Subset Space Public Announcement Logic

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Abstract. The logic of public announcements has received great interest in recent years. In this paper we give an account of public announcements in terms of the semantics of subset space logic (SSL). In particular, we give a natural interpretation of the language of public announcement logic (PAL) in subset models, and show that it embeds PAL. We give sound and complete axiomatisations of different variants of the logic. Unlike in other work combining PAL and SSL, the goal is not to import PAL operators with update semantics into SSL, but to give an alternative semantics for PAL: using neighborhoods *instead* of model updates.

Keywords: subset space logic, epistemic logic, public announcement logic, expressivity, arbitrary announcements, topology.

1 Introduction

Epistemic logics [1, 2] formalise reasoning about knowledge. In recent years there has been a great interest in *dynamic* epistemic logics [3]; extension of epistemic logics for reasoning about the epistemic pre- and post-conditions of different types of events. The simplest, and most well-understood, dynamic epistemic logic is *public announcement logic (PAL)* [4], where events are taken to be truthful public announcements. From the logical point of view, standard epistemic logic is the modal logic S5, and occurrences of events are modeled by updating (i.e., modifying) the S5 models.

A different approach to making epistemic logic dynamic is *subset space logic*, originally due to Moss and Parikh [5]. In this logic the semantics of knowledge modalities, as well as modalities modeling potentially information-changing *effort*, is a topological one in terms of so-called *subset structures*. Different from standard topological semantics for modal logic originating in [6, 7], however, this logic uses a variant of the semantics where a state is not merely a *(full information) point*, but rather what is called an *epistemic scenario* consisting of a point together with an *epistemic range*. The epistemic range is not fixed: it can shrink as the result of an effort made by the agent. The possible consequences of efforts are modeled explicitly in the semantic structures.

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That there are close conceptual relationships between dynamic epistemic logics and subset space logic is obvious: both model knowledge dynamics. In this paper we study one aspect of the relationship between PAL and subset space logic. In particular, we give a natural interpretation of the language of PAL in subset structures, explaining changes made by public announcements in terms of the explicitly modeled subset space rather than using model updates. The resulting logic embeds public announcement logic.

The idea of interpreting public announcement operators in subset structures is not entirely new. Baskent [8, 9] gives an interpretation of the combined PAL and subset logic language in subset structures, and proves completeness of PAL with respect to this semantics. However, the interpretation of public announcement operators is defined in terms of *updates on subset structures*. The goal of the current paper is to give an interpretation of the PAL language in subset structures that is closer to the key conceptual idea of subset space logic, namely that all the possible consequences of efforts are explicitly represented in the semantic structures, as an *alternative* to update semantics, which is very different from extending subset space logic with update semantics for public announcement logic as done in [8, 9]. Recent work in [10] also uses the same type of update semantics as in [8, 9]. We discuss related work further in Section 6.

Our resulting logic is weaker than PAL; not all subset models correspond to PAL models. We give a sound and complete axiomatisation of the resulting logic. Suitably restricting the class of subset models, we also get soundness and completeness of PAL with respect to our interpretation in subset structures. Thus, we obtain a new and alternative semantics for traditional public announcement logic, as well as a weaker and conceptually interesting logic. We also investigate some other variants of the logical language, and discuss expressive power.

In this paper we only consider the single-agent version of PAL (see also Section 6).

The remainder of the paper is organised as follows. In the next section we briefly review public announcement logic and subset space logic. In Section 3 we give the interpretation of the PAL language in subset structures, and discuss the expressivity of different variants of the language. Then, in Section 4, we investigate translations between Kripke semantics and subset space semantics, and, in Section 5, we study axiomatisations of the resulting logics. We conclude with a discussion in Section 6. Some proofs are unfortunately sketched or omitted due to lack of space.

2 Background

Let PROP be a countable set of propositional variables.

2.1 Public Announcement Logic

Public announcement logic (PAL) [4] extends classical (static) epistemic logic (EL) with an operator which can be used to express public announcements. It is one of the simplest dynamic epistemic logics, and has been investigated extensively in the past few decades. We introduce below some basic definitions and results of classical epistemic logic and public announcement logic which we will use later. For a full introduction we refer to [3]. The definition of PAL is normally parameterised by a set of agents, but in this paper we will only be concerned with the single-agent case and when we refer to PAL in the following we implicitly mean that variant.

Definition 1 (Languages). *The languages of (single-agent) classical epistemic logic, $\mathcal{EL}$, and of (single-agent) public announcement logic, $\mathcal{PAL}$, are:*

$$(\mathcal{EL}) \quad \varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$
$$(\mathcal{PAL}) \quad \varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [\varphi]\varphi,$$

where $p \in$ PROP. We write $\hat{K}\varphi$ as a shorthand for $\neg K \neg\varphi$, and $\langle\varphi\rangle\psi$ for $\neg[\varphi]\neg\psi$.

Interpretation of these languages is defined in terms of *epistemic (Kripke, S5) models* $\mathfrak{M} = (M, \sim, V)$ consisting of a set of *states/points* M, an *indistiguishability relation* $\sim$ which is an equivalence relation on M, and a *valuation function* V : PROP $\to M$.

