Distribution of Algebraic Numbers and Metric Theory of Diophantine Approximation

V. Bernik, V. Beresnevich, F. Götze, and O. Kukso

Abstract In this paper we give an overview of recent results regarding close conjugate algebraic numbers, the number of integral polynomials with small discriminant and pairs of polynomials with small resultants.

Keywords Diophantine approximation • approximation by algebraic numbers • discriminant • resultant • polynomial root separation

2010 *Mathematics Subject Classification.* 11J83, 11J13, 11K60, 11K55

1 Introduction

Throughout the paper μA stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ and dim B denotes the Hausdorff dimension of B. Given $\psi : \mathbb{N} \to (0+\infty)$, let $\mathcal{L}(\psi)$ denote the set of $x \in \mathbb{R}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \tag{1}$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. We begin by recalling two classical results in metric theory of Diophantine approximation.

V. Bernik · O. Kukso
Institute of Mathematics, Academy of Sciences of Belarus, Minsk, Belarus

V. Beresnevich (✉)
Department of Mathematics, University of York, Heslington, York, England

F. Götze
Faculty of Mathematics, Bielefeld University, Bielefeld, Germany

P. Eichelsbacher et al. (eds.), *Limit Theorems in Probability, Statistics and Number Theory*, Springer Proceedings in Mathematics & Statistics 42,
DOI 10.1007/978-3-642-36068-8_2,

Khintchine's theorem [35]. *Let $\psi : \mathbb{N} \to (0, +\infty)$ be monotonic and I be an interval in $\mathbb{R}$. Then*

$$\mu(I \cap \mathcal{L}(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ \mu(I), & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases} \tag{2}$$

Jarník–Besicovitch theorem [25, 34]. *Let $v > 1$ and for $q \in \mathbb{N}$ let $\psi_v(q) = q^{-v}$. Then*

$$\dim \mathcal{L}(\psi_v) = \frac{2}{v+1}.$$

The condition that ψ is monotonic can be omitted from the convergence case of Khintchine's theorem, though it is vital in the case of divergence—see [12, 33, 42] for a further discussion. By the turn of the millennium the above theorems were generalised in various directions. One important direction of research has been Diophantine approximation by algebraic numbers and/or integral polynomials, which has eventually grown into an area of number theory known as Diophantine approximation on manifolds.

Given a polynomial $P = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, the number $H = H(P) = \max_{0 \le i \le n} |a_j|$ will be called the (naive) height of P. Given $n \in \mathbb{N}$ and an approximation function $\Psi : \mathbb{N} \to (0, +\infty)$, let $\mathcal{L}_n(\Psi)$ be the set of $x \in \mathbb{R}$ such that

$$|P(x)| < \Psi(H(P)) \tag{3}$$

for infinitely many $P \in \mathbb{Z}[x] \setminus \{0\}$ with $\deg P \le n$. Note that $\mathcal{L}_1(\Psi)$ is essentially the same as the set $\mathcal{L}(\Psi)$ introduced above. Thus, the following statement represents an analogue of Khintchine's theorem for the case of polynomials.

Theorem 1. *Let $n \in \mathbb{N}$ and $\Psi : \mathbb{N} \to (0, +\infty)$ be monotonic. Then for any interval I*

$$\mu(I \cap \mathcal{L}_n(\Psi)) = \begin{cases} 0, & \text{if } \sum_{h=1}^{\infty} h^{n-1}\Psi(h) < \infty, \\ \mu(I), & \text{if } \sum_{h=1}^{\infty} h^{n-1}\Psi(h) = \infty. \end{cases} \tag{4}$$

The case of convergence of Theorem 1 was proved in [17], the case of divergence was proved in [4]. The condition that Ψ is monotonic can be omitted from the case of convergence as shown in [6]. Theorem 1 was generalised to the case of approximation in the fields of complex and p-adic numbers [9, 19], to simultaneous approximations in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ [22, 26] and to various other settings. When $\Psi = \Psi_w$ is given by $\Psi_w(q) = q^{-w}$ Theorem 1 reduces to a famous problem of Mahler [37, 41] solved by Sprindžuk. The versions of Theorem 1 for monic polynomials were established in [27, 40]. For the more general case of Diophantine approximation on manifolds see, for example, [5, 7, 10, 15, 18, 20, 36, 42].

The more delicate Jarník-Besicovitch theorem was also generalised to the case of polynomials and reads as follows.

Theorem 2. *Let* $w > n$ *and* $\Psi_w(q) = q^{-w}$*. Then*

$$\dim \mathcal{L}_n(\Psi_w) = \frac{n+1}{w+1}. \tag{5}$$

The lower bound $\dim \mathcal{L}_n(\Psi_w) \geq \frac{n+1}{w+1}$ was obtained by Baker and Schmidt [2] who also conjectured (5). The conjecture was proved in full generality in [16]. It is worth noting that the generalised Baker-Schmidt problem for manifolds remains an open challenging problem in dimensions $n \geq 3$; the case of $n = 2$ was settled by R.C. Baker [3], see also [1, 8] and [7, 11, 43] for its analogue for simultaneous rational approximations.

The various techniques used to prove Theorems 1 and 2 make a substantial use of the properties of discriminants and resultants of polynomials and to some extent the distribution of algebraic numbers. The main substance of this paper will be to overview some relevant recent developments and techniques in this area.

2 Distribution of Discriminants of Integral Polynomials

The discriminant of a polynomial is a vital characteristic that crops up in various problems of number theory. For example, they play an important role in Diophantine equations, Diophantine approximation and algebraic number theory [41].

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial of degree n and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be its roots. By definition, the discriminant of P is given by

$$D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2. \tag{6}$$

The following matrix formula for $D(P)$ is well known:

$$D(P) = (-1)^{n(n-1)/2} \begin{vmatrix} 1 & a_{n-1} & a_{n-2} & \ldots & a_0 & 0 & \ldots \\ 0 & a_n & a_{n-1} & \ldots & a_1 & a_0 & \ldots \\ & & & \ldots & & & \\ 0 & \ldots & 0 & a_n & \ldots & a_1 & a_0 \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots & 0 & 0 & \ldots \\ 0 & na_n & (n-1)a_{n-1} & \ldots & a_1 & 0 & \ldots \\ & & & \ldots & & & \\ 0 & \ldots & \ldots & 0 & na_n & \ldots & a_1 \end{vmatrix}.$$

Thus, the discriminant is an integer polynomial of the coefficients of P. Consequently, whenever P has rational integer coefficients the discriminant $D(P)$ is also an integer. Furthermore,

$$|D(P)| \geqslant 1 \quad \text{for any } P \in \mathbb{Z}[x] \text{ with } \deg P \geq 1 \text{ and } D(P) \neq 0. \tag{7}$$

Clearly, by (6), $D(P) \neq 0$ if and only if P has no multiple roots.

Fix $n \in \mathbb{N}$. Let $Q > Q_0(n)$, where $Q_0(n)$ is a sufficiently large number. Let $\mathcal{P}_n(Q)$ denote the set of non-zero polynomials $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ and $H(P) \leq Q$. Throughout c_j, $j = 0, 1, \ldots$ will stand for positive constants depending on n only. When it is not essential for calculations we will denote these constants as $c(n)$. Also we will use the Vinogradov symbols: $A \ll B$ meaning that $A \leq c(n)B$. The expression $A \asymp B$ will mean $B \ll A \ll B$. Finally $\#S$ means the cardinality of a finite set S. In what follows we consider polynomials such that

$$c_1 Q < H(P) \leq Q, \; 0 < c_1 < 1. \tag{8}$$

Using the matrix representation for $D(P)$ one readily verifies that $|D(P)| < c(n)Q^{2n-2}$ for $P \in \mathcal{P}_n(Q)$. Thus, by (7), we have that

$$1 \leq |D(P)| < c(n)Q^{2n-2} \tag{9}$$

for polynomials $P \in \mathcal{P}_n(Q)$ with no multiple roots. Further, it is easily verified that

$$\#\mathcal{P}_n(Q) < 2^{2n+2}Q^{n+1}.$$

The latter together with (9) shows that $[1, c(n)Q^{2n-2}]$ contains intervals of length $c(n)Q^{n-3}$ that are not hit by the values of $D(P)$ for any $P \in \mathcal{P}_n(Q)$ whatsoever. For $n \geqslant 4$ these intervals can be arbitrarily large. Thus, the discriminants $D(P)$ are rather sparse in the interval $[1, c(n)Q^{2n-2}]$.

In order to understand the distribution of the values of $D(P)$ as P varies within $\mathcal{P}_n(Q)$, for each given $v \geq 0$ we introduce the following subclass of $\mathcal{P}_n(Q)$:

$$\mathcal{P}_n(Q, v) = \{P \in \mathcal{P}_n(Q) \; : \; |D(P)| < Q^{2n-2-2v} \text{ and (8) holds}\}. \tag{10}$$

These subclasses are of course dependant on the choice of c_1, but for the moment let us think of c_1 as a fixed constant.

We initially discuss some simple techniques utilizing the theory of continued fractions that enable one to obtain non-trivial lower bounds for $\#\mathcal{P}_n(Q, v)$ in terms of Q and v.

The first observation concerns shifts of the variable x by integers. More precisely, if $m \in \mathbb{Z}$ then $D(P(x)) = D(P(x - m))$. The height of $P(x - m)$ changes as m varies. It is a simple matter to see that imposing (8) on $P(x - m)$ restricts m to at most c_2 values. Furthermore, (8) ensures that polynomials of relatively small height cannot be in $\mathcal{P}_n(Q, v)$.

