Many-Valued Logics, Fuzzy Logics and Graded Consequence: A Comparative Appraisal

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Abstract. In this paper a comparative study of many-valued logics, fuzzy logics and the theory of graded consequence has been made focussing on consequence, inconsistency and sorites paradox.

1 Introduction

What brings many-valued logic, fuzzy logic and theory of graded consequence (GCT) on a common platform is that each considers bivalence inadequate and embraces multivalence for interpreting their proposed theories. Traditionally, different systems of many-valued logic admitted values other than truth and falsehood from different motivations. The second and the third decade of the last century saw a sprouting of different systems of many-valued logic. However, treatment of vagueness was far from the original motive behind any of these many-valued systems [18]. Generally the set of truth values admitted in different systems of many-valued logic is some subset of the real interval [0, 1].

For Zadeh's proposed system of fuzzy logic based on fuzzy set theory [21], the unit interval [0, 1] was a natural choice for interpreting vague sentences which occur either as premises or as conclusions of inferences in most cases of human reasoning. A further development of fuzzy logic in a more formal way was carried out in [10–14, 16].

A general set up for generating a system of logics with a notion of graded consequence was proposed by Chakraborty [3, 4] to gauge the strength of 'derivability' of a conclusion from a set of premises which may hold in degrees. Where the approach of GCT differs from other many-valued and fuzzy logical approaches is that it admits degrees of truth not only of predications at the object level, but of predications at the meta-level and if required at a level higher than that also.

In this paper a comparative study of many-valued logics, fuzzy logics and the theory of graded consequence has been made focussing on consequence, inconsistency and sorites paradox. We arrange the content of the paper in three different sections; the fisrt two focus on the notions of 'consequence', 'inconsistency', and the last one on the treatment of 'Sorites paradox'.

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2 Notion of Consequence: A Comparative Appraisal

Existing approaches towards reasoning with imprecise concepts allow the object level to be many-valued, while maintaining meta-level statements to be of yes/no type. A comment made by Pelta [17] is relevant in this regard: "*Until now the construction of superficial many-valued logics, that is, logics with an arbitrary number (bigger than two) of truth values but always incorporating a binary consequence relation, has prevailed in investigations of logical many-valuedness*".

In contrast, the philosophy behind GCT is: '*if object level formulae are having many-valued truth values then it generally cannot be denied that meta-level sentences also happen to be so*' [3].

One may wonder what could be the motivation for incorporating grade in metalanguage. After all metalinguistic sentences are of mathematical nature. One immediate answer may be given from the standard of mathematics itself viz., generalization, which is a usual motivation behind mathematical research, and that has been echoed in the above quote. We may recall the history of many-valued logics itself. Though for incorporation of a third value there had been various extraneous reasons e.g. inclusion of future contingents, or taking 'undecided' and 'unknown' also as (truth) values of sentences, there had been no apparent reason for extension to an infinite set of truth values. This was simply generalization as done in mathematical practice. The infinite value set obtained a meaning only in mid-sixties after the advent of fuzzy set theory [21] and incorporation of vague predicates also in logic and computer applications. But a more down-to-earth reason may be given. Since seventies, primarily because of the needs of computer scientists the classical notion of consistency was being felt to be inadequate for application. Notions of partial consistency, and inconsistency tolerant systems were brought in within the discourse [8, 2]. In traditional logic consistency is a hard notion yielding either 'yes' or 'no' answers. Thus consistency to a certain degree was a concept quite naturally waiting at the door steps in search of a theory. Linking consistency with consequence is a long-practised methodology in classical logic. A similar approach with graded (in)consistency automatically leads to graded consequence. Apart from all these, if we examine the nature of actual (not normative) inferences made by human brain we notice that from certain premises the brain often makes inferences not very strongly. The procedure itself might have weakness, tentativeness and vagueness. Cases of medical decision making offer ample instances.

2.1 Theory of Graded Consequence Relation

A graded consequence relation is a fuzzy relation, say $|\!\sim$ between the power set of formulae $P(F)$ and F satisfying the following set of axioms [3, 4].

(GC1) If $\alpha \in X$ then $gr(X \mid\!\sim \alpha) = 1$ (Reflexivity).

(GC2) If $X \subseteq Y$ then $gr(X \mid\!\sim \alpha) \leq gr(Y \mid\!\sim \alpha)$ (Monotonicity).

(GC3) $\inf_{\beta \in Y} gr(X \mid\!\sim \beta) * gr(X \cup Y \mid\!\sim \alpha) \leq gr(X \mid\!\sim \alpha)$ (Cut).