Definition 2 (Kripke semantics). *Given an epistemic model $\mathfrak{M} = (M, \sim, V)$ and a point $m \in M$, the satisfaction relation, $\Vdash$, is defined as follows. $\mathfrak{M}, m \nVdash \bot$, and:*

$$\begin{array}{lll} \mathfrak{M}, m \Vdash p & \textit{iff} & m \in V(p) \\ \mathfrak{M}, m \Vdash \neg\varphi & \textit{iff} & \mathfrak{M}, m \nVdash \varphi \\ \mathfrak{M}, m \Vdash \varphi \wedge \psi & \textit{iff} & \mathfrak{M}, m \Vdash \varphi \;\&\; \mathfrak{M}, m \Vdash \psi \\ \mathfrak{M}, m \Vdash K\varphi & \textit{iff} & \forall n \in M. \, (m \sim n \Rightarrow \mathfrak{M}, n \Vdash \varphi) \\ \mathfrak{M}, m \Vdash [\varphi]\psi & \textit{iff} & \mathfrak{M}, m \Vdash \varphi \Rightarrow \mathfrak{M}|\varphi, m \Vdash \psi, \end{array}$$

In the above, $\mathfrak{M}|\varphi$ is the submodel of $\mathfrak{M}$ having $[\![\varphi]\!]^{\mathfrak{M}}$, where $[\![\varphi]\!]^{\mathfrak{M}} = \{n \mid \mathfrak{M}, n \Vdash \varphi\}$ is the truth set *of φ in $\mathfrak{M}$, as states and where $\sim$ and V are restricted to $[\![\varphi]\!]^{\mathfrak{M}}$.* Validity *is defined as usual.* ⊣

$\mathcal{PAL}$ is as expressive as $\mathcal{EL}$ [4]. A sound and complete axiomatisation for EL is the well-known Hilbert system **S5** (Fig. 1). The axiomatisation **PAL** for PAL (Fig. 1) is obtained by adding to **S5** reduction axioms for the public announcement operators [4, 11, 12].

(PC)	Instances of tautologies	(MP)	$\vdash \varphi \;\&\; \vdash \varphi \to \psi \Rightarrow \vdash \psi$
(N)	$\vdash \varphi \Rightarrow \vdash K\varphi$	(K)	$K(\varphi \to \psi) \to K\varphi \to K\psi$
(T)	$K\varphi \to \varphi$	(5)	$\neg K\varphi \to K\neg K\varphi$
(AP)	$[\varphi]p \leftrightarrow (\varphi \to p), p \in$ PROP	(AN)	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
(AC)	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	(AK)	$[\varphi]K\psi \leftrightarrow (\varphi \to K[\varphi]\psi)$
(AM)	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$		

Fig. 1. PAL, the axiomatisation of public announcement logic, and the sub-system **S5** consisting of (PC), (MP), (N), (K), (T) and (5). The 4 axiom, i.e., $K\varphi \to KK\varphi$, meaning *positive introspection*, is often also included, but technically redundant — it can be derived in **S5**.

2.2 Subset Space Logic

The study of subset space logic (SSL) was initiated in [5]. One of the main motivations was to characterise epistemic efforts in a reasonably simple framework. Below we briefly introduce the classical subset space logic, and we refer to [13] for more details.

Definition 3 (Language). *The language $\mathcal{SSL}$ has the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi,$$

where $p \in$ PROP. We write $\hat{K}\varphi$ as a shorthand for $\neg K \neg\varphi$, and $\Diamond\varphi$ for $\neg\Box\neg\varphi$.

Intuitively, $K\varphi$ reads as "φ is known in the current situation", while $\Diamond\varphi$ reads as "there is a refinement of knowledge (e.g., a new evidence) under which φ is true". One of the most interesting $\mathcal{SSL}$-sentences is $\neg K\varphi \wedge \Diamond K\varphi$, which reads as "$\varphi$ is not known under the current situation, but there is a refinement of knowledge to make φ known". Formally the semantics is defined as follows.

Definition 4 (Subset structures). *A pair $(X, \mathcal{O})$ is called a* subset space*, if X is a non-empty set and $\mathcal{O} \subseteq \wp(X)$. A* subset model *is a tuple $\mathcal{X} = (X, \mathcal{O}, V)$ where $(X, \mathcal{O})$ is a subset space and V : PROP $\to \wp(X)$ is an evaluation function.*

For historical reasons, elements of $\mathcal{O}$ are called *open sets* or simply *opens*. For any subset model $\mathcal{X} = (X, \mathcal{O}, V)$, we take a point $x \in X$ as a factual state, and an $O \in \mathcal{O}$ as an *epistemic range* or *evidence*. A pair (x, O) is called an *epistemic scenario* (or simply *scenario*) of $\mathcal{X}$ if it holds that $x \in O$. The set of all epistemic scenarios of $\mathcal{X}$ is denoted by $ES(\mathcal{X})$. A *pointed subset space* (resp. *pointed subset model*) is a subset space (resp. subset model) together with an epistemic scenario of it.

Definition 5 (Semantics). *Let $\mathcal{X} = (X, \mathcal{O}, V)$ be a subset model, and $(x, O) \in ES(\mathcal{X})$. The satisfaction relation, $\models$, is given as follows:*

$$\begin{array}{lll}
\mathcal{X}, x, O \models p & \textit{iff} & x \in V(p) \\
\mathcal{X}, x, O \models \neg\varphi & \textit{iff} & \mathcal{X}, x, O \not\models \varphi \\
\mathcal{X}, x, O \models \varphi \wedge \psi & \textit{iff} & \mathcal{X}, x, O \models \varphi \ \& \ \mathcal{X}, x, O \models \psi \\
\mathcal{X}, x, O \models K\varphi & \textit{iff} & \forall y \in O.\, \mathcal{X}, y, O \models \varphi \\
\mathcal{X}, x, O \models \Box\varphi & \textit{iff} & \forall U \in \mathcal{O}.\, (x \in U \subseteq O \Rightarrow \mathcal{X}, x, U \models \varphi).
\end{array}$$

We stress that satisfaction is undefined for a pair (x, O) with $x \notin O$. We write $\mathcal{X} \models \varphi$ (read as "φ is globally true in the subset model $\mathcal{X}$"), if $\mathcal{X}, (x, O) \models \varphi$ holds for all $(x, O) \in ES(\mathcal{X})$. In a similar fashion, we can define the validity of φ in a subset space $(X, \mathcal{O})$ (notation: $X, \mathcal{O} \models \varphi$), global validity (notation: $\models \varphi$), and so on. ⊣

A sound and complete [5, 14] axiomatisation **SSL** of subset space logic is given in Fig. 2. Among the axioms and rules, PC, K•, T•, 5•, N• and MP compose an S5 system for the knowledge operator K, while PC, K○, T○, 4○, N○ and MP compose an S4 system for the refinement operator $\Box$. There are two extra axioms: AP which stands for *atomic persistence* and Cr for *cross*.