By (6), the fact that P belongs to $\mathcal{P}_n(Q, v)$ with $v > 0$ implies that P must necessarily have at least two close roots. This gives rise to a natural path to constructing polynomials P in $\mathcal{P}_n(Q, v)$—we have to make sure that they have close roots. We now describe a very special procedure that enables one to do exactly this.

Take n best approximations (convergents) $\frac{p_j}{q_j}$ to the number $\sqrt{2}$ with $k + 1 \leq j \leq k + n$ for some $k \in \mathbb{N}$. Define the polynomial

$$T_{\sqrt{2}}(x) = \prod_{j=k+1}^{k+n} (q_j x - p_j)$$

of degree n. Clearly the above mentioned best approximations to $\sqrt{2}$ are the roots of T. Also note that the height of T is $\ll a_n$, where

$$a_n = \prod_{k+1 \leq i \leq k+n} q_i .$$

From the theory of continued fractions we know that $q_j \leq 3q_{j-1}$. Thus $a_n \leq c(n) q_{k+1}^n$. On the other hand, we also know that for $i < j$

$$\left| \frac{p_i}{q_i} - \frac{p_j}{q_j} \right| \leq \left| \frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} \right| = \frac{1}{q_i q_{i+1}} < \frac{1}{q_i^2}.$$

Therefore, we can estimate the following product

$$\Pi_1 = \prod_{k+1 \leq i < j \leq k+n} \left| \frac{p_i}{q_i} - \frac{p_j}{q_j} \right|^2 \leq \prod_{k+1 \leq i \leq k+n-1} \left(\frac{1}{q_i^2} \right)^{2(k+n-i)} \ll q_{k+1}^{-\sigma},$$

where

$$\sigma = 2 \sum_{i=k+1}^{k+n-1} 2(k + n - i) = 4 \sum_{\ell=1}^{n-1} \ell = 2n(n - 1).$$

and see that

$$|D(T_{\sqrt{2}}(x))| \leq a_n^{2n(n-1)} \Pi_1 \ll q_{k+1}^{2n(n-1)} q_{k+1}^{-\sigma} = 1.$$

This way we construct a polynomial of degree n with arbitrarily large height and discriminant as small as $c(n)$. However, to get quantitative bounds for $\#\mathcal{P}_n(Q, v)$ more needs to be done. The following lemmas underpin the construction.

Lemma 1. *Let I be an interval, $I \subset \mathbb{R}$, c_3 and c_4 be positive constants such that* $\max\{c_3, c_4\} \leq 1$. *Given a sufficiently large Q, let $\mathcal{L}_{1,Q}(c_3, c_4)$ be the set of $x \in I$ such that the system of inequalities*

$$\begin{cases} |qx - p| < c_3 Q^{-1}, \\ 1 \le q \le c_4 Q, \end{cases} \tag{11}$$

has a solution in coprime $(p, q) \in \mathbb{Z} \times \mathbb{N}$. *Then for* $c_3 c_4 < \lambda$, $0 < \lambda < \frac{1}{3}$, *we have*

$$\mu \mathcal{L}_{1,Q}(c_3, c_4) < 3\lambda |I|.$$

Lemma 2. *Let* $M_n(Q)$ *denote the set of* $x \in I$ *such that the following* n *systems of inequalities*

$$\begin{cases} \frac{Q^{-1}}{3^i (n+1)^i} < |q_i x - p_i| < \frac{Q^{-1}}{3^{i-1}(n+1)^{i-1}} \\ 3^{i-2}(n+1)^{i-2} Q \le q_i \le 3^{i-1}(n+1)^{i-1} Q, \ 1 \le i \le n \end{cases}$$

have solutions in $\{(p_i, q_i)\}_{i=1}^n$. *Then* $\mu M_n(Q) > \frac{|I|}{n+1}$.

Lemma 1 is proved by summing up the measures of intervals given by the first inequality of (11). Lemma 2 is a corollary of Lemma 1 (which should be applied n times) and Minkowski's theorem for convex bodies. See [21] for details.

Now take any point $x_1 \in M_n(Q)$ and define

$$T_1(x) = \prod_{j=1}^{n} (q_j x - p_j) \qquad \text{and} \qquad Q = c(n) \prod_{i=1}^{n} q_i,$$

where (p_j, q_j) arise from Lemma 2. Estimating $|D(T_1)|$ gives

$$|D(T_1)| \ll Q^{2n-2} \prod_{1 \le i < j \le n} \left| \frac{p_i}{q_i} - \frac{p_j}{q_j} \right| \ll 1.$$

We now use the fact that $M_n(Q)$ is a fairly large subset of I to produce other polynomials with this property. For this purpose we choose points $x_2, x_3, \ldots \in M_n(Q)$ that are well separated. As a result we obtain

Theorem 3 ([21]). *For any sufficiently large* Q *there are* $c(n) Q^{\frac{2}{n}}$ *polynomials* $P \in \mathcal{P}_n(Q)$ *such that* $1 \le |D(P)| \le c(n)$.

The above ideas can be generalised to give a similar bound for the number of polynomials $P \in \mathcal{P}_n(Q)$ such that $|D(P)|$ lies in a neighborhood of some K with $c(n) < K < c(n) Q^{2n-2}$.

Theorem 4 ([21]). *For any* θ, $0 \le \theta \le 2n-2$, *there are at least* $c(n) Q^{2/n}$ *polynomials* $P \in \mathcal{P}_n(Q)$ *with discriminants satisfying the inequalities*

$$c_5 Q^{\theta} < |D(P)| < c_6 Q^{\theta}.$$

We proceed by describing a more sophisticated method from [23] that produces lower bounds for $\#\mathcal{P}_n(Q, v)$. The main result is as follows.

Theorem 5 ([23]). *Let $v \in [0, \frac{1}{2}]$. Then there are at least $c(n)Q^{n+1-2v}$ polynomials $P \in \mathcal{P}_n(Q)$ with discriminants*

$$|D(P)| < Q^{2n-2-2v}. \tag{12}$$

Establishing upper bounds for $\#\mathcal{P}_n(Q, v)$ is likely a more difficult task. We expect that if we impose some reasonable conditions on polynomials P from $\mathcal{P}_n(Q)$ (for example excluding reducible polynomials) then the lower bound given by Theorem 5 would become sharp. We now state this formally as the following

Problem 1. Find reasonable constrains on polynomials P that chop a subclass $\mathcal{P}'_n(Q)$ off $\mathcal{P}_n(Q)$ such that $\#\mathcal{P}'(Q) \asymp \#\mathcal{P}_n(Q)$ and for $v \in [0, \frac{1}{2}]$

$$\#\{P \in \mathcal{P}'_n(Q) : |D(P)| < Q^{2(n-1-v)}\} \asymp Q^{n+1-2v}. \tag{13}$$

Obtaining the estimates of this ilk for a larger range of v is another problem. We wish to note that (13) is false for $\mathcal{P}_n(Q)$—see [32] for precise upper and lower bounds in the case $v < 3/5$ and $n = 3$.

Problem 2. For each n find the function $f_n(v)$, if it exists at all, such that for all sufficiently large Q one has the estimates

$$\#\{P \in \mathcal{P}_n(Q) : |D(P)| < Q^{2(n-1-v)}\} \asymp Q^{n+1-f_n(v)}. \tag{14}$$

It was shown in [32] that $f_3(v) = \frac{5}{3}v$ for $0 \le v \le 3/5$.

2.1 Sketch of the Proof of Theorem 5

Underlying the proof of Theorem 5 is the following result, which essentially plays the role of Lemma 1 in this more general context. In what follows, given an interval $I \subset [-1/2, 1/2]$, let $\mathcal{L}_n(I, Q, v, c_7, c_8)$ be the set of $x \in I$ such that

$$\begin{cases} |P(x)| < c_7 Q^{-n+v}, \\ |P'(x)| < c_8 Q^{1-v} \end{cases} \tag{15}$$

holds for some $P \in \mathcal{P}_n(Q)$.

Theorem 6. *Let Q denote a sufficiently large number, $v \in [0, \frac{1}{2}]$ and let c_7 and c_8 be positive constants such that $c_7 c_8 < n^{-1}2^{-n-12}$. Then*

$$\mu\mathcal{L}_n(I, Q, v, c_7, c_8) < \frac{|I|}{4}.$$

We now explain the role of Theorem 6 in establishing Theorem 5. Suppose that $P \in \mathbb{Z}[x]$, $\deg P \le n$, $|a_n| > cH$. If $|a_n| \le cH$ then the polynomial can be transformed into one with a large leading coefficient with the same discriminant—see [41].