The intended meaning of '$gr(X \mid\!\sim \alpha)$' is the truth value of the meta-linguistic sentence 'α is a consequence of X'. This value is not necessarily the topmost (1) or the least (0) of the lattice, the value set, and it is read as 'the degree to which α is a consequence of X'. The '$*$' and 'inf' used in (GC3) are the operators for computing the meta-linguistic '**and**' and '**for all**' present in the statement '**for all** $\beta \in Y$, $X \vdash \beta$ **and** $X \cup Y \vdash \alpha$ *imply* $X \vdash \alpha$', the classical cut condition. $|\!\sim$ is the many-valued counterpart of $\vdash$, the two-valued notion of 'consequence'. To elucidate, let us consider formulae $\alpha, \beta, \ldots$ and sets of formulae $X, Y, \ldots$ as object level entities. Then in meta level, one can form sentences of the form 'X' $|\!\sim$ 'α', 'X'$|\!\sim x$, $x \in$'X', where 'X', 'α' are the terms of the meta level language representing the names of the respective object level entities, x is a meta level variable ranging over object level formulae, and $\in$, $|\!\sim$ are predicates of the meta level language. So, the cut condition, which is a meta-meta level statement, in symbols can be presented as *imply*($\ulcorner \forall x$(**if** $x \in$ 'Y'**then** 'X' $|\!\sim x$) **and** '$X \cup Y$'$|\!\sim$'α' $\urcorner$, $\ulcorner$ 'X'$|\!\sim$'α' $\urcorner$), where expressions within $\ulcorner\ \urcorner$ are meta level sentences and at meta-meta level those sentences are named is in quotes $\ulcorner\ \urcorner$. So, expressions with $\ulcorner\ \urcorner$ are meta-meta linguistic terms and '*imply*' is a meta-meta linguistic predicate. The lattice order relation '$\leq$' takes care of the meta-meta level relation '*imply*'. Hence, in graded context (GC3) ascertains that the truth value of the meta level sentence $\forall x$(**if** $x \in$ 'Y'**then** 'X' $|\!\sim x$) **and** '$X \cup Y$'$|\!\sim$'α' is less or equal to the truth value of the sentence 'X' $|\!\sim$ 'α'. The value set say, L along with the operators for meta linguistic connectives forms a complete residuated lattice $(L, *, \rightarrow, 0, 1)$.

That logic activity, generally, comprises of three levels, namely, object, meta and meta-meta, and proper distinction between them plays a crucial role in establishing well-formedness of a sentence pertaining to a specific level have been discussed elaborately in [7].

From the semantic angle the graded counterpart of the notion of consequence is a generalization of the notion of semantic consequence proposed by Shoesmith and Smiley [19]. Classically, 'α is a semantic consequence of X' is defined by the meta-linguistic sentence 'for all T_i belonging to the collection of all state-of-affairs, if all members of X are true under T_i, then α is true under T_i'. In [19], the definition has been generalized by replacing the constraint of 'all state-of-affairs' by 'any collection of state-of-affairs', say $\{T_i\}_{i \in I}$. A rationale for taking an arbitrary collection of state-of-affairs according to [19] is: '*the necessity with which conclusions follow is relative to the presuppositions of an argument, and different argument may have different presuppositions. But whatever idea of necessity is involved there is a corresponding idea of possibility*'. This leads to the new version viz., '*To say that a conclusion follows from a given set of premises is to say that each possible state-of-affairs in which all the premises are true is one in which the conclusion is true*' [19]. Instead of presuppositions we prefer to use the more general word 'context'. So each $\{T_i\}_{i \in I}$ constitutes a context, which may be treated as a collection of worlds of Kripke semantics. Besides, from the angle of pure mathematics, a passage from all valuations to arbitrary number of valuations is an elegant generalization.

In GCT, $\{T_i\}_{i \in I}$ represents any collection of fuzzy sets assigning values to the formulae. Thus, the proposed logic turns out to be context dependent. The meta-level sentence for semantic consequence, i.e., (Σ) $\forall T_i\{X \subseteq T_i \rightarrow_m \alpha \in T_i\}$, is evaluated by the expression $\inf_i\{\inf_{\gamma\in X} T_i(\gamma) \rightarrow T_i(\alpha)\}$, where $\inf_{\gamma\in X} T_i(\gamma)$, $T_i(\alpha)$ are the respective values of the meta level sentences '$X \subseteq T_i$' and '$\alpha \in T_i$' (see [4, 6]), and $\rightarrow$ is the residuum of $*$, the monoidal operator of $(L, *, \rightarrow, 0, 1)$, computing $\rightarrow_m$, the meta-linguistic 'if-then'. It is to be noted that the value assigned to the meta-linguistic sentence 'α is a semantic consequence of X' can also be explained maintaining the proper distinction between levels [7].