(PC) Instances of tautologies	(MP) $\vdash \varphi \;\&\; \vdash \varphi \to \psi \Rightarrow \vdash \psi$
(K•) $K(\varphi \to \psi) \to K\varphi \to K\psi$	(K∘) $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$
(T•) $K\varphi \to \varphi$	(T∘) $\Box\varphi \to \varphi$
(5•) $\neg K\varphi \to K\neg K\varphi$	(4∘) $\Box\varphi \to \Box\Box\varphi$
(N•) $\vdash \varphi \Rightarrow \vdash K\varphi$	(N∘) $\vdash \varphi \Rightarrow \vdash \Box\varphi$
(Cr) $K\Box\varphi \to \Box K\varphi$	(AP) $(p \to \Box p) \wedge (\neg p \to \Box\neg p), p \in \textsc{prop}$

Fig. 2. The axiomatisation **SSL** of subset space logic

3 Incorporating Public Announcements into SSL

The key question related to incorporating public announcements into subset space logic is: how to model changes made by public announcements in terms of subset models? As discussed in the introduction, the goal is to give an interpretation of the public announcement operators using neighborhood refinement in place of model updates.

3.1 How to Model Public Announcements in SSL?

Observe that the interpretation of the formula $[\varphi]\psi$ in PAL is in the following pattern:

$$\mathfrak{M}, m \Vdash [\varphi]\psi \quad \text{iff} \quad \mathfrak{M}, m \Vdash \mathsf{pre}(\varphi) \Rightarrow \mathfrak{M}', m \Vdash \psi$$

where $\mathsf{pre}(\varphi)$ stands for the *precondition* for announcing φ, while $\mathfrak{M}'$ is the model resulting from publicly announcing φ in the current model $\mathfrak{M}$. In classical public announcement logic, $\mathsf{pre}(\varphi)$ is φ merely itself; only announcements of true formulae can result in a (possible) change of a model. The above pattern has been used in various dynamic epistemic logics, such as arbitrary public announcement logic [15], group announcement logic [16], and action model logic [17]. Subset space logics with public announcements introduced in [9, 10] are in this pattern as well.

Following this pattern, we propose the following definition, using the subset space instead of model updates:

$$\mathcal{X}, x, O \models [\varphi]\psi \quad \text{iff} \quad \mathcal{X}, x, O \models \mathsf{pre}(\varphi) \Rightarrow \mathcal{X}, x, \llparenthesis\varphi\rrparenthesis^O \models \psi,$$

where $\llparenthesis\varphi\rrparenthesis^O = \{y \in O \mid \mathcal{X}, y, O \models \varphi\}$, and $\mathcal{X}, x, O \models \mathsf{pre}(\varphi)$ iff $x \in \llparenthesis\varphi\rrparenthesis^O \in \mathcal{O}$. In other words, an announcement can be made only when the truth set of the announced formula under the current neighborhood is indeed a valid sub-neighborhood.

We have a few more remarks on the formula $\mathsf{pre}(\varphi)$. The definition of the meaning of $\mathsf{pre}(\varphi)$ (i) makes it a precondition stronger than merely φ, (ii) behaves like an executability check of φ in the sense of [18–20], and (iii) is in the flavor of the $\Box$-operator under the classical neighborhood semantics for modal logic.

3.2 Logics and Expressivity

We will work with the languages $\mathcal{EL}$ and $\mathcal{PAL}$, of course, reinterpreted in the *subset semantics* (defined below).

Definition 6 (Subset semantics). *The following is a simultaneous definition of satisfaction for $\mathcal{EL}$ and $\mathcal{PAL}$. Given a subset model $\mathcal{X}$ and a scenario (x, O),*

$$\begin{array}{lll}
\mathcal{X}, x, O \not\models \bot & & \\
\mathcal{X}, x, O \models p & \textit{iff} & x \in V(p) \\
\mathcal{X}, x, O \models \neg\varphi & \textit{iff} & \mathcal{X}, x, O \not\models \varphi \\
\mathcal{X}, x, O \models \varphi \wedge \psi & \textit{iff} & \mathcal{X}, x, O \models \varphi \;\&\; \mathcal{X}, x, O \models \psi \\
\mathcal{X}, x, O \models K\varphi & \textit{iff} & \forall y \in O.\, \mathcal{X}, y, O \models \varphi \\
\mathcal{X}, x, O \models [\varphi]\psi & \textit{iff} & x \in (\!|\varphi|\!)^O \in \mathcal{O} \Rightarrow \mathcal{X}, x, (\!|\varphi|\!)^O \models \psi
\end{array}$$

where $(\!|\varphi|\!)^{\mathcal{X},O} = \{y \in O \mid \mathcal{X}, y, O \models \varphi\}$ *is the* truth set of φ in $\mathcal{X}$ at O. *We often hide the parameter $\mathcal{X}$ of $(\!|\varphi|\!)^{\mathcal{X},O}$ (and simply write $(\!|\varphi|\!)^O$), as it is usually clear from the context. Validity is defined as usual. Note that $(\!|\varphi|\!)^O \subseteq O$ always holds.* ⊣