By Dirichlet's pigeonhole principle, for any point $x \in I$ and $Q > 1$ the following system

$$\begin{cases} |P(x)| < Q^{-n+v}, \\ |P'(x)| < 8nQ^{1-v} \end{cases} \tag{16}$$

holds for some polynomial $P \in \mathcal{P}_n(Q)$. Let $\gamma = n^{-2}2^{-n-15}$ and $I = [-1/2, 1/2]$. Then, by Theorem 6, the set

$$B_1 = I \setminus \mathcal{L}_n(I, Q, v, 1, \gamma) \cup \mathcal{L}_n(I, Q, v, \gamma, 8n)$$

satisfies $\mu B_1 \geqslant \frac{1}{2}$ for all sufficiently large Q. Hence for any $x_1 \in B_1$ the solution $P \in \mathcal{P}_n(Q)$ to the system (16) must satisfy

$$\begin{cases} \gamma Q^{-n+v} < |P(x_1)| < Q^{-n+v}, \\ \gamma Q^{1-v} < |P'(x_1)| < 8nQ^{1-v}. \end{cases} \tag{17}$$

For all x in the interval $|x - x_1| < Q^{-\frac{2}{3}}$, the Mean Value Theorem gives

$$P'(x) = P'(x_1) + P''(\xi_1)(x - x_1) \;\text{ for some } \xi_1 \in [x, x_1]. \tag{18}$$

The obvious estimate $|P''(\xi_2)| < n^3 Q$ implies $|P''(\xi_1)(x - x_1)| < n^3 Q^{\frac{1}{3}}$. But $|P'(x_1)| \gg Q^{\frac{1}{2}}$ for $v \le \frac{1}{2}$ and therefore, by (18) and the second inequality of (17), for sufficiently large Q we have that

$$\tfrac{\gamma}{2} Q^{1-v} < \tfrac{1}{2}|P'(x_1)| < |P'(x)| < 2|P'(x_1)| < 16nQ^{1-v}.$$

There are four possible combinations for signs of $P(x_1)$ and $P'(x_1)$. To illustrate the ideas we consider the case when $P_1(x_1) < 0$ and $P_1'(x_1) > 0$—the others are dealt with in a similar way. Our goal for now is to find a root of P close to x_1. Once again we appeal to the Mean Value Theorem:

$$P(x) = P(x_1) + P'(\xi_2)(x - x_1) \;\text{ for some } \xi_2 \in [x_1, x]. \tag{19}$$

Write $x = x_1 + \Delta$ and suppose that $\Delta > 2\gamma^{-1}Q^{-n-1+2v}$. If $P(x_1) < P(x_1 + \Delta) < 0$ then the first inequality of (17) implies

$$0 < P(x_1 + \Delta) - P(x_1) < Q^{-n+v}.$$

On the other hand we have

$$|P'(\xi_2)\Delta| > \frac{\gamma}{2} Q^{1-v} 2\gamma^{-1} Q^{-n-1+2v} = Q^{-n+v}.$$

Thus in view of (19) we obtain a contradiction. This means that $P_1(x_1 + \Delta) > 0$ and there is a real root α of the polynomial $P(x)$ between x_1 and $x_1 + \Delta$. Once again using the Mean Value Theorem and the estimates for $P(x)$ and $P'(\alpha)$ we get

$$|x_1 - \alpha| < 2\gamma^{-1} Q^{-n-1+2v} = n 2^{n+13} Q^{-n-1+2v}. \tag{20}$$

Note that as well as ensuring that α, a root of P, is close to x_1 inequalities (17) keep α sufficiently away from x_1. We now explain this more formally. Again we consider only one of the four possibilities: $P(x_1) > 0$, $P'(x_1) < 0$. With $x = x_1 + \Delta_1$, by the Mean Value Theorem, we have

$$P(x) = P(x_1) + P'(\xi_3)\Delta_1,\ \xi_3 \in [x_1, x]. \tag{21}$$

If $\Delta_1 < 2^{-4} n^{-1} \gamma Q^{-n-1+2v}$ then in (21) the following holds: $|P(x_1)| > \gamma Q^{-n+v}$ and $|P'(\xi_3)\Delta_1| < \gamma Q^{-n+v}$. It implies that the polynomial $P(x)$ cannot have any root in the interval $[x_1, x_1 + \Delta_1]$ and therefore for any root α, we have

$$n^{-1} 2^{-n-13} Q^{-n-1+2v} < |x - \alpha|.$$

This time let α be the root of P closest to x_1. By the Mean Value Theorem,

$$P'(\alpha) = P'(x_1) + P''(\xi_4)(x_1 - \alpha),\ \xi_4 \in [x, \alpha],$$

the estimate $|P''(\xi)| < n^3 Q$ and (20) for sufficiently large Q we get

$$n^{-1} 2^{-n-13} Q^{1-v} < |P'(\alpha)| < 16 n Q^{1-v}.$$

The square of derivative is a factor of the discriminant of P. Taking into account that for $|a_n| \asymp H(P)$ all roots of the polynomial are bounded, see [41]. Then we can estimate the differences $|\alpha_i - \alpha_j|$, $2 \le i < j \le n$, by a constant $c(n)$. This way we obtain (12). Since $\mu B_1 \ge 1/2$ and x_1 is an arbitrary point in B_1 we must have $\gg Q^{n+1-2v}$ different α's that arise from (20). Since each polynomial P of degree $\le n$ has at most n roots this gives $\gg Q^{n+1-2v}$ polynomials in $\mathcal{P}_n(Q, v)$ satisfying (12)—see [13] for further details.

2.2 *Sketch of the Proof of Theorem 6*

The purpose of this section is to discuss the key ideas of the proof of Theorem 6 given in [13] as they may be useful in a variety of other tasks. We start by estimating the measure of x such that the system

$$\begin{cases} |P(x)| < c_{11} Q^{-n+v}, \\ Q^{1-v_1} < |P'(x)| < c_{12} Q^{1-v} \end{cases} \tag{22}$$

is solvable for $P \in \mathcal{P}_n(Q)$, where v_1 satisfies $v < v_1 \leq 1$ and will be specified later.

We shall see that $P'(x)$ can be replaced with $P'(\alpha)$ in the second inequality of (22), where α denotes the root of P nearest to x. Indeed, using the Mean Value Theorem gives

$$P'(x) = P'(\alpha) + P''(\xi_1)(x-\alpha), \quad \xi_1 \in (\alpha, x).$$

We apply the following inequality for $|x-\alpha|$

$$|x-\alpha| < n\frac{|P(x)|}{|P'(x)|},$$

which was proved in [17, 41]. Then

$$|P'(\alpha)| = |P'(x) - P''(\xi_5)(x-\alpha)|, \ \xi_5 \in (\alpha, x).$$

As

$$|P''(\xi_1)(x-\alpha)| \leq n^3 Q c_{11} n Q^{-n-1+v+v_1} = c_{11} n^4 Q^{-n+v+v_1}$$

for sufficiently large Q we obtain

$$\frac{3}{4} Q^{1-v_1} \leq \frac{3}{4}|P'(x)| \leq |P'(\alpha)| \leq \frac{4}{3}|P'(x)| \leq \frac{4}{3} c_{12} Q^{1-v}$$

and

$$\frac{3}{4}|P'(\alpha)| \leq |P'(x)| \leq \frac{4}{3}|P'(\alpha)|.$$

Therefore for sufficiently large Q inequality (22) implies

$$\begin{cases} |P(x)| < c_{11} Q^{-n+v} \\ \frac{3}{4} Q^{1-v_1} < |P'(\alpha)| < \frac{4}{3} c_{12} Q^{1-v} \\ |a_j| \leq Q. \end{cases} \tag{23}$$

Let $\mathcal{L}_n'(v)$ denote the set of x, for which system (23) is solvable for $P \in \mathcal{P}_n(Q)$. Now we are able to prove that $\mu \mathcal{L}_n'(v) < \frac{3}{8}|I|$.

Consider the intervals:

$$\sigma_1(P) = \{x : |x-\alpha| < \frac{4}{3} c_{11} n Q^{-n+v} |P'(\alpha)|^{-1}\}$$

and

$$\sigma_2(P) = \{x : |x - \alpha| < c_{13} Q^{-1+\nu} |P'(\alpha)|^{-1}\}.$$

The value of c_{13} will be specified below. Of course, each polynomial P has up to n roots and potentially we have to consider all the different intervals $\sigma_1(P)$ and $\sigma_2(P)$ that correspond to each P. However, this will only affect the constant in the estimates. Thus, without loss of generality we confine ourselves to a single choice of $\sigma_1(P)$ and $\sigma_2(P)$. Obviously

$$|\sigma_1(P)| \leq \tfrac{4}{3} c_{11} c_{13}^{-1} n Q^{-n+1} |\sigma_2(P)|. \tag{24}$$

Fix a vector $\overline{b} = (a_n, \ldots, a_2)$ of the coefficients of P. The polynomials $P \in \mathcal{P}_n(Q)$ with the same vector $\overline{b}$ form a subclass of $\mathcal{P}_n(Q)$ which will be denoted by $\mathcal{P}(\overline{b})$.

The interval $\sigma_2(P_1)$ with $P_1 \in \mathcal{P}(\overline{b})$ is called *inessential* if there is another interval $\sigma_2(P_2)$ with $P_2 \in \mathcal{P}(\overline{b})$ such that

$$|\sigma_2(P_1) \cap \sigma_2(P_2)| \geqslant \tfrac{1}{2} |\sigma_2(P_1)|.$$

Otherwise for any $P_2 \in \mathcal{P}(\overline{b})$ different from P_1

$$|\sigma_2(P_1) \cap \sigma_2(P_2)| < \tfrac{1}{2} |\sigma_2(P_1)|$$

and the interval $\sigma_2(P_2)$ is called *essential*.

The case of essential intervals. In this case every point $x \in I$ belongs to at most two essential intervals $\sigma_2(P)$. Hence for any vector $\overline{b}$

$$\sum_{\substack{P \in \mathcal{P}(\overline{b}) \\ \sigma_2(P) \text{ is essential}}} |\sigma_2(P)| \leq 2|I|. \tag{25}$$

The number of all possible vectors $\overline{b}$ is at most $(2Q+1)^{n-1} < 2^n Q^{n-1}$. Then, by (24) and (25), we obtain

$$\sum_{\overline{b}} \sum_{\substack{P \in \mathcal{P}(\overline{b}) \\ \sigma_2(P) \text{ is essential}}} |\sigma_1(P)| < \tfrac{4}{3} c_{11} c_{13}^{-1} n Q^{-n+1} 2|I| 2^n Q^{n-1} = n 2^{n+2} c_{11} c_{13}^{-1} |I|.$$

Thus for $c_{13} = n 2^{n+5} c_{11}$ the measure will be not larger than $\frac{1}{8}|I|$.