So, in graded context 'α is a semantic consequence of X', denoted by $X \mid\approx \alpha$, is not a crisp notion; rather, it is a matter of grade and the value of $X \mid\approx \alpha$, i.e., $gr(X \mid\approx \alpha) = \inf_i\{\inf_{\gamma\in X} T_i(\gamma) \rightarrow T_i(\alpha)\}$. In [4], a soundness-completeness like result has been proved connecting a graded consequence relation $\mid\sim$ axiomatized by (GC1) to (GC3) with $\mid\approx$, the semantic counterpart.

The notion of axiomatic graded consequence [5], is determined with respect to $\mathcal{A}$, a logical base of axioms and $\mathcal{R}$, rules. A context, say $\{T_i\}_{i\in I}$, determining the truth values of the basic formulae needs to be prefixed also. The tautologihood degree of the axioms present in $\mathcal{A}$ and degree of the rules present in $\mathcal{R}$ depend on $\{T_i\}_{i\in I}$. The tautologihood degree of an axiom α is $inf_i T_i(\alpha)$, which is the value of the meta-linguistic sentence '$\forall_T$ $(\alpha \in T)$'. To illustrate the degree of a rule, let us consider the rule Modus Ponens (MP). For all instances of $(\{\alpha, \alpha \supset \beta\}, \beta)$, the degree to which β is related to $\{\alpha, \alpha \supset \beta\}$ is $\inf_{\alpha,\beta} gr(\{\alpha, \alpha \supset \beta\} \mid\approx \beta)$ i.e., $\inf_{\alpha,\beta}\inf_i\{(T_i(\alpha) \wedge T_i(\alpha \supset \beta)) \rightarrow T_i(\beta)\}$, which is the degree of the rule MP. Now, given a pair $(\mathcal{A}, \mathcal{R})$, where say, $\mathcal{R}$ consists of MP only, a derivation of a formula α_n from a set of formulae X, is an ordered pair of sequences viz. $(\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle, \langle \mid \alpha_1 \mid, \mid \alpha_2 \mid, \ldots, \mid \alpha_n \mid\rangle)$. In this pair of sequences the first sequence consists of formulae used in the derivation and the second sequence indicates the values associated with each step. The value associated with i-th step will be 1 if α_i comes from the premise X. The value will be tautologihood degree of α_i if α_i is taken from $\mathcal{A} \setminus X$ and $|\alpha_i|$ will be the degree of MP if α_i is obtained by applying MP on the previous formulae $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$. The value of the meta-linguistic sentence '$\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ is a derivation of α_n from X', formally written as $\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle D(X, \alpha_n)$, is $\mid \alpha_1 \mid * \mid \alpha_2 \mid * \ldots * \mid \alpha_n \mid$. Finally, the value of the meta level sentence 'α_n is an axiomatic consequence of X', that is 'there is a sequence of formulae which is a derivation of α_n from X', is computed by $\sup_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle} \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle D(X, \alpha_n)$. This notion of axiomatic graded consequence satisfies (GC1)-(GC3) [5].

The above discussion explains that in GCT assignment of values to sentences like, 'α is a consequence of X', whether it be axiomatic or semantic, is not arbitrary; it depends on the sentence unfolding the meaning of the concept.

2.2 Notion of Consequence in Many-Valued Logics

In many-valued logics, the notion of semantic consequence is defined usually in two ways; one, say '$\models_1$', is defined in terms of a designated subset of the set of values of the wffs and the other, say '$\models_2$', is defined in terms of the order relation present in the value set and a composition operator conjoining the values of the premises. Let us now concentrate on the definition of $\models_1$.

According to the definition of $\models_1$, $X \models_1 \alpha$ iff for the collection of all valuations, say $\{T_i\}_{i\in I}$ mapping formulae to $[0,1]$, for all $\gamma \in X$, $T_i(\gamma) \in D$ i.e. $T_i(X) \subseteq D$, implies $T_i(\alpha) \in D$, where D, the designated set of values, is a proper subset of $[0,1]$ not containing 0.

There is no ambiguity regarding the two-valued nature of the notion of semantic consequence of many-valued logics. Let us see how this notion of semantic consequence is placed in the proposed scheme for graded semantic consequence. In order to cast the definition in our sense, let us do the following construction. Let for each T_i, T_i^D be a mapping defined by, $T_i^D(\alpha) = 1$, if $T_i(\alpha) \in D$
$= 0$, otherwise.

Identifying, the function T_i^D with the set it determines, the above definition reduces to 'for all T_i^D, $X \subseteq T_i^D$ implies $\alpha \in T_i^D$'.