We shall call the above defined semantics *subset semantics*, in comparison to Kripke semantics. As discussed in Section 3.1, we are interested in the sentence $\mathsf{pre}(\varphi)$, which is interpreted in subset semantics by

$$\mathcal{X}, x, O \models \mathsf{pre}(\varphi) \quad \text{iff} \quad x \in (\!|\varphi|\!)^O \in \mathcal{O}.$$

It is easy to verify that $\mathsf{pre}(\varphi)$ is definable in $\mathcal{PAL}$ by $\neg[\varphi]\bot$. We treat $\mathsf{pre}(\varphi)$ as an abbreviation of $\neg[\varphi]\bot$, as long as it is not primitive in the language.

Proposition 7. *For any $\varphi, \psi \in \mathcal{PAL}$, the following hold:*

1. $\models \neg\mathsf{pre}(\bot)$
2. $\models \mathsf{pre}(\varphi) \to \neg\mathsf{pre}(\neg\varphi)$
3. $\models \mathsf{pre}(\varphi) \to \varphi$
4. $\models \mathsf{pre}(\varphi) \to \mathsf{pre}(\mathsf{pre}(\varphi))$
5. $\models K\varphi \to \mathsf{pre}(\varphi)$
6. $\models \varphi$ *implies* $\models K\varphi$
7. $\models \mathsf{pre}(\varphi) \to K(\varphi \to \mathsf{pre}(\varphi))$
8. $\models \neg(\varphi \to \mathsf{pre}(\varphi)) \to K\neg\mathsf{pre}(\varphi)$
9. $\models (\varphi \leftrightarrow \psi) \to (\mathsf{pre}(\varphi) \leftrightarrow \mathsf{pre}(\psi))$
10. $\not\models \mathsf{pre}(\varphi \to \psi) \to (\mathsf{pre}(\varphi) \to \mathsf{pre}(\psi))$
11. $\models \varphi$ *implies* $\models \mathsf{pre}(\varphi)$

Proof. 1 through 3 are easy. We first show 4 here. For any $\mathcal{X} = (X, \mathcal{O}, V)$ and any epistemic scenario (x, O), suppose $\mathcal{X}, x, O \models \mathsf{pre}(\varphi)$. Then, $(\!|\varphi|\!)^O \in \mathcal{O}$, and therefore $(\!|\mathsf{pre}(\varphi)|\!)^O = \{y \in O \mid \mathcal{X}, y, O \models \mathsf{pre}(\varphi)\} = \{y \in O \mid y \in (\!|\varphi|\!)^O \in \mathcal{O}\} = (\!|\varphi|\!)^O$. Hence, $\mathcal{X}, x, O \models \mathsf{pre}(\mathsf{pre}(\varphi))$ iff $x \in (\!|\mathsf{pre}(\varphi)|\!)^O \in \mathcal{O}$ iff $x \in (\!|\varphi|\!)^O \in \mathcal{O}$ iff $\mathcal{X}, x, O \models \mathsf{pre}(\varphi)$. Thus, we have $\mathcal{X}, x, O \models \mathsf{pre}(\mathsf{pre}(\varphi))$ under the supposition. Now we show 5. $\mathcal{X}, x, O \models K\varphi$ iff $\forall y \in O.\, \mathcal{X}, y, O \models \varphi$. Therefore $(\!|\varphi|\!)^O = O$, and so $x \in (\!|\varphi|\!)^O \in \mathcal{O}$. Hence $\mathcal{X}, x, O \models \mathsf{pre}(\varphi)$. Other proofs are omitted. □

We now move on to discussing the expressive power of the defined languages.

Definition 8 (Partial bisimulation). *Given any two subset models, $\mathcal{X} = (X, \mathcal{O}, V)$ and $\mathcal{X}' = (X', \mathcal{O}', V')$, a non-empty relation $\rightleftarrows^p$ between $ES(\mathcal{X})$ and $ES(\mathcal{X}')$ is called a* partial bisimulation *between $\mathcal{X}$ and $\mathcal{X}'$, if the following hold for all $(x, O) \in ES(\mathcal{X})$ and $(x', O') \in ES(\mathcal{X}')$ such that $(x, O) \rightleftarrows^p (x', O')$:*

Atom *For any propositional variable p, $x \in V(p)$ iff $x' \in V(p)$;*
K-forth *If $y \in O$, then there is $y' \in O'$ such that $(y, O) \rightleftarrows^p (y', O')$;*
K-back *If $y' \in O'$, then there is $y \in O$ such that $(y, O) \rightleftarrows^p (y', O')$.*
We write $(\mathcal{X}, x, O) \rightleftarrows^p (\mathcal{X}', x', O')$, if $\mathcal{X} \rightleftarrows^p \mathcal{X}'$ links (x, O) and (x', O'). ⊣

Proposition 9 **($\mathcal{EL}$-invariance of partial bisimulation).** *Partial bisimulation implies $\mathcal{EL}$-equivalence. Namely, for any subset models $\mathcal{X}$ and $\mathcal{X}'$, any $(x, O) \in ES(\mathcal{X})$ and $(x', O') \in ES(\mathcal{X}')$,*

$$(\mathcal{X}, x, O) \rightleftarrows^{p} (\mathcal{X}', x', O') \quad \Rightarrow \quad \forall \varphi \in \mathcal{EL}.\ (\mathcal{X}, x, O \models \varphi \Leftrightarrow \mathcal{X}', x', O' \models \varphi).$$

Theorem 10. *$\mathcal{PAL}$ is strictly more expressive than $\mathcal{EL}$ (in subset semantics).*