The case of inessential intervals. In this case we need to estimate the values of $|P_j(x)|$, $j = 1, 2$, for $x \in \sigma_2(P_1) \cap \sigma_2(P_2)$. By Taylor's formula,

$$P_j(x) = P_j'(\alpha)(x - \alpha) + \tfrac{1}{2} P_j''(\xi_6)(x - \alpha)^2 \text{ for some } \xi_6 \in (\alpha, x),$$

where α is the root of either P_1 or P_2 as appropriate, and

$$P_j'(x) = P_j'(\alpha) + P_j''(\xi_7)(x-\alpha) \text{ for some } \xi_7 \in (\alpha, x).$$

The second summand is estimated by

$$|P''(\xi_6)(x-\alpha)^2| \le 2n^3 c_{13}^2 Q^{-3+2v+2v_1},$$

while

$$|P'(\alpha)(x-\alpha)| < c_{13} Q^{-1+v}.$$

As $2v_1 < 2 - v$ for an appropriate choice of $v_1 < \frac{3}{4}$ we obtain

$$|P_j(x)| \le \tfrac{7}{6} c_{13} Q^{-1+v}, \;\; j = 1, 2. \tag{26}$$

Similarly we obtain the following estimate for $P_j'(x)$ when $v_1 \le 2 - 2v$:

$$|P_j'(x)| \le \tfrac{4}{3} c_{12} Q^{1-v}, \;\; j = 1, 2. \tag{27}$$

Let $K(x) = P_2(x) - P_1(x) \in \mathbb{Z}[x]$. Obviously $K(x)$ is non-zero and has the form $K(x) = b_1 x + b_0$. By (26) and (27), we readily obtain that

$$|b_1 x + b_0| < \tfrac{8}{3} c_{13} Q^{-1+v} \tag{28}$$

and

$$|b_1| = |K'(x)| < \tfrac{8}{3} c_{12} Q^{1-v}. \tag{29}$$

Thus, the union of inessential intervals can be covered by intervals $\Delta(b_1, b_0) \subset I$ given by (28). For fixed b_0 and b_1 the length of $\Delta(b_1, b_0)$ is bounded by $\frac{16}{3} c_{13} Q^{-1+v} b_1^{-1}$. Given that $x \in I$ and (28) is satisfied we conclude that b_0 takes at most $|I||b_1| + 2$ values. Then

$$\sum_{b_0} |\Delta(b_1, b_0)| \le \tfrac{16}{3} c_{13} Q^{-1+v} b_1^{-1} (|I||b_1| + 2) < 6 c_{13} Q^{-1+v} |I|. \tag{30}$$

Using (29) we further obtain that

$$\sum_{b_1} \sum_{b_0} |\Delta(b_1, b_0)| \le 2^5 c_{12} c_{13} Q^{1-v-1+v} |I| = n 2^{n+8} c_{11} c_{12} |I| = \tfrac{1}{8} |I|$$

for we have that $c_{11} c_{12} < n^{-1} 2^{-n-11}$. Finally, combining the estimates for essential and inessential intervals we obtain $\frac{1}{4}|I|$ as an upper bound for their total measure. The case $v \ge v_1$ can be deal with using methods described in [17] and [36].

3 Divisibility of Discriminants by Prime Powers

Let p be a prime number. Throughout μ_p denotes the Haar measure on $\mathbb{Q}_p$ normalized to have $\mu_p(\mathbb{Z}_p) = 1$. In this section we consider the divisibility of the discriminant $D(P)$, $P \in \mathcal{P}_n(Q)$ by prime powers p^l. This natural arithmetical question has usual interpretation in terms of Diophantine approximation in $\mathbb{Q}_p$, the field of p-adic number. Indeed, $p^l | D(P)$ if and only if $|D(P)|_p \le p^{-l}$, where $|\cdot|_p$ stands for the p-adic norm. Thus, the question we outlined above becomes a p-adic analogue of the problems we have considered in the previous section. Naturally, we proceed with the following p-adic analogue of Theorem 5.

Theorem 7 ([24]). *Let $v \in [0, \frac{1}{2}]$. Then there are at least $c(n)Q^{n+1-2v}$ polynomials $P \in \mathcal{P}_n(Q)$ with*

$$|D(P)|_p < Q^{-2v}. \tag{31}$$

The proof of this result relies on the following p-adic version of Theorem 6.

Theorem 8 ([24]). *Let Q denote a sufficiently large number and c_{14} and c_{15} denote constants depending only on n. Also, let K be a disc in $\mathbb{Q}_p$. Assume that $c_{14}c_{15} < 2^{-n-11}p^{-8}$ and $v \in [0, \frac{1}{2}]$. If $M_{n,Q}(c_{14}, c_{15})$ is the set of $w \in K \subset \mathbb{Q}_p$ such that the system of inequalities*

$$\begin{cases} |P(w)|_p < c_{14}Q^{-n-1+v}, \\ |P'(w)|_p < c_{15}Q^{-v} \end{cases}$$

has solutions in polynomials $P \in \mathcal{P}_n(Q)$, then

$$\mu_p(M_{n,Q}(c_{14}, c_{15})) < \tfrac{1}{4}\mu_p(K).$$

The techniques used in the proof of this theorem are essentially the p-adic analogues of those used for establishing Theorem 6 and draw on the estimates obtained in [39]—see [24] for more details. Skipping any explanation of the proof of Theorem 8, we now show how it is used for establishing Theorem 7.

Let K be a disc in $\mathbb{Q}_p$, $M_{n,Q}$ be the same as in Theorem 8, $\gamma = p^{-11}2^{-n-11}$ and

$$B = K \setminus \big(M_{n,Q}(\gamma, p^3) \cup M_{n,Q}(1, \gamma)\big).$$

Then, by Theorem 8, $\mu_p(B) \geqslant \frac{1}{2}\mu_p(K)$. Take any $w_1 \in B$. Then, using Dirichlet's pigeonhole principle we can find a polynomial $P \in \mathcal{P}_n(Q)$ such that $|P(w_1)|_p < Q^{-n-1+v}$ and $|P'(w_1)|_p < p^3Q^{-v}$. Since $w_1 \in B$ we have that

$$\begin{cases} \gamma Q^{-n-1+v} \le |P(w_1)|_p < Q^{-n-1+v}, \\ \gamma Q^{-v} \le |P'(w_1)|_p < p^3Q^{-v}. \end{cases} \tag{32}$$

Let $w \in K_1 = \{w \in \mathbb{Q}_p : |w - w_1|_p < Q^{-\frac{3}{4}}\}$. By Taylor's formula

$$P'(w) = P'(w_1) + \sum_{i=2}^{n} \frac{P^{(i)}(w_1)(w - w_1)^{i-1}}{(i-1)!}.$$

Since

$$|(i-1)!|_p^{-1} |P^{(i)}(w_1)|_p |w - w_1|_p^{i-1} \ll Q^{-\frac{3}{4}},$$

and

$$|P'(w)|_p \geqslant \gamma Q^{-v} \gg Q^{-\frac{1}{2}},$$

for all $w \in K_1$ we obtain that $|P'(w)|_p = |P'(w_1)|_p$. Let α be the closest root of $P(w)$ to the point w_1. Then, using the Mean Value Theorem, we get that $|w_1 - \alpha|_p \leq |P(w_1)|_p |P'(w_1)|_p^{-1}$. By (32),

$$|w_1 - \alpha|_p \leq \gamma^{-1} Q^{-n-1+2v}. \tag{33}$$

To estimate the distance between w_1 and the root of the polynomial we can also apply Hensel's lemma. Since $|P(w_1)|_p < |P'(w_1)|_p^2$ we obtain that the sequence $w_{n+1} = w_n - \frac{P_1(w_n)}{P_1'(w_n)}$ converges to the root α_1 of P that lies in $\mathbb{Q}_p$ and satisfies the inequality

$$|w_1 - \alpha_1|_p \leq |P(w_1)|_p |P'(w_1)|_p^{-2} \leq \gamma^{-2} Q^{-n-1+3v}. \tag{34}$$

Since $0 < \gamma < 1$ and $v > 0$ the right hand side of (33) is less than that of (34). This implies that the root α belongs to the disc with center w_1 of radius less than the radius for the disc defined for the root α_1. By Hensel's lemma, we find that $\alpha_1 \in \mathbb{Q}_p$ but estimate (33) does not guarantee that $\alpha \in \mathbb{Q}_p$. Suppose that $\alpha \neq \alpha_1$ and consider the discriminant of the polynomial $P \in \mathcal{P}_n(Q)$

$$D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2. \tag{35}$$

From $|\alpha_j|_p \ll 1$ follows that $|\alpha_i - \alpha_j|_p \ll 1$. The product in (35) contains the difference $(\alpha - \alpha_1)$ for some $i \neq j$. We have $D(P) \in \mathbb{Z}$ and $|D(P)| \ll Q^{2n-2}$. Assume for the moment that $D(P) \neq 0$. Then $|D(P)|_p \geq |D(P)|^{-1} \gg Q^{-2n+2}$. From (33) and (34) we further obtain that

$$|\alpha_1 - \alpha|_p = |\alpha_1 - w_1 + w_1 - \alpha|_p \leq \max\{|w_1 - \alpha_1|_p, |w_1 - \alpha|_p\} \leq \gamma^{-2} Q^{-n-1+3v}.$$

Therefore

$$Q^{-2n+2} \ll |D(P)|_p \ll |\alpha_1 - \alpha|_p^2 < \gamma^{-4} Q^{-2n-2+6v}. \tag{36}$$

For $v \leq \frac{1}{2}$ and $Q > Q_0$ the inequality $Q^{-2n+2} \ll \gamma^{-4} Q^{-2n-2+6v}$ is a contradiction. Hence, $\alpha_1 = \alpha$. Thus, $\alpha \in \mathbb{Q}_p$ and $|w_1 - \alpha|_p$ satisfies condition (33).