Theorem. Given the collection of all valuations, say $\{T_i\}_{i\in I}$, $|\!\approx$ generated by $\{T_i^D\}_{i\in I}$ with the operator $\rightarrow_c$, defined by $a \rightarrow_c b = 1$, if $a \leq b$
$= 0$ otherwise,
computing the meta-level implication $\rightarrow_m$ of (Σ) coincides with $\models_1$.

Theorem. $|\!\approx$ generated by $\{T_i^D\}_{i\in I}$, in the above mentioned sense, is a graded consequence relation.

These two theorems give a general scheme for reproducing the completeness theorem of many-valued logics in terms of GCT by fixing $\{T_i^D\}_{i\in I}$ to be the context determining the tautologihood degrees and degrees of rules of $(\mathcal{A}, \mathcal{R})$, the axiomatic base of the particular logic of concern.

2.3 Notion of Consequence in Fuzzy Logics

Most of the mainstream fuzzy logics frequently use the term 'degree of consequence', but does not really mean the notion of consequence is graded. The idea of approximate rule prevalent in fuzzy logics may illustrate the point.

2.3.1 Modus Ponens as a Special Case of Derivation in Fuzzy Logics

As introduced by Goguen [12] the approximate rule Modus Ponens is such that "*If you know P is true at least to the degree a and $P \supset Q$ at least to the degree b then conclude that Q is true at least to the degree $a.b$.*"

Assuming the value set for formulae as [0, 1] and '.' as the usual product operation, the above-mentioned approximate rule takes the form:

$$\frac{\begin{array}{c}(P,\ a)\\(P \supset Q,\ b)\end{array}}{(Q,\ a.b)}$$

Fig. 1.

That is, as proposed by Goguen, a many-valued rule of inference can be viewed as a crisp relation from $P(F \times [0,1])$ to $(F \times [0,1])$.

Pavelka's [16] interpretation of a many-valued rule of inference is as follows. A many-valued rule of inference r consists of two components $\langle r', r''\rangle$, where the first (grammatical) component r' operates on formulae and the second (evaluation) component r'' operates on truth values and says *how the truth value of the conclusion is to be computed from the truth values of the premises.*

So, it is clear from Pavelka's own words that the value, which is being attached to the concluding formula, is the *truth value of the conclusion* - not of the 'so called' many-valued rule. Pavelka, himself puts a many-valued rule MP in a form which is similar to the form presented above.

'From partially true premises partially true conclusion can be deduced'- this is the central idea of Peter Ha$\acute{j}$ek's [14] fuzzy logical system, known as Rational Pavelka Logic (RPL). Identifying the pair (P, a) with the formula $\overline{a} \supset P$, where $\overline{a}$ is the wff denoting the truth value a, it can be shown that the many-valued rule MP mentioned in **Fig. 1** can be obtained as a derived rule in RPL.

So, from the above discussion it is evident that neither Goguen, nor Pavelka, nor Ha$\acute{j}$ek considered a rule of inference in fuzzy context as a fuzzy relation between a set of formulae and a single formula. And this is clearly not the case in the context of graded consequence (see section 2.1). One can argue that presentation of a fuzzy rule in the form of **Fig. 1** is nothing but a variant form of writing the same rule as a fuzzy relation say, $MP(\{(\alpha, a)(\alpha \supset \beta, b)\}\ \beta)$ with relatedness grade $a.b$ or $r''(a, b)$. In this connection readers are referred to [7] where it has been observed that if the principle of 'use and mention' of a symbol is to be maintained then this correspondence gives rise to a difficulty in placing a rule as a well-formed concept of the meta level language.

Rule being a special case of derivation, it can be guessed that 'consequence' in fuzzy logic is such that given a set of formulae, along with their truth values, a formula with certain truth value either can be derived or not derived. However, the way '$C(X)(\alpha)$' is read in fuzzy logic creates a confusion.

This becomes more visible if one goes through the distinction of levels maintaining the principle of 'use' and 'mention' of a symbol in a logical discourse [7]. A formula α, a set of formulae X - all these are linguistic elements of level-0. **'**α**'** is a consequence of **'**X**'**, **'**X**'** is inconsistent - these are level-1 statements, and value of these statements should be computed by a reasonable, definite method as it is maintained for computing values of level-0 formulae. This is exactly lacking in the understanding of the meta level concepts of the existing fuzzy logics.

2.3.2 Notion of Provability in Ha$\acute{j}$ek's Rational Pavelka Logic

According to Ha$\acute{j}$ek [14] the notion of 'provability degree of a formula α from a set of formulae X' is given by the value $\sup\{r : X \vdash (\alpha, r)\} \ldots$(A).