Proof. We show that the $\mathcal{PAL}$-formula $\mathsf{pre}(p)$ is not equivalent to any $\mathcal{EL}$-formula. By Proposition 9, it suffices to show that $\mathsf{pre}(p)$ can distinguish two subset models which are partially bisimilar. Consider two subset models $\mathcal{X} = (\{x, y\}, \{\{x\}, \{x, y\}\}, V)$ and $\mathcal{Y} = (\{x, y\}, \{\{x\}\}, V)$ with $V(p) = \{x, y\}$. The relation $\{((x, \{x, y\}), (x, \{x\})), ((y, \{x, y\}), (x, \{x\}))\}$ reveals $(\mathcal{X}, x, \{x, y\}) \rightleftarrows^{p} (\mathcal{Y}, x, \{x\})$. But $\mathcal{X}, x, \{x, y\} \models \mathsf{pre}(p)$ while $\mathcal{Y}, x, \{x\} \not\models \mathsf{pre}(p)$. □

Theorem 11. *The following $\mathcal{PAL}$-formulae are valid (with $p \in$ PROP):*

$[\varphi]\bot \leftrightarrow \neg \mathsf{pre}(\varphi)$	$[\varphi](\psi \wedge \chi) \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
$[\varphi]p \leftrightarrow \mathsf{pre}(\varphi) \rightarrow p$	$[\varphi]K\psi \leftrightarrow \mathsf{pre}(\varphi) \rightarrow K[\varphi]\psi$
$[\varphi]\neg\psi \leftrightarrow \mathsf{pre}(\varphi) \rightarrow \neg[\varphi]\psi$	$[\varphi][\psi]\chi \leftrightarrow [\mathsf{pre}(\varphi) \wedge [\varphi]\mathsf{pre}(\psi)]\chi.$

4 Translations between Kripke Semantics and Subset Semantics

We work in the language $\mathcal{PAL}$. As a convention, we denote by PAL_K the logic resulting from interpreting $\mathcal{PAL}$ in Kripke models (i.e., standard public announcement logic), and by PAL_S the result of interpreting the same language in subset models. We write $\mathfrak{K}$ for the set of all $S5$ Kripke models, and $\mathfrak{S}$ the set of all subset models.

It is well known that in the single-agent setting, every $S5$ model is equivalent to an $S5$ model whose component relation is a *universal relation*, i.e., a model $(M, \sim, V)$ such that $\sim = M \times M$. In this section we implicitly assume (without loss of generalisation) that all $S5$ models are of this kind. The reason is that it simplifies the presentation, in particular because these $S5$ models are quite similar to standard subset models, as the reader will see.

Definition 12 **($\mathfrak{K}$-$\mathfrak{S}$-translation).** *We define a translation $\kappa : \mathfrak{K} \to \mathfrak{S}$ as follows. Let $\mathfrak{M} = (M, \sim, V)$ be an $S5$ model. Its translation, $\kappa(\mathfrak{M})$, is the subset model $(X, \mathcal{O}, \nu)$, such that $X = M$, $\mathcal{O} = \{[\![\varphi]\!]^{\mathfrak{M}|\psi} \mid \varphi, \psi \in \mathcal{PAL}\}$, and $\nu = V$.* ⊣

Theorem 13 **(κ-equivalence).** *Given an $S5$ model $\mathfrak{M} = (M, \sim, V)$, and $m \in M$, for any $\mathcal{PAL}$-formulae φ and α such that $\mathfrak{M}, m \Vdash \alpha$, it holds that $\mathfrak{M}|\alpha, m \Vdash \varphi$ iff $\kappa(\mathfrak{M}), m, [\![\alpha]\!]^{\mathfrak{M}} \models \varphi$.*

Proof. By induction on φ. Let $\kappa(\mathfrak{M}) = (X, \mathcal{O}, \nu)$. Note that i) for any φ, $[\![\varphi]\!]^{\mathfrak{M}|\top} = [\![\varphi]\!]^{\mathfrak{M}}$, and ii) $X \in \mathcal{O}$, since $X = [\![\top]\!]^{\mathfrak{M}}$. The base case and Boolean cases are easy to verify.

$$\begin{array}{lll} \mathfrak{M}|\alpha, m \Vdash K\psi & \text{iff} & \forall n \in [\![\alpha]\!]^{\mathfrak{M}}.\ \mathfrak{M}|\alpha, n \Vdash \psi \quad (\text{note that } \sim = M \times M) \\ & \text{iff} & \forall n \in [\![\alpha]\!]^{\mathfrak{M}}.\ \kappa(\mathfrak{M}), n, [\![\alpha]\!]^{\mathfrak{M}} \models \psi \qquad (\text{IH}) \\ & \text{iff} & \kappa(\mathfrak{M}), m, [\![\alpha]\!]^{\mathfrak{M}} \models K\psi. \end{array}$$

$$\begin{array}{ll} & \mathfrak{M}|\alpha, m \Vdash [\chi]\psi \\ \text{iff} & \mathfrak{M}|\alpha, m \Vdash \chi \Rightarrow \mathfrak{M}|\alpha|\chi, m \Vdash \psi \\ \text{iff} & \mathfrak{M}|\alpha, m \Vdash \chi \Rightarrow \mathfrak{M}|(\alpha \wedge [\alpha]\chi), m \Vdash \psi \qquad (\text{cf. [3, Proposition 4.17]}) \\ \text{iff} & \kappa(\mathfrak{M}), m, [\![\alpha]\!]^{\mathfrak{M}} \models \chi \Rightarrow \kappa(\mathfrak{M}), m, [\![\alpha \wedge [\alpha]\chi]\!]^{\mathfrak{M}} \models \psi \quad (\text{IH}) \\ \text{iff} & m \in (\!|\chi|\!)^{[\![\alpha]\!]^{\mathfrak{M}}} \in \mathcal{O} \Rightarrow \kappa(\mathfrak{M}), m, (\!|\chi|\!)^{[\![\alpha]\!]^{\mathfrak{M}}} \models \psi \qquad (*) \\ \text{iff} & \kappa(\mathfrak{M}), m, [\![\alpha]\!]^{\mathfrak{M}} \models [\chi]\psi. \end{array}$$