In the case when $D(P) = 0$ one has to use the above argument with P replaced by its factor, say $\tilde{P}$. If α and α_1 are conjugate over $\mathbb{Q}$ one can take $\tilde{P}$ to be the minimal polynomials (over $\mathbb{Z}$) of α. Otherwise, $\tilde{P}$ is taken to be the product of the minimal polynomials for α and α_1.

By Taylor's formula,

$$P'(\alpha) = P'(w_1) + \sum_{i=2}^{n}((i-1)!)^{-1}P^{(i)}(w_1)(\alpha - w_1)^{i-1}. \tag{37}$$

Observe that

$$|(i-1)!|_p^{-1}|P^{(i)}(w_1)|_p|\alpha - w_1|_p^{i-1} \ll Q^{-n-1+2v}.$$

Then, by (33), we obtain

$$|P'(\alpha)|_p = |P'(w_1)|_p < p^3 Q^{-v}.$$

Therefore

$$|D(P)|_p = |a_n^2 \prod_{k=2}^{n}(\alpha_1 - \alpha_k)^2|_p \, |a_n^{2n-4} \prod_{2\le i<j\le n}(\alpha_i - \alpha_j)^2|_p \ll |P'(\alpha)|_p^2 \ll Q^{-2v}. \tag{38}$$

Inequality (33) implies that in the neighborhood of the point $w_1 \in B$ there exists a root α of the polynomial P with discriminant satisfying (38).

By (33), w_1 lies in the disc $K(\alpha, c(n)Q^{-n-1+2v})$. Since w_1 is an arbitrary point of B and $\mu_p(B) \ge \frac{1}{2}\mu_p(K)$, we must have $\ge c(n)Q^{n+1-2v}\mu_p(K)$ discs $K(\alpha, c(n)Q^{-n-1+2v})$ to cover B, where α is a root of some $P \in \mathcal{P}_n(Q)$ satisfying (38). Since each polynomial $P \in \mathcal{P}_n(Q)$ has at most n roots we must have $\ge c(n)Q^{n+1-2v}$ different polynomials $P \in \mathcal{P}_n(Q)$ satisfying (38), that is (31).

4 Close Conjugate Algebraic Numbers

Estimating the distance between conjugate algebraic numbers of degree n (in $\mathbb{C}$) has been investigated over the last 50 years. There are various upper and lower bounds found. However, the exact answers are known in the case of degree 2 and 3 only. Define κ_n (respectively κ_n^*) to be the infimum of κ such that the inequality

$$|\alpha_1 - \alpha_2| > H(\alpha_1)^{-\kappa}$$

holds for arbitrary conjugate algebraic numbers (respectively algebraic integers) $\alpha_1 \ne \alpha_2$ of degree n with sufficiently large height $H(\alpha_1)$. Here and elsewhere $H(\alpha)$ denotes the height of an algebraic number α, which is the absolute height of the minimal polynomial of α over $\mathbb{Z}$. Clearly, $\kappa_n^* \le \kappa_n$ for all n.

In 1964 Mahler [37] proved the upper bound $\kappa_n \leq n-1$, which is the best estimate up to date. It can be easily shown that $\kappa_2 = 1$ (see, e.g. [30]). Evertse [31] proved that $\kappa_3 = 2$. In the case of algebraic integers $\kappa_2^* = 0$ and $\kappa_3^* \geq 3/2$. The latter has been proved by Bugeaud and Mignotte [30].

For $n > 3$ estimates for κ_n are less satisfactory. At first Mignotte [38] showed that $\kappa_n, \kappa_n^* \geq n/4$ for all $n \geq 3$. Subsequently Bugeaud and Mignotte [29, 30] proved that

$$\begin{array}{ll} \kappa_n \geq n/2 & \text{when } n \geq 4 \text{ is even,} \\ \kappa_n^* \geq (n-1)/2 & \text{when } n \geq 4 \text{ is even,} \\ \kappa_n \geq (n+2)/4 & \text{when } n \geq 5 \text{ is odd,} \\ \kappa_n^* \geq (n+2)/4 & \text{when } n \geq 5 \text{ is odd.} \end{array}$$

In a recent paper Bugeaud and Dujella [28] have further shown that

$$\kappa_n \geq \frac{n}{2} + \frac{n-2}{4(n-1)}. \tag{39}$$

On taking an alternative route it has been shown in [14] that there are numerous examples of close conjugate algebraic numbers:

Theorem 9 ([13, 14]). *For any $n \geq 2$ we have that*

$$\min\{\kappa_n, \kappa_{n+1}^*\} \geq \frac{n+1}{3}. \tag{40}$$

There are at least $c(n)Q^{\frac{n+1}{3}}$ pairs of conjugate algebraic numbers of degree n (or conjugate algebraic integers of degree $n+1$) α_1 and α_2 such that

$$|\alpha_1 - \alpha_2| \asymp H(\alpha_1)^{-\frac{n+1}{3}}$$

The proof of Theorem 9 is based on solvability of system of Diophantine inequalities for analytic functions [7] on the set of positive density on any interval $J \subset [-\frac{1}{2}, \frac{1}{2}]$. The interval $[-\frac{1}{2}, \frac{1}{2}]$ is taken to simplify the calculation of constants.

As above μ will denote Lebesgue measure in $\mathbb{R}$ while λ will be a non-negative constant. Given an interval $J \subset \mathbb{R}$, $|J|$ will denote the length of J. Also, $B(x, \rho)$ will denote the interval in $\mathbb{R}$ centered at x of radius ρ.

Let $n \geq 2$ be an integer, $\lambda \geq 0$, $0 < \nu < 1$ and $Q > 1$. Let $\mathbb{A}_{n,\nu}(Q, \lambda)$ be the set of algebraic numbers $\alpha_1 \in \mathbb{R}$ of degree n and height $H(\alpha_1)$ satisfying

$$\nu Q \leq H(\alpha_1) \leq \nu^{-1} Q \tag{41}$$

and

$$\nu\, Q^{-\lambda} \leq |\alpha_1 - \alpha_2| \leq \nu^{-1} Q^{-\lambda} \qquad \text{for some } \alpha_2 \in \mathbb{R}, \text{ conjugate to } \alpha_1. \tag{42}$$

Theorem 10. *For any $n \geq 2$ there is a constant $\nu > 0$ depending on n only with the following property. For any λ satisfying*

$$0 < \lambda \leq \frac{n+1}{3} \tag{43}$$

and any interval $J \subset [-\frac{1}{2}, \frac{1}{2}]$, for all sufficiently large Q

$$\mu\left(\bigcup_{\alpha_1 \in \mathbb{A}_{n,\nu}(Q,\lambda)} B(\alpha_1, Q^{-n-1+2\lambda}) \cap J\right) \geq \tfrac{3}{4}|J|. \tag{44}$$

Remark. The constant $\frac{3}{4}$ in the right hand side of (44) can be replaced by any positive number < 1.

Corollary 1. *For any $n \geq 2$ there is a positive constant ν depending on n only such that for any λ satisfying (43) and any interval $J \subset [-\frac{1}{2}, \frac{1}{2}]$, for all sufficiently large Q*

$$\#\big(\mathbb{A}_{n,\nu}(Q,\lambda) \cap J\big) \geq \tfrac{1}{8} Q^{n+1-2\lambda}|J|. \tag{45}$$

The deduction of the corollary is rather simple. Indeed, if we have that $B(\alpha_1, Q^{-n-1+2\lambda}) \cap \frac{1}{2}J \neq \varnothing$ then $\alpha_1 \in J$ provided that Q is sufficiently large. Then, using (44) we obtain

$$\#\big(\mathbb{A}_{n,\nu}(Q,\lambda) \cap J\big) 2Q^{-n-1+2\lambda} \geq$$

$$\geq \mu\left(\bigcup_{\alpha_1 \in \mathbb{A}_{n,\nu}(Q,\lambda)} B(\alpha_1, Q^{-n-1+2\lambda}) \cap \tfrac{1}{2}J\right) \overset{(44)}{\geq} \tfrac{1}{4}|J|,$$

whence (45) readily follows. Taking the largest possible value of λ gives Theorem 9.