In RPL, (α, r) is a level-0 formula and hence one can place $X \vdash (\alpha, r)$ in level-1. For a crisp set X of formulae, given a crisp set of axioms and crisp rules of inference, a formula of the form '(α, r) is a derivation of X', is a two-valued notion. So what does this 'provability degree of α from X' mean? The definition (A) suggests to compute the supremum of all those r for which (α, r) is derived from X. This leads to a number of problems.

1. It should be natural to think that, the provability degree of a formula α from a set of formulae X would be the truth value of the statement 'α is provable

from X' i.e., 'there is a derivation of α from X'. But expression (A) does not seem to compute this sentence.
2. As 'sup' usually is meant to compute meta-linguistic 'there exists' to make the definition of 'α is provable from X' closer to the expression (A) if we assume that (A) is assumed to compute the sentence 'there is a derivation of (α, r) from X', then also difficulty arises because, $X \vdash (\alpha, r)$ is a two-valued notion. So, how does this 'r' come in the scenario to get counted under 'sup'?
3. Semantically, $X \vdash (\alpha, r)$ means in every model of X, the value of α should be at least r. This fact can not be unfolded at the same level where $X \vdash (\alpha, r)$ lies.

2.3.3 Notion of Proof in the Context of Pavelka's Fuzzy Logic

The notion of proof, as introduced by Pavelka [16], also has some difficulties. Given a fuzzy set $\mathcal{A}$ of formulae, interpreted as axioms and a set $\mathcal{R}$ of rules of inference, an $\mathcal{R}$-proof is defined as a finite non-empty string $\omega = \langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ over $\mathrm{F} \cup (\mathrm{F} \times \{0\}) \cup (\mathrm{F} \times \mathcal{R} \times N^+)$. That is for each ω_i $(i = 1, 2, \ldots, n)$ either ω_i is (x) or $(x, 0)$ or $(x, r, \langle i_1, i_2, \ldots, i_n \rangle)$, where $x = \lceil \omega_i$, the formula under consideration at the i-th term of ω.

If $\omega_i = (x, r, \langle i_1, i_2, \ldots, i_n \rangle)$, $(i = 1, 2, \ldots, n)$ then $x = r'$ $(\lceil \omega_{i_1}, \lceil \omega_{i_2}, \ldots, \lceil \omega_{i_n})$ where r' is the grammatical component of the rule r (see Section 2.3.1). For an $\mathcal{R}$-proof $\omega = \langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ of $\lceil \omega_n$ from a fuzzy set of formulae X there is a function $\widehat{\omega} : L^F \mapsto L$ such that (i) if length of ω is 1, then either $\omega = (x)$ or $(x, 0)$. If $\omega = (x)$ then $\widehat{\omega}(X) = Xx$ i.e. the membership degree of x in the fuzzy subset X and if $\omega = (x, 0)$ then $\widehat{\omega}(X) = \mathcal{A}x$ i.e. the membership degree of x in the fuzzy set of axioms $\mathcal{A}$. (ii) If $\omega = \langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ then

$$\begin{aligned} \widehat{\omega}(X) &= Xx && \text{if } \omega_n = (x) \\ &= \mathcal{A}x && \text{if } \omega_n = (x, 0) \\ &= r''(\widehat{\omega}_{i_1}(X), \widehat{\omega}_{i_2}(X), \ldots, \widehat{\omega}_{i_n}(X)) && \text{if } \omega_n = (x, r, \langle i_1, \ldots, i_n \rangle),\ i_1, \ldots, i_n < n. \end{aligned}$$

So, it can be noticed that the value of $\langle \omega_1, \omega_2, \ldots, \omega_n \rangle$, a proof of $\lceil \omega_n$ from X, only takes care of the value of the last step of the derivation. This does not seem to be the value of the sentence '$\langle \omega_1 \rangle$ is a proof of $\lceil \omega_1$ from X **and** $\langle \omega_1, \omega_2 \rangle$ is a proof of $\lceil \omega_2$ from X **and** ... **and** $\langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ is a proof of $\lceil \omega_n$ from X'.

Like any definition by recursion, in Pavelka's definition of proof also value of each step is computed with the help of the value of a segment of the proof preceeding that particular step. But 'proof' as a whole should refer to the complete chain of steps and the same applies to the value of proof too. In the context of graded consequence value of a proof is not computed by recursion.

2.3.4 Łukasiewicz Fuzzy Propositional Logic

In Łukasiewicz fuzzy propositional logic [1] concepts like $\text{Fuzzy}_{\text{Ł}}$ entailment, degree entailment, n-degree entailment, fuzzy consequence are introduced.

A formula α is a $\text{Fuzzy}_{\text{Ł}}$ entailment of a set of formulae Γ if for every fuzzy truth-value assignment, which is a fuzzy set from the set of all formulae to the set [0, 1], if every member of Γ gets the value 1 then α gets the value 1. It is quite clear that the notion of $\text{Fuzzy}_{\text{Ł}}$ degree entailment is actually the same as the $\models_1$ where the designated set is $\{1\}$, and $\models_1$ is a two-valued concept.