We show $(*)$ in the above. First show that the antecedents match. Bottom up is clear; from top down, $m \in (\!|\chi|\!)^{[\![\alpha]\!]^{\mathfrak{M}}}$ is also clear. From $[\![\chi]\!]^{\mathfrak{M}|\alpha} = (\!|\chi|\!)^{[\![\alpha]\!]^{\mathfrak{M}}}$ which is guaranteed by IH, it follows that $(\!|\chi|\!)^{[\![\alpha]\!]^{\mathfrak{M}}} \in \mathcal{O}$ by the definition of κ. Then we show that the consequents are equivalent. It suffices to show $[\![\alpha \wedge [\alpha]\chi]\!]^{\mathfrak{M}} = (\!|\chi|\!)^{[\![\alpha]\!]^{\mathfrak{M}}}$ under the condition $\mathfrak{M}, m \Vdash \alpha$. This is easy from definitions and IH. □

Corollary 14. *Given an $S5$ model $\mathfrak{M} = (M, \sim, V)$, and $m \in M$, for any $\mathcal{PAL}$-formula φ, $\mathfrak{M}, m \Vdash \varphi$ iff $\kappa(\mathfrak{M}), m, X \models \varphi$, where X is the domain of $\kappa(\mathfrak{M})$.*

Corollary 15. *PAL$_K$ is not weaker than PAL$_S$, i.e., all validities of PAL$_S$ are also validities of PAL$_K$.*

But is PAL$_K$ (strictly) stronger than PAL$_S$, namely, is there a $\mathcal{PAL}$-formula which is valid in PAL$_K$ but not in PAL$_S$? The answer is yes. Recall that the K-axiom for pre, i.e., $\mathsf{pre}(\varphi \to \psi) \to (\mathsf{pre}(\varphi) \to \mathsf{pre}(\psi))$, is not valid in PAL$_S$ (Proposition 7.10). This formula is valid in PAL$_K$, since in PAL$_K$ it is equivalent to $(\varphi \to \psi) \to (\varphi \to \psi)$. This translation illustrates a similarity between the pre-operator under subset semantics and the □-operator under neighborhood semantics (as we already mentioned in Section 3.1), giving weaker logical principles than under Kripke semantics.

We obtain standard public announcement logic by restricting the class of subset models. Let $\mathcal{X} = (X, \mathcal{O}, V)$ be a subset model. We say $\mathcal{X}$ is a *public announcement subset model* (*PASM* for short), if $\mathcal{O} = \{(\!|\varphi|\!)^{(\!|\psi|\!)^X} \mid \varphi, \psi \in \mathcal{PAL}\}$. Namely, $\mathcal{X}$ being a PASM requires that:

– ("All evidence is announceable") for all $O \in \mathcal{O}$, there exists $\mathcal{PAL}$-formulae φ and ψ such that $O = \{x \mid \mathcal{X}, x, (\!|\psi|\!)^X \models \varphi\}$, i.e., $O = (\!|\varphi|\!)^{(\!|\psi|\!)^X}$; and
– ("All announcements are evidence") for all φ and ψ, $(\!|\varphi|\!)^{(\!|\psi|\!)^X} \in \mathcal{O}$.

Theorem 16 (Correspondence). *Every $S5$ model is equivalent to a PASM, and vice versa. That is, given a pointed $S5$ model, there exists a pointed PASM satisfying exactly the same $\mathcal{PAL}$-formulae; and given a pointed PASM, there exists a pointed $S5$ model satisfying exactly the same $\mathcal{PAL}$-formulae.*

Proof. Given an $S5$ model $\mathfrak{M}$, let $\kappa(\mathfrak{M}) = (X, \mathcal{O}, V)$. It suffices to show that $\kappa(\mathfrak{M})$ is a PASM. We show that $[\![\varphi]\!]^{\mathfrak{M}|\psi} = (\!|\varphi|\!)^{(\!|\psi|\!)^X}$ for every $\mathcal{PAL}$-formulae φ and ψ. First, $[\![\psi]\!]^{\mathfrak{M}} = (\!|\psi|\!)^X$, for $m \in [\![\psi]\!]^{\mathfrak{M}}$ iff $\mathfrak{M}, m \Vdash \psi$ iff (by Corollary 14) $\kappa(\mathfrak{M}), m, X \models \psi$ iff $m \in (\!|\psi|\!)^X$. Therefore, $m \in [\![\varphi]\!]^{\mathfrak{M}|\psi}$ iff $\mathfrak{M}|\psi, m \Vdash \varphi$ iff (by Theorem 13) $\kappa(\mathfrak{M}), m, [\![\psi]\!]^{\mathfrak{M}} \models \varphi$ iff $\kappa(\mathfrak{M}), m, (\!|\psi|\!)^X \models \varphi$ iff $m \in (\!|\varphi|\!)^{(\!|\psi|\!)^X}$.