The key element of the proof of Theorem 10 is a far reaching generalisation of the arguments around (17) shown earlier. The appropriate analogue of Theorem 6 is given by Theorem 5.8 from [7]. In order to give a formal statement we first introduce some further notation. In what follows $\xi_0, \ldots, \xi_n \in \mathbb{R}^+$ will be positive real parameters satisfying the following conditions

$$\begin{aligned} &\xi_i \ll 1 \quad \text{when } 0 \leq i \leq m-1, \\ &\xi_i \gg 1 \quad \text{when } m \leq i \leq n, \\ &\xi_0 < \varepsilon, \qquad \xi_n > \varepsilon^{-1} \end{aligned} \tag{46}$$

for some $0 < m \leq n$ and $\varepsilon > 0$, where the constants in the Vinogradov's symbol $\ll$ depend on n only. We will also assume that

$$\prod_{i=0}^{n} \xi_i = 1. \tag{47}$$

Lemma 3. *For every $n \ge 2$ there are positive constants δ_0 and c_0 depending on n only with the following property. For any interval $J \subset [-\frac{1}{2}, \frac{1}{2}]$ there is a sufficiently small $\varepsilon = \varepsilon(n, J) > 0$ such that for any $\xi_0, \dots, \xi_n$ satisfying (46) and (47) there is a measurable set $G_J \subset J$ satisfying*

$$\mu(G_J) \ge \tfrac{3}{4}|J| \tag{48}$$

such that for every $x \in G_J$ there are $n+1$ linearly independent primitive irreducible polynomials $P \in \mathbb{Z}[x]$ of degree exactly n such that

$$\delta_0\xi_i \le |P^{(i)}(x)| \le c_0\xi_i \qquad \text{for all } i = 0, \dots, n\,. \tag{49}$$

We now reprocess the main steps of the proof of this statement. Let $n \ge 2$ and let $\xi_0, \dots, \xi_n$ be given and satisfy (46) and (47) for some m and ε. Let $J \subset [-\frac{1}{2}, \frac{1}{2}]$ be any interval and $x \in J$. Consider the system of inequalities

$$|P^{(i)}(x)| \le \xi_i \qquad \text{when } 0 \le i \le n\,, \tag{50}$$

where $P(x) = a_n x^n + \cdots + a_1 x + a_0$. Let B_x be the set of $(a_0, \dots, a_n) \in \mathbb{R}^{n+1}$ satisfying (50). Note that B_x is a convex body in $\mathbb{R}^{n+1}$ symmetric about the origin. By (47), the volume of B_x equals $2^{n+1}\prod_{i=1}^n i!^{-1}$. Let $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n$ be the successive minima of B_x. By Minkowski's theorem for successive minima,

$$\frac{2^{n+1}}{(n+1)!} \;\le\; \lambda_0 \dots \lambda_n \operatorname{vol} B_x \;\le\; 2^{n+1}.$$

Substituting the value of $\operatorname{vol} B_x$ gives $\lambda_0 \dots \lambda_n \;\le\; \prod_{i=1}^n i!$, whence we get that

$$\lambda_n \le \lambda_0^{-n} \prod_{i=1}^n i!. \tag{51}$$

Further we define certain subsets of J that we will 'avoid'. The avoidance will alow us to find the polynomials P of interest as well as to establish the lower bounds in (49). Let $E_\infty(J, \delta_1)$ be the set of $x \in J$ such that $\lambda_0 = \lambda_0(x) \le \delta_1$, where $\delta_1 < 1$. By the definition of λ_0, there is a non-zero polynomial $P \in \mathbb{Z}[x]$, $\deg P \le n$ satisfying

$$|P^{(i)}(x)| \le \delta_1\xi_i \qquad (0 \le i \le n). \tag{52}$$

Applying Lemma 3 from [7] gives

$$\mu(E_\infty(J, \delta_1)) \ll \left(1 + \frac{1}{\delta_J^\alpha}\max\left\{\frac{\delta_1\xi_0}{\delta_1}, \frac{1}{\xi_n}\right\}^\alpha\right)\delta_1^{\frac{\alpha}{n+1}}|J|,$$

where $\delta_J > 0$ is a constant. By (46), $\max\{\xi_0, \xi_n^{-1}\} < \varepsilon$. Therefore $\mu(E_\infty(J, \delta_1)) \ll \delta_1^{\alpha/(n+1)}|J|$ provided that $\varepsilon < \delta_J$. Then there is a sufficiently small δ_1 depending on n only such that

$$\mu(E_\infty(J, \delta_1)) \le \tfrac{1}{4n+8}|J|. \tag{53}$$

By construction, for any $x \in J \setminus E_\infty(J, \delta_1)$ we have that

$$\lambda_0 \ge \delta_1. \tag{54}$$

Combining (51) and (54) gives

$$\lambda_n \le c_{16} := \delta_1^{-n} \prod_{i=1}^{n} i!, \tag{55}$$

where c_{16} depends on n only. By the definition of λ_n, there are $(n+1)$ linearly independent integer points $\mathbf{a}_j = (a_{0,j}, \ldots, a_{n,j})$ $(0 \le j \le n)$ lying in the body $\lambda_n B_x \subset c_{16} B_x$. In other words, the polynomials $P_j(x) = a_{n,j}x^n + \cdots + a_{0,j}$ $(0 \le j \le n)$ satisfy the system of inequalities

$$|P_j^{(i)}(x)| \le c_{16}\xi_i \qquad (0 \le i \le n). \tag{56}$$

Let $A = (a_{i,j})_{0 \le i,j \le n}$ be the integer matrix composed from the integer points $\mathbf{a}_j$ $(0 \le j \le n)$. Since all these points are contained in the body $c_{16}B_x$, we have that $|\det A| \ll \mathrm{vol}(B_x) \ll 1$. That is $|\det A| < c_{17}$ for some constant c_{17} depending on n only. By Bertrand's postulate, choose a prime number p satisfying

$$c_{17} \le p \le 2c_{17}. \tag{57}$$

Therefore, $|\det A| < p$. Since $\mathbf{a}_0, \ldots, \mathbf{a}_n$ are linearly independent and integer, $|\det A| \ge 1$. Therefore, $\det A \not\equiv 0 \pmod p$ and the following system

$$A\overline{t} \equiv \overline{b} \pmod p \tag{58}$$

has a unique non-zero integer solution $\overline{t} = {}^t(t_0, \ldots, t_n) \in [0, p-1]^{n+1}$, where $\overline{b} := {}^t(0, \ldots, 0, 1)$ and ^{t} denotes transposition. For $l = 0, \ldots, n$ define $\overline{r}_l = {}^t(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^{n+1}$, where the number of zeros is l. Since $\det A \not\equiv 0 \pmod p$, for every $l = 0, \ldots, n$ the following system

$$A\overline{\gamma} \equiv -\frac{A\overline{t} - \overline{b}}{p} + \overline{r}_l \pmod p \tag{59}$$

has a unique non-zero integer solution $\overline{\gamma} = \overline{\gamma}_l \in [0, p-1]^{n+1}$. Define $\overline{\eta}_l := \overline{t} + p\overline{\gamma}_l$ $(0 \le l \le n)$. Consider the $(n+1)$ polynomials of the form

$$P_l(x) = a_n x^n + \cdots + a_0 := \sum_{i=0}^{n} \eta_{l,i} P_i(x) \in \mathbb{Z}[x], \tag{60}$$

where $(\eta_{l,0}, \ldots, \eta_{l,n}) = \overline{\eta}_l$. Since $\overline{r}_0, \ldots, \overline{r}_n$ are linearly independent modulo p, the vectors $-\frac{A\overline{t}-\overline{b}}{p} + \overline{r}_l$ $(l = 0, \ldots, n)$ are linearly independent modulo p. Hence, by (59), the vectors $\gamma_0, \ldots, \gamma_n$ are linearly independent modulo p. Hence, $\gamma_0, \ldots, \gamma_n$ are linearly independent over $\mathbb{Z}$. Since these vectors are integer, they are also linearly independent over $\mathbb{R}$. Therefore, the vectors $\overline{\eta}_l := \overline{t} + p\overline{\gamma}_l$ $(0 \le l \le n)$ are linearly independent over $\mathbb{R}$. Hence the polynomials given by (60) are linearly independent and so are non-zero.

Let $\overline{\eta} = \overline{\eta}_l$. Observe that $A\overline{\eta}$ is the column ${}^t(a_0, \ldots, a_n)$ of coefficients of P. By construction, $\overline{\eta} \equiv \overline{t} \pmod{p}$ and therefore $\overline{\eta}$ is also a solution of (58). Then, since $\overline{b} = {}^t(0, \ldots, 0, 1)$ and $A\overline{\eta} \equiv \overline{b} \pmod{p}$, we have that $a_n \not\equiv 0 \pmod{p}$ and $a_i \equiv 0 \pmod{p}$ for $i = 0, \ldots, n-1$. Furthermore, by (59), we have that $A\overline{\eta} \equiv \overline{b} + p\overline{r}_l \pmod{p^2}$. Then, on substituting the values of $\overline{b}$ and $\overline{r}_l$ into this congruence one readily verifies that $a_0 \equiv p \pmod{p^2}$ and so $a_0 \not\equiv 0 \pmod{p^2}$. By Eisenstein's criterion, P is irreducible.

Since both $\overline{t}$ and $\overline{\gamma}_l$ lie in $\in [0, p-1]^{n+1}$ and $\overline{\eta} = \overline{t} + p\overline{\gamma}_l$, it is readily seen that $|\eta_i| \le p^2$ for all i. Therefore, using (56) and (57) we obtain that

$$|P^{(i)}(x)| \le c_0 \xi_i \qquad (0 \le i \le n) \tag{61}$$

with $c_0 = 4(n+1)c_{16}c_{17}^2$. Without loss of generality we may assume that the $(n+1)$ linearly independent polynomials P constructed above are primitive (that is the coefficients of P are coprime) as otherwise the coefficients of P can be divided by their greatest common divisor. Thus, $P \in \mathbb{Z}[x]$ are primitive irreducible polynomials of degree n which satisfy the right hand side of (49). The final part of the proof is aimed at establishing the left hand side of (49).