On the other hand, α is said to be a degree entailment of Γ if for any fuzzy truth-value assignment, the infimum of the values assumed by all members of Γ is less or equal to the value obtained by α. This coincides with the definition of $\models_2$ when the lattice meet is taken to be the composition operator conjoining the premises. An argument is said to be degree valid if its premises degree entails its conclusion. So, degree entailment and degree valid both are 'yes/no concepts'.

Then in [1], for an argument which is not degree valid, a notion like approximate degree of validity has been introduced. Given an argument having Γ as the set of premises and α as the conclusion, and a fuzzy truth-value assignment V, a function say d_V, called downward distance, has been defined by
$d_V(\Gamma, \alpha) = \inf_{\gamma \in \Gamma} V(\gamma) - V(\alpha)$, if $\inf_{\gamma \in \Gamma} V(\gamma) - V(\alpha) > 0$
$= 0$, otherwise.
Now the argument is said to be n-degree valid if $n = 1 - \sup_V d_V(\Gamma, \alpha)$. So, here an attempt to attach a value to the meta-linguistic notion 'an argument is valid' is found. But this again proposes a method of computation which does not have any connection with the defining criterion of the concept 'an argument is valid'. Also presence of '$-$' implies that the definition only applies in [0, 1].

The notion of fuzzy consequence defined in [1], is exactly the same as the notion of semantic consequence defined by Pavelka [16]. Given a fuzzy set of formulae Γ, the fuzzy consequence of Γ, denoted by $FC(\Gamma)$, is a fuzzy set such that for any formula α, $FC(\Gamma(\alpha)) = \inf_T\{T(\alpha) : \Gamma \subseteq T\}$. In [7], it has been discussed that in presenting $FC(\Gamma(\alpha))$ according to the scheme (Σ), given in section 2.1, one needs to have two implication operators to compute the meta-linguistic connective 'if-then' present in the notion viz., '[**if** for all T, (for all γ (**if** $\gamma \in \Gamma$ **then** $\gamma \in T$)) **then** $\alpha \in T$]'. So, one may be skeptic in accepting $FC(\Gamma)(\alpha)$ as a proper reading for 'degree of consequence of α from Γ'.

So, in GCT the value of 'α is a consequence of X' happens to be the value of its defining sentence, while the other existing fuzzy logics proposed to attach such a value which cannot be claimed as the value of its defining sentence.

3 Notion of Inconsistency: A Comparative Appraisal

If object level formulae are accepted to assume values other than 0 and 1, then it is quite immediate that a formula and its negation need not be false together. So, one can think of a non-zero threshold for a formula of the form $\alpha \wedge \neg\alpha$. For instance, in Łukasiewicz logic [18] this value is $\frac{1}{2}$. Then what about the notions like, 'for any α, $\{\alpha, \neg\alpha\} \vdash \beta$ for any β' or '$\{\alpha, \neg\alpha\}$ is inconsistent'? Let us briefly revisit the notion of inconsistency in the logics of our present concern.

3.1 Inconsistency in the Theory of Graded Consequence

The classical notion of inconsistency, in the graded context, has been assumed [5] as a fuzzy subset $INCONS$ of $P(F)$. Given any set of formulae X, $INCONS(X)$, read as the inconsistency degree of X, is postulated [5] by the following axioms.
(I1) If $X \subseteq Y$ then $INCONS(X) \leq INCONS(Y)$.

(I2) $INCONS(X \cup \{\neg\alpha\}) * INCONS(X \cup Y) \leq INCONS(X)$ for any $\alpha \in Y$.
(I3) There is some $k > 0$ such that for any α, $INCONS(\{\alpha, \neg\alpha\}) \geq k$.

This notion of graded inconsistency is equivalent to the notion of graded consequence, extended with the axioms (GC4) and (GC5)[5].
(GC4) There is some $k > 0$ such that for any α, $inf_\beta\ gr(\{\alpha, \neg\alpha\} \mid\sim \beta) \geq k$.
(GC5) $gr(X \cup \{\alpha\} \mid\sim \beta) * gr(X \cup \{\neg\alpha\} \mid\sim \beta) \leq gr(X \mid\sim \beta)$.

So, the point to be noted here is that the classical connection between consequence and inconsistency is preserved in this context too, where both the notions of consequence and inconsistency are matters of grades.