Given a PASM $\mathcal{X} = (X, \{(\!|\varphi|\!)^{(\!|\psi|\!)^X} \mid \varphi, \psi \in \mathcal{PAL}\}, V)$, it suffices to show that there exists an $S5$ model $\mathfrak{M}$ such that $\kappa(\mathfrak{M}) = \mathcal{X}$. Let $\mathfrak{M} = (X, X \times X, V)$, then $\kappa(\mathfrak{M}) = (X, \{[\![\varphi]\!]^{\mathfrak{M}|\psi} \mid \varphi, \psi \in \mathcal{PAL}\}, V)$. We get $\kappa(\mathfrak{M}) = \mathcal{X}$ as $[\![\varphi]\!]^{\mathfrak{M}|\psi} = (\!|\varphi|\!)^{(\!|\psi|\!)^X}$ is already shown above. □

We immediately get soundness and completeness of the standard axiomatisation of PAL, with respect to public annuouncement subset models:

Corollary 17. *The axiomatisation **PAL** (Fig. 1) is sound and complete with respect to the class of all PASMs.*

In the next section we study axiomatisations of the full class of subset models.

5 Axiomatisations

We introduce axiomatisations for public announcement logic under subset semantics, and show that they are sound and complete with respect to all subset models. Similarly to the notations PAL_K and PAL_S, we denote by EL_K and EL_S the static epistemic logic (for the language $\mathcal{EL}$) interpreted in Kripke semantics and subset semantics, respectively.

5.1 Axiomatisation of $\mathcal{EL}$

In Section 2.1 we noted that **S5** axiomatises EL_K. In this section we show that it also axiomatises EL_S. Namely, we show that **S5** is sound and complete with respect to the class of all subset models.

Theorem 18 (Soundness of S5). ***S5*** *is sound with respect to the class of all subset models. That is, for all $\mathcal{EL}$-formula φ, $\vdash_{\mathbf{S5}} \varphi$ implies $\models \varphi$.*

Proof. $\vdash_{\mathbf{S5}} \varphi$ implies $\vdash_{\mathbf{SSL}} \varphi$, and the theorem follows from soundness of **SSL**. □

Completeness is also straightforward. Consider a subset model $(X, \mathcal{O}, V)$ with a scenario (x, O). Since there is no update operator in the language, the set $\mathcal{O}$ of all opens is equivalent to the singleton set $\{O\}$ for no other opens are accessible. Thus the neighborhood O is simply an equivalence relation. The pointed subset model $((X, \mathcal{O}, V), (x, O))$ can therefore be truth-preservingly translated into a pointed $S5$ model. We leave out the formal proof of the following theorem.

Theorem 19 (Completeness of S5). *The axiomatisation **S5** is strongly complete with respect to the class of all subset models.* □

5.2 Axiomatisation of $\mathcal{PAL}$

We introduce a new language $\mathcal{EL}^+$ for technical reasons. It adds to $\mathcal{EL}$ the clause $\mathsf{pre}(\varphi)$ explicitly, i.e., it has the following grammar rule:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \mathsf{pre}(\varphi).$$

Satisfaction is defined just as that for $\mathcal{EL}$ and $\mathcal{PAL}$ (in subset semantics; Section 3), except that the pre-operator is now primitive. Clearly, $\mathcal{EL}^+$ is as expressive as $\mathcal{PAL}$ (see the reduction principles in Theorem 11).

The axiomatisation $\mathbf{EL}^+$ of $\mathcal{EL}^+$ (Fig. 3) is obtained by adding to **S5** the axioms $\mathrm{T}_{\mathsf{pre}}$, 4_{pre}, Int1, Int2, KP and Cl.

(PC)	Instances of all tautologies	(K)	$K(\varphi \to \psi) \to K\varphi \to K\psi$
(T)	$K\varphi \to \varphi$	(5)	$\neg K\varphi \to K\neg K\varphi$
($\mathrm{T}_{\mathsf{pre}}$)	$\mathsf{pre}(\varphi) \to \varphi$	(4_{pre})	$\mathsf{pre}(\varphi) \to \mathsf{pre}(\mathsf{pre}(\varphi))$
(Int1)	$\mathsf{pre}(\varphi) \to K(\varphi \to \mathsf{pre}(\varphi))$	(Int2)	$\neg(\varphi \to \mathsf{pre}(\varphi)) \to K\neg\mathsf{pre}(\varphi)$
(KP)	$K\varphi \to \mathsf{pre}(\varphi)$	(Cl)	$(\varphi \leftrightarrow \psi) \to (\mathsf{pre}(\varphi) \leftrightarrow \mathsf{pre}(\psi))$
(MP)	$\vdash \varphi \;\&\; \vdash \varphi \to \psi \Rightarrow \vdash \psi$	(N)	$\vdash \varphi \Rightarrow \vdash K\varphi$

Fig. 3. $\mathbf{EL}^+$. Some axioms are redundant. E.g., T can be derived from KP and $\mathrm{T}_{\mathsf{pre}}$.

Theorem 20 (Soundness of $\mathbf{EL}^+$). *$\mathbf{EL}^+$ is sound with respect to the class of all subset models. That is, for any $\mathcal{EL}^+$-formula φ, $\vdash_{\mathbf{EL}^+} \varphi$ implies $\models \varphi$.*

Proof. For any axiom of $\mathcal{EL}^+$, if it is an $\mathcal{EL}$-formula, then we can see that it is also an axiom of **S5**. Therefore, its validity follows from the soundness of **S5**. The validity of all the extra axioms are shown in Proposition 7. □

For completeness of $\mathbf{EL}^+$, given a consistent set Φ of $\mathcal{EL}^+$-formulae, it suffices to find a subset model for it. For **S5** the canonical model method can be used, but this does not work for $\mathbf{EL}^+$ because the pre-operator has impact on the open sets in a subset model. Therefore, we use a more flexible model construction method. Instead of building a canonical model which uses the set of all maximal consistent sets of formulae (MCSs) as it domain, we rather pick up the MCSs that we need, and build a model stepwise. This method is derived from the *step-by-step method* (see, e.g., [21, Chapter 4]), although we construct a model "row by row" rather than a countable series of finite approximations of a desired model. Using this method, we get the following:

Theorem 21 (Completeness of $\mathbf{EL}^+$). *The axiomatisation $\mathbf{EL}^+$ is strongly complete with respect to the class of all subset models.*

We now move on to $\mathcal{PAL}$. The axiomatisation **PAL** is given in Fig. 4. It contains all axioms and rules of $\mathbf{EL}^+$, together with the reduction principles introduced in Theorem 11 as axioms. A subtlety is that we need to make sure that the reduction axioms are not circular. This is shown by the following.

