f = (I - Q)r$ or to $r = (I - Q)^{-1}(I - Q^*Q)^{\frac{1}{2}} f$ which is equivalent to (3). Since $f \in H_{ndet}$ and H_{ndet} is invariant with respect to Q, such $r \in H_{ndet}$ exists if and only if (4) holds. The last assertions follow from (23) and (24). □

Remark 3.6 (Unicity and reality). It is easy to see that the equation $f = g - Qg$ may have at most one solution $g \in H_{ndet}^{ext}$.

Functions we consider are in general complex-valued; so functions and quasi-functions constitute Banach spaces over $\mathbb{C}$. The involutive conjugation in these spaces is well-defined, its fixed points are said to be real. The operators Q, Q^*, their spectral projections and conditional expectations $\mathbb{E}_n (n \in \mathbb{Z})$ preserve the reality of functions and quasi-functions. In view of this, for example, in the orthogonal decomposition $f = f_{det} + f_{ndet}$ the summands f_{det} and f_{ndet} are real-valued provided that so is f. These facts and the unicity imply that the solution of the Poisson equation with a real right-hand side must be real; also for a real function the ingredients of the martingale-coboundary representation must be real. Notice that for the spectral measure ρ_f of a real-valued function f with respect to the operator Q the real axis is the symmetry axis. □

4 The CLT

In addition to assumptions of Sect. 2 (including the normality of the transition operator) we assume that 1 is a simple eigenvalue of the operator Q. It is known [10] that this implies (and is equivalent to) the ergodicity of the shift transformation T.

We give now an alternative proof of a version of the Central Limit Theorem for a stationary normal Markov chain (Thm. 7.1 in [10]). Let $N(m, \sigma^2)$ be the normal law with the mean value m and the variance σ^2, degenerate if $\sigma^2 = 0$. As above, D denotes the closed unit disk in $\mathbb{C}$.

Theorem 4.1. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary homogeneous Markov chain which has a probability measure μ as the one-dimensional distribution and a normal operator $Q : L_2(\mu) \to L_2(\mu)$ as the transition operator. Assume that the eigenvalue 1 of Q is simple. Let a real-valued function $f \in L_2(\mu)$ with the spectral measure ρ_f satisfy the conditions*

(1) $\sigma_f^2 = \int_D \frac{1-|z|^2}{|1-z|^2} \rho_f dz < \infty$,
(2) $\lim_{n\to\infty} n^{-\frac{1}{2}} \left|\sum_{k=0}^{n-1} Q^k f\right|_2 = 0$.

Then the random variables $\left(n^{-\frac{1}{2}} \sum_{k=0}^{n-1} f(X_k)\right)_{n\geq 1}$ *converge in distribution to the normal law* $N(0, \sigma_f^2)$. *Moreover,*

$$\lim_{n\to\infty} n^{-1} \left\|\sum_{k=0}^{n-1} f(X_k)\right\|_2^2 = \sigma_f^2. \tag{25}$$

Proof. Let us first reduce the proof to the case $f \in H_{ndet}$. Since the decomposition $L_2(\mu) = H_{det} \oplus H_{ndet}$ reduces the operator Q, the assumption (2) implies for $f = f_{det} + f_{ndet}$ ($f_{det} \in H_{det}$, $f_{ndet} \in H_{ndet}$) that

$$\lim_{n\to\infty} n^{-\frac{1}{2}} \left|\sum_{k=0}^{n-1} Q^k f_{det}\right|_2 = 0 \tag{26}$$

and

$$\lim_{n\to\infty} n^{-\frac{1}{2}} \left|\sum_{k=0}^{n-1} Q^k f_{ndet}\right|_2 = 0. \tag{27}$$

Since $Q|_{H_{det}}$ is a unitary operator which agrees, after embedding $\widetilde{H_{det}}$ to $L_2(\mathbb{P})$, with U, for every $n \geq 1$ we have

$$\left|\sum_{k=0}^{n-1} Q^k f_{det}\right|_2 = \left\|\sum_{k=0}^{n-1} f_{det}(X_k)\right\|_2,$$

which follows

$$\lim_{n\to\infty} n^{-\frac{1}{2}} \left\|\sum_{k=0}^{n-1} f_{det}(X_k)\right\|_2 = 0. \tag{28}$$

It is therefore clear that the random variables $(n^{-\frac{1}{2}} \sum_{k=0}^{n-1} f_{det}(X_k), n \geq 1)$ converges to 0 both in probability and in the norm $\|\cdot\|_2$. By this reason we will assume $f \in H_{ndet}$ in the rest of the proof.

In view of the assumption (1) and Proposition 3.5, f admits the representation $f = g - Qg$ with some $g \in H_{ndet}^{ext}$, and we have

$$\sum_{k=0}^{n-1} f(X_k) = \sum_{k=0}^{n-1} U^k(U\tilde{g} - \widetilde{Qg}) + \tilde{g} - U^n\tilde{g}.$$

Here $(U^k(U\tilde{g} - \widetilde{Qg}))_{k\geq 0}$ is a stationary ergodic sequence of martingal differences whose variance is, by Proposition 3.5, σ_f^2. Then, in view of the Billingsley-Ibragimov theorem, we only need to show that

$$n^{-1}||\tilde{g} - U^n\tilde{g}||_2^2 \underset{n\to\infty}{\to} 0. \tag{29}$$

We have with an $r \in H_{ndet}$ from (3) in Proposition 3.5

$$\begin{aligned}
n^{-1}||\tilde{g} - U^n\tilde{g}||_2^2 &= n^{-1}\Big|\Big|\sum_{k\leq 0} U^k l^{-1}(Q^{-k}r) - \sum_{k\leq 0} U^{n+k} l^{-1}(Q^{-k}r)\Big|\Big|_2^2 \\
&= n^{-1}\Big|\Big|\sum_{0\leq k\leq n-1} U^{n-k} l^{-1}(Q^k r)\Big|\Big|_2^2 + n^{-1}\Big|\Big|\sum_{k\leq 0} U^k l^{-1}(Q^{-k}r) - \sum_{k\leq 0} U^k l^{-1}(Q^{n-k}r)\Big|\Big|_2^2 \\
&= n^{-1}\sum_{k=0}^{n-1} |Q^k r|_2^2 + n^{-1}\Big|\Big|\sum_{k\leq 0} U^k l^{-1}(Q^{-k}r) - \sum_{k\leq 0} U^k l^{-1}(Q^{n-k}r)\Big|\Big|_2^2 \\
&= n^{-1}\sum_{k=0}^{n-1} |Q^k r|_2^2 + n^{-1}|g - Q^n g|_2^2 = n^{-1}\sum_{k=0}^{n-1} |Q^k r|_2^2 + n^{-1}\Big|\sum_{k=0}^{n-1} Q^k f\Big|_2^2.
\end{aligned} \tag{30}$$

The summands of the last sum tend to zero: the first one because so does $|Q^n r|_2$ and the second one by assumption (2) of the theorem. This completes the proof. □

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