3.2 Notion of Inconsistency in Many-Valued/fuzzy Logics

In Pavelka's fuzzy logical system though $C(X)(\alpha)$ has been regarded as the degree to which α is a consequence of X, the notion of consistency (inconsistency) has been introduced absolutely crisply. According to Pavelka [16], a fuzzy set of formulae X is said to be consistent if $C(X) \neq F$, and inconsistent otherwise. So, no question of grade arises here.

On the other hand, in Hajek's fuzzy logic [14], a set of formulae X is said to be inconsistent if $X \vdash 0$ i.e. $X \vdash \overline{0} \supset 1$ or in other words, $X \vdash (\overline{0}, 1)$.

At this point, the question which arises is that if, for some set X of formulae, $X \vdash (\overline{0}, r)$, where $r \neq 1$, whether the underlying logic of Hajek accepts the set X as a partially inconsistent set. As understood from [14], the answer seems to be 'no'. This definition reflects that Hajek also intended to introduce the notion of inconsistency as a two-valued concept. This attitude towards the notion of inconsistency, found in both of these fuzzy logical systems, poses question. Because there are tacit mentions of the terms like, 'degree of consequence', 'provability degree' etc., in both of these systems and if these terms are really genuine in addressing many-valuedness of 'the notion of consequence' then how can they be commensurable to 'the notion of inconsistency' which is a two-valued concept.

4 Solution to Sorites Paradox: A Comparative Appraisal

A sorites paradox involving a vague predicate P can be stated as follows.
One starts with Px_1 and a collection of conditional premises of the form 'if Px_i, then Px_{i+1}', for $1 \leq i \leq n$. Then, by repetitive application of MP, one arrives at the obviously false conclusion Px_n, for some suitably large n.

4.1 Sorites Paradox: In the Context of Tye's Many-Valued Logic

Michael Tye [20] adopts a three-valued semantics framed after Kleene's three-valued logic. The third value is called 'indefinite'. Tye observed that there are borderline bald men with say, n hairs, who would not cease to be bald by addition of one hair on his head. Thus, there is some n, for which both the statements 'a man with n hairs on his head is bald' and 'a man with $n + 1$ hairs on his head is bald' are indefinite. Hence according to Kleene's three-valued matrix for 'if-then' ('$\supset$'), for such n, the conditional statement 'if a man with n hairs on his head is bald then a man with $n+1$ hairs on his head is bald' will be indefinite.

According to Tye, the initial few statements in the array, having the form 'a man with n hairs on his head is bald', where n ranges from 0 to 1,000,000, are true, also the last few statements of the same form are false, and in between somewhere in the array, there are statements that are indefinite. But one could never say where in the array, the statements of the form 'a man with n hairs on his head is bald' cease to be true and become indefinite, and also at which point in the array indefinite statements end and false statements begin.

Tye's approach respects tolerance of a vague predicate to minute changes. But by admitting one of the premises to be non-true he dissolves the paradox instead of giving any solution to the paradoxical situation where starting from true premises, following a valid rule(s) of inference, one arrives at a false conclusion.

4.2 Sorites Paradox: in the Context of Goguens's Fuzzy Logic

As an example of fuzzy logical approach to the sorites paradox, we present Goguen's [12] approach based on fuzzy set theory. According to Goguen, the conditional statements of the form 'if a man with i hairs on his head is bald then a man with j hairs on his head is bald' should provide a way so that from the truth value of 'a man with i hairs on his head is bald' one can derive the truth value of the statement' a man with j hairs on his head is bald'. He suggested to represent the conditional premise by a fuzzy relation $H(i,j)$, read as, 'the relative baldness of a man with j hairs on his head with respect to the baldness of a man with i hairs on his head' so that $H(i,j)$ satisfies the following equation. $B(j) = H(i,j)\ .\ B(i)$, where 'B' denotes a fuzzy set corresponding to 'bald'. That is, $H(i,j) = \frac{B(j)}{B(i)}$. Now, as the fuzzy set B, representing 'bald' is continuous and monotone decreasing in nature, for some k, $H(k-1,k)$ is non-unit. Hence, if $B(0)$ is 1, then for the series of natural numbers, from 0 to 1,000,000, $B(1,000,000) = \prod_{i=1}^{1,000,000} H(i-1,i)$, which is a result of repetitive product of non-unit numbers and that might be close to zero as the number of steps increases. This explains why the conclusion of the sorites appears to be false.

There are two problems in this solution. The first is the same as in Tye's case, where one of the premises is admitted to be non-true. The second is the case where it is admitted that for some k, $H(k-1,k)$ is non-unit, i.e. $B(k-1) < B(k)$. This goes against the idea that a vague predicate is tolerant to mintue changes.

4.3 Sorites Paradox: In the Context of Graded Consequence

According to GCT, solution to the Sorites paradox can be presented as below.