Definition 22 (Complexity of $\mathcal{PAL}$-formulae). *The complexity $c : \mathcal{PAL} \to \mathbb{N}$ is defined as follows:*

(PC)	Instances of all tautologies	(K)	$K(\varphi \to \psi) \to K\varphi \to K\psi$
(T)	$K\varphi \to \varphi$	(5)	$\neg K\varphi \to K\neg K\varphi$
(T_{pre})	$\mathsf{pre}(\varphi) \to \varphi$	(4_{pre})	$\mathsf{pre}(\varphi) \to \mathsf{pre}(\mathsf{pre}(\varphi))$
(Int1)	$(\mathsf{pre}(\varphi) \to K(\varphi \to \mathsf{pre}(\varphi))$	(Int2)	$\neg(\varphi \to \mathsf{pre}(\varphi)) \to K\neg\mathsf{pre}(\varphi)$
(Cl)	$(\varphi \leftrightarrow \psi) \to (\mathsf{pre}(\varphi) \leftrightarrow \mathsf{pre}(\psi))$	(KP)	$K\varphi \to \mathsf{pre}(\varphi)$
(MP)	$\vdash \varphi \;\&\; \vdash \varphi \to \psi \Rightarrow \vdash \psi$	(N)	$\vdash \varphi \Rightarrow \vdash K\varphi$
($[]p$)	$[\varphi]p \leftrightarrow (\mathsf{pre}(\varphi) \to p)$	($[]\neg$)	$[\varphi]\neg\psi \leftrightarrow (\mathsf{pre}(\varphi) \to \neg[\varphi]\psi)$
($[]\wedge$)	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	($[]K$)	$[\varphi]K\psi \leftrightarrow (\mathsf{pre}(\varphi) \to K[\varphi]\psi)$
($[][]$)	$[\varphi][\psi]\chi \leftrightarrow [\mathsf{pre}(\varphi) \wedge [\varphi]\mathsf{pre}(\psi)]\chi$		

Fig. 4. Axiomatisation **PAL**, where any formula of the form $\mathsf{pre}(\varphi)$ is a shorthand for $\neg[\varphi]\bot$

$$\begin{aligned} c(\bot) &= 0 & c(\varphi \wedge \psi) &= 1 + \max(c(\varphi), c(\psi)) \\ c(p) &= 1 & c(K\varphi) &= 1 + c(\varphi) \\ c(\neg\varphi) &= 1 + c(\varphi) & c([\varphi]\psi) &= (4 + c(\varphi)) \cdot c(\psi). \end{aligned}$$

Proposition 23. *i) $c(\varphi) > c(\psi)$ if ψ is a subformula of a $\mathcal{PAL}$-formula φ; and ii) for all the five reduction axioms of the form $\alpha \leftrightarrow \beta$, $c(\alpha) > c(\beta)$.* □

Now, by Theorem 11 and the soundness of $\mathbf{EL}^+$, we easily get soundness of **PAL**. Completeness of **PAL** is also easy: from the completeness of $\mathbf{EL}^+$, any SSL validity is an $\mathbf{EL}^+$-theorem, and thus also a **PAL**-theorem (of course, in terms of the language $\mathcal{PAL}$). We state these results as follows.

Theorem 24 (Soundness and completeness). ***PAL** is sound and strongly complete with respect to the class of all subset models.* □

6 Discussion

In this paper we defined a natural interpretation of the language of public announcement logic in subset models. The resulting logic is strictly weaker than PAL. We studied the expressivity of some variants of the language, and proved completeness with respect to the complete model class. On a suitably restricted model class it coincides with PAL, giving an alternative semantics for this logic.

As mentioned in the introduction, there is existing work [8–10] which is seemingly close to the work presented in this paper, in that public announcement operators are interpreted in subset space structures, but this closeness is only superfiscial. In particular, the subset space *plays no role* in the interpretation of the public announcement operators in [8–10]; it is only used to interpret the effort modality, while the public announcement operators are interpreted by updates on the current epistemic range. If the language used in these papers is restricted to the PAL language, as in the current paper, the subset space plays no role at all. The goal of the current paper is, on the other hand, exactly to give an account of public announcements in terms of subset spaces.

In this paper we considered only the single-agent variant of PAL. This is both for simplicity of presentation, and because although there have been several proposals for multi-agent extensions of subset space logic [9, 22–24] none of them seem adequate, having problems with the semantics of nested formulae [9, 24] or not being extensions of standard multi-agent epistemic logic with a knowledge operator for each agent [22, 23]. However, we believe that the results of this paper can be relatively easily extended to a suitably defined multi-agent version of subset space logic. The classical subset space logic is known to be decidable [14], or more precisely, PSPACE-complete [10]. We are interested in a complexity result for $\mathcal{PAL}$ (in subset semantics). Also of interest for future work is extensions with arbitrary refinement operators and the relationship to arbitrary public announcement logic [15] and group announcement logic [16], as well as action model logic [17].

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