Let $\delta_0 > 0$ be a sufficiently small parameter depending on n. For every $j = \overline{0, n}$ let $E_j(J, \delta_0)$ be the set of $x \in J$ such that there is a non-zero polynomial $R \in \mathbb{Z}[x]$, $\deg R \le n$ satisfying

$$|R^{(i)}(x)| \le \delta_0^{\delta_{i,j}} c_0^{1-\delta_{i,j}} \xi_i, \tag{62}$$

where $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise. Let $\theta_i = \delta_0^{\delta_{i,j}} c_0^{1-\delta_{i,j}} \xi_i$. Then $E_j(J, \delta_0) \subset A_n(J; \theta_0, \ldots, \theta_n)$. In view of (46) and (47), Lemma 3 from [7] is applicable provided that $\varepsilon < \min\{c_0^{-1}, c_0\delta_0\}$. Then, by the same lemma,

$$\mu(E_j(J, \delta_0)) \ll \left(1 + \frac{1}{\delta_J^{\alpha}} \max\left\{\frac{c_0\xi_0}{c_0^n \delta_0}, \frac{1}{\delta_0 c_0 \xi_n}\right\}^{\alpha}\right) (\delta_0 c_0^n)^{1/(n+1)} |J|,$$

where $\delta_J > 0$ is a constant. It is readily seen that the above maximum is $\le \delta_J$ if $\varepsilon < \delta_J \delta_0 c_0$. Then

$$\mu(E_j(J, \delta_0)) \le \tfrac{1}{4n+8} |J| \tag{63}$$

provided that $\varepsilon < \min\{\delta_J \delta_0 c_0, c_0^{-1}, c_0\delta_0\}$ and $\delta_0 = \delta_0(n)$ is sufficiently small. By construction, for any x in the set G_J defined by

$$G_J := J \setminus \left(\bigcup_{j=0}^{n} E_j(J,\delta_0) \cup E_\infty(J,\delta_1) \right)$$

we must necessarily have that $|P^{(i)}(x)| \ge \delta_0 \xi_i$ for all $i = 0,\dots,n$, where P is the same as in (61). Therefore, the left hand side of (49) holds for all i. Finally, observe that

$$\begin{aligned}\mu(G_J) &\ge |J| - \sum_{i=0}^{n} \mu(E_i(J,\delta_0)) - \mu(E_\infty(J,\delta_1)) \\ &\overset{(53)\,\&\,(63)}{\ge} |J| - (n+2)\,\tfrac{1}{(4n+8)}|J| = \tfrac{3}{4}|J|.\end{aligned}$$

The latter verifies (48) and completes the proof.

The following appropriate analogue of Lemma 3 for monic polynomials can be obtained using the techniques of [27].

Lemma 4. *For every $n \ge 2$ there are positive constants δ_0 and c_0 depending on n only with the following property. For any interval $J \subset [-\frac{1}{2}, \frac{1}{2}]$ there is a sufficiently small $\varepsilon = \varepsilon(n, J) > 0$ such that for any positive $\xi_0,\dots,\xi_n$ satisfying (46) and (47) there is a measurable set $G_J \subset J$ satisfying*

$$\mu(G_J) \ge \tfrac{3}{4}|J| \qquad (64)$$

such that for every $x \in G_J$ there is an irreducible monic polynomials $P \in \mathbb{Z}[x]$ of degree $n+1$ satisfying (49).

5 On the Distribution of Resultants

In this section we discuss the distribution of the resultant $R(P_1, P_2)$ of polynomials P_1 and P_2 from $\mathcal{P}_n(Q)$. It is well known that

$$R(P_1, P_2) = a_n^m b_m^n \prod_{\substack{1\le i\le n \\ 1\le j\le m}} (\alpha_i - \beta_j), \qquad (65)$$

where $\alpha_1,\dots,\alpha_n$ are the roots of P_1 and $\beta_1,\dots,\beta_m$ are the roots of P_2; a_n and b_m stand for the leading coefficients of P_1 and P_2 respectively, where $n = \deg P_1$ and $m = \deg P_2$. The resultant $R(P_1, P_2)$ equals zero if and only if the polynomials P_1 and P_2 have a common root. Since the resultant can be represented as the determinant of the Sylvester matrix of the coefficients of P_1 and P_2 it follows that R is integer. Furthermore,

$$|R(P_1, P_2)| \ll Q^{2n} \tag{66}$$

for $P_1, P_2 \in \mathcal{P}_n(Q)$. Akin to the already discussed results for the distribution of determinants we now state their analogue for resultants.

Theorem 11 ([13]). *Let $m \in \mathbb{Z}$ with $0 \le m < n$. Then there exist $\gg Q^{\frac{2(n+1)}{(m+1)(m+2)}}$ pairs of different primitive irreducible polynomials (P_1, P_2) from $\mathcal{P}_n(Q)$ of degree n such that*

$$1 \le |R(P_1, P_2)| \ll Q^{\frac{2(n-m-1)}{m+2}}. \tag{67}$$

Note that the left had side of (67) is obvious since P_1 and P_2 are primitive and irreducible. There are a few interesting corollaries of the above theorem. For $m = 0$ we have at least $c_1 Q^{n+1}$ pairs (P_1, P_2) that satisfy $|R(P_1, P_2)| \ll Q^{n-1}$. For $m = n-1$ we have at least $c_2 Q^{\frac{2}{n}}$ pairs (P_1, P_2) that satisfy $|R(P_1, P_2)| \le c(n)$.

To introduce the ideas of the proof we first consider the case $m = 0$. By Lemma 3 given in the previous section, for any $x \in G_J$ there are different irreducible polynomials P_1 and P_2 of degree n and height $\ll Q$ such that

$$\begin{aligned} \delta_0 Q^{-n} < |P_i(x_1)| < c_0 Q^{-n},\ i = 1, 2 \\ \delta_0 Q < |P_i'(x_1)| < c_0 Q, \end{aligned} \tag{68}$$

Denote by α_1 the root of P_1 closest to x, and by β_1 the root of P_2 closest to x. Using (68) and the Mean Value Theorem, one can easily find that

$$|x - \alpha_1| \ll Q^{-n-1},\ |x - \beta_1| \ll Q^{-n-1}. \tag{69}$$

By (69), we get $|\alpha_1 - \beta_1| \ll Q^{-n-1}$. This together with (65) gives

$$|R(P_1, P_2)| \ll Q^{n-1}. \tag{70}$$

For a fixed pair of (α_1, β_1) inequalities (69) are satisfied only for a set of x of measure $\ll Q^{-n-1}$. Since $\mu(G_J) \gg |J|$, we must have $\gg Q^{n+1}$ diffract pairs (α_1, β_1) with the above properties. Since each polynomial in $\mathcal{P}_n(Q)$ has at most n root, we must have $\gg Q^{n+1}$ pairs of different irreducible polynomials (P_1, P_2) satisfying (70).

Now let $1 \le m \le n-1$. Let $v_0, \ldots, v_m \ge -1$ and $v_0 + v_1 + \cdots + v_m = n - m$. By Lemma 3, for any $x \in G_J$ there exists a pair of irreducible polynomials P_1, $P_2 \in \mathbb{Z}[x]$ of degree $\le n$ such that for $i = 1, 2$ we have that

$$\begin{aligned} \delta_0 Q^{-v_0} \le |P_i(x)| \le c_0 Q^{-v_0}, \\ \delta_0 Q^{-v_j} \le |P_i^{(j)}(x)| \le c_0 Q^{-v_j},\ 1 \le j \le m, \\ \delta_0 Q \le |P_i^{(j)}(x)| \le c_0 Q, \quad m+1 \le j. \end{aligned} \tag{71}$$

Let $d_0, d_1, \ldots, d_{m+1}$ be a non-increasing sequence of real numbers such that

$$d_j = v_{j-1} - v_j,\ 1 \le j \le m,\ d_{m+1} = v_m + 1. \tag{72}$$

Order the roots α_i with respect to x as follows:

$$|x - \alpha_1| \le |x - \alpha_2| \le \ldots \le |x - \alpha_n|.$$

We claim that the roots α_j with $1 \le j \le m$ satisfy the following inequalities

$$\begin{aligned} |x - \alpha_j| &\ll Q^{-v_{j-1}+v_j}, \quad (1 \le j \le m-1) \\ |x - \alpha_m| &\ll Q^{-v_m-1}. \end{aligned} \tag{73}$$

The $(j-1)$-th derivative of $P(x) = a_n(x-\alpha_1)\cdots(x-\alpha_n)$ is

$$P^{(j-1)}(x) = (j-1)!a_n\Big(\prod_{i=j}^{n}(x-\alpha_i) + \sum_{i_j}(x-\alpha_{i_1})\cdot\cdots\cdot(x-\alpha_{i_{n-j}})\Big), \tag{74}$$

where the sum $\sum_{i_j}$ involves all summands with factor $(x - \alpha_{i_j})$, where $i_j < j$. If for $i < j$ there is a sufficiently large number $s_1 = c(n)$ such that

$$|x - \alpha_j| < s_1|x - \alpha_i| \tag{75}$$

then (72) implies (73) for $|x - \alpha_j|$. Otherwise, (74) implies

$$|P^{(j-1)}(x)| \gg |x - \alpha_j||P^{(j)}(x)|$$

because in this case the summand $(x_1 - \alpha_j)(x_1 - \alpha_{j+1})\cdots(x_1 - \alpha_n)$ in the above expression for $P^{(j-1)}(x_1)$ dominates all the others. Now choose v_j so that

$$v_0 = (m+1)v_m + m \ \text{ and } \ v_0 = (k+1)v_k - kv_{k+1}\ (1 \le k \le m-1). \tag{76}$$

By the first equation of (76), we get

$$v_m = \frac{v_0 - m}{m+1}. \tag{77}$$

By the other equalities of (76) we have that

$$v_{m-1} = \frac{2v_0 - m + 1}{m+1}, \quad v_k = \frac{(m-k+1)v_0 - k}{m+1}\ (1 \le k \le m-2). \tag{78}$$