1. A man with 1 hair on his head is bald. ...1

2. A man with 1 hair on his head is bald $\supset$ A man with 2 hairs on his head is bald. ...1

3. A man with 2 hairs on his head is bald. ...$| MP |$

$\vdots$

2×1,000,000 +1. A man with 1,000,000 hairs on his head is bald. ...$| MP |$

Hence from the given premises, the conclusion ‘A man with 1,000,000 hairs on his head is bald’ can be derived to the degree $\mid MP \mid * \mid MP \mid * \ldots * \mid MP \mid$, taking the value associated with each step under consideration. Now, to compute the truth value of the conditional sentence of the form ‘A man with m hairs on his head is bald $\supset$ A man with $m+1$ hairs on his head is bald’ an implication operator, say $\rightarrow_o$ is needed. Any standard fuzzy implication operator satisfies $a \rightarrow_o b \geq b$. Now, to compute the degree of the rule MP, all possible cases of $\{\alpha, \alpha \supset \beta\}$ implies β have to be considered. As T_i’s are the fuzzy subsets interpreting the vague predicate ‘bald’ for some α representing a sentence of the form ‘A man with m hairs on his head is bald’ and some β representing a sentence, say ‘A man with n hairs on his head is bald’ for some $m < n$, $T_i(\alpha) > T_i(\beta)$. So, as $T_i(\alpha) \rightarrow_o T_i(\beta) \geq T_i(\beta)$ we have $T_i(\alpha) \wedge (T_i(\alpha) \rightarrow_o T_i(\beta)) \geq T_i(\beta)$. It is to be noted that not for all $\rightarrow_o$, $a \wedge (a \rightarrow_o b)$ is necessarily b. Hence for some α, β, $\{T_i(\alpha) \wedge (T_i(\alpha) \rightarrow_o T_i(\beta))\} \rightarrow T_i(\beta)$ is not equals to 1. That is, the calculation for the grade of MP given in Section 2.1 indicates that $\mid MP \mid$ is not necessarily 1. So, in this context, as indicated by the calculation above, the value of the derivation of the conclusion approaches to zero as the number of steps increases. And the point to be noted here is that, the conditional statements of the form ‘A man with m hairs on his head is bald $\supset$ A man with $m+1$ hairs on his head is bald’ need not get a non-unit truth value; more specifically, the truth value 1 may be assigned to them, always.

Edgington [9] also embraces a degree-theoretic approach to give an account of reasoning in vague context. What distinguishes her approach from other degree theories, including GCT, is that it uses probability theory as providing a general structure for calculating logical compositions (not necessarily truth functional) of different degrees of verity of sentences. In her approach, each conditional premise of the form ‘$Px_n \supset Px_{n+1}$ has a degree of verity slightly less than clearly true (1). However, as the deduction proceeds small unverities (1 - degree of verity) of the premises mount up to yield a conclusion which is clearly false (0). Yet each step of the argument is valid; because, the unverity of the conclusion never exceeds the sum of the unverities of the premises. This is how Edgington distinguishes arguments, where fall in the values of the conclusion is constrained by the values of the premises, from ‘genuinely invalid’ arguments, where such constraint does not work.

Hence, in opposition to Tye’s many-valued approach, Goguen’s fuzzy approach, and Edgington’s degree-theoretic approach, GCT neither needs to assume one of the premises to be non-true, nor needs to assume existence of a cut off point violating the linguistic rule for vague predicates.

5 Conclusion

In summing up we can say, in many-valued and fuzzy logics, ‘consequence’ is either a crisp notion or it has been assigned a grade which does not seem to be the ‘truth value’ of the concept underlying it, or it fails to preserve the classical consequence-inconsistency connection. GCT makes a point of difference here.

Sorites, a long chain of arguments involving a vague predicate is a paradoxical phenomenon in the context of reasoning in vague/imprecise context. In section 4, we have seen that the idea of a ‘paradox’ is getting distorted in both Tye’s three valued approach and Goguen’s fuzzy set theoretic approach to sorites paradox. Besides, Goguen’s proposal of assigning a non-unit truth value to some of the conditional premises seems to negate that vague predicates are tolerant to minute changes. The same is the case with Edgington’s approach.

Before ending, we would like to quote a line from Parikh [15] who in a different manner proposed a system of logic dealing with vague sentences. As a point of note against the ‘so called’ fuzzy approach to deal with observationality (property of a vague predicate, whose impreciseness cannot even be removed theoretically) he commented “*... we seem to have come no closer to observationality by moving from two valued logic to real valued, fuzzy logic. A possible solution ... is to use continuous valued logic not only for the object language but also for the metalanguage.*”

In addition to the above, the theory of graded consequence insists that the method of assigning grades to the meta concepts needs to respect the underlying meanings of these concepts.

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