Finally, by (77) and (78), we obtain

$$v_{j-1} - v_j = v_m + 1 = \frac{v_0 + 1}{m + 1} \quad (0 \le j \le m). \tag{79}$$

Taking into account the condition

$$v_0 + v_1 + \cdots + v_m = n - m,$$

by (77) and (78), we have

$$v_0 = \frac{2n - m}{m + 2}.$$

Thus roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ of P_1 lie within $\ll Q^{-(v_0+1)(m+1)^{-1}}$ of x. The same is true for the roots $\beta_1, \beta_2, \ldots, \beta_m$ of P_2. Hence

$$T(m) = \prod_{1 \le i, j \le m+1} |\alpha_i - \beta_j| \ll Q^{-\frac{2(n+1)(m+1)}{m+2}}.$$

Consequently

$$|R(P_1, P_2)| \ll Q^{\frac{2(n-m-1)}{m+2}}. \tag{80}$$

It remains to give a lower bound for the number of pairs of (P_1, P_2) constructed above. Once again we use the fact that $\alpha_1, \ldots, \alpha_m, \beta_1, \beta_m$ lie within $\ll Q^{-(v_0+1)(m+1)^{-1}}$ of x. In other worlds x lies in the interval

$$\Delta(P_1, P_2) = \left\{x : |\max_i\{\max\{\alpha_i - x|, |\beta_i - x|\}\} \ll Q^{-(v_0+1)(m+1)^{-1}}\right\}.$$

Since x is an arbitrary point of G_J and $\mu(G_J) \gg |J|$, we must have $\gg Q^{(v_0+1)(m+1)^{-1}}$ different pairs (P_1, P_2) to cover G_J with intervals $\Delta(P_1, P_2)$. Substituting the value of v_0 we conclude that the number of different pairs (P_1, P_2) as above is at least $c(n)Q^{\frac{2(n+1)}{(m+1)(m+2)}}$ as required.

Acknowledgements The authors are very grateful to the anonymous referee for the very useful comments. The authors are grateful to SFB701 for its support and making this collaborative work possible.

References

1. D. Badziahin, Inhomogeneous Diophantine approximation on curves and Hausdorff dimension. Adv. Math. **223**, 329–351 (2010)
2. A. Baker, W.M. Schmidt, Diophantine approximation and Hausdorff dimension. Proc. Lond. Math. Soc. **21**, 1–11 (1970)

3. R.C. Baker, Dirichlet's theorem on Diophantine approximation. Math. Proc. Camb. Phil. Soc. **83**, 37–59 (1978)
4. V. Beresnevich, On approximation of real numbers by real algebraic numbers. Acta Arith. **90**, 97–112 (1999)
5. V. Beresnevich, A Groshev type theorem for convergence on manifolds. Acta Math. Hungar. **94**(1–2), 99–130 (2002)
6. V. Beresnevich, On a theorem of Bernik in the metric theory of Diophantine approximation. Acta Arith. **117**, 71–80 (2005)
7. V. Beresnevich, Rational points near manifolds and metric Diophantine approximation. Ann. Math. (2) **175**(1), 187–235 (2012)
8. V. Beresnevich, V. Bernik, M. Dodson, On the Hausdorff dimension of sets of well-approximable points on nondegenerate curves. Doklady NAN Belarusi **46**(6), 18–20 (2002)
9. V. Beresnevich, V. Bernik, E. Kovalevskaya, On approximation of p-adic numbers by p-adic algebraic numbers. J. Number Theor. **111**, 33–56 (2005)
10. V. Beresnevich, D. Dickinson, S. Velani, Measure theoretic laws for lim sup sets. Mem. Am. Math. Soc. **179**(846), x+91 (2006)
11. V. Beresnevich, D. Dickinson, S. Velani, Diophantine approximation on planar curves and the distribution of rational points. Ann. Math. (2) **166**, 367–426 (2007)
12. V. Beresnevich, V. Bernik, M. Dodson, S. Velani, *Classical Metric Diophantine Approximation Revisited*. Analytic number theory (Cambridge University Press, Cambridge, 2009), pp. 38–61
13. V. Beresnevich, V. Bernik, F. Götze, On distribution of resultants of integral polynomials with bounded degree and height. Doklady NAN Belarusi **54**(5), 21–23 (2010)
14. V. Beresnevich, V. Bernik, F. Götze, The distribution of close conjugate algebraic numbers. Composito Math. **5**, 1165–1179 (2010)
15. V.V. Beresnevich, V.I. Bernik, D. Kleinbock, G.A. Margulis, Metric diophantine approximation: the Khintchine-Groshev theorem for nondegenerate manifolds. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. Mosc. Math. J. **2**(2), 203–225 (2002)
16. V.I. Bernik, An application of Hausdorff dimension in the theory of Diophantine approximation. Acta Arith. **42**, 219–253 (1983) (In Russian). English transl. in Am. Math. Soc. Transl. **140**, 15–44 (1988)
17. V.I. Bernik, The exact order of approximating zero by values of integral polynomials. Acta Arith. **53**(1), 17–28 (1989) (In Russian)
18. V.I. Bernik, M.M. Dodson, *Metric Diophantine Approximation on Manifolds*, vol. 137. Cambridge Tracts in Mathematics (Cambridge University Press, Cambridge, 1999)
19. V.I. Bernik, D.V. Vasil'ev, Diophantine approximations on complex manifolds in the case of convergence. Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk, **1**, 113–115, 128 (2006) (In Russian)
20. V.I. Bernik, D. Kleinbock, G.A. Margulis, Khintchine-type theorems on manifolds: the convergence case for standard and multiplicative versions. Int. Math. Res. Notices 453–486 (2001)
21. V. Bernik, F. Götze, O. Kukso, Bad-approximable points and distribution of discriminants of the product of linear integer polynomials. Chebyshevskii Sb. **8**(2), 140–147 (2007)
22. V. Bernik, N. Budarina, D. Dickinson, A divergent Khintchine theorem in the real, complex, and p-adic fields. Lith. Math. J. **48**(2), 158–173 (2008)
23. V. Bernik, F. Götze, O. Kukso, Lower bounds for the number of integral polynomials with given order of discriminants. Acta Arith. **133**, 375–390 (2008)
24. V. Bernik, F. Götze, O. Kukso, On the divisibility of the discriminant of an integral polynomial by prime powers. Lith. Math. J. **48**, 380–396 (2008)
25. A.S. Besicovitch, Sets of fractional dimensions (IV): On rational approximation to real numbers. J. Lond. Math. Soc. **9**, 126–131 (1934)
26. N. Budarina, D. Dickinson, V. Bernik, Simultaneous Diophantine approximation in the real, complex and p-adic fields. Math. Proc. Camb. Phil. Soc. **149**(2), 193–216 (2010)
27. Y. Bugeaud, Approximation by algebraic integers and Hausdorff dimension. J. Lond. Math. Soc. **65**, 547–559 (2002)

28. Y. Bugeaud, A. Dujella, Root separation for irreducible integer polynomials, Root separation for irreducible integer polynomials. Bull. Lond. Math. Soc. **43**(6), 1239–1244 (2011)
29. Y. Bugeaud, M. Mignotte, On the distance between roots of integer polynomials. Proc. Edinb. Math. Soc. (2) **47**, 553–556 (2004)
30. Y. Bugeaud, M. Mignotte, Polynomial root separation. Int. J. Number Theor. **6**, 587–602 (2010)
31. J.-H. Evertse, Distances between the conjugates of an algebraic number. Publ. Math. Debrecen **65**, 323–340 (2004)
32. F. Götze, D. Kaliada, O. Kukso, Counting cubic integral polynomials with bounded discriminants. Submitted
33. G. Harman, *Metric Number Theory*. LMS Monographs New Series, vol. 18 (Clarendon Press, Oxford, 1998)
34. V. Jarnik, Diophantische approximationen und Hausdorffsches mass. Mat. Sb. **36**, 371–382 (1929)
35. A.Ya. Khintchine, Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. Math. Ann. **92**, 115–125 (1924)
36. D.Y. Kleinbock, G.A. Margulis, Flows on homogeneous spaces and Diophantine approximation on manifolds. Ann. Math. **148**, 339–360 (1998)
37. K. Mahler, An inequality for the discriminant of a polynomial. Michigan Math. J. **11**, 257–262 (1964)
38. M. Mignotte, Some useful bounds, in *Computer Algebra* (Springer, Vienna, 1983), pp. 259–263
39. A. Mohammadi, A.Salehi Golsefidy, S-arithmetic Khintchine-type theorem. Geom. Funct. Anal. **19**(4), 1147–1170 (2009)
40. N. Shamukova, V. Bernik, Approximation of real numbers by integer algebraic numbers and the Khintchine theorem. Dokl. Nats. Akad. Nauk Belarusi **50**(3), 30–33 (2006)
41. V. Sprindžuk, *Mahler's Problem in the Metric Theory of Numbers*, vol. 25. Translations of Mathematical Monographs (Amer. Math. Soc., Providence, 1969)
42. V.G. Sprindžuk, *Metric Theory of Diophantine Approximations.* Scripta Series in Mathematics (V. H. Winston & Sons, Washington, DC; A Halsted Press Book, Wiley, New York-Toronto, London, 1979), p. 156. Translated from the Russian and edited by Richard A. Silverman. With a foreword by Donald J. Newman
43. R.C. Vaughan, S. Velani, Diophantine approximation on planar curves: the convergence theory. Invent. Math. **166**(1), 103–124 (2006)